Mathematisches
Forschungsinstitut
Oberwolfach

Mitglied der Leibniz Gemeinschaft

Oberwolfach Preprints

OWP 2019 - 20

ARINDAM BISWAS

On a Cheeger Type Inequality in Cayley Graphs of Finite Groups

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

The MFO publishes a preprint series **Oberwolfach Preprints (OWP)**, ISSN 1864-7596, which mainly contains research results related to a longer stay in Oberwolfach, as a documentation of the research work done at the MFO. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

The full copyright is left to the authors. With the submission of a manuscript, the authors warrant that they are the creators of the work, including all graphics. The authors grant the MFO a perpetual, non-exclusive right to publish it on the MFO's institutional repository.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX – XX). Additionally, each preprint will get a Digital Object Identifier (DOI).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI 10.14760/OWP-2019-20

ON A CHEEGER TYPE INEQUALITY IN CAYLEY GRAPHS OF FINITE GROUPS

ARINDAM BISWAS

Abstract. Let G be a finite group. It was remarked in [BGGT15] that if the Cayley graph $C(G,S)$ is an expander graph and is non-bipartite then the spectrum of the adjacency operator T is bounded away from -1. In this article we are interested in explicit bounds for the spectrum of these graphs. Specifically, we show that the non-trivial spectrum of the adjacency operator lies in the interval $\left[-1+\frac{h(\mathbb{G})^4}{\gamma}, 1-\frac{h(\mathbb{G})^2}{2d^2}\right]$, where $h(\mathbb{G})$ denotes the (vertex) Cheeger constant of the d regular graph $C(G,S)$ with respect to a symmetric set S of generators and $\gamma = 2^9 d^6 (d+1)^2$.

1. Introduction

Throughout this article we will consider a finite group G with $|G| = n$. We will denote by $C(G,S)$ for a symmetric subset $S \subset G$ of size $|S| = d$, to be the Cayley graph of G with respect to S. Then $C(G,S)$ is d regular. Given a finite d regular Cayley graph $C(G,S)$, we have the normalised adjacency matrix T of size $n \times n$ whose eigenvalues lie in the interval $[-1,1]$. The normalised Laplacian matrix of $C(G,S)$ denoted by L is defined as

$$L := I_n - T, \tag{1.1}$$

where I_n denotes the identity matrix. The eigenvalues of L lie in the interval $[0,2]$.
It is easy to see that 1 is always an eigenvalue of T and 0 that of L. We denote the eigenvalues of T as $-1 \leqslant t_n \leqslant ... \leqslant t_2 \leqslant t_1 = 1$ and that of L as $\lambda_i = 1 - t_i, i = 1, 2, ..., n$. The graph $C(G,S)$ is connected if and only if $\lambda_2 > 0$ (equivalently $t_2 < 1$). The graph is bipartite if and only if $\lambda_n = 2$.

We recall the notion of Cheeger constant.

Definition 1.1 (Vertex boundary of a set)**.** *Let $\mathbb{G} = (V,E)$ be a graph with vertex set V and edge set E. For a subset $V_1 \subset V$, let $N(V_1)$ denoting the neighbourhood of V_1 be*

$$N(V_1) := \{v \in V : vv_1 \in E \text{ for some } v_1 \in V_1\}.$$

Then the boundary of V_1 is defined as $\delta(V_1) := N(V_1) \setminus V_1$.

Definition 1.2 (Cheeger constant)**.** *The Cheeger constant of the graph $\mathbb{G} = (V,E)$, denoted by $h(\mathbb{G})$ is defined as*

$$h(\mathbb{G}) := \inf\{\frac{|\delta(V_1)|}{|V_1|} : V_1 \subset V, |V_1| \leqslant \frac{|V|}{2}\}.$$

This is also called the vertex Cheeger constant of a graph.

Key words and phrases. Cheeger inequality, expander graphs, finite Cayley graphs.
To appear in the European Journal of Combinatorics.

Definition 1.3 ((n,d,ϵ) expander)**.** *Let $\epsilon > 0$. An (n,d,ϵ) expander is a graph (V,E) on $|V| = n$ vertices, having maximal degree d, such that for every set $V_1 \subseteq V$ satisfying $|V_1| \leqslant \frac{n}{2}$, $|\delta(V_1)| \geqslant \epsilon|V_1|$ holds (equivalently, $h(\mathbb{G}) \geqslant \epsilon$).*

In this article, we are interested in the spectrum of the Laplace operator L for the Cayley graph $C(G,S)$. The Cayley graph is bipartite if and only if there exists an index two subgroup H of G which is disjoint from S. See Proposition 2.6. It was observed in [BGGT15](Appendix E) that if $C(G,S)$ is an expander graph and is non-bipartite, then the spectrum of T is not only bounded away from 1 but also from -1. Here we show that

Theorem 1.4. *Let the Cayley graph $C(G,S)$ be an expander with $|S| = d$ and $h(G)$ denote its Cheeger constant. Then if $C(G,S)$ is non-bipartite, we have*

$$\lambda_n \leqslant 2 - \frac{h(G)^4}{\alpha d^6 (d+1)^2},$$

where λ_n is the largest eigenvalue of the normalised Laplacian matrix and α is an absolute constant (we can take $\alpha = 2^9$).

