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# Arbeitsgemeinschaft
# Ergodic Theory and Heegner Points

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## Abstracts

Guenter Harder, Bonn

### Introduction

In this Arbeitsgemeinschaft mathematicians from two different fields came together. In the first half we discussed results of Marina Ratner, which make strong assertions on the closures of unipotent orbits on $\Gamma\backslash G$, in a sense they say that such orbits are homogeneous. These results had strong implications in number theory (Oppenheim conjecture). The methods use measure and ergodic theory.

But recently Cornout and Vatsal discovered that they also can be applied to the theory of Heegner points. These Heegner points are constructed by a modular interpretation on modular curves. Then we encounter a problem, namely we have to show that there are many of them, they should be non trivial. At this point Ratner's result enter, they imply that the closure of these point has to be of a certain size, which is the conjectured one.

It was of course rather hard for the poeple from one side to follow the talks on the other side. Here we appreciated a lot from the help of Nimish A. Shah and G. Tomanov.

Volker Braungardt, Karlsruhe

### Marina Ratner's results

Given a Lie group $G$, let an Ad-unipotent subgroup $U$ act on a finite volume quotient of $G$. Starting from classical examples (tori, $\mathrm{PSL}_2$), I state some of Ratner's theorems and indicate how they are connected: closures of $U$-orbits are homogeneous; classification of $U$-ergodic Borel probability measures; uniform distribution of unipotent one-parameter flows; the set of groups homogenizing unipotent orbit closures is countable.

Stefan Kühnlein, Karlsruhe

### Application to Oppenheim's Conjecture

Let $q : \mathbb{R}^d \to \mathbb{R}$ be a indefinite, non-degenerate quadratic form which is given by a matrix which is not a multiple of a rational matrix. The Oppenheim conjecture states that for $d \geq 3$ the set of values $q(\mathbb{Z}^d)$ is dense in $\mathbb{R}$.

This conjecture was proved by Margulis (in 1986) using an idea of Raghunathan, and I explained a proof as follows: Define $G := \mathrm{SL}_d(\mathbb{R}\Gamma := \mathrm{SL}_d(\mathbb{Z}), H := \mathrm{SO}(q)$. Then Ratner's result implies that the closure of $H^0 \cdot \Gamma$ in $G$ is $L \cdot \Gamma$ for some closed connected subgroup $L$ of $G$ containing $H$. Hence $L \in \{H^0, G\}$. The case $L = H^0$ is excluded by the irrationality assumption on $q$, hence $L = G$ and $H \cdot \Gamma$ is dense in $G$. Using some vector $v \in \mathbb{R}^d \backslash \{0\}$ as the first column of a matrix $g \in G$ we approximate $g$ by products $h_n \gamma_n$ with $h_n \in H, \gamma_n \in \Gamma$. It follows that $g(v) = \lim_{n\to\infty} q(h_n \gamma_n e_1) = \lim_{n\to\infty} q(\gamma_n e_1) \in q(\mathbb{Z}^d)$, hence the conjecture is true.

In order to get an $S$-arithmetic version of the density of values of a quadratic form, Borel and Prasad used an $S$-arithmetic version of Ratner's density result. Also, for $d \geq 5$ one gets

$$\#\{z \in \mathbb{Z}^d : \|z\| \leq r, a \leq q(z) \leq b\} \sim c_{a,b} d r^{d-2}.$$

Thilo Kuessner, Tübingen

## The case $\mathbf{SL}_2(\mathbb{R})$

We have

**Theorem** (Dani). *Let $\Gamma$ be a lattice in $G = SL_2(\mathbb{R})$ and let $N = \{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) : t \in \mathbb{R}\}$ act from the right on $\Gamma\backslash G$. Then points $x \in \Gamma\backslash G$ are either $N$-periodic ($xn(T) = x$, for some $T \in \mathbb{R}$), or uniformly distributed, i.e. for any bounded uniformly contineous $f : \Gamma\backslash G \to \mathbb{R}$ holds*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^t f(xn(t))dt = \frac{1}{vol(\Gamma\backslash G)} \int_{\Gamma\backslash G)} f(y)dy.$$

As a corollary one gets the classification of $N$-invariant ergodic measures on $\Gamma\backslash G$: they are either multiples of the Haar measure or supported on a periodic $N$-orbit. Another immediate corollary is Hedlund's theorem: $N$-orbits are dense or periodic.

