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Cohomological Aspects of Hamiltonian Group Actions and Toric Varieties

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Introduction by the Organisers

The meeting brought together people with different mathematical background, who all use cohomological methods to study symmetries of manifolds. The main aim was the exchange of ideas, recent results, and the discussion of open problems and questions from diverse viewpoints. Altogether there were 27 talks (including an evening talk on computer programs), and a Hausmusik evening with an artistic juggling intermission.

All talks reflected the central theme of the workshop, namely the use of (equivariant) cohomology in studying Lie group actions on manifolds. Contribution to the following subjects were given:

- classification of G-actions on manifolds,
- equivariant cohomology and cohomology of reduced spaces,
- fixed points and cohomology,
- Hodge theory,
- moment maps and quantization of manifolds,
- new models for the equivariant cohomology of a space,
- toric varieties.

Especially the informal discussions among participants both with similar and with diverse mathematical background were a very important aspect of the workshop. We believe that the meeting has stimulated further cooperation in the study of actions of Lie groups between mathematicians from different areas.

MSC classification: 14, 52, 53, 55, 57, 58

Workshop on Cohomological Aspects of Hamiltonian Group Actions and Toric Varieties

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Abstracts

A deformation of Hodge theory on the cotangent bundle

Jean-Michel Bismut

The purpose of my talk was to introduce a deformation of classical Hodge theory which interpolates between classical Hodge theory and the geodesic flow. The results have been announced in the Comptes Rendus notes [3, 2, 4], in the paper [1], and also in joint work with Lebeau [5].

Let X be a compact manifold of dimension n, and let $\left(F, \nabla^F, g^F\right)$ be a complex flat vector bundle on X. Let $\left(\Omega^{\cdot}(X,F), d^X\right)$ be the de Rham complex of smooth forms on X with coefficients in F, whose cohomology is denoted $H^{\cdot}(X,F)$.

Let g^{TX} be a Riemannian metric on TX, and let g^F be a Hermitian metric on F. Then $\Omega^{\cdot}(X,F)$ is equipped with a corresponding L_2 Hermitian product. Let d^{X*} be the formal adjoint of d^X. Let $D^X = d^X + d^{X*}$ be the corresponding Dirac operator. The associated Laplacian $\square^X = D^{X,2}$ is given by

$$\square^X = d^X d^{X*} + d^{X*} d^X. \tag{1}$$

Let $\mathcal{H} = \ker \square^X$ be the harmonic forms. Then Hodge theory asserts that $\mathcal{H} \simeq H^{\cdot}(X,F)$.

Let $f\colon X \to \mathbf{R}$ be a smooth Morse function. In [11], Witten has introduced a deformation of Hodge theory. Indeed, for $T \in \mathbf{R}$, set $d^X_T = e^{-Tf} d^X e^{Tf}$. Let $d^{X*}_T = e^{Tf} d^{X*} e^{-Tf}$ be the formal adjoint of d^X_T, and let $\square^X_T$ be the corresponding Laplacian. Set $\mathcal{H}_T = \ker \square^X_T$. Still, $\mathcal{H}_T \simeq H^{\cdot}(X,F)$. As $T \to +\infty$, all the eigenvalues except a finite family of them tend to $+\infty$, the other eigenvalues are 0 or are exponentially small as $T \to +\infty$. Let F_T be the finite dimensional complex of eigenbundles associated to small eigenvalues. In [11], Witten shows that F_T localizes near the critical points of f, which is enough for a proof of the Morse inequalities. Assume that ∇f is Morse-Smale. Witten argues that F_T converges in the appropriate sense to the combinatorial Thom-Smale complex associated to ∇f. This was proved rigorously by Helffer-Sjöstrand [8]. The Witten deformation was used in [6] to establish the equality of the Ray-Singer and Reidemeister torsions.

We tried to adapt the above formalism to the loop space LX of X. On the one hand, LX does not have a Hodge theory, in particular because of the lack of a satisfactory L_2 scalar product on the de Rham complex. On the other hand, LX carries many natural S_1-invariant functionals associated to Lagrangians $L(x,\dot{x})$. Prominent among this, there is the energy functional $E(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2 dt$. Morse theory has been used successfully on LX, in particular by Bott [7] in his proof of Bott periodicity.

Our strategy consists in trying to consider the small 'time' asymptotics of the heat kernel $e^{-s\square^{LX}}$ (which does not exist ...) instead, by describing it in terms of classical partial differential operators on T^*X. To make the construction effective, the functional integral approach is most useful. Indeed, let $F = \int_0^1 f(x_t)dt$ be the obvious lift of f to an S_1-invariant function on LX. Then, at least formally, localization of certain eigenforms near the critical points of f as $T \to +\infty$ can be properly understood via the pull-back by ∇F of the Mathai-Quillen forms [10] of TLX . The idea is now to replace F by E. Note that

$$\nabla E = -\ddot{x}. \tag{2}$$

The path integral to be considered takes the form

$$\int_{LX} \exp\left(-\frac{1}{2} \frac{\int_0^1 |\dot{x}|^2 \, dt}{2} - \frac{T^2}{2} \int_0^1 |\ddot{x}|^2 \, dt + \ldots \right). \tag{3}$$

The dynamic interpretation of (3) just says that

$$\dot{x} = p, \qquad \dot{p} = \frac{1}{T} \left(-p + \dot{w} \right), \tag{4}$$

which is equivalent to

$$\ddot{x} = \frac{1}{T} \left(-\dot{x} + \dot{w} \right). \tag{5}$$

In (4) and (5), w is a standard Brownian motion along the fibres of TX. The second order differential operator on T^*X which describes the dynamic in (4) and (5) is given by

$$\frac{1}{2} \left(-\Delta^V + |p|^2 - n \right) + \nabla_p. \tag{6}$$

In (6), ∇_p is the Hamiltonian vector field on T^*X associated to the Hamiltonian $\mathcal{H} = \frac{1}{2} |p|^2$, i.e., the generator of the geodesic flow.

Our problem can then be reformulated as follows. Is there a natural deformation of classical Hodge theory whose Laplacian on T^*X would 'look like' the operator in (6)? The answer to this question is positive. To make the argument simpler, we take $T = 1$ here. Let $\pi \colon T^*X \to X$ be the canonical projection. Let ω be the symplectic form of T^*X. Let η be the bilinear form on T^*X,

$$\eta (U, V) = \langle \pi_* U, \pi_* V \rangle_{g^{TX}} + \omega (U, V). \tag{7}$$

This bilinear form induces a corresponding bilinear form on $\Omega^{\cdot} (T^*X, \pi^* F)$. Then we take the adjoint $\overline{d}^{T^*X}_{\phi,\mathcal{H}}$ that one obtains with respect to this bilinear form, while making a Witten twist with respect to $\mathcal{H}$. The corresponding Laplacian is indeed of the type (6). It is not self-adjoint, but it is hypoelliptic by Hörmander [9]. It is indeed self-adjoint with respect to a hermitian form of signature (∞, ∞). When we introduce a parameter $c = \frac{1}{T}$, the Laplacian interpolates between classical Hodge theory for $c \to +\infty$ and the generator of the geodesic flow for $c \to 0$. It has a number of analytical properties described in joint work with Lebeau [5]. In particular the Hodge theorem holds except maybe for a discrete family of values of c.

References

[1] J.-M. Bismut. The hypoelliptic Laplacian on the cotangent bundle. *To appear*, 2004.

[2] J.-M. Bismut. Le Laplacien hypoelliptique sur le fibré cotangent. *C. R. Math. Acad. Sci. Paris Sér. I*, 338:555–559, 2004.

[3] J.-M. Bismut. Une déformation de la théorie de Hodge sur le fibré cotangent. *C. R. Acad. Sci. Paris Sér. I*, 338:471–476, 2004.

[4] J.-M. Bismut. Une déformation en famille du complexe de de Rham-Hodge. *C. R. Math. Acad. Sci. Paris Sér. I*, 338:623–627, 2004.

[5] J.-M. Bismut and G. Lebeau. The analysis of the hypoelliptic Laplacian. *to appear*, 2004.

[6] J.-M. Bismut and W. Zhang. An extension of a theorem by Cheeger and Müller. *Astérisque*, (205):235, 1992. With an appendix by François Laudenbach.

[7] R. Bott. The stable homotopy of the classical groups. *Ann. of Math. (2)*, 70:313–337, 1959.

[8] B. Helffer and J. Sjöstrand. Puits multiples en mécanique semi-classique. IV. Étude du complexe de Witten. *Comm. Partial Differential Equations*, 10(3):245–340, 1985.

[9] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[10] V. Mathai and D. Quillen. Superconnections, Thom classes, and equivariant differential forms. *Topology*, 25(1):85–110, 1986.

[11] E. Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.

Toric degenerations of spherical varieties

Michel Brion

Let G be a connected complex reductive group, i.e., G is the complexification of a connected compact Lie group K. A *polarized G-variety* is a pair (X, L), where X is a complex projective algebraic variety equipped with an algebraic action of G, and L is an ample line bundle on X. The G-variety X is *spherical*, if it is normal and contains a dense orbit of a Borel subgroup of G.

Nonsingular polarized G-varieties yield quantized Hamiltonian K-varieties, and spherical varieties correspond to *multiplicity-free spaces*, i.e., those Hamiltonian K-varieties for which the preimage of any K-orbit under the moment map $\mu : X \to \mathfrak{k}^*$ (where $\mathfrak{k}$ is the Lie algebra of K) is a unique K-orbit. The intersection of the image of the moment map with a positive Weyl chamber is the *moment polytope* $P(X, L)$, a rational convex polytope which is an important invariant of the pair (X, L).

The simplest spherical varieties are the *toric varieties*, i.e., the normal varieties where a complex torus $(\mathbb{C}^*)^n$ acts with a dense orbit. These correspond to multiplicity-free spaces under the compact torus $(S^1)^n$; they are classified by their moment polytope, an integral polytope in $\mathbb{R}^N$.

In the joint work [1] with Valery Alexeev, we prove that *any spherical polarized G-variety (X, L) degenerates to a toric $\mathbb{Q}$-polarized variety* (X_0, L_0), i.e., L_0 is an ample linearized $\mathbb{Q}$-line bundle on the projective toric variety X_0. The torus acting on X_0 is a quotient of $T \times (\mathbb{C}^*)^N$, where T is a maximal torus of G, and N is the number of positive roots of G.

Such a degeneration was first constructed by Gonciulea and Lakshmibai [5] for Grassmanians and varieties of complete flags, by using standard monomial theory. It was generalized to all flag varieties and their Schubert varieties by Caldero

[4], by a very different method based on properties of the *dual canonical basis* in representation theory. We build on Caldero's work; it allows us to generalize to all groups a result of Kaveh [6] which constructs toric degenerations of spherical varieties under the symplectic group.

Specifically, given any polarized G-variety (X, L), we construct a T-variety $\mathcal{X}$ together with a T-invariant regular function $\pi : \mathcal{X} \to \mathbb{C}$ and an ample linearized $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$, such that
(i) the restriction $\pi^{-1}(\mathbb{C}^*) \to \mathbb{C}^*$ identifies to the second projection $X \times \mathbb{C}^* \to \mathbb{C}^*$, and the restriction of $\mathcal{L}$ identifies to the pull-back of L under the first projection $X \times \mathbb{C}^* \to X$.
(ii) the pair $(\pi^{-1}(0), \mathcal{L}_{\pi^{-1}(0)}) = (X_0, L_0)$ is a polarized variety for the torus $T \times (\mathbb{C}^*)^N$.

We also show that the moment polytope $P(X_0, L_0) \subset \mathfrak{t}^* \times \mathbb{R}^N$ projects onto the moment polytope $P(X, L) \subset \mathfrak{t}^*$, with fiber at any point λ being the *string polytope* $Q(\lambda)$. The latter is the moment polytope of the toric limit of the flag variety G/P associated with the point λ of the positive Weyl chamber.

If, in addition, X is spherical under G, then we show that X_0 is toric under $T \times (\mathbb{C}^*)^N$. Then our construction provides a geometric explanation of a result of Okounkov [8] for Hilbert polynomials of spherical varieties.

As in Caldero's work, our degeneration depends on the choice of a reduced decomposition $\underline{w}_0$ of the longest element of the Weyl group W. In fact, each string polytope $Q(\lambda) = Q_{\underline{w}_0}(\lambda)$ is the intersection of the *string cone* $\mathcal{C}_{\underline{w}_0} \subset \mathfrak{t}^* \times \mathbb{R}^N$ (a rational polyhedral convex cone) with the affine space $\{\lambda\} \times \mathbb{R}^N$.

Explicit linear inequalities defining the string cones, and hence the string polytopes, have been obtained by Littelmann [7] and Berenstein & Zelevinsky [3]. Further, for any dominant weight λ, the string polytope $Q_{\underline{w}_0}(\lambda) \subset \mathbb{R}^N$ admits a linear projection to the convex hull of the orbit $W\lambda \subset \mathfrak{t}^*$, and the number of integer points in the fiber at any weight μ is the multiplicity of μ in the simple G-module with highest weight λ.

If G is the general linear group and $\underline{w}_0 = (s_1, s_2, s_1, s_3, s_2, s_1, s_4, s_3, s_2, s_1, \cdots)$ is the simplest reduced decomposition, the string polytopes $Q_{\underline{w}_0}(\lambda)$ are just the *Gelfand-Cetlin polytopes*. In particular, they are integral polytopes in $\mathbb{R}^N$. In fact, *we conjecture that* $Q_{\underline{w}_0}(\lambda)$ *is an integral polytope for* G *of type A, any reduced decomposition* $\underline{w}_0$*, and any dominant weight* λ. (This fails for other types.)

For arbitrary G and $\underline{w}_0$, we show that $Q_{\underline{w}_0}(\lambda)$ *is a lattice polytope whenever the dominant weight* λ *is minuscule or cominuscule.* Then the toric limit of the corresponding flag variety G/P is a Fano variety; but it is singular unless G/P is a projective space. This is related with results and conjectures of Batyrev et al. [2] concerning mirror symmetry for Calabi-Yau hypersurfaces in flag varieties.

Examples show that the shape of the string polytopes $Q_{\underline{w}_0}(\lambda)$ depends on the choice of the reduced decomposition $\underline{w}_0$; it may also depend on the dominant weight λ, even if it is assumed to be regular. However, these polytopes do have some common features, e.g., the image measure of their linear projection onto

Conv($W\lambda$) is the Duistermaat-Heckman measure for the action of T on the corresponding flag variety G/P.

References

[1] V. Alexeev and M. Brion, *Toric degenerations of spherical varieties*, preprint, arXiv: math.AG/0403379.

[2] V. V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D. van Straten, *Mirror symmetry and toric degenerations of partial flag manifolds*, Acta Math. **184** (2000), no. 1, 1–39.

[3] A. Berenstein and A. Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties*, Invent. Math. **143** (2001), no. 1, 77–128.

[4] P. Caldero, *Toric degenerations of Schubert varieties*, Transform. Groups **7** (2002), no. 1, 51–60.

[5] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transform. Groups **1** (1996), no. 3, 215–248.

[6] K. Kaveh, *SAGBI bases and Degeneration of Spherical Varieties to Toric Varieties*, preprint, arXiv: math.AG/0309413.

[7] P. Littelmann, *Cones, crystals, and patterns*, Transform. Groups **3** (1998), no. 2, 145–179.

[8] V. L. Okounkov, *A remark on the Hilbert polynomial of a spherical variety*, Funct. Anal. Appl. **31** (1997), no. 2, 138–140.

Equivariant symplectic Hodge theory

Reyer Sjamaar

(joint work with Yi Lin)

A fuller account [6] of this work is available on the arXiv and will be published in the Journal of Symplectic Geometry.

Let G be a connected compact Lie group and let (M, ω) be a compact symplectic manifold equipped with a Hamiltonian G-action. It was proved by Kirwan [5] and Ginzburg [3] that the Leray spectral sequence of the fibre bundle $G \to M_G \to B_G$ degenerates at the first term. This implies the equivariant formality theorem, which states that the equivariant cohomology $H_G^*(M)$ is isomorphic to $(S\mathfrak{g}^*)^G \otimes H^*(M)$ as an $(S\mathfrak{g}^*)^G$-module. Equivalently, the restriction map $H_G^*(M) \to H^*(M)$ induced by the inclusion of the fibre $M \to M_G$ is surjective. (We use real coefficients throughout.)

In terms of the Cartan complex $\Omega_G(M) = (S\mathfrak{g}^* \otimes \Omega(M))^G$ of M, the equivariant formality theorem can be restated by saying that every closed form on M admits an equivariantly closed extension. For example, equivariantly closed extensions of the symplectic form ω are of the form $\omega + \phi$, where $\phi\colon \mathfrak{g} \to \Omega(M)$ is a moment map for the action.

Our object is to define a section of the restriction map $H_G^*(M) \to H^*(M)$. We accomplish this by picking a (symplectically) harmonic representative α of a de Rham cohomology class and finding an extension $\alpha_G \in \Omega_G(M)$ which is equivariantly harmonic. (A form or equivariant form β is *harmonic* if $d\beta = d{*}\beta = 0$, where $*$ is Brylinski's [2] symplectic Hodge star operator, and *equivariantly harmonic* if $d_G\beta = d{*}\beta = 0$.) The class of α_G in $H_G(M)$ then turns out to be uniquely determined by the class of α in $H(M)$. For instance, the canonical extension of the symplectic class $[\omega]$ is the class $[\omega + \phi_0]$, where ϕ_0 is the unique moment map satisfying $\int_M \phi_0(\xi)\omega^n = 0$ for all $\xi \in \mathfrak{g}$, where n is one half the dimension of M.

However, the section $[\alpha] \mapsto [\alpha_G]$ exists only under the assumption that every de Rham cohomology class possesses a harmonic representative. As shown by Mathieu [7], this is the case precisely when (M, ω) has the strong Lefschetz property in the sense that the map

$$H^{n-k}(M) \longrightarrow H^{n+k}(M), \qquad c \longmapsto [\omega]^k \wedge c$$

is an isomorphism for each $0 \leq k \leq n$. (Incidentally, under the strong Lefschetz assumption the degeneracy of the equivariant cohomology spectral sequence follows easily from an argument due to Blanchard [1].) We rely on a sharpened version of Mathieu's result due to Merkulov [8] and Guillemin [4], who independently established the symplectic $d\delta$-lemma. Let $\delta = \pm{*}d{*}$ be Koszul's boundary operator and suppose M has the strong Lefschetz property. The $d\delta$-lemma asserts that

$$\ker d \cap \operatorname{im} \delta = \operatorname{im} d\delta = \ker \delta \cap \operatorname{im} d.$$

In words, if α is a harmonic k-form on M that is either exact or coexact then $\alpha = d\delta\beta$ for some k-form β. An equivariantly harmonic extension of a harmonic form can be found by successive applications of the $d\delta$-lemma. As a corollary we establish an equivariant version of the $d\delta$-lemma, to the effect that

$$\ker d_G \cap \operatorname{im} \delta = \operatorname{im} d_G\delta = \ker \delta \cap \operatorname{im} d_G.$$

In view of the Kirwan injectivity and surjectivity theorems these results suggest close relationships among the harmonic forms on M, the fixed-point set M^T (where T is a maximal torus of G), and the symplectic quotients of M. However, because of the poor functorial properties of δ, these relationships remain at present obscure.

After I presented this paper at the Oberwolfach meeting, Christopher Allday kindly pointed out that, in the case of a circle action, a section $s\colon H^*(M) \to H^*_G(M)$ can also be constructed by introducing a Riemannian metric on M and using classical elliptic Hodge theory. A formula for s can then be written down in terms of the Green's function for the Laplacian. Eckhard Meinrenken subsequently showed this can be done equally well for an arbitrary compact connected G. Their argument works for any compact Riemannian G-manifold for which the spectral sequence in equivariant cohomology collapses at E_1. (On the other hand, the section so constructed is perhaps less natural from a symplectic point of view.)

References

[1] A. Blanchard, *La cohomologie réelle d'un espace fibré à fibre kählérienne*, C. R. Acad. Sci. Paris **239** (1954), 1342–1343. MR 16,737a

[2] J.-L. Brylinski, *A differential complex for Poisson manifolds*, J. Differential Geom. **28** (1988), no. 1, 93–114. MR **89m:**58006

[3] V. A. Ginzburg, *Equivariant cohomology and Kähler geometry*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 4, 19–34, 96, English translation in Functional Anal. Appl. **21** (1987), no. 4, 271–283. MR **89b:**58013

[4] V. Guillemin, *Symplectic Hodge theory and the dδ-lemma*, preprint, Massachusetts Institute of Technology, 2001.
[5] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, Princeton, N.J., 1984. MR **86i:**58050
[6] Y. Lin and R. Sjamaar, *Equivariant symplectic Hodge theory and the $d_G\delta$-lemma*, preprint, Cornell University, 2003, math.SG/0310048.
[7] O. Mathieu, *Harmonic cohomology classes of symplectic manifolds*, Comment. Math. Helv. **70** (1995), no. 1, 1–9. MR **96e:**58004
[8] S. Merkulov, *Formality of canonical symplectic complexes and Frobenius manifolds*, Internat. Math. Res. Notices **14** (1998), 727–733. MR **99j:**58078

Theory of toric varieties from a topological point of view

Mikiya Masuda

The theory of toric varieties was established early 70's (see [3], [13]). Among toric varieties, complete non-singular toric varieties are well studied. For instance, their cohomology and Chern classes are determined. However, these are topological invariants and it is possible to reprove these results using only topological technique, and this leads us to develop a topological analogue of the theory of toric varieties. Like the theory has applications to combinatorics (e.g., counting lattice points in convex lattice polytopes and face numbers of simplicial convex polytopes), our topological analogue also has applications to combinatorics, which is similar to the toric case but treats a wider class in combinatorics.

In this talk I reported some development ([7], [8], [10], [11], [12]) on this topological analogue. Our geometrical object is a *torus manifold* which is a closed omnioriented smooth manifold of even dimension, say $2n$, with a smooth action of an n-dimensional torus $T = (S^1)^n$. A complete non-singular toric variety of complex dimension n with restricted T-action provides an example of a torus manifold, but a class of torus manifolds is much wider. To a torus manifold M, one can associate a combinatorial object Δ_M called a *multi-fan* which reduces to an ordinary fan when M is toric. A multi-fan is a collection of cones in which cones may overlap. The correspondence from torus manifolds to multi-fans is not one-to-one, but many topological invariants of M can be described in terms of Δ_M.

