Mathematisches Forschungsinstitut Oberwolfach

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Mini Workshop: Dessin D'Enfants

August 19th – August 25, 2001

The present conference was organized by Gareth A. Jones (University of Southampton, Southampton) and Jürgen Wolfart (Johann Wolfgang Goethe-Universität, Frankfurt). There were 15 participants coming from France, Germany, Russia, Slovakia, Spain and United Kingdom. The aim was to present new results in a profusely and detailed setting and several surveys related with Dessin's d'Enfants (using Gröthendieck's terminology), widely related with other topics coming (or being approach through this meeting) from very different areas in mathematics: maps, hypermaps, trees, knots of trees from Topological Graph Theory; Fuchsian and finite groups from Representation Theory; compact Riemann surfaces from the point of view of Algebraic, Analytic and Hyperbolic Geometry; moduli fields and fields of definition from Field Theory and Belyi functions in positive characteristic from Arithmetic Geometry.

Abstracts

Dessins, curves and knots

Norbert A'Campo

A rooted planar dessin of a certain kind has a slalom curve which is a generic, relative immersion P of the unit interval in the disk. Its knot in $\mathcal{S}^3$ is $K(P) := \{(x,u) \in T\mathbb{R}^2/x \in P,\ u \in T_xP,\ |x|^2 + |u|^2 = 1\}$. The knot is fibered. Its fiber surface has a natural 3-valent unicell map, as do all knots of divides P. For rooted planar trees we get slalom curves P and slalom knots $K(P)$. Let $\varphi_P : \mathcal{S}_P \to \mathcal{S}_P$ be the monodromy, then

Theorem. *For slalom knots $K(P)$ the word length growth of the action of φ_P on $\pi_1(\mathcal{S}_P)$ is equal to the growth of the action of φ_P on $H_1(\mathcal{S}_P, \mathbb{Z})$.*

In general, for a relative generic inmersion of $[0,1]$ in D, the growth of the actions of φ_P on $H_1(\mathcal{S}_P, \mathbb{Z})$ and on $\pi_1(\mathcal{S}_P)$ differ. We have examples with $\lambda^{\perp}_{H_1} = 0$ and $\lambda_{\pi_1} > 0$.

Homological growth equals growth on π_1, so the invariant quadratic differentials are oriented. In fact, these differentials have only boundary singularities. Let $\varphi,\ \psi \in M_{5,1}$ be these monodromies. The mapping classes φ and ψ are not conjugate in $M_{5,1}$, since the knots differ. However, we have evidence for the statement: φ and ψ are conjugate in the profinite completion $\hat{M}_{5,1}$. It would follow that the knot groups are isomorphic after profinite completion.

From plane trees to higher genera

Nikolai Adrianov

The talk recalls some results concerning plane trees with relation to the Gröthendieck's theory of dessins d'enfants. As the Galois action on the dessins preserves the sets of valencies, it is natural to classify them by number R of non-isomorphic dessins having the same valency sets. This question for plane trees was answered in a joint work with G. Shabat: for each R there are several infinite series and only finitely many "sporadic" valency sets which have exactly R realizations as plane trees.

Considering a plane tree as a 1-face dessin of genus 0, one can be interested of obtaining enumeration results for arbitrary genus. I mention here a result by J. Harer and D. Zagier for dessins of genus g with N edges as well as my own result for bicolored dessins with prescribed valency sets were enumerated by A. Goupil and G. Schaeffer. Their results actually generalizes a well-known formula for plane trees by F. Harary and W. T. Tutte. It is conjectured that for each R there are only finitely many valency sets which have exactly R realizations as 1-face dessin of genus > 0.

Classifying automorphism groups of compact Riemann surfaces of small genera, and of complex group characters arising from the action on the holomorphic differentials of a compact Riemann surface

Thomas Breuer

- We view compact Riemann surfaces as orbit spaces of the complex upper half plane, under the action of discrete groups (so called Fuchsian groups). In this context, groups of automorphisms are obtained as certain factor groups Γ/K of Fuchsian groups Γ and K, where K is torsion free and determine the genus of the surface.
- The isomorphism types of finite groups G that occur as Γ/K of that form have been classified for the genera from 2 to 48, this involved combinatorial methods for enumerating the possible isomorphism types of Γ, and methods from computation group and representation theory for reducing the number of cases to deal with a manageable amount.
- When considering the characters of a fixed given finite group that are obtained by the action of G on the space of holomorphic differentials of compact Riemann surfaces, one can often classify this set of characters in terms of combinatorial necessary conditions involving certain fixed point numbers, and a set of exceptional characters (which may be empty, finite, or infinite). The methods used to obtain such a classification are the same as in the second paragraph.