The proof heavily relies on the interplay between geodesic flows (action of $A = \left\{ \left( \begin{smallmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{-\frac{s}{2}} \end{smallmatrix} \right) \right\}$) and horocyclic flows (action of $N$). From ergodicity of the $N$-action and Birkhoff's theorem one gets that almost all points are uniformly distributed. Then one has to check that close to $x$ one finds uniformly distributed points $y$ such that the values $\frac{1}{T}\int_o^T f(xn(t))dt$ and $\frac{1}{T}\int_o^T f(yn(t))dt$ remain close for a sequence $T_n \to \infty$. This can be done if the $A$-orbit of $X$ returns to some compact set for a sequence of times $\tau_1 \to \infty$, and this is the case always if $x$ is not $N$-periodic.

Bruno Klingler, IHES - ETH Zürich

## Measure Rigidity

Two talks in a series of four presenting the proof of the following generalization of Ratner's measure rigidity for real Lie groups, due to Margulis and Tomanov ('94):

**Theorem .** *Let $V$ be a finite set. For every $v \in V$, let $K_v$ be a local field of characteristic zero, $\underline{G}_v$ be a $K_v$-algebraic group and $G = \prod_{v\in V} \underline{G}_v(K_v)$. Let $\Gamma \subset G$ be a discrete subgroup and $H = \prod_{v\in V} \underline{H}_v(K_v)$, where $H_v \subset G_v$ is generated by unipotent subgroups. Then any $H$-invariant, $H$-ergodic Borel probability measure $\mu$ on $G/\Gamma$ is algebraic (i.e. there exists* *a closed subgroup $\Sigma \subset G$ and a point $x \in G/\Gamma$ such that the orbit $\Sigma x$ is closed and $\mu(\Sigma x) = 1$).*

We introduce basic notions on algebraic actions, horospherical subgroups and dynamics of class $\mathcal{A}$ elements. We prove that $\mu$ can be assumed "Zariski-dense", construct quasi-regular mpas and prove the Basic Lemma of the proof.

Dan Fulea, Heidelberg

## Consequences of the Basic Lemma

In this third talk in a series of four, taking advantage on the technical result, the Basic Lemma, of the last talk, one can formulate ([MT]:=[Margulis, Tomanov: Inv. Math., 1994]) intermediate results of algebraic nature, which will be exploited in the next talk using structures and results of ergodic (measure theoretical) nature to prove the Main Theorem (measure rigidity): " Each $U$–ergodic, $U$–invariant measure on $G/\Gamma$ is algebraic ".

These intermediate results are:

(1) [MT, Prop. 8.2] Let $\underline{G}$ be an algebraic group, $\underline{U} < \underline{G}$ an unipotent algebraic subgroup, $\mu$ a probability measure on $G/\Gamma$. Suppose $N < G := \underline{G}(K_V)$ is a subgroup, maximal with the following properties:

$\mu$ is $N$–invariant and $N$ is unipotently generated in $G$.

Assume $U \not\subset N$. Then (using techniques related to the Basic Lemma):

There exists a quasiregular map $\phi : U \to N_G(U)$ with the properties:

(i) $\phi(U)$ invariates the measure $\mu$, and

(ii) The group $F := \langle U, \phi(U)\rangle$ contains an element $s$ of $\mathcal{A}$–class, with the folowing properties:

(a) $U^+(s)$ :=[ maximal algebraic subgroup of $W^+(s)$ preserving $\mu$ ] is not trivial,

(b) $\alpha(s, \mathcal{F}(U^+(s))) \geq 1$,

(c) $N(s)$ := Auslander normal subgroup $\langle W^+(s), W^-(s)\rangle \triangleleft G$ satisfies: $N(s)/N(s) \cap N$ is infinite.

(2) [MT, Prop. 8.3] Given $\epsilon > 0$, there exists a compact $M_\epsilon \subset G/\Gamma$, which is a set of uniform convergence with respect to $U^+(s)$, $\mu(M_\epsilon) > 1 - \epsilon$, such that for all sequences $(g_i)$, $g_i \notin N_G(U)$, $g_i \to e$, $g_i M_\epsilon \cap M_\epsilon \neq \emptyset$ (as in the Basic Lemma), in the decomposition

$$g_i = u^-(g_i)\; v^-(g_i)\; z(g_i)\; u^+(g_i)\; v^+(g_i)$$

the factor $u^-(g_i)$ "dominates" the factor $v^-(g_i)$.