The strategy of the proof closely follows the combinatorial arguments of Breuillard–Green–Guralnick–Tao in [BGGT15].

2. Proofs

There are two notions of expansion in graphs - the vertex expansion as in Definition 1.3 and the edge expansion.

Definition 2.1 (Edge expansion)**.** *Let $\mathbb{G} = (V,E)$ be a d-regular graph with vertex set V and edge set E. For a subset $V_1 \subset V$, let $E(V_1, V\backslash V_1)$ be the edge boundary of V_1, defined as*

$$E(V_1, V\backslash V_1) := \{(v_1, s) \in E : v_1 \in V, v_1 s \in V\backslash V_1\}.$$

Then the edge expansion ratio $\phi(V_1)$ is defined as

$$\phi(V_1) := \frac{|E(V_1, V\backslash V_1)|}{d|V_1|}.$$

Definition 2.2 (Edge-Cheeger constant)**.** *The edge-Cheeger constant denoted by $\mathfrak{h}(\mathbb{G})$ is*

$$\mathfrak{h}(\mathbb{G}) := \inf_{V_1 \subset V, |V_1| \leqslant |V|/2} \phi(V_1).$$

In a d regular graph the two Cheeger constants are related by the following -

Lemma 2.3. *Let $\mathbb{G} = (V,E)$ be a d-regular graph*

$$\frac{h(\mathbb{G})}{d} \leqslant \mathfrak{h}(\mathbb{G}) \leqslant h(\mathbb{G}).$$

Proof. Let $V_1 \subset V$ and we consider the map

$$\psi : E(V_1, V\backslash V_1) \to \delta(V_1) \text{ given by } (v_1, s) \to v_1 s.$$

The map is surjective hence we have the left hand side and at the worst case d to 1 wherein we get the right hand side. □

We have the following inequalities, called the discrete Cheeger-Buser inequality. It is the version for graphs of the corresponding inequalities for the Laplace-Beltrami operator on closed Riemannian manifolds. It was first proven by Cheeger [Che70] (lower bound) and by Buser [Bus82] (upper bound). The discrete version was shown by Alon and Millman [AM85] (Proposition 2.4).

Proposition 2.4 (Discrete Cheeger-Buser inequality)**.** *Let $\mathbb{G} = (V, E)$ be a finite d-regular graph. Let λ_2 denote the second smallest eigenvalue of its normalised Laplacian matrix and $\mathfrak{h}(\mathbb{G})$ be the (edge) Cheeger constant. Then*

$$\frac{\mathfrak{h}(\mathbb{G})^2}{2} \leqslant \lambda_2 \leqslant 2\mathfrak{h}(\mathbb{G}).$$

Proof. See [Lub94] prop. 4.2.4 and prop. 4.2.5 or [Chu96] sec. 3. □

Before proceeding further, let us recall the notion of Cayley graph of a group.

Definition 2.5 (Cayley graph)**.** *Let G be a finite group and S be a symmetric generating set of G. Then the Cayley graph $C(G, S)$ is the graph having the elements of G as vertices and $\forall x, y \in G$ there is an edge between x and y if and only if $\exists s \in S$ such that $sx = y$. If $1 \in S$, then the graph has a loop (which we treat as an edge) going from x to itself $\forall x \in G$.*

A graph is said to be r-regular (where $r \geqslant 1$ is an integer) if there are exactly r half edges connected to each vertex (except for a loop which counts as one half edge). If $|S| = d$, it is clear that $C(G, S)$ will be d-regular (where $|S|$ denotes the cardinality of the set S).

Next, we recall the definition of the adjacency matrix associated to any finite undirected graph. For any finite undirected graph $\mathcal{G}$ having vertex set $V = \{v_1, ..., v_{|\mathcal{G}|}\}$ and edge set E, the adjacency matrix T is the $|V| \times |V|$ matrix having T_{ij} = the number of edges connecting v_i with v_j. The discrete Cheeger inequality applies to all finite regular graphs (the inequality also holds for finite non-regular graphs where we need to consider the maximum of the degrees of the all the vertices - see [Lub94] prop. 4.2.4, but for our purposes we shall restrict to regular graphs).

We show the following proposition -

Proposition 2.6 (Criteria for non-bipartite property)**.** *A finite Cayley graph $C(G, S)$ is non-bipartite if and only if there does not exist an index two subgroup H of G which is disjoint from S.*

Proof. Let $C(G, S)$ be bipartite. Then we can partition the vertex set G into two disjoint sets A and B such that $G = A \sqcup B$. Let $\mathbf{1} \in B$. Let $s \in S \cap B$. Then $s^{-1} \in S$ and so $1 = ss^{-1} \in A$. This is a contradiction. So $S \cap B = \phi$.
Now suppose $x, y \in B$ but $xy \notin B$. So $xy \in A$. Thus there exists $s_1, s_2, \cdots, s_{2r+1} \in S, r \in \mathbb{N}$ such that $s_1 s_2 \cdots s_{2r+1}(xy) = y$. This implies that $s_1 s_2 \cdots s_{2r+1} x = 1 \in B$. But this is impossible because $x \in B$ so $s_1 s_2 \cdots s_{2r+1} x \in A$. Thus we have a contradiction and $xy \in B$. So, B is an index 2 subgroup disjoint from S.
The other direction is clear. □

Lemma 2.7. *Let G be a finite group and $C(G, S)$ denote its Cayley graph with respect to a symmetric set S of size d. Let S be such that*

$$|SA \backslash A| \geqslant \epsilon' |A| \quad (\epsilon'\text{-combinatorial expansion of } S)$$

for every set $A \subseteq G$ with $|A| \leqslant \frac{|G|}{2}$ and some $\epsilon' > 0$. Then we have the estimate

$$|SA\backslash A| \geqslant \frac{\epsilon'}{d}|G\backslash A|$$

for all sets $A \subseteq G$ with $|A| \geqslant \frac{|G|}{2}$.

