Mathematisches Forschungsinstitut Oberwolfach

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Konvexgeometrie

April 22nd – April 28th

The topic of this conference was convex geometry with emphasis on the analytic aspect. The 11 main lectures dealt with the following subjects:

- Geometric probability
- Local theory of normed spaces
- Convexity and affine geometry
- Relations to topology
- Geometric inequalities
- Measures related to convex bodies
- Convex polytopes
- Various problems of convex geometry

A majority of the presented posters and short communications were related to these subjects. It was interesting to see the numerous relations convexity has to other areas of mathematics and that these areas have a strong impact on the development of modern convex geometry.

Paul Goodey
Peter M. Gruber

A survey on recent results and open problems in flexible polyhedra and related topics

Victor Alexandrov

Let $\sum$ be a connected two-dimensional simplicial complex. A continuous mapping $P : \sum \to E^3$ is said to be a polyhedron if it is affine over each simplex of $\sum$. However we call the image $P(\sum)$ a polyhedron, too.

A polyhedron $P = P(\sum)$ is said to be flexible if there exists an analytic (with respect to a parameter) family of polyhedra $P_t = P_t(\sum)$ $(0 \leq t \leq 1)$ such that

i) $P_0 = P$;

ii) the length of each edge of P_t is independent of t;

iii) there exist two vertices of P_t such that the (spatial) distance between them is not constant in t.

The family P_t $(0 \leq t \leq 1)$ is called a nontrivial flex of P.

Similarly one can define the notion of a flexible polyhedra in spherical, hyperbolic and Minkowski spaces of arbitrary dimension.

The most important results in the theory of flexible polyhedra in Euclidean 3-space are as follows:

a) there are no convex flexible polyhedra (A.Cauchy, 1813, A.D.Aleksandrov, 1950);

b) there exist flexible sphere-homeomorphic embedded polyhedra (R.Connelly, 1976, K.Steffen, 1980);

c) each flexible polyhedron preserves its mean curvature during a flex (R.Alexander, 1985);

d) each flexible polyhedron preserves its (generalized) volume during a flex (I.Kh. Sabitov, 1996, R.Connelly, I.Sabitov, A.Walz, 1997).

In the present talk, we will present the following recent results in flexible polyhedra:

1) application of Cauchy's method to studying rigidity of certain non-convex polyhedra in E^3 (rigidity results for polyhedral herissons obtained by L.Rodriguez and H.Rosenberg);

2) existence of flexible polyhedra in Euclidean 4-space E^4 (results by R.Connelly and H.Stachel);

3) existence of flexible polyhedra in Minkowski 3-space and volume conservation for them;

4) flexibility and rigidity of polyhedra via infinitesimal bending.

We are going to discuss also some related results about smooth herissons (by Y.Martinez-Maure) and about invariance of the mean curvature of a smooth surface under infinitesimal bending (by I.Rivin).

The complex plank problem

Keith M. Ball

A plank theorem is a result of the following type. X is a normed space, $\phi_1, \ldots, \phi_n$ are unit functionals on X and $w_1, \ldots w_n$ are positive numbers satisfying a size condition. The conclusion is that there is a vector x in X of norm at most 1 for which $|\phi_k(x)| \geq w_k$ for each k. This talk discusses 5 such results and some applications.

1) The original plank theorem of Bang in which X is Hilbert space and $\sum w_k \leq 1$, which solved the plank problem of Tarski and has an application to sphere-packing.

2) The author's plank theorem in which X is an arbitrary normed space and $\sum w_k \leq 1$, which provides a sharp quantitative version of the Banach-Steinhaus Theorem.

3) Nazarov's solution of the coefficient problem for Fourier series in which X is L_1, $\sum w_k^2 \leq 1$ and the functionals satisfy a Bessel inequality.

4) Lust-Piquard's non-commutative plank theorem which shows that bounded operators on Hilbert space satisfy no non-trivial size estimates on their entries.

5) The author's recent complex plank theorem in which X is complex Hilbert space and $\sum w_k^2 \leq 1$, which is related to topics in control theory.

There are infinitely many irrational values of zeta at the odd integers

Keith M. Ball

(joint work with T. Rivoal)

This talk outlines a proof of the fact stated in the title. The proof depends upon the construction of explicit linear forms for each odd d and each even number m,

$$A_m^{(0)} + A_m^{(3)}\zeta(3) + \ldots + A_m^{(d)}\zeta(d)$$

whose coefficients are integers at most C^{md} for some $C > 1$ and for which the sum is at most $c^{md \log d}$ for some $c < 1$. These forms are derived from series of the following type,

$$\sum_{k=1}^{\infty} \frac{q(k)}{(k(k+1)(k+2)\ldots(k+m))^d}$$

where q is a polynomial (of small enough degree) with rational coefficients. Such series were already used for various purposes (eg. by Nikishin). Here we choose a polynomial q which is an *even* function of $k + m/2$. This parity constraint ensures that the coefficients of the even number values of zeta disappear. The precise series are as follows

$$(m!)^{d-2r} \sum_{k=1}^{\infty} \frac{(k-1)(k-2)\ldots(k-rm).(k+m+1)(k+m+2)\ldots(k+(r+1)m)}{(k(k+1)(k+2)\ldots(k+m))^d}$$

for some integer r. The sums are small because the zeroes in the numerator ensure that the sum doesn't start until the denominator has become large. The size conditions will hold as long as r is at most $d/\log d$ and at least some power of d.

Hyperplane projections of the unit ball of ℓ_p^n

Franck Barthe

(joint work with Assaf Naor)

For $p \geq 1$, let $B_p^n = \{x \in R^n;\ \sum_{i=1}^n |x_i|^p \leq 1\}$. We are interested in the extreme values of the volume of the orthogonal projections of B_p^n onto hyperplanes. For a fixed hyperplane $H \subset R^n$ we show that the ratio $\text{vol}(P_H B_p^n)/\text{Vol}(B_p^{n-1})$ of the volume of the projection onto H to the volume of the canonical projection, is a non-decreasing function of p. This analogue of the Meyer-Pajor theorem for sections immediately gives informations about largest projections for $p \leq 2$, since the latter ratio is 1 for every direction H when $p = 2$. Using Fourier transforms and elementary results on completely monotonic functions, we determine the extremal projections for $p \geq 2$. All this work relies on an explicit formula for the volumes of projections in terms of the first moment of a combination of independent random variables.