(3) [MT, Cor. 8.4] There exists a conull set $M \subset G/\Gamma$, such that the following "algebraic condition" is satisfied: $M \cap W^-(s)x \subset U^-(s)x$, all $x \in M$.

In the given time limit, the basic strategy, some intermediate steps for (1) and the flavour of the involved proof techniques were given. A main use of the quasiregularity of $\phi$ and the strategy of the repeated use of the Basic Lemma were explained "explicitly".

George Tomanov, Lyon

## Entropy and Measure Rigidity

The goal of the talk is to complete the proof of the measure rigidity theorem in the $S$-adic case due to G.A. Margulis and the speaker (Inv. Math. vol 116, pp.347-392, 1994). (See also M. Ratner's paper in Duke Math. J. vol. 77, pp. 275-382, 1995 for an independent

proof of the same result.)
We use the notions $G, \Gamma, \mu, s, W^{\pm}(s)$ as introduced in the talk of Bruno Klingler. Additionally denote by $V$ a closed subgroup of $W^{-}(s)$ normalized by $s$ and by $\alpha$ the module of the automorphisms of $V$ obtained by restricting $\mathrm{Int}(s)$ to $V$.
First we prove the following

**Theorem** (Entropy Theorem). *Assume that $\mu$ is $\langle s\rangle$-invariant and ergodic.*

(1) *If $\mu$ is $V$-invariant then $h(s,\mu) \geq \log \alpha$, where $h(s,\mu)$ is the entropy of $Int(s)|_V$;*
(2) *Assume that there exists a measurable subset $\psi \subset G/\Gamma$ such that $\mu(\psi) = 1$ and $\psi \cap W^{-}(s)x \subset Vx$ for every $x \in \psi$. Then $h(s,\mu) \leq \log \alpha$ and the equality implies that $\mu$ is $V$-invariant.*

Next we deduce the measure rigidity from the Entropy Theorem and from the results exposed in the previous talks by Bruno Klingler and Dan Fulea.

Nimish A. Shah, Tata Inst. Mumbai

## Uniform distribution of orbits of unipotent flows, (two talks)

Using the classification of finite ergodic invariant measures for actions of unipotent subgroups on homogeneous spaces of "Lie groups" (which are products of real Lie groups and linear $p$-adic Lie groups), it was shown that in finite volume homogeneous spaces of these Lie groups, any trajectory of an one-parameter unipotent subgroup is equidistributed (see works of M. Ratner, G.A. Margulis, S.G. Dani, G. Tomanov).
The aim of the talks was to give a proof of the result in a very special case which is quite relevant to the theme of the conference:
Let $G = \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$, $\Gamma$ a cocompact discrete subgroup of $G$ and

$$U = \{u(t) = \left(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\right) : t \in \mathbb{Q}_p\}.$$

Then for any $x \in G/\Gamma$ there exists a closed subgroup $F$ of $G$ containing $U$ such that the orbit $Fx$ is compact, there exists a $F$-invariant probability measure $\lambda_F$ on $Fx$ and the trajectory $\{u(t)x : t \in \mathbb{Q}_p\}$ is uniformly distributed with respect to $\lambda_F$; more precisely: for any continuous $f$ on $G/\Gamma$,

$$\lim_{r\to\infty} \frac{1}{\Theta(I(r))} \int_{I(r)} f(u(t)x)d\Theta(t) = \int_{Fx} f d\lambda_F,$$

where $\Theta$ is a Haar measure on $\mathbb{Q}_p$ and for $r > 0$ define $I(r) = \{t \in \mathbb{Q}_p : |t|_p \leq r\}$.
In fact, if $F \neq G$ then $vFv^{-1}$ is the diagonal embedding of $\mathrm{SL}_2(\mathbb{Q}_p)$ in $G$ for some

$$v = \left(\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix}\right)\right), s \in \mathbb{Q}_p.$$