Uniformization of hyperelliptic surfaces. On Whittaker's conjecture

Ernesto Girondo

The question of stating explicitly the correspondence between Fuchsian groups and algebraic curves has been studied by many authors. In particular, E.T. Whittaker studied the case of the hyperelliptic curves ($y^2 = \prod_{i=1}^{2g+2}(x - a_i)$), and he showed (1898) that the Fuchsian group Γ that uniformizes such a curve is contained in a larger group Γ^*. The determination of Γ^* depends on the explicit knowledge of $2g-1$ coefficients, the *accessory parameters*, which are only theoretically determined in terms of the $a_1, \dots, a_{2g+2}$. Whittaker conjectured (1929) a concrete value for them and the conjecture was shown to be true for many surfaces (all of them with a regular Belyi function) by Whittaker, Dhar, Mursi and Rankin, among others, and to be not true in all its generality (Kra, 1989).

In this talk, based in a joint work with Gabino González-Diez, we give a new proof for the conjecture for every surface with a regular Belyi function, and prove that there are other surfaces beyond these that also verify it: this shows that a second conjecture about Whittaker's one, by Chudnovsky and Chudnovsky (1986), is not true. We also prove that Whittaker's conjecture is not verified for all surfaces uniformized by (not necessarily normal) subgroups inside a triangle group.

One face trivalent dessins

Gabino González-Diez

In his paper " Discs extremaux..." (Ann. de la Fac. des Sc. Toulouse V (1996),No.2" C. Bavard began the study of compact hyperbolic surfaces $S = \mathbb{D}/K$ admitting discs of maximal radius within given genus g. It is readily seen that this phenomenon occurs when the uniformizing group K is an index $N = 12g - 6$ subgroup of the triangle group with signature $(2, 3, N)$ or, equivalently, when S corresponds to a one face trivalent uniform dessin.

In his talk I reported on the following results obtained jointly with E. Girondo

- As in the case of genus 2 (observed by Bavard) one can find extremal surfaces of genus $g = 3$ possessing several extremal discs.
- On the contrary, when $g = 4$ extremal discs are unique.
- When $g = 2$ there are, up to orientation isometry, 9 extremal surfaces (something already known to Fricke and Klein). For each of them we describe all its discs and its automorphism group.

Truncations: some generalizations and applications

Gareth Jones

Dessins (or finite oriented hypermaps) correspond to conjugacy classes of subgroups of finite index in triangle groups, with the dessin of a given type (corresponding to a particular triangle group) forming a category. Any inclusion between triangle groups induces a functor between dessins of the corresponding types. All such inclusions were classified by Singerman in 1972: there are seven infinite families, and seven sporadic examples. The main aim of the talk was to describe the corresponding functors between dessins: in some cases they are easily expressed in terms of simple operators such as truncation of maps, or the Cori, James and Walsh representations of hypermaps, but in other cases more complicated constructions are involved. The talk ended with some speculative comments on the possible applications of truncations of hypermaps of type $(7, 2, 3)$ together with the geometry of minimal surfaces to the construction of negatively curved carbon molecules analogous to the positively curved molecule C_{60} formed by truncating an icosahedron.

Complete bipartite maps

Gareth Jones

This is joint work in progress with Roman Nedela and Martin Škoviera.

One of the main problems of topological graph theory is to find all the regular embeddings (in orientable surfaces) of each graph or class of graphs. One of the few cases where this has been achieved concerns the complete graphs K_n (James and Jones, 1985), a result which has been extended to a few other classes by Nedela and Škoviera. This talk concerns the complete bipartite graphs $K_{n,n}$. For each n this graph has a *standard embedding*; this is normally defined as a Cayley map for the group $\mathbf{Z}_{2n}$ with respect to the generating set $\{1, 3, \ldots, 2n-1\}$ in that cyclic order, but it can also be defined as a bipartite dessin on the Fermat curve $x^n + y^n = 1$ corresponding to the Belyi function $\beta(x, y) = x^n$. Nedela, Škoviera and Zlatoš have recently shown that if n is prime then this is the only regular embedding of $K_{n,n}$. It is natural to ask which other values of n have this uniqueness property. Using a correspondence between regular embeddings of $K_{n,n}$ and certain groups which factorise as a product of two cyclic groups of order n, we show that the standard embedding is unique if and only if n is coprime to $\phi(n)$, where ϕ is Euler's function. Curiously, these are also the values of n for which every group of order n is cyclic.

