Mathematisches Forschungsinstitut Oberwolfach

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Topological and Geometric Combinatorics

April 6th – April 12th, 2003

The 2003 Oberwolfach meeting "Topological and Geometric Combinatorics" was organized by Anders Björner (KTH Stockholm), Gil Kalai (Hebrew University, Jerusalem), and Günter M. Ziegler (TU Berlin).

It has shown an exciting mix of algebraic, topological and geometric methods, tools and intuitions that interact productively on the stage of combinatorics. For example, ideas related to positive as well as to negative curvature have growing influence and impact on purely combinatorial questions. This was visible on this workshop for example on the lectures by Jon McCammond, Ezra Miller, Julian Pfeifle, and John Shareshian.

Similarly, the combinatorial construction of characteristic classes as well as obstruction classes provide both great challenges as well as effective tools. The survey lectures by Laura Anderson and by Robin Forman presented very different approaches in this area, the first one very "high-power," the other one distinctly and intentionally elementary. The report by Dmitry Kozlov on his resolution (joint with Eric Babson) of Lovász' conjecture on Hom-complexes is a striking instance of the obstruction classes methodology in combinatorics.

There is no way to compress the richness of the conference's program into a one page report. Enclosed you will find the abstracts from both the 10 one hour lectures (with a greater survey component) as well as the 21 half-hour lectures of the conference. All the additional smaller presentations, group discussions and blackboard meetings, some of them until deep into the night, cannot be recorded here at all — many of them will bear fruit in future research, publications and collaborations.

We are very grateful to the Oberwolfach institute, its director and all of its staff for providing the facilities and an ideal environment for high-power, intensive work during this week.

Anders Björner, Gil Kalai, Günter M. Ziegler
Stockholm / Jerusalem / Berlin, May 2003

Abstracts

Monday, April 7, 2003

Topological lower bounds for the chromatic number

Jiří Matoušek
(joint work with Günter M. Ziegler)

The Kneser conjecture is concerned with the chromatic numbers of certain graphs. These graphs are quite special, but the proof methods that are known for it, starting with Lovász's breakthrough from 1978, extend beyond the original examples: Each of them yields, explicitly or implicitly, a lower bound for the chromatic number of any graph (although these bounds are rather weak for some classes of graphs). The underlying ideas of all the proofs are topological, based on the Borsuk–Ulam theorem and its extensions. We briefly survey the various lower bounds (based on the work of Lovász, Bárány, Dol'nikov, Sarkaria, Kříž, and others) and we describe relations among them. It turns out that the known bounds are almost linearly ordered by strength, the strongest one being essentially Lovász' original bound in terms of a neighbourhood complex. A recent concept, the Hom-complex due to Lovász, may be still more powerful.

New results on combinatorial differential manifolds and matroid bundles

Laura Anderson

Combinatorial differential (CD) manifolds and matroid bundles are combinatorial analogs to differential manifolds and real vector bundles, in which the role of real vector spaces is played by oriented matroids. They support natural combinatorial analogs to many fundamental tools in real vector bundle theory. Most notably, there are natural "combinatorial Grassmannians," called *MacPhersonians*, leading to a classifying space for matroid bundles.

CD manifolds and matroid bundles were first developed by Gelfand and MacPherson, as the key tool in their combinatorial formula for the rational Pontrjagin classes of differential manifolds. Since then research has centred on clarifying the relation between real vector bundles and matroid bundles. This question was recently settled spectacularly by Daniel Biss, who proved that the combinatorial Grassmannians are homotopy equivalent to their real counterparts, and thus *the theory of matroid bundles is equivalent to the theory of real vector bundles.*

This talk surveyed CD manifolds, matroid bundles, and recent results in the area, including Biss's result and related progress on smoothing CD manifolds.

Wreath products of polytopes and spheres

Frank Lutz
(joint work with Michael Joswig)

We define a wreath product $P \wr Q$ for polytopes P and Q with the property that the wreath product $\mathrm{Aut}P \wr \mathrm{Aut}Q$ of the automorphism groups $\mathrm{Aut}P$ and $\mathrm{Aut}Q$ of P and Q acts as a (large) group of automorphisms on $P \wr Q$.

An analogous construction can be carried out for (finite) simplicial complexes, in particular for simplicial spheres, respecting various properties such as vertex-decomposability, shellability, and constructibility.

Moreover, wreath products of vertex-transitive triangulations of homology spheres with simplices yield examples of vertex-transitive non-PL spheres.