Victor Rotger, Barcelona

## The Arithmetic of definite quaternion algebras and Gross curves

In this talk we introduce some of the key ingredients that are crucial in the proofs of Mazur's conjecture by Cornut and Vatsal.
Namely, we introduce first of all ring class fields $H$ of an imaginary quadratic field $K$ and we will be interested on the Mordell-Weil group $E(H)$ of $H$-rational points of an elliptic curve $E/\mathbb{Q}$. In order to study $E(H)$ and their associated $L$-function we introduce definite quaternion algebras $B$ and we construct a Gross curve $X_B/\mathbb{Q}$ attached to it. The curve

$X_B$ is a disjoint union of spheres and we describe a family of special points $X_{p^n} \subset X_B(K)$ (where $p$ any prime, $n \geq 1$) that are rational over $K$ on $X_B$. The main result of this talk is Vatsal's description of a Galois action of $\mathrm{Pic}(O_{p^n})$, where $O_{p^n} \subset K$ has conductor $p^n$, on the family of Gross points $X_{p^n}$: this describtion is given in terms of a related tree $\mathcal{T}_p$, the Bruhat-Tits building of $\mathrm{PSL}_2(\mathbb{Q}_p)$.

Ignazio Longhi, Münster

## Gross' formula

Goal of the talk was to explain Gross' formula relating special values of $L$-functions to Gross points on the curve $X = X_B$ which in the previous talk was associated to a definite quaternion algebra $B$ with discriminant $N^-$ (recall that $N = N^- N^+$).

**Preliminaries.** We started by defining the action of a Hecke algebra $\mathbb{T}$ as a ring of correspondences on $X$ and hence an action of $\mathbb{T}$ on $\mathcal{P}ic(X) = \oplus \mathbb{Z} e_i$ (recall that $X$ is the disjoint union of curves of genus 0, indexed by $Cl(B)$, the set of classes of oriented Eichler orders of level $N^+$).

Then, following [2], we introduced a positive definite height pairing on $\mathcal{P}ic(X)$, $< e_i, e_j > := w_i \delta_{ij}$, with the property that the Hecke action is self-adjoint, i.e. $< T_m e, e' > = < e, T_m e' >$ for all $T_m \in \mathbb{T}$.

Let $\mathcal{M}$ denote the lattice of weight 2 modular forms for $\Gamma_0(N)$ with integral coefficients; by the multiplicity one theorem, $\mathcal{M}$ is a free module of rank 1 over $\mathbb{T} \otimes \mathbb{Q}$. Define a $\mathbb{T}$-module homomorphism $\Phi : \mathcal{P}ic(X) \otimes_{\mathbb{T}} \mathcal{P}ic(X)^\vee \to \mathcal{M}$ by

$$e \otimes e^\vee \mapsto \frac{1}{2} \deg e \deg e^\vee + \sum_{m \geq 1} < T_m e, e^\vee > q^m.$$

$\Phi$ becomes an isomorphism over $\mathbb{T} \otimes \mathbb{Q}$.

**The map $\psi$.** Let $g = \sum a_n q^n \in S_2(N)$ be a newform: one knows that the coefficients $a_n$ are all real. We use $g$ to define a homomorphism $\psi_g : \mathbb{T} \to \mathbb{R}$, $T_m \mapsto a_m$ (this is an instance of the Jacquet-Langlands correspondence). Thanks to the multiplicity one theorem, one has $\mathcal{P}ic(X) \otimes_{\mathbb{T}} \mathbb{R} \simeq \mathbb{R}$; fix such an identification to get a map $\psi : Cl(B) \to \mathbb{R}$. We extend $\psi$ to a map on Gross points by putting $\psi(P) := \psi([R])$ for $P = (f, R)$.

**The formula.** Let $\chi$ be a primitive character for $\mathcal{P}ic(\mathcal{O}_n)$ and $P$ any Gross point of conductor $n$ on the curve $X$. Gross' formula then says:

$$| \sum_{\sigma \in \mathcal{P}ic(\mathcal{O}_n)} \chi(\sigma) \psi(P^\sigma) |^2 = *(g, g) L(g, \chi, 1) \tag{1}$$

where $(g, g)$ denotes the Petersson inner product and $*$ is an explicit, non-zero fudge factor.

