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Algebraic Surfaces

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# On a group functor describing invariants of algebraic surfaces

Heiko Dietrich and Primož Moravec

ABSTRACT. Liedtke (2008) has introduced group functors  $K$  and  $\tilde{K}$ , which are used in the context of describing certain invariants for complex algebraic surfaces. He proved that these functors are connected to the theory of central extensions and Schur multipliers. In this work we relate  $K$  and  $\tilde{K}$  to a group functor  $\tau$  arising in the construction of the non-abelian exterior square of a group. In contrast to  $\tilde{K}$ , there exist efficient algorithms for constructing  $\tau$ , especially for polycyclic groups. Supported by computations with the computer algebra system GAP, we investigate when  $K(G, 3)$  is a quotient of  $\tau(G)$ , and when  $\tau(G)$  and  $\tilde{K}(G, 3)$  are isomorphic.

## 1. Introduction

In the study of complex algebraic surfaces it is of interest to find strong invariants which are not too complicated to be useful. Towards this aim, Liedtke [10] introduced group theoretical functors  $K$  and  $\tilde{K}$  that are related to the fundamental groups of the associated Galois closures. More precisely, let  $X$  be a smooth projective surface, fix a generic projection  $f: X \rightarrow \mathbb{P}^2$  of degree  $n$ , and let  $f_{\text{gal}}: X_{\text{gal}} \rightarrow \mathbb{P}^2$  be its Galois closure. Let  $\mathbb{A}^2$  be the complement of a fixed generic line in  $\mathbb{P}^2$ , and set  $X^{\text{aff}} = f^{-1}(\mathbb{A}^2)$  and  $X_{\text{gal}}^{\text{aff}} = f_{\text{gal}}^{-1}(\mathbb{A}^2)$ . It is proved in [10, Theorems 5.1 & 5.2] that  $\pi_1(X_{\text{gal}}^{\text{aff}})$  has images isomorphic to  $\tilde{K}(\pi_1(X^{\text{aff}}), n)$  and to  $K(\pi_1(X^{\text{aff}}), n)$ . It is the constructions of  $K(-, n)$  and  $\tilde{K}(-, n)$  that are central to Liedtke's investigation in [10, 11]. As pointed out in these papers, it is important to have a better understanding of  $\tilde{K}$  in order to describe the above mentioned fundamental groups.

The aim of this work is to extend the group theoretical analysis of the functors  $\tilde{K}$  and  $K$ , and to relate these to a functor  $\tau$  associated with Brown and Loday's construction of the non-abelian tensor square of a group [3]. The latter has applications in topology and K-theory, and can efficiently be computed for several classes of groups, such as polycyclic groups.

In Section 2, we set the notations and give the definitions of  $K(G, n)$ ,  $\tilde{K}(G, n)$ , and  $\tau(G)$ . In Section 3, we elaborate on these and provide explicit descriptions that enable efficient computations for polycyclic groups. In Section 4, we introduce the concept of an AI-automorphism and show that the existence of such an automorphism for a group  $G$  yields a central extension

$$1 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow \tau(G) \longrightarrow K(G, 3) \longrightarrow 1,$$

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similar to the one proved in [10, Theorem 2.2]:

$$1 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow \tilde{K}(G, 3) \longrightarrow K(G, 3) \longrightarrow 1.$$

It is therefore natural to ask when  $\tau(G)$  and  $\tilde{K}(G, 3)$  are isomorphic. In Section 5, we explore this question for several classes of groups. For example, we show that if  $G$  is a finite group and its Schur cover  $H/M = G$  admits an AI-automorphism which acts as inversion on  $M$ , then  $\tau(G) \cong \tilde{K}(G, 3)$ .

In Section 6, we show that  $K(G, 3)$  and  $\tilde{K}(G, 3)$  are closely related to the unramified Brauer group of the field of  $G$ -fixed points in a complex function field. This group is also known as the Bogomolov multiplier  $B_0(G)$ , and has various applications in algebraic geometry, in particular, to Noether's Problem. In Section 7 we comment on our computational experiments with the system GAP [7].

## 2. Definitions and preliminary results

Unless stated otherwise, all groups are finite and written multiplicatively. For a group  $G$  and integer  $n > 0$  we denote by  $G^n$  the direct product of  $n$  copies of  $G$ . We write  $C_n$  for the cyclic group of size  $n$ . The commutator subgroup  $G'$  is the subgroup of  $G$  generated by all commutators  $[g, h] = g^{-1}h^{-1}gh = g^{-1}h^g$  with  $g, h \in G$ . A free presentation for  $G$  is a free group  $F$  with normal subgroup  $N \trianglelefteq F$  such that  $G \cong F/N$ . A polycyclic presentation  $\text{pc}\langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$  for  $G$  is a group presentation with abstract generators  $g_1, \dots, g_n$  and relations  $r_1, \dots, r_m$  that are power or conjugate relations, with the convention that trivial conjugate relations are omitted; see [6, Section 2.1] for details. For example,  $\text{pc}\langle g_1, g_2 \mid g_1^2, g_1^2 \rangle$  describes the Klein 4-group  $\langle g_1, g_2 \mid g_1^2, g_2^2, g_2^{g_1} = g_2 \rangle$ . A group extension of  $A$  by  $B$  is written  $G = B.A$ , meaning that  $A \trianglelefteq G$  with quotient  $G/A = B$ .

**2.1. Liedtke's constructions.** For a group  $G$  and positive integer  $n$ , the group  $K(G, n)$  is the kernel of the map  $G^n \rightarrow G/G'$  that sends an  $n$ -tuple  $(g_1, \dots, g_n)$  to the product of its components modulo the commutator subgroups, that is,

$$K(G, n) = \{(g_1, \dots, g_n) \in G^n : g_1 \cdots g_n \in G'\}.$$

To define the group  $\tilde{K}(G, n)$ , choose a free presentation  $G = F/R$  for  $G$ , and set

$$\tilde{K}(G, n) = K(F, n)/K(N, n)^{F^n},$$

where  $K(N, n)^{F^n}$  is the normal closure of  $K(N, n)$  in  $F^n$ ; if  $n \geq 3$ , then this is simply the normal closure of  $K(N, n)$  in  $K(F, n)$ , see [10, p. 248]. It is shown in [10, Theorem 2.2] that the definition of  $\tilde{K}(G, n)$  does not depend on the choice of presentation for  $G$ .

**2.2. Non-abelian exterior square.** Let  $G$  and  $G^*$  be groups, with isomorphism  $G \rightarrow G^*$ ,  $g \mapsto g^*$ ; we continue to use “ $*$ ” to denote elements and subsets of  $G^*$ . Let  $G \star G^*$  be the free product of  $G$  and  $G^*$ , and, following [16], define  $\nu(G)$  as a quotient group of  $G \star G^*$  via

$$\nu(G) = (G \star G^*) / \langle \{ [x, y^*]^z [x^z, (y^z)^*]^{-1}, [x, y^*]^{(z^*)} [x^z, (y^z)^*]^{-1} : x, y, z \in G \} \rangle^{G \star G^*}.$$

To simplify notation, we identify elements in  $\nu(G)$  with elements in  $G \star G^*$ , keeping in mind that further relations hold in  $\nu(G)$ . If we want to emphasise the parent group, then we sometimes use subscripts at generated groups  $\langle - \rangle_A$  or at commutators  $[-, -]_A$  to indicate that the corresponding structures are to be considered in the group  $A$ . For example, if  $g \in G$  and  $g^* \in G^*$ ,

then  $[g, g^*]_{\nu(G)}$  denotes their commutator in  $\nu(G)$ , not in  $G * G^*$ . With this convention, consider  $\nabla(G) = \langle [x, x^*]_{\nu(G)} : x \in G \rangle$  as a subgroup of  $\nu(G)$ , and define

$$\tau(G) = \nu(G) / \nabla(G).$$

Note that the homomorphism  $G * G^* \rightarrow G^2$ ,  $g_1 h_1^* g_2 h_2^* \dots g_k h_k^* \mapsto (g_1 \dots g_k, h_1 \dots h_k)$ , maps commutators  $[x, y^*]$  to 1, hence it induces short exact sequences

$$\begin{aligned} 1 &\longrightarrow G \otimes G \longrightarrow \nu(G) \xrightarrow{c_\nu} G \times G \longrightarrow 1 \\ 1 &\longrightarrow G \wedge G \longrightarrow \tau(G) \xrightarrow{c_\tau} G \times G \longrightarrow 1 \end{aligned}$$

where  $G \otimes G$  and  $G \wedge G$  are the *non-abelian tensor square* and *non-abelian exterior square* of  $G$ , respectively, see [3]. We conclude with a lemma that will be useful later.