Proof. Let $A^c = G\backslash A$. The proof is based on the fact that $|SA\backslash A| \geqslant \frac{1}{d}|SA^c\backslash A^c|$ for all subsets $A \subseteq G$ and $S = S^{-1} \subset G$.
Let $s \in S$,

$$|sA^c \cap A| = |s^{-1}(sA^c \cap A)| = |A^c \cap s^{-1}A| \leqslant |A^c \cap SA|$$
$$\Rightarrow |SA^c\backslash A^c| = |SA^c \cap A| = |\cup_{s\in S} sA^c \cap A| \leqslant \Sigma_{s\in S}|SA \cap A^c| = d|SA\backslash A|.$$

Hence, we have

$$|SA\backslash A| \geqslant \frac{1}{d}|SA^c\backslash A^c| \geqslant \frac{\epsilon'}{d}|A^c| = \frac{\epsilon'}{d}|G\backslash A|.$$

(Using the property of combinatorial expansion of S and noting that $|A| \geqslant \frac{|G|}{2} \Rightarrow |A^c| \leqslant \frac{|G|}{2}$). □

To prove Theorem 1.4 we have to show that, under the given assumptions, we have

$$t_n \geqslant -1 + \frac{h(G)^4}{\alpha d^6(d+1)^2},$$

for some absolute constant α (which we shall precise).

Method of Proof : The proof is based on the following strategy. We shall first fix a small real number ζ (depending on the degree d and expansion ϵ) and suppose that the Cayley graph $C(G,S)$ has an eigenvalue less than $-1+\zeta$. Under this condition, we shall obtain a set A, such that $|A|$ is close to $\frac{|G|}{2}$ and which satisfies certain properties when we take translates of A by elements $s \in S$. This is Lemma 2.8. Using this set A, we shall construct a subgroup H of G whose index will be 2 (when ζ is small enough) and we shall show that this H cannot intersect the generating set S of G. This will give the required contradiction with Proposition 2.6, since the Cayley graph $C(G,S)$ was non-bipartite.

Lemma 2.8. *Let G be a finite group, $k \geqslant 1$ and $S = S^{-1} = \{s_1, ..., s_d\}$ be a symmetric generating set of G. Let S be ϵ-combinatorially expanding, i.e.,*

$$|SX\backslash X| \geqslant \epsilon|X|$$

for every set $X \subseteq G$ with $|X| \leqslant \frac{|G|}{2}$ and some $\epsilon > 0$.[1] *Suppose, there exists a sufficiently small $\zeta, 0 < \zeta \leqslant \frac{\epsilon^2}{4d^4}$, such that the adjacency matrix T of $C(G,S)$ has an eigenvalue in $[-1, -1+\zeta)$. Fix $\beta = d^2\sqrt{2\zeta(2-\zeta)}$. Then, there exists a set A with the following properties*

(1) $(\frac{1}{2+\beta+\frac{d\beta}{\epsilon}})|G| \leqslant |A| \leqslant \frac{1}{2}|G|$,

(2) $|SA \cap A| \leqslant \frac{1}{\epsilon}\beta|A|$,

(3) $\forall s \in S, g \in G, |sAg\Delta(Ag)^c| \leqslant \beta\Big(1 + \frac{d}{\epsilon} + \frac{2}{\epsilon}\Big)|A|$.

[1]It is clear that $d \geqslant \epsilon$ and in fact, considering $X \subset S, |X| \leqslant \frac{|G|}{2}$ we get that $d > \epsilon$, so that $\frac{\epsilon}{d}$ always remains strictly less than 1 for finite Cayley graphs.

Proof. We have

$$\epsilon|X| \leqslant |SX\backslash X|, \tag{2.1}$$

whenever $X \subset G$ with $|X| \leqslant \frac{|G|}{2}$ and using Lemma 2.7 with $|S| = d$

$$\frac{\epsilon}{d}|G\backslash X| \leqslant |SX\backslash X|, \tag{2.2}$$

whenever $|X| \geqslant \frac{|G|}{2}$.

Since T has an eigen-value in $[-1, -1+\zeta)$, T^2 has a non-trivial eigenvalue (say) t' in $((1-\zeta)^2, 1]$.[2]

Now consider the set S^2 (obtained by identifying all equal elements in the multi-set $S.S$) and the muti-set $S' = S.S$ (without identification). T^2 is the adjacency matrix associated with S' and $|S^2| \leqslant |S'| = d^2$. Let $h(G, S')$ denote the vertex Cheeger constant (Definition 1.2) and $\mathfrak{h}(G, S')$ denote the edge-Cheeger constant (Definition 2.2) for G with respect to the multi-set S'.