Short rational generating functions for lattice point problems

Alexander Barvinok
(joint work with Kevin Woods)

We consider a hierarchy of sets of lattice points defined by formulas of the Presburger arithmetics. Thus a formula without quantifiers defines the set of lattice points in a polyhedron and a formula without quantifier alternations defines a projection of such a set. Examples include integer semigroups, Hilbert bases of rational cones, etc. For a set S of lattice points we consider the generating function $f(S; x) = \sum_{m \in S} x^m$. For a fixed dimension d, for any set S of lattice points defined by a formula with no quantifier alternations, the function $f(S; x)$ can be written as a "short rational function" (the motivating example: if $S = \{0, 1 \ldots, n\}$ then $f(S; x) = \sum_{m=0}^{n} x^m = (1 - x^{n+1})/(1 - x)$). Thus various algorithmic questions for such sets admit polynomial time algorithms.

Discrete Morse theory on poset order complexes

Patricia Hersh

Discrete Morse theory was introduced by Robin Forman in the 90's as a way of studying CW complexes by collapsing them onto smaller, simpler-to-understand complexes of critical cells with the same homotopy type. In analogy to traditional Morse theory, the number of i-dimensional critical cells, denoted m_i, gives an upper bound on the i-th Betti number. Discrete Morse functions have proven quite useful for studying simplicial complexes arising in combinatorics (e.g. various graph complexes), yielding properties such as Cohen-Macaulayness, homotopy equivalence to a wedge of spheres of a single dimension, connectivity lower bounds, etc.

In joint work with Eric Babson, we showed how to construct a discrete Morse function with a relatively small number of critical cells for the order complex of any finite poset, based on any lexicographic order on its saturated chains. Applications include:

(1) (joint with E. Babson) Π_n/S_λ is homotopy equivalent to a wedge of spheres of top dimension, at least for hook shapes λ. It remains open whether this is true for all λ.
(2) A new simple proof that all intervals in the weak order for the symmetric group are homotopy equivalent to balls or spheres.

(3) (joint with V. Welker) A lower bound on the connectivity of intervals in monoid posets, in terms of the degree d of a Gröbner basis for a related toric ideal of syzygies (connectivity is essentially bounded below by the rank of the interval divided by $d-1$).
(4) Cohen-Macaulayness of the poset Π_{S_n} of partitions of $\{1,\ldots,n\}$ into cycles. (This poset was recently defined by Jeff Remmel.)

All of these applications came from lexicographic discrete Morse functions, and all but the second one also involved optimization by gradient path reversal.

Tuesday, April 8, 2003

The role of curvature in combinatorics

JON MCCAMMOND

Geometric combinatorics and geometric group theory share a common fascination with the topological properties of polyhedral complexes, and in particular the topology of order complexes of posets. In addition, quite a bit of recent work in geometric group theory has focused on adding a metric structure to polyhedral complexes in order to simplify the analysis of their structure. The resulting theory of 'non-positively curved' or CAT(0) spaces is now one of the central themes in present day geometric group theory.

Unfortunately, most of the existing results rely on cubes or other polyhedra with uniformly large dihedral angles in order to get started with the analysis. In a series of recent articles, my various co-authors and I (John Meier, Noel Brady, Tom Brady and Murray Elder) have been working to extend these technical results to simplicial complexes, and in particular to order complexes of posets. At this point there are two results that might be of interest to geometric combinatorialists. The first is that once you decide on a method of metrizing an order complex of a graded poset, there exists some combinatorial list of conditions which determines whether a particular poset leads to metric order complex with good curvature properties, and second that for order complexes of small dimension, and a 'natural' metric involving Coxeter shapes, we can explicitly describe these combinatorial conditions.

Topological obstructions to graph colourings

DMITRY KOZLOV

(joint work with Eric Babson)

For any two graphs G and H Lovász has defined a cell complex $\mathtt{Hom}(G,H)$, having in mind the general program that the algebraic invariants of these complexes should provide obstructions to graph colourings. Here we announce the proof of a conjecture of Lovász concerning these complexes with G a cycle of odd length.

More specifically, we show that

If $\mathtt{Hom}(C_{2r+1},G)$ *is k-connected, then* $\chi(G)\geq k+4$.

Our actual statement is somewhat sharper, as we find obstructions already in the non-vanishing of powers of certain Stiefel-Whitney characteristic classes.

Symmetric products of amoebas; Arrangements, diagrams, end spaces, commutative $(m+k,m)$-groups

Rade Živaljević

(joint work with Pavle Blagojević and Vladimir Grujić)

We study the combinatorics and topology of general arrangements of subspaces of the form $D + SP^{n-d}(X)$ in symmetric products $SP^n(X)$ where $D \in SP^d(X)$. Symmetric products $SP^m(X) := X^m/S_m$ appear in different areas of mathematics and mathematical physics as orbit spaces, divisor spaces, particle spaces, Eilenberg Mac Lane spaces etc. Our approach is based on the topological technique of *diagrams of spaces*. Along the lines of Ziegler-Živaljević '93 and Welker-Ziegler-Živaljević 99, we calculate the homology of the union and the complement of these arrangements. As an application we include a computation of the (co)homology of the *homotopy end space* of the open manifold $SP^n(M_{g,k})$, where $M_{g,k}$ is a (g,k)-amoeba, i.e. a Riemann surface of genus g punctured at k points. For example if $\Delta_d^{n,g}$ is the $(2n-d)$-dimensional *end cohomology* group of $SP^n(M_{g,k})$, then

$$\Delta_d^{n,g} = \begin{cases} \binom{2g+k-1}{d-1}, & d \leq n-1 \\ \binom{2g+k}{n} - \binom{2g}{n}, & d = n \text{ or } d = n+1 \\ \binom{2g+k-1}{2n-d}, & d \geq n+2. \end{cases} \tag{1}$$

