Mathematisches Forschungsinstitut Oberwolfach

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Reelle Algebraische und Analytische Geometrie

March 17th – March 23rd, 2002

The meeting was organized by Eberhard Becker (Dortmund), Ludwig Bröcker (Münster) and Michel Coste (Rennes). In 27 talks, many different aspects of real algebraic geometry were covered. In the following we include the abstracts in alphabetical order.

Abstracts

Variations of Lipschitz-Killing curvatures

A. Bernig

The normal cycle associated to convex sets, subanalytic sets or submanifolds allows a unified treatment of curvature questions for these spaces. All of them admit Lipschitz-Killing curvatures which are related to volume growth. We consider what happens when varying the metric of the ambient space.

Theorem 1. *Let $\tilde{X}$ be the normal cycle of one of the above spaces, $\Lambda_i(X,g)$ the i^{th} Lipschitz-Killing curvature. Then for each symmetric bilinear form h on the ambient space, there are canonical (and explicitely known) differential forms ϕ_i^h such that*

$$\left.\frac{d}{dt}\right|_{t=0} \Lambda_i(X,g_t) = \tilde{X}(\phi_i^h)$$

where $g_t = g + th$ is a variation of the metric.

As a corollary, we get some well-known formulas from smooth theory, like variation formulas for the total scalar curvature. Also we give a new proof of Schläfli's formula and a generalization to subanalytic sets.

Characteristic exponents of nonisolated semialgebraic singularities

L. Birbrair

Let X be a semialgebraic set and x_0 be a singular point of X. We consider semialgebraic cycles defined in some neighborhood of x_0. All these cycles can be contracted to x_0. Let c be a cycle and η be a chain such that $c = \partial\eta$. We prove that there exists a constant μ_k depending only on X and on $k \leq \dim X$ such that if $\mu(\operatorname{supp}\eta, x_0) \geq \mu_k$ then $[c] = 0$ in a neighborhood of x_0 and c can be contracted without going to x_0. The μ_k are invariant under bi-Lipschitz maps.

On approximation of smooth submanifolds of real algebraic varieties by real algebraic subvarieties

J. Bochnak

Let X be a nonsingular compact real algebraic variety. We denote by $H_d^{alg}(X,\mathbb{Z}/2)$ the subgroup of $H_d(X,\mathbb{Z}/2)$ of homology classes represented by d-dimensional algebraic subvarieties of X. Let $M \subset X$ be a smooth compact submanifold of X. We say that M admits an *algebraic approximation* in X if for each neighbourhood U of the inclusion $i : M \hookrightarrow X$ there exists an embedding $\phi : M \mapsto X$ such that $\phi(M)$ is a nonsingular algebraic subvariety of X (we consider the space $C^\infty(M,X)$ of all C^∞ mappings from M into X equipped with the C^∞-topology). Clearly if a d-dimensional C^∞ submanifold M of X admits an algebraic approximation, then the class $[M]_X$ represented by M is in $H_d^{alg}(X,\mathbb{Z}/2)$.

Problem A: Given X, when does every compact smooth submanifold M of X with $[M]_X \in H_d^{alg}(X,\mathbb{Z}/2), d = \dim M$, admit an algebraic approximation in X?

This problem is very difficult as the answer is not fully known even for simple varieties like the n-sphere S^n. It is known that the answer is positive if $\dim M = \dim X - 1$ (i.e. for hypersurfaces).

Theorem 1. *Let X be a compact nonsingular real algebraic variety of dimension 3 and let M be a compact smooth curve with $[M]_X \in H_1^{alg}(X, \mathbb{Z}/2)$. Then M admits an algebraic approximation in X.*

It is very likely that Theorem 1 is still valid for X of dimension > 3.

Define Ω to be the set of all $(n, d) \in \mathbb{N} \times \mathbb{N}, 0 < d < n$ such that for every compact nonsingular real algebraic variety X of dimension n and every compact smooth submanifold M of X of dimension d with $[M]_X \in H_d^{alg}(X, \mathbb{Z}/2)$, M admits an algebraic approximation in X.

The results mentioned above say that $(n, n-1) \in \Omega$ and $(3, 1) \in \Omega$.

Conjecture 2. $(n, d) \in \Omega \Longleftrightarrow n - d = 1$ *or* $d = 1$.