The covariogram problem

Gabriele Bianchi

The covariogram $C_K(x)$ of a convex body $K \subset \mathbb{R}^n$ is defined for $x \in \mathbb{R}^n$ as

$$C_K(x) = Vol(K \cap (K + x)) = \chi_K * \chi_{(-K)}(x)\,. \tag{1}$$

where λ_n denotes the Lebesgue measure on $\mathbb{R}^n$. The covariogram is invariant with respect to translations or reflections of K. G. Matheron asked whether the covariogram determine a convex body among all convex bodies up to translations and reflections.

The covariogram C_K coincides, up to a constant, with the distribution of the difference $X - Y$ of two independent random variables X, Y which are uniformly distributed over K. Knowing C_K is equivalent to knowing for each direction $u \in S^1$ the distribution of the lengths of the chords of K which are parallel to u.

Passing to the Fourier transforms formula (1) becomes $\hat{C}_K = |\hat{\chi}_K|^2$: thus the problem of determining K from the knowledge of C_K is equivalent to the phase retrieval problem in Fourier analysis, restricted to the class of characteristic functions of convex bodies.

The following theorem has been proved by G. Bianchi, F. Segala and A. Volčič in the case of C^2 bodies with strictly positive curvature, and by G. Bianchi alone in the general case.

Theorem: *Let K be a plane convex body whose boundary is the union of a finite number of C^2 arcs which meet in points where the boundary is not C^1. Let us assume moreover that ∂K contains at most a finite number of segments. Its covariogram determines K uniquely (up to translations and reflections) among all convex bodies.*

Random polytopes: expected values, variances, distributions

Christian Buchta

Given a convex body $K \subset I\!R^d$, a random polytope is defined as the convex hull of n independent random points distributed uniformly in K. Denote by K_n the convex hull of n such points, by V_n the volume of K_n, by D_n the volume of $K \backslash K_n$, and by N_n the number of vertices of K_n.

In the first part of the talk a survey on results about the expected values EV_n, ED_n, and EN_n is given. It is pointed out that an identity due to Efron relates the expected volume ED_n — and thus EV_n — to the expected number EN_{n+1}.

In the second part of the talk it is described how this identity can be extended from expected values to higher moments. The planar case of the arising identity for the variances provides in a simple way the corrected version of a result by Cabo and Groeneboom and an improvement of a result by Hsing. Estimates of $\mathrm{var}D_n$ and $\mathrm{var}N_n$ are obtained in the cases that K is a d-dimensional convex polytope or a d-dimensional smooth convex body, respectively.

The identity for moments of arbitrary order shows that the distribution of N_n determines $EV_{n-1}, EV_{n-2}^2, \ldots, EV_{d+1}^{n-d-1}$. Conversely it is proved that these $n-d-1$ moments determine the distribution of N_n entirely. The resulting formula for the probability that $N_n = k$ ($k = d+1, \ldots, n$) appears to be new for $k \geq d+2$ and yields a solution to a problem raised by Baryshnikov. For $k = d+1$ the formula reduces to an identity which has repeatedly been pointed out, e.g., by Henze, Schneider, and Weil and Wieacker.

Applications and developments of a theorem of Rogers and Shephard on moving convex sets

Stefano Campi

Given a compact set $K \in I\!\!R^d$, a direction v and a bounded real valued function α defined on K, the family of convex sets

$$K_t = \mathrm{conv}[\{x + \alpha(x)tv \,:\, x \in K\}],\, t \in I\!\!R$$

is called a *shadow system* along v with speed function α. A theorem proved by Rogers and Shephard in 1958 says that if K_t is a shadow system, then the volume $V(K_t)$ is a convex function of t. The theorem can be used successfully for determining minimizers of several functionals in Convex Geometry, as functionals of Sylvester-type, of Busemann-type and functionals involving the volume of p-centroid bodies. In all these cases the theorem implies, via Steiner's symmetrization, that ellipsoids are minimizers. The theorem also provides a selection criterion for searching maximizers of the same functionals. Such a criterion leads to partial results and enforces the conjecture that simplices are solutions of the maximum problems.

An Infinite Set of Solid Packings on the Sphere

August Florian

The simple concept of solid packing and solid covering was introduced by L. Fejes Tóth. We consider the following example. We place $2n$ congruent non-overlapping circles with their centers at the vertices of the spherical Archimedean tiling $(3, 3, n)$ such that each circle touches four others. The system is supplemented by two additional circles, each touching n circles of the system. It can be proved that this packing is solid, for all $n \geq 4$. On the other hand, it can be conjectured that the incircles of $(3, 3, 3, n)$ do not form a solid packing, and the circumcircles do not form a solid covering of the sphere. This is subject of current joint work of A. Heppes and A. Florian, and is open at the moment.

Shape reconstruction from brightness functions

Richard J. Gardner

The talk outlined joint work in progress with Peyman Milanfar. The problem is to reconstruct a planar or 3-dimensional convex body from noisy, and possibly sparse, measurements of its brightness function. The approach taken comprises two steps that result in a polygon or polyhedron that is supposed to approximate the unknown convex body. In Step 1, an approximation to the surface area measure (sometimes also called the extended Gaussian image) is obtained by using Cauchy's projection formula and solving a least squares problem. (A similar but different method was employed and implemented independently by M. Kiderlen in his 1999 PhD thesis; see also his paper, "Nonparametric estimation of the directional distribution of stationary line and fibre processes," *Adv. Appl. Probab.* **33** (2001), 6–24.) In Step 2, a convex polygon or polyhedron is constructed whose surface area measure is the output of Step 1. For polygons this is trivial, and for polyhedra, an algorithm for this purpose, based on an earlier one of J. Little, was published by J. Lemordant, P. Tao, and H. Zouaki in *RAIRO Modél. Math. Anal. Numér.* **27** (1993), 349–374. This involves maximizing over l the concave function $V(P(l))^{1/3}$ subject to linear constraints, where $V(P(l))$ is the volume of the convex polyhedron $P(l)$ and l denotes the vector of distances of its facets from the origin. The output l^*, together with the unit normals from the input for Step 2, give the so-called H-representation of the reconstructed polyhedron. This must be converted to its V-representation, the set of its vertices, and the convex hull produced from this set. Our implementation of the whole reconstruction process is not yet complete, but will utilize Matlab's optimization toolbox to solve the nonlinear programming problems in Steps 1 and 2, the C++ program Vinci by B. Büeler, A. Enge, and K. Fukuda to compute the function $V(P(l))$ and to convert the H-representation to the V-representation, the program Qhull from The Geometry Center at the University of Minnesota to calculate the convex hull, and Mathematica to display the reconstructed polyhedron.

