Mathematisches Forschungsinstitut Oberwolfach

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Miniworkshop: Quantization of Poisson Spaces with Singularities

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According to a philosophy going back to Dirac, the correspondence between a classical theory and its quantum counterpart should be based on an analogy between their mathematical structures. This idea has led to the problem of *quantization*, which is central in modern mathematical physics. Mathematically, quantization relates such diverse areas as differential geometry, functional analysis, algebraic topology, Lie theory, and representation theory.

A classical theory is described by its phase space and an algebra of functions defined on it, the classical observables, together with a Lie bracket which turns the algebra into a Poisson algebra. Dirac's fundamental observation was that the Poisson bracket is the classical analogon of the quantum mechanical commutator. An important mathematical insight of the last decades has been that the notion of quantization could be made mathematically precise in the language of deformation theory. A quantization of a Poisson algebra is then understood as a noncommutative deformation of the underlying commutative algebra of classical observables such that the commutator in the deformed algebra reproduces the Poisson bracket up to terms of higher order. A further aspect, namely how one should define and how one could construct Hilbert space representations of deformed Poisson algebras (possibly only in an asymptotic sense) is the objective of current research.

Traditionally, the phase space of a classical theory is assumed to be a smooth symplectic manifold, or, more generally, a smooth Poisson manifold, and various quantization schemes have been developed for that particular situation. In many cases of interest, however, the correct phase space has singularities. This is true for simple classical mechanical systems, as well as for the solution spaces of classical field theories with gauge or diffeomorphism symmetries. In particular, when the phase space is obtained by symplectic reduction, its singularity structure may be described by means of the notion of a stratified symplectic space. The natural question then arises what it means to quantize such a stratified symplectic space and whether general quantization schemes for spaces with singularities exist.

The workshop in Oberwolfach was aimed at recent results on the different approaches to quantization in connection with their relevance to the quantization of singular spaces. The goal of the workshop was to encourage cross-fertilization among various approaches, to bring seemingly independent lines of research together, and to achieve progress in the quantization theory of singular Poisson spaces. In 15 talks presented by participants and lively discussions the current state of research in the field has been evaluated and further developments have been initiated.

Abstracts

Asymptotic faithfulness of the quantum $SU(n)$ representations of the mapping class group

JØRGEN ANDERSEN

We prove that the sequence of projective quantum $SU(n)$ representations of the mapping class group obtained from the projective flat $SU(n)$-Verlinde bundles over Teichmüller space is asymptotically faithful, that is the intersection over all levels of these representations is trivial, whenever the genus of the underlying surface is at least 3. For the genus 2 case, we prove that this intersection is exactly the order of the subgroup generated by the hyperelliptic involutions, in case $n = 2$, and for $n > 2$ the intersection is trivial. The proof uses the BMS construction of the $\times$-products on Kähler manifolds generalized to the singular moduli spaces.

Morphisms, representations and reduction of star-products

MARTIN BORDEMANN

We give the definitions of morphisms, representations and reduction of star-products, and their relations to Poisson morphisms, coisotropic maps and commutants in the framework of deformation quantization. We show that a Poisson map between two symplectic manifolds can be quantized to a morphism of star-products in case the Atiyah-Molino class of the horizontal foliation defined by that map vanishes. This result implies that a symplectic star-product admits a representation on the functions on a coisotropic submanifold in case the Atiyah-Molino class of the corresponding foliation on that manifold vanishes. This is the case when a classical reduced phase space exists, in which case the star-product can be reduced under certain conditions for the Deligne classes.

The reduced phase space of the Poisson sigma model

ALBERTO CATTANEO

The Poisson sigma model is a topological string theory (or a topological two-dimensional field theory) whose perturbative path-integral quantization (around trivial critical points) yields Kontsevich's formula for the deformation quantization of a Poisson manifold (while the whole formality map appears as Ward identities of the model). In this talk I will describe the classical Hamiltonian approach to the same model. It turns out that the reduced phase space has naturally a groupoid structure that is compatible with the symplectic structure. In the nonsingular situations this is the "symplectic groupoid" introduced by Weinstein. This construction immediately suggests how to integrate any Lie algebroid to a topological groupoid. The possibility of getting a Lie groupoid has later been investigated by Crainic and Fernandes who, via this construction, obtained an if-and-only-if integrability criterion, thus solving a question open for over 40 years.

