MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Geometry and Analysis on Loop Space

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The conference was organized by Michael Röckner (Bielefeld) and Stephan Stolz (Notre Dame) and was attended by 23 participants. The goal of the conference was to bring topologists and stochastic analysts closer together in order to investigate the geometry of the loop space of a closed Riemmannian manifold. The participating topologists have become interested in stochastic analysis since it conjecturally will lead to an analytically rigorous definition of the so-called Dirac operator on loop space. The latter is frequently used in mathematical physics, and it led Witten to define a genus, which should be thought of as the S^1-equivariant index of the Dirac operator on loop space. A mathematically rigorous definition of the Dirac operator is expected to give new relations between Witten's genus and the differential geometry of the underlying manifold. The stochastic analysts on the other hand appreciated that their research might eventually lead to nice applications outside their field.
The conference started out with some introductory lectures on differential geometry, Dirac bundles, characteristic classes and elliptic cohomology, as well as lectures on Wiener measure, stochastic integration by parts and stochastic exterior forms on Hilbert space bundles. After this various people gave talks on related research projects. During the whole conference there was a good atmoshpere dominated by the spirit of learning from each other.

Wolfgang Lück
The Laplace-Beltrami operator and Bochner's formula

We introduce basic notions like deRham complex, deRham cohomology and Laplace operator and explain the Hodge-deRham theorem, which identifies the singular cohomology of a closed Riemannian manifold with the space of harmonic forms. Then we introduce the curvature tensor and explain how it yields the Ricci tensor and the scalar curvature by contractions. We explain that the curvature tensor carries the same information as the sectional curvature and how one can interprete Ricci curvature as a function on the sphere tangent bundle. We state how one gets Ricci curvature from sectional curvature and scalar curvature by integration. We give geometric interpretations of all these notions, for instance we explain how scalar curvature and Ricci curvature is linked to the infinitesimal growth of volumes of balls and sectors respectively. Finally we state the Bochner-Lichnerowicz formula and present the consequence that positive Ricci curvature implies the vanishing of the first Betti number. We give a list of known topological obstruction for a closed smooth manifold to carry a Riemannian metric with certain sign-conditions on one of the curvature notions.

Michael Joachim
The Atiyah-Singer operator and Lichnerowicz's formula

We started off with the definition of Clifford algebras, Clifford algebra bundles and modules over those. We introduced the notion of a Dirac bundle over a Riemannian manifold and defined its associated Dirac operator. We then presented the theorem, that the Dirac operator associated to the Dirac bundle $Cl(M)$ of a given Riemannian manifold M agrees with the Hodge Laplacian. Next we discussed spin structures on manifolds, the notion of a spinor bundle and investigated the natural Dirac bundle structures on spinor bundles. Finally we presented the general Bochner formula for Dirac bundles and the Lichnerowicz formula which is a special case.

Andreas Eberle
Wiener measures on path and loop spaces

In this introductory talk, Wiener measure on the space of continuous loops over a Riemannian manifold M is defined in terms of its marginals. These are absolutely continous w.r.t. the volume element on M^N, and the density is a product of heat kernels. The definition is motivated by showing that one is led more or less directly to a measure closely related to Wiener measure, if one tries to find a measure on the loop space that satisfies certain basic assumptions.

Brownian motion and Brownian bridges on manifolds are defined as stochastic processes having Wiener measure on path or loop space respectively as their sample path distribution. After reviewing standard results on regularity of Brownian paths, the *"rolling without slipping"* construction of Brownian motion on a manifold from Brownian motion on $\mathbf{R}^n$ is discussed briefly. For this purpose, a very short introduction to Ito and Stratonovich integrals is given.

Finally, a simple approximation scheme for Wiener measures over manifolds is described on a heuristic level.

David Elworthy
Introduction to integration by parts on path spaces

This expository talk described some of the problems and successes in obtaining integration by parts formulae for measures on infinite dimensional spaces such as loop spaces of Riemannian manifolds.The starting point was the fact that on an infinite dimensional Banach space there is no non-trivial Radon measure which is quasi-invariant ,let alone invariant,by all translations.Gaussian measures were described:for them there is a Hilbert space of directions for which quasi-invariance holds with a consequent restricted class of vector fields for which there is an integration by parts formula,with the Ito integral coming in as a divergence operator in the case of classical Wiener space. The extensions of this to spaces of paths and loops on manifolds ,with measures derived from Brownian motion or other diffusions by Driver and others were then rather briefly indicated.