A complete non-singular toric variety together with an equivariant ample line bundle L associates a moment map whose image is a lattice convex polytope P_L. Using this fact, one can count a number of lattice points in P_L by applying Hirzebruch-Riemann-Roch Theorem to the bundle L. This well-known story can be generalized to an *arbitrary* T-line bundle over a *torus* manifold. However, the moment map image is no longer convex ([9]) and this leads us to the notion of *multi-polytope*, which generalizes the notion of simple convex polytope, and allows us to generalize results on counting lattice points in convex polytopes to multi-polytopes.

I also discussed the equivariant cohomology of M and the topology of the orbit space M/T. When M is toric, M/T is (often) a simple convex polytope, the dual $\partial(M/T)^*$ of the boundary of M/T is a simplicial complex and the equivariant cohomology of M is the face ring (or Stanley-Reisner ring) of the simplicial complex. When M is a torus manifold, M/T is not necessarily a simple convex polytope, $\partial(M/T)^*$ is not necessarily a simplicial complex (but a simplicial poset) and the equivariant cohomology of M is the face ring of the simplicial poset when $H^{odd}(M) = 0$. These topological ideas or generalization enable us to characterize h-vectors of simplicial cell decompositions of spheres (more generally, Gorenstein* simplicial posets), which can be viewed as a topological version of the so-called g-theorem characterizing h-vectors of simplicial convex polytopes.

References

[1] V. M. Buchstaber and T. E. Panov, *Torus Actions and Their Applications in Topology and Combinatorics*, University Lecture, vol. 24, Amer. Math. Soc., Providence, R.I., 2002.

[2] M. W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991), 417–451.

[3] W. Fulton, *An Introduction to Toric Varieties*, Ann. of Math. Studies, vol. 113, Princeton Univ. Press, Princeton, N.J., 1993.

[4] M. Grossberg and Y. Karshon, *Equivariant index and the moment map for completely integrable torus actions*, Adv. in Math. 133 (1998), 185–223.

[5] V. Guillemin, Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces, Progress in Math. **122**, Birkhäuser 1994.

[6] V. W. Guillemin and C. Zara, *One-skeleta, Betti numbers and equivariant cohomology*, Duke Math. J. 107 (2001), 283–349.

[7] A. Hattori, *Almost complex toric manifolds and positive line bundles*, Homotopy and Geometry, Banach Center Publications, 45 (1998), 95-114.

[8] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math. 40 (2003), 1–68.

[9] Y. Karshon and S. Tolman, *The moment map and line bundles over presymplectic manifolds*, J. Diff. Geom. 38 (1993), 465–484.

[10] M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. 51 (1999), 237–265.

[11] M. Masuda, *h-vectors of Gorenstein* simplicial posets*, arXiv math.CO/0305203.

[12] M. Masuda and T. Panov, *On the cohomology of torus manifolds*, arXiv math.AT/0306100.

[13] T. Oda, Convex Bodies and Algebraic Geometry, Springer-Verlag 1988.

[14] R. P. Stanley, *Combinatorics and Commutative Algebra*, second edition, Progress in Math., vol. 41, Birkhäuser, Boston, 1996.

Rational homotopy theory of toric spaces

Taras Panov

(joint work with Nigel Ray)

Since the pioneering work of Davis and Januszkiewicz [2], algebraic topologists have been drawn increasingly towards the study of spaces which arise from well-behaved actions of the torus T^n. Investigations are no longer confined to the properties of Davis and Januszkiewicz's toric manifolds, but have been extended to related geometrical structures, such as moment-angle complexes [1], subspace arrangements, and torus manifolds of Hattori and Masuda [4], as well as the homotopy types of associated spaces [8] and their rationalisations and localisations [6]. We refer to this enlarged field of activity as *toric topology.*

The classical models for good T^n-actions are provided by the nonsingular projective toric varieties of algebraic geometry. The quotient space of any such variety M is a simple convex n-polytope Q. The polar polytope of Q is necessarily simplicial, and its boundary is a simplicial complex K. By duality, Q may be decomposed as the cone $C(K')$ on the barycentric subdivision of K, and the isotropy subgroups of the action may be recorded by assigning certain combinatorial data to the vertices of K. It then becomes possible to reconstruct M from T^n and K, and much of toric topology stems from generalising and extending this relationship. For example, K may first be weakened to a simplicial sphere, and ultimately to an arbitrary simplicial complex. General toric spaces are then defined by analogy as quotients of $C(K') \times T^n$.

We use the following notation: $T := T^n$, $M := M^{2n}$ a T-manifold, $Q := M/T$ the orbit quotient. Particular examples are

- non-singular compact toric varieties:
  the T-action is a part of an algebraic $(\mathbb{C}^*)^n$-action with a dense orbit;
- (quasi)toric manifolds of Davis-Januszkiewicz:
  they are "locally standard" (i.e., locally look like $\mathbb{C}^n$ with the standard T-action) and Q is combinatorially a simple polytope;
- torus manifolds of Hattori–Masuda:
  the T-fixed point set is non-empty.

Let K be a simplicial complex on $V = \{v_1, \ldots, v_m\}$ (e.g., K is the dual to the boundary of Q). If the only missing faces have dimension 1, then K is known as a *flag complex.* Denote by $S(V)$ the symmetric algebra on V, $\deg v_i = 2$. Given $\omega \subseteq V$, set $v_\omega := \prod_{i\in\omega} v_i$. The *Stanley-Reisner algebra* [9] (or the *face ring*) of K is given by

$$\mathbb{Z}[K] := S(V)/(v_\omega \colon \omega \notin K).$$

A key example is Davis and Januszkiewicz's space $DJ(K)$, whose integral cohomology $H^*(DJ(K);\mathbb{Z})$ is isomorphic to $\mathbb{Z}[K]$. The space $DJ(K)$ is homotopy equivalent to the colimit, or nested union, of the classifying spaces $BT^\sigma \subseteq BT^m$ over the faces $\sigma \in K$. We therefore have

- $(\mathbb{C}P^\infty)^{\vee m} \subseteq DJ(K) \subseteq (\mathbb{C}P^\infty)^{\times m}$;
- $DJ(K) \simeq ET \times_T M$ for $K = (\partial Q)^*$;
- $H^*(DJ(K);\mathbb{Z}) \cong H^*_T(M;\mathbb{Z}) \cong \mathbb{Z}[K]$.

Another important toric space, the *moment-angle complex* $\mathcal{Z}_K$, is defined as the homotopy fibre of the inclusion $DJ(K) \hookrightarrow BT^m$. We therefore have two homotopy pullback diagrams

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & ET^m \\ \downarrow & & \downarrow \\ DJ(K) & \longrightarrow & BT^m \end{array} \quad \text{and} \quad \begin{array}{ccc} M^{2n} & \longrightarrow & ET^n \\ \downarrow & & \downarrow \\ DJ(K) & \longrightarrow & BT^n \end{array}$$

The map $DJ(K) \to BT^n$ is determined by a choice of a regular sequence in the Cohen–Macaulay algebra $\mathbb{Z}[K] = H^*(DJ(K))$.

Our aim is to relate

- the topology of M, $\mathcal{Z}_K$, $DJ(K)$ and their loop spaces,
- the combinatorics of Q and K, and
- the commutative and homological algebra of $\mathbb{Q}[K]$

through rational homotopy theory.

According to a result of [6], the space $DJ(K)$ is formal and $\mathbb{Q}[K]$ (with zero differential) is a rational model. In [7] we

(1) describe rational models for M and $\mathcal{Z}_K$ as free extensions of the face ring $\mathbb{Q}[K]$ and recover the cohomology calculations of [2] and [1] accordingly;
(2) show that all toric manifolds and those torus manifolds whose cohomology is concentrated in even dimensions are formal;
(3) identify the Pontrjagin homology ring of the loop space $\Omega DJ(K)$ with $\mathrm{Ext}_{\mathbb{Q}[K]}(\mathbb{Q}, \mathbb{Q})$;
(4) for flag complexes K appeal to results of Fröberg [3] which establish the *Koszul property* for the quadratic algebra $\mathbb{Q}[K]$, and we deduce an explicit presentation of $H_*(\Omega\, DJ(K), \mathbb{Q})$ with generators and relations, as well as calculate its Poincaré series in terms of the h-vector of K;
(5) for arbitrary K describe rational models for $\Omega\, DJ(K)$ in terms of its homotopy colimit [10] decomposition, mirroring the results of [8], which present $\Omega\, DJ(K)$ as a homotopy colimit of a diagram of tori in the category of topological monoids.

The last item requires a careful analysis of the *model category structure* [5] in the related algebraic categories (differential graded algebras, coalgebras, Lie algebras, etc.), as well as an explicit construction of appropriate homotopy colimits.

References

[1] Victor M Buchstaber and Taras E Panov. *Torus Actions and Their Applications in Topology and Combinatorics.* Volume 24 of *University Lecture Series*, Amer. Math. Soc., Providence, R.I., 2002.

[2] Michael W Davis and Tadeusz Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.*, 62:417–451, 1991.

[3] Ralph Fröberg. Determination of a class of Poincaré series. *Math. Scand.*, 37:29–39, 1975.

[4] Akio Hattori and Mikiya Masuda. Theory of multi-fans. *Osaka J. Math.*, 40(1):1–68, 2003.

[5] Philip S Hirschhorn. *Model Categories and Their Localizations.* Volume 99 of *Mathematical Surveys and Monographs*, Amer. Math. Soc., Providence, R.I., 2003.

[6] Dietrich Notbohm and Nigel Ray. On Davis Januszkiewicz Homotopy Types I; Formality and Rationalisation. Preprint math.AT/0311167, 2003.

[7] Taras Panov and Nigel Ray. Rational aspects of toric topology. Preprint, University of Manchester, 2004.

[8] Taras Panov, Nigel Ray and Rainer Vogt. Colimits, Stanley-Reisner algebras, and loop spaces. In *Categorical Decomposition Techniques in Algebraic Topology*, volume 215 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, Switzerland, 2003, pp. 261–291.

[9] Richard P Stanley. *Combinatorics and Commutative Algebra*, second edition. Volume 41 of *Progress in Mathematics.* Birkhäuser, Boston, 1996.

[10] Rainer M Vogt. Homotopy limits and colimits. *Math. Z.*, 134:11–52, 1973.

Equivariant vector fields and cohomology

Jim Carrell

1. Opening comments

The purpose of this talk is to give a survey of some results, both old and new, on the connection between zeros of vector fields and cohomology. The starting point is the Koszul complex of a holomorphic vector field V on a smooth projective variety X. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. The vector field V defines a derivation $V\colon \mathcal{O}_X \to \mathcal{O}_X$, which extends to give a contraction operator $i(V)\colon \Omega^p \to \Omega^{p-1}$ on the sheaves of holomorphic p-forms on X such that $i(V)^2 = 0$. In addition, for all ϕ, $\omega \in \Omega^*$,

$$i(V)(\phi \wedge \omega) = i(V)\phi \wedge \omega + (-1)^p \phi \wedge i(V)\omega$$

if $\phi \in \Omega^p$. This gives a complex K^* of sheaves

$$0 \to \Omega^n \to \Omega^{n-1} \to \cdots \to \Omega^1 \to \mathcal{O}_X \to 0 \tag{1}$$

where $n = \dim X$, and, in turn, a spectral sequence whose first term is $E_1^{-p,q} = H^q(X, \Omega^p)$ with first differential $i(V)$. The key point, which was proved in a joint paper with David Lieberman [5] is that if V has zeros, then every differential in this spectral sequence is zero. This is an application of the Deligne degeneracy criterion and a lemma due to Lichnerowicz. Consequently $E_1 = E_\infty$, and we obtain a $\mathbb{C}$-algebra isomorphism

$$\bigoplus_s H^{q+s}(X, \Omega^q) \cong \bigoplus_s F_s H^q(K^*)/F_{s-1}H^q(K^*), \tag{2}$$

where $H^q(K^*)$ denotes the hypercohomology of this Koszul complex and $F.$ is its canonical filtration.

A second point proved in [5] (also see [7]) is that the hypercohomology groups $H^q(K^*)$ vanish when $q >$ dim zero(V). In fact, zero(V) can be viewed as the scheme Z defined by the sheaf of ideals $i(V)\Omega^1 \subset \mathcal{O}_X$, so when this scheme is finite (and non trivial), we get the following result:

Theorem 1. *When V has isolated zeros, then $H^p(X, \Omega^q) = \{0\}$ for all $p \neq q$ (hence $H^p(X, \Omega^p) = H^{2p}(X, \mathbb{C})$), and there exists a $\mathbb{C}$-algebra isomorphism*

$$\bigoplus_p H^{2p}(X, \mathbb{C}) \cong \bigoplus_s F_p\mathbb{C}[Z]/F_{p-1}\mathbb{C}[Z],$$

where $\mathbb{C}[Z]$ is the coordinate ring of the scheme Z.

For further discussion, see [7].

2. The semi-simple and nilpotent cases

For the rest of the talk, we will assume that $H^p(X,\Omega^q)$ is trivial when $p \neq q$. This implies, as above, that $H^{2k+1}(X,\mathbb{C}) = 0$ and $H^{2k}(X,\mathbb{C}) = H^k(X,\Omega^k)$ for all k. Then there are two situations of particular interest. The first is where V is generated by a torus action on X. In this case Z is reduced and smooth. If Z is finite, then $\mathbb{C}[Z] = H^0(Z,\mathbb{C})$ and $H^*(X,\mathbb{C})$ is the associated graded of a certain filtration of the ring of all $\mathbb{C}$-valued functions on Z. More generally, under the above vanishing assumption, $H^p(K^*) = H^p(Z,\Omega^p)$, so we can write

$$H^*(X,\mathbb{C}) \cong \mathrm{Gr}\big(H^*(Z,\mathbb{C})\big),$$

for some filtration of $H^*(Z,\mathbb{C})$. Unfortunately, this filtration is difficult to describe in general, although in the finite case we can state a recent result of K. Kaveh.

Theorem 2. *Suppose Z is finite and V is generated by a $\mathbb{C}^*$-action on X. Then the localization map $i_Z^*\colon H^*_{\mathbb{C}^*}(X,\mathbb{C}) \to H^*_{\mathbb{C}^*}(Z,\mathbb{C})$ is injective. Thus, for any $\alpha \in H^*_{\mathbb{C}^*}(X,\mathbb{C})$, we can write $i_Z^*(\alpha) = (f_1, f_2, \ldots, f_N)$, where $N = |Z|$ and each $f_i \in \mathbb{C}[u]$ for a certain indeterminate u. If $\alpha \in H^{2d}_{\mathbb{C}^*}(X,\mathbb{C})$, put $\phi(\alpha) = (f_1^{(d)}(0), \ldots, f_N^{(d)}(0))$. Then $\phi\big(H^{2d}_{\mathbb{C}^*}(X,\mathbb{C})\big) = F_d\big(\mathbb{C}[Z]\big)$.*

This describes an explicit connection between equivariant cohomology and Theorem 1. I was also informed by Volker Puppe that a similar result holds in the topological setting. For this, see *Cohomological Methods in Transformation Groups* by C. Allday and V. Puppe, Cambridge Univ. Press (1993).

Before giving examples, let us describe the second case, which has some truly surprising features. Here V is generated by an algebraic $\mathbb{C}$-action $\varphi\colon \mathbb{C} \to Aut(X)$ on X, and there is a $\mathbb{C}^*$-action (say $\lambda\colon \mathbb{C}^* \to Aut(X)$) on X such that

$$\lambda(a)\varphi(z)\lambda(a^{-1}) = \varphi(a^2 z)\,.$$

Equivalently, X admits an action of the group $\mathfrak{B}$ of all upper triangular 2×2 matrices over $\mathbb{C}$ having determinant 1. By a theorem of Horrocks, zero(V) is connected, so if zero(V) is also finite, then V has a unique zero. Moreover, the $\mathbb{C}^*$-action determines a grading on $H^0(K^*)$. The remarkable fact is that even if zero(V) is not finite, this grading has the property that $H^0(K^*) \cong \mathrm{Gr}\,H^0(K^*)$. Hence, we get

Theorem 3. *Suppose $H^p(X,\Omega^q)$ vanishes if $p \neq q$ and X admits an action of the group $\mathfrak{B}$ such that $X^{\mathfrak{B}}$ is non empty. Then there is an isomorphism of graded rings between $H^*(X,\mathbb{C})$ and the hypercohomology algebra $H^0(K^*)$.*

Recently, Behrend and O'Halloran [1] used this result to describe the cohomology of the stable map space $X = \overline{M}_{0,0}(\mathbb{P}^n, d)$ of genus zero curves of degree d in $\mathbb{P}^n$. Incidentally, it is unlikely that one could have $H^p(X,\Omega^q) = 0$ if $p \neq q$ and $X^{\mathfrak{B}} = \emptyset$.

3. Examples and applications

Let us begin with the simplest possible example. Let $X = \mathbb{P}^n$ and consider the $\mathbb{C}$-action on X defined by $\exp tJ$, where J is the $(n+1)\times(n+1)$ matrix in Jordan canonical form with one Jordan block. Clearly Z is supported by the point $[e_0]$ defined by the first coordinate vector. Now V is generated by $\exp tJ$. Using affine coordinates $w_1 = z_1/z_0, \ldots, w_n = z_n/z_0$ near $[e_0]$, we find that

$$\begin{aligned} V &= (w_2 - w_1^2)\partial/\partial w_1 + (w_3 - w_1 w_2)\partial/\partial w_2 + \\ &\quad + \cdots + (w_n - w_1 w_{n-1})\partial/\partial w_{n-1} - w_1 w_n \partial/\partial w_n. \end{aligned}$$

Hence

$$\mathbb{C}[Z] = \mathbb{C}[w_1, \ldots, w_n]/(w_2 - w_1^2,\ w_3 - w_1 w_2, \ldots, -w_1 w_n)$$

After simplifying we realize that $\mathbb{C}[Z]$ is the graded ring

$$\mathbb{C}[Z] = \mathbb{C}[w_1]/(w_1^{n+1}).$$

Consequently, there is an isomorphism $\mathbb{C}[Z] \cong H^*(\mathbb{P}^n, \mathbb{C})$. To see how the isomorphism is obtained, we need to study the localization of the Chern classes of a V-equivariant vector bundle to Z. We refer to [6] where this computation is explicitly carried out.

The most useful applications of the above results seem to occur in representation theoretical settings. Thus let G denote a semi-simple algebraic group over $\mathbb{C}$ and let B be a Borel subgroup. The flag variety G/B parameterizes the set of all Borel subgroups of G. Suppose T is a maximal torus in B, and let W be the Weyl group $N_G(T)/T$. If we let W be the semi-simple vector field on G/B generated by a generic one dimensional torus in T, then W has isolated zeros, and it turns out that the picture that one gets from Theorem 1 is that

$$H^*(G/B, \mathbb{C}) \cong \operatorname{Gr} \mathbb{C}[W \cdot h],$$

where $h \in \operatorname{Lie}(T)$ is the element defining W and $W \cdot h$ is the orbit of h under W. (For details, see [3].) The only thing we have to describe is the filtration on $\mathbb{C}[W \cdot h]$. But $W \cdot h$ is closed in $\operatorname{Lie}(T)$ so its coordinate ring $\mathbb{C}[W \cdot h]$ is the quotient of a graded ring by an ideal, hence it has a natural filtration which defines Gr. This description can be easily identified with the famous Borel picture of $H^*(G/B, \mathbb{C})$ as the co-invariant algebra of W (see [3]).

On the other hand, a $\mathfrak{B}$-action on G/B with the unique fixed point property is obtained by starting with a regular nilpotent in $\operatorname{Lie}(B)$. We refer the reader to [4] for the complete construction. The point is that V vanishes exactly at the Borel B, and what one obtains by following through the isomorphism of Theorem 1 is the famous result of Kostant that asserts $H^*(G/B, \mathbb{C}) \cong \mathbb{C}[\mathcal{N} \cap \operatorname{Lie}(T)]$ [3].

4. Equivariant cohomology and equivariant vector fields

Finally, we will discuss a recent joint result with M. Brion [2]. Suppose X is a smooth projective variety admitting a $\mathfrak{B}$-action satisfying the fixed point assumption that V has exactly one zero o, and let T denote the maximal torus of $\mathfrak{B}$ on the diagonal. Now X has a finite number of $\mathfrak{B}$-stable curves, namely the closures of the orbits of the T-fixed points in X distinct from o. On the other hand, $\mathfrak{B}$ acts on $\mathbb{P}^1$, so it acts naturally on $X \times \mathbb{P}^1$ with unique fixed point $(o, 0)$. Hence the closure of the orbit $\mathfrak{B}(x,\infty)$, where $x \in X^T$, is a T-stable curve $\mathcal{Z}_x$ in $X \times \mathbb{P}^1$. Let

$$\mathcal{Z} = \bigcup_{x \in X^T} \big(\mathcal{Z}_x - (x,\infty)\big).$$

An interpretation of this curve is given in the following

Theorem 4. *The curve $\mathcal{Z}$ is an affine T-stable subvariety of $X \times \mathbb{C}$ such that every pair of irreducible components of $\mathcal{Z}$ meet exactly at $(o,0)$. More importantly, the coordinate ring $\mathbb{C}[\mathcal{Z}]$ is isomorphic as a graded $\mathbb{C}$-algebra with the equivariant cohomology algebra $H_T^*(X,\mathbb{C})$.*

This result can be viewed as an analogue of the Goresky-Kottitz-MacPherson picture of T-equivariant cohomology in terms of the momentum graph associated to a torus action with finitely many T-curves [8]. In fact, the variety $\mathcal{Z}$ is well defined independently of whether X is smooth, as long as we have the condition that the unipotent radical of $\mathfrak{B}$ has a unique fixed point. This viewpoint allows one to extend many of the above results to the case where X is singular.

References

[1] K. Behrend and A. O'Halloran. On the cohomology of stable map spaces. *Invent. Math.* **154** (2004), 385–450.

[2] M. Brion and J. B. Carrell. The equivariant cohomology ring of regular varieties. *Michigan Math. J.* **52** (2004), 189–203.

[3] J. B. Carrell. Some remarks on regular Weyl group orbits and the cohomology of Schubert varieties. Pages 33–41 in *Kazhdan-Lusztig theory and related topics* (Chicago, IL, 1989). Contemp. Math. **139**, Amer. Math. Soc., Providence, RI, 1992.

[4] J. B. Carrell. Bruhat cells in the nilpotent variety and the intersection rings of Schubert varieties. *J. Differential Geom.* **37** (1993), 651–668.

[5] J. B. Carrell and D. I. Liebermann. Holomorphic vector fields and Kaehler manifolds. *Invent. Math.* **21** (1973), 303–309.