We have $t' > (1-\zeta)^2$. Let $\mathbb{L}$ denote the Laplacian matrix of the graph of G with respect to S', with the adjacency operator T^2 and let its eigenvalues be denoted by $0 = \mathbf{L}_1 \leqslant \mathbf{L}_2 \leqslant \ldots \leqslant \mathbf{L}_n \leqslant 2$. We know that

$$\mathbf{L}_2 = 1 - t' < 1 - (1-\zeta)^2 = \zeta(2-\zeta).$$

By the discrete Cheeger-Buser inequality (Proposition 2.4) for the graph of G with respect to S' we have

$$\frac{\mathfrak{h}^2(G, S')}{2} \leqslant \mathbf{L}_2 < \zeta(2-\zeta).$$

Hence by Lemma 2.3,

$$\frac{h(G, S')}{d^2} \leqslant \mathfrak{h}(G, S') < \sqrt{2\zeta(2-\zeta)}.$$

This implies that $\exists A \subset G$ with $|A| \leqslant \frac{|G|}{2}$ such that

$$\frac{|S^2A\backslash A|}{|A|} \leqslant \frac{|S'A\backslash A|}{|A|} < d^2\sqrt{2\zeta(2-\zeta)} = \beta. \tag{2.3}$$

Claim 2.9. $|A \cup SA| \geqslant \frac{|G|}{2}$ *for* $\zeta \leqslant \frac{\epsilon^2}{4d^4}$.

Proof of claim. We know that for arbitrary sets $X, Y, Z \subset G$, $X(Y \cup Z) = XY \cup XZ$. Hence

$$|S(A \cup SA)\backslash(A \cup SA)| = |S^2A\backslash A| < d^2\sqrt{2\zeta(2-\zeta)}|A|.$$

Let $|A \cup SA| \leqslant \frac{|G|}{2}$. This implies (using equation 2.1 and 2.3) that

$$\epsilon|A| \leqslant \epsilon|A \cup SA| \leqslant |S(A \cup SA)\backslash(A \cup SA)| < d^2\sqrt{2\zeta(2-\zeta)}|A|,$$

[2] actually we only need the fact that $t' > (1-\zeta)^2$. That $t' \neq 1$ follows when we consider non-bipartite graphs, since a graph is bipartite iff T has -1 as an eigenvalue.

which cannot hold for $\zeta \leqslant \frac{\epsilon^2}{4d^4}$.

$\square$

This means that under the assumption $\zeta \leqslant \frac{\boldsymbol{\epsilon^2}}{\mathbf{4d^4}}$ we have $|A \cup SA| \geqslant \frac{|G|}{2}$.

We can apply Lemma 2.7 to $|A \cup SA| \geqslant \frac{|G|}{2}$ and use equation 2.2 and equation 2.3 to get

$$\frac{\epsilon}{d}|G \backslash (A \cup SA)| \leqslant |S(A \cup SA) \backslash (A \cup SA)| = |S^2 A \backslash A| < \beta |A|.$$

Noting the fact that $|G \backslash (A \cup SA)| = |G| - |A \cup SA|$, we have

$$|G| - \frac{d\beta}{\epsilon}|A| \leqslant |A \cup SA| \leqslant |A| + |SA|.$$

We use the fact that,

$$|SA| \leqslant |S^2 A| \leqslant |A| + \beta |A|$$

to conclude that,

$$\left(\frac{\mathbf{1}}{\mathbf{2} + \beta + \frac{\mathbf{d}\beta}{\epsilon}} \right) |\mathbf{G}| \leqslant |\mathbf{A}|. \tag{2.4}$$

For arbitrary sets $X, Y, Z \subset G$ we have $X(Y \cap Z) \subset XY \cap XZ$.

Hence

$$|S(A \cap SA) \backslash (A \cap SA)| \leqslant |S^2 A \backslash A| \leqslant \beta |A|.$$

As $|A| \leqslant \frac{|G|}{2}$ clearly $|A \cap SA| \leqslant \frac{|G|}{2}$. So, the hypothesis of ϵ-combinatorial expansion applies to $A \cap SA$ (i.e., $\epsilon |A \cap SA| \leqslant |S(A \cap SA) \backslash (A \cap SA)| \leqslant \beta |A|$) and we have

$$|\mathbf{A} \cap \mathbf{SA}| \leqslant \frac{\mathbf{1}}{\epsilon} \beta |\mathbf{A}|. \tag{2.5}$$

Our next aim is to compute the bounds on $|sA \Delta A|, |sA \Delta A^c|, |sAg \Delta Ag|, |sAg \Delta (Ag)^c|$ for $g \in G$.