**Lemma 2.1.** *Let  $H/M = G$ . The projection  $H \rightarrow G$  induces epimorphisms  $\beta: \nu(H) \rightarrow \nu(G)$  and  $\gamma: \tau(H) \rightarrow \tau(G)$  whose kernels are*

$$\langle M, M^* \rangle_{\nu(H)} [M, H^*]_{\nu(H)} [H, M^*]_{\nu(H)} \quad \text{and} \quad \langle M, M^* \rangle_{\tau(H)} [M, H^*]_{\tau(H)} [H, M^*]_{\tau(H)}.$$

PROOF. For  $\beta$  this is [16, Proposition 2.5]. Since  $\beta$  maps  $\nabla(H)$  to  $\nabla(G)$ , this induces  $\gamma$ . Note that  $\ker \gamma = \{x \nabla H : x \in \beta^{-1}(\nabla(G))\}$ , and  $\beta^{-1}(\nabla(G)) = \ker \beta \nabla(H)$ , so the claim follows.  $\square$

**2.3. Schur multiplier.** We recall some facts about the Schur multiplier of a group. A *Schur cover* of  $G$  is a group  $H$  such that  $H/M \cong G$  for some  $M \leq H' \cap Z(H)$  isomorphic to the *Schur multiplier*

$$M(G) = H^2(G, \mathbb{C}^\times),$$

see [9, p. 16]. Hopf's formula [9, Theorem 2.4.6] says that if  $F/R = G$  is a free presentation for the finite group  $G$ , then  $M(G)$  is isomorphic to the torsion subgroup of  $(F' \cap R)/[F, R]$ . In particular, if  $G$  is finite, then [9, Theorem 2.7.3] shows that

$$M(G) \cong H_2(G, \mathbb{Z});$$

if  $G$  is abelian, then  $M(G) \cong G \wedge G$ , see [9, Theorem 2.6.7]. By [9, Theorem 2.5.1], the isomorphism type of  $H'$  depends only on  $G$ , and not on the chosen cover  $H$ . By [2, Corollary 2], if  $M(G)$  is finitely generated, then there is an isomorphism

$$G \wedge G \rightarrow H', \quad g \wedge h \rightarrow [g', h'],$$

where  $g', h' \in H$  are lifts of  $g, h \in G$ ; if  $G$  is abelian, then  $H' = M$ . If  $G = \langle g_1, \dots, g_n \rangle$ , then  $H = \langle g'_1, \dots, g'_n \rangle$ : clearly,  $\langle g'_1, \dots, g'_n \rangle M / M = G$ , so every  $x \in H$  can be written as  $x = w_x m_x$  where  $w_x \in \langle g'_1, \dots, g'_n \rangle$  and  $m_x \in M$ . Each  $m \in M$  is a product of commutators  $[x, y]$  in  $H$ ; since  $M \leq Z(H)$ , we have  $[x, y] = [w_x, w_y]$ , so  $M \leq \langle g'_1, \dots, g'_n \rangle$  as well.

### 3. Explicit description

As a first step towards investigating the relation between  $\tau(G)$  and  $\tilde{K}(G, 3)$  we provide a more concrete description of these groups.

**3.1. An explicit description of  $\tau$ .** First we summarise some facts about  $\tau(G)$  and  $\nu(G)$ .

**Lemma 3.1.** *Every  $w \in \nu(G)$  can be written uniquely as  $w = gh^*w'$  with  $w' \in [G, G^*]_{\nu(G)}$ ; similarly in  $\tau(G)$ . Moreover,  $\ker c_\nu = [G, G^*]_{\nu(G)}$  and  $\ker c_\tau = [G, G^*]_{\tau(G)}$ .*

PROOF. Let  $g = g_1h_1^* \cdots g_nh_n^* \in \nu(G)$ . The identities  $h^*g = gh^*[h^*, g]$  and

$$\begin{aligned} [h^*, g]k &= k[h^*, g][h^*, g]^{-1}[h^*, g]^k = k[h^*, g][h^*, g]^{-1}[(h^k)^*, g^k], \\ [h^*, g]k^* &= k^*[h^*, g][h^*, g]^{-1}[h^*, g]^{k^*} = k^*[h^*, g][h^*, g]^{-1}[(h^k)^*, g^k]. \end{aligned}$$

can be used to collect  $g = g_1h_1^* \cdots g_nh_n^* = (g_1 \cdots g_n)(h_1 \cdots h_n)^*w$  with  $w \in [G, G^*]$ . The formula for the kernel of  $c_\nu$  follows from [16, Proposition 2.6]. Clearly,  $c_\tau$  maps commutators  $[x, y^*]$  to 1, so we have  $[G, G^*]_{\tau(G)} \leq \ker c_\tau$ . Conversely, a representative  $w = g_1h_1^* \cdots g_kh_k^* \in G \star G^*$  of an element in the kernel of  $c_\tau$  satisfies  $g_1 \cdots g_k = 1$  in  $G$  and  $h_1^* \cdots h_k^* = 1$  in  $G^*$ . Writing  $w = g_1 \cdots g_k(h_1 \cdots h_k)^*w' = w'$  for some  $w' \in [G, G^*]$ , we get  $\ker c_\tau = [G, G^*]_{\tau(G)}$ . The uniqueness now follows from the exact sequences associated with  $c_\tau$  and  $c_\nu$ .  $\square$

We identify  $G \otimes G = [G, G^*]_{\nu(G)}$  via  $g \otimes h \rightarrow [g, h^*]$ , and  $[G, G^*]_\tau = G \wedge G$  via  $g \wedge h \rightarrow [g, h^*]$ .

**Proposition 3.2.** *The group  $\tau(G)$  is isomorphic to  $G^2.(G \wedge G)$  with multiplication*

$$(a, b; c)(g, h; d) = (ag, bh; (b^h \wedge g^h)c^{gh}d),$$

and derived subgroup  $\tau(G)' \cong (G' \times G').(G \wedge G)$ .

PROOF. By Lemma 3.1, the element  $gh^*w \in \tau(G)$  corresponds to  $(g, h; w) \in G^2.(G \wedge G)$ , and this correspondence defines the multiplication in  $G^2.(G \wedge G)$ . Note that  $c \in G \wedge G$  corresponds to an element of the form  $\prod_i [x_i, y_i^*]$ , and so  $c^g$  and  $c^{(g^*)}$  both correspond to  $\prod_i [x_i^g, (y_i^g)^*]$ . The last claim is [16, Theorem 3.1].  $\square$

**3.2. An explicit description of  $\tilde{K}$ .** The following result is based on [10, Theorem 3.2].

We use the convention that the components of a tuple  $g$  are written  $g_1, g_2, \dots$ , that is,  $g \in G^{n-1}$  is  $g = (g_1, \dots, g_{n-1})$ .

**Proposition 3.3.** *Let  $G$  be a group with Schur cover  $H$  and  $H/M = G$ . The following hold for  $n \geq 3$ .*

- a) *We have  $K(G, n) \cong G^{n-1}.G'$ , where the product of  $u = (g; c)$  and  $v = (h; d)$  in  $G^{n-1}.G'$  is defined as*

$$uv = (gh; cd\alpha(u, v))$$

with  $\alpha(u, v) = (h_{n-1}^{-1}g_{n-1}^{-1} \cdots h_1^{-1}g_1^{-1}cd)^{-1} \cdot (g_{n-1}^{-1} \cdots g_1^{-1}ch_{n-1}^{-1} \cdots h_1^{-1}d) \in G'$ .

- b) *We have  $\tilde{K}(G, n) \cong G^{n-1}.H'$ , where the product of  $u = (g; c)$  and  $v = (h; d)$  in  $G^{n-1}.H'$  is defined as*

$$uv = (gh; cd\alpha(u', v'));$$

here  $\alpha$  is the map defining  $K(H, n)$  as in a) and  $u', v' \in H^{n-1}.H'$  map onto  $u, v \in G^{n-1}.H'$ ; in particular,  $\tilde{K}(G, n) \cong G^2.(G \wedge G)$ .