[6] J. B. Carrell and D. I. Lieberman. Vector fields, Chern classes, and cohomology. Pages 251–254 in: *Several complex variables* (Proc. Sympos. Pure Math., Vol. XXX, Part 1, Williams Coll., Williamstown, Mass., 1975), Amer. Math. Soc., Providence, R.I., 1977.

[7] J. B. Carrell and D. I. Liebermann. Vector fields and Chern numbers. *Math. Ann.* **225** (1977), 263–273.

[8] M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.* **131** (1998), 25–83.

Intersection numbers in reduced spaces of q-Hamiltonian spaces

Lisa Jeffrey

(joint work with Joon-Hyeok Song)

The definition of a quasi-Hamiltonian (abbreviated q-Hamiltonian) G-space was given by Alekseev, Malkin and Meinrenken in [1]: it is a manifold M equipped with the action of a compact Lie group G, a 2-form ω and a map $\Phi\colon M \to G$ satisfying the following properties:

(1) $d\omega = -\Phi^*\chi$ where $\chi \in \Omega^3(G)$ is given by $\chi = \frac{1}{12}\langle\theta,[\theta,\theta]\rangle$

(2) $i_{\xi^{\#}}\omega = \frac{1}{2}\Phi^*\langle\theta + \bar{\theta}, \xi\rangle$
(3) $\mathrm{Ker}(\omega_x) = \{\xi^{\#}(x), \xi \in \mathrm{Ker}(\mathrm{Ad}(\Phi(x)) + 1)$

Here, θ (resp. $\bar{\theta}$) is the left-invariant (resp. right-invariant) Maurer-Cartan form, and ξ is an arbitrary element of the Lie algebra $\mathfrak{g}$ which generates a Hamiltonian vector field $\xi^{\#}$ on M via the action of G.

Examples of q-Hamiltonian spaces include conjugacy classes, the double $G \times G$ of a compact Lie group G, and S^4 [1, 2]. A fundamental property is that if M_1 and M_2 are both q-Hamiltonian G-spaces, then $M_1 \times M_2$ is also a q-Hamiltonian G-space (via the fusion product).

If $c \in Z(G)$, the reduced space M_c of M at c is defined by $M_c = \Phi^{-1}(c)/G$. If c is a regular value of Φ, then M_c is a symplectic orbifold (it is a symplectic manifold if in addition G acts freely on $\Phi^{-1}(c)$). In [2] (see also [3]) a formula was given for intersection numbers in M_c, in terms of fixed point data on M (the components F of fixed point sets of subgroups of the maximal torus T of G, the values of $\Phi(F)$, and the equivariant Euler classes of the normal bundles to F in M).

In [7] Jeffrey and Kirwan gave formulas for intersection numbers in reduced spaces of Hamiltonian G-spaces $\widetilde{M}$ in terms of fixed point data. In [8] they adapted these methods to give formulas for intersection numbers in the moduli space $M(n, d)$ of semistable holomorphic vector bundles of rank n, degree d and fixed determinant over a compact Riemann surface, when n and d are coprime. The space $M(n, d)$ is the motivating example for q-Hamiltonian reduced spaces, since it is the q-Hamiltonian reduced space M_c of the q-Hamiltonian space G^{2g} where $G = SU(n)$ and the action of G is by conjugation, and $c = e^{2\pi i d/n}\mathbb{I}$. In this talk (which describes the results in [9]) we adapt the methods of [8] to give formulas for intersection numbers in reduced spaces of q-Hamiltonian spaces. By this means we recover the formulas of [2].

The key steps may be summarized as follows.

Step 1: For a q-Hamiltonian G-space M we define a Hamiltonian G-space $\widetilde{M}$ via the following construction:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mu} & \mathfrak{g} \\ \pi_1 \big\downarrow & & \big\downarrow c\exp \\ M & \xrightarrow{\Phi} & G \end{array}$$

The space $\widetilde{M}$ has a nondegenerate closed 2-form $\Omega = \pi_1^*\omega + \mu^*\sigma$ where $\sigma \in \Omega^2(\mathfrak{g})$ satisfies $d\sigma = \exp^*\chi$.

Step 2 (equivariant Poincaré dual): The space $\widetilde{M}$ is singular. This leads us to introduce $\alpha \in H^*_G(M \times \mathbf{g})$ with the property that for all forms $\eta \in \Omega^*_G(\widetilde{M})$ supported on the smooth locus of $\widetilde{M}$, we have

$$\int_{\widetilde{M}} \eta = \int_{M \times \mathfrak{g}} \eta\alpha.$$

This enables us to replace the singular space $\widetilde{M}$ by the smooth manifold $M \times \mathfrak{g}$. The form α is the equivariant Poincaré dual of $\widetilde{M}$.

Step 3 (periodicity): Guillemin-Kalkman [6] and Martin [10] proved that if $a, b \in \mathfrak{t}^*$ are two regular values of the moment map for the action of a torus T of rank r on a Hamiltonian T-space $\widetilde{M}$, one has

$$\int_{\mu^{-1}(a)/T} \kappa_r(\eta e^{\bar{\omega}}) - \int_{\mu^{-1}(b)/T} \kappa_r(\eta e^{\bar{\omega}}) = \sum_i \int_{M_i} \kappa_{r-1}\left(\mathrm{Res}_{Y=0} \frac{\eta e^{\bar{\omega}}}{e_{M_i}}\right)$$

where $\kappa_r \colon H_T^*(M) \to H^*(\mu^{-1}(a)/T$ is the Kirwan map in rank r, and M_i is a component of the fixed point set of a one parameter subgroup $S \cong S^1$ (whose Lie algebra is generated by $\hat{e}_1 \in \mathfrak{t}$ and with $Y = \langle \hat{e}_1, X\rangle$ for $X \in \mathfrak{t}$), for which the image of M_i under the moment map intersects a ray between a and b. The final result expresses $\kappa(\eta e^{\bar{\omega}})$ as the iterated residue of a sum over the fixed point set of the maximal torus T acting on $\widetilde{M}$.

If $\lambda \in \mathfrak{t}$ (the Lie algebra of T) satisfies the condition that $\exp \lambda = 1$, then

$$\mu^{-1}(\lambda + t)/T \cong \mu^{-1}(t)/T$$

for all $t \in \mathfrak{t}$ as symplectic manifolds. The only difference is that the (constant) value of the moment map has been shifted by the addition of λ. We combine this with the previous paragraph to obtain (when 0 is a regular value for μ)

$$\int_{\mu^{-1}(0)/T} \kappa_r(\eta e^{\bar{\omega}}) = \sum_{M_i \subset M^S : -|\hat{e}_1|^2 < \langle \hat{e}_1, \mu(M_i)\rangle < 0} \int_E \kappa_{r-1} \mathrm{Res}_{Y=0} \frac{\eta e^{\bar{\omega}}}{e_{M_i}(1 - e^Y)}$$

where $Y = \langle \hat{e}_1, X\rangle$ for $X \in \mathfrak{t}$. We use this, combined with a key lemma which asserts that the fixed point set of a circle subgroup acting on a q-Hamiltonian G-space is itself a q-Hamiltonian H-space where H is a subgroup of G of lower rank, to enable us to make an argument by induction on the rank of G.

Step 4 (Szenes' theorem): To recover the formula of Alekseev-Meinrenken-Woodward [2], we make use of a result of Szenes [11] (see also Brion-Vergne [4, 5]) which expresses an iterated residue as the sum of a meromorphic function evaluated at points of the weight lattice.

References

[1] A. Alekseev, A. Malkin, E. Meinrenken, Lie group valued moment maps. *J. Differential Geometry* **48** (1998), no. 3, 445-449.

[2] A. Alekseev, E. Meinrenken, C. Woodward, Group-valued equivariant localization. *Inventiones Mathematicae* **140** (2000) 327-350.

[3] A. Alekseev, E. Meinrenken, C. Woodward, Duistermaat-Heckman measures and moduli spaces of flat bundles over surfaces. *Geometric and Functional Analysis* **12** (2002), 1–31.

[4] M. Brion, M. Vergne, Arrangement of hyperplanes I: Rational functions and Jeffrey-Kirwan residue. *Ann. Scient. Ec. Norm. Sup.* **32** (1999) 715-741.

[5] M. Brion, M. Vergne, Arrangement of hyperplanes II: Szenes formula and Eisenstein series. *Duke Math. J.* **103** (2000) 279-302.

[6] V. Guillemin, J. Kalkman, The Jeffrey-Kirwan localization theorem and residue operations in equivariant cohomology. *J. reine angew. Math.* **470** (1996) 123-142.

[7] L. C. Jeffrey, F. C. Kirwan, Localization for nonabelian group actions. *Topology* **34** (1995), 291-327.
[8] L. C. Jeffrey, F. C. Kirwan, Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface. *Annals of Mathematics* **148** (1998) 109-196.
[9] L.C. Jeffrey, J.-H. Song, A new proof of formulas for intersection numbers in q-Hamiltonian reduced spaces. Preprint math.SG/0312345 (2003).
[10] S.K. Martin, Symplectic quotients by a nonabelian group and by its maximal torus, preprint math.SG/0001002; Transversality theory, cobordisms, and invariants of symplectic quotients, preprint math.SG/0001001; *Ann. Math.*, to appear.
[11] A. Szenes, Iterated residues and multiple Bernoulli polynomials. *International Mathematics Research Notices* **18** (1998) 937-956.
[12] E. Witten, Two-dimensional gauge theories revisited. *J. Geom. Phys.* **9** (1992), 303 -368.

The Hermitian eigenvalue problem and a new product in the cohomology of flag varieties

Shrawan Kumar

Let G be a complex semisimple algebraic group and let K be a maximal compact subgroup with their Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Choose a maximal subalgebra (which is necessarily abelian) $\mathfrak{a} \subset \mathfrak{p}$ and let $\mathfrak{a}_+$ be a dominant chamber in $\mathfrak{a}$. Then any K-orbit in $\mathfrak{p}$ intersects $\mathfrak{a}_+$ in a unique point.

For any $n \geq 2$, the celebrated *Hermitian eigenvalue problem* concerns determining the following subset Δ_n of $\mathfrak{a}_+^n$:

$$\Delta_n := \big\{(a_1, \ldots, a_n) \in \mathfrak{a}_+^n : \exists (x_1, \ldots, x_n) \in \mathfrak{p}^n \text{ with } \textstyle\sum x_i = 0 \text{ and } x_i \in \operatorname{Ad} K.a_i\big\}.$$

By works of several mathematicians including Klyachko, Berenstein-Sjamaar, and Belkale, Δ_n is given by certain inequalities parametrized by standard maximal parabolic subgroups P of G and n Schubert cohomology classes $\epsilon_{w_1}^P, \ldots, \epsilon_{w_n}^P$ such that

$$\epsilon_{w_1}^P \cdot \cdots \cdot \epsilon_{w_n}^P = \epsilon^P,$$

where ϵ^P is the top cohomology class of G/P.

But, as shown by Kumar-Leeb-Millson, these sets of inequalities are, in general, not irredundant.

Now, the main topic of this talk is a recent joint work with Belkale. We give a new commutative and associative product in the cohomology $H^*(G/P)$ of any flag variety G/P (which still satisfies the Poincaré duality) and show that the ineqalities determining Δ_n are given in terms of this new product in $H^*(G/P)$ for maximal parabolics P. This results in general in far fewer inequalities determining Δ_n. We show that for simple groups of rank 3, our new set of inequalities is an irredundant system.

We believe that similar results can be obtained for the cone determining when the product of n elements in K is 1 in terms of the modified product in the quantum cohomology of G/P's.

References

[AMW] A. Alekseev, E. Meinrenken and C. Woodward, *Linearization of Poisson actions and singular values of matrix products*, Ann. Inst. Fourier (Grenoble), vol. **51** (2001), no. 6, p. 1691–1717.

[Bel] P. Belkale, *Local systems on $\mathbb{P}^1$-S for S a finite set*, Compositio Math., vol. **129** (2001) p. 67–86.

[BeSj] A. Berenstein and R. Sjamaar, *Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, Journ. Amer. Math. Soc., vol. **13** (2000), no. 2, p. 433–466.

[BGG] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, *Schubert cells and cohomology of the spaces G/P*, Russian Math. Surveys, vol. **28** (1973), p. 1–26.

[Fu] W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.), vol. **37** (2000), no. 3, p. 209–249.

[He] S. Helgason, "Differential Geometry, Lie groups, and Symmetric Spaces", Academic Press, 1978.

[KLM1] M. Kapovich, B. Leeb, J. J. Millson, *Polygons in symmetric spaces and buildings*, Preprint, 2002.

[KLM2] M. Kapovich, B. Leeb, J. J. Millson, *The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra*, Preprint, 2002.

[Kly98] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Mathematica, vol. **4** (1998), p. 419–445.

[Kly99] A. Klyachko, *Random walks on symmetric spaces and inequalities for matrix spectra*, Linear Algebra Appl. Special Issue: Workshop on Geometric and Combinatorial Methods in the Hermitian Sum Spectral Problem (Coimbra, 1999), vol. **319** (2000), no. 1-3, p. 37–59.

[KTW] A. Knutson, T. Tao, C. Woodward *The honeycomb model of $GL(n)$ tensor products II: Puzzles determine the facets of the Littlewood-Richardson cone*, to appear in J.A.M.S.

[Ku] S. Kumar, "Kac-Moody Groups, their Flag Varieties and Representation Theory", Progress in Mathematics,vol. **204**, Birkhäuser, 2002.

[KLM] S. Kumar, B. Leeb, J. J. Millson, *The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type*, Contemp. Math., vol. **332** (2003), p. 171–195.

[LM] B. Leeb, J. J. Millson, *Convex functions on symmetric spaces and geometric invariant theory for spaces of weighted configurations on flag manifolds*, Preprint.

Equivariant cohomology of supervector bundles

Pascal Lavaud

1. Introduction

The motivation of this work is the generalization of the Berline-Vergne's localization formula (cf. [3]) to the supergeometric situation (cf. [6, 7]). The first step consist to construct an equivariant Thom form for an Euclidean oriented supervector bundle. Since the odd part of an Euclidean supervector bundle is a symplectic vector bundle, in a first part we study some aspects of equivariant cohomology of symplectic vector bundles. Let $\mathcal{V} \to M$ be an equivariant symplectic vector bundle. We construct an equivariant form with generalized coefficients α which is integrable along the fibres of $\mathcal{V}$ and such that its restriction to M is 1. We give some of its properties.

In a second part we recall some basic definitions of supergeometry. We define change of parity Π on a supervector space V and a "Fourier transform" between forms on V and forms on ΠV. When ΠV is symplectic and and $V_{\mathbf{0}}$ is oriented, we show that the "Fourier transform" of the above form α on ΠV gives the Mathai-Quillen construction of a Thom form on the Euclidean supervector space V.

2. Symplectic case

General situation. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\pi\colon \mathcal{V} \to M$ be a G-equivariant vector bundle of rank k. Let $\Gamma_{\mathcal{V}}$ the sheaf of sections of $\mathcal{V}$. Let B a G-invariant symplectic structure on the fibers. It induces an orientation of $\mathcal{V}$. For $X \in \mathfrak{g}$, we denote by X_M the vector field on M generated by the infinitesimal action of G in the direction of X. For a vector field ζ on M we denote by $\iota(\zeta)\colon \Omega(M) \to \Omega(M)$ the operator of contraction by ζ. Let (Z_i) be a basis of $\mathfrak{g}$ and (z^i) be its dual basis. We put: $\iota = \sum_i z^i \iota(Z_{iM})$.

Integrable forms. We say that $\omega \in \Omega(\mathcal{V})$ is integrable (resp. integrable along the fibres) if ω is compactly supported on M and rapidly decreasing in the direction of the fibers (resp. is rapidly decreasing in the direction of the fibers). We denote by $\Omega_{\int}(\mathcal{V})$ (resp. $\Omega_{\pi_*}(\mathcal{V})$) the set of integrable forms (resp. forms integrable in the fibers). For $\omega \in \Omega_{\pi_*}(\mathcal{V})$, we denote by $\int_{\mathcal{V}/M} \omega$ its integral along the fibers.

Equivariant forms with generalized coefficients. We denote by

$$\Omega_G^{-\infty}(M) = \Big(\mathcal{C}^{-\infty}(\mathfrak{g})\widehat{\otimes}\Omega(M)\Big)^G$$

the set of equivariant forms with generalized coefficients introduced by Kumar-Vergne [5]. We denote by $\Omega_{G,\int}^{-\infty}(\mathcal{V})$ (resp. $\Omega_{G,\pi_*}^{-\infty}(\mathcal{V})$) the set of integrable (resp. integrable along the fibres) equivariant forms with generalized coefficients on $\mathcal{V}$. Let $\alpha \in \Omega_{G,\int}^{-\infty}(\mathcal{V})$ (resp. $\alpha \in \Omega_{G,\pi_*}^{-\infty}(\mathcal{V})$). Let dX be a Lebesgue measure on $\mathfrak{g}$ and $f(X) \in \mathcal{C}_c^{\infty}(\mathfrak{g})$. Then, by definition: $\int_{\mathfrak{g}} dX f(X)\alpha(X) \in \Omega_{\int}(\mathcal{V})$ (resp. $\in \Omega_{\pi_*}(\mathcal{V})$).

We define the equivariant differential by $d_{\mathfrak{g}} = d - \boldsymbol{i}\iota$. We use the notion of superconnection $\mathbb{A}$ on $\mathcal{V}$ defined by Mathai-Quillen [9] and of equivariant connection $\mathbb{A}_{\mathfrak{g}}$, equivariant curvature $F_{\mathfrak{g}}$ and equivariant moment μ as defined in [2].

A special condition. We assume that there exists a covering of M by open subsets $W \subset M$ such that

$$O(W) = \Big\{X \in \mathfrak{g} \Big/ \forall m \in W, \forall v \in \mathcal{V}_m \setminus \{0\}, B(v, \mu(X)v) > 0\Big\}$$

contains a non-empty open subset.

Definition of the form α. Let (e_j) be a local basis of sections of $\mathcal{V}$ and (y^j) be its dual basis. We put $\epsilon = \sum_j e_j y^j \in \Big(\Gamma_{\mathcal{V}}(M) \underset{\mathcal{C}^\infty(M)}{\otimes} \mathcal{C}^\infty(\mathcal{V})\Big)^G = \Gamma_{\mathcal{V}}(\mathcal{V} \underset{M}{\times} \mathcal{V})^G$. We lift $\mathbb{A}$ to a connection of $\mathcal{V} \underset{M}{\times} \mathcal{V} \to \mathcal{V}$. We put

$$\alpha = \exp\Big(-\frac{1}{2}d_{\mathfrak{g}}\Big(B(\epsilon, \mathbb{A}_{\mathfrak{g}}\epsilon)\Big)\Big) = \exp\Big(-\frac{1}{2}B(\epsilon, \mathbb{A}_{\mathfrak{g}}^2\epsilon) - \frac{1}{2}B(\mathbb{A}_{\mathfrak{g}}\epsilon, \mathbb{A}_{\mathfrak{g}}\epsilon)\Big) \in \Omega_G(\mathcal{V}).$$

We denote by $\boldsymbol{j}\colon M \hookrightarrow \mathcal{V}$ the zero section. Thus $\boldsymbol{j}^*\colon \Omega(\mathcal{V}) \to \Omega(M)$ is the restriction morphism. We have $\alpha \in \Omega^{-\infty}_{G,\pi_*}(\mathcal{V})$, $d_{\mathfrak{g}}\alpha = 0$ and $\boldsymbol{j}^*\alpha = 1$.
Remark: We have $\alpha \in \Omega_G(\mathcal{V})$ and $\alpha \in \Omega^{-\infty}_{G,\pi_*}(\mathcal{V})$, but $\alpha \notin \Omega_{G,\pi_*}(\mathcal{V})$.
Injection in cohomology. Let $\omega \in \Omega_{G,\int}(\mathcal{V})$. Then in $H^{-\infty}_{G,\int}(\mathcal{V})$ we have the following equality: $\omega \equiv \pi^*(\boldsymbol{j}^*\omega)\alpha$. It follows that the map $H_{G,\int}(\mathcal{V}) \to H_{G,\int}(M)$, $\omega \mapsto \boldsymbol{j}^*\omega$ is injective and $\int_{\mathcal{V}} \omega(X) = \int_M \Big(\int_{\mathcal{V}/M} \alpha(X)\Big) \boldsymbol{j}^*\omega(X)$.
Inverse Euler form. Assume that $\mathcal{V}$ has an Euclidean structure Q which is G- and $\mathbb{A}$-invariant. Let $\mathcal{E}_{\mathfrak{g}} \in \Omega_G(M)$ be an element of the equivariant Euler class of $\mathcal{V}$. Let $W \subset M$ and $U(W) \subset \mathfrak{g}$ be open subsets such that for $X \in \mathcal{U}(W)$, the form $\mathcal{E}_{\mathfrak{g}}(X)$ is invertible on W. Then $\mathcal{E}_{\mathfrak{g}}^{-1} \equiv \frac{1}{(2\boldsymbol{i}\pi)^{\frac{k}{2}}} \int_{\mathcal{V}/M} \alpha$ in $H\big(\mathcal{C}^\infty(U(W), \Omega(W))^G, d_{\mathfrak{g}}\big)$.
"Super" remark: All what has been said above can easily be generalized to super objects with one technical supplementary condition on the connection $\mathbb{A}$.