Lie group valued moment maps and presymplectic groupoids

Marius Crainic

This is based on joint work with H. Bursztyn, A. Weinstein, C. Zhu, and is strongly related to the work of Cattaneo–Felder–Xu and Crainic–Fernandes. The main motivation is to understand, relate and explain the relation between the following:

- the Lie group valued moment maps i.e. the 2-hamiltonian $\mathfrak{g}$-spaces of Alekseev–Malkin–Meinrenken (G - Lie group, $\mathfrak{g}$ -its Lie algebra, $(\cdot,\cdot)_{\mathfrak{g}}$ - invariant metric);
- the AMM-from $\omega_{\mathrm{AMM}} \in \Omega^2(G \times G)$ central to the theory above; this, and the fact that ω_{AMM} has the important property of being multiplicative in a certain sense was explained by Xu;
- the Cartan–Dirac structure $\mathfrak{h}_{\mathrm{Cartan}} \subset TG \oplus T^*G$ of Severa–Weinstein, and the Cartan 3-form $\varphi_{\mathrm{Cartan}} = \frac{1}{2}(\theta, [\theta, \theta])_{\mathfrak{g}} \in \Omega^3(G; \mathfrak{g})$
($\theta = g^{-1}dg \in \Omega^1(G; \mathfrak{g})$ is the Maureer–Cartan form).

The underlying structure is that of (twisted) Dirac-structures, which are generalizations of (twisted) Poisson structures. Inspired by the Poisson case we

(1) discuss (define and motivate) presymplectic realizations,
(2) find the global objects behind (i.e. integrate) Dirac structures.

The conclusions are:

- 2-hamiltonian $\mathfrak{g}$-spaces correspond to presymplectic realizations of the Cartan–Dirac structure,
- the integration of the Cartan–Dirac structure equals $(G \times G, \omega_{\mathrm{AMM}})$.

The main theorem is more general and refers to the infinitesimal counterpart of multiplicative two-forms. In particular, we conclude that at the root of the theory is

(1) the equivariant form $\varphi_{\mathrm{Cartan}} + \frac{1}{2}(v_r + v_l) \in \Omega^3_G(G)$,
(2) the guiding principle of multiplicativity and symplectic groupoids.

Symplectic connections of Ricci-type and reduction

Simone Gutt

The first part of the talk is devoted to the link between deformation quantization on a symplectic manifold and symplectic connections.

A star product $\star = \sum_{\nu \leq 0} \nu^k c_k$ is said to be natural, when the c_k are bidifferential operators of order at most k in each argument. To such a natural star product is associated exactly one symplectic connection. One studies equivalence, parametrization, invariance and quantum moment maps for natural star products.

The second part of the talk is devoted to the study of a class of symplectic connections, those of Ricci type, i.e. whose curvature is given only in terms of the Ricci tensor. It is shown that local models for analytic symplectic manifolds admitting an analytic Ricci type symplectic connection are given by a local version of the reduced space of a quadratic surface in flat vector space (this result is joint work with M. Cahen and L. Schwachhöfer).

Quantization in the presence of singularities

Johannes Huebschmann

The program of quantization turns the correspondence principle around and seeks to construct the quantum observables from the classical Poisson algebras. The standard treatment of symmetries as well as that of constraints entails that, on the classical side, singularities can hardly be avoided. In the talk, the significance of singularities for the standard quantization schemes has been discussed in view of the scientific program of the meeting, with a special emphasis whether and how singularities may appear directly on the quantum side, that is, not just as shadow of classical singularities.

Quantization as a functor

Klaas Landsman

Quantization is defined as a functor from the category of Weinstein dual pairs in the theory of symplectic and Poisson manifolds to the category KK in the theory of C*-algebras. It is shown how the Guillemin-Sternberg "[Q,R]=0" conjecture would follow from the functoriality of quantization. The formalism of KK-theory immediately shows how this conjecture can be generalized to noncompact groups.

Fourier integral projections, deformations and index theorems

Ryszard Nest

Given (X, ω) a symplectic manifold, suppose that $\mathcal{A}(X)$ is its formal deformation and Γ is a pseudogroup of automorphisms of $\mathcal{A}(X)$. Let τ be the unique trace on $\mathcal{A}(X)$ localized at the identity of Γ. The following holds for all $e \in K_0(\mathcal{A}(X) \rtimes \Gamma)$:

$$\langle \mathrm{ch}(e), \tau \rangle = \int_X \mathrm{ch}_\Gamma(e_0)\, \mathrm{Td}_\Gamma(X) e^{\theta_\Gamma},$$

where e_0 is the $\hbar = 0$-component of the projection, $\mathrm{Td}_\Gamma(X)$ is the equivariant Todd-genus and θ_Γ is a characteristic class of the deformation, both in $H^*_\Gamma(X)$.