The Riemann surface of a uniform dessin

Gareth Jones, d'après David Singerman and Robert Syddall

This talk is a report on work of David Singerman (who was unable to attend the workshop) and Robert Syddall, concerning the function R which sends each dessin (or compact oriented hypermap) $\mathcal{H}$ to its associated Riemann surface $R(\mathcal{H})$. By Belyi's Theorem, R is not an epimorphism, but on what classes of dessins is it injective? In other words, when does the Riemann surface determine the dessin uniquely? (To avoid trivialities, we will consider only uniform dessins, and will ignore dualities, interchanging vertices, edges and faces). It is easy to see that R is not injective on dessins of genus $g = 0$ or 1, so will assume that $g \geq 2$. Inclusions between triangle groups give one source of non-injectivity: a subgroup of finite index in the smaller triangle group also has finite index in the larger one, so these inclusions give non-isomorphic dessins on the same surface. Such inclusions between triangle groups were classified by Singerman (JLMS 1972), and in the case of maps they have simple geometric interpretations (for instance, corresponding to truncations or median maps). A related source of non-injectivity arises from commensurability between triangle groups. Takeuchi has listed the finitely many types of arithmetic triangle groups, and a number of examples of non-injectivity are given; in the non-arithmetic case, however, it is proved that for dessins associated with maximal triangle groups, R is injective.

Belyi's Theorem Revised

Bernard Koeck

We give an elementary, self-contained and quick proof of the so-called "obvious part" of the following famous theorem by Belyi:

Theorem. *A complex smooth projective curve X is defined over a number field, if and only if there exists a non-constant morphism $t : X \to \mathbb{P}^1_{\mathbb{C}}$ with at most 3 critical values.*

After introducing the moduli field of a curve X and of a non-constant morphism $t : X \to \mathbb{P}^1_{\mathbb{C}}$, we show that any such t is defined over a finite extension of its moduli field and that the moduli field of t is a number field, if the critical values of t are algebraic integers. This immediately proves the "obvious part", i.e. the if-direction, of Belyi's theorem. As a by-product of our proof we obtain an explicit bound for the degree of the defining number field.

Weierstrass points at genus 2 surfaces

Tapani Kuusalo

These observations come from a joint work with Dr. Marjatha Näätänen (Helsinki). We found a way to give a geometric determination of the 6 Weierstrass points of a closed genus 2 surface, based on the following theorem of Haas and Susskind (Proc. Amer. Math. Soc. 1989): On a genus 2 surface X a simple closed geodesic α is always mapped by the hyperelliptic involution j onto itself, dividing the surface X when the orientation of α is preserved and, on the other hand, α is non-dividing when the hyperelliptic involution j reverses its orientation. As the Weierstrass points of X are the fixed points of the involution j, one thus has exactly two Weierstrass points on any non-dividing simple closed geodesic. Thus every Weierstrass point P can be determined as the single intersection point of two properly chosen non-dividing simple closed geodesics α and β. We have made practical determination for all genus 2 surfaces admitting regular fundamental polygons.

Regular maps and congruence subgroups of orthogonal groups

Joref Širáň

It is known that finite regular maps (i.e., 2-cell graph embeddings whose groups of orientation-preserving automorphisms act transitively on darts) of type $\{m, n\}$ are in a $1-1$ correspondence with finite-index torsion-free normal subgroups of the $(2, m, n)$-triangle groups. A particularly interesting class of such subgroups are congruence subgroups of certain faithful representations of triangle groups in 3-dimensional orthogonal groups. We give a survey of known results in this area, focusing on the related regular maps whose automorphism groups are projective linear groups over finite fields. Applications to problems in the theory of symmetric graphs were discussed as well.

Regular maps with multiple edges

Martin Škoviera

A map $\mathcal{M} : G \to S$ on a surface is an embedding of a connected graph G in an orientable surface S such that its complement consists of simply connected regions; it is said to be regular if its orientation preserving automorphism group acts regularly on darts (oriented edges).

We dealt with the problem of classifying regular maps with multiple edges in terms of regular maps without multiple edges. Each regular map $\mathcal{M} : G^{(k)} \to S$ with k parallel edges between any two adjacent vertices of a single graph G induces three important invariants: its *shadow*, the unique regular embedding of G covered by $\mathcal{M}$; its *twisting number*, which corresponds to the derived regular embedding of the dipole with k parallel edges and it is characterized by a number $e \in \mathbb{Z}_k$ such that $e^2 \equiv 1 (\text{mod } k)$; and the *rotation gradient*, a function on G with values on $\mathbb{Z}_k$.