These results allow us to discuss the following questions:

(A) To what extent is the topology of an amoeba determined by the topology of its symmetric product $SP^m(M)$ for a given m?

(B) Are there examples of non-homeomorphic amoebas M and N such that the associated symmetric products $SP^m(M)$ and $SP^m(N)$ are homeomorphic?

(C) Which surfaces support the structure of a continuous, commutative $(m+k,m)$-group?

Constructing toric flag diagrams

Pavle Blagojević

The toric variety X_Σ can be seen as a homotopy co-limit of an appropriate diagram $X : L_\Sigma \to Top$. Working with the homotopy lemma allows us to modify original diagrams. Instead of working with affine toric varieties of dual cones we work with affine toric varieties of its co-spans. The central property, which is responsible for the good behaviour of toric variety is the following.

For every "boolean subdiagram" of an affine toric variety $X|_{\mathcal{B}_k} : \mathcal{B}_k \to Top$ there exists a diagram morphism

$$\Gamma : X|_{\mathcal{B}_k} \to J \times (\mathbb{C}^*)^m$$

which is a homotopy equivalence on each level. The first consequence of this fact is that homotopy co-limit of toric variety hocolim$X$ can be recovered from the Λ-tree of the form

$$\begin{array}{ccccccc} pt \longleftarrow & S^{2s_2-1} \text{ or } S(2s_2-1) & \longleftarrow & \ldots & \longleftarrow & S^{2s_k-1} \text{ or } S(2s_k-1) \\ & \downarrow & & & & \downarrow \\ & pt & & & & pt \end{array}$$

where $S(i)$ denotes some i-dimensional $\mathbb{Q}$-sphere.

The aim of this talk is to explain how one can construct topological spaces with a torus action for broad families of geometric objects. For example, if K is a shellable d-dimensional polytopal complex in $\mathbb{R}^d$, we can construct a space X_K with the property

$$H_i(X_K, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{h_{i/2}}, \text{ if } i \text{ even} \\ 0, \text{ if } i \text{ is odd} \end{cases}$$

where $(h_0, h_1, .., h_i, ..)$ is the generalized h-vector of the first barycentric subdivision of K.

Question: Can these spaces help us in proving the generalized h-hypothesis? It appears that they have the right homology.

Topological representations of matroids

Edward Swartz

Topological representations of matroids usually consist of realizing the geometric lattice of the matroid as the intersection poset of an arrangement of real hyperplanes, complex hyperplanes or pseudospheres. We examine two more recent types of representations: 1) arrangements of homotopy spheres, 2) linear quotients of spheres by real and $\mathbb{Z}_p$-tori.

A S-homotopy sphere is a d-dimensional CW-complex homotopy equivalent to S^d. Surprisingly, the class of intersection posets of arrangements of homotopy spheres coincides with the class of geometric lattices. In addition, these arrangements retain many of the combinatorial properties of hyperplane and pseudosphere arrangements.

A completely different use of matroids appears in linear quotients of spheres by real and $\mathbb{Z}_p$-tori. Let Γ be a subgroup of $O(n)$ isomorphic to $(\mathbb{Z}_p)^r$ or $(S^1)^r$, and consider the quotient space S^{n-1}/Γ. It is possible to assign a natural matroid M_X to X which carries much of the geometric and topological properties of X. For instance, when p is 2 or 3 then M_X determines X up to isometry. In all cases M_X determines $H_*(X)$.

The polytope of non-crossing graphs on a planar point set

Francisco Santos

(joint work with David Orden)

Theorem. *Let $A = \{p_1, \dots, p_n\}$ be a finite set of n points in the plane, not all contained in a line. Let n_i, n_s and n_v be the number of interior, semi-interior and extremal points of A, respectively.*

There is a simple polytope $Y_f(A)$ of dimension $2n_i + n - 3$, and a face F of $Y_f(A)$ (of dimension $2n_i + n_v - 3$) such that the complement of the star of F in the face-poset of $Y_f(A)$ equals the poset of non-crossing graphs on A that use all the convex hull edges.

We explicitly give facet equations for the polytope. It lives in $\mathbb{R}^{3n}$ and is defined by the following equations, where we represent an element of $\mathbb{R}^{3n}$ as $(v_1, \dots, v_n; t_1, \dots, t_n)$, $v_i \in \mathbb{R}^2$ and $t_i \in \mathbb{R}$. The v_i's are interpreted as infinitesimal velocities of the points p_i.