The above has direct applications for the computation of the index of Fredholm Fourier integral operators associated to coisotropic cones in cotangent bundles.

Manifolds with a Lie structure at infinity

Victor Nistor

A manifold with a Lie structure at infinity is a manifold whose geometry is described by a compactification to a manifold with corners and a Lie algebra of vector fields on this compactification, satisfying a series of axioms. These axioms were suggested by Melrose's work on geometric scattering theory. In my talk I will present some results on the geometry and analysis on manifolds with a Lie structure at infinity, beginning with some motivation from analysis on singular spaces. The analytic results depend on a "quantization" of the defining Lie algebra of vector fields. The results are based on a joint work with B. Amann and R. Lauter.

On the deformation quantization of symplectic orbispaces

Markus Pflaum

In the first part of the talk we provide a geometrically oriented approach to the theory of orbispaces originally introduced by G. Schwarz and W. Chen. We explain the notion of a vector orbibundle and characterize the good sections of a reduced vector orbibundle as the smooth stratified sections. In the second part we elaborate on the quantizability of a symplectic orbispace. By adapting Fedosov's method to the orbispace setting we show that every symplectic orbispace has a deformation quantization. As a byproduct we obtain that every symplectic orbifold possesses a star product.

On the Berezin transform

Martin Schlichenmaier

For compact Kähler manifolds the Berezin transform relating the suitable extended contravariant symbols with the suitable extended covariant symbols is introduced. This is done via Berezin-Toeplitz quantum operators and coherent states à la Berezin-Rawnsley. In joint work with Alexander Karabegov I showed that the Berezin transform $I^{(m)}$ has a complete asymptotic expansion in terms of the tensor power m of the quantum line bundle, i.e.

$$I^{(m)}f(x) \sim \sum_{k=0}^{\infty} I_k(f)(x)m^{-k}, \quad m \to \infty,$$

with differential operators I_k. It turns out that $I_0 = id$ and $I_1 = \Delta$ (the Laplacian). This has consequences for the properties of the Berezin-Toeplitz deformation quantization. Some comments on the relation between the quantum Hilbert space with the projective coordinate ring of the via the coherent state embedded Kähler manifold are given. Some remarks on the singular situations are given.

Stratification of the gauge orbit space for gauge group $SU(n)$

Matthias Schmidt

The configuration space of a pure Yang-Mills theory is given by the space of the orbits of the gauge group acting on connections. For non-Abelian gauge group, this space is stratified by orbit types. I will present a method to determine the orbit types in terms of characteristic classes. Moreover, I will propose a framework for a systematic investigation of the (quantum) physical effects of the stratification.

Spin-c quantization and reduction

Reyer Sjamaar

This is a survey of work by Meinrenken, Vergne and others on the equivariant index of the Dolbeault-Dirac operator on a Hamiltonian G-manifold, where G is a compact Lie group. In particular, for $G = S^1$, the circle, I give a simple proof of their result that the index of the symplectic quotient is equal to the invariant part of the equivariant index ("quantization commutes with reduction"). I also show how to extend this result to the case of a singular symplectic quotient (joint work with Eckhard Meinrenken).

Poisson Reduction and Quantization for Proper Actions

Jedrzej Sniatycki

I shall compare the Poisson reduction technique due to Arms, Cushman and Gotay, and Sjamaar and Lerman with that of Sniatycki and Weinstein. I shall also discuss quantization of an example, due to Arms, Gotay and Jennings, in which both reduction techniques give inequivalent results.

Strong Morita Equivalence in Deformation Quantization

Stefan Waldmann

In my talk I will review the ideas of strong Morita equivalence in the algebraic framework of *-algebras over a ring $\mathsf{C} = \mathsf{R}(\mathrm{i})$ where R is ordered and $\mathrm{i}^2 = -1$. I shall discuss the classification of strongly Morita equivalent star products on a symplectic manifold and point out some more recent results concerning the Picard groups of deformed algebras.

Edited by Markus Pflaum

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