The conjecture is partially confirmed by the following theorem:

Theorem 3. *Let $N \subset S^m$ be a compact smooth submanifold of S^m, $0 < \dim N < m$. Assume that N is not a boundary of a smooth compact manifold with boundary. Then for each integer $k > 0$, there exist a nonsingular real algebraic variety X and a smooth submanifold M of X, such that X is diffeomorphic to $S^m \times S^k$, M is diffeomorphic to $N \times S^k$, the class $[M]_X = 0$ (thus in $H_*^{alg}(X, \mathbb{Z}/2)$) and M does not admit an algebraic approximation in X.*

Corollary 4. *For n and d with $n - d \geq 2, d \geq 3$, one has $(n, d) \notin \Omega$.*

Proof: Apply Theorem 3 with $k = d - 2, m = n - d + 2$ and $N = \mathbb{RP}^2$.

Conjecture 2 is therefore reduced to

Conjecture 5. $(n, 1) \in \Omega$ *and* $(n, 2) \notin \Omega$*, for all* $n \geq 4$.

The proof of Theorem 3 is based on the following result:

Theorem 6. *Let X and Y be nonsingular real algebraic varieties, X compact and Y irreducible. Let $f : X \mapsto Y$ be a regular mapping and let y_1, y_2 be regular values of f. Then the smooth manifolds $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are cobordant.*

References: J. Bochnak, W. Kucharz: On smooth submanifolds of real algebraic varieties admitting algebraic approximation, preprint 2002.

The Hörmander-Łojasiewicz inequality in global semianalytic geometry

F. Broglia

(joint work with F. Acquistapace and M. Shiota)

We want to investigate to what extent a Łojasiewicz type inequality holds true for global semianalytic sets, that is semianalytic sets described by finitely many global analytic functions. We find the following result (that was known only for dimension ≤ 2, cf A. Diaz-Cano's thesis).

Theorem 1. *Let A be a closed global semianalytic set and f, g be analytic. Then there is $q \geq 0$ global analytic such that*

- $Sign(f + qg) = Sign(f)$ *on* A.
- $q^{-1}(0) = \overline{f^{-1}(0) \cap \bar{A}}^Z$

This result is completely analogous to the so called Hörmander-Łojasiewicz inequality for semialgebraic sets.

As an application we get the finiteness property for global semianalytic sets, that is any open (closed) global analytic set is a finite union of basic open (closed) global semianalytic sets.

Connectedness for symmetric K3-surfaces

A. Degtyarev

(joint work with I. Itenberg, V. Kharlamov)

We discuss strong quasi-simplicity of $K3$-surfaces, i.e., whether the deformation class of a finite group Klein action (i.e., an action by both holomorphic and anti-holomorphic maps) on a $K3$-surface is determined by its equivariant homeomorphism type. We consider the 'obvious' topological invariants of an action, namely, the induced actions in the homology and on the class of a holomorphic form. The latter is called the *fundamental representation*.

The principal results are the following:

Theorem 1. *If the action is holomorphic or the fundamental representation is real, then the obvious invariants determine the action up to deformation and identification* $X \sim \bar{X}$.

Corollary 2. *The deformation type of a real action on a real* $K3$*-surface is determined by its obvious invariants.*

In conclusion, we construct an example to show that a surface X does not need to be equivariantly deformation equivalent to $\bar{X}$, and another example, to show that, in the case of a properly Klein action with non-real fundamental representation, the obvious invariants might not determine the action.

Bad points of positive semidefinite polynomials, and their effect on continuously varying sum-of-square representations

C. Delzell

Let $x = (x_1, \dots, x_n)$, $\mathcal{P}_n = \{f \in \mathbb{R}[x] : \forall x \in \mathbb{R}^n, f(x) \geq 0\}, \Sigma = \sum \mathbb{R}[x]^2, f \in \mathbb{R}[x]$ and $(\Sigma : f) = \{h^2 : h \in \mathbb{R}[x], h^2 f \in \Sigma\}$.

Definition 1. *The bad set of* f *is* $Z(\Sigma : f) := \{x \in \mathbb{R}^n : \forall h^2 \in (\Sigma : f), h(x) = 0\}$.

For background, see Delzell, Bad points for positive semidefinite polynomials, Abstracts of Papers Presented to the AMS, Vol. 18, no3, issue 104(1997), p.482.

Theorem 2. $\forall f \in \mathbb{R}[x], \exists h^2 \in (\Sigma : f)$ *such that* $Z(h) = Z(\Sigma : f)$.