PROOF. a) Clearly,  $K(G, n) = \{(g_1, \dots, g_{n-1}, g_{n-1}^{-1} \cdots g_1^{-1}d) : g_1, \dots, g_{n-1} \in G, d \in G'\}$ . The isomorphism from  $G^{n-1}.G'$  to  $K(G, n)$  maps  $(g; c) \in G^{n-1}.G'$  to  $(g, g_{n-1}^{-1} \cdots g_1^{-1}c) \in K(G, n)$ ; the definition of  $\alpha$  guarantees that this is an isomorphism.

b) It is shown in [10, Theorem 3.2] that  $\tilde{K}(G, n) \cong K(H, n)/K(M, n)$ , independent of the chosen Schur cover. By a), we have  $K(H, n) \cong H^{n-1}.H'$ , and  $K(M, n) \cong M^{n-1}$  corresponds to the central subgroup  $M^{n-1}.1$  of  $H^{n-1}.H'$ . Note that the multiplication is well-defined since  $M \leq Z(H)$ . Recall that we assume that all groups are finite, so  $G$  is finite and  $H' \cong G \wedge G$  by Section 2.3.  $\square$

**Corollary 3.4.** *If  $H$  has nilpotency class 2, then  $K(H, n) \cong H^{n-1}.H'$  with multiplication*

$$(g; c)(h, d) = (gh; cd \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} [g_i, h_j]).$$

PROOF. Consider  $K(H, n) = H^{n-1}.H'$  with multiplication defined by  $\alpha$  as in Proposition 3.3a), that is, the product of  $u = (g; c)$  and  $v = (h; d)$  in  $H^{n-1}.H'$  is  $uv = (gh; cd\alpha(u, v))$  where

$$\begin{aligned} \alpha(u, v) &= (h_{n-1}^{-1}g_{n-1}^{-1} \cdots h_1^{-1}g_1^{-1}cd)^{-1} \cdot (g_{n-1}^{-1} \cdots g_1^{-1}ch_{n-1}^{-1} \cdots h_1^{-1}d) \\ &= d^{-1}c^{-1}g_1h_1 \cdots g_{n-1}h_{n-1} \cdot g_{n-1}^{-1} \cdots g_1^{-1}ch_{n-1}^{-1} \cdots h_1^{-1}d \\ &= \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} [h_j, g_i^{-1}][c, h_i] = \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} [h_j, g_i^{-1}]; \end{aligned}$$

for the last equations note that  $c \in H' \leq Z(H)$  and  $[h_j, g_i^{-1}] = [h_j, g_i^{-1}]^{h_j} = [g_i, h_j]$ .  $\square$

**3.3. Abelian groups.** For a group  $G$  let  $Z^\wedge(G) = \{g \in G : g \wedge x = 1 \text{ for all } x \in G\}$  be the *epicentre* of  $G$ . Note that  $Z^\wedge(G)$  is equal to the projection of the center of a Schur cover of  $G$  on  $G$ , see [5, p. 254], therefore the next result agrees with [10, Proposition 4.7]. It is shown in [5, Proposition 16(vii)] that there exists  $H$  with  $H/Z(H) \cong G$  if and only if  $Z^\wedge(G) = 1$ .

**Proposition 3.5.** *If  $G$  is an abelian group, then  $\tilde{K}(G, n)$  is isomorphic to the group  $G^{n-1}.(G \wedge G)$  with multiplication*

$$(g; c)(h; d) = (gh; cd \prod_{i=1}^{n-1} g_i \wedge h_i \cdots h_{n-1}).$$

Under this identification,

$$\begin{aligned} Z(\tilde{K}(G, n)) &= \{(u, uy_2, \dots, uy_{n-1}; c) \in G^{n-1}.(G \wedge G) : y_2, \dots, y_{n-1}, u^n \in Z^\wedge(G)\} \\ &\cong Z^\wedge(G)^{n-1} \times (G \wedge G) \times \{u \in G : u^n \in Z^\wedge(G)\} / Z^\wedge(G). \end{aligned}$$

PROOF. Let  $H$  be a Schur cover of  $G$  with  $H/M = G$ . It follows from Corollary 3.4 and Proposition 3.3b) that  $\tilde{K}(G, n) \cong G^{n-1}.H'$  with multiplication

$$(g; c)(h, d) = (gh; cd \prod_{i=1}^{n-1} \prod_{j=i}^{n-1} [g'_i, h'_j]),$$

where each  $g'_i$  and  $h'_j$  is a lift of  $g_i, h_j \in G$  to  $H$ ; note that  $H' = M \leq Z(H)$  and  $H' = M \cong G \wedge G$  since  $G$  is abelian. Recall that  $G \wedge G = \ker c_\tau$ , that is,  $G \wedge G = \langle g \wedge h : x, y \in G \rangle$  with the convention  $g \wedge h = [g, h^*]_{\tau(G)}$ . In particular, if  $[g', h']_H \in H$  where  $g', h' \in H$  are lifts of  $g, h \in G$ , then  $H' \cong G \wedge G$  via  $[g', h'] \mapsto g \wedge h$ . The first claim follows.

If  $(a; c) \in Z(\tilde{K}(G, n))$ , then the following is equal for all  $(g; d) \in \tilde{K}(G, 3)$ :

$$\prod_{i=1}^{n-1} a_i \wedge g_i \cdots g_{n-1} = \prod_{i=1}^{n-1} g_i \wedge a_i \cdots a_{n-1}.$$

Assuming  $g$  has only one nontrivial entry  $g_i = h$ , this forces

$$a_1 \cdots a_{i-1} a_i^2 a_{i+1} \cdots a_{n-1} \wedge h = 1 \quad \text{for all } h \in G \text{ and } i \in \{1, \dots, n-1\}.$$

Thus, each  $z_i = a_1 \dots a_{i-1} a_i^2 a_{i+1} \dots a_{n-1}$  lies in  $Z^\wedge(G)$ ; now  $z_{i-1}^{-1} z_i = a_{i-1}^{-1} a_i$  shows that each  $a_i = a_1 y_i$  for some  $y_i \in Z^\wedge(G)$ . Now  $z_1 \in Z^\wedge(G)$  yields  $a_1^n \in Z^\wedge(G)$ . Conversely, it is easy to check that every such element yields a central  $(a; c)$ ; note that  $a^{n-i} = a^i z$  for some  $z \in Z^\wedge(G)$ .  $\square$

**Proposition 3.6.** *If  $G$  is an abelian group, then  $\tau(G)$  is isomorphic to the group  $G^2 \cdot (G \wedge G)$ , where the multiplication is given by  $(g_1, g_2; c)(h_1, h_2; d) = (g_1 h_1, g_2 h_2; cd(g_2 \wedge h_1))$ . Under this identification,  $Z(\tau(G)) = \{(a, b; c) : a, b \in Z^\wedge(G), c \in G \wedge G\} \cong Z^\wedge(G)^2 \times (G \wedge G)$ .*

PROOF. The first claim follows from Proposition 3.2. As above,  $(a, b; c) \in Z(\tau(G))$  if and only if  $b \wedge g = h \wedge a$  for all  $g, h \in G$ . In particular, it follows that  $ab \in Z^\wedge(G)$ , so  $b = a^{-1}z$  for some  $z \in Z^\wedge(G)$ . Now  $b \wedge g = h \wedge a$  implies  $a \wedge hg^{-1} = 1$  for all  $g, h \in G$ , thus  $a \in Z^\wedge(G)$ , and so also  $b \in Z^\wedge(G)$ . Conversely, every such  $(a, b; c)$  lies in the centre; the claim follows.  $\square$

#### 4. Relating $\tau(G)$ with $\tilde{K}(G, 3)$ and $K(G, 3)$

The aim of this section is to relate  $\tau(G)$  with  $\tilde{K}(G, 3)$ . As a first step, we first consider a construction of an epimorphism  $\tau(G) \rightarrow K(G, 3)$ . Our construction requires an automorphism of  $G$  which acts as inversion on the abelianisation of  $G$ .

**4.1. AI-automorphisms.** An automorphism  $\alpha \in \text{Aut}(G)$  of a group  $G$  is an *AI-automorphism* if it induces the inversion automorphism on the abelianisation  $G/G'$ ; this is not to be confused with an *IA-automorphism* introduced by Bachmuth (1966), which is an automorphism that induces the identity on the abelianisation. Clearly, the composition of two AI-automorphisms is an IA-automorphism; for abelian groups the only AI-automorphism is inversion.

