MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 14/2000

Nichtkommutative Geometrie
26.3.–1.4.2000

The conference was organized by Alain Connes (Paris), Joachim Cuntz (Münster), and Marc Rieffel (Berkeley).

There were 24 lectures altogether. A topic of great interest was the Baum-Connes conjecture. Recently, several counterexamples have been found to more general formulations of the conjecture. They were explained in the talks of Yu and Lafforgue. Moreover, the left hand side of the assembly map can be computed explicitly up to torsion using appropriate Chern characters introduced in the talks by Baum and Lück. Another recent breakthrough is the discovery of the relationship between exactness of the group C^*-algebra and the Novikov conjecture explained in Kaminker's talk.

A central issue of the conference was the relationship between non-commutative geometry and quantum field theory. The recent work of Connes and Kreimer seems to turn renormalization theory from a list of cooking recipes into a part of solid mathematics. Cattaneo's talk explained the relationship between quantum field theory and Kontsevich's formula for the deformation quantization of a Poisson manifold.

Several talks dealt with Hopf algebras—in particular the Connes-Kreimer Hopf algebra of rooted trees—and the relationship between Hopf algebra cyclic cohomology and characteristic classes.

Submitted abstracts

Non-commutative Hardy spaces and Toeplitz operators over $\mathrm{SL}(2,\mathbb{R})$

Alexander Alldridge

For the semi-simple Lie group $S = \mathrm{SL}(2,\mathbb{R})$, $L^2(S)$ admits a decomposition

$$L^2(S) = H^2(S_+^{\mathbb{C}}) \oplus H^2(S_0^{\mathbb{C}}) \oplus H^2(S_-^{\mathbb{C}})$$

into 'non–commutative Hardy spaces'. These are associated to a decomposition

$$S^{\mathbb{C}} = S_+^{\mathbb{C}} \,\dot{\cup}\, S_0^{\mathbb{C}} \,\dot{\cup}\, S_-^{\mathbb{C}} \quad \text{(up to sets of zero measure)}$$

into complex domains, of which $S_\pm^{\mathbb{C}}$ are pseudoconvex.

We give a structure theory of the C*-algebra $\mathcal{T}(S_+^{\mathbb{C}})$ generated by Toeplitz operators $T(f) = E^+ f E^+$, $f \in C_0(S)$, where E^+ is the Szegö projection onto $H^2(S_+^{\mathbb{C}})$. The main result is a composition series $\mathcal{LC}(H^2(S_+^{\mathbb{C}})) \lhd I_1 \lhd \mathcal{T}(S_+^{\mathbb{C}})$, where $\mathcal{T}(S_+^{\mathbb{C}})/I_1 \cong C_0(S)$ and

$$I_1/\mathcal{LC}(H^2(S_+^{\mathbb{C}})) \cong C_0\big(S \times S/N \times N \cdot \mathrm{diag}(AM)\big) \otimes \mathcal{LC}(H^2(N_+^{\mathbb{C}}))\ .$$

Path integrals and deformation quantization

Alberto Cattaneo

(joint work with Giovanni Felder)

Kontsevich's formula for the deformation quantization of Poisson manifolds can be obtained as the perturbative expansion of a certain expectation value in a topological open string theory called the Poisson sigma model.

The fields in this model are morphisms from the tangent bundle of the oriented disk (or more generally a Riemann surface, possibly with boundary) to the cotangent bundle of the given Poisson manifold. The associativity of the $*$-product follows in this picture from the topological nature of the theory.

On the other hand, the Hamiltonian approach reveals that the phase space of the Poisson sigma model is an explicit realization of the symplectic groupoid (in general, with singularities) for the given Poisson manifold. This is an object introduced independently by Weinstein and Karasev as a part of a program to quantize Poisson manifolds. So, in a sense, the Poisson sigma model provides a realization of their program.

Equivariant Chern character

Paul Baum

Let Γ be a countable discrete group. Let X be a locally finite simplicial complex with a proper action of Γ such that the quotient space X/Γ is compact and: if v, v' are vertices of a simplex σ in X with $v \neq v'$, then there does not exist $\gamma \in \Gamma$ with $\gamma v = v'$. We let $\hat{X}$ be the space $\hat{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$ equipped with the Γ-action $g(\gamma, x) = (g\gamma g^{-1}, gx)$. Then $\hat{X}$ satisfies the above requirements for X.

Let F be the cyclotomic field. In this talk (using a result of W. Lück) a simple explicit Chern character is constructed:

$$\mathrm{ch}\colon K_j(C_0(X) \rtimes \Gamma) \to \bigoplus_l H^{j+2l}(\hat{X}/\Gamma; F), \qquad j = 0, 1.$$

Upon tensoring the left side by F, this Chern character becomes an isomorphism. By duality, one then obtains

$$\mathrm{ch}\colon KK_\Gamma^j(C_0(X), \mathbb{C}) \to \bigoplus_l H_{j+2l}(\hat{X}/\Gamma; F), \qquad j = 0, 1.$$

Again this character becomes an isomorphism after tensoring the left hand side with F. In particular, we obtain a Chern character

$$\mathrm{ch}\colon K_j^\Gamma(\underline{E}\Gamma) \to \bigoplus_l H_{j+2l}(\Gamma; F\Gamma),$$

where $F\Gamma$ is the free Γ-module over the Γ-space $S\Gamma$ of elements of finite order in Γ acted upon by conjugation. This Chern character becomes an isomorphism upon tensoring with F.

Problem: Find ? to fill in the diagram

$$\begin{array}{ccc} K_j^\Gamma(\underline{E}\Gamma) & \longrightarrow & K_j(C_r^*\Gamma) \\ \downarrow & & \downarrow \\ \bigoplus_l H_{j+2l}(\Gamma; F\Gamma) & \longrightarrow & ? \end{array}$$

What does this have to do with the possible existence of discrete groups which do not satisfy the Baum-Connes conjecture?