Definition 3. *For* $f \in \mathcal{P}_n$ *and* $x \in \mathbb{R}^n$, *write* $\rho_*(f) =$ *least* $l \in \mathbb{N}$ *such that* $\exists h^2 \in (\Sigma : f)$ *with* $h(x) \neq 0, deg(h) = l$, *if such* l *exists,* ∞ *otherwise.*

Definition 4. *For* $f \in \mathcal{P}_n$ *write* $\rho(f) =$ *least* l *such that* $\exists h^2 \in (\Sigma : f)$ *such that* $Z(h) = Z(\Sigma : f), deg(h) = l$.

Note: $\forall f \in \mathcal{P}_n, \sup_{x \text{ good}} \rho_x(f) \leq \rho(f) \leq 2 \sup_{x \text{ good}} \rho_x(f)$.

For $d \in \mathbb{N}$, write $\mathcal{P}_{n,d} = \{f \in \mathcal{P}_n : deg(f) \leq d\}$.

Open: True or false: $\forall n, d \exists l \forall f \in \mathcal{P}_{n,d}, \rho(f) \leq l$? (known for n=1, 2).

(*) Continuity: (1980) $\forall n, d \exists l, s \in \mathbb{N}$ such that $\exists$ a family of polynomials $h := h_f \in \mathbb{R}[x]$ whose coefficients depend continuously on the coefficients of $f \in \mathcal{P}_{n,d}$, and such that

$h_f^2 f = \sum_{i=1}^s g_{i,f}^2$ for some $g_{i,f} \in \mathbb{R}[x]$ whose coefficients also depend continuously on f with $deg(h_f) \leq l$ and $Z(h_f) \subseteq Z(f)$.

Question 1 (H. Lombardi, 1999): In (*) can we arrange, in addition, that $\forall f \in \mathcal{P}_{n,d}, Z(h_f) = Z(\Sigma : f)$?

Question 2 (Ishai Oren, 1980): $\forall n, d$, given a particular $f_0 \in \mathcal{P}_{n,d}$ and $h_0^2 \in (\Sigma : f)$, can we find a continuously varying h_f as in (*) such that when f is f_0, h_f is h_0?

First answer to (1): Yes for $n = 1$ (Martin Ziegler, 1988).

Second answer to (1) and (2): No for $n \geq 2$.

Torsors under abelian varieties over $R((t))$

A. Ducros

Let R be a real closed field and A an abelian variety over the field $k = R((t))$. Denote by B the dual variety, and by $\tilde{B}^\circ$ the Zariski neutral-component of the special fibre of the Neron model of A.

We compute the cohomology group $H^1(k, R)$ (which classifies k-torsors under A) in terms of $\tilde{B}^\circ$ and of the semialgebraic topology of $A(k^+)$ and $A(k^-)$, k^+, k^- being the real closures of k.

On the curvature of real polynomial fibres

N. Dutertre

We give a Gauss-Bonnet type formula for a smooth non-compact polynomial fibre. For this we have to study topological properties of a general hyperplane section of this fibre.

Polynomial images of $\mathbb{R}^n$

J. F. Fernando

(joint work with J. M. Gamboa)

Let R be a real closed field and m, n be positive integers. A map $f = (f_1, \ldots, f_n) : R^m \to R^n$ is said to be *polynomial* if the functions $f_i \in R[x_1, \ldots, x_m]$.

A celebrated Theorem of Tarski-Seidenberg says that the image of any polynomial map $f : R^m \to R^n$ is a semialgebraic subset of R^n, i.e. it can be written as a finite union of subsets defined by a finite conjunction of polynomial equalities and inequalities. We study some kind of converse of this result.

In the 1990 *Oberwolfach* week, J.M. Gamboa proposed to characterize the semialgebraic sets of R^n which are polynomial images of R^m. In particular, the open ones deserve a special attention. In his talk, he proposed two problems: (1) to decide whether or not the open quadrant $Q = \{x > 0, y > 0\}$ is a polynomial image of $\mathbb{R}^2$; and (2) to decide whether or not the exterior of the unit closed disc $S = \{x^2 + y^2 > 1\}$ is a polynomial image of $\mathbb{R}^2$.

In relation with this we prove that: (1) for every finite subset F of R^n, $n \geq 2$, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n; and (2) for any independent linear forms $l_1, \ldots, l_r$ of R^n, $n \geq 2$, the semialgebraic set $\{l_1 > 0, \ldots, l_r > 0\} \subset R^n$ is a polynomial image of R^n.