- Three linear equalities on the v's ruling out any non-zero trivial motion of the whole set. For example, set $v_1 = 0$ and require v_2 to be parallel to $p_1 - p_2$.
- For each i, the inequality $t_i \geq 0$, with equality if p_i is in the boundary of conv(A).

- For each pair $p_i, p_j \in A$, the inequality

$$\langle p_i - p_j, v_i - v_j \rangle - |p_i - p_j|(t_i + t_j) \geq f_{ij},$$

  with equality if the segment $p_i p_j$ lies in the boundary of $\mathrm{conv}(A)$. Here, the f_{ij}'s are real numbers which have to be chosen appropriately. One valid choice is $f_{ij} = (\det(O, p_i, p_j))^2$.

It follows from our description that the polytope has $\binom{n}{2} + n - 2n_v$ facets. The proof of our statement uses some infinitesimal rigidity theory and the combinatorial properties of pseudo-triangulations, a generalization of two-dimensional triangulations which is becoming a standard tool in Computational Geometry. Actually, our methods also prove that:

Theorem. *The vertex set of the polytope $Y_f(A)$ is in bijection to the set of all pseudo-triangulations of A. The 1-skeleton of $Y_f(A)$ is the graph of flips between them.*

Projectivities in simplicial complexes

Michael Joswig

(joint work with Ivan Izmestiev)

We start by associating a finite group to each facet of a finite-dimensional simplicial complex, the *group of projectivities*. For strongly connected complexes the isomorphism class of the group does not depend on the facet chosen. It can be shown that, e.g., for combinatorial manifolds, in order to determine the group of projectivities, it suffices to have combinatorial information about the fundamental group plus local combinatorial data. This approach generalizes results by Heawood, Edwards et al., Rybnikov, and others. Specializing this result to the (boundary complexes) of simplicial polytopes says:

Theorem. *The dual graph of simplicial d-polytope is d-colourable if and only if each face of codimension 2 is contained in an even number of facets.*

It turns out that the group of projectivities of a simplicial complex K can be interpreted as the monodromy group of a branched covering, called the *partial unfolding*, with base space K. The *complete unfolding* is the regularization of the partial unfolding; it can also be defined combinatorially. It is essential that the unfoldings depend on the combinatorial properties of K, not only on the topology.

The main result is the following:

Theorem. *Each closed oriented 3-manifold arises as the partial unfolding of some triangulation of the 3-sphere.*

New features in *Cinderella* 2.0

Jürgen Richter-Gebert

Presentation of the interactive geometry software *Cinderella*.
See `http://www.cinderella.de/`

Wednesday, April 9, 2003

Using matchings to compute intersection cohomology

Francesco Brenti

The purpose of this talk has been to introduce a new kind of complete matchings of a partially ordered set, which we have called special, and to show how these can be used to compute the intersection cohomology of Schubert and toric varieties. More precisely, a complete matching M of the Hasse diagram of a partially ordered set P is *special* if, for all $x, y \in P$, such that $M(x) \neq y$, we have that $x \lhd y \Rightarrow M(x) \leq M(y)$. We have shown how these matchings give a completely combinatorial (poset theoretic) procedure to compute the intersection cohomology of Schubert varieties. In particular, our result shows that the local intersection cohomology (with complex coefficients) of a Schubert variety depends only on the closure relations of the Schubert cells contained in it, considered as an abstract poset. We have then shown how completely analogous results hold also for toric varieties.

Narayana numbers for Weyl groups

Christos Athanasiadis

The classical Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ count the number of lattice paths in $\mathbb{R}^2$ from $(0,0)$ to (n,n) which stay below the diagonal $y = x$ and have k North-East corners. We discuss a generalization of these numbers to any irreducible crystallographic root system Φ. These numbers have at least four interesting combinatorial interpretations. They count antichains in the root poset of Φ by cardinality, orbits of the action of the Weyl group W in the finite torus by stabilizer rank, elements of the W-noncrossing partition lattice by rank and appear as the entries of the h-vector of the generalized associahedron corresponding to Φ. We explain a case-free proof of the equivalence of the first two interpretations, conjectured by S. Fomin and J. Stembridge and, independently, by F. Chapoton.

Lattice points of polyhedra for tableaux and plane partitions

Jesús A. De Loera

(joint work with T. B. McAllister)

It has been already observed by many authors that polyhedra play a special role in combinatorial representation theory. In this lecture we answer two open questions about semi-standard Young tableaux and plane partitions from the point of view of polyhedral geometry. In particular, we show explicit counterexamples to a conjecture raised by Berenstein and Kirillov.

