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from Oberwolfach

№ 8/2019

Nonlinear Acoustics

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Nonlinear acoustics has been a topic of research for more than 250 years. Driven by a wide range and a large number of highly relevant industrial and medical applications, this area has expanded enormously in the last few decades. Here, we would like to give a glimpse of the mathematical modeling techniques that are commonly employed to tackle problems in this area of research, with a selection of references for the interested reader to further their knowledge into this mathematically interesting field.

1 Introduction

High intensity ultrasounds, that is, sound waves with frequencies higher than the upper audible limit of human hearing, are used in numerous applications, ranging from lithotripsy and thermotherapy to sonochemistry.[1] At the pressure levels relevant for these applications, sound propagation does not follow the standard *linear acoustic wave equation*

$$\ddot{u} = c^2 \Delta u \,, \tag{1}$$

[1] Lithotripsy is a medical procedure involving the physical destruction of hardened masses like kidney stones, bezoars or gallstones. Thermotherapy is the use of heat for pain relief and other healthcare applications. Ultrasound waves can be used to heat tissues deep inside the body. Sonochemistry is the application of ultrasound to chemical reactions and processes.

any more, and higher-order nonlinear terms have to be taken into account.[2] Here $\ddot{u} = \frac{\partial^2}{\partial t^2} u$ denotes the partial derivative with respect to time and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ the *spatial Laplace operator*. The real function $u(t, x, y, z)$ represents the *wave profile*, such as a mechanical displacement or an acoustic pressure, as a function of space and time and c is a parameter describing the speed of wave propagation.

A central theme of this snapshot is the description of the propagation of acoustic waves through space (as a function of time) by appropriate partial differential equations (PDEs), that will be generalizations of the linear wave equation (1). For a complete physical description and conditions for the unique mathematical solvability, these PDEs would need to be equipped with initial and boundary conditions, for example, describing what happens when they reflect on a surface or encounter an obstacle. This issue will not be addressed here, though.

The history of the physical investigation and mathematical description of nonlinear sound propagation already started in the 1750s with the formulation of Leonard Euler (1707–1783) of the fundamental underlying equations and involved not only physicists, but a remarkable number of famous mathematicians, such as Georg Friedrich Bernhard Riemann, Pierre-Simon Laplace, Joseph-Louis Lagrange (born as Giuseppe Lodovico Lagrangia), Siméon Denis Poisson, Sir George Gabriel Stokes (1st Baronet, PRS), Sir George Biddell Airy, and Guido Fubini, to name just a few examples. For a nice historical overview we refer to Chapter 1 in [12].

2 Models of acoustic waves

2.1 Models of acoustic waves: basic principles

Waves propagate through space as a function of time. Any quantity that describes the waves must therefore be, in principle, a function of the spatial coordinates $\boldsymbol{x} \equiv (x, y, z)$ and time t. The key physical quantities in the description of acoustic wave propagation are:

- The acoustic particle velocity $\mathbf{v}(t, \boldsymbol{x})$;
- The acoustic pressure $p(t, \boldsymbol{x})$;
- The mass density $\varrho(t, \boldsymbol{x})$.

From now on, for the sake of simplicity, we drop the explicit dependence of these functions (or any new function) on the coordinates $(t, \boldsymbol{x})$ unless stated otherwise.

[2] For an introduction to the linear acoustic wave equation see Snapshot 006/2018 *Fast Solvers for Highly Oscillatory Problems* by Alex Barnett.

In most applications, except extreme events such as explosions, the main variables velocity $\mathbf{v}$, pressure p and density ρ have values that fluctuate around a *mean*. This means that they can typically be decomposed into their average values, which we denote by, for instance, $\mathbf{v}_0$ and fluctuating components, denoted by $\mathbf{v}_\sim$.[3] We have

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_\sim , \qquad p = p_0 + p_\sim , \qquad \varrho = \varrho_0 + \varrho_\sim . \tag{2}$$

We note however that, in this snapshot, we will only consider applications where the mean value $\mathbf{v}_0$ of the velocity is equal to zero. Therefore, we can replace $\mathbf{v}$ with $\mathbf{v}_\sim$. Furthermore, we will consider applications where the remaining average quantities are time independent, which means that $\dot{\varrho}_0 = 0$ and $\dot{p} = 0$.

