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Diffenrentialgeometrie im Großen

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The conference was organized by Werner Ballmann (Bonn), Jean-Pierre Bourguignon (Bures-sur-Ivette) and Wolfgang Ziller (Philadelphia). About 50 participants from all over the world took part in it.

There were 22 talks given on new developments in (global) differential geometry, which covered a big scope of current research. One main topic was the study of manifolds of positive and nonnegative curvature. Recently new obstructions to manifolds of positive sectional curvature with a certain amount of symmetries were found, and it was possible to show that each cohomogeneity one manifold admits a metric of almost nonnegative sectional curvature. Other talks focussed on the question which vector bundles admit metrics of positive or nonnegative curvature. R. Bryant gave an introduction to calibrations and opened the dicussion of Kähler manifolds, which constituted another focus of the conference. Other speakers presented results on the existence (or non-existence) of Einstein metrics, discussed rigidity phenomena, bounds for eigenvalues of the Dirac operator, hyperbolic metrics and hyperbolization, applications of comparison geometry and geometric aspects of random walks.

Beside the talks, which were attended by nearly all participants, the discussions, bringing together mathematicians from different areas of research were very important.

The pleasent atmosphere of the Institute, the friendlyness and helpfulness of the staff also contributed to the success of the conference.

Abstracts

Ricci curvature, Einstein metrics and the Seiberg-Witten equations

Claude LeBrun

Riemannian geometry is constrained by smooth topology in dimension 4 in a manner quite unlike the situation occurring in high dimensions. For example consider the invariant $I_S(M) = \inf_g \int_M |S_g|^{n/2} d\mu_g$ of a smooth, compact n-manifold M. Here the infimum is taken over all Riemannian metrics g, and S_g denotes the scalar curvature of g. Then there are many 4-manifolds for which $I_S \neq 0$, whereas I_S is zero for any simply connected manifold of dimension ≥ 5. In this talk I attempted to describe some results concerning I_S and the related invariant $I_r(M) = \inf_g \int |r|^2 dr$ for smooth compact 4-manifolds, here r denotes the Ricci curvature.

Theorem. *Let M be the underlying 4-manifold of a complex surface of Kodaira dimension $\neq -\infty$. Let X be the minimal model of M, and let l denote the number of points at which X must be blown up in order to obtain M; thus $M \cong_{diffeo} X \# l\overline{\mathbb{C}P}_2$ (l maximal). Then*

$$I_S(M) = 32\pi^2 c_1(X)$$

$$I_r(M) = 8\pi^2[c_1(X) + l].$$

The same method also implies non-existence results for Einstein metrics

Theorem. *Let X be a symplectic 4-manifold with $b^+(X) > 1$. Then*

$$M = X \# l\overline{\mathbb{C}P}_2$$

does not admit Einstein metrics if $l \geq \frac{1}{3}c^2{}_1(X)$.

By example, these results concern the diffeotype, rather then the homeotype, of M.

Group actions on manifolds of positive sectional curvature

Burkhard Wilking

There are only few known examples of manifolds of positive sectional curvature. In dimensions above 24 the known simply connected examples are diffeomorphic to rank one symmetric spaces. On the other hand there are also only very few obstructions known for these manifolds.

Grove '91 proposed to study isometric group actions on manifolds with positive sectional curvature in order to find obstructions and perhaps new examples.

The aim of the talk is to give some new obstructions for isometric group actions on manifolds of positive sectional curvature. One of the main new tool in the study of this problem is the following

Theorem. *Let (M^n, g) be a compact Riemannian manifold of positive sectional curvature. Suppose there is a totally geodesic embedded submanifold $N^{n-k} \subset M^n$ of codimension $k < \frac{n}{3}$. Then there is an element $e \in H^k(M, \mathbb{Z})$ such that*

$$\cup e: \quad H^i(M, \mathbb{Z}) \to H^{i+k}(M, \mathbb{Z})$$

is surjective for $k - 1 \leq i < n + 1 - 2k$ and injective for $k - 1 < i \leq n + 1 - 2k$.

