MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 9/2001

Recent Developments in the Mathematical Theory of Water Waves

The mini workshop was organized by W. Craig (Hamilton), M. D. Groves (Loughborough), G. Schneider (Bayreuth) and J. F. Toland (Bath).

The water-wave problem formulated by Euler is a paradigm for most modern methods in nonlinear functional analysis and nonlinear dispersive wave theory. Its mathematical study calls upon many different approaches (iteration methods, bifurcation theory, dynamical systems theory, complex variable methods, PDE methods, the calculus of variations, positive operator theory, topological degree theory, KAM theory, and symplectic geometry). The last decade has seen vigorous activity in the mathematical theory of water waves by several independent international research groups. The aim of this workshop was to bring these groups together and to focus attention upon certain famous and still outstanding open problems in water waves, particularly stability, three-dimensional waves, justification of model equations, multiscale aspects such as modulational dynamics and variational methods for global bifurcation theory. These aspects pose considerable challenges and are of importance far beyond the hydrodynamic application in which they emerge.

The water-wave problem is the study of the two- and three-dimensional irrotational flow of a perfect fluid bounded above by a free surface subject to the forces of gravity and surface tension. The governing equations for this famous problem in the field of nonlinear dispersive waves were first written down by Euler, and their mathematical study began with Stokes in the nineteenth century. At about the same time Scott Russell made extensive experimental observations of ‘solitary’ water waves. These two strands came together in the systematic use of approximate equations to describe the fluid motion, for example the celebrated equations named after Boussinesq and Korteweg & deVries. The use of approximate equations led to rapid advances in engineering and were of great technological significance.

In contrast to the well-developed theories for model equations, there has been comparatively little progress on the exact equations for water waves written down by Euler. Indeed, some of the most interesting aspects remain completely open and pose considerable challenges. In contrast to other meetings on water waves, which usually focus upon modelling and numerical issues, this miniworkshop was devoted to the rigorous mathematical theory for the exact equations of Euler. As well as the classical case of gravity-capillary surface waves described above, issues such as external forcing, diffusive effects and dispersive stability were considered.

The following topics were chosen as priority research areas for the workshop since they represent issues which have been almost fully settled for model equations, but remain almost fully open for the exact water-wave problem. They pose mathematical challenges whose resolution is likely to prove beneficial in a wide range of situations beyond the water-wave problem.

- The stability of steady-wave solutions of the time-dependent water-wave problem;
- The development of variatonal methods and global bifurcation theory for large-amplitude gravity-capillary waves;
- The transition from 2D to 3D waves ('dimension breaking phenomena');
- In multiscale analysis and modulational dynamics, the existence of modulated waves in which pulses move within an envelope of permanent form.

In addition, the workshop operated around the following central themes:

- Comparison of the diverse mathematical formulations of the hydrodynamic equations which have been used by different researchers;
- Connections between the water-wave problem and other nonlinear PDEs in the applied sciences and mathematical physics discussed in a related topics session.

At the meeting 15 participants from 6 countries discussed their work and 14 lectures were presented. The schedule allowed for many discussions in the afternoons and the evenings. Researchers with different approaches to similar problems met and the resulting collaborations lead to significant progress in many of the areas outlined above. The stimulating atmosphere of the Forschungsinstitut obviously had a positive effect and all participants agreed that that the workshop was extremly benefical.

ABSTRACTS

Comparison of model equations for water waves

Jerry Bona, University of Texas at Austin

After general remarks about hierarchies of models for water waves, a discussion was initiated of the Euler-Boussinesq systems - Korteweg-deVries hierarchy. Classes of Boussinesq systems were introduced and means of distinguishing then discussed. Several one-way modules were also models were also introduced. Comparison theorems for these one-way models were stated. Especially the analysis of the equation $u_t + u_x + uu_x + u_{xxt} = 0$ turn out to be very helpful in understanding the comparison between the two-way and one-way models.