Moreover, the second of the two problems above has a negative answer. This is easily obtained, via certain result of Jelonek (1982), since the Zariski closure of *exterior boundary* $\delta S = \overline{S} \setminus S$ of S is not a polynomial image of R.

Finally, we state again some questions. Let $S \subset R^2$ be a semialgebraic open set which is a polynomial image of R^2:

(1) Is there a bound on the number of semialgebraic connected components of the exterior boundary δS of S?, and in particular, is the set $\{y > 0, x^2 + 1 - y > 0\} \subset R^2$ a polynomial image of R^2?

(2) Given C a semialgebraic connected component of the exterior boundary δS of S and $\overline{C}^{\text{zar}}$ its Zariski closure, which can be written by a result of Jelonek as $\cup_{i=1}^{r} \overline{L_i}^{\text{zar}}$ where the L_i's are polynomials images of R, is there a bound on the number of L_i's?, and in particular, is the set $\{y > 0, x > 0, x + y - 1 > 0\} \subset R^2$ a polynomial image of R^2?

Real plane curves and ramified coverings of the dual real projective plane

J. Huisman

Let C be a real algebraic plane curve. We show how to associate to C a ramified covering of the dual real projective plane $\mathbb{P}^2(\mathbb{R})\check{}$. The source of f is the topological surface $X = Div^2_{\geq 0}(C)^0$ of effective divisors of degree 2 on C that have even degree on each connected component of $C(\mathbb{R})$. If all connected components of $C(\mathbb{R})$ are convex, we get a new proof of Klein's equation and Rokhlin's Inequality. We show also how one can read of from the covering, the arrangements of the ovals of C in $\mathbb{P}^2(\mathbb{R})$.

Finiteness for symmetric $K3$-surfaces

I. Itenberg

(joint work with A. Degtyarev and V. Kharlamov)

We discuss several finiteness results concerning Klein actions of finite groups on $K3$-surfaces. Let G be a group. A G-action on a complex manifold X is called a *Klein action* if any element of G acts either by a holomorphic or by an antiholomorphic map.

The principal results are the following.

Theorem 1. *There exist at most finitely many (up to isomorphism) Klein actions of finite groups on a given $K3$-surface.*

In particular, on a given $K3$-surface (or a given Enriques surface), there exist at most finitely many real structures (up to conjugation by automorphisms of the surface).

Theorem 2. *There exist only finitely many G-equivariant deformation classes of $K3$-surfaces, where G is a finite group.*

On classification of Markowitz mappings

P. Jaworski

In my talk I dealt with so called Markowitz mappings $(V, E) : \mathbb{R}^n \supset H \supset \Delta \mapsto \mathbb{R}^2$, where H is a hyperplane $e^t \circ x = 1, e = (1, \dots, 1)^T$, Δ is a simplex spanned by unit vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, V is quadratic positive-definite, $V(X) = x^T \circ C \circ x$ and E is linear, $E(x) = \mu^T \circ x$.

The interest in studying this subject comes from the fact that such mappings are the basic part of Markowitz model which is a cornerstone of the modern portfolio analysis.

We denote by Σ the critical line, i.e. the set of critical points of $(V, E)|_H$ and by Σ_Δ the set of points of relatively minimal V.

$$\Sigma_\Delta := \{x \in \Delta : \forall y \in \Delta \cap E^{-1}(E(x)), V(x) \leq V(y)\}$$

Definition 1. *A mapping (V, E) is called generic if*

- *$E(e_i)$ are pairwise different.*
- *Σ neither intersects nor is parallel to the faces of Σ of codimension 2.*
- *(V, E) restricted to $\{x_i = 0\}$ is still generic.*

The set of generic mappings is open semialgebraic and we say that two generic mappings are equivalent if they belong to the same connected component. There are several invariants of this equivalency.

The set of nongeneric mappings is a union of zeros of n-minors of the $n \times (n+2)$ matrix (e, μ, C, Id_n) containing the first two columns (e, μ) and not containing the columns of the same index of C and Id_n. Therefore the signs of these minors are invariants of the equivalence relation.

The other invariant is the equivalence class of the mapping $Sim : \Sigma_\Delta \mapsto \{0, 1\}^n$ up to the right PL-homeomorphism, where Sim is given by the rule: for $x \in \Sigma_\Delta$

$$Sim(x) = (sgn(x_1), sgn(x_2), \dots, sgn(x_n))$$

i.e. it associates to a point x the subsimplex containing it.

