Mathematisches Forschungsinstitut Oberwolfach

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Geometrie

September 29th – October 5th, 2002

The workshop was organized by V. Bangert (Freiburg), Yu.D. Burago (St. Petersburg) and U. Pinkall (Berlin).

The 47 participants came from 10 countries, about half of them from Germany and larger groups from the U.S.A., from Switzerland and from Great Britain.

The official program consisted of 22 lectures, and included three lectures by I. Babenko (Montpellier) on the "Topological nature of systolic rigidity" and two lectures by M. Bonk (Ann Arbor) on "Quasisymmetric and bi-Lipschitz parametrizations". The program covered a wide range of new developments in geometry. To name some of them we mention the topics "Submanifolds and integrable systems, including discrete ones and numerics", "Finsler geometry" and "Alexandrov geometry". Each participant had prepared an abstract of his/her recent research. These abstracts were posted and stimulated lively discussions.

Abstracts

Topological nature of systolic rigidity I-III

I. Babenko

The course is devoted to the recent progress in systolic geometry. First results (of the modern form) in systolic geometry were obtained more than 50 years ago (Loewner, Pu). Essential progress in this domain of geometry in the large was made 20 years ago (Gromov) who found the sufficient topological condition of systolic rigidity for 1-systoles (1-rigidity). The necessary condition for 1-rigidity was obtained 10 years later (Babenko). Higher-dimensional systols were introduced in the beginning of the 70-th (Berger). During more than 10 years there were no results in this direction. In the middle of the 80-th a few of particular results for stable higher-dimensional systols have been obtained (Gromov, Hebda). The general result for stable higher-dimensional systols was obtained only recently (Bangert, Katz). The problem of rigidity for higher-dimensional unstable systoles is completely different from the same one in stable case. The general situation here is systolic (or intersystolic) freedom. This case was developed significantly during the last 4 years in the series of works (Babenko, Katz, Suciu).

1. I. Babenko, *Asymptotic invariants of smooth manifolds,* (*Russian Acad. Sci. Izv. Math.*, Vol. 41, 1993, pp. 1-38). **2. I. Babenko, M. Katz**, *Systolic freedom of orientable manifolds,* (*Ann. scient. Éc. Norm. Sup.*, 4^e série, T 31, 1998, pp. 787-809). **3. I. Babenko, M. Katz, A. Suciu**, *Volumes, middle-dimensional systoles, and Whitehead products,* (*Math. Res. Lett.*, Vol 5, 1998, pp. 461-471). **4. I. Babenko**, *Forte souplesse intersystolique de variétés fermées et de polyèdres,* (*Annales de l'Institut Fourier*, Vol 52, 2002, fasc. 5). **5. I. Babenko**, *Nature topologique des systoles. Z_2-systoles unidimensionnelles,* (*Université Montpellier-II*, preprint n14, 2002). **6. V. Bangert, M. Katz**, *Stable systolic inequalities and cohomology products,* (*Communications on Pure and Appl. Math.* Vol 56, 2003). **7. C. Bavard**, *Inegalité isoperimétrique pour la bouteille de Klein,* (*Math. Annalen.*, Vol 274, 1986, pp. 439-441). **8. M. Berger**, *A l'ombre de Loewner,* (*Ann. scient. Éc. Norm. Sup.*, T 5, 1972, pp. 241-260). **9. M. Berger**, *Du côté de chez Pu,* (*Ann. scient. Éc. Norm. Sup.*, T 5, 1972, pp. 1-44). **10. M. Berger**, *Systoles et applications selon Gromov,* exposé 771 Séminaire N. Bourbaki 1992/93 (*Asterisque*, Vol 216, 1993, pp. 279-310). **11. M. Berger**, *Riemannian geometry during the second half of the twentieth century,* (*Jahresber. Deutsch. Math. -Verein*, Vol 100, 1998, pp. 45-208). **12. H. Federer**, *Real flat chains, cochains and variational problems,* (*Indiana Math. Journal*, Vol 24, 1974, pp. 351-407). **13. M. Gromov**, *Systoles and intersystolic inequalities,* (Actes de la table ronde de géométrie différentielle en l'honeur de Marcel Berger, *Collection SMF* n 1, 1996, pp. 291-362). **14. M. Gromov**, *Filling Riemannian manifolds,* (*J. Diff. Geom.* Vol 18, 1983, pp. 1-147). **15. M. Gromov**, *Metric structures for Riemannian end non-Riemannian spaces,* Birkhäuser, 1999. **16. J.J. Hebda**, *The collars of Riemannian manifolds and stable isosystolic inequalities,* (*Pacific J. of Math.*, Vol 121, 1986, pp. 339-356). **17. M. Katz**, *Counter-examples to isosystolic inequalities,* (*Geometriae Dedicata*, Vol 57, 1995, pp. 195-206). **18. M. Katz**, *Systolically free manifolds*, Appendix D to [15]. **19. M. Katz, A. Suciu**, *Volume of Riemannian manifolds, geometric inequalities, and homotopy theory*, in Rothenberg Festschrift, Contemporary Mathematics, AMS, 1999. **20 .M. Katz, A. Suciu**, *Systolic freedom of loopspaces*, (*Geometric and Functional Analysis*) *à paraitre.* **21. P.M. Pu**, *Some inequalities in certain non-orientable Riemannian manifolds,* (*Pacific J. of Math.*, Vol 2, 1952, pp. 55-71).