Our main results are theorems on manifolds with $sec > 0$ and with a large amount of symmetry. In order to measure the amount of symmetry of a Riemannian manifold we consider the following three constants

$$\begin{array}{rcl} symrank(M.g) & := & rank(Iso(M,g)) \\ symdeg(M,g) & := & dim(Iso(M,g)) \\ cohom(M,g) & := & dim(M/Iso(M,g)), \end{array}$$

where $Iso(M,g)$ is the isometry group.

Under the general assumption that (M^n, g) is a simply connected manifold of positive sectional curvature we prove

- if $symrank(M^n, g) \geq \begin{cases} \frac{n}{4}+2 & \text{for n} \equiv 1 \bmod 4 \text{ or n} = 24, 25, 26 \\ \frac{n}{4}+1 & \text{otherwise} \end{cases}$ and $n \geq 10$, then M^n is homotopically equivalent to $\mathbb{S}^n$, $\mathbb{C}P^{n/2}$ or $\mathbb{H}P^{n/4}$.
- if $symdeg(M^n, g) \geq 2n-5$, then M^n is homotopically equivalent to $\mathbb{S}^n$, $\mathbb{C}P^{n/2}$ or $\mathbb{H}P^{n/4}$ or CaP^2 or (M^n, g) is isometric to a homogeneous space.
- if $cohom(M^n, g) = k$ and $n \geq 900(k+1)^2$, then M is homotopically equivalent to $\mathbb{S}^n$, $\mathbb{C}P^{n/2}$ or $\mathbb{H}P^{n/4}$.

Algebraic geometry and Einstein manifolds in dimension five

Krzysztof Galicki

On simply connected five-manifolds Sasakian-Einstein metrics coincide with Riemannian metrics admitting real Killing spinors which are of great interest as models of near horizon geometry for threebrane solutions in superstring theory. (M^5, g) is Sasakian-Einstein if $(\mathbb{R}_+ \times M, dt^2 + t^2 g)$ is Calabi-Yau (Kähler, Ricci-flat). We expand on the recent work of Demailly and Kollár and Johnson and Kollár who give methods for constructing Kähler-Einstein metric on log del Pezzo surfaces. It is known that the circle V-bundle (orbifold bundle) over a log del Pezzo surface with Kähler-Einstein metric has a Sasakian-Einstein metric on the total space of the bundle. Here, these 1-connected 5-manifolds arise as links of isolated hypersurface singularities which by the well known work of Smale must be diffeomorphic to $\mathbb{S}^5 \# l(\mathbb{S}^2 \times \mathbb{S}^3)$. More precisely, using methods of algebraic geometry we show the existence of 14 inequivalent Sasakian-Einstein structures on $\mathbb{S}^2 \times \mathbb{S}^3$ and infinite families of such structures on $\mathbb{S}^5 \# l(\mathbb{S}^2 \times \mathbb{S}^3)$ with $2 \leq l \leq 9$. In the case of $l = 2$, $l = 9$ these are new examples of differentiable manifolds which are now known to admit Einstein metrics of positive scalar curvature. In other case we show the existence of non-regular Sasakian-Einstein structures on manifolds which were known to admit regular structures of this type.

Almost non–negative curvature on cohomogeneity one manifolds

Lorenz Schwachhöfer

(joint work with Wilderich Tuschmann)

Let M be a closed manifold on which a (compact) Lie group G acts with cohomogeneity one. According to a conjecture of Grove and Ziller, any such manifold supports a Riemannian metric of non-negative sectional curvature. This is elementary if the orbit space is a circle, so one may restrict to the only other possibility, namely that the orbit space $M/G = [a, b]$. This means, in particular, that there are exactly two singular orbits M_+ and M_-. According to Grove and Ziller, the conjecture holds if $M_\pm$ both have codimension two, and moreover, any cohomogeneity-one manifold carries a metric of non-negative Ricci curvature (even positive Ricci curvature if the fundamental group ist finite). Our main result is the following

Theorem. *Every cohomogeneity-one manifold is almost non–negatively curved, meaning that for any $\varepsilon > 0$ there exists a Riemannian metric g_ε on M whose diameter is 1, and whose sectional curvature is bounded below by $-\varepsilon$.*

The proof of this result is constructive, i.e. the metrics g_ε are given explicitely. Similar to the constructions of Grove and Ziller, we "glue" the metric along a principal orbit which is hence totally geodesic.