Capacity-density of subanalytic sets

T. Kaiser

Let $X \subset \mathbb{R}^n, n \geq 2$ be a subanalytic set. We show that the capacity-density of X at x defined by

$$\theta_c(X, x) := \lim_{R \to 0} \frac{c(X \cap \bar{B}_R(x))}{c(\bar{B}_R(x))}$$

exists for every $x \in \overline{X}$. Here $c(E)$ is the capacity of some Borel-set $E \in \mathbb{R}^n$. Moreover, we give some connections to the volume density.

Deformation inequivalent complex conjugated complex structures

V. Kharlamov

(joint work with V. Kulikov)

We give examples of complex varieties X of general type and any dimension $n \geq 2$ such that X and $\bar{X}$ are not deformation equivalent (in particular, X can't be deformed to a real variety or to a variety with an anti-holomorphic automorphism). Such 2-dimensional examples provide new counter-examples to the Dif=Def conjecture: the complex conjugation exchanges two distinct components of the moduli space. As another application, by means of such examples, we are constructing diffeomorphic plane cuspidal curves which are not equisingular deformation equivalent and even not isotopic.

The examples are: some Galois coverings of the projective plane and fake projective planes, their products and products with a curve.

Łojasiewicz's gradient inequality for analytic (o-minimal) maps

K. Kurdyka

Theorem 1. (Łojasiewicz 60's) *Let* $f : U \mapsto \mathbb{R}$ *analytic,* $0 \in U \subset \mathbb{R}^n, f(0) = 0, \nabla f(0) = 0$*. Then there exists* $\rho < 1, c > 0$ *such that*

$$|\nabla f| \geq c|f|^\rho \textit{ in a neighborhood of } 0$$

Theorem 2. O-minimal version, Kurdyka 1998 *Assume* f *as above but instead of analytic we suppose that* f *is* C^1*, definable in an o-minimal structure* $\mathcal{M}$*. Then there exists* C^1 *function* $\psi : (-\epsilon, 0) \cup (0, \epsilon) \mapsto \mathbb{R}$ *bounded, increasing, definable in* $\mathcal{M}$ *such that*

$$|\nabla(\psi \circ f)| \geq 1 \textit{ in a neighborhood of } 0$$

We consider now $f : U \mapsto \mathbb{R}^k$, analytic (o-minimal), with $k \leq n$. For a linear map $A : \mathbb{R}^n \mapsto \mathbb{R}^k$, we put $v(A) = dist(A, \Sigma)$, where $\Sigma = \{B : \mathbb{R}^n \mapsto \mathbb{R}^k,\ \text{linear, nonsurjective}\}$ (cf. Kurdyka, Orro, Simon, J. Diff. Geometry 56(2000)). We proved the following:

Theorem 3. *Let* $f : U \mapsto \mathbb{R}^k$ *be analytic (*C^1*, definable in* $\mathcal{M}$*),* $U \subset \mathbb{R}^n$*,* K *compact in* U*, assume that the rank of* $d_x f$ *is* k *for* x *in an open and dense subset of* U*. Then there exists* $\psi : W \mapsto \mathbb{R}^k$ *subanalytic (analytic, resp. definable in* $\mathcal{M}$*,* C^1*) injective and bounded such that*

$$v(d_x(\psi \circ f)) \geq 1, \quad x \in f^{-1}(W) \cap K$$

where $W = f(U) \setminus (K_f \cup \Gamma)$*,* K_f *critical values of* f *on* K*,* Γ *subanalytic (definable in* $\mathcal{M}$*) nowhere dense in* $\mathbb{R}^k$*.*

Corollary 4. *(An analogue of finiteness of length of trajectories of the gradient field) Let* $S \subset U$ *be a* k*-dimensional submanifold,* $\bar{S} \subset U$ *compact. Assume that* f *is injective on* S *and* $0 < \alpha \leq \angle(T_x S, Ker(d_x f)))$*,* $\quad x \in S$*. Then*

$$\mathrm{vol}_k(S) \leq C\,\mathrm{vol}_k(\psi(f(S)) < \infty, \quad C = C(\alpha)$$

Theorem 5. *Let* f *be like in Theorem 3, assume that* f *is analytic, and that* ψ *extends continuously on* $\bar{W}$ *(this in particular is the case when* f *is finite on its singular locus). Then there exists* $\rho < 1, C > 0$ *such that*

$$v(d_x f) \geq C\,\mathrm{dist}(f(x), \Gamma)^\rho, \quad x \in K$$

where Γ *is a subanalytic subset of* $\mathbb{R}^k$ *containing critical values* K_f *(in general* $\Gamma \neq K_f$*).*

Finiteness property implies o-minimality

J. M. Lion

We consider a family of differential algebras of real functions on real euclidean space, stable under right-composition by affine maps. We prove that under a weak finiteness property these functions are definable in some o-minimal structure.