Local Theory and Convex Geometry

Apostolos Giannopoulos

Let $\mathcal{K}_n$ denote the class of all convex bodies in $\mathbb{R}^n$. For every $K \in \mathcal{K}_n$ we consider the family $P(K) = \{T(K) : T \text{ affine}\}$ of all *positions* of K. We write $w(K)$ for the mean width of K and $A(K)$ for the surface area of K. The Banach-Mazur distance of the convex bodies K and L is defined by $d(K,L) = \min\{t \geq 1 : \exists K_1 \in P(K), L_1 \in P(L)$ such that $K_1 \subseteq L_1 \subseteq tK_1\}$ and the volume ratio of K and L is the quantity $\mathrm{vr}(K,L) = \min\{(|K|/|L_1|)^{1/n} : L_1 \in P(L), L_1 \subseteq K\}$. In this talk we discuss basic extremal problems and results about these quantities.

(i) The reverse isoperimetric inequality of K. Ball:

$$a_n := \max_K \min_{K_1 \in P(K)} \frac{A(K_1)}{|K_1|^{(n-1)/n}} = A(S_n) \simeq n$$

where S_n is an n-dimensional simplex of volume 1 (the cube is the extremal body in the symmetric case).

(ii) The "reverse Urysohn inequality" of G. Pisier (following work of Lewis and Figiel-Tomczak):

$$w_n := \max_K \min_{K_1 \in P(K)} \frac{w(K_1)}{|K_1|^{1/n}} \leq c\sqrt{n}\log n$$

(the lower bound $w_n \geq c_1\sqrt{n \log n}$ follows by considering a simplex or the cross-polytope).

(iii) The estimate of Rudelson on the Banach-Mazur distance of an arbitrary pair of convex bodies: $d_n := \max_{K,L} d(K,L) \leq cn^{4/3}\log^9 n$.

(iv) An estimate for the volume ratio (Giannopoulos-Hartzoulaki):

$v_n := \max_{K,L} \mathrm{vr}(K,L) \leq c\sqrt{n}\log n$.

The main point of the talk is the following: although precise "isotropic" geometric descriptions are available for the extremal positions inside each affine class, giving estimates for the quantities above requires diverse and deep analytic and probabilistic tools. Local theory and convex geometry meet here in a natural way.

Mixed support functions of convex bodies

Paul R. Goodey and Wolfgang Weil

In 1995, the following translative integral formula for (centred) support functions of convex bodies $K,\ M \subset \mathbb{R}^d$ was established,

$$\int_{\mathbb{R}^d} h^*(K \cap (M+x), \cdot)\lambda_d(dx)$$

$$= V_d(M)h^*(K,\cdot) + \sum_{j=2}^{d-1} h_j^*(K,M,\cdot) + V_d(K)h^*(M,\cdot)\,.$$

In addition to the support functions $h^*(K,\cdot)$ of K and $h^*(M,\cdot)$ of M, it contains mixed functions $h_j^*(K,M,\cdot)$ which (amongst other properties) have the property that they depend homogeneously of degree j on K (and of degree $d+1-j$ on M). Here we show that the mixed functions $h_2^*(K,M,\cdot),\ldots,h_{d-1}^*(K,M,\cdot)$ are all convex, hence they are support functions of convex bodies. For the proof, we replace support functions by first surface area measures and assume that K and M are polytopes. The main tool is then the solution of the Christoffel problem for polytopes obtained by Schneider in 1977.

Aspects of Random Variables in Convexity Theory

Yehoram Gordon, Carsten Schütt and Elisabeth Werner

(joint work with Alexander Litvak)

Let f_i, $i=1,\ldots,n$, be copies of a random variable f and N be an Orlicz function. We show that for every $x\in I\!\!R^n$ the expectation $\mathbf{E}\|(x_if_i)_{i=1}^n\|_N$ is maximal (up to an absolute constant) if f_i, $i=1,\ldots,n$, are independent. In that case we show that the expectation $\mathbf{E}\|(x_if_i)_{i=1}^n\|_N$ is equivalent to $\|x\|_M$, for some Orlicz function M depending on N and on distribution of f only. We provide applications of this result.

Geometry of Spaces Constructed between Polytopes and Zonotopes

Yehoram Gordon, Carsten Schütt and Elisabeth Werner

(joint work with Alexander Litvak)

Let $\{a_i\}_{i=1}^N\subset I\!\!R^n$ span the space, $1\le k,n\le N$. Define on $I\!\!R^n$ the norm $\|x\|_k=\sum_{i=1}^k|<x,a_i>|^*$, where $\{\lambda_i^*\}$ denotes the decreasing rearrangement of a sequence of scalars $\{\lambda_i\}$. Denote the unit ball of the space $(I\!\!R^n,\|.\|_k)$ by B_k. Clearly $B_1^*=conv(\{-a_i,a_i\}_{i=1}^N)$ defines a symmetric polytope and $B_N^*=\sum_{i=1}^N[-a_i,a_i]$ defines a zonotope. If $T:R^N\to R^n$ is the operator defined by $T(e_i)=a_i$, then $B_k=(T(kB_1^N\cap B_\infty^N))^*=T^{*-1}(conv(B_1^N,\frac{1}{k}B_\infty^N))$, where B_p^N denotes the unit ball of ℓ_p^N.

We investigate the following topics:

1) Volume estimates for B_k, B_k^* and their $l-$ dimensional sections, using new probabilistic results, with sharp estimates when $\{a_i\}\subset S^{n-1}$, thus extending the classical estimates $k=1$ and $k=N$.

2) Various forms of Dvoretzky's theorem on existence of best isomorphic copies of ℓ dimensional spherical sections for the balls B_k, B_k^*.

3) Volume ratios, type and cotype, projection constants.

4) All the above for $\{a_i\}\subset S^{n-1}$, and in the special cases when $N=n$ and $a_i=e_i, i=1,...,n$.

Affine inequalities involving p-centroid bodies

Paolo Gronchi

Here a joint work with Stefano Campi is presented. If K is a convex body in $I\!R^d$, then $\Gamma_p K$, the p-centroid body of K, is defined by

$$h_{\Gamma_p K}(u) = \left(\frac{1}{c_{d,p} V(K)} \int_K |\langle u, z \rangle|^p \, dz \right)^{\frac{1}{p}}$$

for every real number $p \geq 1$. We call parallel chord movement of K along the direction v a family of convex sets defined by

$$K_t = \{z + \alpha(x)tv \, : \, z \in K, \, x = z - \langle z, v \rangle v \in v^\perp\}], \, t \in I\!R$$

where the "speed function" α is a continuous real function on $v^\perp$.