3. Super

Supermanifolds. Now we assume that M is a point and $\mathcal{V} = V = V_{\mathbf{0}} \oplus V_{\mathbf{1}}$ is a supervector space. We put $\dim(V) = (\dim(V_{\mathbf{0}}), \dim(V_{\mathbf{1}})) = (m, n)$. For an homogenous element $v \in V$ we denote by $p(v) \in \mathbb{Z}/2\mathbb{Z}$ its parity. A supermanifold structure on V is given by a sheaf $\mathcal{C}_V$ of superalgebras such that for any open subset $\mathcal{U} \subset V_{\mathbf{0}}$, $\mathcal{C}_V(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U}) \otimes \Lambda(V_{\mathbf{1}}^*)$. Let $(x^1, \ldots, x^m)$ be a basis of $V_{\mathbf{0}}^*$ and $(\xi^1, \ldots, \xi^n)$ be a basis of $V_{\mathbf{1}}^*$. Let $\mathcal{U} \subset V_{\mathbf{0}}$ be open and $f \in \mathcal{C}_V(\mathcal{U})$. Then we write $f = \sum_{I \in \{0,1\}^n} \xi^I f_I(x^1, \ldots, x^m)$ where $f_I \in \mathcal{C}^\infty(\mathcal{U})$.
Reverse parity. Let ΠV the supervector space with $(\Pi V)_{\mathbf{0}} = V_{\mathbf{1}}$ and $(\Pi V)_{\mathbf{1}} = V_{\mathbf{0}}$. We denote by $\pi\colon V \to \Pi V$ the "odd identity". For $\phi \in V^*$ non-zero and homogeneous and $v \in V$ we put $(\pi\phi)(\pi v) = (-1)^{p(\phi)}\phi(v)$. This can be linearly extended to an odd isomorphism $V^* \to (\Pi V)^*$. Similarly for any non-zero homogeneous $\phi \in \mathfrak{gl}(V)$ and any $v \in V$ we put $\phi(\pi v) = (-1)^{p(\phi)}\pi\phi(v)$. This induces an even isomorphism $\mathfrak{gl}(V) \to \mathfrak{gl}(\Pi V)$. This way we identify the two algebras.
Pseudodifferential forms. Let V be a supervector space. We put $\widehat{V} = V \oplus \Pi V$. Since the tangent space of V is $TV = V \oplus V$, the space $\widehat{V}$ is TV with reverse parity in the fibers. We define the pseudodiffential forms on V as the functions on $\widehat{V}$. We denote by $\widehat{\Omega}(V) = \mathcal{C}_{\widehat{V}}((\widehat{V})_{\mathbf{0}})$ the algebra of pseudodifferential forms on V.

We put $dx^i = \pi x^i$ and $d\xi^j = \pi\xi^j$ ($p(dx^i) = 1$ and $p(d\xi^j) = 0$). For $\omega \in \widehat{\Omega}(V)$ we have $\omega = \sum_{I,J} dx^I \xi^J \omega_{I,J}(x, d\xi)$ where the $\omega_{I,J}$ are smooth functions of the variables x^i and $d\xi^j$. The exterior differential is the vector field on $\widehat{V}$ defined by $d = \sum_i dx^i \frac{\partial}{\partial x^i} + \sum_j d\xi^j \frac{\partial}{\partial \xi^j}$. Other differential operators as defined in the same way.
Integration. We say that $\omega \in \widehat{\Omega}(V)$ is integrable if all $\omega_{I,J}$ are rapidly decreasing in the x^i and in the $d\xi^j$. We denote by $\widehat{\Omega}_{\int}(V)$ the set of integrable differential forms on V. We assume that V is globally oriented, which means

that V is oriented as a vector space (without grading). Let $\omega \in \widehat{\Omega}_{\int}(V)$. We put $\int_V \omega = (-1)^{\frac{(n+m)(n+m-1)}{2}} \int_{\widehat{V}} |d_{(x,d\xi)}| \omega_{(1,\ldots,1),(1,\ldots,1)}(x, d\xi)$ where $|d_{(x,d\xi)}|$ is the Lebesgue measure on $(\widehat{V})_{\mathbf{0}}$. This does not depends on the globally oriented coordinate system (x, ξ) (cf. [1]).

We refer for example to [1, 4] for definitions of supergroups. The definition of equivariant forms is similar to the classical one (cf. [6, 8]).

"Fourier transform." We put: $\eta_i = \pi x^i$ and $y_j = \pi \xi^j$. Then (y^j, η^i) are standard coordinates on ΠV. We define a map $\chi\colon \widehat{\Omega}_{\int}(V) \to \widehat{\Omega}_{\int}(\Pi V)$ by

$$\chi(\omega) = \int_V \omega(x^i, \xi^j, dx^i, d\xi^j) \exp -\boldsymbol{i}\Big(\sum_j y^j d\xi^j + dy^j \xi^j + \sum_i \eta^i dx^i + d\eta^i x^i\Big).$$

We denote by $\overline{\chi}\colon \widehat{\Omega}(\Pi V) \to \widehat{\Omega}(V)$ the map obtained by exchanging V and ΠV and replacing $-\boldsymbol{i}$ by $\boldsymbol{i}$. Let $\boldsymbol{j}_\Pi\colon \{0\} \hookrightarrow \Pi V$ be the canonical injection.

We have in particular for $\omega \in \widehat{\Omega}_{\int}(V)$: $\int_V \omega = \boldsymbol{j}_\Pi^*(\chi(\omega))$ and $\overline{\chi}\big(\chi(\omega)\big) = (2\pi)^{m+n}\omega$. When V is a G-vector space, χ induces an isomorphism in equivariant cohomology.

Thom form. We assume that V has an invariant Euclidean structure Q and is globally oriented. This means that $V_{\mathbf{0}}$ is oriented, $Q|_{V_{\mathbf{0}}}$ is a scalar product and $Q|_{V_{\mathbf{1}}}$ is a symplectic form. We define a symplectic structure ΠQ on ΠV by $\Pi Q(\pi v, \pi w) = (-1)^{p(v)} Q(v, w)$. We assume that there exists an $X \in \mathfrak{g}_{\mathbf{0}}$ such that $Q(v, Xv) > 0$ for any non-zero $v \in V_{\mathbf{1}}$. As in the fist part, we can construct the form $\alpha \in \widehat{\Omega}^{-\infty}_{G,\int}(\Pi V)$ such that $d_{\mathfrak{g}}\alpha = 0$ and $\boldsymbol{j}_\Pi^*\alpha = 1$. Then the form $\theta = \overline{\chi}(\alpha)$ is a Thom form on V. This means that $\theta \in \widehat{\Omega}^{-\infty}_{G,\int}(V)$, $d_{\mathfrak{g}}\theta = 0$ and $\int_V \theta(X) = 1$. This gives the same form as the Mathai-Quillen representative of a Thom form (cf. [9]).

Remark: The use of generalized coefficients is necessary. Otherwise we could evaluate $\theta(0) \in \Omega_{\int}(V)$ which should satisfy $d\theta(0) = 0$. But is easy to see that if $V \neq V_{\mathbf{0}}$, this implies $\int_V \theta(0) = 0$.

References

[1] F.A. Berezin. *Introduction to Superanalysis.* MPAM D. Reidel Pub. Comp., 1987.

[2] N. Berline, E. Getzler, M. Vergne. *Heat Kernels and Dirac Operators.* Springer, 1992.

[3] N. Berline, M. Vergne. Zéros d'un champ de vecteurs et classes caractéristiques équivariantes. *Duke Math. J.*, 539–549, 1983.

[4] B. DeWitt. *Supermanifolds.* Cambridge University Press, 1984.

[5] S. Kumar, M. Vergne. Equivariant cohomology with generalized coefficients. *Astérisque*, 215, 1993.

[6] P. Lavaud. Formule de localisation en supergéométrie. Thèse, Paris VII, 1998.

[7] P. Lavaud. Localization formula in supergeometry. e-print math.DG/0402068, 2004.

[8] P. Lavaud. Superpfaffian. e-print math.GR/0402067, 2004.

[9] V. Mathai, D. Quillen. Superconnections, thom classes, and equivariant differential forms. *Topology*, 25, No 1:85–110, 1986.

Morse theory on Hamiltonian G-spaces and bases of equivariant K-theory

Mikhail Kogan
(joint work with Victor Guillemin)

This talk is based on the joint work with Victor Guillemin [GK].

Let M be a compact symplectic manifold, G an n-dimensional compact torus and $\sigma\colon G \times M \to M$ a Hamiltonian action. Assume the fixed point set M^G is finite. Let T be a circle subgroup of G with the property that $M^T = M^G$ and let $\phi\colon M \to \mathbb{R}$ be the T moment map. This function is a Morse function and all its critical points are of even index; so, by standard Morse theory, the unstable manifolds of ϕ with respect to a G-invariant Riemannian metric define a basis of $H_*(M,\mathbb{R})$ and by Poincare duality a basis for $H^*(M,\mathbb{R})$ consisting of the Thom classes of the closures of unstable manifolds. Moreover, these unstable manifolds are G-invariant so they also define a basis for $H^*_G(M)$ as a module over $H^*_G(\mathrm{pt})$.

In K-theory the situation is a little more complicated. The critical points of ϕ carry a natural partial order, which is defined by setting $p \leq q$ if q is inside the closure of the unstable manifold of ϕ at p and then completing this order by transitivity. So, for any unstable manifold U of ϕ at p one can consider the union

$$W_U = \bigcup U_q$$

of unstable manifolds U_q for $q \geq p$. It is known that there exist classes in K-theory which are supported on this set. However, except in certain special cases (e.g., algebraic torus actions), it is not known whether there is a genuine (Thom) class in K-theory associated with U. (For algebraic torus actions such classes can be defined using the structure sheaf of the closure of U, see [BFM] for details).

We can show, however, that there is another way of attaching to the Morse decomposition of M a basis of $K_G(M)$ which works even in the case of nonalgebraic torus actions. (In the algebraic case our classes will be different from those constructed using structure sheaves.) The key idea in our approach is a notion of *local index* for a K-class $a \in K_G(M)$ at a critical point p of ϕ. This is defined as follows: Let S be the stable manifold of ϕ at p, and for small $\varepsilon > 0$ let S_ε be the compact symplectic orbifold obtained from S by the symplectic cutting operation of Lerman [Ler]. We recall that S_ε is obtained from the manifold with boundary

$$\tilde{S}_\varepsilon = \{x \in S, \phi(x) \geq \phi(p) - \varepsilon\} \tag{1}$$

by collapsing to points the T-orbits on the boundary. Then there is a naturally defined map $\kappa_\varepsilon\colon K_G(M) \to K_G(S_\varepsilon)$.

Now let *the local index* of $a \in K_G(M)$ at p,

$$I_p(a) \in K_G(\mathrm{pt})$$

be the Atiyah-Segal index of $\kappa_\varepsilon(a)$, that is, the pushforward of $\kappa_\varepsilon(a)$ with respect to the map $S_\varepsilon \to \mathrm{pt}$. Recall that $K_G(\mathrm{pt})$ is just the representation ring $R(G)$ of the torus G, so that each local index is just a virtual representation of G. Our main result is the following theorem.

Theorem 1. *Let p be a critical point of ϕ and U the unstable manifold of ϕ at p. Then there exists a unique K-theory class $\tau_p \in K_G(M)$ with the properties:*

(i) $I_p(\tau_p) = 1$,
(ii) $I_q(\tau_p) = 0$ *for all critical points q of ϕ except p,*
(iii) The restriction of τ_p to a critical point q is zero unless $q \in W_U$.

Moreover, the τ_p's generate $K_G(M)$ freely as a module over $K_G(\mathrm{pt})$.

Let $\mathcal{I}\colon K_G(M) \to K_G(M^G)$ be the map which takes the value I_p at p. This we will call the *total index map.* (Note that the total index is not an $R(G)$-module homomorphism but it is a homomorphism with respect to the subring, $R(G/T)$, of $R(G)$.) Theorem 1 implies

Corollary 2. *The total index map, $\mathcal{I}$, is an $R(G/T)$ module isomorphism.*

Remark 3. *Notice that local indices can also be defined in the setting of equivariant cohomology. Namely, for $a \in H^*_G(M)$, we let $I_p(a)$ be the pushforward (the integral over S_ε) of $\kappa'_\varepsilon(a)$, where $\kappa'_\varepsilon\colon H^*_G(M) \to H^*_G(S_\varepsilon)$. An analogue of Theorem 1 holds for equivariant cohomology, and the cohomological analogues of the τ_p's are "the equivariant Poincare duals" of the closures of the unstable manifolds.*

We can also prove a constructive version of Theorem 1 for GKM spaces, that is, an explicit computation of the classes τ_p. Let us recall some facts about GKM spaces. The *one-skeleton* of M

$$\{x \in M,\ \dim G \cdot x = 1\} \tag{2}$$

is a union of symplectic submanifolds of M. The action σ is defined to be *a GKM action* and M *a GKM space* if each component of the one-skeleton is exactly of dimension 2 and hence a symplectic two-sphere. Then there is a graph Γ associated to σ, whose vertices are given by the fixed points and edges are given by the two-spheres of the one-skeleton. Moreover, Γ is equipped with a function which labels each oriented edge $e = (p \to q)$ of Γ by the weight α_e of the isotropy representation of G on the tangent space at p of the two-sphere corresponding to e.

One knows that the restriction map

$$K_G(M) \to K_G(M^G) = \bigoplus_{i=1}^{\ell} K_G(p_i) \tag{3}$$

is injective. Since $K_G(\mathrm{pt}) = R(G)$, an element of the ring $K_G(M^G)$ is just a map on M^G,

$$\chi\colon M^G \to R(G) \tag{4}$$

Theorem 4 ([At, KR][1]). *For each $e \in E$ connecting $p, q \in M^G$ the homomorphisms*

$$e^{2\pi\sqrt{-1}\alpha_{e_p}}: G \to S^1 \text{ and } e^{2\pi\sqrt{-1}\alpha_{e_q}}: G \to S^1$$

[1]An analogous statement for equivariant cohomology is proved in [GKM].

have the same kernel. Denote this kernel by G_e. Then the element (4) of $K_G(M^G)$ is in the image of (3) if and only if for every $e \in E$

$$r_e(\chi_p) = r_e(\chi_q) \tag{5}$$

p and q being the vertices of e and r_e the restriction map $R(G) \to R(G_e)$.

For GKM manifolds one can translate some aspects of Morse theory into the language of graphs. Recall that ϕ is the moment map on M with respect to the circle T-action. Think of each edge e of the graph connecting vertices p and q as two oriented edges e_p and e_q. Then if $\phi(p) > \phi(q)$ we say that the edge e_p going from p to q is *descending* and e_q from q to p *ascending.* If U is the unstable manifold of ϕ at p then every fixed point, q, inside W_U^G is the terminal point of a path on Γ starting at p and consisting of ascending edges; and this gives one a way of describing W_U in terms of Γ. In particular, we prove an explicit formula for the image of τ_p under the imbedding (3), which expresses the restriction of τ_p to $q \in M^G$ as a sum of combinatorial expressions associated with the ascending paths in Γ going from p to q. (An analogous formula for the cohomological counterpart of τ_p can be found in [GZ].) This formula follows from the following theorem, which allows one to compute local indices in terms of restrictions of K-theory classes to fixed points and vice versa.

Theorem 5. *For $p \in V = M^G$, let $e_1, \ldots, e_m$ be the descending edges with initial vertex at p. Let the edge e_i connect p to q_i and be labeled by the weight α_i. Then for any $a \in K_G(M)$ we have*

$$I_p(a) = \sum_{i=1}^{m} \tilde{\pi}_i \tilde{r}_i \Big(\frac{a_{q_i}}{(1-\zeta)\prod_{j\neq i}\big(1-e^{2\pi\sqrt{-1}\alpha_j}\big)} \Big) + \frac{a_p}{\prod_{i=1}^{m}(1-e^{2\pi\sqrt{-1}\alpha_i})}, \tag{6}$$

where a_q is the restriction of a to q, ζ is the generator of the character ring $R(T)$, $\tilde{r}_i$ is the restriction $R(G \times T) \to R(G_{e_i} \times T)$ and $\tilde{\pi}_i \colon R(G_{e_i} \times T) \to R(G)$ is the pushforward map produced by averaging along the fibers of the map $G_{e_i} \times T \to G$.

This theorem is proved by applying Atiyah-Segal localization formula [AS] for the cut space S_ε. We can also prove a combinatorial version of this theorem which is a natural generalization of Lagrange interpolation formula.

References

[At] M. F. Atiyah, *Elliptic operators and compact groups.* Lecture Notes in Mathematics, Vol. 401. Springer-Verlag, Berlin-New York, 1974

[AS] M. F. Atiyah and G. B. Segal, *The index of elliptic operators: II.* Ann. of Math. (2) 87 1968 531–545, MR0236951, Zbl 0164.24201.

[BFM] P. Baum, W. Fulton, R. MacPherson, *Riemann-Roch and topological K theory for singular varieties.* Acta Math. 143 (1979), no. 3-4, 155–192, MR0549773, Zbl 0474.14004.

[GKM] M. Goresky, R. Kottwitz and R. MacPherson, *Equivariant cohomology, Koszul duality and the localization theorem.* Invent. Math. **131** (1998), no. 1, 25-83, MR1489894, Zbl 0897.22009.

[GK] V. Guillemin and M. Kogan, *Morse theory on Hamiltonian G-spaces and equivariant K-theory*, to appear in Journal of Differential Geometry.

[GZ] V. Guillemin and C. Zara, *Combinatorial formulas for products of Thom classes.* Proceedings of the Geometry, Mechanics, and Dynamics Workshop, Toronto, August 7-12, 2002, Springer-Verlag, NY, MR1919836, Zbl 1009.57034.

[KR] A. Knutson and I. Rosu, Appendix of *Equivariant K-theory and Equivariant Cohomology.* Mathematische Zeitschrift 243 (2003), 423-448, MR1970011, Zbl 1019.19003.

[Ler] E. Lerman, *Symplectic cuts.* Math. Res. Lett. 2 (1995), no. 3, 247–258, MR1338784, Zbl 0835.53034.

Integrals of equivariant forms in the setting of non-compact group actions

Matvei Libine

This lecture is based on my expository article [L6]. Let G be a real Lie group acting on a manifold M, and let $\mathfrak{g}$ be the Lie algebra of G. Consider an equivariantly closed form $\alpha(X)$ on M depending on $X \in \mathfrak{g}$. For $X \in \mathfrak{g}$, we denote by M^X the set of zeroes of the vector field on M induced by the infinitesimal action of X. Then the integral localization formula says that the integral of $\alpha(X)$ can be expressed as a sum over the set of zeroes M^X of certain *local* quantities of M and α:

$$\int_M \alpha(X) = \sum_{p \in M^X} \text{local invariant of } M \text{ and } \alpha \text{ at } p. \tag{1}$$

For compact groups this result was proved by N. Berline and M. Vergne [BV] and independently by M. Atiyah and R. Bott [AB] more than twenty years ago, but practically no progress had been made until very recently in generalizing it to non-compact group actions.

I use an interplay between recent results from representation theory and algebraic geometry to find such a generalization (3). This generalization provides, for instance, a geometric proof of the integral character formula from representation theory. These results strongly suggest that many theorems which were previously known in the compact group setting only can be generalized to non-compact groups.

Let G be a real linear reductive Lie group, denote by $G_{\mathbb{C}}$ its complexification, and set $\mathfrak{g}_{\mathbb{C}} = Lie(G_{\mathbb{C}})$. Let M be a smooth complex projective variety on which $G_{\mathbb{C}}$ act algebraically. The main result is a duality theorem between a certain class of G-invariant Borel-Moore homology cycles in the holomorphic cotangent space T^*M and a certain class of forms depending on $X \in \mathfrak{g}_{\mathbb{C}}$ for which the localization formula holds. Let σ be the canonical complex algebraic holomorphic symplectic form on T^*M, and let $\mu\colon T^*M \to \mathfrak{g}_{\mathbb{C}}^*$ be the ordinary holomorphic moment map. We pick another subgroup U of $G_{\mathbb{C}}$ such that, letting $\mathfrak{u}$ be the Lie algebra of U, we have an isomorphism $\mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}_{\mathbb{C}}$. For instance, U may equal G, but in most interesting situations U is a compact real form of $G_{\mathbb{C}}$ (i.e. a maximal compact subgroup).

The Borel-Moore cycles $\Lambda \subset T^*M$ over which we integrate are subject to the following three properties:

- Λ is G-invariant;
- Λ is real Lagrangian, i.e. $\mathrm{Re}\,\sigma|_\Lambda \equiv 0$ and $\dim_{\mathbb{R}} \Lambda = \dim_{\mathbb{R}} M$;
- Λ is *conic*, i.e. invariant under the scaling action of positive reals $\mathbb{R}^{>0}$ on T^*M (but not necessarily under the actions of $\mathbb{C}^\times$ or $\mathbb{R}^\times$).

For example, let $N \subset M$ be a closed G-invariant real submanifold, and let Λ be the conormal space $T^*_N M$ equipped with some orientation.

Any such cycle Λ can be realized as a characteristic cycle $Ch(\mathcal{F})$ of some G-equivariant constructible sheaf $\mathcal{F}$ (see [KS], [SV1]).

We denote by $\Omega^{(p,q)}(M)$ the space of complex-valued differential forms of type (p,q) on M. We consider forms $\alpha\colon \mathfrak{g}_{\mathbb{C}} \to \Omega^*(M)$ which satisfy the following three conditions:

- The assignment $X \mapsto \alpha(X) \in \Omega^*(M)$ depends holomorphically on $X \in \mathfrak{g}_{\mathbb{C}}$;
- For each $k \in \mathbb{N}$ and each $X \in \mathfrak{g}_{\mathbb{C}}$,
$$\alpha(X)_{[2k]} \in \bigoplus_{\substack{p+q=2k \\ p \geq q}} \Omega^{(p,q)}(M);$$
- The restriction of α to $\mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$ is an equivariantly closed form with respect to U.

For example, U-equivariant characteristic forms associated to U-equivariant vector bundles over M (see Section 7.1 of [BGV]) satisfy these conditions.

The integrals are defined as distributions on $\mathfrak{g}$, so let $\varphi \in \mathcal{C}^\infty_c(\mathfrak{g})$ be a test function, and let dX denote the Lebesgue measure on $\mathfrak{g}$. The new localization formula applies to integrals of the following kind:

$$(2) \qquad \int_\Lambda \Big(\int_{\mathfrak{g}} e^{\langle X, \mu(\xi)\rangle + \sigma} \wedge \varphi(X)\alpha(X)\, dX \Big)_{[\dim_{\mathbb{R}} M]}, \qquad X \in \mathfrak{g},\ \xi \in |\Lambda| \subset T^*M.$$

The idea to consider integrals of this kind was inspired by the shape of the integral character formula due to W. Schmid and K. Vilonen [SV1]. These integrals converge if, say, the moment map μ is proper on the support $|\Lambda|$.

Now the main result of [L4] says that if the support of φ lies in $\mathfrak{g}'$ ($\mathfrak{g}$ without a finite number of certain hypersurfaces) then integral (2) can be rewritten as

$$\int_\Lambda \Big(\int_{\mathfrak{g}} e^{\langle X, \mu(\xi)\rangle + \sigma} \wedge \varphi(X)\alpha(X)\, dX \Big)_{[\dim_{\mathbb{R}} M]} = \int_{\mathfrak{g}} F_\alpha(X)\varphi(X)\, dX,$$

where F_α is an $Ad(G \cap U)$-invariant function on $\mathfrak{g}'$ given by the formula

$$(3) \qquad F_\alpha(X) = \sum_{p \in M^X} m_p(X) \cdot \begin{pmatrix} \text{same term which appeared in the} \\ \text{classical localization formula (1)} \end{pmatrix},$$

where $m_p(X)$ is a certain integer multiplicity which is exactly the local contribution of p to the Lefschetz fixed point formula, as generalized to sheaf cohomology $H^*(M, \mathcal{F})$ by M. Goresky and R. MacPherson [GM]. These multiplicities are determined in [L4] in terms of local cohomology of $\mathcal{F}$, where $\mathcal{F}$ is any sheaf with characteristic cycle $Ch(\mathcal{F}) = \Lambda$.