An Invariant based on the Yamabe operator

Christian Bär
(joint work with M. Dahl)

Let M^n be a closed differentiable manifold of dimension $n \geq 3$. Given a Riemannian metric g on M one defines the Yamabe Operator

$$L_g = \Delta_g + \frac{n-2}{4(n-1)} \, S\mathrm{cal}_g.$$

Denote the eigenvalues of L_g by $\mu_0(L_g) < \mu_1(L_g) \leq \mu_2(L_g) \leq \to +\infty$. Define $\kappa(M)$ to be the smallest $k \in \mathbb{N}_0$ such that for each $\epsilon > 0$ there is a Riemannian metric g_ϵ such that $|\mu_i(L_{g_\epsilon})| \leq \epsilon$, $i = 0, \ldots, k-1$ and $\mu_k(L_{g_\epsilon}) \geq 1$. Then $\kappa(M)$ is a differential topological invariant of M. It is positive if and only if M admits a metric of positive scalar curvature. If M is connected and has a scalar flat metric, then $\kappa(M) \leq 1$.

We bound $\kappa(M)$ from below and (in the simply connected case) from above by the α-genus of M. One crucial result is the inequality

$$\kappa(\tilde{M}) \leq \kappa(M)$$

if $\tilde{M}$ is obtained from M by surgery in codimension ≤ 3. This is not true for surgeries in lower codimension. The inequality allows to apply results from bordism theory.

Answering a question by Gromov we show that positivity of the operator

$$\Delta_g + c\, S\mathrm{cal}_g$$

has no topological significance for $0 < c < \frac{n-2}{4(n-1)}$.

Eigenvalues and holonomy

Werner Ballmann
(joint work with J. Brüning and G. Carron)

Let $\mathcal{M}$ be a closed Riemannian manifold of dimension n, let $E \to \mathcal{M}$ be a Hermitian vector bundle with a flat Hermitian connection ∇^E. Assume that the holonomy of E is irreducible and nontrivial. Recall that at each point of $\mathcal{M}$, the fundamental group at that point is generated by loops of length ≤ 2 diam $\mathcal{M}$. It follows that there is a constant $\alpha > 0$ such that the holonomy along each such loop satisfies for all $v \in E_x$

$$|H_c(v) - v| \geq \alpha \cdot |v|,$$

where x is the given point.

Theorem. *If $\lambda \geq 0$ is an eigenvalue of the connection Laplacien $(\nabla^E)^*\nabla^E$, then*

$$\sqrt{\lambda} \exp\{c_o \sqrt{\lambda + (n-1)\kappa} \; diam\mathcal{M}\} \geq \frac{\alpha}{2 \; diam\mathcal{M}},$$

where $c_0 = c_0(n, \sqrt{\kappa} \; diam\mathcal{M})$ and $Ric_{\mathcal{M}} \geq -(n-1)\kappa$.

In applications, the important feature is that diam$\mathcal{M}$ is in the denominator on the RHS. There is a similar version in the case where $R^E \neq 0$, but in the talk I only explained the proof in the flat case $R^E = 0$.

Projective planes, Severi varieties and spheres

Jürgen Berndt
(joint work with Michael Atiyah)

There is an elementary but very striking result which asserts that the quotient of the complex projective plane CP^2 by complex conjugation is the 4-dimensional sphere S^4. This result has first appeared in the literature without proof in a paper by Arnold in 1971, but Arnold himself attributes this result to Pontryagin. Various proofs of this result were given later by Kuiper, Massey, and others.