**Example 4.1.** Let  $F$  be a free group on  $X$ . The map  $X \rightarrow X$  given by  $x \mapsto x^{-1}$  for all  $x \in X$  induces an AI-automorphism  $\iota_F$  of  $F$ . If a group  $G$  is given by a free presentation  $G = F/N$  and  $\iota_F(N) = N$ , then  $\iota_F$  induces an AI-automorphism of  $G$ . Note that if  $F/N$  is abelian, then  $F' \leq N$ , hence  $\iota_F(N) = N$  and  $\iota_F$  induces inversion on  $G$ . If  $\iota_F(N) \neq N$ , then define  $M = \iota_F(N)N \trianglelefteq F$ . By definition,  $\iota_F(M) = M$ , and  $F/M$  is the largest quotient of  $G$  on which  $\iota_F$  induces an AI-automorphism. In particular, every group  $G$  has such a quotient since  $\iota_F$  induces inversion on  $F/F'N \cong G/G'$ . We give two examples. First, the dihedral group of order  $2n$  can be defined as  $D_{2n} = F/R$  where  $F$  is free on  $\{r, m\}$  and  $N$  is the normal closure of  $\{r^n, m^2, r^m r\}$ . Clearly,  $\iota_F(r^n) = (r^{-1})^n$  and  $\iota_F(m^2) = m^{-2}$  lie in  $N$ ; moreover,  $(\iota_F(r^m r)^{-1})^m = (r r^{m-1})^m = r^m r \in N$ , hence  $\iota_F$  induces an AI-automorphism on  $F/R$ . Second, consider  $G = F/N$  where  $F$  is free on  $\{g, h\}$  and  $N$  is the normal closure of  $\{g^4, h^5, h^9 h^2\}$ , that is,  $G$  is a semidirect product  $C_4 \rtimes C_5$ . A direct computation shows that  $G$  does not admit an AI-automorphism, which implies that  $\iota_F(N) \neq N$ . If  $M$  is the normal closure of  $\{g^4, h^5, (h^{-1})^{(g^{-1})} h^{-2}\}$ , then  $\iota_F(M) = M$ , and  $G/M \cong C_4$  is the largest quotient of  $G$  on which  $\iota_F$  induces an AI-automorphism.

**Example 4.2.** Let  $\alpha \in \text{Aut}(G)$  be an automorphism which inverts every element of a generating set  $X$  of  $G$ . Such an automorphism is called *GI-automorphism* in [1], where GI can be interpreted as ‘‘generator inversion’’. (Another interpretation is that GI stands for ‘‘generating-involutions’’ because  $\langle \alpha \rangle \times G$  is generated by involutions  $\{(\alpha, x) : x \in X\}$ .) Clearly, every GI-automorphism is an AI-automorphism. To give an example, consider the alternating group  $\text{Alt}_n$  of rank  $n \geq 3$ : Conjugation by the transposition  $(1, 2)$  defines an automorphism  $\alpha$  of  $\text{Alt}_n$



that inverts every element of the generating set  $\{(1, 2, 3), (1, 2, 4), \dots, (1, 2, n)\}$ ; thus  $\alpha$  is a GI- and AI-automorphism.

**4.2. An epimorphism.** Suppose  $G$  has an AI-automorphism  $\alpha$ ; we use  $\alpha$  to construct  $K(G, 3)$  as a quotient of  $\tau(G)$ . Note that the homomorphism

$$G \star G^* \rightarrow G^3, \quad g_1 h_1^* \dots g_k h_k^* \mapsto (g_1 \dots g_k, h_1 \dots h_k, \alpha(g_1 h_1 \dots g_k h_k))$$

maps commutators  $[x, x^*]$  to 1; since the above map forgets “\*”, it also maps the relations of  $\tau(G)$  to 1. Thus there is an induced homomorphism

$$\Phi_\alpha: \tau(G) \rightarrow G^3.$$

**Remark 4.3.** Recall from above that  $G \wedge G = [G, G^*]_{\tau(G)}$ , and now let

$$\kappa: [G, G^*]_{\tau(G)} \rightarrow G', \quad \prod_i [x_i, y_i^*] \mapsto \prod_i [x_i, y_i].$$

It is shown in [12] that  $\ker \kappa$  is central in  $[G, G^*]_{\tau(G)}$  and isomorphic to  $H_2(G, \mathbb{Z})$ .

**Theorem 4.4.** *If  $\alpha \in \text{Aut}(G)$  is an AI-automorphism, then*

$$\text{im } \Phi_\alpha = K(G, 3) \quad \text{and} \quad \ker \Phi_\alpha = \ker \kappa = H_2(G, \mathbb{Z}).$$

PROOF. The inclusion  $\text{im } \Phi_\alpha \leq K(G, 3)$  follows immediately from the definition and the fact that  $\alpha$  is an AI-automorphism. If  $(g, h, k) \in K(G, 3)$ , then  $k = h^{-1}g^{-1}c$  for some  $c \in G'$ . Note that  $\Phi_\alpha$  maps  $gh^*$  to  $(g, h, \alpha(gh)) \in K(G, 3)$ , and  $\alpha(gh) = h^{-1}g^{-1}d$  for some  $d \in G'$ , thus

$$\Phi_\alpha(gh^*)^{-1}(g, h, k) = (1, 1, d^{-1}c);$$

now  $d^{-1}c = \prod_i [x_i, y_i] \in G'$ , and so  $(1, 1, d^{-1}c) = \Phi_\alpha(\prod_i [\alpha^{-1}(x_i), (\alpha^{-1}(y_i))^*])$ . This shows that  $(g, h, k) \in \text{im } \Phi_\alpha$ , thus  $K(G, 3) \leq \text{im } \Phi_\alpha$ . Now we consider the kernel. Note that

$$\ker \Phi_\alpha = \{g_1 h_1^* \dots g_k h_k^* : g_1 \dots g_k = h_1 \dots h_k = (g_1 h_1) \dots (g_k h_k) = 1\}.$$

If  $w = g_1 h_1^* \dots g_k h_k^* \in \ker \Phi_\alpha$ , then Lemma 3.1 allows us to rewrite  $w$  as

$$w = g_1 \dots g_k (h_1 \dots h_k)^* w' = w'$$

for some  $w' = \prod_i [x_i, y_i^*] \in [G, G^*]$ ; mapping this under  $\kappa$  yields  $\kappa(w) = \kappa(w') = \prod_i [x_i, y_i]$ . If we use the above rewriting process of  $w$  in the opposite direction on  $\kappa(w)$ , then we get  $w$  without all “\*”, that is,  $\kappa(w) = g_1 h_1 \dots g_k h_k$ ; since this is 1 by assumption,  $w \in \ker \kappa$ . Conversely, let  $w \in \ker \kappa$ , that is,  $w = \prod_i [g_i, h_i^*] \in [G, G^*]_{\tau(G)}$  with  $\prod_i [g_i, h_i] = 1$ . Writing  $w$  as  $w = \prod_i g_i^{-1} (h_i^{-1})^* g_i h_i^*$  and applying  $\Phi_\alpha$  shows that  $\Phi_\alpha(w) = (1, 1, \alpha([g_1, h_1] \dots [g_k, h_k])) = (1, 1, 1)$ , hence  $\ker \kappa \leq \ker \Phi_\alpha$ . In conclusion,  $\ker \Phi_\alpha = \ker \kappa = H_2(G, \mathbb{Z})$ , as claimed.  $\square$

We have proved:

**Corollary 4.5.** *The existence of an AI-automorphism of  $G$  yields a central extension*

$$1 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow \tau(G) \longrightarrow K(G, 3) \longrightarrow 1.$$

**Remark 4.6.** It is proved in [10, Theorem 2.2] that there is a central extension

$$1 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow \tilde{K}(G, 3) \longrightarrow K(G, 3) \longrightarrow 1.$$

It seems natural to ask when  $\tau(G) \cong \tilde{K}(G, 3)$ . The next proposition shows that the lack of AI-automorphisms may prevent this, see Example 7.1 below for more evidence supporting this:

**Proposition 4.7.** *If  $G = C_n \rtimes C_m$  with  $n \geq 3$  is a Frobenius group with Frobenius kernel  $C_m$ , then  $G$  does not have AI-automorphisms and  $\tau(G) \not\cong \tilde{K}(G, 3)$ . In particular, there is no epimorphism  $\tau(G) \rightarrow K(G, 3)$  as in Corollary 4.5.*

PROOF. Let  $G = \langle g, u \rangle$ , where  $g$  and  $h$  generate  $C_n$  and  $C_m$ , respectively. By [8, Satz V.8.5], every nontrivial element in  $C_n$  acts fixed-point freely on  $C_m$ , meaning that only the identity is fixed. Now [8, Satz 8.3 & 8.10] imply that  $m$  is a prime with  $n \mid m - 1$ , and  $G' = C_m$ . Assume, for a contradiction, that  $\alpha$  is an AI-automorphism of  $G$ . Write  $\alpha(u) = u^y$  and  $\alpha(g) = g^{-1}v$  with  $v \in C_m$ . Moreover, let  $u^g = u^x$  and  $u^{(g^{-1})} = u^{\bar{x}}$  where  $x\bar{x} \equiv 1 \pmod{m}$ . Note that  $[g, u] = u^{1-x}$ , and mapping this under  $\alpha$  yields  $\alpha(u^{1-x}) = u^{y(1-x)}$  and  $\alpha([g, u]) = [g^{-1}v, u^y] = [g^{-1}, u^y] = u^{y(1-\bar{x})}$ . This forces  $y(1-x) \equiv y(1-\bar{x}) \pmod{m}$  and so  $x \equiv \bar{x} \pmod{m}$  since  $m$  is prime. Now  $x\bar{x} \equiv 1 \pmod{m}$  implies that  $g^2$  has nontrivial fixed points, which is not possible since  $g^2 \neq 1$ .