Permanence properties of the Baum-Connes conjecture (with coefficients)

Jérôme Chabert

(joint work with Siegfried Echterhoff)

We are considering, for G a locally compact group and B a G-C^*-algebra, the assembly map $\mu_G\colon K_*^{\mathrm{top}}(G;B) \to K_*(B \rtimes_r G)$ and investigate the permanence properties of the bijectivity of μ_G with respect to G. We have the following results:

When H is a closed subgroup of G and μ_G is bijective (surjective), then μ_H is also bijective (surjective).

Let $N \subset G$ be a closed normal subgroup, $q\colon G \to G/N$ the quotient map. Suppose that μ_C is bijective and that C has a γ-element for all $C = q^{-1}(\dot{C})$ with $\dot{C} \subset G/N$ compact. If $\mu_{G/N}$ is bijective, then μ_G is bijective.

As a consequence, if N is amenable or has the Haagerup property, and $\mu_{G/N}$ is bijective, then μ_G is bijective. If $G = G_1 \times G_2$, then μ_G is bijective (surjective) iff μ_{G_1} and μ_{G_2} are bijective (surjective).

Using these results, the problem posed by the Baum-Connes conjecture for closed subgroups of almost connected groups can be reduced to the class of semi-simple Lie groups.

Mathematical structures in perturbative quantum field theory I+II

Alain Connes and Dirk Kreimer

We show how renormalization in quantum field theory is a special instance of a general mathematical procedure of extraction of finite values based on the Riemann Hilbert problem. The combinatorics of Feynman graphs gives rise to a commutative Hopf algebra $\mathcal{H}$. It is the dual Hopf algebra of a Lie algebra G whose basis is labeled by one-particle irreducible graphs. The corresponding Lie group G is the group of characters of the Hopf algebra. Using dimensional regularization, the theory gives rise to a loop $\gamma(z) \in G$, $z \in C$, where C is a small circle of complex dimensions around the integer dimension D of space time. The main result is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition $\gamma = \gamma_-^{-1}\gamma_+$.

The group G acts naturally on the complex space X of dimensionless coupling constants. The formula for the effective coupling constant viewed as a power series in the bare coupling constant defines a Hopf algebra homomorphism from the Hopf algebra of coordinates on the group of formal diffeomorphisms G_0 to H. The bare and renormalized coupling constants can be obtained from the Birkhoff decomposition of the unrenormalized coupling constant. This relates renormalization to the theory of non-linear complex bundles on the Riemann sphere. $\mathcal{H}$ allows to lift both the renormalization group and the β-function as the asymptotic scaling in the group G_0. Thus we obtain a scattering formula in G for the full higher pole structure of minimal-subtracted counterterms in terms of the residue.

Cyclic cohomology of Hopf algebras

Marius Crainic

The aim of this talk is to explain how Cuntz-Quillen's formalism applies to the cyclic cohomology $HC^*_\delta(\mathcal{H})$ of Hopf algebras. For simplicity, we restrict ourselves to the unimodular case. Thus $\mathcal{H}$ is a Hopf algebra endowed with a character δ such that the twisted antipode $S_\delta = \delta * S\colon \mathcal{H} \to \mathcal{H}$ is an involution. The leading principle is that $HC^*_\delta(\mathcal{H})$ should be the target of a characteristic map

$$k_\tau\colon HC^*_\delta\colon HC^*_\delta(\mathcal{H}) \to HC^*(A)$$

associated to pairs (A, τ), where A is an $\mathcal{H}$-algebra and τ is a δ-invariant trace on A. We prove that $CC_\delta(\mathcal{H}) \cong X_\delta(T\mathcal{H})$ and $C_{\lambda,\delta}(\mathcal{H}) \cong T\mathcal{H}_{\natural,\delta}$, where $C_{\lambda,\delta}(\mathcal{H})$ is the cyclic complex computing $HC^*_\delta(\mathcal{H})$, $CC_\delta(\mathcal{H})$ is the cyclic bicomplex, X_δ is the quotient of the Cuntz-Quillen X-complex by the coinvariants $h(x) - \delta(h)x$; $T\mathcal{H}$ is the tensor DG-algebra of $\mathcal{H}$; and $T\mathcal{H}_{\natural,\delta}$ is the quotient of $T\mathcal{H}$ by the linear span of graded commutators and coinvariants. This immediately yields that the Connes-Moscovici formulas work out under the minimal assumption that S_δ is an involution. We also prove that $CC_\delta(\mathcal{H}, n) \cong \mathcal{X}^{2n+1}_\delta(W\mathcal{H}, I\mathcal{H})$, where $W\mathcal{H}$ is a non-commutative analogue of the classical Weil complex, $I\mathcal{H}$ is the ideal generated by the curvatures, $\mathcal{X}$ is Cuntz-Quillen's tower of X-complexes, and $CC_\delta(\mathcal{H}, n)$ is a (level n) cyclic bicomplex.

We obtain the following non-trivial isomorphisms:

$$HC^*_\delta(\mathcal{H}) \cong H^*(W_n(\mathcal{H})_{\natural,\delta}) \cong H^*(I_n(\mathcal{H})_{\natural,\delta}),$$

where $I_n(\mathcal{H}) = I(\mathcal{H})^{n+1}$ and ${}_{\natural,\delta}$ denotes the quotient by commutators and co-invariants. Inspired by the classical construction of characteristic classes for foliations due to Bott and Haefliger, and using the truncations of the non-commutative Weil complex, we show that $HC^*_\delta(\mathcal{H})$ is also the target of a characteristic map k_τ associated to δ-invariant τ's that are higher traces in Quillen's sense or Connes's closed cocycles $(\Omega, \int)$. In the case $\mathcal{H} = \mathbb{C}$, we rediscover the cyclic cocycles constructed by Quillen. If $\mathcal{H}$ is the universal enveloping algebra of a Lie algebra, we obtain characteristic maps whose target is Lie algebra homology.

Cyclic cohomology of Hopf algebras and secondary characteristic classes

Alexander Gorokhovsky

We derive higher dimensional analogues of Connes's Godbillon-Vey cocycle.