Recently Arnold and independently Atiyah and Witten proved that the quotient of the quaternionic projective plane HP^2 by a certain $U(1)$-action is a 7-dimensional sphere. Atiyah and Witten were motivated by the fact that the quaternionic projective plane contains several hypersurfaces with G_2-holonomy, which they used in their studies of M-theory dynamics on manifolds with this exceptional holonomy. Arnold's motivation has been of algebraic nature.

In the first part of our work we extend the above two results to the Cayley projective plane and provide a unifying proof for all three projective planes. Actually we have three different proofs: a theoretical one using group theory, and two proofs giving explicit diffeomorphisms using real Jordan algebras resp. real projective geometry.

Every projective plane over a normed real division algebra has a natural complexification. These complexified projective planes are precisely the four Severi varieties in complex projective spaces. For the real projective plane RP^2 this is naturally the complex projective plane CP^2 sitting inside CP^5 as the Veronese surface. In the second part of our work we extend the result relating CP^2 and S^4 to the other Severi varieties. The resulting fibrations exhibit some interesting interplay between complex algebraic geometry and differential geometry.

Literature:

M. Atiyah, J. Berndt: Projective planes, Severi varieties and spheres. math.DG/0206135.

Minimal surfaces from circle patterns: geometry from combinatorics

Alexander I. Bobenko
(joint work with T. Hoffmann and B. Springborn)

A circle packing is a configuration of disjoint discs which may touch but not intersect. It is a classical result by Koebe saying that for every triangulation of the sphere there is a packing of circles in the sphere such that circles correspond to vertices and two circles touch if and only if the corresponding vertices are adjacent. This circle pattern is unique up to Möbius transformations of the sphere. In [A. Bobenko, B. Springborn. Variational principles for circle patterns and Koebe's theorem. Arxiv: math.GT 0203250] this theorem is generalized in two directions. First we consider patterns of circles intersecting at arbitrary angles. Second, we consider not only circle patterns in the sphere but also in other surfaces of constant curvature. We present new variational principles for the circle patterns in Euclidean and hyperbolic surfaces. The functionals are given explicitly in terms of the dilogarithm function Li_2 of the radii of the circles. The convexity of the functional allows us to prove the existence and uniqueness.

Circle patterns in the sphere with intersection angles $\pi/2$ can be treated as Gauss maps of discrete minimal surfaces. This allows us to use our variational description of circle patterns to recover minimal surfaces from the combinatorics of their curvature lines. Examples include discrete catenoid, Enneper, Schwarz, Scherk surfaces.

Quasisymmetric and bi-Lipschitz parametrizations I+II

M. Bonk

A homeomorphism f between two metric spaces X and Y is called bi-Lipschitz if it distorts distances by an at most multiplicative amount. The homeomorphism is called quasisymmetric if relative distances are distorted in a controlled manor, more precisely, we require that there exists an increasing function $\eta : [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ such that for all distinct points $x, y, z \in X$ we have

$$\frac{|f(x) - f(z)|}{|f(x) - f(y)|} \leq \eta \left(\frac{|x - z|}{|x - y|} \right).$$

If there exists a bi-Lipschitz or quasisymmetric homeomorphism between two metric spaces, then they are called bi-Lipschitz or quasisymmetric equivalent, respectively. The basic question of this survey is how to characterize standard metric spaces like S^n or $\mathbb{R}^n$ up to bi-Lipschitz or quasisymmetric equivalence. This is related to Cannon's conjecture in the theory of Gromov hyperbolic groups or the Jacobian problem for quasiconformal maps. Even for $n = 2$ our basic question seems to be a very hard problem and only partial results are available. For example, in joint work with U. Lang the author settled a conjecture of J. Fu by proving the following

Theorem. *Let X be a complete surface homeomorphic to $\mathbb{R}^2$ with (possibly singular) Riemannian metric. If we have $\int_X K^+ dA < 2\pi$ and $\int_X K^- dA < \infty$ for the Gaussian curvature K, then X is bi-Lipschitz equivalent to $\mathbb{R}^2$.*