By [9, Theorem 2.11.3], together with [8, Satz V.8.9b], we have that  $M(G) = 1$ . Thus, Remarks 4.6 and 4.3 show that  $\tilde{K}(G, 3) = K(G, 3)$  and  $G \wedge G \cong G' = \langle u \rangle$ . Note that  $\tau(G) = G^2 \cdot G'$  with  $G' = G \wedge G$  is generated by  $g_1 = (g, 1; 1)$ ,  $h_1 = (h, 1; 1)$ ,  $g_2 = (1, g; 1)$ ,  $h_2 = (1, h; 1)$ ,  $k = (1, 1; h)$ , which allows us to determine a polycyclic presentation

$$\begin{aligned} \tau(G) = \text{pc} \langle g_1, h_1, g_2, h_2, k \mid & g_1^n, g_2^n, h_1^m, h_2^m, k^m, k^{g_1} = k^x, h_1^{g_1} = h_1^x, k^{g_2} = k^x, \\ & h_2^{g_2} = h_2^x, g_2^{h_1} = g_2 k \rangle; \end{aligned}$$

recall that unspecified commutators between generators are trivial. Using the generating set  $g_1 = (g, 1, g^{-1})$ ,  $h_1 = (h, 1, h^{-1})$ ,  $g_1 = (1, g, g^{-1})$ ,  $h_2 = (1, h, h^{-1})$ ,  $k = (1, 1, h)$  of  $K(G, 3)$  shows that  $K(G, 3)$  is given by the following presentation

$$\begin{aligned} \text{pc} \langle g_1, h_1, g_2, h_2, k \mid & g_1^n, g_2^n, h_1^m, h_2^m, k^m, h_1^{g_1} = h_1^x k^{x-\bar{x}}, h_2^{g_1} = h_2 k^{1-\bar{x}}, k^{g_1} = k^{\bar{x}}, \\ & g_2^{h_1} = g_2 k^{\bar{x}-1}, h_2^{g_2} = h_2^x k^{x-\bar{x}}, k^{g_2} = k^{\bar{x}} \rangle. \end{aligned}$$

In both cases, the derived subgroup is elementary abelian, generated by  $\{h_1, h_2, k\}$ . It follows from the presentations that  $\tau(G)$  and  $K(G, 3)$  act on their derived subgroups as  $\rho_T, \rho_K \leq \text{GL}_3(m)$ , where

$$\rho_T = \left\langle \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \right\rangle \quad \text{and} \quad \rho_K = \left\langle \begin{pmatrix} x & 0 & x-\bar{x} \\ 0 & 1 & 1-\bar{x} \\ 0 & 0 & \bar{x} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1-\bar{x} \\ 0 & x & x-\bar{x} \\ 0 & 0 & \bar{x} \end{pmatrix} \right\rangle.$$

Since  $x^2 \not\equiv 1 \pmod{m}$ , we have  $\rho_K \leq \text{SL}_3(m)$  but  $\rho_T \not\leq \text{SL}_3(m)$ . Thus these groups are not conjugate in  $\text{GL}_3(m)$ , which implies that  $\tau(G) \not\cong K(G, 3)$ . Indeed, one can also verify that  $\langle k \rangle$  is characteristic in  $\tau(G)$ , but not in  $K(G, 3)$ .  $\square$

## 5. Some isomorphisms

On the positive side, there is a strong evidence that  $\tau(G)$  is closely related to  $\tilde{K}(G, 3)$ , see also Section 7. The next theorem is a useful tool for establishing various isomorphisms.

**Theorem 5.1.** *Let  $H$  be a Schur cover of  $G$  with  $H/M = G$ . If  $H$  admits an AI-automorphism whose restriction to  $M$  is inversion, then  $\tau(G) \cong \tilde{K}(G, 3)$ .*

PROOF. If  $\alpha$  is the said AI-isomorphism, then  $\Phi_\alpha: \tau(H) \rightarrow K(H, 3)$  is an epimorphism with kernel  $H_2(H, \mathbb{Z})$ , see Corollary 4.5. It is shown in [10, Theorem 3.2] that  $\tilde{K}(G, n)$  is isomorphic to  $K(H, n)/K(M, n)$ , so we obtain an epimorphism  $\tau(H) \rightarrow \tilde{K}(G, n)$ . By Lemma 2.1, the projection  $H \rightarrow G$  induces an epimorphism  $\gamma: \tau(H) \rightarrow \tau(G)$  with kernel

$(\langle M, M^* \rangle [M, H^*] [H, M^*])_{\tau(H)}$ . We can construct an induced epimorphism  $\tau(G) \rightarrow \tilde{K}(G, 3)$  if  $\Phi_\alpha(\ker \gamma) \leq K(M, 3)$ . If  $m \in M$ , then  $\Phi_\alpha(m) = (m, 1, \alpha(m))$ , which lies in  $K(M, 3)$  since  $\alpha(m) = m^{-1}$  by assumption; similarly for  $m^* \in M^*$ . If  $[m, h^*]$  is a generator of  $[M, H^*]$ , then this is mapped under  $\Phi_\alpha$  to  $(1, 1, \alpha([m, h^*])) = (1, 1, 1)$  since  $M \leq Z(H)$ ; similarly for elements in  $[H, M^*]$ . This implies the claim.  $\square$

**Proposition 5.2.** *Let  $G$  be an extra-special  $p$ -group.*

- a) *If  $|G| = p^3$ , and  $p = 2$  or  $\exp(G) = p$ , then  $\tau(G) \cong \tilde{K}(G, 3)$ .*
- b) *If  $p > 2$  and  $\exp(G) = p^2$ , then  $G$  does not have an AI-automorphism.*

PROOF. a) For  $p = 2$  the claim can be checked by a direct computation, so let  $p > 2$  and suppose  $G$  is given by the polycyclic presentation  $G = \text{pc}\langle g_1, g_2, g_3 \mid g_1^p, g_2^p, g_3^p, [g_2, g_1] = g_3 \rangle$ . As the Schur multiplier is isomorphic to  $C_p^2$ , it is straightforward to verify that the group

$$H = \text{pc}\langle h_1, h_2, h_3, h_4, h_5 \mid h_1^p, h_2^p, h_3^p, h_4^p, h_5^p, [h_2, h_1] = h_3, [h_3, h_1] = h_4, [h_3, h_2] = h_5 \rangle,$$

is a Schur cover of  $G$  with  $H/M = G$  for  $M = \langle h_4, h_5 \rangle \cong C_p^2$ . The elements  $h_1^{-1}h_3$ ,  $h_2^{-1}h_3^{-1}$ ,  $h_3$ ,  $h_4^{-1}$ ,  $h_5^{-1}$  satisfy the relations of  $H$ , so von Dyck's Theorem [15, 2.2.1] shows that  $(h_1, h_2, h_3, h_4, h_5) \mapsto (h_1^{-1}h_3, h_2^{-1}h_3^{-1}, h_3, h_4^{-1}, h_5^{-1})$  extends to an automorphism  $\alpha$  of  $H$ . This is an AI-automorphism of  $H$  that inverts elements of  $M$ , so Theorem 5.1 proves the claim.

b) First consider  $|G| = p^3$ . We can define  $G = \text{pc}\langle g_1, g_2, g_3 \mid g_1^p = g_3, g_2^p, g_3^p, [g_2, g_1] = g_3^{-1} \rangle$  with  $G' = \langle g_3 \rangle$ . Suppose  $\alpha \in \text{Aut}(G)$  is an AI-automorphism. Since  $G' = Z(G)$ , we have  $\alpha(g_3^{-1}) = [\alpha(g_2), \alpha(g_1)] = [g_2^{-1}, g_1^{-1}] = [g_2, g_1]^{(-1)^2} = g_3^{-1}$ , so  $\alpha(g_3) = g_3$ . Now  $g_3 = \alpha(g_1)^p = g_1^{-p} = g_3^{-1}$  forces  $|g_3| = 2$ , a contradiction. Thus,  $G$  has no AI-automorphism. If  $|G| = p^{1+2n}$ , then  $G$  is a central product of  $n$  extra-special groups of size  $p^3$ , at least one of exponent  $p^3$ , see [8, Satz III.13.7]. The same argument shows that  $G$  has no AI-automorphisms.  $\square$