Connes's construction gives a characteristic class associated with an orientation preserving action of a discrete pseudogroup Γ on an oriented manifold M. If ω is a volume form on M, one constructs for each $g \in \Gamma$ the map $\delta_g\colon M \to \mathbb{R}^+$ by the formula $\delta_g = \omega^g/\omega$. This induces a 1-parameter group of automorphisms of $C_0^\infty(M) \rtimes \Gamma$ by $\phi_t(aU_g) = a\delta(g)^t U_g$. Using also the transverse fundamental class, we obtain the Godbillon-Vey cocycle. If we also have a rank n Γ-bundle over M, we have an action of the Hopf algebra $\mathcal{H} = C^\infty(\mathrm{Gl(n, \mathbb{R})})$ on the cross product algebra. We construct an extension of this action to the differential graded algebras. Using the Connes-Moscovici theory of cyclic cohomology of Hopf algebras, we get a construction of all the higher secondary classes.

Relations of exactness of $C^*_r(\Gamma)$ to the Novikov conjecture

Jerry Kaminker

(joint work with Erik Guentner)

Let Γ be a finitely presented group with $C^*_r\Gamma$ exact. Then we show that Γ is uniformly embeddable in a Hilbert space. This has several consequences. By Yu's theorem, one has that Γ satisfies the Novikov conjecture. Gromov has shown the existence of groups which do not uniformly embed. Thus, there exist non-exact groups. By refining the methods of the theorem, Ozawa showed that if $C^*_r(\Gamma)$ is exact then Γ acts amenably on a compact space. One thus has the following equivalences:

$C_r^*(\Gamma)$ is exact iff Γ acts amenably on $\beta\Gamma$ iff Γ has Yu's property A. Each of these properties imply uniform embeddability. Moreover, one now knows that the set of groups which act amenably on a compact space is closed under group extensions because this is true for exact groups by results of Kirchberg and Wasserman.

Some examples of field theories in non-commutative geometry

Thomas Krajewski

(joint work with Ludwik Dabrowski and Giovanni Landi)

In physics, classical and quantum field theories are used to describe the interaction of infinitely many degrees of freedom, as it happens in relativistic quantum field theory or statistical physics. More precisely, one defines an infinite dimensional manifold $\mathcal{C}$ called the configuration space and a positive function S on $\mathcal{C}$ called the action functional. Then classical field theory deals with the extremal points of the functional S, whereas quantum field theory tries to make sense of the functional integral $S[\phi] = \int [D\phi] \exp(-S[\phi]/\hbar)$. In general both $\mathcal{C}$ and S are geometric objects, e.g., $\mathcal{C}$ is the space of connections on a bundle and S is the Yang-Mills action. We wish to extend these to the non-commutative case.

We emphasize the case of nonlinear σ-models, where the configuration space is the space of maps between a 2-dimensional surface Σ with metric g and a target M with metric G. The action functional is

$$S[X] = \int_\Sigma \sqrt{g} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j G_{ij}(X)$$

and is conformally invariant. To define the non-commutative analogue, we dualize this picture, replacing Σ and M by $A = C(M)$ and $B = C(\Sigma)$. The conformal structure of Σ is encoded in a positive Hochschild cocycle ϕ and the metric G on M is described by a positive form $G \in \Omega^2 B$. The configuration space is the space of algebra homomorphisms $\pi\colon B \to A$ and the action functional is $S[\pi] = \phi \circ (\pi \otimes \pi \otimes \pi)(G)$. We can now replace A by a non-commutative torus A_θ and let $B = \mathbb{C}^2$. Then algebra homomorphisms $B \to A$ correspond to projections $p \in A_\theta$ and $S[\pi] = \int \partial p \overline{\partial} p$. Analogously one can construct models with $B = C(S^1)$ and add the Wess-Zumino term to the action.

Counter-examples to the Baum-Connes conjecture

Vincent Lafforgue

(joint work with George Skandalis and Nigel Higson)

In September 1999, Gromov constructed a group whose Cayley graph does not embed uniformly in a Hilbert space. Because of the work of Yu this group was a candidate for a counterexample to the coarse Baum-Connes conjecture. Nigel Higson then constructed a counterexample to the surjectivity of the coarse Baum-Connes assembly map. Then Skandalis showed that surjectivity of the Baum-Connes assembly map fails for the bundle of groups $\mathrm{Sl}(3, \mathbb{Z}/n\mathbb{Z})$, $n \in \mathbb{N} \cup \{\infty\}$, viewed as a (Hausdorff) groupoid. The reason for the failure of surjectivity is that the assembly map factors through $K_0(C^*_{\max}(G))$. One shows that a particular projection in $C_r^*(G)$ does not come from $C^*_{\max}(G)$. Skandalis has found an example of a foliation with non-Hausdorff holonomy groupoid where both injectivity and surjectivity of the assembly map fail; and an example of a Hausdorff groupoid where the assembly map is not injective.

If G is a groupoid with compact basis, are there a sequence (f_n) in $C_0(G)$ with $f_n(g) \to 1$ uniformly on compact subsets of G and a dense Banach subalgebra $A \subset C_r^*(G)$, stable under holomorphic functional calculus, such that the Schur multiplication by f_n, viewed as a map from A to $C_r^* G$, has norm ≤ 1? Essentially, this property is known, unknown, or false in the same cases as the injectivity of the Baum-Connes map (without coefficients).