Cohomogeneity one manifolds with positive curvature

Luigi Verdiani

Homogeneous compact simply connected Riemannian manifolds are classified (Berger, Wallach, Berard-Bergery). Looking for new examples it seems natural to consider the class of cohomogeneity one manifolds. It has been recently proved that two of the three classes of inhomogeneous known examples admit in some cases a cohomogeneity one action (Grove, Ziller). More precisely those in dimension 7 and 13. Our main result is that new examples may arise just in the odd dimensional case. More precisely:

Theorem. *Let M be a compact cohomogeneity one manifold with $\pi_1(M) = 0$. If dim(M) is even and the sectional curvature of M is positive then M is equivariantly diffeomorphic to a rank one symmetric space.*

Extremal cycles in Hermitian symmetric spaces

Robert Bryant

This talk was in two parts: A brief survey of calibration theory and then a summary of some new results obtained by an extension of this idea.

Let (M, g) be a Riemannian manifold. A calibration φ on M is a closed p-form with the property that its restriction to any oriented p-plane E is at most the (Riemannian) volume element on $E \subset T_xM$. A p-dimensional, oriented submanifold $N \subset M$ is said to be *calibrated* by a calibration φ if φ pulls back to N to be the induced volume form on N The Fundamental Lemma of Calibrated Geometry is that a closed p-dimensional submanifold that is calibrated by a calibration φ is volume-minimizing in its homology class. This criterion provides the most common method of proving that a (possibly singular) submanifold is homologically volume minimizing.

The basic examples of calibrations and calibrated geometries are surveyed, following the fundamental work of Harvey and Lawson on the subject. The Kähler calibration (due to the Wirtinger inequality) and its role in complex analysis, the special Lagrangian calibrations and their role in special Lagrangian geometry, particularly in mirror symmetry on Calabi-Yau manifolds, and other examples related to invariant forms on homogeneous spaces are discussed. In the latter case, I discuss both existence and rigidity of calibrated cycles in Lie groups and real Grassmannians. I also announce the result that any real-analytic arc in a Calabi-Yau 3-fold is the singular locus of a (real-analytic) special Lagrangian 3-fold.

In the case of Hermitian symmetric spaces, the theory of calibrations is particularly rich. When M is an Hermitian symmetric space of compact type, the homology groups are freely generated by a special family of subvarieties called the (generalized) Schubert cycles. In addition to being calibrated by the Kähler form, each such Schubert cycle σ of complex dimension p is dual to an invariant (p,p)-form φ_σ that has two important properties: The first is that φ_σ is positive as a (p,p)-form (i.e., it restricts to every p-dimensional complex subvariety to be a non-negative multiple of the volume form). The second is that φ_σ vanishes identically on each of the p-dimensional Schubert cycles $\sigma' \neq \sigma$. In particular, any p-dimensional complex subvariety $V \subset M$ that is homologous to σ must satisfy the first-order condition that $\varphi_{\sigma'}$ vanishes identically on V for all p-dimensional Schubert cycles $\sigma' \neq \sigma$. I show that these first order conditions are actually a set of holomorphic differential equations on the subvariety V and that these local conditions can often be integrated to yield striking rigidity theorems for subvarieties that are homologous to a multiple of a Schubert cycle.

These rigidity results have a number of applications: First, they prove that many subvarieties in Grassmannians and other Hermitian symmetric spaces cannot be smoothed (i.e., are not homologous to a smooth subvariety). Second, they provide characterizations of holomorphic bundles over compact Kähler manifolds that are generated by their global sections but that have certain polynomials in their Chern classes vanish (for example, $c_2 = 0$, $c_1c_2 - c_3 = 0$, $c_3 = 0$, etc.). Third, they provide a complete description of the moduli space of algebraic cycles in certain homology classes in Grassmannians or other Hermitian symmetric spaces.

Kähler metrics on toric orbifolds

Miguel Abreu

In the first part of this talk I describe a symplectic approach to Kähler geometry on toric orbifolds, based on the existence of action-angle coordinates. The main result is an effective parametrization of all toric Kähler metrics via smooth functions on the moment-polytope.