Theorem: *If K_t is a parallel chord movement with speed function α, then the volume of $\Gamma_p K$ is a strictly convex function of t unless α is linear.*

This theorem implies the L_p-Busemann-Petty centroid inequality, which characterizes centered ellipsoids as the only minimizers of $V(\Gamma_p K)/V(K)$. Such a result was recently proved also by Lutwak, Yang and Zhang in a different way. An other consequence of the theorem is that triangles are maximizers of the same functional.

Optimal distribution of points on Riemannian manifolds

Peter M. Gruber

We give the following extension of the theorem of Fejes Tóth on sums of moments in $I\!E^2$ to d-dimensional Riemannian manifolds:

Theorem: *Let $f : [0, +\infty) \to [0, +\infty)$ be an increasing function satisfying suitable growth conditions. Then there exists a constant $\mathrm{div}_{f,d} > 0$ such that for any Jordan measurable set J on a Riemannian d-manifold M (with metric ϱ_M and volume ω_M) we have*

$$\inf_{\substack{S \subset M \\ \#S = n}} \{ \int_J \min_{p \in S} \{ f(\varrho_M(p, x)) \} d\omega_M(x) \sim \mathrm{div}_{f,d} \, \omega_M(J) f\left(\frac{\omega_M(J)^{\frac{1}{d}}}{n^{\frac{1}{d}}}\right) \text{ as } n \to \infty.$$

This result has applications to asymptotic best approximations of convex bodies, to the isoperimetric problem in Minkowski spaces, to numerical integration, and to Gauss channels.

Isomorphic Dvoretzky's theorem for some classes of convex bodies

Olivier Guédon

Let K be a symmetric convex body in $I\!R^n$, we define the Banach Mazur distance between K and the Euclidean ball B_2^n as

$$d(K, B_2^n) = \inf_{\mathcal{E} \text{ symmetric ellipsoid}} \{\lambda > 0, \text{ such that } \mathcal{E} \subset K \subset \lambda \mathcal{E}\}.$$

Does there exist a $k-$dimensional subspace F_k, an ellipsoid $\mathcal{E}$ in F_k such that

$$\mathcal{E} \subset K \cap F_k \subset f(n, k) \, \mathcal{E} \ ?$$

We need to find a good estimation (up to some numerical constant) of $f(n, k)$ for all range of $k \in \{1, \ldots, n\}$ and for all convex body in some particular geometrical classes.

The original "almost isometric" Dvoretzky's question was to replace $f(n,k)$ by $1+\varepsilon$ and to study the largest possible $k_0 = k(\varepsilon, n)$ so that we could answer positively to this question.

The most simple estimation of $f(n,k)$ is by John's theorem that $f(n,k) \le \sqrt{k}$. But an answer was given by Milman and Schechtman when K is a general convex body :

$$f(n,k) \le c\Big(1 + \sqrt{\frac{k}{\log\left(1+\frac{n}{k}\right)}}\Big).$$

In this talk, I have presented results of a joint paper with Y. Gordon, M. Meyer and A. Pajor, *Random Euclidean sections of some classical Banach spaces* (Mathematica Scandinavica). In this paper, we were interested in the above question for some special classes of unit balls of an $n-$dimensional normed space E. We say that a family $u_1, \dots, u_N$ of vectors of E, with $N \le n$, satisfies a (C,s)-*estimate* for $C > 0$ and $s > 0$, if for all $(t_i)_{i=1}^N \in I\!R^N$ and all $m = 1, \dots, N$, one has

$$\frac{C}{m^{1/s}}\Big(\sum_{i=1}^m (t_i^\star)^2\Big)^{1/2} \le \|\sum_{i=1}^N t_i u_i\| \le \Big(\sum_{i=1}^N t_i^2\Big)^{1/2},$$

where $(t_i^\star)_{i=1}^N$ denotes the decreasing rearrangement of the sequence $(|t_i|)_{i=1}^N$. By a result of Bourgain and Szarek, there exists a constant $C > 0$ such that for any n, any n-dimensional normed space contains a sequence $u_1, \dots, u_N$, with $N \ge \frac{n}{2}$, satisfying a $(C,2)$-estimate. It is easy to see that for $q \ge 2$, the canonical basis of ℓ_q^n satisfies a $(1,s)$-estimate, with $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. Under this geometric assumption, we obtain precise estimate of the function $f(n,k)$, which is optimal in the case of ℓ_q^n (and where q may depend on the dimension n).

The second part of the paper is devoted to the study of the Schatten classes and the estimate of $f(n,k)$ is again optimal up to a constant.

Trefoil knots with tritangent support planes

Erhard Heil

There are examples of trefoil knots without tritangent planes. But if a wire model of familiar shape is placed on a plane, it will rest on 3 points. Precisely:

Let C be a trefoil knot which can be projected into a plane in such a way that the following is true: The image is locally convex, has rotation number 2 and exactly 3 double points. Apart from the double points the projection is injective. Then C has at least 2 tritangent support planes.

The proof starts with a polygon approximating C. Sperner's lemma is applied to its upper and lower part, which gives triangles lying in planes supporting the polygon and, by a limit process, 2 supporting tritangent planes of C.

This investigation was motivated by the search for the minimum number of vertices.

Measures, curvatures and currents in convex geometry

Daniel Hug

Support measures of convex bodies play an important role in convex and integral geometry. Moreover, there are many applications in stochastic geometry. In the first part of this talk, we focus on results related to the absolute continuity of curvature and surface area measures (the image measures of support measures under projection maps), since such results provide a common framework for a better understanding of various separate contributions. Starting with characterizations of absolute continuity of curvature and surface area measures we describe two transfer principles which allow one to translate properties connected with the absolute continuity of the rth curvature measure of a convex body (containing the origin in its interior) to dual properties related to the absolute continuity of the $(d-1-r)$th surface area measure of the polar body, and conversely. Various new results are deduced as applications of these transfer principles. The second part describes extensions to Minkowski spaces of recent results on curvature measures in Euclidean spaces. Here characterization and stability results and a splitting theorem are obtained. The final part outlines ideas related to projection functions and mixed volumes. In particular, a common framework for representing these functionals in terms of generalized curvatures and the normal bundles of the bodies involved is suggested.