Xue-Mei Li
Integration by parts formulae for differential forms

Let $C_{x_0}M$ be the space of continuous paths on a Riemannian manifold. I described the approach to integration by parts formulae for functions, using stochastic differential equations instead of using the development map. This method extends for forms to give integration by parts formula of the following type:

$$\int_{C_{x_0}M} d\phi\,(V)\,d\mu_{x_0} = \int_{C_{x_0}M} \phi\,(divV)\,d\mu_{x_0},$$

where ϕ is a BC^1 q-form on $C_{x_0}M$, μ_{x_0} the Brownian motion measure and V a suitable q+1 vector field with $divV$ a q-vector field. For $q=1$, it is natural to take $V \in \Gamma(H.)$ where $H_\sigma \subset T_\sigma C_{x_0}M$ is the Bismut tangent space. However for higher order differential forms, $\Gamma(\Lambda^q H.)$ is not a suitable space of q-vector fields and the formulae we obtained apply to sections of rather different 'Hilbert bundle' with fibre $H^q_\sigma \subset T_\sigma C_{x_0}M$. The use of these H^q_σ appears to get rid of the problem of the definition of exterior differentiation on some Hilbert space of forms. This well known problem comes from the fact that for 1-form ϕ,

$$d\phi\left(v^1, v^2\right) = L_{v_1}\left(\phi\left(v^2\right)\right) - L_{v_2}\left(\phi\left(v^1\right)\right) - \phi\left(\left[v^1, v^2\right]\right)$$

and if $v^1, v^2 \in \Gamma(H.)$, it is well known that $[v^1, v^2]$ may not belong to $\Gamma(H.)$.

Remi Leandre
Stochastic cohomology of Chen-Souriau

We define a diffeological structure in the Holderian loop space by using a system of stochastic plots. This allows to define stochastic cohomology groups of the Holderian loop space. They are equal to the deterministic cohomology groups of the Holderian loop space: it can be shown by using the Weil proof from the fact that the Cech cohomology is equal to the de Rham cohomology, because there are partitions of unity associated to a certain cover of the Holderian loop space. It can be shown too by using the general theory of cohomology of sheaves.

Anand Dessai
The Atiyah-Singer Index Theorem

Given an elliptic differential operator on a closed manifold its index may be computed from topological data involving the symbol of the operator and the characteristic classes of the manifold. This is the content of the Atiyah-Singer index theorem. For the Dirac operator D^+ on a *Spin*-manifold M the index theorem gives

$$ind(D^+) = \hat{A}(M) = \int_M \hat{A}(p_1(M), p_2(M), \dots),$$

where $\hat{A}$ is the genus and $\hat{\mathcal{A}}$ is the multiplicative sequence associated to the characteristic power series $\frac{x/2}{\sinh x/2}$. More generally, if E is a complex vector bundle over M then the index of the twisted Dirac operator D_E^+ is given by

$$ind(D_E^+) = \int_M \hat{\mathcal{A}}(p_1(M), p_2(M), \dots) \cdot ch(E).$$

In this talk, being part of a series of introductory lectures, we recall the Chern-Weil definition of characteristic classes and sketch the heat kernel proof of the index theorem for the Dirac operator.

JOHN JONES
The Witten genus and loop spaces

A genus is a cobordsim invariant of closed manifolds. Genera are studies via certain generating functions. A genus is elliptic if its generating series is the power series expansion of an elliptic function with a simple pole at the origin. The first part of the talk was an outline of the general theory of elliptic genera.
The Witten genus is a particular example of an elliptic genus. The second part of the talk was an account of the formal calculation relating the Witten genus to free loop space - essentially the Witten genus is the character valued index of an S^1-equivariant Dirac operator on the free loop space. Unfortunately it has not been shown rigorously that this Dirac operator exists so at this time, this is nothing more than a formal, but very suggestive calculation.
In the third part of the talk I gave an account of two calculations of my student Andrew Stally. Both calculations are along the following lines: to what extend can you calculate the Witten genus by taking finite dimensional approximations to the loop space and then take a limit.