The second part of the talk is devoted to two applications:

- simple explicit description of recent work of Robert Bryant that produces extremal Kähler metrics on any weighted projective space;
- description of $U(n)$–invariant extremal Kähler metrics, which in this symplectic approach arise naturally as solutions to a linear second order ODE.

Contact and quaternionic geometry

Jesus Gonzalo Perez

(joint work with Hansjörg Geiges)

We study the notions of *contact sphere* and *contact circle.*

Recall that a 1–form α on a 3–manifold M is a *contact form* if $\alpha \wedge d\alpha$ is a volume form on M.

A contact sphere on a 3–manifold is intrinsically defined as a triple of 1–forms $\omega_1, \omega_2, \omega_3$ such that their linear combinations with constant coefficients are all contact forms (except of course the zero combination). This notion is extrinsically motivated by a multiple pseudoconvexity: quaternion space $\mathbb{H}$ has an $\mathbb{S}^2$–worth of complex structures, and a real hypersurface $M^3 \hookrightarrow \mathbb{H}$ is pseudoconvex for all those complex structures if and only if $\langle i\nu, \cdot\rangle, \langle j\nu, \cdot\rangle, \langle k\nu, \cdot\rangle$ is a contact sphere on M, where ν is the unit normal of M^3.

A contact circle is a pair of 1–forms ω_1, ω_2 satisfying the analogous condition, and is thus also an intrinsic notion. We consider the study of these as a necessary step toward the study of contact spheres. Also, contact circles are more abundant than contact spheres and in this respect they provide a richer theory.

The first example of contact circle is the tautological pair of 1–forms on the orthonormal frame bundle $F\Sigma$ of a surface Σ with a Riemann metric. This motivates the notion of a *taut* contact circle, which is a contact circle ω_1, ω_2 such that as $\lambda = (\lambda_1, \lambda_2)$ ranges over S^1 the contact form $\lambda_1\omega_1 + \lambda_2\omega_2$ defines a volume form independent of λ. We define taut contact spheres similarly.

A contact circle on M^3 induces an almost complex structure J on $M^3 \times \mathbb{R}$, and when M^3 is closed the integrability condition for J amounts to the contact circle being taut. This makes the notion of a taut contact circle a very natural one.

Theorem. *A closed, connected 3–manifold admits a taut contact circle if and only if it is a left–quotient of one of the following Lie groups:* $\mathrm{SU}(2)$*,* $\widetilde{\mathrm{SL}}_2$*, or* $\widetilde{\mathrm{E}}_2$*.*

These Lie groups are the universal coverings of the (orientation–preserving) isometry groups of the 2–dimensional space forms.

Those left–quotients are all Seifert manifolds, and a characterization in terms of the Seifert invariants is known for them (due to Raymond and Vasquez). We have clarified such characterization and made it fully synthetic.

The ingredients that go into the proof of this Theorem are the Enriques–Kodaira classification of complex surfaces, Wall's study of geometric complex surfaces, and a famous theorem of Bogomolov (stating that a class VII surface with $c_2 = 0$ and $b_2 = 0$ is a Hopf or Inoue surface) for which the first completely correct proof was obtained by A.Teleman using Seiberg–Witten theory.
Recently, we have constructed the moduli spaces of taut contact circles for all those manifolds.

Theorem. *Non–taut contact circles exist on all compact orientable 3–manifolds. Moreover, each orientation is realized.*

Theorem. *Every left–quotient of* $\mathrm{SU}(2)$ *admits taut contact spheres. The taut contact circles on the left–quotients of* $\widetilde{\mathrm{SL}}_2$ *and* $\widetilde{\mathrm{E}}_2$ *do not extend to contact spheres, not even to non–taut contact spheres.*

Obstructions to non-negative curvature

Igor Belegradek

(joint work with Vitali Kapovitch)

According to the Soul theorem of Cheeger and Gromoll any open nonnegatively curved manifold is a vector bundle over a compact manifold with $sec \geq 0$. It is a natural question which bundles admit $sec \geq 0$. In a joint work with Vitali Kapovitch we show that this question can be often reduced to the case when the base is simply-connected. In particular, a majority of $\mathbb{R}^2$-bundles over compact manifolds (with $sec \geq 0$ and infinite fundamental group) admit no metric with $sec \geq 0$. Some similar but weaker results are proved for higher rank bundles. For example, if the base is the product of a torus and a closed simply connected manifold whose rational cohomology algebra has no derivations of negative degree, then a majority of $\mathbb{R}^k$-bundles over this base do not carry $sec \geq 0$.