Let λ be a partition and μ be a composition of the same size and whose lengths do not exceed n. Let $GT(\lambda,\mu)$ be the convex polytope in the space $\mathbb{R}^{\frac{n(n+1)}{2}}$ of all points $\mathbf{x} = (x_{ij})_{1\leq i\leq j\leq n}$ satisfying the following conditions

- $x_{in} = \lambda_i$, $1 \leq i \leq n$;
- $x_{i,j+1} \geq x_{ij}$, $x_{ij} \geq x_{i+1,j+1}$, for all $1 \leq i \leq j \leq n-1$;
- $x_{11} = \mu_1$, and if $2 \leq j \leq n$, then $\sum_{i=1}^{j} x_{ij} - \sum_{i=1}^{j-1} x_{ij-1} = \mu_j$.

The polytope $GT(\lambda,\mu)$ is called *Gelfand–Tsetlin polytope.*

Theorem. *Dimension* $\dim V_\lambda(\mu)$ *of the weight* μ *subspace* $V_\lambda(\mu)$ *of the irreducible highest weight* λ *representation* V_λ *of the general linear algebra* $gl(n)$ *is equal to the number of integer points in the Gelfand–Tsetlin polytope* $GT(\lambda, \mu)$*:*

$$\dim V_\lambda(\mu) = \#|GT(\lambda, \mu) \cap \mathbb{Z}^{n(n+1)/2}|.$$

Conjecture: (Berenstein & Kirillov 1995) Let λ and μ be partitions. All vertices of the Gelfand–Tsetlin polytope $GT(\lambda, \mu)$ have integer coordinates, i.e. $GT(\lambda, \mu)$ is a convex integral polytope.

In this talk we present a counterexample for this conjecture for $n \geq 5$ and sketch the proof for smaller values. The conjecture seemed to have been motivated by the fact that, for an integer parameter ℓ, the Kostka number $K_{\ell\lambda,\ell\mu}$ is a polynomial in ℓ with integer coefficients, which was proved by using some fermionic formulas. Thus we get some fascinating implication for the theory of Ehrhart functions that count lattice points in polytopes.

Lifting inequalities for polytopes

Richard Ehrenborg

The f-vector enumerates the number of faces of a convex polytope according to dimension. The flag f-vector is a refinement of the f-vector since it enumerates face incidences of the polytope. To classify the set of flag f-vectors of polytopes is an open problem in discrete geometry. This was settled for 3-dimensional polytopes by Steinitz a century ago. However, already in dimension 4 the problem is open.

I will discuss the known linear inequalities for the flag f-vector of polytopes. These inequalities include the non-negativity of the toric g-vector, that the simplex minimizes the *cd*-index, and the Kalai convolution of inequalities.

I will introduce a method of lifting inequalities from lower-dimensional polytopes to higher dimensions. As a result we obtain two new inequalities for 6-dimensional polytopes.

Wednesday afternoon: Excursion to St. Roman

Thursday, April 10, 2003

Recent progress on algebraic shifting

Isabella Novik

(joint work with Eric Babson and Rekha Thomas)

Algebraic shifting introduced by Gil Kalai is an algebraic operation that given a simplicial complex Γ produces a shifted complex $\Delta(\Gamma)$. This new complex has a simpler combinatorial structure, yet it shares with Γ several combinatorial, topological, and algebraic properties such as face numbers, (topological) Betti numbers, extremal (algebraic graded) Betti numbers, etc. In the talk I will survey existing results and will present several new ones on algebraic shifting and their applications to combinatorics.

Abelianizing the real permutation action via blowups

Eva-Maria Feichtner
(joint work with Dmitry N. Kozlov)

We present an abelianization of the permutation action of the symmetric group $\mathcal{S}_n$ on $\mathbb{R}^n$ in analogy to the Batyrev abelianization construction for finite group actions on complex manifolds. The abelianization is provided by a particular De Concini-Procesi wonderful model for the braid arrangement. In fact, we show a stronger result, namely that stabilizers of points in the arrangement model are isomorphic to direct products of $\mathbb{Z}_2$. To prove that, we develop a combinatorial framework for explicitly describing the stabilizers in terms of automorphism groups of set diagrams over families of cubes.

We observe that the natural nested set stratification on the arrangement model is not stabilizer distinguishing with respect to the $\mathcal{S}_n$-action, that is, stabilizers of points are not in general isomorphic on open strata. Motivated by this structural deficiency, we furnish a new stratification of the De Concini-Procesi arrangement that distinguishes stabilizers.

Unfolding convex polyhedra in many dimensions

Ezra Miller

Let S be the d-dimensional boundary of a convex polyhedron P of dimension $d+1$. There exists a polyhedral complex $\overline{K}$ inside of S such that $S \backslash \overline{K}$ is isometric to an open topological ball in $\mathbb{R}^n$ ($\overline{K}$ might not consist of ridges.) In particular, $\overline{K}$ can be chosen as the cut locus of a point in S. Such complexes $\overline{K}$ can be constructed algorithmically. The complexity is polynomial in the number of facets (but exponential in d), assuming our conjecture that the number of combinatorial types of shortest paths in S is polynomial in the number of facets. There results a canonical subdivision of S into regions: points v and v' lie in the same region if their cut loci have isomorphic combinatorial structures.