Instantons on non-commutative spaces

GIOVANNI LANDI

(joint work with Ludwik Dabrowski and Thomas Krajewski)

We study σ-model type theories in noncommutative geometry. If the target space consists of two points we obtain a theory of projectors in the algebra of the source space. Actions are of the form $S = (2\pi)^{-1} \int \partial_\mu p \partial_\nu p g^{\mu\nu}$ with p a projector and $g^{\mu\nu}$ is a constant metric. If $p \in A_\theta$, the noncommutative 2-torus, there is an inequality $S(p) \geq \pm 2Q(p)$ with the topological quantity

$$Q(p) = \frac{1}{2\pi i} \int p(\partial_1 p \partial_2 p - \partial_2 p \partial_1 p).$$

Equality occurs iff p is a solution of the self-duality or anti-self-duality equations $\overline{\partial} pp = 0$, $\partial pp = 0$. We seek self dual solutions of the form $p = |\psi\rangle\, (\langle\psi \mid \psi\rangle)^{-1} \langle\psi|$ with $|\psi\rangle \in \mathcal{S}(\mathbb{R})$ (the latter space being viewed as an A_θ-$A_{-1/\theta}$-bimodule) such that $\langle\psi \mid \psi\rangle$ is an invertible element in $A_{-1/\theta}$. Then ψ is a solution of the equations $\overline{\nabla}\psi - \psi\lambda = 0$, where $\overline{\nabla} = \nabla_1 + i\nabla_2$ and ∇_j are constant curvature connections on $\mathcal{S}(\mathbb{R})$ as constructed by Connes and Rieffel. The parameter $\lambda \in A_{-1/\theta}$ can be gauged away to a constant. For constant λ, the solutions are Gaussian functions $\psi = A\exp(-\pi\theta s^2 - 2i\lambda s)$ and the corresponding projector is of rank θ. and has topological charge $Q(p) = 1$. We call it an instanton. The moduli space of solutions modulo gauge transformations is $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. With a more general constant metric parametrized by a constant $\tau \in \mathbb{C}$, $\Im\tau > 0$, the moduli space is $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

The Novikov conjecture for manifolds with boundary

JOHN LOTT

(joint work with Eric Leichtnam and Paolo Piazza)

Theorem: If M is a compact manifold with boundary and $\pi_1(M)$ is virtually nilpotent or hyperbolic in the sense of Gromov, then the higher signatures of M are oriented homotopy invariants.

To define the higher signatures of a manifold with boundary, let $\nu\colon M \to B\Gamma$ be a continuous map, where Γ is a finitely generated discrete group. Put a Riemannian metric on M which is a product near the boundary. Define

$$\sigma_M = \int_M L(TM) \wedge \omega - \tilde{\eta}_{\partial M} \in \overline{H}_*(B^\infty).$$

Here B^∞ is a smooth subalgebra of $C_r^*\Gamma$ with $\mathbb{C}\Gamma \subset B^\infty \subset C_r^*\Gamma$. $\omega \in \Omega^* M \otimes \overline{\Omega}_*(\mathbb{C}\Gamma)$ is a certain closed biform. $\tilde{\eta}_{\partial M} \in \overline{\Omega}_*(B^\infty)$ is the higher eta-invariant. $\overline{H}_*(B^\infty)$ is the non-commutative de Rham homology as considered by Connes and Karoubi.

Let ν_0 be the canonical flat $C_r^*\Gamma$-bundle on ∂M.

Assumption 1: The surjection $H^k(\partial M; \nu_0) \to \overline{H}^k(\partial M; \nu_0)$ is an isomorphism for $k = [\frac{1}{2}(\dim \partial M + 1)]$. Here $H^k = \ker d / \operatorname{im} d$ and $\overline{H}^k = \ker d / \overline{\operatorname{im} d}$.

If Assumption 1 holds, then σ_M is well-defined and is an oriented homotopy invariant of the pair (M, ν).

Assumption 2: Each element of $H^*(\Gamma, \mathbb{C})$ extends to a cyclic cocycle on B^∞.

Under Assumptions 1 and 2, $\langle \sigma_M, \tau\rangle \in \mathbb{C}$ is an oriented homotopy invariant of (M, ν) for each $\tau \in H^*(\Gamma, \mathbb{C})$.

Chern characters for proper equivariant homology theories and applications to K- and L-theory

WOLFGANG LÜCK

Let $\mathcal{H}_*^?$ be an equivariant homology theory. It assigns to each (discrete) group G a G-homology theory $\mathcal{H}_*^G$. For any group homomorphism $\alpha\colon H \to G$ and H-CW-complex with free action of

$\ker\alpha$ we have an isomorphism $\mathrm{ind}_\alpha\colon \mathcal{H}_p^H(X) \cong \mathcal{H}_p(G\times_\alpha X)$. If the coefficient system $H \mapsto \mathcal{H}_p^H(*) = \mathcal{H}_p^G(G/H)$ for finite groups H is a Mackey functor, we construct an isomorphism

$$\mathrm{ch}\colon \bigoplus_{p+q=n} H_p(C_GH\backslash X^H;\mathbb{Q}) \otimes_{\mathbb{Q}[N_GH/C_GH]} S_H\mathcal{H}_q^H(*) \to \mathcal{H}_n^G(X)$$

for proper G-CW-complexes X, where $S_H\mathcal{H}_q^G(*)$ is

$$\mathrm{cok}(\bigoplus_{K\subset H, K\neq H} \mathcal{H}_q^K(*) \to \mathcal{H}_q^H(*))$$

and $H_p(C_GH\backslash X^H;\mathbb{Q})$ denotes the ordinary homology with rational coefficients. This gives explicit computations of the sources of the assembly maps appearing in the Farrell-Jones and Baum-Connes conjectures and thus, provided these conjectures are true, of $K_n(F[G])\otimes_{\mathbb{Z}}\mathbb{Q}$, $L_n(F[G])\otimes_{\mathbb{Z}}\mathbb{Q}$, and $K_n(C_r^*(G))\otimes_{\mathbb{Z}}\mathbb{Q}$ for F a field of characteristic 0.

Operads and the Connes-Kreimer Hopf algebra

Ieke Moerdijk

Let $\mathbb{P}$ be an operad in an additive symmetric monoidal category $(\mathcal{C},\otimes)$. Recall from Getzler-Jones that $\mathbb{P}$ is called a Hopf operad if it is equipped with a coalgebra structure, compatible with the operad structure. For a Hopf operad $\mathbb{P}$, the $\mathbb{P}$-algebras are closed under $\otimes$, and one can speak of coalgebras in $\mathbb{P}$-algebras, briefly Hopf $\mathbb{P}$-algebras.