The condition about derivations is true for any compact homogeneous space, any 1-connected Kähler manifold, and is closed under taking fiber bundles.

Conditions for non-negative curvature on bundles

Kristopher Tapp

(joint work with Detlef Gromoll)

I propose a strategy for studying two related questions:

- which vector bundles admit nonnegative sectional curvature?
- which sphere bundles admit positive sectional curvature?

These questions partially translate into questions about whether the vector bundle admits a connection and a tensor satisfying a certain differential inequality. For nonnegatively curved metrics on $\mathbb{S}^2 \times \mathbb{R}^2$, the inequality forces rigidity of the metric at the soul, which motivates the following classification:

Theorem. *Any metric of nonnegative curvature on* $\mathbb{S}^2 \times \mathbb{R}^2$ *is isometric to a Riemannian quotient of the form* $((\mathbb{S}^2, g_0) \times (\mathbb{R}^2, g_1) \times \mathbb{R})/\mathbb{R}$

Boundary and conjugacy rigidity

Christopher Croke

In this talk we survey the known results concerning the boundary rigidity problem: For a compact manifold M with smooth boundary ∂M and Riemannian metric g we let

$$d_g : \partial M \times \partial M \to \mathbb{R}$$

be the boundary distance function, i.e. for $p, q \in \partial M$ $d_g(p,q)$ is the g distance between p and q. The question is to determine to what extent d_g determines g. After surveying work we show how the problem is related to the conjugacy rigidity problem: compact manifolds M and N have C^k-conjugate geodesic flows if there is a C^k diffeomorphism $F : UM \to UN$ between the unit tangent bundles so that ${g_N}^t \circ F = F \circ {g_M}^t$ (where g^t is the geodesic flow). We then survey the known results in this area. We include the new result that if two 3-dimensional graph manifolds have C^0- conjugate geodesic flows then they must be isometric.

Asymptotic geometry of negatively curved manifolds and quasisymmetric parametrization

Bruce KLeiner

(joint work with Mario Bonk)

The boundary $\partial_\infty G$ of any non-elementary Gromov hyperbolic group G carries a natural family of metrics which are Ahlfors regular and "approximately self-similar" . These metrics are pairwise quasisymmetrically homeomorphic by the identity map $id : \partial_\infty G \to \partial_\infty G$. If G_1 ans G_2 are two non-elementary hyperbolic groups, then a homeomorphism $\partial_\infty G_1 \to \partial_\infty G_2$ is quasi-symmetric iff it is induced by a quasi-isometry $G_1 \to G_2$. Hence questions about the existence of quasi-isometries between hyperbolic groups are equivalent to questions about the existence of quasisymmetric homeomorphisms. In the latter part of the lecture, several theorems about quasisymmetric homeomorphisms were presented.

Theorem. *If Z is an Ahlfohrs 2-regular metric 2-sphere, then Z is quasisymmetric to the standard 2-sphere $\mathbb{S}^2$ iff Z is linearly locally contractible.*

Theorem. *If $Q \geq 2$, and Z is an Ahlfors Q-regular metric 2-sphere admitting a (1,Q)-Poincare inequality (in the sense of Heinonen-Koskela), then $Q = 2$ ans Z is quasisymmetric to $\mathbb{S}^2$.*

Using similar techniques one can prove that many fractal 2-spheres are quasisymmetric to $\mathbb{S}^2$.

Fuchsian affine actions of surface groups

Françoise Labourie

The Auslander conjecture asserts that if Γ acts properly on the affine space $\mathbb{A}^n$ (for Γ a discrete group of affine transformations) in such a way that $\mathbb{A}^n/\Gamma$ is compact, then Γ does not contain a free group with 2 generators.