Next, for  $n \geq 1$  we consider the generalised quaternion group  $Q_n$  and dihedral group  $D_n$  of order  $4n$  and  $2n$ , respectively, which are defined as

$$Q_n = \langle a, b \mid a^{2n}, b^2 = a^n, a^b = a^{-1} \rangle \quad \text{and} \quad D_n = \langle a, b \mid a^n, b^2, a^b = a^{-1} \rangle.$$

**Proposition 5.3.** *We have  $\tau(Q_n) \cong \tilde{K}(Q_n, 3)$  and  $\tau(D_n) \cong \tilde{K}(D_n, 3)$ .*

PROOF. For  $Q_1 = C_4$  and  $D_1 = C_2 \times C_2$  the claim follows from a direct computation, so let  $n \geq 2$ . It follows from [9, Example 2.4.8] that  $M(Q_n) = 1$ , thus  $Q_n$  is a Schur cover of  $Q_n$ . Note that  $\{a^{-1}, b^{-1}\}$  also satisfies the relations of  $Q_n$ , so  $(a, b) \mapsto (a^{-1}, b^{-1})$  extends to a GI-automorphism of  $Q_n$  by von Dyck's Theorem. Now  $\tau(Q_n) \cong \tilde{K}(Q_n, 3)$  by Theorem 5.1. Let  $H$  be a Schur cover of  $D_n$  with  $H/M = D_n$ . By [9, Proposition 2.11.4], we have  $M = 1$  and  $H = D_n$  if  $n$  is odd, and  $M = C_2$  and  $H = Q_n$  otherwise. As seen above and in Example 4.1, the group  $H$  admits an AI-automorphism which necessarily stabilises  $M$ ; recall that  $n \geq 2$ , so  $M \cong Z(Q_n) = \langle a^n \rangle$  or  $M = 1$ . Again, the claim follows with Theorem 5.1.  $\square$

**Proposition 5.4.** *Let  $G$  be a perfect group. If the exponent of  $M(G)$  divides 2, then  $\tau(G) \cong \tilde{K}(G, 3)$ .*

PROOF. Let  $H$  be a Schur cover of  $G$  with  $H/M \cong G$ . Let  $\pi: H \rightarrow G$  be the natural projection. Note that  $H$  is perfect as well: this follows by observing that every  $\pi(h)$  with  $h \in H$  can be written as a product of commutators in  $G$ , hence the same holds for  $h$  modulo  $M$ . Since  $M \leq H'$ , we have  $H = H'$ . The identity automorphism of  $H$  is an AI-automorphism. Clearly it acts as inversion on  $M$  since  $\exp(M)$  divides 2. Now Theorem 5.1 proves the claim.  $\square$

**Proposition 5.5.** *We have  $\tau(\text{Sym}_n) \cong \tilde{K}(\text{Sym}_n, 3)$  and  $\tau(\text{Alt}_n) \cong \tilde{K}(\text{Alt}_n, 3)$ .*

PROOF. By [9, Theorem 2.12.3], the Schur multiplier of  $\text{Sym}_n$  is cyclic of order 2 for  $n \geq 4$ , and trivial otherwise, and a Schur cover for  $\text{Sym}_n$  is

$$H_n = \langle g_1, g_2, \dots, g_{n-1}, z \mid g_i^2 = (g_j g_{j+1})^3 = (g_k g_l)^2 = z, z^2 = [g_i, z] = 1 \\ \text{for } 1 \leq i \leq n-1, 1 \leq j \leq n-2, k \leq l-2 \rangle.$$

Note that the generators  $g_1^{-1}, \dots, g_{n-1}^{-1}, z^{-1}$  also satisfy the relations of  $H_n$ , so von Dyck's Theorem shows that there is a corresponding GI-automorphism of  $H_n$ . Note that  $M = \langle z \rangle$  satisfies  $M \cong M(\text{Sym}_n) \cong C_2$  and  $H_n/M \cong \text{Sym}_n$ . The given GI-automorphism acts as inversion on  $M$ , so the claim for  $\text{Sym}_n$  follows by Theorem 5.1. The proof for the alternating groups follows along the same lines using [9, Theorem 2.12.5].  $\square$

The next result shows that Theorem 5.1 cannot be applied to abelian groups  $G$  in general. Recall that if  $M$  is a trivial  $G$ -module of exponent 2, then a 2-coboundary in  $B^2(G, M)$  is a function  $\delta: G \times G \rightarrow M$  such that  $\delta(g, h) = \kappa(gh)\kappa(g)\kappa(h)$  for all  $g, h \in G$  and some map  $\kappa: G \rightarrow M$ .

**Proposition 5.6.** *Let  $G$  be an abelian group and let  $H$  be a Schur cover of  $G$  with  $H/M = G$ . Then  $H$  admits an AI-automorphism whose restriction to  $M$  is inversion if and only if  $G$  is 2-group,  $M$  has exponent 2, and the map  $G \times G \rightarrow G \wedge G$  defined by  $(g, h) \mapsto g \wedge h$  is a 2-coboundary; in particular, any such AI-automorphism has order 2.*

PROOF. Since  $G$  is abelian,  $H' \leq M$ . Now  $M \leq H' \cap Z(H)$  implies  $M = H' \leq Z(H)$ . First suppose that  $H$  admits an AI-automorphism, say  $\alpha$ , whose restriction to  $M$  is inversion. Then every  $h \in H$  can be written as  $\alpha(h) = h^{-1}c_h$  for some  $c_h \in H'$ . Now

$$h^{-1}g^{-1}c_{gh} = \alpha(gh) = \alpha(g)\alpha(h) = g^{-1}c_g h^{-1}c_h = h^{-1}g^{-1}[g^{-1}, h^{-1}]c_g c_h$$

implies that  $c_{gh} = [g^{-1}, h^{-1}]c_g c_h$  for all  $g, h \in H$ . Note that  $[g, h] = [g^{-1}, h^{-1}]^{gh} = [g^{-1}, h^{-1}]$  since  $H'$  is central, so  $c_{gh} = c_g c_h [g, h]$ . Moreover,  $1 = c_1 = c_{gg^{-1}}$  yields  $c_{g^{-1}} = (c_g)^{-1}$ . This can be used to show that  $\alpha^{2n+1}(g) = g^{-1}c_g^{2n+1}$  and  $\alpha^{2n}(g) = g c_g^{-2n}$  for all  $g \in H$  and  $n \geq 1$ . If  $G$  has odd order, then  $m = |M|$  is odd, so  $\alpha^m(g) = g^{-1}$  describes an isomorphism of  $H$ . This is not possible as  $H$  is non-abelian. By [9, Lemma 2.9.1], the same contradiction can be reached if  $G$  has even order but a nontrivial Sylow subgroup of odd order. So  $G$  is an abelian 2-group, and since

$$[h, g] = \alpha([g, h]) = [\alpha(g), \alpha(h)] = [g^{-1}c_g, h^{-1}c_h] = [g^{-1}, h^{-1}] = [g, h]^{h^{-1}g^{-1}} = [g, h]$$

for all  $g, h \in H$ , we must have that  $H' = M$  has exponent 2. Thus,  $\alpha$  is the identity on  $M$ , and so  $\alpha^2(h) = \alpha(h^{-1}c_h) = h c_{h^{-1}} c_h = h$  for all  $h \in H$  proves that  $\alpha$  has order 2. Note also that  $[g, h] = c_{gh} c_g c_h$ . The map  $\gamma: H \times H \rightarrow H'$ ,  $(g, h) \mapsto [g, h]$ , is a 2-cocycle in  $Z^2(H, H')$  since for all  $g, h, k \in H$  we have  $\gamma(g, hk)\gamma(h, k) = \gamma(gh, k)\gamma(g, h)$ . Since  $H'$  is central,  $\gamma$  induces a 2-cocycle  $\delta \in Z^2(G, H')$ . Since  $G$  is abelian, an isomorphism  $G \wedge G \rightarrow H'$  is given by  $g \wedge h \rightarrow [g', h']$ , where  $g', h' \in H$  are lifts of  $g, h \in G$ . This shows that the induced 2-cocycle  $\delta$  lies in  $Z^2(G, G \wedge G)$  and  $\delta(g, h) = g \wedge h$  for all  $g, h \in G$ . Recall that if  $h \in H$  and

$z \in H'$ , then  $\alpha(h) = h^{-1}c_h$  and  $(hz)^{-1}c_{hz} = \alpha(hz) = \alpha(h)\alpha(z) = h^{-1}c_h z$ , which shows that  $c_{hz} = c_h$ . Thus for  $g \in G$  we can define  $\kappa(g) = c_{g'}$  where  $g' \in H$  is a lift of  $g$ . This shows that  $\delta(g, h) = \kappa(gh)\kappa(g)\kappa(h)$ , that is,  $\delta$  is a 2-coboundary in  $B^2(G, G \wedge G)$ .