Equipartition of measures by hyperplanes

Siniša Vrecića
(joint work with Peter Mani and Rade T. Živaljević)

This is a report on work started some time ago and especially on some more recent results. The problem we deal with is the following:

Problem. *Determine triples (d, j, k) (we call them admissible) such that for every collection of j continuous measures in $\mathbb{R}^d$, there exist k hyperplanes so that 2^k orthants determined by them form an equipartition of each of the given measures.*

The problem could be rephrased as to determine $\Delta(j, k) = \min\{d \mid (d, j, k) \text{ admissible}\}$ for given j and k.

Of course, the case $k = 1$ gives $\Delta(j, 1) = j$, which is exactly the ham-sandwich theorem. It is well known (and easy) that any measure in $\mathbb{R}^2$ admits an equipartition by two lines and that any measure in $\mathbb{R}^3$ admits an equipartition by three planes. Also, it is known that in general a measure in $\mathbb{R}^d$ could not be equipartitioned by d hyperplanes for $d \geq 5$. The question whether any measure in $\mathbb{R}^4$ could be equipartitioned by 4 hyperplanes is open and serves as one of motivations for this work.

The known facts are mostly due to Edgar Ramos and could be described (except for some special cases) by the inequalities:

$$j\frac{2^k-1}{k} \leq \Delta(j,k) \leq j2^{k-1}.$$

We reduce the question to the topological one and then deal with it using the obstruction theory and the cohomological index theory. We reduce the gap between the lower and upper bound for $\Delta(j,k)$ by showing:

$$\Delta(2^q + r, k) \leq 2^{k+q-1} + r.$$

This upper bound is for $r \neq 0$ strictly better then the previous one and in the case $r = 2^q - 1$ it equals the lower bound. We also obtain some special cases.

On combinatorial formulas for characteristic numbers

Robin Forman

This talk is primarily a historical survey of mathematics developed by Euler, Gauß, Bonnet, Allendoerfer, Fenchel, Weil, Chern, Stiefel, Whitney, Pontrjagin, Thom, Hirzebruch, Rohlin, Svarič, Lievitt, Rourke, Cheeger, Gabrielov, Gelfand and Losik leading up to:

Problem. *Find explicit locally-defined combinatorial formulas for the rational Pontrjagin classes of a combinatorial manifold.*

We will briefly discuss previous work on this question, particularly that by Gelfand and MacPherson. We will then present the following theorem:

Theorem. *Any real-valued locally defined combinatorial invariant of combinatorial manifolds is a sum of a multiple of the Euler characteristic and a Pontrjagin number.*

(This is related to some unpublished work of E. Y. Miller.)
Finally, we will present some partial results towards explicit formulas for such invariants.

Generalized triangulations of the n-gon

Jakob Jonsson

Let $k \geq 1$ and $n \geq 2k+1$, and let $\Omega_{n,k}$ be the set of two-sets (*edges*) $\{i,j\}$ such that $0 \leq i < j \leq n-1$ and $k+1 \leq j-i \leq n-k-1$. Two edges *intersect* if their representations as open line segments in the (convex) n-gon intersect; the vertices in the n-gon are denoted as $0, 1, \ldots, n-1$ in (say) clockwise direction. For $j \geq 1$, a *j-intersection* is a set of j mutually intersecting edges in $\Omega_{n,k}$. Let $\Delta_{n,k}$ be the simplicial complex of edge sets $\sigma \subseteq \Omega_{n,k}$ not containing any $(k+1)$-intersection. For example, $\Delta_{n,1}$ is the boundary complex of the dual polytope of the associahedron with one maximal face for each triangulation of the n-gon. We show that $\Delta_{n,k}$ is a piece-wise linear sphere; this has also been proved (at least for $k \leq 3$) by A. Dress, V. Moulton, and S. Grunewald. Moreover, we demonstrate that the number of maximal faces in $\Delta_{n,k}$ is counted by a $k \times k$ determinant of Catalan numbers.

Many triangulated 3-spheres

Julian Pfeifle

(joint work with Günter M. Ziegler)

We construct $2^{\Omega(n^{5/4})}$ combinatorial types of triangulated 3-spheres on n vertices. Since by a result of Goodman and Pollack (1986) there are no more than $2^{O(n \log n)}$ combinatorial types of simplicial 4-polytopes, this proves that asymptotically, there are far more combinatorial types of triangulated 3-spheres than of simplicial 4-polytopes on n vertices. This complements results of Kalai (1988), who had proved a similar statement about d-spheres and $(d+1)$-polytopes for fixed $d \geq 4$.