Notice that the cycle Λ is invariant with respect to the action of the group G which need not be compact, while the form $\alpha\colon \mathfrak{g}_{\mathbb{C}} \to \Omega^*(M)$ is required to be equivariant with respect to a different group U only, but U may not preserve the cycle Λ.

References

[AB] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1-28.

[BGV] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag, 1992.

[BV] N. Berline and M. Vergne, *Classes caractéristiques équivariantes. Formules de localisation en cohomologie équivariante*, C. R. Acad. Sci. Paris **295** (1982), 539-541.

[GM] M. Goresky and R. MacPherson, *Local contribution to the Lefschetz fixed point formula*, Inventiones Math. **111** (1993), 1-33.

[KS] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Springer, 1990.

[L1] M. Libine, *A localization argument for characters of reductive Lie groups*, Jour. Func. Anal. **203** (2003), 197-236; also math.RT/0206019.

[L2] M. Libine, *A Localization argument for characters of reductive Lie groups: an introduction and examples*, Non-Commutative Harmonic Analysis: In Honor of Jacques Carmona, Progress in Mathematics, vol. 220, Birkhäuser, 2004, pp. 375-394; also math.RT/0208024.

[L3] M. Libine, *A fixed point localization formula for the Fourier transform of regular semisimple coadjoint orbits*, math.DG/0302352, 2003, to appear in the Jour. Func. Anal.

[L4] M. Libine, *Integrals of equivariant forms and a Gauss-Bonnet theorem for constructible sheaves*, math.DG/0306152, 2003.

[L5] M. Libine, *Riemann-Roch-Hirzebruch integral formula for characters of reductive Lie groups*, math.RT/0312454, 2003.

[L6] M. Libine, *On equivariant cohomology, integrals of equivariant forms and Duistermaat-Heckman measures for non-compact group actions*, math.SG/0403506, 2004.

[SV1] W. Schmid and K. Vilonen, *Characteristic cycles of constructible sheaves*, Inventiones Math. **124** (1996), 451-502.

[SV2] W. Schmid and K. Vilonen, *Two geometric character formulas for reductive Lie groups*, Jour. AMS **11** (1998), 799-876.

Reduction of Goresky-Kottwitz-MacPherson graphs, an analogue of symplectic reduction

Charles Cochet

In 1988, Thomas Delzant ([Del]) built a bridge between Hamiltonian geometry and the world of convex polytopes. For any symplectic compact connected manifold with Hamiltonian effective action, the dimension of the manifold is at least twice the dimension of the torus and the image of the manifold by the moment map is a convex polytope. Moreover, if this dimension is exactly twice that of the torus, then the convex polytope (named *Delzant polytope*) characterizes up to isomorphism the Hamiltonian manifold. In other terms, all data from the Hamiltonian manifold is stored in this polytope.

Demazure ([Dem]) introduced the notion of toric manifold (see also [Au] and [G]). A certain subcategory of these manifolds, containing in particular projective spaces, satisfies the hypotheses of Delzant's theorem. Unfortunately, many interesting

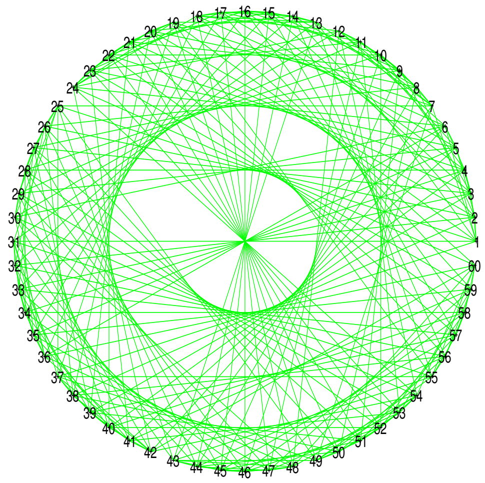


Figure 1. GKM graph of the manifold $GL(3,\mathbb{C})/B \times G_{2,5}(\mathbb{C})$ (screenshot from our program)

manifolds do not fulfill these drastic conditions, for example Grassmannians and flag manifolds. Thus, during the passage from these manifolds to their associated polytopes there is loss of data. Hence how to encode all data from a compact connected manifold with an action of a torus of arbitrary dimension?

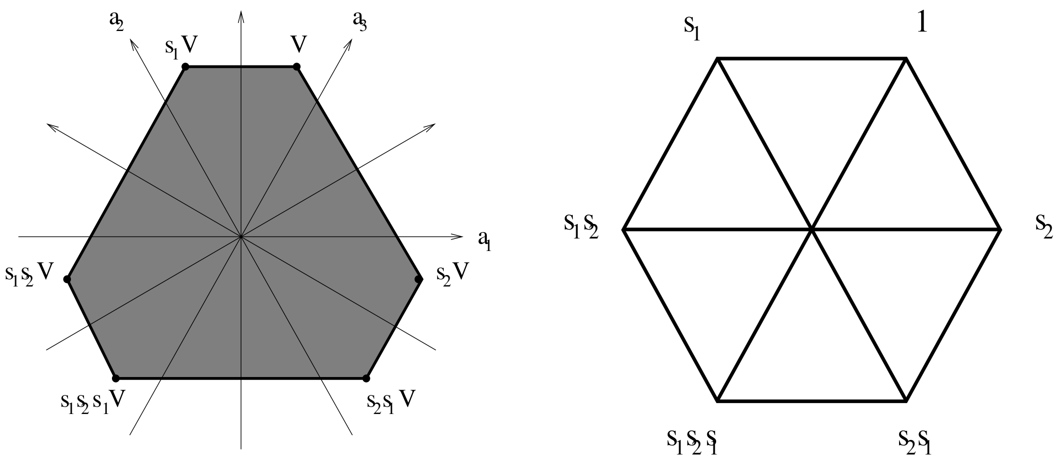


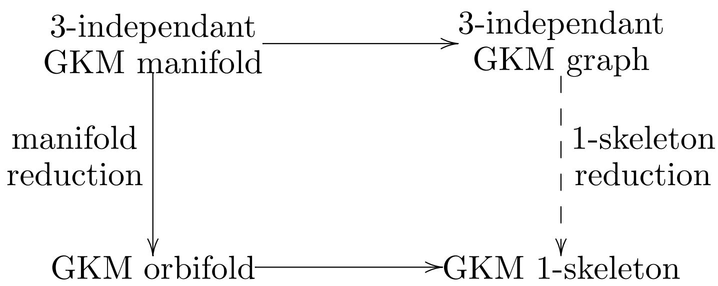
Figure 2. Image by the moment map of the coadjoint orbit $U(3)\cdot V$ and GKM graph of the flag manifold $U(3)/\{\text{diagonal matrices}\}$

Goresky, Kottwitz and MacPherson proved ([GKM]) that the ring of equivariant cohomology of certain compact connected manifolds — *GKM manifolds*, which we will discuss later — can be computed with tools from graph theory. Among GKM manifolds are toric manifolds and homogeneous spaces G/H, where G is a compact connected group and H a subgroup of G of the same rank.

Guillemin and Zara have highlighted a graph associated to each GKM manifold. This graph, oriented and with edges labeled by an *axial vector*, is called *Goresky-Kottwitz-MacPherson* (GKM) *graph* or 1-skeleton ([GZ2]). For example, in the Hamiltonian case, this graph takes into account the fact that images of fixed points by the moment map are linked not only by edges in the sense of polytopes (intersection of facets), but also sometimes by edges "inside" of the polytope.

Guillemin and Zara then "forgot" the underlying manifold: the *abstract* 1-*skeleton* was born. Since then, they have investigated properties of this new object ([GZ1], [GZ3], [GZ4]). They found many analogues of notions from symplectic geometry in graph theory: orientation, cohomology, K-theory, quantization. They also discovered that several classical theorems from symplectic geometry can be proved with only GKM graphs, like the Atiyah-Bott-Berline-Vergne localization theorem ([BV] and [AB]) and the Jeffrey-Kirwan theorem ([JK]).

Under certain hypotheses, one can compute the *reduction* of an abstract 1-skeleton by a 1-dimensional torus and at a regular value of a moment map of the graph. The reduction is still an abstract 1-skeleton. This graph operation imitates the reduction of a manifold, so that the reduction of the graph of a manifold is the graph of the reduced manifold (when this makes sense).



The probably most fascinating part of their research was the following. In the framework of the reduction of a GKM graph by a 1-dimensional torus, the invariant character of a K-theory element is in fact equal to a character built only from data coming from the reduced graph (and from the K-theory element), called the *reduced character*. This result is the analogue in graph theory of the assertion "quantization and reduction commute" from the symplectic world ([MS]). In addition to this, while the invariant character of a K-theory element is a rather big polynomial, the reduced character is a condensed rational fraction.

Reduction of a 1-skeleton is a fastidious task. If we go beyond low-dimension examples, we have to face intractable computations. For example the graph of the Grassmannian of complex 2-planes in $\mathbb{C}^5$ is 6-valent and has 10 vertices (hence 30 edges). Its reductions by the torus generated by $\xi = (0, 1, 2, 3, -6)$ are 5-valent and possess 6, 10, 12 and 14 vertices (hence 15, 25, 30 and 35 edges). Computer science can be of great help in order to study non-trivial examples.

Consequently I implemented in Maple the reduction of a 1-skeleton. The output is not only the data of the reduced graph (vertices, edges and axial vectors), but also a graphical representation of the result. This permitted to validate the concept of multiple reductions (work in progress). This program is able to calculate

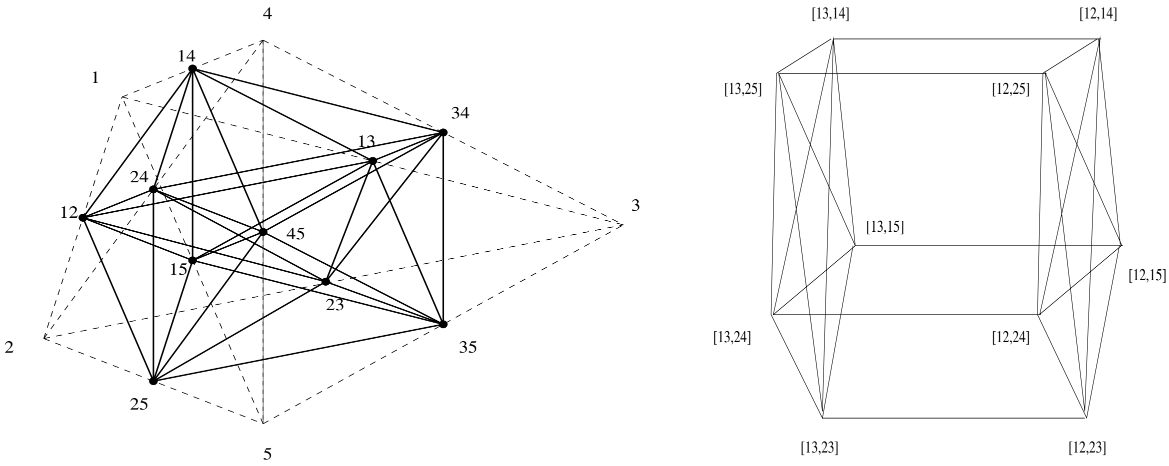


FIGURE 3. GKM graphs of $G_{1,5}(\mathbb{C}) = \mathbb{P}^4(\mathbb{C})$ (left, dashed) and $G_{2,5}(\mathbb{C})$ (left, plain), and its reduction along $\xi = (0, 1, 2, 3, -6)$ at $c = 7/2$ (right)

for example the reduction of Grassmannian manifolds of dimension 40 and with hundreds of fixed points, with standard computers.

Similarly, the computation and the storage of the invariant character are intractable even for small examples. For instance the dilatation of the K-theory element implies an impressive growth of the number of terms of the character. For the K-theory element $\Theta(p) = e^{2i\pi\theta_p}$ of the manifold $\mathbb{P}^3(\mathbb{C})$ and for the 1-dimensional torus whose infinitesimal generator is $\xi = (1, 2, -1, -2)$, invariant characters $\chi(\Theta^n)^H$ for $n = 1$, 10, 100 and 1000 possess 1, 12, 867 and 83667 monomials, respectively. This is why I also implemented the computation of the reduced character of a K-theory element of a GKM graph. The output is a sum of rational fractions whose size remains constant for any dilatation of the K-theory element.

My two programs, called `reduction.mws` and `caractere.mws`, come with many examples. Procedures generate Grassmannians $G_{k,n}(\mathbb{C})$ and the cycle with $4N$ vertices. The flag manifold $U(3)/\{\text{diagonal matrices}\}$ (whose reduction is a GKM *hypergraph*) is also available. A procedure performs the product of 1-skeleta.

The aim of these two programs is to better understand GKM graphs. We implemented them with MAPLE, a widespread software whose language is quite understandable. There are lots of comments inside of my programs, so that a curious user may understand internal procedures. The sourcecode is freely available and may be modified. The independence of subroutines permits to adapt the programs to one's needs. They are actually the only ones performing these tasks.

The programs can be downloaded at `http://www.math.jussieu.fr/~cochet/`.

References

[AB] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[Au] Michèle Audin. *The Topology of Torus Actions on Symplectic Manifolds.* Birkhäuser Verlag, Basel, 1991.

[BV] Nicole Berline and Michèle Vergne. Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(9):539–541, 1982.

[Del] Thomas Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France*, 116(3):315–339, 1988.

[Dem] Michel Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup. (4)*, 3:507–588, 1970.

[GKM] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.

[G] Victor Guillemin. *Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces*, volume 122 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1994.

[GZ1] V. Guillemin and C. Zara. Morse theory on graphs. *math.CO/0007161.*

[GZ2] V. Guillemin and C. Zara. Equivariant de Rham theory and graphs. *Asian J. Math.*, 3(1):49–76, 1999. Sir Michael Atiyah: a great mathematician of the twentieth century.

[GZ3] V. Guillemin and C. Zara. 1-skeleta, Betti numbers, and equivariant cohomology. *Duke Math. J.*, 107(2):283–349, 2001.

[GZ4] V. Guillemin and C. Zara. *G*-actions on graphs. *Internat. Math. Res. Notices*, (10):519–542, 2001.

[JK] Lisa C. Jeffrey and Frances C. Kirwan. Localization for nonabelian group actions. *Topology*, 34(2):291–327, 1995.

[MS] Eckhard Meinrenken and Reyer Sjamaar. Singular reduction and quantization. *Topology*, 38(4):699–762, 1999.

Combinatorial intersection cohomology: a survey

Karl-Heinz Fieseler

Projective toric varieties and lattice polytopes may be considered as two faces of the same coin. Accordingly, in the last 25 years, investigations related with toric varieties and their cohomology have played an increasingly important role in studying the combinatorics of convex polytopes. This started around 1980 with Stanley's spectacular proof of the necessity of McMullen's conditions (characterizing the face numbers of simple polytopes) using the cohomology of "rationally smooth" projective toric varieties. It continued with his introduction of a generalized h-vector for non-simple polytopes, modeled on the properties of the intersection cohomology Betti numbers of general projective toric varieties. In the last five years, attempts to prove the conjectured properties of this generalized h-vector led to the introduction of a purely combinatorial "virtual" intersection cohomology for polytopes, inspired by equivariant intersection cohomology of projective toric varieties. This work culminated in the recent proof of a "combinatorial Hard Lefschetz" theorem, which provides the keystone to proving Stanley's conjectures. – The aim of the present talk is to survey these developments.

The most basic combinatorial data of a convex polytope in $\mathbf{R}^n$ are the numbers f_i of i-dimensional faces, collected in the f(ace)-vector $(f_0, \ldots, f_n)$ or, equivalently, encoded in the f(ace)-polynomial $f(t) := \sum_{i=0}^n f_i t^i$. For *simple* polytopes, i.e., where each vertex lies on exactly n facets, the possible f-polynomials are characterized by *McMullen's conditions*. These are most conveniently stated in terms of the *h-vector* $(h_0, \ldots, h_n)$, i.e., the coefficient vector of the "*h-polynomial*" $h(t) := f(t{-}1) =: \sum_{i=0}^n h_i t^i$: The integers h_i are strictly *positive*, they satisfy the *symmetry* relation $h_i = h_{n-i}$, the "*unimodality* property" $h_i \leq h_{i+1}$ holds for $i \leq n/2 - 1$, and there are specific estimates for the growth of the differences $h_{i+1} - h_i$. By duality, there is a corresponding characterization for the class of simplicial polytopes.

The h-polynomial occurs in quite a different context if a simple polytope P is *rational*: The outer normal fan $\Delta(P)$ determines a projective toric variety $X_{\Delta(P)}$. Since the fan is simplicial, this variety is a rational homology manifold. It turns out that its Poincaré polynomial agrees with $h(t^2)$. This yields Stanley's "topological" proof for the necessity of McMullen's conditions: Symmetry corresponds to Poincaré duality, positivity and unimodality come from the Hard Lefschetz theorem, and the growth conditions follow from the fact that the cohomology algebra $H^*(X_{\Delta(P)})$ – and hence also its factor algebra $H^*(X_{\Delta(P)})/(\omega)$ with the hyperplane class ω – is generated by elements of degree 2.

On the other hand, if the simple polytope P is *non-rational*, then there is no longer an associated toric variety and thus, no cohomology algebra. Nevertheless, the above argument for the h-vector still can be used: Regarding P as an intersection of half-spaces, any nearby rational polytope has the same combinatorial type. But there is a more fundamental approach, namely, to associate to P itself – or rather to the simplicial fan $\Delta(P)$ – a "virtual" cohomology algebra $H^*(\Delta(P))$ with Hilbert polynomial $h(t^2)$ as follows: Let V denote the ambient vector space of $\Delta(P)$, so P "lives" in V^*. Let us consider $A := S(V^*)$, the algebra of polynomial functions on V, graded by $V^* =: A^{(2)}$, and the homogeneous maximal ideal $\mathfrak{m}$ of all polynomials vanishing at 0. For a graded A-module M, we denote with $\overline{M} := (A/\mathfrak{m}) \otimes_A M$ the graded real vector space obtained by reduction modulo $\mathfrak{m}$. In this setting, we associate to $\Delta(P)$ the graded A-module $A_{\Delta(P)}$ of all cone-wise polynomial functions, and then define $H^*(\Delta(P)) := \overline{A}_{\Delta(P)}$. This approach is motivated by the equivariant cohomology of the toric variety $X_{\Delta(P)}$ associated to P in the rational case: There is a natural action of an algebraic torus T. If P is simple, the variety $X_{\Delta(P)}$ is T-equivariantly formal, i.e., the ordinary cohomology $H^*(X_{\Delta(P)})$ is obtained from $H^*_T(X_{\Delta(P)}) \cong A_{\Delta(P)}$ by reduction modulo the homogeneous maximal ideal $\mathfrak{m}$ in $H^*(BT) \cong A$.

We now consider polytopes that are *not simple*, so their outer normal fan fails to be simplicial. If such a polytope P is *rational*, the associated projective toric variety $X_{\Delta(P)}$ fails to be a rational homology manifold. Neither its Betti numbers nor the h-vector of P do enjoy the properties mentioned above. Considering intersection cohomology instead of the "usual" theory, however, yields an even Poincaré polynomial $\mathcal{Q}$ with "good" properties since both, Poincaré duality and the Hard Lefschetz theorem, hold for $IH^*(X_{\Delta(P)})$. One may thus assign the polynomial h with $\mathcal{Q}(t) = h(t^2)$ to the polytope P as *generalized* h-polynomial. The corresponding generalized h-vector then enjoys three of the properties that hold for simple polytopes, namely, *positivity, symmetry, and unimodality.* In contrast to the simple case, however, there is no natural algebra structure on $IH^*(X_{\Delta(P)})$, so the proof of the growth estimates does not carry over; furthermore, there is no immediate connection between this new h-polynomial and the face polynomial. On the other hand, there is a recursion method to compute h from combinatorial data of P, so the same recursion allows to assign a generalized h-polynomial also to *non-rational* polytopes, cf. [St].

In contrast to the situation for simple polytopes, nearby polytopes in general do not necessarily have the same combinatorial type. So the following question is natural: In the non-rational case, does the new h-vector still have the same three properties: positivity, symmetry, and unimodality? It motivated the search for a "virtual" intersection cohomology theory $IH^*(\Delta(P))$, as in the case of simple polytopes. In fact, the investigation of the "sheafified" equivariant intersection cohomology of toric varieties leads to the following construction entirely in terms of the fan Δ: To apply sheaf theory, the fan is endowed with the "fan topology", where the subfans $\Lambda \subset \Delta$ are the open subsets. On that fan space, there is a natural structure sheaf $\mathcal{A}$ of graded rings given by the assignment $\Lambda \mapsto A_\Lambda$, so in particular $A_\sigma := \mathcal{A}(\sigma) = S(V_\sigma^*)$ with $V_\sigma := \mathrm{span}(\sigma)$. A sheaf $\mathcal{F}$ of graded $\mathcal{A}$-modules is called *pure* if it is flabby and satisfies the following condition:

(*) For each $\sigma \in \Delta$, the A_σ-module $F_\sigma := \mathcal{F}(\sigma)$ is finitely generated and free.

A sheaf $\mathcal{F}$ on Δ is flabby iff for each cone σ, the restriction map $F_\sigma \to F_{\partial\sigma}$ is surjective; if $\mathcal{F}$ even satisfies (*), then this surjectivity is equivalent to that of $\overline{F}_\sigma \to \overline{F}_{\partial\sigma}$. The structure sheaf $\mathcal{A}$ clearly satisfies condition (*); it is flabby iff Δ is simplicial, which holds for a polytopal fan $\Delta(P)$ iff P is simple. Up to isomorphism, among the pure sheaves $\mathcal{F}$ on Δ with $F_o = \mathbf{R}$, there is a unique minimal object $\mathcal{E}$ determined by the condition that $\overline{E}_\sigma \xrightarrow{\cong} \overline{E}_{\partial\sigma}$ even is an isomorphism for each $\sigma \neq o$. It is called the "equivariant intersection cohomology sheaf" of Δ, and $IH^*(\Delta) := \overline{E}_\Delta$ is the "virtual" intersection cohomology sought after, cf. [BBFK$_2$, BreLu$_1$].