Approximation of smooth maps by regular maps and real Del Pezzo surfaces of degree 2

F. Mangolte

We give a solution to the problem of approximating smooth maps by regular maps when the targent space is the standard sphere and the source space is a geometrically rational real algebraic surface. When the source space is a real algebraic surface rational over $\mathbb{R}$, the approximation results are due to J. Bochnak and W. Kucharz. We complete the results for real geometrically rational surfaces. We give a detailed description of the most interesting case which is a real Del Pezzo surface of degree 2. We show a picture of these surfaces.

Tropical Algebraic Geometry

G. Mikhalkin

Real numbers with two operations, taking the maximum and addition, form a semiring. This semiring is called tropical. Algebra of tropical semiring is a subject well-studied by computer scientists. This talk will concern geometric objects described by tropical equations. It turns out that the tropical objects are handier than their classical algebro-geometric counterparts. For instance, tropical algebraic curves are graphs. A curious fact is that, even though this geometry is based only on a semiring, it resembles in many ways complex algebraic geometry (which is based on an algebraically closed field).

Cubic resolvent as a tool to distinguish real algebraic and real pseudo-holomorphic curves

S. Orevkov

We give an example of an arrangement on $\mathbb{RP}^2$ which can not be realized as the union of curves of degree 3 and 4. But this arrangement can be easily constructed pseudo-holomorphically.

Differentiable functions defined on closed subsets of $\mathbb{R}^n$: Whitney's problem

W. Pawlucki

(joint work with E. Bierstone and P. D. Milman)

In 1934, Whitney raised the question how to recognize whether a function f defined on a closed subset of $\mathbb{R}^n$ is the restriction of a function of class C^k (k given positive integer). A necessary and sufficient criterion was given in the case $n = 1$ by Whitney using limits of divided differences of the function, and in the case $k = 1$ by Glaeser (1958), using limits of secants of the set. Developing the idea of Glaeser, we introduce a necessary geometric criterion, for general n and k, involving limits of finite differences that we conjecture to be sufficient at least if X has a "tame topology". We prove that if X is a compact subanalytic subset, then there exists $l = l_X(k)$ such that the criterion for l implies that f has C^k-extensions to $\mathbb{R}^n$. The result gives a new approach to higher-order tangent bundles on singular spaces.

Curvature of the Milnor fibre

J.-J. Risler

Let f be a germ of an analytic curve at $0 \in \mathbb{R}^2$, f_t a smooth deformation, B_ϵ a ball of radius ϵ, $|t| << \epsilon$. Then

Theorem 1.

$$\lim_{\epsilon\to 0} 2\int_{C_t\cap B_\epsilon} |k| \leq \lim_{\epsilon\to 0}\int_{(C_t\cap B_\epsilon)^{\mathbb{C}}} |K| = 2\pi(\mu+n-1) \qquad (*)$$

where k is the real curvature, K the complex Gauss curvature, $C_t := \{f_t = 0\}$, μ is the Milnor number and n the multiplicity.

Theorem 2. *If f is real irreducible, then there exists an M-smoothing for which (*) is sharp.*

Theorem 3. *If $C \subset \mathbb{R}^2$ is an algebraic curve of degree d, then*

$$2\int_C |k| \leq \int_{C^{\mathbb{C}}} K = 2\pi(d^2-2d+p) \qquad (**)$$

where p is the number of asymptotic directions.

Theorem 4. *For all d and for all $\epsilon > 0$, there exists an M-curve for which (**) is sharp.*

Iterating the real holomorphy ring and generalising Schmüdgen's Positivstellensatz

M. Schweighofer

Let A be a commutative $\mathbb{R}$–algebra of finite transcendence degree $d \in \mathbb{N}$. We denote by Sper A the real spectrum of A.

We investigate the relationship between the subring of *(geometrically) bounded* elements

$$H(A) := \{a \in A \mid \exists \nu \in \mathbb{N} : |a| \leq \nu \text{ on Sper } A\}$$

and the subring of *arithmetically bounded* elements

$$H'(A) := \{a \in A \mid \exists \nu \in \mathbb{N} : \nu + a \text{ and } \nu - a \text{ are sums of squares in } A\}.$$

Obviously, $H'(A) \subseteq H(A)$.