Given a shadow, a twisting number and a rotation gradient, there exists a regular embedding with these invariants if and only if a certain system of equations in $\mathbb{Z}_k$ has a solution. This result enables one, among other things, to characterize all the multiplicities with which complete graphs, complete bipartite graphs and other classes of graphs admit regular embeddings.

Computational Problems in Riemann Surfaces

Manfred Streit

This talk is based mainly in a join work with Yolanda Fuertes. A family of Riemann surfaces X_t, genus $g = 3$, depending by a one dimensional parameter t can be obtained by considering the following construction:

Define a surjective homomorphism φ from a fuchsian group of type $[2, 2, 2, 3]$ to the symmetric group S_4. The torsion free kernel $\Gamma := \mathrm{Ker}\varphi$ defines a Riemann surface $X = \mathbb{H}/\Gamma$, where $\mathbb{H}$ is the upper half plane. The parameter t comes from the fact that there is a one dimensional family of groups of type $[2, 2, 2, 3]$. During the talk the number of kernels were calculated, the representation on the holomorphic 1-differentials was constructed and used to give the family of surfaces explicitly as smooth plane quartics:

$$X_t \cong \{x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0\}.$$

The groups $[2, 2, 2, 3]$ are uniquely contained in the following triangle groups:

$$[2, 4, 6], \quad [2, 3, 9], \quad [2, 3, 8] \quad \text{and} \quad [2, 3, 7].$$

The values of the parameter t for which such an inclusion occurs can be calculated. Finally, the wronskian was calculated and questions concerning the jacobians of the curves were discussed.

Three point curves with bad redution

Stefan Wewers

Let $t : X \to \mathbb{P}^1$ be a three point cover, i.e. a nonconstant morphism between smooth projective curves over $\mathbb{C}$, ramified at 0, 1 and ∞. Let G be the monodromy group of t and $M/\mathbb{Q}$ the field of moduli. Then,

Theorem. *Suppose $p > 2$ is a prime such that p^2 does not divide the order of G. Then p is at most tamely ramified in the extension $M/\mathbb{Q}$. Moreover, the ramification index of any prime ideal $\mathcal{P}$ of M over p can be bounded by an explicit constant depending only on p and on the group G; more explicitly*

$$e(\mathcal{P}/p) \leq C(p) \cdot |Centr_G P|,$$

where $P \leq G$ is a p-Sylow subgroup.

This theorem extends the well known result of Beckmann, which says that p is unramified in $M/\mathbb{Q}$ if p does not divide the order of G.

The proof is based on recent work of Raynaund, which analyses the stable reduction of the curve X.

Cartographic groups of cyclic projective planes

Jürgen Wolfart

The talk describes part of a recent joint work with Manfred Streit (Documenta Math. 6 (2001) 67-95). Let $\mathbb{P}$ be a finite projective plane of order $n = q - 1$, i.e. with q points on every line. Suppose that $\mathbb{P}$ is cyclic, i.e. that there is a cyclic "Singer" group Z_l of automorphisms of $\mathbb{P}$ acting transitively on all $l = n^2 + n + 1$ points of $\mathbb{P}$ what is true at least for the standard projective planes over finite fields with n elements. The usual bipartite graph associated to $\mathbb{P}$ can be embedded into orientable compact surfaces giving a dessin D (D. Singerman, A. White). Moreover, the dessin can be chosen such that Z_l acts as an automorphism group of D and that D/Z_l is a genus zero dessin. In that case, the cartographic (or monodromy) group of D can be shown to be a semidirect product $A \rtimes Z_q$ for a quotient A of Z_l^n. If l is prime, we can determine A and the action of Z_q on A more precisely. It depends on the cyclic ordering of a difference set for $\mathbb{P}$ which determines D, its algebraic curve equations and the Galois actions on D as well.

Several facets of Belyi functions

Alexander Zvonkin

The talk consisted of five mini-talks, the general guide-line being to demonstrate that Belyi functions are a very interesting class of functions and have many beautiful applications even outside Galois theory.

- The first mini-talk presented an incredibly simple "combinatorial" proof of the Davenport-Stothers-Zannier bound (a conjecture that remained open for 30 years).
- The second one presented a class of maps whose Belyi functions are expressed in terms of Jacobi polynomials.
- In the third one the dessins d'enfants were used in order to construct complex dynamic systems with Julia set $J = \mathbb{C}$.
- The fourth mini-talk was dedicated to a series of dessins in which the splitting into two orbits finds its origin in rational points on a specific elliptic curve.
- The final one was a numerical remark concerning the 3rd step of the "only if" part of Belyi theorem.

Edited by Yolanda Fuertes

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