Weakly Monotonic Endomorphisms of The Space of Convex Bodies

Markus Kiderlen

Let $\mathcal{K}$ be the family of convex bodies (nonempty, compact, convex subsets) in d-dimensional Euclidean space. A mapping $A : \mathcal{K} \to \mathcal{K}$ is called an *endomorphism of* $\mathcal{K}$, if it is Minkowski-additive, continuous, and intertwines the action of the group of rigid rotations that fix the origin. A classical result states that for $d = 2$ every endomorphism can be written as a suitable defined mixing of 'prototypes', namely the rotations. This is not true for higher dimensions. For $d \geq 3$ we define prototypes of endomorphisms using the generalized spherical Radon transform. We then show that any endomorphism of $\mathcal{K}$ can be written as a mixing of these prototypes if the so-called 'mixing distribution' is used. The mixing distributions associated to certain subclasses of endomorphisms can be characterized: Monotonic (i.e. inclusion-preserving) endomorphisms correspond to positive mixing measures, weakly monotonic endomorphisms (i.e. those that are inclusion-preserving for centered bodies) correspond to essentially positive mixing measures. We conclude with two applications concerning fixed points of endomorphisms and an injectivity result for the m-th mean projection body.

On the derivatives of X-ray functions

Alexander Koldobsky

Let K be an origin-symmetric star body in $\mathbb{R}^n$. For every $\xi \in S^{n-1}$, we define the *X-ray function* $z \to A_{K,\xi}(z),\ z \in \mathbb{R}$ by

$$A_\xi(z) = vol_{n-1}\big(K \cap (\xi^\perp + z\xi)\big),$$

where $\xi^\perp = \{x \in \mathbb{R}^n : (x,\xi) = 0\}$ is the central hyperplane orthogonal to ξ.

Denote by

$$AV_m(K) = \int_{Gr(n,m)} vol_m(K \cap H)\ dH,$$

where $Gr(n,m)$ is the Grassmanian of m-dimensional subspaces of $\mathbb{R}^n$ equipped with the probability Haar measure. Then $AV_m(K)$ is the average volume of m-dimensional central sections of K. We prove the following result of the Busemann-Petty type:

Theorem: *Let $n \geq 4$, K and L be $(n-4)$-smooth origin symmetric convex bodies in $\mathbb{R}^n$. Suppose that n is an even number and that, for every $\xi \in S^{n-1}$,*

$$(-1)^{(n-4)/2} A_{K,\xi}^{(n-4)}(0) \leq (-1)^{(n-4)/2} A_{L,\xi}^{(n-4)}(0),$$

(for odd n, replace the derivatives in (3) by the corresponding expressions from (2)). Then, for every integer $3 \leq m \leq n$, we have $AV_m(K) \leq AV_m(L)$.

As a consequence we get that for even n, $3 \leq m \leq n$ and any $(n-4)$-smooth origin-symmetric convex body K in $\mathbb{R}^n$

$$\max_{\xi \in S^{n-1}} (-1)^{(n-4)/2} A_{K,\xi}^{(n-4)}(0) \geq \frac{2^{n-2}}{3} \big(\Gamma(m/2+1)\big)^{3/m} \pi^{n/2-3} \Gamma(\frac{n-3}{2}) (AV_m(K))^{3/m}$$

with equality for the Euclidean ball.

The proof of Theorem 1 is based on a connection between the derivatives of X-ray functions and the Fourier transform and on a fact that if f is a function on $\mathbb{R}^n$ that is a positive definite distribution on all hyperplanes passing through the origin then $\|x\|_2^{-1} f(x)$ is a positive definite distribution on $\mathbb{R}^n$.

A problem in convex tomography

David G. Larman

The following conjectures are considered and partial results given:-

Conjecture: *Suppose K and L are convex bodies in $\mathbb{R}^n$ with $L \subset \mathrm{int}K$. Suppose, for every hyperplane H supporting L, the $n-l$–volume $f_{K,L}(H)$ of $H_n K$ is known. Then K is uniquely determined by $f_{K,L}$ and L.*

Conjecture: *Suppose that K, L are convex bodies in $\mathbb{R}^n$, $n = 3$ and $L \subset \mathrm{int}K$. Suppose further that whenever H is a hyperplane supporting L the section $H_n K$ of K is centrally symmetric. Then K is an ellipsoid.*

Partial results for conjecture 1 include, for example, if L is a ball and $f_{K,L}(H)$ is constant then K is also a ball.

Affine surface area

Monika Ludwig

For a convex body K, i.e. a compact convex set, in d-dimensional Euclidean space, the *affine surface area* is defined as

$$\Omega(K) = \int_{\mathrm{bd}\, K} \kappa(x)^{\frac{1}{d+1}}\, d\sigma(x),$$

where $\mathrm{bd}\, K$ is the boundary of K, $\kappa(x)$ is the (generalized) Gaussian curvature of $\mathrm{bd}\, K$ at x, and σ is the $(d-1)$-dimensional Hausdorff measure. We give a geometric interpretation of affine area and describe the definitions of affine surface area for general convex bodies as proposed by Leichtweiß, Lutwak, and Schütt and Werner. Using these definitions, it was shown that affine surface area is an upper semicontinuous, translation and SL(d)-invariant functional defined for all convex bodies. We discuss applications, especially in problems of asymptotic approximation of convex bodies, and a way of characterizing affine surface area (joint work with Matthias Reitzner).

Face-numbers of simple polytopes

Peter McMullen

The g-theorem describes the possible f-vectors (numbers of faces of each dimension) of simple polytopes; it was conjectured by McMullen in 1970. The sufficiency of its conditions was established by Billera & Lee in 1979. In the same year, Stanley found the first proof of their necessity, appealing to a deep result in algebraic geometry (the hard Lefschetz theorem in the cohomology ring of the toric variety associated with a rational simple polytope). In 1992, McMullen discovered an alternative proof; this used the polytope algebra, and so worked within convexity. In 1994, McMullen further simplified this proof, employing the weight algebra instead. This talk presented a yet shorter proof, found very recently by McMullen's research student, Harry Paterson. This proof also uses the weight algebra, but avoids the need of the previous approach to prove the quadratic Hodge-Riemann-Minkowski inequalities for mixed volumes as well.

A geometric proof of some inequalities involving mixed volumes

Mathieu Meyer and Shlomo Reisner

A new, "spherical harmonics free" proof of mixed-volume inequalities due to Schneider and to Goodey and Groemer, is presented.