Conversely, let  $G$  be an abelian 2-subgroup with  $G \wedge G$  of exponent 2 such that  $\delta(g, h) = g \wedge h$  defines a 2-coboundary in  $B^2(G, G \wedge G)$ , say  $g \wedge h = \delta(g, h) = \kappa(gh)\kappa(g)\kappa(h)$  for some map  $\kappa: G \rightarrow G \wedge G$ . Let  $H$  be a Schur cover of  $G$  with natural projection  $\pi: H \rightarrow G$ , such that  $M = \ker \pi$  satisfies  $M = H' \leq Z(H)$ . Note that under the isomorphism  $H' \rightarrow G \wedge G$ ,  $[h, k] \mapsto \pi(h) \wedge \pi(k)$  we have  $[h, k] = \delta(\pi(h), \pi(k)) = \kappa(\pi(hk))\kappa(\pi(h))\kappa(\pi(k))$ . Now define  $\alpha \in \text{Aut}(H)$  by  $\alpha(h) = h^{-1}c_h$  where  $c_h = \kappa(\pi(h))$ ; note that

$$\alpha(hk) = k^{-1}h^{-1}c_{hk} = h^{-1}k^{-1}[k^{-1}, h^{-1}]c_{hk} = h^{-1}k^{-1}[k, h]c_{hk} = h^{-1}c_h k^{-1}c_k = \alpha(h)\alpha(k),$$

so  $\alpha$  is indeed a homomorphism. Clearly,  $\alpha$  acts as inversion (that is, as identity) on  $M$ , and as inversion on  $H/M$ . This proves the claim.  $\square$

**Proposition 5.7.** *If  $G$  is an abelian 2-group such that  $G \wedge G$  has exponent 2, then  $\tau(G) \cong \tilde{K}(G, 3)$ .*

PROOF. We use Propositions 3.5 and 3.6 and identify

$$\begin{aligned} \tilde{K}(G, 3) &= G^2.(G \wedge G) \quad \text{with} \quad (a, b; c)(d, e; f) = (ad, be; cf(a \wedge de)(b \wedge e)), \\ \tau(G) &= G^2.(G \wedge G) \quad \text{with} \quad (a, b; c)(d, e; f) = (ad, be; cf(b \wedge d)). \end{aligned}$$

Let  $G = C_{2^{k_1}} \times \dots \times C_{2^{k_n}}$  and write  $a \in G$  as  $a = x_1^{a_1} \dots x_n^{a_n}$ , where each  $x_i$  generates  $C_{2^{k_i}}$ ; then

$$N = \{(x_1, 1, 1), \dots, (x_n, 1, 1), (1, x_1, 1), \dots, (1, x_n, 1), (1, 1, x_i \wedge x_j) : i < j\}$$

generates  $\tilde{K}(G, 3)$  and  $\tau(G)$ . We now show that mapping the generating set  $N$  of  $\tau(G)$  to the generating set  $N$  of  $\tilde{K}(G, 3)$  defines an isomorphism  $\psi: \tau(G) \rightarrow \tilde{K}(G, 3)$ . Note that the image of  $(a, b; c) \in \tau(G)$  under  $\psi$  can be computed by decomposing  $(a, b; c)$  in  $\tau(G)$  as

$$(a, b; c) = \prod_i (x_i, 1, 1)^{a_i} \cdot \prod_j (1, x_j, 1)^{b_j} \cdot (1, 1; c),$$

and then considering this product in  $\tilde{K}(G, 3)$ . In  $\tilde{K}(G, 3)$  we have  $(x_i, 1, 1)^{a_i} = (x_i^{a_i}, 1, 1)$ , and

$$\prod_i (x_i, 1, 1)^{a_i} = (a, 1; \prod_{i < j} (x_i \wedge x_j)^{a_i a_j}) \quad \text{and} \quad (a \wedge b) = \prod_{i < j} (x_i \wedge x_j)^{(a_i b_j - a_j b_i)},$$

which shows that

$$\psi: (a, b; c) \mapsto (a, b; c \prod_{i < j} (x_i \wedge x_j)^{(a_i a_j + b_i b_j + a_i b_j - a_j b_i)}).$$

Now consider a product  $(a, b; c)(d, e; f) = (ad, be; cf(b \wedge d))$  in  $\tau(G)$ . We have

$$\psi((a, b; c))\psi((d, e; f)) = (ad, be; cf \prod_{i < j} (x_i \wedge x_j)^{p_{i,j}})$$

where

$$\begin{aligned} p_{i,j} &= a_i a_j + b_i b_j + a_i b_j - a_j b_i + d_i d_j + e_i e_j \\ &\quad + d_i e_j - d_j e_i + b_i e_j - b_j e_i + a_i (d_j + e_j) - a_j (d_i + e_i), \end{aligned}$$

and

$$\psi((ad, be; cf(b \wedge d))) = (ad, be; cf \prod_{i < j} (x_i \wedge x_j)^{r_{i,j}})$$

where

$$\begin{aligned} r_{i,j} &= (a_i + d_i)(a_j + d_j) + (b_i + e_i)(b_j + e_j) + (a_i + d_i)(b_j + e_j) \\ &\quad - (a_j + d_j)(b_i + e_i) + b_i d_j - b_j d_i. \end{aligned}$$

It follows that  $p_{i,j} - r_{i,j} = -2(b_j e_i + a_j d_i) \equiv 0 \pmod{2}$ , that is, if  $G \wedge G$  has exponent 2, then  $\psi$  is an isomorphism, as claimed.  $\square$

**Example 5.8.** The assumptions on  $G$  in Proposition 5.7 hold if  $G \cong C_{2^m} \times C_2^n$  for some  $n, m$ . On the other hand, for  $A = C_4^3$ , we determine  $\tau(A) \not\cong \tilde{K}(A, 3)$  by computing that  $\text{Aut}(\tau(A))$  and  $\text{Aut}(\tilde{K}(G, 3))$  have orders 94575592174780416 and 283726776524341248, respectively. Since  $A \wedge A$  has exponent 4, this is also an example showing that the assumptions in Proposition 5.7 cannot be relaxed. Similarly, it shows that the next result, Proposition 5.9, cannot be extended to higher rank. A comparison of the automorphism group orders also shows that  $\tau(B) \not\cong \tilde{K}(B, 3)$  for  $B = C_5^3$ .

**Proposition 5.9.** *Let  $G$  be an abelian group.*

- a) *Suppose all Sylow  $p$ -subgroups of  $G$  have rank at most 2. Then  $\tau(G) \cong \tilde{K}(G, 3)$  if and only if the Sylow 3-subgroup of  $G$  is cyclic.*
- b) *If the Sylow 3-subgroup of  $G$  has rank at least 2, then  $\tau(G) \not\cong \tilde{K}(G, 3)$ .*

PROOF. Let  $G = \prod_p G_p$  be the decomposition of  $G$  into its Sylow subgroups. By [10, Proposition 4.1] and [16, Corollary 3.7], we can also decompose  $\tau(G) = \prod_p \tau(G_p)$  and  $\tilde{K}(G, n) = \prod_p \tilde{K}(G_p, n)$ , and every isomorphism  $\tau(G) \rightarrow \tilde{K}(G, n)$  induces an isomorphism from  $\tau(G_p)$  to  $\tilde{K}(G_p, n)$ . Thus it is sufficient to assume that  $G$  is an abelian  $p$ -group.

a) We have  $G \cong C_m \times C_n$  with  $m = p^a$  and  $n = p^b$  for  $a \geq b$ . Let  $g$  and  $h$  be generators of  $C_m$  and  $C_n$ , respectively. Considering the description of  $\tau(G)$  as in Proposition 3.6, set  $g_1 = (g, 1; 1)$ ,  $h_1 = (h, 1; 1)$ ,  $g_2 = (1, g; 1)$ ,  $h_2 = (1, h; 1)$ , and  $k = (1, 1; g \wedge h)$ . These elements form a polycyclic generating sequence of  $\tau(G)$ , with corresponding polycyclic presentation

$$\tau(G) = \text{pc}\langle g_1, h_1, g_2, h_2, k \mid g_1^m, g_2^m, h_1^n, h_2^n, k^n, g_2^{h_1} = g_2 k^{-1}, h_2^{g_1} = h_2 k \rangle.$$