Gross' proof (in [2], under the assumptions that $N = N^-$ is prime and $\chi$ is a character of $\mathcal{P}ic(\mathcal{O})$) works in four steps:

- decompose $L(g, \chi, s) = \sum_{\sigma \in \mathcal{P}ic(\mathcal{O})} \chi(\sigma) L(g, \sigma, s)$;
- use Rankin's method to obtain $L(g, \sigma, 1) = *(g, F_\sigma)$, for a certain modular form $F_\sigma$;
- compute the coefficients of $F_\sigma$ and show that they are equal to $\sum_\tau < P^\sigma, T_m P^{\sigma\tau} >$, so that $F_\sigma = \sum_\tau \Phi(P^\sigma \otimes P^{\sigma\tau})$;
- put everything together to get (1).

Gross' result was later extended by Daghigh to the generality above ([1]).

The final part part of the talk (skipped for lack of time) should have explained how Gross' formula is applied in [3], [4].

### Literatur


[1] H. Daghigh: Modular forms, quaternion algebras and special values of $L$-functions. McGill University Ph.D. thesis, 1997.

[2] B. H. Gross: Heights and the special values of $L$-series. In: Number theory (H. Kisilevski and J. Labute, eds.) Can. Math. Soc. Conf. Proc. **7**, AMS 1987, 115-187.

[3] V. Vatsal: Uniform distribution of Heegner points. Preprint, 2000.

[4] V. Vatsal: Special values of anticyclotomic $L$-functions. Preprint, 2001.



Nike Vatsal, UBC


## Special Values of Anticyclotomic $L$-functions

In this talk we state Gross' special value formula for anticyclotomic $L$-functions and show how the distribution of Gross points of conductor $p^n$ (with $n \to \infty$) is related to the resolution of Mazur's conjecture in the definite case. Using a result of Shimura, we show how to reduce Mazur's conjecture to studying the distribution of the vectors $(P_n^\tau)_{\tau \in G_0}$, where $P_n$ is a Gross point of conductor $p^n$, $G_0$ is the so-called "tame" subgroup of $\mathrm{Gal}(K(p^\infty)/K)$ and $n \to \infty$. Roughly speaking, one needs to show that the distribution is uniform in the components of the Gross curve; this will follow from Ratner's theorems.


Hilmar Hauer, Nottingham


## Heegner Points on Modular Curves

We reviewed some facts about elliptic curves over fields. I.e. the structure of endomorphism rings, classical theory of complex multiplication in characteristic zero and supersingular elliptic curves. Then we introduced the modular curve $X_0(N)$ over $\mathbb{C}$ and gave its interpretation as moduli scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$. As a technical tool, we defined a scheme-theoretic version of $\mathfrak{a}$-transforms.
In the first main part of the lecture, we determine the fields of definition of CM points on $X_0(N)(\mathbb{C})$. Furthermore we gave a formula for the Galois action on these CM points in terms of $\mathfrak{a}$-transforms.
Finally we found a describtion of the supersingular locus $X_0^{ss}(N)(\overline{\mathbb{F}_l})$ as the coset space $\mathrm{Cl}(R, N)$ of $(N, N)$-inclusions of left $R$-ideals. Here $R$ is a maximal order in a suitable definite quaternion algebra $B$. We also formulated an adelic version:

$$O_r\widehat{(J_0/I_0)}^* \widehat{\mathbb{Q}}^* \backslash \widehat{B}^* / B^* \cong X_0^{ss}(N)(\overline{\mathbb{F}_l}).$$


Sigrid Wortmann, Heidelberg


## Mazur's Conjecture for classical Heegner points

The aim of this talk was to present Cornut's proof of the following conjecture by Mazur (1983): Let $E/\mathbb{Q}$ be a (modular) elliptic curve, $K$ an imaginary quadratic field. Then for some $n \geq 0$ the trace $\mathrm{Tr}_{K[p^\infty]/H_\infty}(y_{p^n}) \notin E(H_\infty)_{\mathrm{tors}}$, where $K[p^\infty] = \bigcup_{n \geq 0} K[p^n]$, $K[p^n]$ the ring class field of conductor $p^n$, $H_\infty$ the anticyclotomic $\mathbb{Z}_p$-extension of $K$ and $y_{p^n}$ a Heegner point of conductor $p^n$ on $E$. As $E(H_\infty)_{\text{tors}}$ itself is a finite group it suffices to produce "many" points in the image of the trace, i.e. more than $\#E(H_\infty)_{\text{tors}}$. This in turn is shown by constructing "many" points in the reduction $E(\mathbb{F}_l)$ for some inert prime $l$. For this one writes $\mathrm{Tr}_{G_0}(y_{p^n}) = \mathrm{Tr}_{G_0/G_1}(\mathrm{Tr}_{G_1}(y_{p^n}))$ for the genus subgroup $G_1 \subset G_0 = \mathrm{Gal}(K[p^\infty]/H_\infty)$. The "geometric" part $\mathrm{Tr}_{G_1}(y_{p^n})$ can be described by using CM points coming from the modular curve $X_0(NM)$, with $M = \prod_{q|d_k} q$. Now one uses the reduction map