We now have to analyse how far Poincaré duality and, in the case of polytopal fans $\Delta = \Delta(P)$, the Hard Lefschetz theorem continue to hold. As to Poincaré duality, we note that for any oriented fan Δ, the category of pure sheaves admits an involutive duality functor $\mathcal{F} \mapsto \mathcal{D}\mathcal{F}$. After fixing a volume form on V, that provides a natural isomorphism $\mathcal{D}\mathcal{E} \cong \mathcal{E}$. In fact, the naturality is not immediate since it relies on the Hard Lefschetz theorem for polytopal fans of lower dimensions. This duality isomorphism provides a natural intersection product "$\cap$" on $IH^*(\Delta)$. In particular, this yields Poincaré duality between homogeneous subspaces of complementary dimensions, cf. [BBFK$_3$, BreLu$_2$].

As to the Hard Lefschetz Theorem, one assigns to a polytope P a natural strictly convex conewise linear function ψ on its outer normal fan $\Delta := \Delta(P)$ as follows: For each n-dimensional cone σ, the restriction $\psi|_\sigma \in V^*$ is precisely the corresponding vertex of the polytope $P \subset V^*$. The multiplication endomorphism

$\mu_\psi \colon E_\Delta \to E_\Delta$ induces the "Lefschetz operator" $L := \overline{\mu}_\psi \colon IH^*(\Delta) \to IH^{*+2}(\Delta)$, and the Hard Lefschetz Theorem states that each $L^k \colon IH^{n-k}(\Delta) \to IH^{n+k}(\Delta)$ is an isomorphism for $k \geq 0$. Its proof follows easily from the "Hodge-Riemann bilinear relations" (HRR), according to which the pairing

$$IH^{n-k}(\Delta) \times IH^{n-k}(\Delta) \longrightarrow \mathbf{R}\,, \quad (\xi, \eta) \mapsto \xi \cap L^k(\eta)$$

is $(-1)^{(n-k)/2}$-definite on the "primitive" subspace $IP^{n-k}(\Delta) := \ker(L^{k+1})$. For a simple polytope P, these relations have been proved in [Mc], to which the general case can be reduced according to [Ka].

Let us sketch a geometric idea for such a reduction: We successively cut off "bad" faces from the polytope P, starting with non-simple vertices, and then proceeding in stages according to the dimension. Since a polytope without bad faces is simple, this procedure eventually yields the starting point for an induction. We describe the typical step: We call a face $F \subset P$ "bad" if its link is not a cone $C(Q)$ over some polytope Q. A bad face F of minimal dimension is itself a simple polytope and admits a "tubular neighbourhood" in P. To cut off F, we write $F = P \cap H$ with a hyperplane H and move H slightly towards the interior of P. Intersecting P with the two corresponding half-spaces yields a decomposition $P = P_1 \cup P_2$ into polytopes, with P_2 containing F and P_1 on the other side of the hyperplane. By induction hypothesis, HRR holds for P_1 since it has less bad faces than P. The fact that HRR also holds for P_2 can be derived from the lower-dimensional case: The polytope P_2 is "hip-roofed" with ridge F, and a transversal cross-section is a cone $C(Q)$ over a polytope Q of dimension $n - 1 - \dim F$. Now HRR for Q implies HRR for $C(Q)$, and for $\dim F > 0$ the polytope P_2 is "trivialized" by moving the ridge to infinity. Patching together the HRR for P_1 and P_2 by a Mayer-Vietoris argument yields the result for P.

Hence, even for non-rational polytopes, the generalized h-vector satisfies the three properties: positivity, symmetry, and unimodality.

References

[BBFK$_1$] G. Barthel, J.-P. Brasselet, K.-H. Fieseler and L. Kaup, *Equivariant Intersection Cohomology of Toric Varieties*, Algebraic Geometry, Hirzebruch 70, 45–68, Contemp. Math. **241**, Amer. Math. Soc., Providence, R.I., 1999.

[BBFK$_2$] G. Barthel, J.-P. Brasselet, K.-H. Fieseler and L. Kaup, *Combinatorial Intersection Cohomology for Fans*, Tôhoku Math. J. **54** (2002), 1–41.

[BBFK$_3$] G. Barthel, J.-P. Brasselet, K.-H. Fieseler and L. Kaup, *Combinatorial Duality and Intersection Product: A Direct Approach*, (pr)e-print `math.AG/0309352v1` (21 pages), 2003, to appear in Tôhoku Math. J.

[BreLu$_1$] P. Bressler and V. Lunts, *Intersection cohomology on nonrational polytopes*, (pr)e-print `math.AG/0002006` (33 pages), 2000, to appear in Compositio Math.

[BreLu$_2$] P. Bressler and V. Lunts, *Hard Lefschetz theorem and Hodge-Riemann relations for intersection cohomology of nonrational polytopes*, (pr)e-print `math.AG/0302236 v2` (46 pages), 2003.

[Ka] K. Karu, *Hard Lefschetz Theorem for Nonrational Polytopes*, (pr)e-print `math.AG/0112087` (25 pages), 2002.

[Mc] P. McMullen, *On simple Polytopes*, Invent. Math. **113** (1993), 419–444.

[St] R. Stanley, *Generalized h-vectors, intersection cohomology of toric varieties and related results*, M. Nagata, H. Matsumura, eds., *Commutative Algebra and Combinatorics*, 187–213, Adv. Stud. Pure Math. **11**, Kinokunia, Tokyo, and North Holland, Amsterdam/New York, 1987.

The combinatorics of Hamiltonian circle actions

Susan Tolman

Let a compact torus T act on a compact symplectic manifold (M, ω) with a moment map $\Phi\colon M \to \mathfrak{t}^*$. We would like to be able to associate to (M, ω, Φ) a simple combinatorial object, such as a labelled graph. Ideally, we should be able to say which graphs arise, and also to use the graphs to calculate the invariants of the manifolds. When the torus is sufficiently large, it is possible to do this.

Assume, for example, that (M, ω, Φ) is a *symplectic toric manifold*, that is, that the dimension of T is half the dimension of M. In this case, the fixed point set M^T is finite. Delzant [De] proved that symplectic toric manifolds are classified by the discrete set $\Phi(M)$.

Suppose, instead, that M is four-dimensional, that T is a circle, and that M^T is finite. Given $H \subseteq T$, let M_H denote the set of points in M with stabilizer exactly H. Given any finite subgroup $H = \mathbb{Z}/(k\mathbb{Z}) \subset T$, the closure of each component of $M_H \subset M$ is a symplectic sphere with two fixed points, which we will call a *k-isotropy sphere*. We define a graph as follows: The vertices are the fixed points $p \in M^T$, labelled by their moment images $\Phi(p)$. The edges of the graph are the isotropy spheres, labelled by their generic stabilizer $\mathbb{Z}/(k\mathbb{Z})$. This graph determines (M, ω, Φ) up to equivariant symplectomorphism [K].

However, in general it is probably not realistic to classify manifolds up to equivariant symplectomorphism. For example, even in the case of a two-dimensional torus acting on a six-dimensional manifold, this classification can involve more complicated homotopy type invariants which are not naturally captured by a labelled graph [KT].

This leads us to consider invariants of M. For instance, the *equivariant cohomology* of M, denoted $H_T^*(M)$, is the ordinary cohomology of the space $M \times_T ET$, where ET is any contractible space on which T acts freely. The equivariant cohomology of M is extremely well-behaved. For example, if $i\colon M^T \to M$ is the inclusion of the fixed point set, the restriction $i^*\colon H_*^T(M) \to H_*^T(M)$ is injective. Moreover, the natural restriction map $j^* H_T^*(M) \to H^*(M)$ is surjective.

We say that two symplectic manifolds M and $\widetilde{M}$ with Hamiltonian T actions are *cohomologically equivalent* if their exists a diffeomorphism from M^T to $\widetilde{M}^T$ with the following two properties:

(1) It induces an isomorphism between the image of i^* in $H_T^*(M^T)$ and the image of $\tilde{i}^*$ in $H_T^*(\widetilde{M}^T)$.
(2) It induces an equivariant isomorphism between the normal bundles of M^T and $\widetilde{M}^T$.

This equivalence implies that the equivariant (and ordinary) cohomology of M and $\widetilde{M}$ are isomorphic as rings, and that this isomorphism respects the equivariant (and ordinary) Chern classes.

Assume that M is a *GKM space*, that is, M^T is finite and the closure of M_H is a disjoint union of 2-spheres for every codimension-one subgroup $H \subset T$. We define a graph as follows: The vertices are fixed points, labelled by their moment image. The edges are components of M_H, labelled by their principal isotropy group. Up to cohomological equivalence, (M, ω, Φ) is determined by its graph.

While there are many interesting GKM spaces, if T is a circle, the only GKM space is a 2-sphere. Henceforth, we will consider the case that T is a circle. Recall that, in this case, Φ is a Morse-Bott function. Moreover, for any isolated fixed point p, there is a unique cohomology class $\alpha_p \in H_T^*(p)$ whose restriction to p is the product of the negative weights at p, and which vanishes when restricted to every other fixed point q whose index is at most the index of p.

Assume that the fixed point set M^T consists of isolated fixed points, and that the action is *semi-free*, that is, that the stabilizer of every point is either trivial or the whole circle. In this case, M must be cohomologically equivalent to $(S^2)^n$, where n is half the dimension of M [TW].

Although this seems to indicate that very little information is required to determine the equivariant cohomology of M, a word of caution is needed: it is not true that every automorphism of M^T takes the image of i^* to itself. For example, if $n = 3$ and p is any fixed point of index 2, the restriction $\alpha_p|_q$ is non-zero for two of the fixed points of index 4, but not for the third.

The result above has been extended to the case where the fixed set M^T contains components of dimension 2, under the assumption that the second Betti number is small, by Hui Li. Godinho has extended it in a different direction by proving a related result when every fixed point has the same weights, up to sign. (She also needs additional technical restrictions.)

Now assume that M is a six-dimensional manifold, and that M^T consists of precisely four points. For example, this occurs if M is complex projective three-space, or if M is the Grassmannian of oriented two-planes in $\mathbb{R}^5$, and T is a subgroup of the natural torus action. I am currently trying to show that, up to cohomological equivalence, these are the only two possible examples. Surprisingly, the proof is much easier if you assume that no pair of edges can be joined by two different isotropy spheres. In this case, it follows from two arguments. First, the integral of any low-dimensional Chern class must be zero and can be computed using the Atiyah-Bott-Berline-Vergne localization formula. Second, the weights at the north and south pole of a k-isotropy sphere must agree modulo k. In general, one needs to consider slightly more complicated possibilities. For instance, suppose that the minimum and the maximum are joined by both a 2-isotropy sphere and a 3-isotropy sphere, but the action is otherwise semi-free. This possibility cannot be eliminated by the conditions above, but can be ruled out by looking carefully at the cohomology ring of the reduced space.

R. Goldin and I are beginning a project which we hope will allow us to compute the equivariant cohomology in a much more general context. Assume that M is six-dimensional, that the fixed point set M^T is finite, and that there exists a S^1-invariant Palais-Smale metric on M.

Consider the stable and unstable manifolds of each fixed point with respect to the associated gradient flow. Note that these manifolds are S^1-invariant. Define a graph G as follows: The vertices are the fixed points, labelled by their moment image. For each pair of fixed points p and q with $\Phi(p) \leq \Phi(q)$, and each finite subgroup $H \subset T$ (including the trivial subgroup), there is an edge between p and q for each sphere in the closure of the unstable manifold of p, the stable manifold of q, and M_H itself. The edge is labelled by H and by an orientation. We have fairly good evidence that, in this case, the equivariant cohomology of M is determined by the graph. Interestingly, the computation of $\alpha_p|_q$ involves a sum over paths ascending from p to q, much like that found by Guillemin and Zara.

We hope that it will be possible to prove a similar result in arbitrary dimensions, and also that the result will still hold even if their is no S^1-invariant Palais-Smale metric.

References

[De] T. Delzant, *Hamiltoniens périodiques et image convexe de l'application moment*, Bull. Soc. Math. France **116** (1988), 315–339.

[K] Y. Karshon, *Periodic Hamiltonian flows on four dimensional manifolds*, Mem. Amer. Math. Soc. **672** (1999).

[KT] Y. Karshon and S. Tolman, *Complete invariants for Hamiltonian torus actoins with two dimensional quotients*, J. Sympl. Geom., to appear.

[TW] S. Tolman and J. Weitsman, *On semifree symplectic circle actions with isolated fixed points*, Topology **39** (2000) no. 2, 299–310.

Tori in symplectomorphism groups

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(joint work with Liat Kessler and Martin Pinsonnault)

Finiteness theorem: *Let (M, ω) be a four dimensional compact symplectic manifold and $T \cong (S^1)^2$ a two dimensional torus. Then the set of effective Hamiltonian T-actions on (M, ω) modulo equivariant symplectomorphisms and modulo automorphisms of T is finite.*

Remarks.

(1) This is a "95% theorem," as its complete proof has not yet been LaTeXed.
(2) The image of T in the symplectomorphism group $\mathrm{Sympl}(M, \omega)$ is a maximal torus. This follows from the fact that the orbits of a Hamiltonian torus action are isotropic.

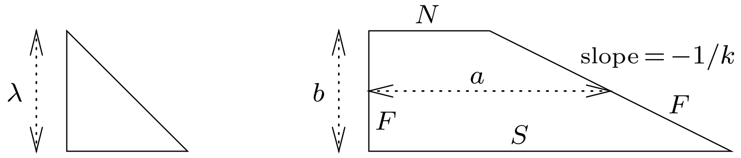

FIGURE 1. Delzant triangle and Hirzebruch trapezoid

(3) If (M, ω) admits a Hamiltonian T-action, then every symplectic T-action on (M, ω) is Hamiltonian. In this case, the theorem asserts that the number of conjugacy classes of two-dimensional tori in $\mathrm{Sympl}(M, \omega)$ is finite.
(4) In contrast, Eugene Lerman has constructed a compact contact manifold that admits infinitely many non-conjugate toric actions.

As a consequence of a theorem of Delzant, a Hamiltonian T-action on M with moment map Φ is equivariantly symplectomorphic to the symplectic toric manifold $(M_\Delta, \omega_\Delta)$ associated to the Delzant polygon $\Delta = \Phi(M) \subset \mathfrak{t}^*$. We need to show that the number of Delzant polygons Δ such that $(M_\Delta, \omega_\Delta)$ is symplectomorphic to (M, ω), modulo translations and $\mathrm{GL}(2, \mathbb{Z})$-congruence, is finite.

The "size" of an edge of a Delzant polygon is measured by its "rational length," which it characterized by being invariant under $\mathrm{GL}(2, \mathbb{Z})$-congruence and translations and being standard along the coordinate axes. The moment map preimage of an edge is a symplectic sphere whose symplectic area is 2π times the rational length of the edge.

Examples of Delzant polygons are a "Delzant triangle," which corresponds to $\mathbb{CP}^2$, and a "Hirzebruch trapezoid," which corresponds to a Hirzebruch surface. See Figure 1. Up to translations and $\mathrm{GL}(2, \mathbb{Z})$-congruence, a Delzant triangle is determined by the rational length λ of each side, and a Hirzebruch trapezoid is determined by parameters (a, b, k) where b is its height, a is the average of the lengths of its top and bottom edges, and k is a non-negative integer such that the right edge has slope $-1/k$ (or is vertical if $k = 0$). A Hirzebruch surface is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$. The moment map preimages of the top and bottom edges are the "north pole section" and the "south pole section"; the moment map preimages of the side edges are fibers.

The perimeter and area of a Delzant polygon Δ are symplectic invariants of the underlying toric variety $(M_\Delta, \omega_\Delta)$: the perimeter is equal to the pairing of ω_Δ with the first Chern class $c_1(TM_\Delta)$, and the area is equal to the Liouville volume $\frac{1}{2\pi} \int_M \omega_\Delta^2 / 2!$.

An equivariant symplectic blowup of size δ of a toric manifold amounts to "chopping" off a corner of size δ of its polygon. This reduces the perimeter by δ and the area by $\frac{1}{2}\delta^2$. The preimage of the new edge is the exceptional divisor. A homology class which is represented by the moment map preimage of an edge gives a homology class in the blown up manifold which is represented by the preimage of at most two edges of the "chopped" polygon. After s blowups, the symplectic area of such a homology class is bounded by 2^s times the perimeter.

Each Delzant polygon is either a Delzant triangle or is obtained from a Hirzebruch trapezoid by a sequence of "corner choppings," so each symplectic toric manifold is either $\mathbb{CP}^2$ or is obtained from a Hirzebruch surface by a sequence of equivariant symplectic blow-ups.

Fix a symplectic manifold (M, ω). To prove the finiteness theorem for this manifold, it is enough to show that the number of tuples $(a, b, k; \delta_1, \ldots, \delta_s)$ such that (M, ω) is symplectomorphic to a symplectic toric manifold $(M_\Delta, \omega_\Delta)$ that is obtained from a Hirzebruch surface with parameters (a, b, k) by equivariant symplectic blow-ups of sizes $\delta_1, \ldots, \delta_s$ is finite.

Suppose that (M, ω) is symplectomorphic to a toric manifold $(M_\Delta, \omega_\Delta)$ that is obtained from a Hirzebruch surface with parameters (a, b, k) by a sequence of equivariant symplectic blowups of sizes $\delta_1, \ldots, \delta_s$. Let

$$E_1, \ldots, E_s \in H_2(M)$$

be the homology classes of the exceptional divisors. Then

(1) $E_i \cdot E_i = -1$;
(2) E_i can be represented by an embedded symplectic sphere;
(3) $\langle \omega, E_i \rangle$ is smaller than 2^s times $\langle \omega, c_1(TM) \rangle$.

As a consequence of Gromov's compactness, there exist only finitely many cohomology classes with these properties. Because $\delta_i = \langle \omega, E_i \rangle$, the set of possible s-tuples $(\delta_1, \ldots, \delta_s)$ is finite.

The perimeter of the Delzant polygon Δ is $2(a+b) - \sum_{j=1}^{s} \delta_j$ and its area is $ab - \frac{1}{2} \sum_{j=1}^{s} \delta_j^2$. Because these are symplectic invariants of (M, ω), we can recover $a + b$ and ab from $\delta_1, \ldots, \delta_s$, so the set of possible values for a and b is finite. Let

$$N, S, F \in H_2(M)$$

be the homology classes coming from the north pole section, south pole section, and fiber, respectively. Then

(1) $S = N + kF$;
(2) $\langle \omega, S \rangle$ is smaller than 2^s times $\langle \omega, c_1(TM) \rangle$;
(3) $\langle \omega, N \rangle$ is positive;
(4) $\langle \omega, F \rangle = b$.

It follows that the non-negative integer k is bounded from above by $2^s \langle \omega, c_1(TM) \rangle / b$. Because there are finitely many possibilities for the value of b, there are finitely many possibilities for the value of k. This completes the outline of the proof of the finiteness theorem.

References

[D] T. Delzant, *Hamiltoniens périodiques et image convexe de l'application moment*, Bulletin de la Société Mathématique de France **116** (1988), 315–339.

[F] W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993.

[K] Y. Karshon, *Maximal tori in the symplectomorphism groups of Hirzebruch surfaces*, Math. Res. Lett. **10** no. 1 (2003), 125–132.

[L] E. Lerman, *Maximal tori in the contactomorphism groups of circle bundles over Hirzebruch surfaces*, Math. Res. Lett. **10** no. 1 (2003), 133–144.

[G] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Inv. Math. **82** (1985), 307–347.
[MS] D. McDuff and D. Salamon, *J-holomorphic curves and quantum cohomology*, Amer. Math. Soc. Univ. Lect. Ser. **6**, 1994.

Signature quantization

Jonathan Weitsman
(joint work with Victor Guillemin and Shlomo Sternberg)

Let (M, g) be a Riemannian manifold, and let L be a Hermitian line bundle with connection on M. We consider an elliptic operator given by twisting the signature operator on M with the line bundle L. We define the *signature quantization* of M to be the index of this elliptic operator. We show that this quantization (which we call $Q(M)$) satisfies analogs of many of the theorems proved for classical geometric quantization of symplectic manifolds. For example, we give analogs of the Borel-Weil-Bott theorem, the Khovanski theorem, the Kostant multiplicity formula, and the principle that "quantization commutes with reduction." We also show how signature quantization behaves under an analog of symplectic cutting. Finally we review some recent work of Guillemin and Rassart showing how the Steinberg formula and the Gelfand-Cetlin formulas extend to signature quantization.

References

[1] V. Guillemin, S. Sternberg, J. Weitsman. Signature quantization. *Proc. Nat. Acad. Sci. USA* **100**, 12559-12560 (2003)
[2] V. Guillemin, S. Sternberg, J. Weitsman. Signature quantization. *J. Diff. Geom.*, in press.
[3] V. Guillemin, E. Rassart. To appear.

Kirwan surjectivity for orbifold cohomology

Rebecca Goldin & Allen Knutson
(joint work with Tara Holm)

1. The definition of preorbifold cohomology

Let Y be an almost complex manifold with a torus action by T. As a vector space (indeed, as an $H_T^*(pt)$-module), we define the preorbifold cohomology by

$$PH_T^{*,\diamond}(Y) := \bigoplus_{g \in T} PH_T^{*,g}(Y),$$

where $PH_T^{*,g}(Y) = H_T^*(Y^g)$ is the equivariant cohomology of $Y^g = \{y \in Y \mid g \cdot y = y\}$. The product and grading on $PH_T^{*,\diamond}(Y)$ are more subtle. We will come to these presently.

For a class $a \in PH_T^{*,\diamond}(Y)$, let a_g denote the component of a in the summand $PH_T^{*,g}(Y)$. We will say that $a \in PH_T^{*,g}(Y)$ or a is *supported* on Y^g if $a_h = 0$ for

$h \neq g$. Suppose that Y is compact and T acts on Y locally freely: that is, $\mathrm{Stab}(y)$ is finite for all $y \in Y$. Notice that in this case, $PH_T^{*,\diamond}(Y)$ reduces to a finite direct sum over the finitely many elements in T that appear as stabilizers.

The product on preorbifold cohomology. The definition of the product in preorbifold cohomology requires the introduction of a new space and an associated (union of) bundle(s) over its connected components. Let

$$\tilde{Y} := \coprod_{g,h \in T} Y^{g,h}$$

where $Y^{g,h} = (Y^g)^h$.