A generalisation of conformal maps for 3D water waves

Boris Buffoni, Swiss Federal Institute of Technology, Lausanne

Let the domain $\Omega = \{(x, \eta) \in \mathbb{R}^2 : y < \eta(x)\}$ be filled by a stationary 2D travelling wave (in a moving frame that follows the wave). Consider the conformal map

$$\Omega \ni (x, \eta) \to (u, v) = K(x, y) := (\phi(x, y), \psi(x, y)) \in \mathbb{R}^2$$

where ϕ is the velocity potential and ψ is the stream function. It transforms the unknown domain Ω to the half-plane $\{(u, v) \in \mathbb{R}^2,\ v < 0)\}$ (if ψ is normalised in the appropriate way) and the elevation $x \to \eta(x)$ is transformed into $u \to \xi(u)$, with $\eta(x) \equiv \xi(\phi(x, \eta(x)))$. We propose a generalisation of the functional map $\xi \to K^{-1}$ to 3D water waves: to each $\xi \in C^\infty(\mathbb{R}^2)$ that is doubly perodic (say) corresponds a map

$$K^{-1} : \{(u, v, w) : v < 0\} \to \mathbb{R}^3 = \{(x, y, z) \in \mathbb{R}^3\}$$

such that

- $\xi \to K^{-1} - I$ is linear and continuous for the appropriate topologies;
- there is no preferred direction in the (u, w)-plane;
- when ξ is independent of w, K^{-1} is conformal and commutes with translations in the transverse directions (i.e. w direction and z direction).

The notation K^{-1} can be justified: if ξ is of small enough amplitude, K^{-1} is indeed invertible. The talk ended with an application to gravity-capillary gravity waves (i.e. description of the equations in the (u, v, w)-variables).

Travelling water waves
Walter Craig, Hamilton

This talk is an existence theory for doubly periodic travelling waves in the free surface of an ideal fluid in three dimensions. For this work it is important that one includes the effects of surface tension. The proof of existence involves the harmonic analysis of the Dirichlet-Neuman operator for the fluid domain, and a Hamiltonian formulation of the equations of motion. The count of the number of solutions depends upon an appropriate variational principle. There is a pleasing analogy with Moser's proof of the resonant Lyapunov centre theorem of dynamical systems.

Open problems related to interfacial waves
F. Dias, ENS-Cachan

Various open problems in solitary waves in stratified fluids (essentially two-layer flows) are described.

Modulating pulse solutions for a class of nonlinear wave equations
Mark Groves, Loughborough

We consider modulating pulse solutions for a nonlinear wave equation on the infinite line. Such a solution consists of a permanent pulse-like envelope steadily advancing in the laboratory frame and modulating an underlying wave-train. The problem is formulated as an infinite-dimensional dynamical system with one stable, one unstable and infinitely many neural directions. Using a partial normal form and invariant-manifold theory we establish the existence of modulating pulse solutions which decay to small-amplitude disturbances at large distances.

(Joint work with G. Schneider, Bayreuth)

A dimension-breaking phenomenon in the theory of steady gravity-capillary water waves
Mark Groves, Loughborough

The existence of a line solitary-wave solution to the water-wave problem with large surface tension was predicted on the basis of a model equation in the celebrated paper by D.J. Korteweg & G. de Vries (1895) and rigorously confirmed a century later by C.J. Amict & K. Kirchgässner (1989). A model equation derived by B.B. Kadomtsev & V.I. Petviashuiti (1970) suggests that the Korteweg-de Vries line solitary wave belongs to a family of *periodically modulated solitary waves* which have a solitary-wave profile in the direction of motion and are periodic in the transverse direction. This prediction is rigorously confirmed for the full water-wave problem in the present talk. It is shown that the Korteweg-de Vries solitary wave undergoes a *dimension-breaking bifurcation* which generates a family of periodically modulated solitary waves. The term *dimension-breaking phenomenon* describes the spontaneous emergence of a spatially inhomogeneous solution which is homogeneous in one or more spatial dimensions.

(Joint work with M. Haragus, Bordeaux and S. M. Sun, Virginia Tech.)

Finite-wavelength stability of capillary-gravity solitary waves
M. Haragus, Bordeaux

We consider the Euler equations describing nonlinear waves on the free surface of a two-dimensional inviscid, irrational fluid layer of finite depth under the influence of gravity and surface tension. In the case of large surface tension, we use bifurcation theory to deduce spectral stability of small-amplitude solitary waves. We formulate the stability problem in terms of the spectrum of a generalised eigenvalue problem and show that the solitary waves are stable with respect to perturbations finite wave-number. In particular, we exclude possible unstable eigenvalues in the long-wavelength regime, and unstable eigenvalues arising from non-adiabatic interaction of the infinite-wavelength soliton with finite-wavelength perturbations.