Together with these key quantities, we need a few physical constants characterizing any fundamental wave equation, namely:

- The speed of sound c, that is, the speed at which sound waves propagate through a medium (for example air or water);
- The dimensionless nonlinearity coefficient $\beta_a = 1 + \frac{B}{2A}$, where we anticipate that A and B are the coefficients of the first and second order terms of the Taylor series expansion of the equation relating the acoustic pressure p with the mass density ϱ, see (10). To understand this snapshot it is enough to think of β_a as a constant;
- The viscosities $\mu_1 = \frac{4\mu_V}{3} + \zeta_V$ and $\mu_2 = \frac{c_p}{c_V}(\frac{1}{c_V} - \frac{1}{c_p})$, where μ_V is the dynamic viscosity, ζ_V is the bulk viscosity, and c_V, c_p are the specific heat capacity at constant volume and constant pressure, respectively. The viscosity is a quantity that measures the resistance of the fluid to flow or, more precisely, its resistance to gradual deformation by shear stress or tensile stress. The heat capacity expresses the amount of (heat) energy which is needed to increase the temperature of a medium by 1 degree Celsius.

2.2 Models of acoustic waves: *balance* and *material* laws

The equations that govern the changes of $\mathbf{v}$, p and ϱ, and their relationship, consist of *balance* and *material* laws, which can be summarized as follows:

- **Continuity equation.** This equation encodes the conservation of mass.

$$\dot{\varrho} + \nabla \cdot (\varrho \mathbf{v}) = 0. \tag{3}$$

[3] An example of this is the pressure inside of a hot sealed pot. The air inside has a certain temperature and a certain pressure *on average*, but if we had a thermometer and barometer that were sensitive enough, we would see that the pressure and temperature oscillate around a mean value.

Here the dot on top of a quantity means its derivative with respect to time and $\boldsymbol{x} \cdot \boldsymbol{y}$ is the standard vector product between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Also, we recall that the gradient ∇ is a vector defined by $\nabla := (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, and the divergence $\nabla \cdot \mathbf{v}$ of a vector $\mathbf{v} = (v_x, v_y, v_z)$ is defined to be $\nabla \cdot \mathbf{v} := \partial_x v_x + \partial_y v_y + \partial_z v_z)$, and the Laplacian Δ of a function u is defined to be $\Delta u := (\partial_x^2 + \partial_y^2 + \partial_z^2)\, u$.

- **Euler equation.** This equation encodes the conservation of momentum.

$$\varrho\Big(\dot{\mathbf{v}} + \tfrac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v})\Big) + \nabla p = \mu_1 \Delta \mathbf{v}. \tag{4}$$

- **Equation of state.** This equation encodes the relation between pressure and density.

$$\varrho_\sim = \tfrac{1}{c^2} p_\sim - \tfrac{\beta_a - 1}{\varrho_0 c^4} p_\sim^2 - \tfrac{\mu_2}{\varrho_0 c^4} \dot{p}_\sim. \tag{5}$$

- **Conservation of energy.** In case one is interested in the derivation of higher order models, such as those described by the Blackstock-Crighton equation (16) which we present later on, one needs to employ an equation expressing conservation of energy. We will elaborate on this later on.

2.3 Models of acoustic waves: an example

Models of linear and nonlinear acoustics are obtained by appropriately combining the equations presented above and by dropping "unnecessary" higher order terms from the equations obtained. We demonstrate this procedure for the basic case of the standard linear acoustic wave equation, which results from retaining only linear and non-dissipative terms. This means that we will drop terms like $\nabla \cdot (\varrho_\sim \mathbf{v})$, $\varrho_\sim \dot{\mathbf{v}}$, $\frac{1}{2}\varrho\nabla(\mathbf{v}\cdot\mathbf{v})$, $\mu_1 \Delta \mathbf{v}$.