In 1991, Schmüdgen proved the remarkable theorem that

$$A = H(A) \implies A = H'(A)$$

if A is finitely generated.

In 1996, Becker and Powers considered the chain

$$A \supseteq H(A) \supseteq H^2(A) \supseteq H^3(A) \supseteq \dots$$

of iterated rings of bounded elements where $H^k(A)$ is inductively defined by $H^0(A) := A$ and $H^{k+1}(A) := H(H^k(A))$ for $k \in \mathbb{N}$. Their main result was that this chain becomes stationary, more exactly,

$$H^d(A) = H^{d+1}(A).$$

In 1998, Monnier related both results and conjectured

$$H^d(A) = H'(A)$$

which generalises both of them at the same time. We prove this conjecture and develop tools to study $H'(A)$.

One of the applications is the following: If $a \in A$ is "small at infinity" and $a \geq 0$ on Sper A, then $a + \varepsilon$ is a sum of squares in A for every $\varepsilon > 0$.

A geometric proof of the definability of Hausdorff limits in o-minimal structures

P. Speissegger

(joint work with J. M. Lion)

Let $\mathcal{S} = (\mathbb{R}, +, \cdot, 0, 1, \dots)$ be an o-minimal expansion of the real field, and let $A \subseteq \mathbb{R}^{m+n}$ be definable and bounded in $\mathcal{S}$. A theorem by Marker and Steinhorn implies that every Hausdorff limit of the family $(A_a)_{a \in \mathbb{R}^m}$ is definable in $\mathcal{S}$. We prove the following strengthening of this result, using geometric methods.

Theorem 1. *There are $N \in \mathbb{N}$, a compact definable $K \subset R^N$ and a definable function $f : \Pi_m(A) \mapsto K$ such that*

1. *$cl(f(\Pi_m(A))) = K$ and*
2. *whenever (a_i) and (b_i) are subsequences of $\Pi_m(A)$ such that $\lim a_i = \lim b_i = a$ and $\lim f(a_i), \lim f(b_i), \lim A_{a_i}$ and $\lim A_{b_i}$ exist, we have $\lim f(a_i) = \lim f(b_i) \Rightarrow \lim A_{a_i} = \lim A_{b_i}$.*

As a corollary, we recover the uniform versions of the Marker-Steinhorn theorem that were obtained by model-theoretic means by Pillay and van den Dries.

Root count bounds via differential inequalities

G. Stengle

Familiar examples of such bounds are

- $f^{(n)}(x) > 0 \Rightarrow f$ has no more than n zeros (Rolle's Theorem).
- $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$ positive definite $\Rightarrow f = c$ meets any line at most twice ($f \leq c$ is convex).

We give another example using the following scheme of Khovanski. The number of non degenerate solutions of $F = G = 0$ is no more than the number of solutions of $F = z, G = 0, \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} = 0$ plus the number of unbounded components of $F = z, G = 0$. Let $\mathcal{C}$ be the family of conics $Ax^2 + By^2 = 1$. Applying Khovanski's scheme twice with general F and $G = Ax^2 + By^2 - 1$ gives a system in which the equation $A^2x^2F_{yy} - AB(2F_{xy} + xF_x + yF_y) + B^2y^2F_{xx} = 0$ appears as one of the equations. Moreover in each stage of deriving a new system each curve is a graph over the curve G. This gives a bound of $1+1$ plus number of solutions of this derived equation. If this quadratic form is positive definite than this latter contribution to the count is 0 and we have a bound of two intersections in $(\mathbb{R}^+)^2$. The differential inequalities are $F_{xx}, F_{yy} \geq 0$ and $4x^2y^2F_{xx}F_{yy} \geq (2xyF_{xy} + xF_x + yF_y)^2$. To show that these conditions are not vacuous (not all sets of differential inequalities have solutions) we show that $u_\lambda(x, y) = x^{-\frac{1}{2}+\frac{\lambda}{2}}y^{-\frac{1}{2}+\frac{1}{2\lambda}}$ satisfies $2xyF_{xy} + xF_x + yF_y = 0$ and for $\lambda \notin [\frac{1}{3}, 3], (u_\lambda)_{xx}$ and $(u_\lambda)_{yy}$ are positive. Thus the cone generated by the $(u_\lambda)_{\lambda \notin [\frac{1}{3}, 3]}$ gives a family of functions with level sets in $(\mathbb{R}^+)^2$ which intersect conics in $\mathcal{C}$ no more than twice.