Let $\mathbb{P}[t]$ be the operad whose algebras are pairs (A,α), where A is a $\mathbb{P}$-algebra and $\alpha\colon A\to A$ is a map in the category $\mathcal{C}$, not necessarily an algebra map. Let $(\mathcal{H},\lambda)$ be the initial $\mathbb{P}[t]$-algebra. That is, $(\mathcal{H},\lambda)=\mathbb{P}[t](0)$. Suppose that $\mathbb{P}$ is unitary, so that $\mathcal{H}$ has a unit $u\colon k\to\mathcal{H}$.

By the universal property of $(\mathcal{H},\lambda)$, there is a unique augmentation $\epsilon\colon\mathcal{H}\to k$ such that $\epsilon\lambda=0$, and any two $\sigma_1,\sigma_2\colon\mathcal{H}\to\mathcal{H}$ give a unique "coproduct" $\Delta\colon\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$, an algebra map satisfying $\Delta\lambda=(\sigma_1\otimes\lambda+\lambda\otimes\sigma_2)\Delta$. Now this makes $\mathcal{H}$ into a Hopf $\mathbb{P}$-algebra whenever $\epsilon\sigma_i=\epsilon$ and $(\sigma_i\otimes\sigma_i)\Delta=\Delta\sigma_i$ for $i=1,2$. There are many maps satisfying these conditions. In the special case where $\mathcal{C}$ is the category of vector spaces and $\mathbb{P}(n)=k$ for all n, so that $\mathbb{P}$-algebras are just commutative unitary algebras, and $\sigma_1=\mathrm{id}$, $\sigma_2=u\epsilon$, one finds that $(\mathcal{H},\Delta)$ is the Connes-Kreimer Hopf algebra of rooted trees.

Modular Pairs and the Modular Square

Henri Moscovici

(joint work with Alain Connes)

We have recently adapted cyclic cohomology to the treatment of Hopf symmetry in non-commutative geometry. The resulting theory of characteristic classes for Hopf algebras and their actions on algebras allows to expand the range of applications of cyclic cohomology.

The goal of this talk was to illustrate the remarkable agreement between the framework which was used to define the cyclic (co)homology for Hopf algebras and the algebraic as well as the analytic theory of quantum groups. In particular, this accord is manifest in the construction of the modular square associated to a locally compact quantum group.

Hochschild and cyclic homology of Hecke algebras of reductive p-adic groups

Victor Nistor

There is a description of the periodic cyclic homology of the Hecke algebras of reductive p-adic groups in terms of the representation theory of the stabilizers of simplices of the building (Schneider, Higson, and Nistor). This description is useful, among other things, for the Baum-Connes conjecture (proved for $GL(n)$ by Baum, Higson, and Plymen). It is interesting then to identify these periodic cyclic homology groups as explicitly as possible. In my talk, I give a description of these groups using conjugacy classes and their stabilizers in the group, in the spirit of Burghelea's computation for discrete groups. The result is in terms of certain commutative subgroups, called "standard", their regular elements, their Weyl groups, and the continuous cohomology of the space of locally constant functions on the unipotent variety.

The bivariant Chern character of the γ-element

Michael Puschnigg

The basic properties of the local cyclic cohomology HC^l_* bifunctors on the category of Banach algebras were discussed and an explicit calculation was presented which leads to the following theorem:

If $\Gamma = \pi_1(M)$ for a compact manifold M of nonpositive curvature, then the cohomological assembly map provides an isomorphism $H_*\big(\Gamma, HC^l_*(\mathbb{C})\big) \cong HC^l_*\big(\ell^1(\Gamma)\big)$.

Using work of Connes-Moscovici and Cowling-Haagerup, this can be applied to study the K-theoretic assembly map. Let $\Gamma = \pi_1(M)$ for a compact manifold M of strictly negative curvature. Then:

1. The local cyclic homology of $C^*_r\Gamma$ decomposes into a homogeneous part isomorphic to $H_*\big(\Gamma, HC^l_*(\mathbb{C})\big)$ and an inhomogeneous part.
2. We have $HC^*_l(C^*_r\Gamma, C^*_r\Gamma) = \mathrm{End}\big(HC^l_*(C^*_r\Gamma)\big)$.
3. Let $\gamma \in KK^\Gamma(\mathbb{C},\mathbb{C})$ be the γ-element of Kasparov, $\gamma \rtimes \Gamma \in KK(C^*_r\Gamma, C^*_r\Gamma)$. Then $\mathrm{ch}_{\mathrm{biv}}(\gamma \rtimes \Gamma)$ equals the canonical projection onto the homogeneous part of $HC^l_*(C^*_r\Gamma)$.

As a consequence, $\mathrm{ch}_{\mathrm{biv}}(\gamma \rtimes \Gamma)[\tau] = [\tau]$, where $\tau\colon C^*_r\Gamma \to \mathbb{C}$ is the canonical trace. Therefore the idempotent conjecture holds for $C^*_r\Gamma$ (this was previously proved by Lafforgue).

Gromov-Hausdorff distance for non-commutative metric spaces

Marc A. Rieffel

By a non-commutative metric space we mean an order-unit space A (e.g., the real vector space of self-adjoint elements of a unital C^*-algebra), together with a semi-norm L on A. The semi-norm will usually be unbounded with respect to the norm of A. We use L to define a metric ρ_L on the state space $S(A)$ of A by

$$\rho_L(\mu,\nu) = \sup\{|\mu(a) - \nu(a)| \mid L(a) \le 1\}.$$

The main requirement on L is that the topology on $S(A)$ from this metric should agree with the weak-$*$ topology.