The Alexandrov-Fenchel inequality implies that if K_1 and K_2 are two convex bodies in $\mathbb{R}^n$, and if for $i, j = 1, 2$, $V_{ij} = V(K_i, K_j, B, \dots, B)$ is the mixed volume of K_i, K_j with $n-2$ copies of the Euclidean unit ball B, then $V_{12}^2 \geq V_{11}V_{22}$. Moreover, it was proved by Schneider and by Goodey and Groemer that one can control the difference $V_{12}^2 - V_{11}V_{22}$ in the following way :

(1) If $K_2 = B$, $V_{12} = V_1 = V(K_1, B, \dots, B)$, B_1 is the Steiner ball of K_1 (see the definitions below) and v_n is the volume of B then

$$V_1^2 - v_n V_{11} \geq \frac{n+1}{n(n-1)} v_n \int_{S_{n-1}} \left(h_{K_1} - h_{B_1}\right)^2 \, d\sigma,$$

where σ denotes the surface measure on the sphere S_{n-1}, and for a convex body K, h_K denotes its support function.

(2) If the Steiner balls of K_1, K_2 are the same, then

$$V_{12}^2 - V_{11}V_{22} \geq \frac{n+1}{n(n-1)} v_n \int_{S_{n-1}} \left(h_{K_1} - h_{K_2}\right)^2 \, d\sigma\,.$$

The results (1) and (2) may be interpreted as stability results in specific cases of the Alexandrov - Fenchel inequality. Using them one can derive further inequalities between intrinsic volumes of different orders of a given convex body. The proofs of Schneider and of Goodey and Groemer make use of spherical harmonics and of a representation of mixed volumes which involves the action of differential operators on support functions of convex bodies. We present here a new proof which is "spherical harmonics free" and which has a more geometric flavor. This proof is based on a variational argument involving Santaló's inequality. We believe that this variational method may prove useful for the treatment of other problems as well. In fact we prove the following more general result (which also admits a proof using spherical harmonics):

(3) Let $K_1, \ldots, K_p$ be convex bodies in $\mathbb{R}^n$ and for $1 \leq i \leq p$, let $V_i = V(K_i, B, \ldots, B)$, $h_i = h_{K_i}$ and let B_i be the Steiner ball of K_i. Then the quadratic form $q : \mathbb{R}^p \mapsto \mathbb{R}$ defined by $q(s_1, \ldots, s_p) = \sum_{i,j=1}^{p} a_{ij} s_i s_j$, where

$$a_{ij} = V_i V_j - v_n V_{ij} - \frac{n+1}{n(n-1)} v_n \int_{S_{n-1}} (h_i - h_{B_i})(h_j - h_{B_j}) d\sigma \,,$$

is non-negative. (*Beitr. z. Alg. u. Geom.* **41** (2000), 335-344.)

Some applications of topology to convex geometry

Luis Montejano-Peimbert

The purpose of the talk is to speak about k-polars of Convex Bodies. This simple concept will allow us to put several classic results in the same setting and from that to develop new results, generalizations and conjectures. Let Γ be a k-plane of n- projective space, P^n, and let Δ be a $(n-k-1)$-plane of P^n that does not intersect Δ. $0 \leq k \leq n-1$. We say that Γ is a polar k-plane of a strictly convex body K if for every line L that meets Γ, Δ and int$K$, we have that $[A, B; P, Q] = -1$, where $L \cap \partial K = A, B$, $L \cap \Gamma = P$ and $L \cap \Delta = Q$. If Γ is a polar k-plane of a convex body K with dual polar the $(n-k-1)$-plane Δ and $\Gamma \cap K = \emptyset$, then $\Delta \cap \partial K$ is shadow boundary of K with respect Γ.

Theorem: *Let $K \subset E^{n+1}$ be a strictly convex body and let $H \subset P^{n+1}$ be a non-supporting hyperplane. Let $0 \leq k \leq n-1$. If every k-plane $\Delta \subset H - K$ is a polar k-plane of K, then K is an ellipsoid.*

If $k = 0$ and $H \cap K = \emptyset$, then our theorem restates the classic characterization of ellipsoids due to Blaschke and Brunn, regarding the middle points of parallel chords. If $k = 1$ and $H \cap K = \emptyset$, then our theorem restates the classic characterization of ellipsoids regarding the planarity of shadow boundaries. The next theorem is also interesting.

Shaked Rogers Theorem: *Let K_1 and K_2 be two convex bodies. If we can choose continuously, for every direction a section of K_1 which is a translation of a section of K_2, then K_1 is a translation of K_2.*

On Minkowski decompositions of polytopes

Gajane Panina

Virtual polytope group was introduced originally by Khovanskii, Pukhlikov (see also McMullen and Morelli). Virtual polytopes in the real space $\mathbb{R}^n$ form a group $\mathcal{P}^*$ with respect to the Minkowski summation.

A virtual polytope $K \in \mathcal{P}^*$ is called a k *-cylinder* ($k = 1, \ldots, n+1$), if it is representable as the Minkowski sum of $n-k+1$ -dimensional polytopes: $K = \otimes_i K_i$, $dim K_i \leq n-k+1$. Consider the following problem: *Given a polytope K, to find whether K belongs to Cyl_k, i.e., whether K is decomposable into the Minkowski sum of k -dimensional polytopes.* Its solution is the following: We construct (explicitly) a collection of mutually orthogonal projectors, group homomorphisms $\delta_k : \mathcal{P}^* \to Cyl_k$, whose sum is the identity operator.

Theorem: *A polytope K belongs to Cyl_k iff $\delta_i K = E$ for all $i = 1, ..., k-1$, where E is the unite element of $\mathcal{P}^*$.*

These projectors induce the following direct sum decompositions. $\mathcal{P}^* = \delta_1 \mathcal{P}^* \oplus \delta_2 \mathcal{P}^* \oplus \cdots \oplus \delta_n \mathcal{P}^*$ and $Cyl_k = \delta_k \mathcal{P}^* \oplus \cdots \oplus \delta_n \mathcal{P}^*$.

'Central' points of finite sets

Pier Luigi Papini

Let X be a normed space of finite or infinite dimension. Given a finite set A, or the convex hull of such a set, the research of some particular points is interesting, also with respect to applications: for example, the (Chebyshev) center; the Fermat-Weber point; the minimizing point of some convex function of the points.

Apart from the case of two-dimensional Minkowski spaces and from Hilbert spaces, also for rather "nice" norms (when dim(X)3) we can have the following "pathological" situations: the solution(s) of the above problems do(es) not belong to the convex hull of A, and/or give points far from each other.