For any connected component Z of $Y^{g,h}$, the group $\langle g, h\rangle$ generated by g and h acts on the almost complex vector bundle νZ, the normal bundle to Z in Y, fixing Z itself. Thus as a representation of $\langle g, h\rangle$, νZ breaks up into isotypic components

$$\nu Z = \bigoplus_{\lambda \in \widehat{\langle g,h\rangle}} I_\lambda$$

where I_λ is a bundle over Z on which $\langle g, h\rangle$ acts with representation given by λ. We define the *time of* I_λ to be the sum

$$\mathrm{time}(I_\lambda) = \mathrm{yr}(g) + \mathrm{yr}(h) + \mathrm{yr}((gh)^{-1}).$$

Note that $\mathrm{time}(I_\lambda)$ is 0, 1, or 2.

Definition 1. *For each connected component Z of $\tilde{Y}$, let $E|_Z$ be the vector bundle given by*

$$E|_Z = \bigoplus_{\mathrm{time}(I_\lambda)=2} I_\lambda.$$

The obstruction bundle *E is the union of $E|_Z$ over all connected components Z in $\tilde{Y}$. Note that the rank of E may change on different connected components.*

Remark 2. *Each component Z is T-invariant and hence $E|_Z \to Z$ is a T-equivariant bundle. Thus there is a well-defined equivariant Euler class ε of E: for every component Z, let ε restricted to Z be the equivariant Euler class of $E|_Z$. The class ε is called the* virtual class *of E.*

Consider the three inclusion maps given by

$$\begin{aligned} e_1 :& Y^{g,h} \hookrightarrow Y^g \\ e_2 :& Y^{g,h} \hookrightarrow Y^h, \quad \text{and} \\ \overline{e}_3 :& Y^{g,h} \hookrightarrow Y^{gh}. \end{aligned}$$

The maps $e_1, e_2, \overline{e}_3$ clearly extend to maps on $\tilde{Y}$. They therefore induce the pullbacks

$$e_1^*, e_2^* : PH_T^{*,\diamond}(Y) \to \bigoplus_{g,h \in T} H_T^*(Y^{g,h})$$

and the pushforward map

$$(\overline{e}_3)_* : \bigoplus_{g,h\in T} H_T^*(Y^{g,h}) \to PH_T^{*,\diamond}(Y).$$

Definition 3. *For $a_1, a_2 \in PH_T^{*,\diamond}(Y)$, we define*

$$a_1 \smile a_2 := (\overline{e}_3)_*(e_1^*(a_1) \cdot e_2^*(a_2) \cdot \varepsilon),$$

where ε is the virtual class of the obstruction bundle E over $\tilde{Y}$, and the product occurring on the right hand side is the usual product in the equivariant cohomology of each piece $Y^{g,h}$ of $\tilde{Y}$.

Remark 4. *If $a_1 \in PH_T^{*,g}(Y)$ and $a_2 \in PH_T^{*,h}(Y)$, then $e_1^*(a_1) \cdot e_2^*(a_2) \in H_T^*(Y^{g,h})$. After multiplying by ε, the pushforward map $(\overline{e}_3)_*$ sends this class to $H_T^*(Y^{gh})$, which implies $a_1 \smile a_2 \in PH_T^{*,gh}(Y)$.*

The $\mathbb{R}$-grading on $PH_T^{*,\diamond}(Y)$. Clearly multiplication in $PH_T^{*,\diamond}(Y)$ is not graded if the degree is assigned in the naive way. However there is a different definition of degree for $PH_T^{*,\diamond}(Y)$ making it into a graded algebra. Let $g \in T$, and $y \in Y^g$. Then $T_yY = \oplus_j L_j$ under the g action. The sum of the years of g on each of these lines is called the *age* of g at y. Since this number depends only on the connected component Z of y in Y^g, we let

$$\operatorname{age}(Z, g) = \sum_j \operatorname{yr}_j(g).$$

Let $r : Z \to Y^g$ be the inclusion map. Let a be a class $a \in H_T^*(Y^g)$ such that $r^*(a) \in H_T^i(Z)$ and a restricts to 0 on other connected components of Y^g. Then considered as an element of $PH_T^{*,\diamond}(Y)$, we assign $\deg(a) = i + 2\operatorname{age}(Z, g)$. Note that this grading is *real* rather than integral.

2. The case of Hamiltonian T-spaces

Suppose that Y is a Hamiltonian T-space with proper moment map Φ. Let F be a fixed component of Y^T. Then T acts on νF and it breaks up into isotypic components

$$\nu F = \bigoplus_\lambda I_\lambda,$$

where the sum is over weights $\lambda \in \hat{T}$. For each component I_λ, we define the *experience* of I_λ under the elements $g_1, \ldots, g_n$ to be

$$\operatorname{exper}(I_\lambda, g_1, \ldots, g_n) = \operatorname{yr}(g_1) + \cdots + \operatorname{yr}(g_n) - \operatorname{yr}(g_1 g_2 \cdots g_n).$$

We denote this by $\operatorname{exper}(I_\lambda)$ when the group elements are understood. Note that $\operatorname{exper}(I_\lambda)$ is an integer between 0 and $n-1$.

Definition 5. *Let Y be a Hamiltonian T-space. The product $\star$ on $PH_T^{*,\diamond}(Y)$ is given as follows. Let $a_1, \ldots, a_n \in PH_T^{*,\diamond}(Y) = \bigoplus_{g\in T} PH_T^{*,g}(Y)$ be elements such that $a_i \in PH_T^{*,g_i}(Y)$ for $i = 1, \ldots, n$. We define $a_1 \star \cdots \star a_n$ by its restriction on* *each piece of the fixed point set. For F a connected component of $(Y^{g_1 g_2 \cdots g_n})^T$, define*

$$(a_1 \star \cdots \star a_n)|_F = \prod_i (a_i|_F) \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{\mathrm{exper}(I_\lambda)}, \tag{1}$$

where $e(I_\lambda) \in H_T^*(F)$ is the equivariant Euler class of I_λ. For any fixed point component F' of Y^g with $g \neq g_1 g_2 \cdots g_n$, we define $(a_1 \star \cdots \star a_n)|_{F'} = 0$.

3. Results

Theorem 6. *Let Y be an almost complex manifold with a T action preserving the almost complex structure. Then $(PH_T^{*,\diamond}(Y), \smile)$ is a graded, associative ring. If Y is a Hamiltonian T space, then the identity map is a ring isomorphism between the rings $(PH_T^{*,\diamond}(Y), \smile)$ and $(PH_T^{*,\diamond}(Y), \star)$.*

Theorem 7. *Suppose Y is compact and T acts on Y locally freely. Then*

$$PH_T^{*,\diamond}(Y) = H_{orb}^*(Y/T),$$

where $H_{orb}^*(Y/T)$ is the orbifold cohomology as defined by Chen and Ruan.

Using this and the surjectivity results of Kirwan [K], we obtain:

Corollary 8. *Let Y be a Hamiltonian T-space, with moment map $\Phi : Y \to \mathfrak{t}^*$. Suppose that 0 is a regular value of Φ. Then*

$$PH_T^{*,\diamond}(\Phi^{-1}(0)) \cong H_{orb}^*(Y//T),$$

where the cohomology is taken with coefficients in any ring. Moreover, the natural map

$$\kappa : PH_T^{*,\diamond}(Y;\mathbb{Q}) \longrightarrow H_{orb}^*(Y//T;\mathbb{Q})$$

induced by inclusion of the level set on each piece Y^g is a surjection.

The kernel of κ can be easily computed in many examples.

4. Computational facility

We now introduce a subring of $PH_T^{*,\diamond}(Y)$ which is more computable, but in many cases contains all information necessary to compute $PH_T^{*,\diamond}(Y)$. Let Γ be the group generated by all elements of T occurring as finite stabilizers. We assume Γ is finite.

Definition 9. *The* Γ-subring $PH_T^{*,\Gamma}(Y)$ *is a subring of $PH_T^{*,\diamond}(Y)$ given as a vector space by*

$$PH_T^{*,\Gamma}(Y) := \bigoplus_{g\in\Gamma} PH_T^{*,g}(Y).$$

In the case that T acts on Y locally freely, this ring equals $PH_T^{*,\diamond}(Y)$. In the case that Y is a Hamiltonian T-space with moment map Φ, we note that $PH_T^{*,g}(Y) \to H^*(Y^g//T)$ is a surjection for each g. If g is not a finite stabilizer, then $Y^g \cap \Phi^{-1}(0) = \emptyset$. In other words, all elements in $PH_T^{*,g}(Y)$ are in the kernel when $g \notin \Gamma$. Thus the orbifold cohomology of the reduced space may computed with the Γ-subring alone.

References

[AGV] D. Abramovich, T. Graber, and A. Vistoli, Algebraic orbifold quantum products. Preprint `math.AG/0112004`.

[CR] W. Chen and Y. Ruan, A New Cohomology Theory for Orbifold. Preprint `math.AG/0004129`.

[BCS] L. Borisov, L. Chen, and G. Smith, The orbifold Chow ring of toric Deligne-Mumford stacks. Preprint `math.AG/0309229`.

[FG] B. Fantechi and L. Göttsche, Cohomology for global quotients. *Duke Math. J.* **117** (2003), no. 2, 197–227. Preprint `math.AG/0104207`.

[K] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry.* Mathematical Notes, **31**. Princeton University Press, Princeton, NJ, 1984.

Equivariant cohomology and the Maurer-Cartan equation

Eckhard Meinrenken

(joint work with Anton Alekseev)

Let G be a compact, connected Lie group, acting smoothly on a manifold M. In [6] Goresky-Kottwitz-MacPherson described the following "small Cartan model" for the equivariant cohomology of M,

$$(S\mathfrak{g}^*)_{\mathrm{inv}} \otimes \Omega(M)_{\mathrm{inv}}, \quad 1 \otimes \mathrm{d} - \sum_j p^j \otimes \iota(c_j). \tag{1}$$

Here c_j are primitive generators of $(\wedge\mathfrak{g})_{\mathrm{inv}}$, and the p^j are generators of $(S\mathfrak{g}^*)_{\mathrm{inv}}$ corresponding to the dual basis by Chevalley's transgression theorem. One of the results in [6] states that the small Cartan complex is quasi-isomorphic to the standard (large) Cartan complex of equivariant differential forms. Our main result is an explicit cochain map from the small Cartan model into the standard Cartan model, intertwining the $(S\mathfrak{g}^*)_{\mathrm{inv}}$-module structures and inducing an isomorphism in cohomology. This construction involves the solution of an interesting Maurer-Cartan equation, and leads to a refinement of Chevalley's transgression theorem. We will also address similar questions for the Chevalley-Koszul complex [7, 9], viewed as a "small model" for the cohomology of principal bundles.

References

[1] C. Allday and V. Puppe, *On a conjecture of Goresky, Kottwitz and MacPherson*, Canad. J. Math. **51** (1999), no. 1, 3–9. MR **2000e:**18008

[2] H. Cartan, *La transgression dans un groupe de Lie et dans un fibré principal*, Colloque de topologie (espaces fibrés) (Bruxelles), Centre belge de recherches mathématiques, Georges Thone, Liège, Masson et Cie., Paris, 1950, pp. 73–81.

[3] ———, *Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie.*, Colloque de topologie (espaces fibrés) (Bruxelles), Georges Thone, Liège, Masson et Cie., Paris, 1950.

[4] M. Franz, *Koszul duality and equivariant cohomology*, arXiv math.AT/0307115.

[5] ———, *Koszul duality and equivariant cohomology for tori*, Int. Math. Res. Not. (2003), no. 42, 2255–2303. MR 1 997 219

[6] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83.

[7] W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology*, Academic Press, New York, 1976, Volume III: Cohomology of principal bundles and homogeneous spaces.
[8] J. Huebschmann, *Homological perturbations, equivariant cohomology, and Koszul duality*, arXiv math.AT/0401160.
[9] J. L. Koszul, *Sur un type d'algèbres différentielles en rapport avec la transgression*, Colloque de topologie (espaces fibrés), Bruxelles, 1950, Georges Thone, Liège, 1951, pp. 73–81. MR 13,109a
[10] T. Maszczyk and A. Weber, *Koszul duality for modules over Lie algebras*, Duke Math. J. **112** (2002), no. 3, 511–520.

The singular Cartan model

Matthias Franz

Let G be a compact connected Lie group with classifying space BG. We denote by $x_1, \ldots, x_r$ the generators of the exterior algebra $\mathbf{\Lambda} = H(G)$, and by $\xi_1, \ldots, \xi_r$ the generators of the symmetric algebra $\mathbf{S}^* = H^*(BG)$ which correspond under transgression to the generators of $\mathbf{\Lambda}^* = H^*(G)$ dual to the x_i. (For the moment, all (co)homology is with real coefficients.) Recall the "small Cartan model"

$$\mathbf{S}^* \otimes \Omega^*(X)^G, \qquad d = 1 \otimes d + \sum_{i=1}^{r} \xi_i \otimes x_i \tag{1}$$

which computes the equivariant cohomology of a G-manifold X, and the Chevalley–Koszul complex

$$\mathbf{\Lambda}^* \otimes \Omega^*(Y), \qquad d = 1 \otimes d + \sum_{i=1}^{r} x_i \otimes \xi_i, \tag{2}$$

which computes the cohomology of a G-principal bundle over a manifold Y. (Cf. Eckhard Meinrenken's talk in this report.) The complexes (1) and (2) are actually the Koszul-dual modules of the differential graded (dg) $\mathbf{\Lambda}$-module $\Omega^*(X)^G$ and the dg $\mathbf{S}^*$-module $\Omega^*(Y)$, respectively.

I will show how to generalise this to cohomology with coefficients in an arbitrary principal ideal domain R. (See [1], [2], [4].) In order to do so, we will replace differential forms by (normalised) singular cochains. This will also allow us to work with arbitrary topological spaces instead of manifolds.

So let G be a topological group whose homology $H(G) = H(G; R)$ is an exterior algebra $\mathbf{\Lambda} = \bigwedge(x_1, \ldots, x_r)$ on finitely many generators of odd degrees. This is equivalent to $H^*(BG)$ being a symmetric algebra $\mathbf{S}^* = R[\xi_1, \ldots, \xi_r]$ on finitely many generators of even degrees. In characteristic 0 it suffices that G be connected and $H(G)$ free and finite-dimensional. This holds, for example, for $G = (S^1)^r$, $U(n)$, $SU(n)$ or $Sp(2n)$ and $R = \mathbb{Z}$. Recall that the singular chain complex $C(G) = C(G; R)$ of a group G is a dg algebra by the Pontryagin product induced by the group multiplication. (Passing to homology, we get the product in $H(G)$.)

Instead of G-manifolds, we allow arbitrary topological G-spaces. The singular chain complex $C(X)$ of a G-space X is a dg module over $C(G)$, hence also its dual $C^*(X)$.

For a generalisation of the Chevalley–Eilenberg complex, we consider spaces over BG, i.e., maps $Y \to BG$ from some topological space Y to BG. (Recall that any principal G-bundle $P \to Y$ is induced from the universal G-bundle $EG \to BG$ by some map $Y \to BG$, unique up to homotopy.) The map of dg algebras $C^*(BG) \to C^*(Y)$ gives $C^*(Y)$ the structure of a dg $C^*(BG)$-module.

In order to imitate the constructions (1) and (2), we would like to define a $\mathbf{\Lambda}$-action on the $C(G)$-module $C^*(X)$ and an $\mathbf{S}^*$-action on the $C^*(BG)$-module $C^*(Y)$. But it is not clear how to do this because representatives $c_i \in C(G)$ of the generators $x_i \in \mathbf{\Lambda}$ will not commute in general (unless G is commutative), nor do representatives ξ_i' of the generators $\xi_i \in \mathbf{S}^*$ (unless $r = 1$). This implies that the naive imitations of the maps (1) and (2) are not differentials any more.

The key idea is to introduce higher order terms in both differentials to compensate for the lack of strict commutativity. In other words, on $\mathbf{S}^* \otimes C^*(X)$, we look for a differential of the form

$$d = 1 \otimes d + \sum_{0 \neq \alpha \in \mathbb{N}^r} \xi^\alpha \otimes c_\alpha, \tag{3}$$

where α is a multi-index and $c_\alpha \in C(G)$. (The c_i used above corresponds to c_α for α, the i-th canonical basis vector.) This (necessarily $\mathbf{S}^*$-equivariant) map is a differential for all X if and only if the c_α satisfy the relations

$$\forall\, 0 \neq \alpha \in \mathbb{N}^r \qquad dc_\alpha = \sum_{\beta+\gamma=\alpha} c_\beta \cdot c_\gamma, \tag{4}$$

Similarly, on $\mathbf{\Lambda}^* \otimes C^*(Y)$ we consider the differential

$$d = 1 \otimes d + \sum_{\emptyset \neq \pi \subset [r]} x_\pi \otimes \gamma_\pi, \tag{5}$$

where $[r] = \{1, \ldots, r\}$ and (x_π) is the canonical R-basis of $\mathbf{\Lambda}$ induced by the x_i. The condition on the γ_π reads

$$\forall\, \emptyset \neq \pi \subset [r] \qquad d\gamma_\pi = -\sum_{\mu \dot\cup \nu = \pi} (-1)^{|\mu|} \operatorname{sign}(\nu, \mu)\, \gamma_\mu \cup \gamma_\nu, \tag{6}$$

where $|\mu|$ denotes the size of the set μ and $\operatorname{sign}(\nu, \mu)$ the sign of the permutation defined by the partition $\mu \,\dot\cup\, \nu = \pi$.

Theorem 1. *The equations (4) and (6) have solutions with* $[c_i] = x_i$ *and* $[\gamma_i] = \xi_i$ *for all* i.

For equation (6), there is actually an explicit formula for the γ_π in terms of repeated cup-1-products of arbitrarily chosen representatives γ_i (Gugenheim–May [3]).

Theorem 2. *With these differentials, the complex $\mathbf{S}^* \otimes C^*(X)$ computes the equivariant cohomology of X as module over $\mathbf{S}^*$, and the complex $\mathbf{\Lambda}^* \otimes C^*(Y)$ the cohomology of the pull-back P of $EG \to BG$ along $Y \to BG$ as module over $\mathbf{\Lambda}$.*

Actually a stronger statement is true: these complexes are quasi-isomorphic to $C^*(X_G)$ and $C^*(P)$ as modules 'up to homotopy' over $\mathbf{S}^*$ and $\mathbf{\Lambda}$, respectively.

References

[1] M. Franz. Koszul duality and equivariant cohomology for tori. *Int. Math. Res. Not.* **42**, 2255-2303 (2003).

[2] M. Franz, Koszul duality and equivariant cohomology, arXiv e-print math.AT/0307115.

[3] V. K. A. M. Gugenheim and J. P. May. On the theory and applications of differential torsion products. *Mem. Am. Math. Soc.* **142** (1974).

[4] J. Huebschmann, Homological perturbations, equivariant cohomology, and Koszul duality, arXiv e-print math.AT/0401160.

The equivariant cohomology of hypertoric varieties

Megumi Harada

In order to construct a toric variety as a Kähler quotient of $\mathbb{C}^n$ by a torus, one begins with the combinatorial data of an arrangement $\mathcal{H}$ of n cooriented, rational, affine hyperplanes in $\mathbb{R}^d$. The normal vectors to these hyperplanes determine a subtorus $T^k \subset T^n$ ($k = n - d$), and the affine structure determines a value $\alpha \in (\mathfrak{t}^d)^*$ at which to reduce, so that we may define $X = \mathbb{C}^n /\!\!/_{\alpha} T^k$. Using the same combinatorial data, one can also construct a *hypertoric variety* as studied in [1, 6, 7, 5], which is defined as the hyperkähler quotient $M = \mathbb{H}^n /\!\!/\!\!/\!\!/_{(\alpha,0)} T^k$ of $\mathbb{H}^n \cong T^*\mathbb{C}^n$ by the induced action of the same subtorus $T^k \subset T^n$ [1]. These are non-compact varieties containing as a subvariety the Kähler variety X. It is well known that the toric variety X does not retain all of the information of $\mathcal{H}$; indeed, it depends only on the polyhedron Δ obtained by intersecting the half-spaces associated to each of the cooriented hyperplanes. Thus it is always possible to add an extra hyperplane to $\mathcal{H}$ without changing X. In contrast, the hypertoric variety M remembers the number of hyperplanes in $\mathcal{H}$, but its equivariant diffeomorphism type depends neither on the coorientations nor on the affine structure of $\mathcal{H}$.

In joint work with Nicholas Proudfoot [4] and Tara Holm [3], we have studied the equivariant topology of the hypertoric variety M equipped with a $T^d \times S^1$ action, where the T^d is the standard quotient T^d action on a toric or hypertoric variety, and the extra S^1 action descends from the scalar action of S^1 on the fibers of $T^*\mathbb{C}^n$. This extra S^1-action turns out to be a key ingredient that encodes additional combinatorial structure of $\mathcal{H}$. Namely, this S^1-action is sensitive to both the coorientations and the affine structure of $\mathcal{H}$, even on the level of equivariant cohomology. One can also recover the toric variety X as the minimum of the S^1 moment map, so in some sense the structure of a hypertoric variety M *along with this circle action* is the universal geometric object from which both M and X can be recovered.

In both [4] and [3] we give a computation of the $T^d \times S^1$-equivariant cohomology of M. Both uses the full combinatorial data of $\mathcal{H}$. In [4] the description uses a Kirwan surjectivity argument and the Chern classes of certain natural equivariant line bundles over M. Thus, $H^*_{T^d\times S^1}(M)$ is presented as a quotient of a polynomial ring. On the other hand, in [3] we give a description of $H^*_{T^d\times S^1}(M)$ using a Kirwan injectivity argument, thus describing it as a subring of the equivariant cohomology of the isolated fixed points M^T. This computation requires us to generalize to the non-compact setting a theorem of Goresky, Kottwitz, and MacPherson, which computes T-equivariant cohomology rings of compact Hamiltonian T-spaces satisfying some technical conditions [2]. The essential observation in [3] is that the Morse-theoretic arguments for the GKM theorem given in, e.g., [8] go through with only slight modifications in the setting when there is a direction of the moment map which is proper and bounded below. This is the case for smooth hypertoric varieties. We finish by giving a combinatorially explicit isomorphism [3] between the quotient and GKM descriptions of $H^*_{T^d\times S^1}(M)$.

References

[1] R. Bielawski and A. Dancer. The geometry and topology of toric hyperkähler manifolds. *Comm. Anal. Geom.*, 8:727–760, 2000.

[2] M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131:25–83, 1998.

[3] M. Harada and T. Holm. The equivariant cohomology of hypertoric varieties and their real loci. *In preparation*, 2004.

[4] M. Harada and N. Proudfoot. Properties of the residual circle action on a hypertoric variety. *math.DG/0207012*, 2002.

[5] T. Hausel and B. Sturmfels. Toric hyperkähler varieties. *Doc. Math.*, 7:495–534, 2002.