We start by inserting the decomposition (2) into the conservation of mass equation (3), and we obtain

$$\dot{\varrho}_0 + \dot{\varrho}_\sim + \nabla \cdot (\varrho_0 \mathbf{v}) + \nabla \cdot (\varrho_\sim \mathbf{v}) = 0\,.$$

Here, recall that we are assuming that the (time) derivative $\dot{\varrho}_0$ of the constant mean mass density ϱ_0 vanishes. The term $\nabla \cdot (\varrho_\sim \mathbf{v})$, which contains the product of two fluctuating parts, is an example of a *nonlinear term* and will therefore be dropped. Proceeding similarly for the conservation of momentum and the pressure-density relation, we end up with the following linearized versions of the equations (3), (4), (5):

$$\dot{\varrho}_\sim + \varrho_0 \nabla \cdot \mathbf{v} = 0, \tag{6a}$$

$$\varrho_0 \dot{\mathbf{v}} + \nabla p_\sim = 0, \tag{6b}$$

$$\varrho_\sim = \tfrac{1}{c^2} p_\sim. \tag{6c}$$

We now take the second derivative of (6c) with respect to time, which gives us $\ddot{\varrho}_\sim = \frac{1}{c^2}\ddot{p}_\sim$, the time derivative of (6a), obtaining $\ddot{\varrho}_\sim + \varrho_0 \nabla \cdot \dot{\mathbf{v}} = 0$. This allows us to eliminate $\ddot{\varrho}_\sim$ from the first equation, obtaining $\frac{1}{c^2}\ddot{p}_\sim + \varrho_0 \nabla \cdot \dot{\mathbf{v}} = 0$. Now we take the divergence of (6b), which gives $\varrho_0 \nabla \cdot \dot{\mathbf{v}} = -\Delta p_\sim$ and insert this into the result of our previous manipulations, which allows us to eliminate $\mathbf{v}$. These algebraic manipulations yield as a result the *linear* wave equation

$$\ddot{p}_\sim - c^2 \, \Delta p_\sim = 0. \tag{7}$$

We emphasize here that this equation is *linear* because all terms that appear contain the fluctuating quantities $\boldsymbol{v}, p_\sim$ and $\rho_\sim$ at most to first order. All other combinations, such as $p_\sim^2$ or $\boldsymbol{v}\,\rho_\sim$ are nonlinear terms.

2.4 Models of acoustic waves: Blackstock's scheme

The derivation of models for *nonlinear* acoustics follows the same guidelines, but requires some additional "sophistication" when choosing to neglect a term. Here, Blackstock's scheme, which was first introduced by Sir Michael James Lighthill (1924-1998) [18] and subsequently described in more detail by Blackstock [2], plays an essential role. This scheme distinguishes between the following categories:

- **First order.** First order terms are linear with respect to the fluctuating quantities and are not related to any dissipative effect;
- **Second order.** Terms of this order are obtained as the union of quadratic and dissipative linear terms (that is, those terms that contain the viscosities μ_1, μ_2 as prefactors);
- **Higher order.** All remaining terms.

Blackstock's scheme prescribes that one should retain only first and second order terms. Additionally, a result called the *substitution corollary* allows us to replace any quantity in a second or higher order term by its first order approximation, that is, for the term $\dot{p}_\sim \Delta p_\sim$, we can employ the (linear) wave equation (7) to replace $\Delta p_\sim$ by $\frac{1}{c^2}\ddot{p}_\sim$ to obtain $\frac{1}{c^2}\dot{p}_\sim \ddot{p}_\sim$. This step is needed for the derivation of, for example, the Kuznetsov and Westervelt equations, which we discuss below. These equations have important implications for real-life applications.