Curvature and global rigidity in Finsler manifolds

Patrick Foulon

We present some strong global rigidity results for Finsler manifolds. Following E Cartan's definition a locally symmetric Finslermetric is one whose curvature is parallel. These spaces strictly contain the spaces such that the geodesic reflections are local isometrics and also constant curvature manifolds. In case of negative curvature we prove that the locally symmetric Finsler metrics on compact manifolds are in fact Riemannian and this therefore extends Akbar Zadeh's rigidity result. We also survey some results on positive constant curvature where the situation is still far from being understood. We prove that on S^2 a reversible Finsler metric of $K = +1$ has its geodesic flow conjugate to the standard one. We also introduce an integral geometric condition for projective structures. We show that if the (axiomatic) projective structure on the space of geodesics fulfils this condition then the metric is Riemannian. But the (non-)existence of a non-Riemannian reversible metric with constant flag curvature on S^2 remains open. It should be quoted that there are many such non reversible metrics due to Bryant's work on S^n.

Decomposing hypersurfaces by mean curvature flow

Gerhard Huisken

(joint work with Carlo Sinestrari)

The motion of a hypersurface $F : M^n \to \mathbb{R}^{n+1}$ by mean curvature flow $\frac{d}{dt}F = \vec{H}$, leads to a large variety of finite time singularities. The lecture describes results on the structure of singularities under various curvature assumptions, in particular

(i) If the initial mean curvature of the surface is positive, then any smooth rescaling of a singularity is convex and can be split into a Euclidean part times a strictly convex solution of the flow. The strictly convex flow (in the limit) is homothetically shrinking if the singularity is type I, it is translating of the singularity is type II, i.e. if $\sup|\lambda_i|(T-t)^{\frac{1}{2}}$ is unbounded; here λ_i are the principal curvatures and T is final time.

(ii) If the $(n-1)$st elementary symmetric polynomial of the principal curvatures is positive, this remains so during the flow; in this case, if one of the principal curvatures is small compared to the mean curvature, i.e. $|\lambda_1| \leq \mu H$, the remaining principal curvatures must be close together near a singularity, i.e. $\sum\limits_{i,j\neq 1} |\lambda_i - \lambda_j| \leq \mu H + C_\mu$.

Using this description of singularities at the curvature level, we show that there is an axially symmetric "neck" present in the surface just before the first singular time. It is described how such a neck can be parametrised in a canonical way and a surgery can be performed which respects the a priori estimates above. It is an open problem to prove that only finitely many "necks" occur for a given initial surface.

Willmore spheres in $\mathbb{HP}^n$

Katrin Leschke

We are interested in transformations of conformal maps $f : M \to S^4$ which preserve Willmore surfaces. Using a quaternionic approach we consider, more generally, transformations of holomorphic curves in $\mathbb{H}P^n$ where a holomorphic curve in $\mathbb{H}P^n$ can be thought of as a family of branched conformal immersions. A Willmore surface in $\mathbb{H}P^n$ is a critical point of the Willmore energy under variations by holomorphic curves. By the Kodaira correspondence a holomorphic curve in $\mathbb{H}P^n$ corresponds to a basepoint free linear system of a quaternionic holomorphic line bundle. Thus conformal maps in S^4 with high-dimensional space of holomorphic sections arise from projections of holomorphic curves in $\mathbb{H}P^n$. In the talk I present two transformations (joint work with F. Pedit) of holomorphic curves. The Bäcklund transforms of a holomorphic curve arise from new linear systems which we get by integrating closed $(1,0)$-forms. In case of Willmore surfaces the Bäcklund transforms are again Willmore. Tangent curves of holomorphic curves arise by intersecting the tangent planes with a fixed hyperplane at infinity. The tangent curve of a Willmore sphere is Willmore. For Willmore spheres in $\mathbb{H}P^n$ we get a generalization of Bryant's theorem on Willmore surfaces in $\mathbb{R}^3$: Willmore spheres in $\mathbb{H}P^n$ arise from complex holomorphic data, and have integer Willmore energy.

Isoperimetric inequalities and Tits buildings

Enrico Leuzinger

Let $\mathcal{B}$ be a euclidean or hyperbolic building and let $G \subset \mathrm{Aut}\,\mathcal{B}$ be a locally compact unimodular group, which acts strongly transitively on $\mathcal{B}$. We use graphs $\mathcal{G}$, quasi-isometric to $\mathcal{B}$, to study asymptotic properties of quotients $\Gamma\backslash\mathcal{B}$, where Γ is a discrete subgroup of $\mathcal{G}$. If $\mathcal{G}$ has Kazhdan's property (T) we show that such quotients satisfy strong isoperimetric inequalities. This yields new examples of graphs with positive Cheeger constant. Such graphs cannot be bi-Lipschitz embedded into Hilbert space if they are infinite. Moreover, simple random walks on such quotients are shown to be recurrent if and only if Γ is a lattice in $\mathcal{G}$.