For this,

$$\begin{aligned} |sA \Delta A| &= |sA \cup A \backslash sA \cap A| \\ &= |sA \cup A| - |sA \cap A| \\ &= |sA| + |A| - 2|sA \cap A| \\ &= 2|A| - 2|sA \cap A| \\ &\geqslant 2|A| - 2|SA \cap A| \\ &\geqslant \left(2 - \frac{2\beta}{\epsilon} \right) |A|. \end{aligned}$$

This implies,

$$\begin{aligned}|sA\Delta A^c| &= |G \setminus (sA\Delta A)| \\ &= |G| - |sA\Delta A| \\ &\leqslant |G| - (2 - \frac{2}{\epsilon}\beta)|A| \\ &\leqslant \Big(\beta + \frac{d\beta}{\epsilon} + \frac{2}{\epsilon}\beta\Big)|A|.\end{aligned}$$

Thus we have

$$|sA\Delta A^c| \leqslant \beta\left(1 + \frac{d}{\epsilon} + \frac{2}{\epsilon}\right)|A| \tag{2.6}$$

and by the symmetricity of S,

$$|sA^c\Delta A| \leqslant \beta\left(1 + \frac{d}{\epsilon} + \frac{2}{\epsilon}\right)|A|.$$

Now let $g \in G$ be arbitrary. Then we have,

$$\begin{aligned}|sAg\Delta Ag| &= |sAg| + |Ag| - 2|sAg \cap Ag| \\ &= 2|A| - 2|sA \cap A| \\ &\geqslant \left(2 - \frac{2}{\epsilon}\beta\right)|A|.\end{aligned}$$

(Since for fixed $g \in G$, $X, Y \subset G$, $(X \cap Y)g = Xg \cap Yg$).

Similarly, we get

$$|\mathbf{sAg\Delta(Ag)^c}| \leqslant \beta\left(\mathbf{1} + \frac{\mathbf{d}}{\epsilon} + \frac{\mathbf{2}}{\epsilon}\right)|\mathbf{A}| \tag{2.7}$$

and

$$|s(Ag)^c\Delta Ag| \leqslant \beta\left(1 + \frac{d}{\epsilon} + \frac{2}{\epsilon}\right)|A|.$$

□

We shall now use the set A which we obtained from the lemma to prove our main theorem.

Theorem 2.10. *Let G be a finite group, $k \geqslant 1$ and $S = S^{-1} = \{s_1, ..., s_d\}$ be a symmetric generating set of G. Suppose that G does not have an index two subgroup H disjoint from S. Let S be ϵ-combinatorially expanding, i.e.,*

$$|SX\setminus X| \geqslant \epsilon|X|$$

for every set $X \subseteq G$ with $|X| \leqslant \frac{|G|}{2}$ and some $\epsilon > 0$. Then all the eigenvalues of the operator T are $\geqslant -1 + \frac{\epsilon^4}{\alpha d^6(d+1)^2}$ where α is an absolute constant (we can take $\alpha = 2^9$).

Proof. The proof will be by contradiction. Keeping the notations of Lemma 2.8, we shall show that if T has an eigenvalue in $[-1, -1 + \zeta)$, where ζ is chosen to be small (precised in Claim 2.11 and satisfying the condition on ζ in Lemma 2.8), there exists an index 2 subgroup, H in G which is disjoint from S. This will give the required contradiction.

First we use Lemma 2.8 to conclude that (under the assumption $\zeta \leqslant \frac{\epsilon^2}{4d^4}$) there exists a set A with the following properties

(1) $(\frac{1}{2+\beta+\frac{d\beta}{\epsilon}})|G| \leqslant |A| \leqslant \frac{1}{2}|G|$,
(2) $|SA \cap A| \leqslant \frac{1}{\epsilon}\beta|A|$,
(3) $\forall s \in S, |sA\Delta A^c| \leqslant \Big(\beta + \frac{d\beta}{\epsilon} + \frac{2}{\epsilon}\beta\Big)|A|, |sA^c\Delta A| \leqslant \Big(\beta + \frac{d\beta}{\epsilon} + \frac{2}{\epsilon}\beta\Big)|A|$,
(4) $\forall s \in S, g \in G, |sAg\Delta(Ag)^c| \leqslant \beta\Big(1 + \frac{d}{\epsilon} + \frac{2}{\epsilon}\Big)|A|, |s(Ag)^c\Delta Ag| \leqslant \beta\Big(1 + \frac{d}{\epsilon} + \frac{2}{\epsilon}\Big)|A|$.

Using the above set A, we want to construct the subgroup H of index 2. The method will be to translate A using the elements $g \in G$ and check which of them have large intersection with the original set A (i.e, $|A \cap Ag|$ is "almost" $|A|$).

Take $A_g := A \cap Ag, A'_g := (A \cup Ag)^c$. Let $B = A_g \sqcup A'_g$ (it is a disjoint union). Then

$$G \setminus B = B^c = A\Delta Ag$$

and

$$B = (A\Delta Ag)^c = A\Delta(Ag)^c.$$

Also note that $X\Delta Y = X^c\Delta Y^c$ for all $X, Y \subseteq G$.