Given two such non-commutative metric spaces (A, L_A), (B, L_B), we consider such L's on $A \oplus B$ whose quotients on A and B are L_A and L_B. Since $S(A), S(B) \subset S(A \oplus B)$, we can use ρ_L to define the usual Hausdorff distance $\mathrm{dist}^L_{\mathrm{H}}\big(S(A), S(B)\big)$ between $S(A)$ and $S(B)$. We define the Gromov-Hausdorff distance between (A, L_A) and (B, L_B) by

$$\mathrm{dist}_{\mathrm{GH}}\big((A,L_A),(B,L_B)\big) = \inf\big\{\mathrm{dist}^L_{\mathrm{H}}\big(S(A),S(B)\big) \;\big|\; L|_{S(A)} = L_A,\ L|_{S(B)} = L_B\big\}.$$

We prove the following theorem:

Let Θ be the vector space of skew-symmetric $d \times d$ matrices. For $\theta \in \Theta$, let A_θ be the corresponding non-commutative torus. Let $\|\ \|$ be any norm on Θ. In terms of a length function on T^d, and the action of T^d on each A_θ, we define semi-norms L^θ on A_θ. Then for every $\epsilon > 0$ there is $\delta > 0$ such that if $\|\theta - \theta'\| < \delta$, then $\mathrm{dist}_{\mathrm{GH}}\big((A_\theta, L^\theta),(A_{\theta'}, L^{\theta'})\big) < \epsilon$.

Deformation theory and non-commutative geometry

Yan Soibelman

Degeneration of a complex structure gives rise to a foliation with affine structure on the leaves. It is argued that the derived category of coherent sheaves "degenerates" into a derived category of certain modules over the algebra of the foliation.

For example, the derived category of coherent sheaves on the elliptic curve $\mathcal{E}_\tau$ corresponds to the category of modules over the non-commutative torus, generated by unitaries x, y with $xy = \exp(2\pi i\phi)yx$, $\phi = \mathrm{Re}\,\tau$, which are projective over the subalgebra generated by x.

I suggest to consider the (derived) category of modules as a non-commutative stratum in the compactification of the universal covering of the moduli space of complex structures. I discuss this idea from the point of view of dualities. For instance, Morita equivalences of non-commutative tori are related to the manifest $\mathrm{Sl}(2,\mathbb{Z})$-equivalence of elliptic curves. I conjecture that the duality group for the quantized coordinate rings of Poisson-Lie groups is the Galois group of the maximal Abelian extension of $\mathbb{Q}$. I discuss an approach to the homological mirror symmetry of Kontsevich, which uses degeneration in the above sense of both sides of the homological mirror symmetry to the same category of modules over the foliation algebra. The question about Morita equivalence of quantized Poisson manifolds is raised.

Index theorem for Poisson manifolds

Boris Tsygan

Let M be a manifold. Let P be a formal Poisson structure on M, that is, a formal series $P = tP_0 + t^2P_1 + t^3P_2 + \cdots$, where P_i are bivector fields on M and $\{f, g\} = \langle P, df \wedge dg\rangle$ is a Lie bracket on $C^\infty(M)[[t]]$. Thus P_0 is a Poisson structure on M. Kontsevich constructed a $*$-product associated to P. We proved with D. Tamarkin that the space of traces on the associative algebra $A_p = (C^\infty(M)[[t]], *)$ with values in $\mathbb{C}[[t]]$ is isomorphic to the space of classical traces $\tau\colon C^\infty(M)[[t]]/\{,\}_P \to \mathbb{C}[[t]]$. For any τ, let Tr_τ be the corresponding trace. Let $i_P\colon \Omega^\cdot(M)[[t]] \to \Omega^{\cdot-2}(M)[[t]]$ be the contraction operator and $L_p = [d, i_P]$. The map $\tau \circ \exp(ti_P)\colon \Omega^\cdot(M)[[t]] \to \mathbb{C}[[t]]$ is a morphism of complexes.

Let $e, f \in \mathrm{Mat}_N(A_p)$ be idempotents with $e - f$ compactly supported. There is an even cohomology class $\hat{A}(P_0) \in H^{\mathrm{ev}}(M)$ depending on a Poisson structure P_0 such that

$$\mathrm{Tr}_\tau(e - f) = \big(\tau \circ \exp(ti_p)\big)\Big(\big(\mathrm{ch}(\sigma e) - \mathrm{ch}(\sigma f)\big)\hat{A}(P_0)\Big),$$

where $\sigma x = x \bmod t$. This was proved by D. Tamarkin and myself.

Assume that $(\mathcal{E}, \rho, [,], \omega)$ is a symplectic Lie algebroid on M, that is, a Lie algebroid with a symplectic non-degenerate closed $\mathcal{E}$-2-form $\omega \in \Lambda^2\mathcal{E}^*$. Its image under the identification $\omega\colon \mathcal{E}^* \to \mathcal{E}$ composed with $\Lambda^2\rho\colon \Lambda^2\mathcal{E}^* \to \Lambda^2 T$ defines a Poisson structure on M. We prove with R. Nest and P. Bressler that if P_0 comes from a symplectic Lie algebroid $\mathcal{E}$, then $\hat{A}(P_0) = \hat{A}(\mathcal{E})$. One of the approaches to the proof of the above results is founded on the ideas of Tamarkin. They would follow from the following conjecture (of which Tamarkin proved a very particular case):

Let A be an associative algebra with a trace Tr such that $\mathrm{Tr}(ab)$ is a non-degenerate pairing. Then the Hochschild cochain complex $C^\cdot(A, A)$ is a strong homotopy Batalin-Vilkovisky algebra.

A partial case of the general index theorem leads to a local index formula for a Fourier integral operator whose wave front is the graph of a contact isomorphism $\phi\colon T^*X \setminus X \to T^*Y \setminus Y$. With R. Nest and E. Leichtnam we proved that $\mathrm{ind}\,\Phi = \mathrm{Tr}_\tau 1$ for a trace Tr_τ on the deformed algebra $C^\infty(M)$, where M is a certain Poisson manifold. The local index theorem for Poisson manifolds then yields a local index formula for Φ.

Quantum subgroups?

Antony Wassermann

We explain how subfactors can be studied and constructed using the notion of algebra in a braided category of bimodules. Subfactors associated with finite groups $H \subset G$ can all be related to G-actions using the imprimitivity algebra $A = \ell^\infty(G/H)$, an ergodic Abelian G-algebra. The structure of A can be described just as an object in the category of G-modules.

More generally, given a bimodule ${}_N X_M$ over von Neumann factors N, M, of finite index, we say X has finite depth if $X \boxtimes \overline{X} \boxtimes \cdots$ decomposes into only finitely many irreducibles under Connes fusion. In this case $A = X \boxtimes \overline{X}$ is a finite dimensional ergodic algebra in the category of N-N-bimodules if X is irreducible. The left, right, and two-sided A-modules correspond exactly to M-N, N-M, and M-M bimodules.