(Joint work with A. Scheel, Berlin)

The long wave limit for the water wave problem
Guido Schneider, Universität Beyreuth

We consider the long wave limit for the water wave problem and prove that in this limit the solutions split into two wave trains, one moving to the left and one to the right. Each of the wave trains can be described by a KdV equation. The problem is considered with and without surface tension.

(Joint work with C. E. Wayne, Boston)

A new type of bifurcation for ∞-dimensional reversible vector fields: application to travelling waves for two superposed fluid layers, one being deep.
G. Iooss, Nice

This problem is formulated as a spatial dynamical system, 0 being the rest state, and depends on two parameters (λ, ε). The linearised operator has a spectrum consisting in the entire real line (essential spectrum), with in addition isolated eigenvalues. A single pair of eigenvalues $\pm i\lambda$ and a single eigenvalue 0 lie on the imaginary axis, while another pair of single imaginary eigenvalue disappears into the real line while ε crosses 0. We show

- the existence of a 2-parameter family of periodic solutions, lying on a 3-dimensional manifold (despite the resonance due to 0 in the spectrum);
- the existence of a homoclinic solitary wave of Benjamin-Ono (B-O) type, which is an approximate solution bifurcating for $\varepsilon > 0$.
- finally, we prove the existence of a family of solutions close to the above (B-O) wave, and homoclinic to each of the periodic solutions above found, provided that their amplitude is not too small.

(Joint work with E. Lombardi, Nice and S. M. Sun, Virginia Tech.)

Phenomenon beyond all orders and bifurcations of solitary waves near "$\sharp i\omega$" resonances
E. Lombardi, Nice

In this talk we give tools to compute the size of the oscillating integrals involving solutions of nonlinear differential equations. Then we use these tools to prove that for reversible vector fields admitting a "$\sharp i\omega$" resonance at the origin, there is generically no homoclinic connection to the origin with one bump wheras there are always homoclinic connections to exponentially small periodic orbits close to the origin.

Higher order models for water waves
Jean-Claude Saut, Orsay.

We consider higher order nonlinear Schödinger equations, proposed by Dysthe, Hogan and others, to model gravity or capillary-gravity waves on deep water. We first derive various dispersive estimates leading to Strichartz estimates for the linearised equations at 0. We then present new inhomogeneous local smoothing estimates for a rather general class of linear dispersive equations. As a consequence we establish the local well-posedness of the Cauchy problem for the Dysthe-Hogan systems for data in $H^2\ (\mathbb{R}^2)$.

(Joint work with M. Ben-Artzi, Jerusalem and H. Koch, Dortmund)

Transverse linear instability of solitary waves in water with large surface tension
Shu-Ming Sun

It concerns the linear stability of two dimensional solitary waves using exact governing equations in three dimensional space. The problem is determined by two nondimensional constants. F, wave speed and τ, surface tension. Assume that the perturbation part of the solution is oscillating with wave number k in the transverse direction. It is shown that for $k \in (0, \varepsilon^{1/2}k_m)$ where k_m is a known constant, the perturbation will grow exponentially in t. Here the linearised equations from the exact governing equations around the two dimensional solitary wave are used and τ is assumed to be greater than $1/3$.

(Joint work with R. L. Pego, Maryland)

Invariant manifolds and the long-time asymptotics of the Navier-Stokes equation
Riemann-Hilbert formulation of the Stokes wave problem
John Toland, Bath

It is discussed how the classical two-dimensional Stokes wave problem can be written as a nonlinear Riemann-Hilbert problem for a holomorphic function in the unit disc in $\mathbb{C}$. The relation between regularity up to the boundary of solutions of the Riemann Hilbert problem and the Bernoulli water wave condition is discussed. Open questions remain.

Invariant manifolds and the long-time asymptotics of the Navier-Stokes equation
Gene Wayne

I described the construction of finite dimensional invariant manifolds which control long-time asymptotics of small solutions of the Navier-Stokes equation in two and three dimensions. We use the vorticity formulation of these equations which we rewrite in scaling variables. In this form the equations have finite dimensional invariant manifolds which all solutions approach at a calculable rate.

(Joint work with T. Gallay, Orsay)

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