Milnor asked the question if free groups (with 2 generators) can act properly on $\mathbb{A}^n$. Margulis proved that the answer to this question is positive: he constructed free groups acting properly in $\mathbb{A}^3$. On the other hand, Mess in 1990 proved that a surface group ($= \pi_1$ of a surface) cannot act properly on $\mathbb{A}^3$. In 1999, Goldman and Margulis gave another proof of this result. Inspired by their proof, I prove the following theorem.

Theorem. *If Γ is a subgroup of $Aff(\mathbb{A}^n)$ such that the linear part of Γ is a cocompact group in an irreducible $Sl(2,\mathbb{R})$ in $Sl(n,\mathbb{R})$ then Γ do not act properly on $\mathbb{A}^3$*

Obstructions to positive curvature with symmetry

Anand Dessai

We consider the following two questions:

- What can one say about the topology of a simply connected closed manifold with positive sectional curvature ($sec > 0$)?
- How to distinguish $sec > 0$ from weaker curvature properties? (in this talk: positive Ricci curvature, $Ric > 0$)

a) Gromovs Betti–number theorem and the examples of Sha–Yang with $Ric > 0$ give nice answers to both questions. What is if Gromovs Betti–number theorem does not apply (if Betti numbers are "small")?
b) Lichnerowicz showed that $\hat{A}(M)$, the index of the Dirac operator, vanishes if M is spin with $scal > 0$. Are there other chracteristic numbers which vanish if $sec > 0$?

We consider these questions under mild symmetry assumptions.

Theorem. *Let M be a closed spin manifold of dimension $> 12r - 4$. If M admits a metric with sec > 0 and symrank $\geq 2r$ then the first $r+1$ coefficients of the elliptic genus $\varphi_{ell}(M)$ vanish.*

$\varphi_{ell}(M) = \hat{A}(M) - \hat{A}(M, TM_{\mathbb{C}})q + \hat{A}(M, \Lambda^2 TM_{\mathbb{C}} + TM_{\mathbb{C}})q^2 + \ldots$ a series of indices of twisted Dirac operators.
Special case: M as above with $sec > 0$ and $symrank \geq 2$ and $dim(M) > 8$ then $\hat{A}(M) = 0$ and $\hat{A}(M, TM_{\mathbb{C}}) = 0$.

The theorem allows to distinguish $Ric > 0$ ans $sec > 0$ for manifolds with "small" Betti numbers under mild symmetry assumptions (e.g. $symrank \geq 6$).

Proof of the theorem uses the following extension of a theorem of Hirzebruch and Slodowy as well as Frankels theorem in $sec > 0$.

Theorem. *Let M be a spin manifold with $\mathbb{S}^1$- action, $\sigma \in \mathbb{S}^1$ of order ≥ 2. If $codim(M^\sigma) > 2\sigma r$ then the first $r+1$ coefficients of $\varphi_{ell}(M)$ vanish.*

Using in addition a recent theorem of Burkhard Wilking this leads to

Corollary. *For every $k \geq 2$ and $d \in N$ there exists a simply connected closed manifold M of dimension $\geq d$ such that*

- *M carries a metric with $Ric > 0$ which admits an isometry σ of order k which is contained in a compact connected subgroup of the diffeomorphism group of M.*
- *M does not have this property if one replaces $Ric > 0$ by $sec > 0$.*

Dirac eigenvalue estimates on surfaces

Bernd Amman

Most classical eigenvalue estimates for the Dirac operator use only local data, e.g. scalar curvature. We present several estimates for Dirac eigencalues depending on global data. Let λ_1^2 be the first eigenvalue of the square of the Dirac operator. We prove

$$\inf \lambda_{h+1}^2 vol^{2/n} > 0, \quad h := dim(ker D_{g_0})$$

where the infimum runs over all Riemmanian metrics in a fixed conformal class $[g_0]$ on a fixed manifold with fixed spin structure. In the special case of the sphere in dimension 2, Christian Bär derived the explicit value of the infimum, i.e. on $\mathbb{S}^2$ we have

$$\inf \lambda_1^2 area = 4\pi.$$

We specialize now to the 2-dimensional torus. To any spin structure we define an invariant of the spin structure, the Arf invariant, $Arf \in \{-1, 1\}$. If $Arf = -1$, then $\lambda_1 = 0$.
If $Arf = 1$, then we obtain two explicit lower estimates for λ_1.