`polymake` software presentation

Michael Joswig

Presentation of the software system `polymake` for the construction, analysis and visualization of convex polytopes. See `http://www.math.tu-berlin.de/polymake/`

Friday, April 11, 2003

g- and *h*-polynomials of non-rational polytopes — recent progress

Tom Braden

The "toric" g and h-polynomials, defined and first studied by Stanley, generalize the definition for simplicial complexes. In case the polytope has rational coefficients, it defines a toric variety, and the coefficients of h (respectively g) are intersection cohomology (resp. primitive intersection cohomology) Betti numbers of these varieties.

Karu has recently shown that a purely combinatorial construction of equivariant intersection cohomology defined by Barthel-Brasselet-Fieseler-Kaup and Bressler-Lunts satisfies the Hard Lefschetz theorem. This is the key result in the computation of IH^* for toric varieties that resulted in the original definition of Stanley, and we can now show purely combinatorially a number of results which were previously only known with the artificial assumption of rationality. In particular, g and h have nonnegative coefficients, and we have a lower bound $g_P \geq g_F \cdot g_{P/F}$ coefficient by coefficient (this was a conjecture of Kalai, previously known for rational polytopes by our work with MacPherson).

Other applications of Karu's result include a "polytopal Morse theory" which, given a shelling of P, expresses h_P as a sum of nonegative polynomials attached to each face, and Koszul duality, which relates intersection cohomology and perverse sheaves between a polytope P and its polar P^*.

Partially ordered sets with $\mathbb{Z}_2$-action

Mark de Longueville

In the spirit of the recent combinatorial proof of the Kneser-Lovász theorem by J. Matoušek we discuss the notion of a $\mathbb{Z}_2$-index for partially ordered sets with a $\mathbb{Z}_2$-action.

Lower bounds for simplicial covers and triangulations of cubes

Francis Edward Su
(joint work with Adam Bliss)

A *simplicial cover* of a convex polytope P is a collection of simplices such that (i) the vertices of the simplices are chosen from vertices of P and (ii) the union of the simplices is P. The simplices may overlap; if they do not the simplicial cover is a *triangulation* of P.

What is *covering number* of P, i.e., the minimal number of simplices needed for a simplicial cover? While it is of interest in its own right, the covering number also gives a lower bound for the size of the minimal triangulation of P (even triangulations with extra vertices). This can in turn be bounded from below the *pebble sets* of the "polytopal Sperner lemma" of De Loera, Peterson and Su (2002).

Our work establishes new lower bounds for the covering number of d-cubes, which improve the bounds of Smith in all dimensions that we could calculate, up through $d = 12$. We avoid the use of hyperbolic volume techniques and instead focus on examining the

Dimension	Smith (2000)	Bliss-Su (2003)
3	5	5
4	15	16
5	48	60
6	174	250
7	681	1,117
8	2,863	4,680
9	12,811	21,384
10	60,574	88,172
11	300,956	494,547
12	1,564,340	2,681,790
d	asymptotic bound $\frac{6^{\frac{d}{2}} d!}{2(d+1)^{\frac{d+1}{2}}}$	(?)

Table 1. Lower bounds for the covering number of the cube.

relationship between exterior faces of simplices in the cube and exterior faces of the cube. This yields a linear program which is much smaller than that of Hughes and Anderson (1996), and our results compare favourably with their bounds for vertex triangulations of the cube. Continuing work includes the development of an asymptotic bound for our linear program for large d.

On Sylvester's four point problem

Emo Welzl
(joint work with László Lovász, Katalin Vesztergombi, and Uli Wagner)

In 1864 J.J. Sylvester posed the problem of determining the probability of a random 4-gon to be convex; shortly after that, he noted that "This problem does not admit of a determinate solution," since the answer depends on the underlying distribution. So let μ be some (sufficiently generic) probability distribution in the plane, let P_1, P_2, P_3, and P_4 be i.i.d. points from μ, and consider

$$s_\mu := \mathrm{Prob}(P_i\text{'s are in convex position}).$$

W. Blaschke (1917) showed that if μ is uniform in some convex domain, then

$$2/3 = s_\Delta \leq s_\mu \leq s_\bigcirc = 1 - 35/(12\pi^2) < 0.705,$$

where Δ and $\bigcirc$ stand for the uniform distribution in a triangle and circular disk, respectively.

We are interested in $s := \inf_\mu s_\mu$ (over general sufficiently generic μ) and show that

$$s > 3/8 + 10^{-5}$$

(improving on a sequence of previous bounds, $s > 0.328$ by U. Wagner being the latest; the best upper bound is $s < 0.3807$ by O. Aichholzer, F. Aurenhammer, and F. Krasser). The problem relates to the minimum number $\overline{\mathrm{cr}}(K_n)$ of crossings in a generic straight line embedding of the complete graph K_n on n vertices via

$$\lim_{n\to\infty} \overline{\mathrm{cr}}(K_n)/\binom{n}{4} = s.$$

It is interesting to note that if we are not restricted to straight line edges, then K_n can be drawn with $(3/8)\binom{n}{4} + O(n^3)$ crossings.