2.4.1 Kuznetsov's equation

The prescriptions given in Blackstock's scheme lead to *Kuznetsov's equation*, see [17, 16], which consists of the two parts

$$\ddot{p}_\sim - c^2 \Delta p_\sim - b \Delta \dot{p}_\sim = \frac{d^2}{dt^2}\left(\tfrac{\beta_a - 1}{\varrho_0 c^2} p_\sim^2 + \varrho_0 |\mathbf{v}|^2 \right) \tag{8}$$

and

$$\varrho_0 \dot{\mathbf{v}} = -\nabla p_\sim \,, \tag{9}$$

that is, a coupled system of differential equations for the acoustic particle velocity $\mathbf{v}$ and the acoustic pressure p, where $b = \frac{\mu_1 + \mu_2}{\varrho_0}$ is the *diffusivity* of sound.[4] This system can be rewritten as a single equation for the acoustic velocity potential in the following way. First, we know that $\nabla \times \mathbf{v} = 0$ and hence

$$\mathbf{v} = -\nabla \psi$$

where we have introduced an acoustic velocity potential ψ that satisfies

$$\varrho_0 \psi_t = p_\sim \,, \tag{10}$$

due to the equation (9). We can then rewrite (8) and (9) as a single differential equation for the potential ψ, which reads

$$\ddot{\psi} - c^2 \Delta \psi - b \Delta \dot{\psi} = \frac{d}{dt} \left(\tfrac{\beta_a - 1}{c^2} (\dot{\psi})^2 + |\nabla \psi|^2 \right) . \tag{11}$$

2.4.2 Westervelt's equation

If we now use the approximation

$$\varrho_0^2 |\mathbf{v}|^2 \approx \tfrac{1}{c^2} {p_\sim}^2 \,,$$

which is equivalent to neglecting nonlinear effects in (8), we arrive at *Westervelt's equation* [26], which reads

$$\ddot{p}_\sim - c^2 \Delta p_\sim - b \Delta \dot{p}_\sim = \tfrac{\beta_a}{\varrho_0 c^2} \ddot{p}_\sim^2 . \tag{12}$$

It can be written in terms of the velocity potential ψ by plugging equation (10) and its first and second derivatives into (12), leaving us with

$$\ddot{\psi} - c^2 \Delta \psi - b \Delta \dot{\psi} = \tfrac{\beta_a}{c^2} \, \ddot{\psi}^2 \,. \tag{13}$$

2.4.3 Khokhlov-Zabolotskaya-Kuznetsov's equation

If we assume further that there exists a preferred direction of propagation, say the x direction, then we obtain the *Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation* [27], which reads

$$2\, c \, \frac{\partial}{\partial x} \dot{p}_\sim - c^2 \Delta_{yz} p_\sim - \tfrac{b}{c^2} \dddot{p}_\sim = \tfrac{\beta_a}{\varrho_0 c^2} \ddot{p}_\sim^2 \,, \tag{14}$$

where $\Delta_{yz} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator with respect to the coordinates orthogonal to the propagation direction.

[4] Diffusion or diffusivity, in acoustics and architectural engineering, is the efficacy by which sound energy is spread evenly in a given environment.

2.4.4 Burgers' equation

Reduction of the setting to one space dimension yields the well-known *Burgers equation*, which has the expression

$$\frac{\partial}{\partial x} p_{\sim} - \frac{b}{2c^3} \frac{\partial^2}{\partial \tau^2} p_{\sim} = \frac{\beta_a}{\varrho_0 c^3} p_{\sim} \frac{\partial}{\partial \tau} p_{\sim} \,, \tag{15}$$

where $\tau = t - \frac{x}{c}$ is the *retarded time*, which measures the position of the propagation of a perturbation (or value of $p_{\sim}$). More concretely, the perturbation at $t - \frac{x}{c} = \tau_0$ is the same for any values of t and x that satisfy $\tau = \tau_0$. For example, it has *propagated* from $x = -c\,\tau_0$ at time $t = 0$ to $x = c\,(t - \tau_0)$ at time t.[5]

To give a visual impression of nonlinear acoustic wave propagation, in Figure 1 we plot the approximate solutions of the 1-dimensional Burgers equation at different distances x from a source that excites the system in a *sinusoidal manner* – that is, it excites the system periodically in time. The corresponding explicit (in the form of inifinite series) solution formulas for the cases of low and high distance from the sound source have been found by Fubini [11] and Fay [10], respectively, see also [3].