On integrability of geodesic flows on Riemannian manifolds

Alexey Bolsinov

(joint work with Iskander Taimanov)

Let M be a smooth compact closed manifold. The question we discuss in our talk is whether M admits a Riemannian metric with integrable geodesic flows. One knows several topological obstructions to integrability of such flows (Koglov '78, Taimanov '84). In 1992-94 G. Paternain suggested an approach to finding new obstructions based on the topological entropy of the flow and formulated the following

Conjecture. *If M admits integrable geodesic flows then its fundamental group has polynomial growth.*

and the

Question. *Is it true that the topological entropy of an integrable geodesic flow vanishes?*

We prove the following result

Theorem. *There is a 3-dimensional real-analytic Riemannian manifold (M^3, g) such that*

- *the geodesic flow of g is integrable in C^∞-sense*

- *the topological entropy of this flow is positive*
- *the growth of $\pi_1(M^3)$ is exponential*
- *this flow is not integrable in real-analytical sense.*

Thus we show that in the smooth case the answer to the above question is negative and the conjecture is not true. However in the real-analytical case the answer to the question is still unknown (the conjecture in this situation was proved by Taimanov much earlier in 1984).

Geometric aspects of large deviations for random walks on a crystal lattice

Motoko Kotani

(joint work with Toshikazu Sunada)

A crystal lattice is an infinite graph which admits an action of a free abelian group P with a finite quotient graph X_0. We consider X as a metric space with the graphdistance d. The Gromov–Hausdorff limit $\lim_{\varepsilon\to 0}(X, \varepsilon d)$ is a vector space with a metric whose unit ball is a compact convex polyhedra. This convex polyhedra turns out to be closely related with the LDP (Large deviation principle) of a random walk on X. The polyhedra coincides with the domain on which the entropy funtion in LDP is defined.

A variational approach to homogeneous Einstein metrics

Christoph Böhm

(joint work with McKenzie Wang and Wolfgang Ziller)

Let $M = G/H$ be a compact, homogeneous space. Suppose G, H connected, $|\pi_1(M^n)| < \infty$.

Definition. *Let $\lambda \in \mathbb{R}$. We call a sequence (g_i) of G-invariant, volume 1 metrics on $M^n = G/H$ a λ-Palais-Smale-sequence, if*

- $scal(g_i) \to \lambda$
- $||Ric^0(g_i)||_{g_i} \to 0.$

Theorem. *Let $\lambda > 0$. Then every λ-Palais-Smale-sequence of G-invariant, volume 1 metrics has a convergent subsequence.*

Corollary. *The set $\mathcal{E}(G/H)$ of G-invariant, volume 1 Einstein metrics on G/H is compact.*

Theorem. *There exists a 0-Palais-Smale-sequence (g_i) iff G/H is a homogeneous torus bundle, i.e. $\exists K, H$ $H \subsetneq K \subsetneq G$ with $K/H = T^z$.*

We assign to G/H a graph $\Gamma_{G/H}$, purely defined by Lie theoretic data, related to the intermediate subgroups of G and H.

Theorem. *Let G/H be a compact homogeneous space. If the graph $\Gamma_{G/H}$ of G/H has at least two non–toral components, then G/H carries an Einstein metric of coindex ≥ 1.*

Definition. *We call a compact homogeneous space G/H, $(G, H$ connected$)$, a homogeneous space of finite type, if there exist only finitely many connected subgroups in between G and H.*

As above we assign to such a homogeneous space a simplicial complex $\Delta_{G/H}$, purely defined by Lie theoretic data.