Most interesting is when the category of N-N-bimodules is braided. Then an Abelian ergodic algebra A can be used to imitate all the constructions of subfactors for finite group and subgroups, so can be regarded as a "quantum subgroup". If $LG_r \supset LH_s$ is a conformal inclusion, the vacuum representation of LG defines such an algebra A in the category of LH positive energy representations under Connes fusion. Taking $LSO(3)_1 \supset LSU(2)_{10}$ and $LG_2 \supset LSU(2)_{28}$, one gets the E_6 and E_8 Jones subfactors by the analogue of my "shift" construction for classical groups.

We explain Lyubashenko's Hopf algebra construction in any braided tensor category. In the modular case, the Fourier transform S and the square of the antipode T give a projective representation of $\mathrm{Sl}(2,\mathbb{Z})$ on this algebra. Using the induced modules $\bigoplus X_i \otimes X \otimes X_i^*$ in any category of N-N-bimodules, and their manifest braiding, we explain the "quantum double construction" of Ocneanu-Drinfeld. Its modularity can easily be read off using Lyubashenko's Hopf algebra.

Renormalization of Yang-Mills theory on non-commutative $\mathbb{R}^4$

Raimar Wulkenhaar

It has turned out to be impossible to formulate a consistent quantum field theory of gravity, strong, weak, and electro-magnetic interactions based upon an ordinary Riemannian manifold. This raises the problem to formulate quantum field theories on non-commutative spaces. The simplest example inspired by quantum mechanics is obtained by assuming that the commutators of coordinates satisfy $[x^\mu, x^\nu] = -2i\theta^{\mu\nu}$ with constants $\theta^{\mu\nu}$. On such a space the Yang-Mills action including BRS symmetry can be written down and it is straightforward to derive the Feynman rules. They are those of an ordinary Yang-Mills theory with the structure constants given by trigonometric functions of the momenta. The 1-loop calculation leads to the surprise of a quadratic infrared divergence which destroys the classical limit $\theta^{\mu\nu} \to 0$. The model leads to a confinement of size $\sqrt{|\theta^{\mu\nu}|}$ and can therefore not be interpreted as an approach to quantum gravity. This indicates that further terms must be added to the action, for instance, super-symmetry or something which establishes a symmetry $p^\mu \leftrightarrow \theta^{\mu\nu} p^\nu$.

Expanding graphs and the (rough) Baum-Connes conjecture

Guoliang Yu

The rough Baum-Connes conjecture for bounded geometry metric spaces is a close relative of the coarse Baum-Connes conjecture. In the case of a finitely generated discrete group with a word metric, the rough Baum-Connes conjecture is equivalent to the Baum-Connes conjecture for the group with a certain coefficient.

In this talk, we explain how the rough Baum-Connes conjecture fails for an expanding sequence of finite graphs. This implies that the rough Baum-Connes conjecture is almost always false in a certain probabilistic sense. The same argument is used to show that Gromov's recent groups, which contain an expanding sequence of graphs, are counterexamples to the Baum-Connes conjecture with coefficients.

This work was inspired by Higson's earlier counterexample to the coarse Baum-Connes conjecture for bounded metric spaces.

This report was written by Ralf Meyer, Münster.

email addresses of the participants

name	email
Alexander Alldridge	alldridg@mathematik.uni-marburg.de
Paul Frank Baum	baum@math.psu.edu
Paul Bressler	bressler@mpim-bonn.mpg.de
Alberto Cattaneo	asc@math.unizh.ch
Jérôme Chabert	chabert@math.uni-muenster.de
Alain Connes	connes@ihes.fr
Marius N. Crainic	crainic@math.uu.nl
Joachim Cuntz	cuntz@math.uni-muenster.de
Sergio Doplicher	dopliche@mat.uniroma1.it
Siegfried Echterhoff	echters@math.uni-muenster.de
Alexander Gorokhovsky	gorokhov@math.lsa.umich.edu
Olivier Grandjean	grandj@mpim-bonn.mpg.de
Dale Husemoller	dale@mpim-bonn.mpg.de
Pierre Julg	julg@math.u-strasbg.fr
Jerry Kaminker	kaminker@math.iupui.edu
Max Karoubi	karoubi@math.jussieu.fr
Masoud Khalkhali	masoud@julian.uwo.ca
Thomas Krazewski	krajew@fm.sissa.it
Dirk Kreimer	dirk.kreimer@uni-mainz.de
Vincent Lafforgue	vlafforg@dmi.ens.fr
Giovanni Landi	landi@univ.trieste.it
Nicolaas Landsman	npl@wins.uva.nl
Robert Lauter	lauter@mathematik.uni-mainz.de
Eric Leichtnam	leicht@dmi.ens.fr
John Lott	lott@math.lsa.umich.edu, lott@mpim-bonn.mpg.de
Wolfgang Lück	lueck@math.uni-muenster.de
Ralf Meyer	rameyer@math.uni-muenster.de
Ieke Moerdijk	moerdijk@math.uu.nl
Henri Moscovici	henri@math.ohio-state.edu
Ryszard Nest	rnest@math.ku.dk
Victor Nistor	nistor@math.psu.edu
Roger J. Plymen	rogermath@msn.com
Raphael Ponge	ponge@topo.math.u-psud.fr
Michael Puschnigg	puschni@math.uni-muenster.de
Marc A. Rieffel	rieffel@math.berkeley.edu
Peter Schneider	pschnei@math.uni-muenster.de
Yan Soibelman	soibel@ihes.fr
Harold Steinacker	harold.steinacker@physik.uni-muenchen.de
Nicola Teleman	teleman@popcsi.unian.it
Boris Tsygan	tsygan@math.psu.edu
Harald Upmeier	upmeier@mathematik.uni-marburg.de
Mathai Varghese	vmathai@maths.adelaide.edu.au
Elmar Vogt	vogt@math.fu-berlin.de
Christian Voigt	cvoigt@math.uni-muenster.de
Markus Walze	walze@math.uni-muenster.de
Antony Wassermann	ajw@pmms.cam.ac.uk
Raimar Wulkenhaar	raimar@doppler.thp.univie.ac.at
Guoliang Yu	gyu@euclid.colorado.edu