We wish to estimate $|B|$ when $g \in G$. For this, we first estimate $|SB\Delta B|$ and $|SB^c\Delta B^c|$.

$$\begin{aligned}
|SB\Delta B| &\leqslant \Sigma_{s\in S}|sB\Delta B| \\
&= \Sigma_{s\in S}|s(A\Delta(Ag)^c)\Delta(A\Delta(Ag)^c)| \\
&= \Sigma_{s\in S}|(sA\Delta s(Ag)^c)\Delta(A\Delta(Ag)^c)| \\
&= \Sigma_{s\in S}|(sA\Delta s(Ag)^c)\Delta(A^c\Delta Ag)| \\
&= \Sigma_{s\in S}|(sA\Delta A^c)\Delta(s(Ag)^c\Delta(Ag))| \\
&\leqslant d(|sA\Delta A^c| + |sAg\Delta(Ag)^c|) \\
&\leqslant 2d\beta\Big(1 + \frac{d}{\epsilon} + \frac{2}{\epsilon}\Big)|A|.
\end{aligned}$$

(where we use the fact that, all the above sets are defined inside G, $X\Delta Y = X^c\Delta Y^c$, $sX^c = (sX)^c$, $s(X\Delta Y) = (sX\Delta sY)$ [3] and Δ is both associative and commutative).

[3]These do not hold for sets $S \subset G$ in general, i.e., $S.X^c \neq (SX)^c$ and $SX\Delta SY \subset S(X\Delta Y)$ for arbitrary sets $S, X, Y \subset G$. This is one of the main reasons why we had to estimate translates of A by elements $s \in S$ rather than translate of A by S.

Similarly,

$$\begin{aligned}|SB^c\Delta B^c| &\leqslant \Sigma_{s\in S}|sB^c\Delta B^c|\\ &= \Sigma_{s\in S}|s(A\Delta Ag)\Delta(A\Delta Ag)|\\ &= \Sigma_{s\in S}|(sA\Delta sAg)\Delta(A\Delta Ag)|\\ &= \Sigma_{s\in S}|(sA\Delta sAg)\Delta(A^c\Delta(Ag)^c)|\\ &= \Sigma_{s\in S}|(sA\Delta A^c)\Delta(sAg\Delta(Ag)^c)|\\ &\leqslant d(|sA\Delta A^c| + |sAg\Delta(Ag)^c|)\\ &\leqslant 2d\beta\Big(1+\frac{d}{\epsilon}+\frac{2}{\epsilon}\Big)|A|.\end{aligned}$$

We now have the following two cases depending on the size of the set B.

(1) $|B| \leqslant \frac{|G|}{2}$ in which case,

$$|B| \leqslant \frac{2d\beta}{\epsilon^2}\Big(\epsilon+d+2\Big)|A| \tag{2.8}$$

(using the fact that $\epsilon|B| \leqslant |SB\backslash B| \leqslant |SB\Delta B| \leqslant 2d\beta\Big(1+\frac{d}{\epsilon}+\frac{2}{\epsilon}\Big)|A|$).
From this it follows that,

$$|A\cap Ag| \leqslant \frac{d\beta}{\epsilon^2}\Big(\epsilon+d+2\Big)|A|. \tag{2.9}$$

(There are two ways to see it. $B = A_g \sqcup A'_g$ and $|A'_g| \geqslant |A_g|$ when $|A| \leqslant \frac{|G|}{2} \Rightarrow |A\cap Ag| = |A_g| \leqslant \frac{|B|}{2}$, is one way. The other way is to use $G\setminus B = A\Delta Ag$. Hence after taking the cardinalities and expanding we have $|B| = |G| - 2|A| + 2|A\cap Ag|$. Then use the fact that $2|A| \leqslant |G|$, to get that $|A\cap Ag| \leqslant \frac{|B|}{2}$.)

OR

(2) $|B| > \frac{|G|}{2}$ in which case $|B^c| \leqslant \frac{|G|}{2}$ and then

$$|G\backslash B| \leqslant \frac{2d\beta}{\epsilon^2}\Big(\epsilon+d+2\Big)|A| \tag{2.10}$$

(using the fact that $\epsilon|B^c| \leqslant |SB^c\backslash B^c| \leqslant |SB^c\Delta B^c| \leqslant 2d\beta\Big(1+\frac{d}{\epsilon}+\frac{2}{\epsilon}\Big)|A|$).
From this it follows that,

$$\Big(1-\frac{d\beta}{\epsilon^2}\Big(\epsilon+d+2\Big)\Big)|A| \leqslant |A\cap Ag|. \tag{2.11}$$

(Again, using $G\setminus B = A\Delta Ag$, taking the cardinalities and expanding the expression we have $|G\setminus B| = 2|A| - 2|A\cap Ag|$).

Thus for any $g\in G$, we have either

(i) $|A\cap Ag| \leqslant \frac{d\beta}{\epsilon^2}\Big(\epsilon+d+2\Big)|A|$,

OR

(ii) $|A \cap Ag| \geqslant \Big(1 - \frac{d\beta}{\epsilon^2}\Big(\epsilon + d + 2\Big)\Big)|A|$.

The trick now is to use the method of Freiman in [Fm73] to find a subgroup H of G. We prove it in the following claim.

Claim 2.11. *If $H := \{g \in G : |A \cap Ag| \geqslant r|A|\}$ where $r = 1 - \frac{d\beta}{\epsilon^2}\Big(\epsilon + d + 2\Big)$ and $\beta \leqslant \frac{1}{2^3\sqrt{2}} \times \frac{\epsilon^2}{d(d+1)}$, then H is a subgroup of G of index 2.*

Proof of claim. We have $H = H^{-1}$, $1 \in H$ and $r > \frac{1}{2} + \frac{d\beta}{\epsilon^2}\Big(\epsilon + d + 2\Big)$. Also for $g, h \in H$ we have by the triangle inequality

$$\begin{aligned} |A \backslash Agh| &\leqslant |A \backslash Ah| + |Ah \backslash Agh| \\ &\leqslant 2(1-r)|A|. \end{aligned}$$

This implies,

$$|A \cap Agh| \geqslant (2r-1)|A|.$$

Hence, gh cannot belong to case (i), gh belongs to case (ii), i.e., $gh \in H$. So H is a subgroup of G.