Theorem. *Let G/H be a compact homogeneous space of finite type, $(G, H$ connected). If $\Delta_{G/H}$ is not contractible then G/H admits an Einstein metric. Furthermore, if $H_q(\Delta_{G/H}, F) \neq 0$ for a field F, then we obtain an Einstein metric of coindex $\geq q$.*

Geometrization in dimension 3

Bernhard Leeb

(joint work with Michel Boileau and Joan Porti)

We explain Thurston's conjecture that, after a canonical topological decomposition process is applied to an arbitrary, say, closed orientable 3-manifold the resulting pieces are geometric. There, a 3-manifold is called geometric if it admits a complete locally homogeneous Riemannian metric on its interior. There are 8 relevant types of model geometries, the richest of which is hyperbolic geometry. The Geometrization Program extends to orbifolds. The Orbifold Theorem states that a compact, connected and (for simplicity) oriented 3-orbifold which is irreducible and atoroidal with non-empty singular set is geometric. A complete written proof for this was obtained in 2000 by Boileau, Porti and myself. One of the consequences is the Generalized Smith Conjecture that any smooth non-free action by a finite group of orientation preserving diffeos on $\mathbb{S}^3$ is conjugate to a linear action.

Hyperbolic metrics on manifolds with boundary

Jean Marc Schlenker

A well-known theorem of Pogorelov describes the convex isometric embeddings of surfaces in $\mathbb{H}^3$. It can be read as describing the hyperbolic metrics on $\mathbb{B}^3$ with convex boundary: the induced metric on the boundary has $K > -1$, and each such metric on $\partial\mathbb{B}^3$ is induced by a unique hyperbolic metric on $\mathbb{B}^3$. A conjecture of Thurston asserts that this situation remains when one replaces $\mathbb{B}^3$ by a 3–manifold with boundary admitting a convex-cocompact hyperbolic metric. The existence part has been proved by Labourie. A parallel theory takes place when one replaces the induced metric on the boundary by its third fundamental form.

We show here that a third chose is possible, namely to consider the "horospherical metric" of the boundary, I+2II+III. The analog of the Thurston conjecture is here true, and fairly easy to check. The proof relies on a stuking duality between $\mathbb{H}^3$ and the space of its horospheres.

Lower bounds on K–energy on Kähler manifolds

Xiuxiong Chen

We pose a new type Monge-Ampere equation:
For every positive $(1,1)$ form χ, does there exist a Kähler metric $\omega_\varphi \in [\omega_0]$, such that χ is harmonic with respect to this new metric ω_φ? The Euler–Lagrange equations
(*) $g^{\alpha\overline{\beta}}\chi_{\alpha\overline{\beta}} = const$, $g_{\alpha\overline{\beta}} = g_{0\alpha\overline{\beta}} + \varphi_{\alpha\beta}$ are necessary conditions. If we normalized the constant to 1, then

(**) $(g_{\alpha\overline{\beta}}) > (\chi_{\alpha\overline{\beta}})$.

Conjecture (Donaldson). *If (**) holds, then (*) has a solution.*

If $\chi = c_1(M) > 0$, then a solution of (*) leads to the lower bound of K–energy $E(\omega_\varphi)$ through the following formula

$$E(\omega_\varphi) = \int \ln(\frac{\omega_\varphi^n}{\omega^n})\omega_\varphi^n + \mathcal{J}_\chi(\omega_\varphi) + \underline{R}I(\omega_\varphi)$$

$\mathcal{J}_\chi$ is convex of $\chi > 0$ thus has a lower bound if $\mathcal{J}_\chi$ has a critical point. Note: the critical points of $\mathcal{J}_\chi$ satisfy (*).

Spines and topology of thin Riemannian manifolds

Stephanie Alexander

(joint work with Richard Bishop)

We show, that if a complete Riemannian manifold M with nonempty boundary has sufficiently small inradius relative to curvature, then the cut locus of ∂M has a canonical polyhedral structure of arbitrarily low branching number. This establishes a connection between Riemannian comparison geometry and the PL notion of collapse to a simple polyhedral spine. We examine topological and geometric congruences. For example, suppose ∂M is connected and sectional curvatures of the interior and second fundamental form of ∂M satisfy $|K_M| \leq 1$ and $|II_{\partial M}| \leq 1$ respectively. Then if $\pi_1(\partial M)$ and $\pi_1(M)$ are isomorphic under the inclusion map, M has inradius at least 0.108 (sharp to within a factor of 2). In dimension 3, we classify up to homeomorphism all compact M with simply connected ∂M having $|K_M| \leq 1$ and $|II_{\partial M}| \leq 1$ and inradius $< a_3$, where a_3 is a universal constant known to lie between 0.1 and 0.2.

Edited by Anna Wienhard

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