The result is derived via a relation on k-sets of an n-point set in the plane. We show (and then use) that for any set S of n points in general position in the plane, there are at least $3\binom{k+1}{2}$ subsets of size at most k which can be separated from their complements in S by a straight line (for $k < n/2$).

A bound of $s \geq 3/8$ has been shown independently by S. Fernández and B. M. Abrego.

Hyperbolic 3-manifolds with no Reebless foliation

John Shareshian
(joint work with Rachel Roberts and Melanie Stein)

A foliation of a 3-manifold M is a decomposition of M into surfaces (called "leaves") which is locally modelled on $\mathbb{R}^2 \times \mathbb{R}$. A foliation is Reebless if it contains no torus leaf T such that $M \backslash T$ has two connected components, at least one of which is decomposed into planes by the foliation. The presence of a Reebless foliation in M provides significant information about its topological structure and fundamental group.

We prove that infinitely many hyperbolic 3-manifolds admit no Reebless foliation, thereby disproving a conjecture of W. Thurston. Our result is provided by showing that infinitely many hyperbolic 3-manifolds have fundamental group which cannot act without a global fixed point on any simply connected (not necessarily Hausdorff) 1-manifold.

Bruhat intervals of length 4 in Weyl groups

Axel Hultman

Combining theoretical arguments and computer calculations, we obtain all isomorphism classes of intervals of length 4 in the Bruhat order on the Weyl groups A_4, B_4, D_4 and F_4. It turns out that there are 24 of them (some of which are dual to each other). Work of Dyer allows us to conclude that these are the only intervals of length 4 that can occur in the Bruhat order on any Weyl group. We also determine the intervals that arise already in the smaller classes of simply-laced Weyl groups and symmetric groups.

Torsion in the matching complex and chessboard complex

Michelle Wachs

(joint work with John Shareshian)

To every finite collection of graphs that is closed under removal of edges, one can associate an abstract simplicial complex whose faces are the edge sets of the graphs in the collection. Graph complexes have provided an important link between combinatorics and algebra, topology and geometry. Here we consider the simplicial complex associated with the collection of subgraphs of a graph G that are partial matchings on the vertices of G. In particular, we deal with the matching complex (G is a complete graph) and the chessboard complex (G is a complete bipartite graph).

Topological properties of the matching complex were first examined by Bouc in connection with Quillen complexes, and topological properties of the chessboard complex were first examined by Garst in connection with Tits coset complexes. Björner, Lovász, Vrécica and Živaljević established bounds on the connectivity of the matching complex and chessboard complex and conjectured that these bounds are sharp. Computer data indicated that the bottom nonvanishing integral homology (i.e., in the degree given by these bounds) has the form $\mathbb{Z}_3^r$ for some $r > 0$. In this paper we show that the conjecture is true, and moreover that integral homology is indeed $\mathbb{Z}_3^r$.

Mixed fibre polytopes

Peter McMullen

With a slight modification of the original definition by Billera and Sturmfels, it is shown that the theory of fibre polytopes extends to one of mixed fibre polytopes. If the short exact sequence

$$\mathbb{O} \longrightarrow \mathbb{X} \xrightarrow{\Phi} \mathbb{V} \xrightarrow{\Psi} \mathbb{Y} \longrightarrow \mathbb{O}$$

consists of euclidean spaces, with
Φ a linear isometric injection and Ψ orthogonal projection, for a polytope P in $\mathbb{V}$ define its *fibre polytope* $\mathrm{fib}(P;\Phi)$ by

$$\mathrm{fib}(P;\Phi) := \int_{P\Psi} ((P-y)\cap \mathbb{X})dy.$$

It turns out that $\mathrm{fib}(P\cdot;\Phi)$ is a weakly continuous translation covariant valuation, which immediately implies the existence of polynomial expansions. Further, there is induced a natural (surjective) homomorphism from the space of tensor weights on polytopes in $\mathbb{V}$ to the corresponding space of such weights on polytopes in $\mathbb{X}$. Moreover, these homomorphisms compose in the correct way; this is in contrast to the situation of the fibre polytope construction, which does not iterate as one would wish. Direct calculations then explain that the general *mixed fibre polytope* $\mathrm{fib}(P_0,\dots,P_m;\Phi)$ can be defined as the coefficient of $(m+1)!\lambda_0\cdots\lambda_m$ in the polynomial expansion of $\mathrm{fib}(\lambda_0 P_0+\cdots+\lambda_m P_m;\Phi)$, where $\lambda_0,\dots,\lambda_m \geq 0$ and $m := \dim \mathbb{Y}$.

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