On the normal geometry of stratified spaces

D. Trotman

(joint work with P. Orro)

Let Z be a closed stratified subset of $\mathbb{R}^n$ and Y a stratum. The normal cone of Z along Y,

$$C_Y Z = \overline{\{(x, \mu(x-\pi(x)) : x \in Z \setminus Y\}}|_Y \subset Y \times S^{n-1}$$

where $\mu(v) = \frac{v}{\|v\|}$ and $\pi : T_Y \mapsto Y$ is the projection of a tubular neighborhood. We say that Z is normally situated along Y if (*) the fibre $(C_Y Z)_y = C_y(Z_y)$, the tangent cone to the fibre $Z_y = \pi^{-1}(y)$. Following Hironaka, we say Z is normally pseudo-flat (npf) along Y if $p : C_Y Z \mapsto Y$ is open.

Lemma 1. *If Z is subanalytic and $(a+n)$ regular or satisfies (npf), then* $\dim(C_Y Z)_y \leq \dim Z - \dim Y - 1$.

A theorem of Ferrarotti, Fortuna and Wilson (2000) says that any closed semialgebraic semicone can be realized as the tangent cone of a real algebraic variety. Hironaka (1969) proved for analytic Z that (n) and (npf) are satisfied by any Whitney (b)-regular stratification. J.P. Henry and M. Merle proved that $(b) \Rightarrow (n)$ for subanalytic Z. An example due to K. Bekka and myself show weak Whitney regularity is not enough for real algebraic strata. We give a new regularity condition, a quantitative version of Kuo's ratio test:

$$(r^e) : \frac{\|y_0 - \pi(x)\|^e \alpha(x)}{\|x - \pi(x)\|} \text{ is bounded near } y_0 \in Y,$$

where $\alpha(x) = \mathrm{dist}(T_x X, T_{\pi(x)} Y)$.

Then $(r^0) = (w)$, Verdier regularity. $(r^e) \not\Rightarrow (a)$ even for real algebraic sets.

Theorem 2. *If Z is $(a + r^e)$-regular, $0 \leq e < 1$, then Z is normally situated along Y.*

Note that $(a+n) \not\Rightarrow (npf)$: G. Valette gave a real algebraic example ($r = (z^2 + \sin^2\theta)\cos\theta$. Also $(a + npf) \not\Rightarrow (n)$: $y^2 = z^2x^2 + x^3$.

Theorem 3. *If Z is $(a + r^e)$-regular, $0 \leq e < 1$, then Z is normally pseudo-flat along Y.*

The proof uses a lift v of a vector field on a C^1 curve γ replacing Y (using that r^0 is preserved by transverse intersection), and study of the trajectories of v. In particular we show that each trajectory reaches $\pi^{-1}(y_0)$ without touching y_0, and that the lateral movement of secants above y_1, y_2 tends uniformly to 0 as y_1, y_2 tend to y_0. These results were useful in the Univ. of Provence thesis of G. Comte (Nice), and in his proof that the density (or Lelong number) of a subanalytic set is continuous along strata of a subanalytic (w)-regular stratification. In fact his theorem remains valid for a subanalytic $(a + r^e)$-regular stratification (cf his paper in the Annales de l'ENS Paris 2001).

Betti numbers of sub-Pfaffian sets

N. Vorobjov

(joint work with A. Gabrielov and T. Zell)

We suggest a method of proving upper bounds on Betti numbers of sets defined by first order formulae with quantifiers as functions of formats of the formulae. For semialgebraic sets the method gives an improvement of previously known bounds in case of one or two quantifier blocks. In the case of sub-Pfaffian sets we obtain a "singly-exponential" bound provided the number of quantifier alternations is fixed. The method uses a spectral sequence converging to the homology of the image of a surjective simplicial map.

Some applications of semialgebraic geometry in control theory

Y. Yomdin

(joint work with M. Briskin)

We consider a control problem

$$\dot{x} = f(x,u), x(0) = x_0, u \in U \tag{$*$}$$

in the plane. Using semialgebraic tools we show that the number of vertices of the time T-reachable set $\Omega(T)$ for $(*)$ grows at most exponentially in time. (A vertex is a boundary point such that locally Ω is contained in a cone of angle $< \pi$).

Edited by Andreas Bernig

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