Let $z = \frac{d\beta}{\epsilon^2}\Big(\epsilon + d + 2\Big)$. Using the estimate,

$$\begin{aligned} |A|^2 &= \Sigma_{g \in G}|A \cap Ag| \\ &\leqslant |H||A| + \frac{d\beta}{\epsilon^2}\Big(\epsilon + d + 2\Big)|A||G \backslash H|, \end{aligned}$$

we have

$$|A| \leqslant |H| + z(|G| - |H|),$$

which implies that,

$$\left(\frac{1}{2 + \beta + \frac{d\beta}{\epsilon}}\right)|G| - z|G| \leqslant (1-z)|H|.$$

(Using the fact that $\left(\frac{1}{2+\beta+\frac{d\beta}{\epsilon}}\right)|G| \leqslant |A|$).

The index of H in G is 2 if $|H| > \frac{|G|}{3}$ and thus, to conclude that H is a subgroup of G of index 2, it suffices to show that [4]

$$\begin{aligned} &\left(\frac{1}{2 + \beta + \frac{d\beta}{\epsilon}} - z\right) > \frac{1-z}{3} \\ \Leftrightarrow\ &\frac{1}{\left(2 + \beta + \frac{d\beta}{\epsilon}\right)} > \frac{1+2z}{3}. \end{aligned}$$

Substituting the expression for z, it suffices to show that,

$$\left(2 + \beta + \frac{d\beta}{\epsilon}\right) + \frac{2d\beta}{\epsilon^2}\left(\epsilon + d + 2\right)\left(2 + \beta + \frac{d\beta}{\epsilon}\right) < 3,$$

[4]Note that $H \neq G$ since there are elements $g \in G$ such that, $g \in G \setminus H$.

i.e.,

$$\left(\beta + \frac{d\beta}{\epsilon}\right) + \frac{2d\beta}{\epsilon^2}\left(\epsilon + d + 2\right)\left(2 + \beta + \frac{d\beta}{\epsilon}\right) < 1.$$

Now, using the fact that $\beta < \frac{1}{8\sqrt{2}}$, $\frac{d\beta}{\epsilon} < \frac{1}{8\sqrt{2}}$, $\frac{2d\beta}{\epsilon^2} < \frac{1}{4\sqrt{2}(d+1)}$, $\epsilon < d$, $\frac{1}{4\sqrt{2}} < 0.177$, we have,

$$\begin{aligned}
&\left(\beta + \frac{d\beta}{\epsilon}\right) + \frac{2d\beta}{\epsilon^2}\Big(\epsilon + d + 2\Big)\left(2 + \beta + \frac{d\beta}{\epsilon}\right) \\
&< \left(\frac{1}{8\sqrt{2}} + \frac{1}{8\sqrt{2}}\right) + \frac{1}{4\sqrt{2}(d+1)}(2d+2)\left(2 + \frac{1}{8\sqrt{2}} + \frac{1}{8\sqrt{2}}\right) \\
&= \frac{1}{4\sqrt{2}} + \frac{d+1}{d+1}.\frac{2 + \frac{1}{4\sqrt{2}}}{2\sqrt{2}} \\
&< 0.177 + 0.77 \\
&< 1.
\end{aligned}$$

Hence the index of H in G is 2 if $d^2\sqrt{2\zeta(2-\zeta)} = \beta \leqslant \frac{\epsilon^2}{2^3\sqrt{2}.d(d+1)}$. This gives us the fact that, for all $\zeta \leqslant \frac{\epsilon^4}{2^9 d^6 (d+1)^2}$, the index of H in G is 2. □

Up until now, we have argued formally that under the condition on ζ (equivalently β) being small enough for the set A to exist (essentially $\beta < \epsilon$ or $\zeta \leqslant \frac{\epsilon^2}{4d^4}$). From the above claim, we see that all $\zeta \leqslant \frac{\epsilon^4}{2^9 d^6 (d+1)^2}$ satisfies this condition (since $\frac{\epsilon}{d} < 1$). From now on fix $\zeta (> 0)$ to be any real number $\leqslant \frac{\epsilon^4}{2^9 d^6 (d+1)^2}$.

We have found an index two subgroup H in G. We shall now show that this subgroup H is disjoint from S.

Suppose, on the contrary that $t \in S \cap H$. This means the following

- $t \in S$. Therefore, $|tA \cap A| \leqslant |SA \cap A| \leqslant \frac{\beta}{\epsilon}|A|$ (see Lemma 2.8).
- $t \in H$. Therefore, by definition of H, $|tA \cap A| \geqslant r|A|$, where $r = 1 - \frac{d\beta}{\epsilon^2}\Big(\epsilon + d + 2\Big)$.