TILMANN WURZBACHER
Some remarks on loop space geometry

Given a finite dimensional smooth riemannian manifold (M, g) the geometry on the loop space $LM = C^\infty(S^1, M)$ hinted at by "quantum mechanics of strings" appears to have several surprising features. Notably, the natural metric on LM seems to be rather the so-called $H^{\frac{1}{2},2}$-metric than the energy metric $H^{1,2}$, leading to a different stochastic set-up than the more traditional couple Wiener measure plus Brownian motion arising from the first quantisation of point mechanics, and the rotation-invariance shows up not only as a remainder of the larger invariance under the group of all diffeomorphism of the circle but as a salient ingredient of the geometry and the analysis on free loop spaces. After recalling briefly the fact that the $H^{\frac{1}{2},2}$-metric is induced on loop groups by their natural embeddings in the restricted Grassmannian, the talk concentrated on three – partly conjectural – topics in the geometry of loop spaces of general riemannian manifolds:

* the classical symplectic geometry and geometric quantisation of loop spaces with special emphasis on diffeomorphism invariance,
* the hamiltonian description and quantisation of non-linear sigma-models (i.e. harmonic maps) on cylinders with values in M,
* the hypothetical rotation-equivariant Dirac-Ramond operator on a loop space and its S^1-space equivariant index.

Michael Röckner
Dirichlet operators on loop spaces

In part one of the talk we review the construction of Laplace–type operators Δ^μ (a particular class of Dirichlet operators) on a Riemannian manifold M, but with the rôle of the volume measure replaced by an arbitrary nonnegative Radon measure on M (positively charging every non–empty open set). The crucial assumption made on μ is that it satisfies an integration by parts formula, i.e., has a logarithmic derivative. Then it is proved that the gradient operator is closable on $L^2(M;\mu)$, so a corresponding Sobolev space $H_0^{1,2}(M;\mu)$ is defined. Hence Δ^μ can be defined directly as a self–adjoint (*not only* symmetric) operator on $L^2(M;\mu)$. In part two of the talk the rôle of M is replaced by the pinned loop space $\mathcal{L}_{x_0}(M)$ over M (where $x_0 \in M$ is the base point) and μ by the pinned Wiener measure μ_{x_0}. After reviewing the basic definitions of the tangent space ("Bismut tangent space") and the gradient on $\mathcal{L}_{x_0}(M)$, it is shown that a *completely analogous* construction as for finite dimensional manifolds gives rise to a Sobolev space $H_0^{1,2}(\mathcal{L}_{x_0}(M);\mu_{x_0})$ and a μ_{x_0}–Laplacian $\Delta^{\mu_{x_0}}$ on $\mathcal{L}_{x_0}(M)$ which is self–adjoint on $L^2(\mathcal{L}_{x_0}(M);\mu_{x_0})$. The same construction apart from some technical complications works for the free loop space $\mathcal{L}(M)$ with μ_{x_0} replaced by the Bismut–Høegh Krohn measure.

Igor Kriz
Elliptic cohomology

I explained the definition of complex-oriented elliptic cohomology using the formal group law and universal elliptic genus. I discussed Ando's interpretation of the Kac character formula which identifies projective representations of LS^1 with sections of line bundles over the Tate curve, and related attempts to define elliptic cohomology of X by means of bundles on the free loop space LX. I explained the flaw of these attempts related both to representability issues, and also the fact that the formal group law of the Tate curve is multiplicative. I proposed an approach to this problem by means of conformal field theories: Let $\Phi(X)$ be the space of all maps $(e, \partial e) \to (X, *)$ for a conformal subsurface of $\mathbb{C}$. Then $\Phi(X)$ has the structure of an operad over and under $\Phi(*)$. Such operads I call string semigroups. Following the path-integral approach to bundles, I defined a String bundle with structure string semigroup G as the set of all maps $\Phi(X) \to G$ in the derived category of string semigroups. This is a representable functor in the category of simply connected spaces. I also proposed a possible choice of G as a certain space of "invertible" elements (in a certain special sense) in the space of Hilbert-Schmidt operators $\mathcal{H}^{\otimes n} \to \mathcal{H}$.

Karl-Theodor Sturm
Nonlinear Markov Operators and Harmonic Maps with Values in Metric Spaces of Nonpositive Curvature

We present a new, elementary theory of harmonic maps $f : M \to N$ between singular spaces M and N. The target will be a complete geodesic space (N, d) of nonpositive curvature in the sense of A. D. Alexandrov. The domain will be a measurable space $(M, \mathcal{M})$ with a given Markov chain $(X_n, \mathbb{P}_x)$ on it. Our theory is a nonlinear generalization of the theory of Markov kernels and Markov chains on M. It allows to construct harmonic maps by an explicit nonlinear Markov chain Monte Carlo algorithm.