Uniqueness theorems for convex bodies in non-Euclidean spaces

Carla Peri

(joint work with Paolo Dulio)

We present a unified approach to X-rays of order i ($i \in \mathbb{R}$) of measurable sets in spaces of constant curvature and generalize uniqueness results for convex bodies obtained by Falconer, Gardner and Volčič in the Euclidean space.
As a consequence we characterize centrally symmetric convex bodies by means of their section functions, by extending to arbitrary dimension a result obtained by G. Fejes Tóth and Kemnitz in the plane.
Some of these results extend locally to Riemannian 2-manifolds.

Stochastical approximation of smooth convex bodies

Matthias Reitzner

Choose n points from the interior of a given convex body K in $\mathbb{R}^d$, randomly, independently, and according to the uniform distribution. The convex hull of these random points is a random polytope. We are interested in the expected values $\mathbb{E}_n(V_i)$ of the intrinsic volumes of the random polytope.

We give an asymptotic series expansion for $\mathbb{E}_n(V_i)$ as $n \to \infty$ if the convex body K is sufficiently smooth, i.e., with boundary of differentiability class $\mathcal{C}^k$ and with positive Gaussian curvature for all boundary points.

$$\mathbb{E}_n(V_i) = V_i(K) + c_2^{(i,d)}(K)\, n^{-\frac{2}{d+1}} + c_3^{(i,d)}(K)\, n^{-\frac{3}{d+1}} + \cdots + O(n^{-\frac{k-2}{d+1}})$$

as $n \to \infty$. The coefficient $c_2^{(i,d)}(K)$ can be given explicitly.

The approximation of a convex body by random polytopes is improved if the vertices of the random polytope are on the boundary of the convex body. Thus we are also interested in random polytopes with vertices chosen according to a given density function on the boundary of the convex body. The expected values of the i–th intrinsic volumes of the random polytope are investigated, and we give an asymptotic series expansion for $\mathbb{E}_n(V_i)$ as $n \to \infty$ if the convex body is sufficiently smooth.

One some recent results from geometric topology related to convex geometry

Dušan Repovš

The Banach–Mazur (-Minkowski) compactum $Q(n)$ is usually defined as the space of all isometry classes of n–dimensional Banach spaces, equipped with the metric $\log\rho$, where $\rho(X,Y) = \inf\{\|T\|\cdot\|T^{-1}\| : T : X \to Y \text{ an isomorphism}\}$ is the classical Banach–Mazur distance. Equivalently, one can define $Q(n)$ as the orbit space $C(n)/GL(n)$ of the $GL(n)$–action on the space $C(n)$ of all compact convex, centrally symmetric bodies in R^n, equipped with the Hausdorff metric ρ_H, the homeomorphism between $C(n)/GL(n)$ and $Q(n)$ being induced by the Minkowski functional $p_V(x) = \inf\{\frac{1}{t} \mid tx \in V\}$. Furthermore, by associating to every convex body $V \in C(n)$ its minimal (Löwner) ellipsoid E_V, one gets a continuous correspondence $\mathcal{L} : C(n) \to \mathcal{E}$ to the space of all ellipsoids, which preserves the $GL(n)$–action. Clearly, $Q(n)$ is homeomorphic to $\mathcal{L}^{-1}(B^n)/O(n)$. Geometric topologists have for a long time been seeking a simple characterization of $Q(n)$. The first success came when few years ago Ageev–Bogatyi–Fabel and independently Antonyan, showed that $Q(n)$ is an AR. Hopes were raised that perhaps even more is true – that $Q(n)$ might be homeomorphic to the Hilbert cube I^∞. Then Ageev–Bogatyi surprisingly proved that at least for $n = 2$ this is not the case. Recently, the following result has been obtained by Ageev-Repovš (*On Banach-Mazur compacta*, J. Austral. Math. Soc. A 69 (2000), 316–335).

Theorem: $Q(2)\setminus\{\text{Eucl.}\}$ *is a Hilbert cube manifold, where Eucl. is the Euclidean point, i.e.* $[\mathcal{L}^{-1}(B^n)]$.

In our attempts to generalize this theorem for all n, we have encountered a difficulty, summarized as follows:

Conjecture: *Let D_i be an H_i–orbit, where H_i is proper subgroup of $O(n)$ and let $\sum_{i=1}^m \lambda_i = 1,\ \lambda_i \geq 0$. Then the body* $\text{Conv}(\sum_{i=1}^m \lambda_i D_i)$ *"essentially" differs from the ball, in the sense that its boundary does not contain any open subset of the sphere.*

For explanation and details please, see pp. 332–333 of our paper (op. cit.). If Conjecture 1 is confirmed, then our proof of the theorem above will immediately yield the following:

Conjecture: *For every $n \geq 3$, $Q(n)\setminus\{\text{Eucl.}\}$ is a Hilbert cube manifold.*

This would then be the ultimate of what one could expect since the following is believed to be true:

Conjecture: *$Q(n)$ is not homeomorphic to the Hilbert cube for any $n \geq 3$.*

Perhaps all that is missing is an input from convex geometry and this quest might be over.

Distances between non-symmetric convex bodies.

Mark Rudelson

We discuss the estimate of the maximal Banach–Mazur distance between two convex bodies in $\mathbb{R}^n$. In the symmetric case John's theorem implies that the distance between the two bodies is bounded by n. Gluskin proved that this estimate is essentially sharp. Without the symmetry assumption, the bound following from John's theorem is n^2, while no examples of bodies, where the distance would be more than cn are known. Using random rotations we reduce the distance estimate to the MM^*-problem of Milman: let K be a convex body in $\mathbb{R}^n$. What is the upper bound for

$$\min w(TK)w((TK)^\circ),$$

where the minimum is taken over all affine images of K. Here $w(K)$ is the mean width.

We show that the MM^*-estimate can be further reduced to evaluating the volume of a section of the difference body $K-K$ by a subspace E via the maximal volume of sections of K parallel to E. Using such estimate, previously obtained by the author, we show that the distance between two n-dimensional convex bodies does not exceed $n^{4/3}$ up to a logarithmic factor.

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The Solution of Polyhedra

Idjad Kh. Sabitov

Let P be a polyhedron in $I\!R^3$ with triangle faces and let F_i, F_j be two faces with a common edge. The diagonal of P joining the vertices of F_i and F_j which are not incident to their common edge is called *small diagonal.*

Theorem: *Every small diagonal d of any polyhedron P is a root of a polynomial equation of view*

$$A_0(l,V)d^{2M} + A_1(l,V)d^{2M-2} + \cdots + A_M(l,V) = 0,$$

where coefficients $A_i(l,V)$ are, in their turn, polynomials in the set l of squares of lengths of P's edges and $V = vol(P)$, with some rational numerical coefficients depending, as well as the degree $2M$, on the combinatorial structure of P and the choice of faces defining the searched small diagonal.