Spectral curves of tori in S^4

Franz Pedit

Over the past 15 years special surface classes, i.e., constant mean curvature tori, minimal tori, Willmore tori etc., have been studied from the view point of integrable systems. Each such torus came with an auxiliary Riemann surface, the so called *spectral curve*, and the torus could be constructed from theta functions on the spectral curve.

In this talk I explained two things:

1. Every conformally immersed torus in S^4 has a spectral curve (of possibly infinite genus) invariant under the Möbius group of S^4. If the spectral curve has finite genus, the corresponding immersed torus is given by theta functions.
2. The spectral curve is described geometrically as all closed "Darboux transforms" of the given immersed torus.

More precisely: let

$$f : T^2 \to S^4$$

be a conformally immersed torus in S^4. Then there is a marked Riemann surface (Σ, o) of possibly infinite genus and a map

$$F : T^2 \times \Sigma \to S^4$$

so that

1. For all $p \in T^2$ the map $F(p, -) : \Sigma \to S^4$ is a Willmore surface, in fact, the twistor projection of a holomorphic curve into $\mathbb{CP}^3$. The immersed torus and the spectral curve touch at $f(p) = F(p, o)$.
2. For all $\sigma \in \Sigma$ the map $F(-, \sigma) : T^2 \to S^4$ is a conformally immersed torus, a Darboux transform of f, whose Willmore energy equals that of f.

For the simplest example of the Clifford torus in S^3 the spectral curve is the Riemann sphere and all the maps $F(p, -)$ are Veronese embeddings in S^4.

Unstable periodic discrete minimal surfaces

Konrad Polthier

We use discrete surfaces to study the index of unstable minimal surfaces by evaluating the spectra of their Jacobi operators. Our numerical estimates confirm known results on the index of some smooth minimal surfaces like the trinoid and the Costa surface, and provide new estimates for a large number of triply period minimal surfaces. Also it gives additional information on the geometry of their area-reducing variations [2].

The investigation of the index requires the numerical computation of unstable discrete minimal surfaces with *excellent numerical qualities*, which is among the very challenging problems. Good quality means that the numerical solution is sufficiently accurate to allow further investigations of the numerical data set. For example, to study the second variation of the area functional and the index of minimal surfaces. Currently, good data sets of unstable discrete minimal surfaces are very hard to produce, even worse, they are hardly available to the community.

The essential ingredient in the algorithm is the introduction of the new *alignment energy* for non-conforming triangle meshes [1]. It turns out that minimizing the alignment energy in the class of *non-conforming* triangle meshes leads to non-conforming discrete minimal surfaces whose discrete conjugate surfaces are then solutions of free-boundary value problems for *conforming* triangle meshes. The new algorithm allows us the computation of many unstable periodic discrete minimal surfaces of highest numerical precision.

[1] K. Polthier. Unstable periodic discrete minimal surfaces. In S. Hildebrandt and H. Karcher, editors, *Geometric Analysis and Nonlinear Partial Differential Equations*, pages 127-143. Springer Verlag, 2002

[2] K. Polthier and W. Rossman. Discrete constant mean curvature surfaces and their index. *J. reine angew. Math.* 549(47-77), 2002.

The sphere theorem in Finsler geometry

Hans-Bert Rademacher

For a non-reversible Finsler metric F on a compact manifold we introduce the *reversibility*

$$\lambda = \max\{F(-X)|F(X) = 1\} \geq 1.$$

If $\lambda = 1$ then the metric is reversible, i.e. then $F(X) = F(-X)$ for all X. We show the following generalization of the classical sphere theorem in Riemannian geometry:

Theorem. *A simply-connected and compact Finsler manifold of dimension $n \geqq 3$ with reversibility λ and with flag curvature*

$$\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leqq 1$$

is homotopy equivalent to the n-sphere.