Using the identification of  $\tilde{K}(G, 3) = G^2.(G \wedge G)$  as in Proposition 3.5, we obtain

$$\begin{aligned} \tilde{K}(G, 3) &= \text{pc}\langle g_1, h_1, g_2, h_2, k \mid g_1^m, g_2^m, h_1^n, h_2^n, k^n, h_1^{g_1} = h_1 k^2, \\ &\quad h_2^{g_1} = g_2 k^{-1}, h_2^{g_1} = h_2 k, h_2^{g_2} = h_2 k^2 \rangle. \end{aligned}$$

If  $p \neq 3$ , then a short calculation confirms that  $(g_1, h_1, g_2, h_2, k) \mapsto (g_1 g_2^2, h_1, g_2 g_1^2, h_2, k)$  extends to an isomorphism  $\tau(G) \rightarrow \tilde{K}(G, 3)$ . If  $p = 3$  and  $G$  has rank 2, then  $\tau(G) \not\cong \tilde{K}(G, 3)$ , see part b). If  $G$  is a cyclic 3-group, then  $\tau(G) = K(G, 3) = \tilde{K}(G, 3)$  follows from Corollary 4.5 and Remark 4.6; note that  $M(G) = 1$ .

b) As  $G_3$  is not cyclic, hence  $Z^\wedge(G_3) \neq G_3$ , it follows that there exists  $u \in G_3 \setminus Z^\wedge(G_3)$  with  $u^3 \in Z^\wedge(G)$ . Now Propositions 3.5 and 3.6 imply  $\tau(G_3) \not\cong \tilde{K}(G_3, 3)$ , hence  $\tau(G) \not\cong \tilde{K}(G, 3)$ .  $\square$

## 6. Bogomolov multiplier

Let  $G$  be a group with AI-automorphism  $\alpha$ , and let  $\Phi_\alpha: \tau(G) \rightarrow K(G, 3)$  be the epimorphism defined above. Set

$$M^b(G) = \langle [x, y^*] : x, y \in G, [x, y] = 1 \rangle_{\tau(G)};$$

note that  $M^b(G)$  is contained in the kernel of the commutator map  $\kappa: [G, G^*]_{\tau(G)} \rightarrow G'$ . Define

$$\tau^b(G) = \tau(G)/M^b(G).$$

If  $x$  and  $y$  commute in  $G$ , then  $\Phi_\alpha([x, y^*]) = (x^{-1}x, y^{-1}y, \alpha([x, y])) = (1, 1, 1)$ , therefore  $\Phi_\alpha$  induces an epimorphism  $\Phi_\alpha^b: \tau^b(G) \rightarrow K(G, 3)$ . Theorem 4.4 implies that the kernel of this map is  $(\ker \kappa)/M^b(G)$ , which is isomorphic to the *Bogomolov multiplier*  $B_0(G)$  of  $G$ , see [14].

**Corollary 6.1.** *The existence of an AI-automorphism of  $G$  yields a central extension*

$$1 \longrightarrow B_0(G) \longrightarrow \tau^b(G) \longrightarrow K(G, 3) \longrightarrow 1.$$

**Proposition 6.2.** *Let  $H$  be a Schur cover of a group  $G$  with  $H/M = G$ . If  $\alpha$  is an AI-automorphism of  $H$ , then the map*

$$\iota: M^2 \rightarrow \tau^b(H), \quad (m_1, m_2) \mapsto m_1 m_2^* \prod_i [\alpha^{-1}(h_i), (\alpha^{-1}(k_i))^*],$$

where  $m_1 m_2 \alpha(m_1 m_2) = \prod_i [h_i, k_i]$ , is a monomorphism. Moreover,  $\tilde{K}(G, 3) \cong \tau^b(H)/\text{im } \iota$ .

PROOF. Since  $M$  is abelian,  $M^2 \cong K(M, 3)$  via  $(m_1, m_2) \mapsto (m_1, m_2, m_1^{-1}m_2^{-1})$ . Note that  $K(M, 3)$  is naturally embedded in  $K(H, 3)$ . From [13, Proposition 6.12] we conclude that  $B_0(H)$  is trivial, therefore  $\Phi_\alpha^b: \tau^b(H) \rightarrow K(H, 3)$  is an isomorphism by Corollary 6.1. It is easy to see that  $\iota$  is an embedding; now the result follows from taking quotients in the following commutative diagram:

$$\begin{array}{ccc} M^2 & \xrightarrow{\cong} & K(M, 3) \\ \downarrow \iota & & \downarrow \\ \tau^b(H) & \xrightarrow{\Phi_\alpha^b} & K(H, 3). \end{array}$$

□

## 7. Computations

If  $G$  is a finite polycyclic group, then also  $\tilde{K}(G, 3)$  is polycyclic, see [10, Proposition 1.5]. In this situation, the algorithms described in [6] can be used to compute  $\tau(G)$ ; these algorithms are implemented in the software package Polycyclic, distributed with the computer algebra system GAP [7]. Our explicit formulas in Section 3 can be used to compute a polycyclic presentation for  $\tilde{K}(G, 3)$ . We have done this to test whether  $\tau(G)$  and  $\tilde{K}(G, 3)$  are isomorphic for certain examples of groups (abelian, Frobenius, extra-special, ...). Even though there exist powerful algorithms for working with polycyclic groups, approaching this isomorphism problem with conventional methods poses a serious computational challenge. This is due to the fact that if  $G$  is an abelian group of order  $p^n$ , then  $\tilde{K}(G, 3)$  and  $\tau(G)$  are both large central extensions of  $G \wedge G$  by  $G^2$ ; they have class 2, order  $p^{2n}|G \wedge G|$ , and often seem indistinguishable. The latter is not a surprise, given the folklore conjecture that *most*  $p$ -groups have class 2: for example, note that among the 49499125314 groups of order at most 1024 (up to isomorphism), 99.976% of

TABLE 1. Statistics for solvable non-abelian groups of cubefree order at most 100

$\tau \cong \tilde{K}$	has AI	# groups
yes	yes	96
yes	no	0
no	yes	2
no	no	25

these are 2-groups and 98.595% are 2-groups of class 2, see [4, Section 4]. A computational isomorphism test for these groups reduces to orbit calculations of huge matrix groups on very large vector spaces; often these computations turn out to be infeasible. For example, the powerful implementations of the  $p$ -group algorithms for automorphism groups and isomorphisms (provided by the GAP package Anupq) struggle to compute automorphisms and isomorphisms for  $\tau(G)$  and  $\tilde{K}(G, 3)$  already for moderately sized  $p$ -groups such as  $G = C_7^3$ . Most of our computer experiments have therefore focused on groups of cubefree order, that is, groups whose order is not divisible by any prime power  $p^3$ .

**Example 7.1.** In Table 1 we report on some example computations: there are 237 cubefree groups of order at most 100. Of these, 113 groups are abelian, 123 groups are non-abelian solvable, and 1 group is simple. Every abelian  $G$  admits AI-automorphisms and, being cubefree,  $\tau(G) \cong \tilde{K}(G, 3)$  if and only if  $G$  has a cyclic Sylow 3-subgroup, see Proposition 5.9. Our computations show that, with two exceptions,  $\tau(G) \cong \tilde{K}(G, 3)$  if and only if  $G$  has AI-automorphisms. The exceptions are  $G = C_3 \times \text{Alt}_4$  and  $H = C_3^2 \times D_{10}$ ; we have  $Z(\tilde{K}(G, 3)) = C_6 \times C_3$  and  $Z(\tau(G)) = C_6$ , and non-isomorphism of  $\tau(H)$  and  $\tilde{K}(H, 3)$  follows from Proposition 5.9.

**Example 7.2.** Running over GAP’s group data base, there are 6505 non-abelian solvable groups of order  $< 256$ ; of these groups, 6127 have AI-automorphisms. Note that every simple and every abelian group admits AI-automorphisms. This computation suggests that for many groups we can apply Corollary 4.5 to describe  $\tau(G)$  as a central extension of  $H_2(G, \mathbb{Z})$  by  $K(G, 3)$ . Table 1 and Proposition 4.7 suggest that the existence of AI-automorphisms for  $G$  is connected to the property  $\tau(G) \cong \tilde{K}(G, 3)$ . Proposition 5.2b) shows that an extra-special group  $G$  of exponent  $p^2$  (with  $p$  odd) has no AI-automorphisms; a calculation of several examples suggests that  $\tau(G) \not\cong \tilde{K}(G, 3)$  as well.

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