Combining, we see that $r \leqslant \frac{\beta}{\epsilon}$. This is clearly a contradiction since $0.82 < (1 - \frac{1}{4\sqrt{2}}) \leqslant r$ and $\frac{\beta}{\epsilon} < \frac{1}{8\sqrt{2}} < 0.09$.

This implies that $S \subset G \backslash H$, contradicting the hypothesis.

To summarise, we have shown that - for any fixed $\zeta \leqslant \frac{\epsilon^4}{2^9 d^6 (d+1)^2}$, if there exists an eigenvalue of the normalised adjacency matrix of $C(G,S)$ less than $-1+\zeta$, then $C(G,S)$ must be bipartite (equivalently it has an index 2 subgroup disjoint from S). That means, for non-bipartite $C(G,S)$, we must have all eigenvalues of the normalised adjacency matrix $\geqslant -1 + \frac{\epsilon^4}{\alpha d^6 (d+1)^2}$ with $\alpha = 2^9$. We are done.

□

Since, by definition, the vertex Cheeger constant $h(G)$ is the infimum of $\frac{|SX \backslash X|}{|X|}$, we can replace ϵ by $h(G)$ in the above arguments, thus proving Theorem 1.4.

3. Concluding Remarks

The above bound is dependent on the Cayley graph structure and does not hold for general non-bipartite finite, regular graphs. In the setting of arbitrary finite regular graphs some recent works are worth mentioning. Bauer and Jost in [BJ13] introduced a dual Cheeger constant $\bar{h}$ which encodes the bipartiteness property of finite regular graphs. The dual Cheeger constant $\bar{h}$ of a d regular graph is defined as

$$\bar{h} := \max_{V_1, V_2, V_1 \cup V_2 \neq \phi} \frac{2|E(V_1, V_2)|}{vol(V_1) + vol(V_2)},$$

for a partition V_1, V_2, V_3 of the vertex set V, $vol(V_k) = d|V_k|$ and $|E(V_1, V_2)|$ denotes the number of edges going from V_1 into V_2. For a general regular graph it was shown by Bauer-Jost (and independently by Trevisan [Tre09]) that

Theorem 3.1 (Bauer-Jost [BJ13]). *Let λ_n be the largest eigen-value of the graph laplace operator. Then λ_n satisfies*

$$\frac{(1-\bar{h})^2}{2} \leqslant 2 - \lambda_n \leqslant 2(1-\bar{h})$$

and the graph is bipartite if and only if $\bar{h} = 1$.

There is also the concept of higher order Cheeger constants introduced by Miclo in [Mic08].

Some recent works treating higher order Cheeger inequalities for general finite graphs are those by Lee–Gharan–Trevisan in [LGT14] and Liu [Liu15] (for the dual case) etc.

Acknowledgements

I wish to thank Emmanuel Breuillard for the problem and also for a number of helpful discussions and advice on the subject. I wish to thank the anonymous referees for their comments and suggestions, thus improving the article. I also wish to acknowledge the support of the OWLF programme of the Mathematisches Forschunginstitut Oberwolfach.

References

[AM85] N. Alon and V. D. Milman, λ_1, *isoperimetric inequalities for graphs, and superconcentrators*, J. Combin. Theory Ser. B **38** (1985), no. 1, 73–88.

[BGGT15] E. Breuillard, B. Green, R. Guralnick, and T. Tao, *Expansion in finite simple groups of Lie type*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 6, 1367–1434.

[BJ13] F. Bauer and J. Jost, *Bipartite and neighborhood graphs and the spectrum of the normalized graph Laplace operator*, Comm. Anal. Geom. **21** (2013), no. 4, 787–845.

[Bus82] P. Buser, *A note on the isoperimetric constant*, Annales scientifiques de l'École Normale Supérieure **15** (1982), no. 2, 213–230 (eng).

[Che70] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis (Papers dedicated to Salomon Bochner), Princeton Univ. Press, Princeton, N. J.,1970, pp. 195–199.

[Chu96] F. R. K. Chung, *Laplacians of graphs and Cheeger's inequalities*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 157–172.

[Fm73] G. A. Freĭ man, *Groups and the inverse problems of additive number theory*, Number-theoretic studies in the Markov spectrum and in the structural theory of set addition (Russian), Kalinin. Gos. Univ., Moscow, 1973, pp. 175–183.

[LGT14] J.R. Lee, S.O. Gharan and L. Trevisan, *Multiway Spectral Partitioning and Higher-Order Cheeger Inequalities*, J. ACM, November 2014, pp. 37:1–37:30.

[Liu15] S. Liu, *Multi-way dual Cheeger constants and spectral bounds of graphs*, Adv. Math. **268** (2015), 306–338.

[Lub94] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel, 1994, With an appendix by Jonathan D. Rogawski.

[Mic08] L. Miclo, *On eigenfunctions of Markov processes on trees*, Probab. Theory Related Fields **142** (2008), no. 3-4, 561–594.

[Nil91] A. Nilli, *On the second eigenvalue of a graph*, Discrete Mathematics **91** (1991), no. 2, 207 – 210.

[Tre09] L. Trevisan, *Max cut and the smallest eigenvalue*, STOC'09—Proceedings of the 2009 ACM International Symposium on Theory of Computing, ACM, New York, 2009, pp. 263–271.

Universität Wien, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria.

Email address: biswasa43@univie.ac.at