Products of hyperbolic metric spaces

Viktor Schroeder

(joint work with Thomas Foertsch)

Let $(X_i, d_i), i = 1, 2$ be proper, geodesic metric spaces which are hyperbolic in the sense of Gromov, and let $z_i \in X_i$ be chosen basepoints.

Define $Y = \{(x_1, x_2) \in X_1 \times X_2 | d_1(x_1, z_1) = d_2(x_2, z_2)\}$ and consider on Y the induced interior metric d.

The following result is joined work with Thomas Foertsch:

Theorem. *(Y, d) is also a proper, geodesic Gromov hyperbolic metric space and $\partial_\infty Y = \partial_\infty X_1 \times \partial_\infty X_2$.*

The result can be carried over to the limit case with $z_i \in \partial_\infty X_i$. There are applications of this result in the study of the "hyperbolic rank" of a metric space.

Quantization of curvature for compact surfaces in $S^n(1)$.

Udo Simon

(joint work with Haizhong Li)

Calabi's classification of all isometric minimal immersions of $S^2(K) \to S^{2+p}(1)$ stated, that $K = \frac{2}{s(s+1)}$ for some $s \in \mathbb{N}$ (1967). Lawson (1970) proved than a closed surface with Gauß curvature K satisfying $\frac{1}{3} \leqq K \leqq 1$, immersed minimally into $S^4(1)$, is totally geodesic or the Veronese surface. This led to the so called

QUANTIZATION CONJECTURE. If (M_2, g) is closed and isometrically, minimally immersed into $S^{2+p}(1)$ with $\frac{2}{(s+1)(s+2)} \leqq K \leqq \frac{2}{s(s+1)}$, then $K =$ const and the immersion belongs to Calabi's classification (U. Simon, 1979).

The conjecture is true for $s = 1, 2$ (Kozlowski-Simon, M.Z. 1984); for $s \geqq 3$ there are many results under additional assumptions. We prove a general integral formula which admits to extend the above result for $s = 1$ to large classes of immersions (e.g. Willmore surfaces; surfaces with parallel mean curvature vector). A typical result is:

Theorem. *Let $(M_2, g) \to S^{2p}(1)$ be a closed Willmore immersion satisfying a curvature control of the (intrinsic) Gauß curvature in terms of the (extrinsic) mean curvature normal $\vec{H}$, namely with $H^2 := \| \vec{H} \|^2$:*

$$\frac{1}{3} + H^2 \leqq K \leqq 1 + H^2.$$

Then either $p = 1, K = 1 + H^2$ and the surface is totally umbilical, or $p = 2, K = \frac{1}{3}$ and the surface is Veronese. A more general classification is based on the curvature control $(\vec{H} \neq 0)$

$$\frac{1}{4} \frac{|\text{ grad } \vec{H}|^2}{|\vec{H}|^2} \leqq 2K - 1 \leqq 1.$$

Essential for the proofs is the construction of a totally symmetric, traceless, vector valued $(3,0)$-form, W, using the position vector x of $M_2 \to S^{2+p} \hookrightarrow \mathbb{R}^{3+p}$.

$$\begin{aligned} W_{ijk} := x_{ijk} + \tfrac{1}{2}(1+K)g_{ij}x_k \quad &+ \quad \tfrac{1}{2}(1-K)(g_{ik}x_j + g_{jk}x_i) \\ &- \quad \tfrac{1}{2}(\vec{H}_k\, g_{ij} + H_i g_{jk} + \vec{H}_j\, g_{ik}) \end{aligned}$$

where (g_{ij}) denote local coefficients of the first fundamental form and x_{ijk} a local notation for covariant derivates. The integral formula and the assumptions lead to $W \equiv 0$; this gives PDEs for the classifications. In all but one case the results are optimal.

Constant mean curvature trinoids with bubbletons

Ivan Sterling

The first main result, which is for trinoids, states that for any trinoid it is possible to add bubbletons. The result is obtained as follows. We start with a known trinoid input data to the DPW method, gauging to a setting where the DPW integration process produces loops which can be explicitly calculated via hypergeometric functions and whose monodromies are given in terms of exponentials and Γ functions. Then we find the dressing matrix which dresses this loop to a loop which produces a closed trinoid. The next step is to compute the set of points in the loop parameter at which all the end monodromies have a common eigenline. Simple factor matrices can be constructed which will dress the given closed trinoid to a closed trinoid with a bubbleton on it.