Tagungsteilnehmer

Prof. Dr. Alexander Alldridge
Fachbereich Mathematik
Universität Marburg
Hans-Meerwein-Str.
35043 Marburg

Prof. Dr. Paul Frank Baum
Department of Mathematics
Pennsylvania State University
218 McAllister Building
University Park , PA 16802
USA

Prof. Dr. Paul Bressler
Max-Planck-Institut
für Mathematik
Vivatsgasse 7
53111 Bonn

Prof. Dr. Alberto Cattaneo
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Dr. Jerome Chabert
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Alain Connes
Institut des Hautes Etudes
Scientifiques
Le Bois Marie
35, route de Chartres
F-91440 Bures-sur-Yvette

Prof. Dr. Marius N. Crainic
Mathematisch Instituut
Rijksuniversiteit te Utrecht
P. O. Box 80.010
NL-3508 TA Utrecht

Prof. Dr. Joachim Cuntz
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Sergio Doplicher
Dipartimento di Matematica
Universita degli Studi di Roma I
"La Sapienza"
Piazzale Aldo Moro, 2
I-00185 Roma

Siegfried Echterhoff
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Alexander Gorokhovsky
Dept. of Mathematics
The University of Michigan
525 East University Avenue
Ann Arbor , MI 48109-1109
USA

Dr. Olivier Grandjean
Max-Planck-Institut
für Mathematik
Vivatsgasse 7
53111 Bonn

Prof. Dr. Dale Husemoller
Max-Planck-Institut für
Mathematik
Vivatgasse 7
53111 Bonn

Prof. Dr. Pierre Julg
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Prof. Dr. Jerry Kaminker
Dept. of Mathematics
Indiana University -
Purdue University
Indianapolis , IN 46205
USA

Prof. Dr. Max Karoubi
U.F.R. de Mathematiques
Case 7012
Universite de Paris VII
2, Place Jussieu
F-75251 Paris Cedex 05

Prof. Dr. Masoud Khalkhali
The Fields Institute
185 Columbia Street, West
Waterloo, Ontario N2L 5Z5
CANADA

Dr. Thomas Krazewski
Centre de Physique Theorique
CNRS
Luminy - Case 907
F-13288 Marseille Cedex 09

Prof. Dr. Dirk Kreimer
Fachbereich 18 - Physik
Universität Mainz
Staudinger Weg 7
55128 Mainz

Dr. Vincent Lafforgue
Departement de Mathematiques et d'Informatique
Ecole Normale Superieure
45, rue d'Ulm
F-75005 Paris Cedex

Prof. Dr. Giovanni Landi
Dipartimento di Scienze Matematiche
Universita di Trieste
Piazzale Europa 1
I-34100 Trieste (TS)

Dr. Nicolaas Landsman
Fac. of Math. & Computer Sciences
University of Amsterdam
Plantage Muidergracht 24
NL-1018 TV Amsterdam

Robert Lauter
Fachbereich Mathematik
Universität Mainz
55099 Mainz

Prof. Dr. Eric Leichtnam
12 Quai Saguet
F-94700 Maisons-Alfort

Prof. Dr. John Lott
Dept. of Mathematics
The University of Michigan
525 East University Avenue
Ann Arbor , MI 48109-1109
USA

Prof. Dr. Wolfgang Lück
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Ralf Meyer
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Ieke Moerdijk
Mathematisch Instituut
Rijksuniversiteit te Utrecht
P. O. Box 80.010
NL-3508 TA Utrecht

Prof. Dr. Henri Moscovici
Department of Mathematics
Ohio State University
231 West 18th Avenue
Columbus , OH 43210-1174
USA

Prof. Dr. Ryszard Nest
Matematisk Afdeling
Kobenhavns Universitet
Universitetsparken 5
DK-2100 Kobenhavn

Prof. Dr. Victor Nistor
Department of Mathematics
Pennsylvania State University
218 McAllister Building
University Park , PA 16802
USA

Prof. Dr. Roger J. Plymen
Department of Mathematics
The University of Manchester
Oxford Road
GB-Manchester M13 9PL

Dr. Raphael Ponge
Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Dr. Michael Puschnigg
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Marc A. Rieffel
Department of Mathematics
University of California
at Berkeley
Berkeley , CA 94720-3840
USA

Prof. Dr. Peter Schneider
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Yan Soibelman
IHES
35 route des Chartres
F-91440 Bures-Sur-Yvette

Dr. Harold Steinacker
Sektion Physik der
Ludwig-Maximilian-Universität
München
Theresienstr. 37 III
80333 München

Prof. Dr. Nicola Teleman
Dipartimento di Matematica
"V. Volterra"
Universita degli Studi di Ancona
I-60131 Ancona

Prof. Dr. Boris Tsygan
Department of Mathematics
Pennsylvania State University
McAllister Building
University Park , PA 16802
USA

Prof. Dr. Harald Upmeier
Fachbereich Mathematik
Universität Marburg
35032 Marburg

Prof. Dr. Mathai Varghese
Department of Pure Mathematics
University of Adelaide
Adelaide S.A. 5005
AUSTRALIA

Prof. Dr. Elmar Vogt
Institut für Mathematik II (WE2)
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Christian Voigt
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Markus Walze
Departement Mathematiques
Universite P. et M. Curie
4, place Jussieu
F-75252 Paris Cedex 05

Prof. Dr. Raimar Wulkenhaar
Centre de Physique Theorique
CNRS
Luminy - Case 907
F-13288 Marseille Cedex 09

Prof. Dr. Antony Wassermann
Department of Mathematics
University of Cambridge
16 Mill Lane
GB-Cambridge CB2 1SB

Prof. Dr. Guoliang Yu
Dept. of Mathematics
University of Colorado
Campus Box 395
Boulder , CO 80309-0395
USA