Every Markov chain $(X_n, \mathbb{P}_x)$ on M defines in a canonical way a *nonlinear Markov operator* P acting on the set of measurable maps $f : M \to N$. The map $Pf : M \to N$ is defined pointwise: For each $x \in M$ the point $Pf(x) \in N$ is the unique minimizer of the strictly convex, continuous function

$$z \mapsto \mathbb{E}_x d^2(z, f(X_1)) < \infty$$

on N, i.e. $Pf(x)$ is the barycenter of the distribution of the random variable $f(X_1)$ on N. In the linear case $N = \mathbb{R}$, the operator P coincides with the usual linear Markov operator defined by $Pf(x) = \mathbb{E}_x f(X_1) = \int f(X_1) d\mathbb{P}_x$. We say that a map $f : M \to N$ is *harmonic on* $M_0 \subset M$ iff $Pf = f$ on M_0.

In the same spirit as above (in the definition of expectations as barycenters) one can define conditional expectations for N-valued random variables. This allows to develop a rich *nonlinear martingale theory* for (time discrete) processes with values in N. It turns out that a map $f : M \to N$ is harmonic if and only if the process $(f(X_n))_n$ is a N-valued martingale under $\mathbb{P}_x$ for each $x \in M$, i.e. iff $\mathbb{E}_x [f(X_{n+1})|\mathcal{F}_n] = f(X_n)$ a.s. for all $n \geq 0$.

STEPHAN STOLZ

A conjecture concerning positive Ricci curvature and the Witten genus

We give evidence for the following

Conjecture *Let M be a closed spin manifold with $\frac{1}{2}p_1(M) = 0$. If M admits a metric of positive Ricci curvature, then the Witten genus of M vanishes.*

We recall that for spin manifolds the first Pontrjagin class $p_1(M)$ is divisible by 2. The Witten genus is a formal powerseries, whose coefficients are characteristic numbers of M. Following Witten, this genus should be thought of as the S^1-equivariant index of a hypothetical 'Dirac operator' on the free loop space of M.

SYLVIE PAYCHA

A Bochner-Weitzenböck formula on loop groups

On a Riemannian manifold, the Laplacian does not in general commute with the Levi-Civita connection and the obstruction can be expressed in terms of the Ricci curvature of the manifold. When trying to make sense of a similar statement in infinite dimensions, one first has to make sense of the notion of Laplacian and the notion of Ricci curvature in infinite dimensions. There are many ways to do so and we suggest here to use zeta type renormalized traces in order to define a Laplacian on functions as a renormalized trace of the Hessian (and similarly for tensors) and the Ricci curvature as a renormalized trace of the Ricci tensor. This makes sense on a class of infinite dimensional Hilbert manifolds including current (and hence loop groups) investigated by Freed and we can derive a Bochner-Weitzenböck type formula which involves an extra term expressed in terms of a Wodzicki residue. This purely infinite dimensional term vanishes on loop groups and we recover an ordinray Bochner-Weitzenböck formula (which coincides on cylinder set functions with a formula derived by Driver and Lhorenz).

STEVEN ROSENBERG

Ricci and Scalar Curvature on Loop Spaces

The infinite dimensional manifold LM of smooth loops on a closed Riemannian manifold has a natural L^2 metric and Levi-Civita connection. The sectional curvature for this metric is well-defined, but the classical definitions of Ricci curvature Ric and scalar curvature s involve traces which produce divergent infinite series on LM. In this talk we use zeta function regularization to define the Ricci and scalar curvature on LM. We produce a regularization scheme which satisfies $\mathrm{Ric}_M \geq 0 \Rightarrow s_{LM} \geq 0$, which is a part of Stolz's "wish list" for the vanishing of the Witten genus.

Recalling the Lichnerowicz formula in finite dimensions, this suggests defining the Dirac operator and its square on LM by a similar regularization scheme $D\psi \equiv \mathrm{Tr}^{\mathrm{reg}}(e \mapsto e \cdot \nabla\psi)$, $e \in T^*LM$, which generalizes the Dirac operator on a finite dimensional manifold. It is unknown at present if this regularized operator has a Lichnerowicz formula

Gerard Miosolek

Fredholmness of the exponential map on the free loop space

As a first step toward a better understanding of global geometry of Hilbert Riemannian manifolds one studies singularities of its exponential map. Singular values of exp are the conjugate points. In contrast with finite dimensions two types of conjugate points can occur on a Hilbert manifold depending on whether dexp fails to be 1-1 or onto. Further, there exist complete Hilbert Riemannian manifolds with finite geodesic segments containing infinitely many conjugate points of either type and of possibly infinite multiplicity. In the talk I described a result stating that the exponential map on the free loop space $L(M)$ of a compact oriented Riemannian manifold M and equipped with its natural Sobolev H^1 metric is a nonlinear Fredholm map of index zero. From this result it follows that none of the situations described above occur on the free loop space. In particular, there can be only one type of conjugate points on $L(M)$, which answers a question of W. Klingenberg.

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