Rope length criticality

John Sullivan

The rope length of a link L is ratio of length to thickness, where thickness $\tau(L)$ is the radius of the largest embedded normal tube. We have $\tau(L) = \inf_{x,y,z \in L} r(x,y,z)$, where $r =$ radius of circle through x, y, z.

Theorem. [Cantarella, Kusner, Sullivan: 2002 Inventiones Math.] *There is a rope length minimizer in any link type. It is $C^{1,1}$ but not necessarily C^2.*

All known minimizers come from:

Theorem. *If one component K of a link of thickness 1 is linked to n others, then len $(K) \geq C_n$.*

Examples: Hopf link - Each component has Len $= 4\pi$. Chain - Middle components are stadium curves.

But for a clasp (one rope attached to ceiling, linked to another attached to floor) the minimizer does not use semicircles! To understand it (and the related minimizer for Borromean rings) consider a notion of criticality for rope length: A link L is tight if for any variation v, $\delta_v \text{len}(L) < 0 \Rightarrow \delta_v \tau(L) < 0$. Assuming L has $\tau = 1$ and curvature $\kappa < 1$, its thickness is determined by Strut $(L) := \{(x,y) \in L \times L : (x-y) \perp T_x L, (x-y) \perp T_y L, \| x-y \| = 2\}$. Given any nonnegative Radon measure μ on Strut (L), we consider a vector-valued measure $\frac{1}{2}(x-y)d\mu(x,y)$ and then project this $L \times L \to L$. Then our (C+K+S+Joe Fu + N.Wrinkle) main theorem says L is tight $\Leftrightarrow \exists \mu$ on Struts such that the projected measure on L equals $-\kappa N ds$. That is, the tension force shrinking L can be balanced by compression forces on the struts.

Affine differential geometry in higher codimension

Martin Wiehe

Suppose that $x : M^n \to A^{n+p}$ is an immersion into the affine space A^{n+p} endowed with a fixed volume form. Furthermore σ is - at first - an arbitrary transversal space. We get

$$\overline{\nabla}_u \overline{\nabla}_v x = dx(\nabla_u v) + \overbrace{h(u,v)}^{\in\sigma}.$$

Generalizations of Blaschkes (Sl - or unimodularly invariant) theory for hypersurfaces to higher codimension were considered by

a) Burstin and Mayer 1927 ($n = 2, p = 2$)
b) Weise 1939: later Klingenberg 1951/52
c) Nomizu and Vrancken 1993 ($n = 2, p = 2$)

among others. The attempts as well as ours are based on regularity assumptions on the immersion. Unfortunately b) needs an additional regularity assumption. Using the regularity we introduce so-called pseudoinverse-elements $\tilde{h}$ of the fundamental form h. Their basic property is:

$$\tilde{h}^{ij}_{\rho} h^{\rho}_{j\eta} = p\delta^i_{\eta}, \tilde{h}^{ij}_{\rho} h^{\gamma}_{ij} = n\delta^{\gamma}_{\rho}.$$

It turns out that the condition trace $(\overset{\circ}{\nabla}\ \tilde{h}) = 0$ fixes uniquely a unimodular invariant transversal space $\sigma(u)$. (Here $\overset{\circ}{\nabla}$ is the "van der Waerden-Bartolotti" connection).

Moreover: Introducing a fundamental unimodular invariant $2p$-tensor field g with its volume form $\omega(g)$ we can show

(i) $\nabla\omega(g) = 0$ for the connection ∇ induced by $\sigma(u)$ (Hence ∇ is Ricci-symmetric).
(ii) The unimodular mean curvature vanishes iff the immersion is a critical point of the area functional induced by $\omega(g)$.

The transversal spaces of a) and b) do not obey these properties.

Torus actions on positively curved manifolds

Burkhard Wilking

The symmetry rank of a Riemannian manifold is one possible way to measure the amount of symmetry. It is defined as the rank of the isometry group

$$\text{symrank } (M,g) = \text{ rank } (\text{Iso}(M,g)).$$

We prove several structure results for positively curved manifolds with a large symmetry rank. Among them is the following

Theorem. *Let* (M^n, g) *be a positively curved manifold with* symrank $(M^n, g) \geq \frac{n}{4}+1, n \geq 10$. *Then* M^n *is homeomorphic to* S^n *or* $\mathbb{H}P^{n/4}$ *or homotopically equivalent to* $\mathbb{C}P^{n/2}$.

Edited by Victor Bangert

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