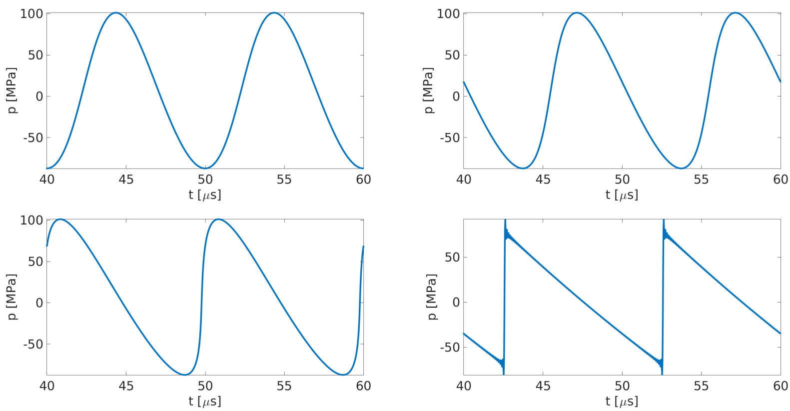

Figure 1: Left to right and top to bottom: Fubini (with $\sigma = 0.2, 0.5, 0.9$) and Fay (with $\sigma = 3$) solutions. Here σ is the so-called "shock formation distance", a parameter describing the spatial distance from the source.

[5] Some material about retarded time can be found in [5, 9, 21].

3 Models of acoustic waves: modern extensions

We emphasize that modeling nonlinear acoustic wave propagation is still an active area of research. Let us just mention two examples of extensions of the most general of the equations above, in particular of (11), namely the *Blackstock-Crighton* equation

$$\dddot{\psi} - c^2\Delta\dot{\psi} - (b+a)\,\Delta\ddot{\psi} + \beta_a a\,(b-(\beta_a-1)a)\,\Delta^2\dot{\psi} + c^2 a\Delta^2\psi = \left(\tfrac{\beta_a-1}{c^2}(\dot{\psi}^2) + |\nabla\psi|^2\right)_{tt}, \tag{16}$$

see [4, 2, 8], where a is the thermal conductivity, and the *Jordan-Moore-Gibson-Thompson* equation

$$\tau_r\dddot{\psi} + \ddot{\psi} - c^2\Delta\psi - b\Delta\dot{\psi} = \frac{d}{dt}\left(\tfrac{\beta_a-1}{c^2}(\dot{\psi})^2 + |\nabla\psi|^2\right), \tag{17}$$

where the *relaxation time* τ_r expresses the characteristic time which a fluid needs to reach its equilibrium after excitation, see [6, 13, 25]. These equations reduce to Kuznetsov's equation (11) for $a = 0$ and $\tau_r = 0$, respectively.

Further interesting models can be found, e.g., in [1, 7, 14, 15, 19, 20, 22, 23, 24] and the references therein.

3.1 Final considerations

Due to the numerous applications of high intensity ultrasound, which is used, for example, for cancer treatment, nonlinear acoustics is a highly active area of applied mathematics. In particular, a mathematical analysis of the modeling equations is a crucial prerequisite for reliable numerical simulation and optimization. Although some of the mentioned models have been analyzed with respect to their well-posedness and asymptotic behaviour,[6] there are still many open problems to be solved. This makes the area of nonlinear acoustics very attractive for mathematicians working, for example, in the areas of partial differential equations, numerics or optimization.

Acknowledgments

The authors wish to thank Manfred Kaltenbacher (Technische Universität Wien) and Vanja Nikolić (Technische Universität München) for providing a Matlab script to plot the Fubini and Fay solutions.

[6] A mathematical problem is well posed if it has a unique solution which depends continuously on the given (initial and boundary) data. The asymptotic behavior refers to the behavior of solutions of differential equations for large parameters or large times.

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DOI
10.14760/SNAP-2019-008-EN

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ISSN 2626-1995

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