By preceding results of author the volume of P also can be found as a root of some polynomial equation, so in general we have a finite number of possible values of any small diagonal and this permits to propose an algorithmical solution to the problem of isometrical realisation in $I\!R^3$ of a given metrical simplicial complex.

Crofton formulas and zonoids

Rolf Schneider

In the course of his investigations on *"Geometries in which the planes minimize area"*, Busemann suggested to study the question whether a given k-dimensional area, vol_k, in affine space A_n satisfies a Crofton formula, that is, whether there exists a (positive) measure μ_{n-k} on the space $A(n, n-k)$ of $(n-k)$-flats in A_n so that

$$\mathrm{vol}_k(M) = \int_{A(n,n-k)} \#(F \cap M)\, d\mu_{n-k}(F)$$

holds for every smooth k-dimensional submanifold M. Such formulas and more general ones have been established for the *Holmes-Thompson area* in hypermetric Minkowski spaces (*Adv. Math.* 129 (1997), 222 - 260). More recently, I investigated Minkowski spaces in which the *Busemann area* does not satisfy a Crofton formula (*Beitr. Algebra Geom.* **42** (2001), 263 – 273). I also extended the Crofton formula for the Holmes-Thompson area to hypermetric (not necessarily smooth) Finsler spaces (*Arch. Math.* **77** (2001), 85 – 97). Through the local norms, such investigations depend on the study of (generalized) projection bodies and intersection bodies.

Geometric probabilities for convex bodies of revolution in the Euclidean space E_3

Marius I. Stoka

(joint work with Andrei Duma)

Let K be an arbitrary convex body of revolution with centroid S and oriented axis of rotation d. Clearly, the axis d is determined by the angle ϑ between d and the z-axis and by the angle φ between the projection of d on the xy-plane and the x-axis and we express this by writing $d = d(\vartheta, \varphi)$. If for a given $d = d(\vartheta, \varphi)$, the body K is tangent to the xy-plane in such a way that the centroid S lies in the upper halfspace, we dentoe by $p(\vartheta, \varphi)$ the distance from S to the xy-plane. Then the length of the projection of K on the z-axis is given by $L(\vartheta, \varphi) = p(\vartheta, \varphi) + p(\pi - \vartheta, \varphi)$. Note that $p(\vartheta, \varphi)$ does actually depend only on the angle ϑ and moreover, since K is a body of revolution about the axis d the value $p(\vartheta, \varphi)$ is invariant to any rotation about this axis, say by any ϕ. Now let $\mathcal{F}$ be a fundamental cell of the lattice $\mathcal{R}$ and assume that the two 3-dimensional random variables defined by the coordinates of S and by the triple $(\vartheta, \varphi, \phi)$ are uniformly distributed in the cell $\mathcal{F}$ and in $[0, \pi] \times [0, 2\pi] \times [0, 2\pi]$ respectively. We investigate the probability $p_{K,\mathcal{R}}$ that the body K intersects the lattice $\mathcal{R}$.

Intrinsic Volumes and Gaussian Processes

Richard A. Vitale

As renormalized versions of the classic quermassintegrals, intrinsic volumes play an important role in the theory of convex bodies, notably in the Steiner volume formula and in the celebrated characterization theorem of Hadwiger. More recently, they have been seen to have a remarkable connection with Gaussian processes through the work of Sudakov, Chevet, and Tsirelson, among others. This has led to novel insights in both areas. The talk sketched some recent results in this vein, including (i) extension of intrinsic volumes to infinite dimensional convex bodies, (ii) bounds and estimates for Gaussian processes, (iii) Ito-Nisio oscillation and Gaussian black holes, (iv) the Wills functional, and (v) the Brownian motion body.

Determination of convex bodies and reconstruction of polyoptes by certain section functions

Aljoša Volčič

(joint work with Alexander M. Lindner)

For any convex body K in $\mathbb{R}^d$ containing the unit sphere S^{d-1} in its interior, and for $1 \leq i \leq d-1$, the *spherical* i-section function $l_i(K)$ is defined as the function associating to any i-dimensional affine subspace H tangent to S^{d-1} the i-dimensional volume of $H \cap K$.

In a recent paper Barker and Larman asked if whether $l_i(K)$ determines K uniquely, among all convex bodies, giving a substantial positive answer:

Theorem: *Under the conditions described above, K is uniquely determined if* $1 \leq i \leq d-2$.

The following question remains open, apart from a partial result in the planar case:

Problem: *Is this true for $i = d-1$?*

We have shown that the class of plane convex bodies which are uniquely determined by the function l_1 is of second Baire category and that it contains the triangles. It contains the circles, too, by a result of Barker and Larman. This does not mean however that *most* convex bodies are uniquely determined by l_1.

Our main result is that for any $d \geq 2$, l_{d-1} determines any convex polytope *among convex polytopes.*

The proof for $d \geq 3$ is different from the proof for the case $d = 2$ (and surprisingly more straightforward). Both proofs are, in principle, reconstructive.

The Cramer–Rao inequality for star bodies

Deane Yang and Gaoyong Zhang

(joint work with Erwin Lutwak)

Associated with each body K in Euclidean n-space $\mathbb{R}^n$ is an ellipsoid $\Gamma_2 K$ called the Legendre ellipsoid of K. It can be defined as the unique ellipsoid centered at the body's center of mass such that the ellipsoid's moment of inertia about any axis passing through the center of mass is the same as that of the body. In an earlier paper the authors showed that associated with each convex body $K \subset \mathbb{R}^n$ is a new ellipsoid $\Gamma_{-2} K$ that is in some sense dual to the Legendre ellipsoid. The Legendre ellipsoid is an object of the dual Brunn–Minkowski theory, while the new ellipsoid $\Gamma_{-2} K$ is the corresponding object of the Brunn–Minkowski theory. The present paper has two aims. The first is to show that the domain of Γ_{-2} can be extended to star-shaped sets. The second is to prove that the following relationship exists between the two ellipsoids: If K is a star shaped set, then

$$\Gamma_{-2} K \subset \Gamma_2 K,$$

with equality if and only if K is an ellipsoid centered at the origin. This inclusion is the geometric analogue of one of the basic inequalities of information theory – the Cramer-Rao inequality.

Edited by Paul Goodey, Peter M. Gruber & Matthias Reitzner

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