$$\mathrm{RED} : \mathcal{L}_p \to \left(X_0^{ss}(NM)(\overline{\mathbb{F}_l}\right)^{(G_0/G_1)},$$

which associates a tuple $(\mathrm{red}_l(x^\sigma))_{\sigma\in G_0/G_1}$ of supersingular points over $\mathbb{F}_{l^2}$ to an CM point $x$ on $X_0(NM)(K[p^\infty])$. Using Ratner's theorem (resp. a consequence of it shown in Vatsal's talk) it was shown that this map is surjective. But the number of supersingular curves in characteristic $l$ grows $\sim \left[\frac{l}{12}\right]$. So choosing a sufficiently big inert prime $l$ in $K$ gives the result.

Christophe Cornut, Strasbourg
Michael Spiess, Nottingham

## On the parity of ranks of Selmer groups

The aim of these talks is to present Jan Nekovář's proof of the parity conjecture for Selmer groups of elliptic curves over $\mathbb{Q}$, to the effect that: the corank of $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has the same parity as the order of vanishing at $s = 1$ of $L(E/\mathbb{Q}, s)$. This follows from the analog statement for $E/K$, where $K = \mathbb{Q}(\sqrt{D}), (D < 0)$ is an imaginary quadratic field, in which all prime factors of conductor$(E)$ are split. The parity conjecture in this case corresponds to the statement that $\mathrm{corank}(\mathrm{Sel}_{p^\infty}(E/K))$ is odd. Let $H_\infty$ be the anticyclotomic $\mathbb{Z}_p$-extension of $K$. The Pontryagin dual of $\mathrm{Sel}_{p^\infty}(E/H_\infty) = S_\infty$ is a $\Lambda$-module of finite type. By Mazur's control theorem the parity conjecture for $E/K$ follows from:

(1) $\mathrm{rank}_\Lambda(S_\infty^\vee) = 1$
(2) $(S_\infty^\vee)_{\text{tors}} = Y^2 \oplus Z$ (up to finite modules), where $Y$ is free over $\mathbb{Z}_p$ and $Z$ is killed by $p^n, n \gg 0$.

The first statement was already known to be a consequence of Mazur's conjecture on higher Heegner points.

The second statement is proved by constructing a symplectic pairing on $(S_\infty^\vee)_{\text{tors}}$ (with values in $\mathrm{Frac}(\Lambda)/\Lambda$). This is done by using the theory of Selmer complexes and a very general duality theorem for these complexes.

Andrei Yafaev, Rennes

## The André-Oort conjecture and its applications to Mazur's conjecture (with an appendix by Christophe Cornut)

The aim of this talk is to introduce the the André-Oort conjecture on the Zariski closure of sets of special points on Shimura varieties. This conjecture predicts that irreducible components of such a closure are subvarieties of Hodge type. Partial results on this conjecture have been obtained by André, Edixhoven, Moonen and Yafaev. The statements of the results and appropriate explanations have been given in this talk. Eventually we stated a case of this conjecture that comes up in the proof of Mazur's conjecture by Cornut and Vatsal. Namely, this is the case of varieties of the form $X_0(N) \times ... \times X_0(N)$. Such a varity is a moduli space for $n$-tuples of elliptic curves over $\mathbb{C}$ (with appropriate level structure). One considers a set $\Sigma$ of CM points such that for any $(E_1, ..., E_n) \in \Sigma$ with $\mathrm{End}(E_i) = \mathbb{Z} + p^m \mathcal{O}_K$, where $K$ is some fixed CM field and $m$ is an integer that tends to infinity as $(E_1, ..., E_n)$ ranges through $\Sigma$. The fact that irreducible components of $\Sigma$ are of Hodge type follows from the result by Moonen or Edixhoven.

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