[6] H. Konno. Equivariant cohomology rings of toric hyperkähler manifolds. *Quaternionic structures in mathematics and physics (Rome)*, pages 231–240 (electronic), 1999.

[7] H. Konno. Cohomology rings of toric hyperkähler manifolds. *Int. J. of Math.*, 11(8):1001–1026, 2000.

[8] S. Tolman and J. Weitsman. On the cohomology rings of Hamiltonian T-spaces. *Proc. of the Northern California Symplectic Geometry Seminar, AMS Translations Series 2*, 196:251–258, 1999.

Projective flatness, Fourier transform, and Maslov index

Siye Wu
(joint work with William Kirwin)

In quantum mechanics, the momentum space and position space wave functions are related by the Fourier transform. We investigate how the Fourier transform arises in the context of geometric quantization. We consider a Hilbert space bundle over the space of compatible complex structures on a symplectic vector. This bundle is equipped with a projectively flat connection. The parallel transport is obtained by solving a partial differental equation and can be expressed by an integral kernel operator. We show that parallel transport along a geodesic in the

Hilbert space bundle is a rescaled orthogonal projection or Bogoliubov transformation. We then construct the kernel for the integral parallel transport operator. Finally, by extending geodesics to the boundary (for which the metaplectic correction is essential), we obtain the Bargmann and Fourier transforms as parallel transport in suitable limits. The above has been a joint work with William Kirwin.

The space of real Lagrangian subspaces is the Shilov boundary of Siegel's upper-half-space, which has a natural Kähler form. When there are three points on the Shilov boundary and three geodesics connecting them pairwise, there is a surface bounded by the three geodesics. It turns out that the triple Maslov index is, up to a factor, the integral of the Kähler form on this surface. This result can be generalized to other Hermitian symmetric spaces (or symmetric domains) classified by E. Cartan.

In the fermionic setting, one considers a Euclidean structure on a real vector space. Each maximal complex isotropic subspace is a polarization that defines a spinor representation of the Clifford algebra. There is also a projectively flat connection on the bundle of representaion spaces over the space of polarizations. The analogue of the Maslov triple index can also be considered.

Equivariant integration over the Higgs moduli

András Szenes

In joint work with Tamás Hausel, I proved a conjecture of Moore, Nekrasov and Shatashvili (hep-th/9712241) on the equivariant volumes of Higgs moduli in the rank-2 case, and formulated a precise conjecture in the higher rank case.

The theme of this work is "integration commutes with reduction" in a novel way. As our test example, consider the Grassmannian $\mathrm{Gr}(2,4)$ of 2-planes in complex 4-space, and denote by E the dual of the tautological rank-2 bundle on it. The topological intersection number $\int_{\mathrm{Gr}(2,4)} c_1(E)^4$ has the geometric interpretation of the number of lines in 3-space intersecting 4 given generic lines. There are 2 different localization principles which one can apply to computing this number (which happens to equal to be equal to 2).

One of them is the Bott fixed point formula. Denote the linear weights of the $U(1)^4$ action on $\mathrm{Gr}(2,4)$ by a, b, c, d. Then the fixed point formula is a sum over the fixed point set $F = \binom{4}{2}$:

$$\sum_{\sigma\in\binom{4}{2}} \sigma \cdot \frac{(a+b)^4}{(a-c)(a-d)(b-c)(b-d)}, \tag{1}$$

where σ acts on a function of a, b, c, d by permuting the variables and we mean $\binom{4}{2} = S_4/(S_2 \times S_2)$.

The other is the Jeffrey-Kirwan-Witten reduction principle. It uses the fact that $\mathrm{Gr}(2,4)$ may be obtained as a (GIT) quotient of the linear space of 2-by-4 matrices by the group $GL(2)$. One needs to take the compact diagonal torus $U(1)^2 \subset GL(2)$; denote the linear weights of this torus by t and v. The fixed point set of this commutative subgroup consists of a single point: the zero matrix. The contribution at this point is given by a residue:

$$\operatorname*{Res}_{t=0}\operatorname*{Res}_{v=0}\frac{-(t-v)^2(t+v)^4\ dt\,dv}{2t^4v^4}. \tag{2}$$

To formulate our results, we need to consider generalizations of this example in several directions:

- to the Berline-Vergne equivariant localization, i.e to integration of equivariant forms;
- to non-compact manifolds with proper moment maps;
- to hyperkähler reduction instead of the usual GIT/symplectic reduction.

Then the conjecture of Moore, Nekrasov and Shatashvili appears as an equality similar to that between (1) and (2), but in a more complex context. In this case, the Grassmanian is replaced by a more complicated space: the moduli space of rank-2 stable Higgs bundles on a Riemann surface. This space is not compact; however it has a dual structure similar to the one described above for the Grassmanian:

- It maybe obtained as an infinite-dimensional hyperkähler quotient.
- It has a circle action with a proper moment map.

Using non-rigorous path integral methods, Moore, Nekrasov and Shatashvili computed the equivariant volume of this moduli space and arrived at a formula, which, in the rank-2 case may be formulated as follows: There exists a rational function of one variable $R(n)$ such that the equivariant volume is

$$\sum R(p),\ e^p = \pm\frac{u-p}{u+p},$$

where u is the equivariant parameter. The sum is taken over the solutions of the so-called Bethe-Ansatz equations,

Our results: studying the circle action,

- we prove a generalization of the conjecture for the rank-2 case;
- formulate a precise conjecture for the higher rank case.

The details of our computations will be published in a forthcoming publication.

Jump formulas in equivariant cohomology

Paul-Émile Paradan

Let (M,ω) be a symplectic manifold equipped with a Hamiltonian action of a torus T, with Lie algebra $\mathfrak{t}$. We denote by $\Phi\colon M\to\mathfrak{t}^*$ the moment map of this action. Let us assume that Φ is *proper*, and that the T-action on M is *effective.* For every regular value ξ of Φ, we consider the reduction $\mathcal{M}_\xi := \Phi^{-1}(\xi)/T$, which is a compact symplectic orbifold. Let $\mathcal{H}^*_T(M)$ be the T-equivariant cohomology of M. Associated to the data (M,T,Φ) we have, for every regular value ξ of Φ, the Kirwan morphism [4]

$$\mathbf{Kir}_\xi\colon \mathcal{H}^*_T(M)\to\mathcal{H}^*(\mathcal{M}_\xi).$$

and the Kumar-Vergne isomorphism [5]

$$\mathbf{kv}_\xi \colon \mathcal{H}^*(\mathcal{M}_\xi) \to \mathcal{H}_T^{-\infty}(\Phi^{-1}(\xi))$$

associated to the T-principal bundle $\Phi^{-1}(\xi) \to \mathcal{M}_\xi$. Here " $\mathcal{H}_T^{-\infty}$ " denotes the equivariant cohomology with generalized coefficients defined by Kumar and Vergne in [5]. So for every regular value ξ of Φ one defines a map

$$p_\xi \colon \mathcal{H}_T^*(M) \to \mathcal{H}_{T,c}^{-\infty}(M)$$

as the composition of $\mathbf{kv}_\xi \circ \mathbf{Kir}_\xi$ with the direct image morphism $\mathcal{H}_T^{-\infty}(\Phi^{-1}(\xi)) \to \mathcal{H}_{T,c}^{-\infty}(M)$ related to the inclusion of the compact submanifold $\Phi^{-1}(\xi) \subset M$.

For every $\eta \in \mathcal{H}_T^*(M)$ the integral $\int_M p_\xi(\eta)$ is a generalized function on $\mathfrak{t}$ supported on 0 such that

$$\int_{\mathfrak{t}} \left(\int_M p_\xi(\eta) \right)(X) dX = \frac{1}{|S_\xi|} \int_{\mathcal{M}_\xi} \mathbf{Kir}_\xi(\eta). \tag{1}$$

Here dX is normalized by $\mathrm{Vol}(T, dX) = 1$, and $|S_\xi|$ is the cardinality of the generic stabilizer of T on $\Phi^{-1}(\xi)$. We denote by $\mathrm{I}(M, \eta, \xi)$ the rhs of (1).

Proposition 1. *There exists a cohomology class $P_\xi \in \mathcal{H}_{T,c}^{-\infty}(M)$ such that $p_\xi(\eta) = \eta \mathrm{P}_\xi$. The cohomology class P_ξ is well defined for every ξ.*

Idea of the proof: In [6, 7], we have defined a notion of *partition of unity in equivariant cohomology.* Let $\mathcal{H}_\xi$ be the Hamiltonian vectors field of the function $\|\Phi - \xi\|^2$, and consider the T-invariant 1-form $\lambda_\xi = (\mathcal{H}_\xi, -)_M$ defined with the help of a T-invariant Riemannian metric $(-,-)_M$. The cohomology class $\mathrm{P}_\xi \in \mathcal{H}_{T,c}^{-\infty}(M)$ is defined by the following closed equivariant form:

$$\frac{1}{(2i\pi)^{dimT}} \left(\chi_\xi + d\chi_\xi [D\lambda_\xi]^{-1} \lambda_\xi \right).$$

Here $[D\lambda_\xi]^{-1}$ is an inverse of the equivariant 1-form $D\lambda_\xi$ defined on the open subset $M - \mathrm{Cr}(\|\Phi - \xi\|^2)$, and χ_ξ is a smooth T-invariant function on M with compact support, equal to 1 in a neighborhood of $\Phi^{-1}(\xi)$) and with the condition that $\mathrm{Support}(f) \cap \mathrm{Cr}(\|\Phi - \xi\|^2) = \Phi^{-1}(\xi)$. We have proved in [6] that $p_\xi(1) = \mathrm{P}_\xi$.

Now the study of the map $\xi \to \mathrm{P}_\xi$ gives a new way to recover the properties of the map $\xi \mapsto \mathrm{I}(M, \eta, \xi) = \frac{1}{|S_\xi|} \int_{\mathcal{M}_\xi} \mathbf{Kir}_\xi(\eta)$ [8].

Proposition 2. *The map $\xi \mapsto \mathrm{P}_\xi$ is locally constant on the open subset of regular values of Φ.*

So $\mathrm{I}(M, \eta, \xi) = \mathrm{I}(M, \eta, \xi')$ if ξ and ξ' belong to the same connected component of regular values of Φ.

Let Δ be an hyperplane of $\mathfrak{t}^*$, equipped with an orientation o, and which separates two connected components of regular values of Φ. Let $T_\Delta \subset T$ be the subtorus of dimension 1 with Lie algebra $\mathfrak{t}_\Delta := \{X \in \mathfrak{t} |\ \langle \xi - \xi', X \rangle = 0,\ \forall \xi, \xi' \in \Delta\}$. Let M^{T_Δ} be the submanifold of points fixed by T_Δ, and let M_Δ be the open subset of $M^{T_\Delta} \cap \Phi^{-1}(\Delta)$ on which T/T_Δ acts locally freely. The symplectic manifold M_Δ

carries a Hamiltonian action of T/T_Δ with moment map $\Phi_\Delta\colon M_\Delta \to \Delta$ equal to the restriction of Φ on M_Δ.

We choose a decomposition $T = T_\Delta \times T/T_\Delta$, where T/T_Δ denotes a subtorus of T. Associated to this decomposition we have $\mathcal{H}^-_T(M_\Delta) \simeq \mathcal{H}^-_{T/T_\Delta}(M_\Delta) \otimes \mathcal{H}^-_{T_\Delta}(M_\Delta)$ where $- \in \{*, -\infty\}$. Let $\xi \in \Delta$ be a regular value of Φ_Δ and let $\xi^\pm \in \mathfrak{t}^*$ be two regular values of Φ belonging respectively to two connected components of regular values of Φ separated by Δ, with the condition that the line (ξ^+, ξ^-) intersects Δ at ξ.

Let N_Δ be the T-equivariant normal bundle of M^{T_Δ} in M, and let $\mathrm{Eul}(N_\Delta) \in \mathcal{H}^*_T(M^{T_\Delta})$ be the T-equivariant Euler class of N_Δ. When restricted to M_Δ, $\mathrm{Eul}(N_\Delta)|_{M_\Delta}$ can be seen as a polynomial function with values in the subalgebra $\mathcal{H}^*(M_\Delta)^{\mathrm{bas}}$ of basic elements of $\mathcal{H}^*(M_\Delta)$. Following [6], we defined inverses $\mathrm{Eul}^{-1}_\pm(N_\Delta) \in \mathcal{C}^{-\infty}(\mathfrak{t}_\Delta, \mathcal{H}^*(M_\Delta)^{\mathrm{bas}})$ by

$$\mathrm{Eul}^{-1}_\pm(N_\Delta)(X) = \lim_{s\to+\infty} \frac{1}{\mathrm{Eul}(N_\Delta)|_{M_\Delta}(X \pm is\beta)},$$

where $\beta \in \mathfrak{t}_\Delta - \{0\}$ is compatible with the orientation o of Δ. Since the polynomial $\mathrm{Eul}(N_\Delta)|_{M_\Delta}$ is invertible in a smooth way on $\mathfrak{t}_\Delta - \{0\}$, the difference

$$\delta^o_\Delta := \mathrm{Eul}^{-1}_-(N_\Delta) - \mathrm{Eul}^{-1}_+(N_\Delta) \tag{2}$$

is a generalized function on $\mathfrak{t}_\Delta$ supported on 0. We will consider δ^o_Δ as a cohomology class in $\mathcal{H}^{-\infty}_{T_\Delta}(M_\Delta)$. Following Proposition 1, one has a cohomology class $\mathrm{P}^\Delta_\xi \in \mathcal{H}^{-\infty}_{T/T_\Delta,c}(M_\Delta)$ associated to the value $\xi \in \Delta$.

Proposition 3. *We have*

$$\mathrm{P}_{\xi^+} - \mathrm{P}_{\xi^-} = (i_\Delta)_*\left(\mathrm{P}^\Delta_\xi \delta^o_\Delta\right) \qquad \text{in} \quad \mathcal{H}^{-\infty}_{T,c}(M) \tag{3}$$

where $\delta^o_\Delta \in \mathcal{H}^{-\infty}_{T_\Delta}(M_\Delta)$ *and* $(i_\Delta)_*\colon \mathcal{H}^{-\infty}_{T,c}(M_\Delta) \to \mathcal{H}^{-\infty}_{T,c}(M)$ *is the direct image map.*

With δ^o_Δ one defines a *residue map* $\mathrm{Res}^o_\Delta\colon \mathcal{H}^*_T(M) \to \mathcal{H}^*_{T/T_\Delta}(M_\Delta)$ (see [8]), and from (3) we get the formulas of Guillemin-Kalkman [2]

$$\mathrm{I}(M, \eta, \xi^+) - \mathrm{I}(M, \eta, \xi^-) = \mathrm{I}(M_\Delta, \mathrm{Res}^o_\Delta(\eta), \xi),$$

for every $\eta \in \mathcal{H}^*_T(M)$

References

[1] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren, vol. 298, Springer, Berlin, 1991.

[2] V. Guillemin and J. Kalkman, The Jeffrey-Kirwan localization theorem and residue operations in equivariant cohomology, *J. Reine Angew. Math.*,**470**, (1996), p. 123-142.

[3] V. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory.* With an appendix containing two reprints by Henri Cartan. Mathematics Past and Present. Springer-Verlag, Berlin, 1999.

[4] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Princeton Univ. Press, Princeton, 1984.

[5] S. Kumar and M. Vergne, Equivariant cohomology with generalized coefficients, *Astérisque*, 215, 1993, p. 109-204.
[6] P.-E. Paradan, Formules de localisation en cohomologie équivariante, *Compositio Mathematica*, **117**, 1999, p. 243-293.
[7] P.-E. Paradan, The moment map and equivariant cohomology with generalized coefficients, *Topology*, **163**, 2000, p. 401-444.
[8] P.-E. Paradan, Note sur les formules de saut de Guillemin-Kalkman, À paraître aux *C.R.A.S.*.

Conjugation spaces

Jean-Claude Hausmann
(joint work with Tara Holm and Volker Puppe)

Let τ be a continuous involution on a space X, seen as an action of the cyclic 2-group $C = \{I, \tau\}$. Let $\rho\colon H^{2*}_C(X) \to H^{2*}(X)$ and $r\colon H^*_C(X) \to H^*_C(X^\tau)$ be the restriction homomorphisms (cohomology with $\mathbb{Z}_2$-coefficients). Suppose that $H^{odd}(X) = 0$. An H^**-frame* for (X, τ) is a pair (κ, σ), where

(a) $\kappa\colon H^{2*}(X) \to H^*(X^\tau)$ is an additive isomorphism dividing the degrees in half, and
(b) $\sigma\colon H^{2*}(X) \to H^{2*}_C(X)$ is an additive section of ρ.

Moreover, κ and σ should satisfy the equation

$$r\circ\sigma(a) = \kappa(a)u^m + \ell t_m \tag{1}$$

for all $a \in H^{2m}(X)$ and all $m \in \mathbb{N}$, where ℓt_m stands for any polynomial of degree $< m$ in the variable u. An involution admitting an H^*-frame is called a *conjugation* and a space together with a conjugation is called a *conjugation space.* We prove the following properties for conjugation spaces.

Proposition 1. *Let (σ, κ) be an H^*-frame for an involution. Then σ and κ are ring homomorphisms.*

Proposition 2 (Naturality and uniqueness of H^*-frames)**.** *Let $f\colon Y \to X$ be an equivariant map between spaces with involution. Let (σ_X, κ_X) and (σ_Y, κ_Y) be H^*-frames for the involutions on X and Y. Then $H^*_C f\circ\sigma_X = \sigma_Y\circ H^*f$ and $H^*f^\tau\circ\kappa_X = \kappa_Y\circ H^*f$. In particular, the H^*-frame for a conjugation is unique.*

By the Leray-Hirsch theorem, the section σ gives rise to an isomorphism of $\mathbb{Z}_2[u]$-modules $\hat\sigma\colon H^*(X)[u] \xrightarrow{\approx} H^*_C(X)$. As σ is a ring homomorphism by Proposition 1, one has the following complete describtion of the ring $H^*_C(X)$ in terms of $H^*(X)$.

Corollary 3. *Let (κ, σ) be the H^*-frame for a conjugation on X. Then*

$$\hat\sigma\colon H^*(X)[u] \xrightarrow{\approx} H^*_C(X)$$

is an isomorphism of $\mathbb{Z}_2[u]$-algebras. Moreover, $\hat\sigma$ is functorial for equivariant maps.

Our main examples of conjugation spaces are spherical conjugation complexes. Let Y be a topological space with an involution τ. Let D^{2k} be the closed disk of radius 1 in $\mathbb{R}^{2k}$, equipped with a linear involution with exactly k negative eigenvalues. Let $\alpha\colon S^{2k-1} \to Y$ be an equivariant map. Then the involutions on Y and on D^{2k} induce an involution on the space $X = Y \cup_\alpha D^{2k}$. We say that X is obtained from Y by attaching a *conjugation cell* of dimension $2k$. For $k = 0$, this amounts to the disjoint union with a point. More generally, one can attach to Y a set Λ of $2k$-conjugation cells, via an equivariant map $\alpha\colon \amalg_\Lambda S^{2k-1}_\lambda \to Y$.

Proposition 4. *Let Y be a conjugation space and let X be obtained from Y by attaching a collection of conjugation cells of dimension $2k$. Then X is a conjugation space.*

A *spherical conjugation complex* is a space (with involution) obtained from $\emptyset$ by successive adjunction of collection of conjugation cells. The adjective "spherical" emphasizes that the collections of conjugation cells do not need to occur in increasing dimensions. Proposition 4 implies the following

Corollary 5. *A spherical conjugation complex is a conjugation space.*

Example 6. *The complex projective space $\mathbb{C}P^k$ with the involution being the complex conjugation.* Its standard cell decomposition makes $\mathbb{C}P^k$ a spherical conjugation complex and therefore a conjugation space. Let a be the generator of $H^2(\mathbb{C}P^k)$ and $b = \kappa(a)$ that of $H^1(\mathbb{R}P^k)$. One can show that Equation (1) is here $r \circ \sigma(a^m) = (bu + b^2)^m$.

Example 6 generalizes in the following way. Let X be a space together with an involution τ and a continuous action of a torus T. We say that τ is *compatible* with this torus action if $\tau(g \cdot x) = g^{-1} \cdot \tau(x)$ for all $g \in T$ and $x \in X$. It follows that τ induces an involution on on the fixed point set X^T. We are interested in the case where X is a compact symplectic manifold for which the torus action is Hamiltonian and the compatible involution is smooth and anti-symplectic. Using a Morse-Bott function obtained from the moment map for the T-action, we prove the following

Proposition 7. *Let X be a compact symplectic manifold equipped with a Hamiltonian action of a torus T and a smooth anti-symplectic compatible involution. If X^T is a spherical conjugation complex, then X is a spherical conjugation complex.*

Examples of such Hamiltonian spaces include:

(a) co-adjoint orbits of any semi-simple compact Lie group, with the Chevalley involution,
(b) smooth toric manifolds, and
(c) polygon spaces.

Consequently, these examples are conjugation spaces. The existence of a ring isomorphism κ is classical for Grasmannians, and known for toric manifolds [DJ] and polygon spaces [HK]. When X is a GKM-space with isotropy weights pairwise independent over $\mathbb{Z}_2$, it has been proved in [S] and in [BGH] that κ is induced from a ring isomorphism $\hat{\kappa}\colon H^{2*}_T(X) \to H^*_{T_2}(X^\tau)$, ($T_2$ being the 2-torus of T). This covers the case of co-adjoint orbits of $SU(n)$ and of toric manifolds. We hope that this refined equivariant result may be also be reproved and generalized using conjugation spaces.

For more examples, one can prove that there are infinitely many C-equivariant homotopy types of spherical conjugation complexes with three conjugation cells, for instance in dimension 0, 2 and 4. Torus manifolds of [HM] are likely to produce other families of conjugation spaces. Finally, the category of conjugation spaces is closed under various operations, including direct products and connected sums.

Natural bundles over conjugation spaces are the *conjugate equivariant bundles* introduced by Atiyah [A] under the name of "real bundles". These are complex vector bundles $\eta = (E \xrightarrow{p} X)$ together with an involution $\hat{\tau}$ on E which covers τ and is conjugate linear on each fiber. Then $E^{\hat{\tau}}$ is a real bundle η^τ over X^τ. Using the naturality of H^*-frames and the Schubert cells in Grasmannians, one proves that $\kappa(c(\eta)) = w(\eta^\tau)$, where $c()$ denotes the (mod 2) total Chern class and $w()$ the total Stiefel-Whitney class. This is true provided some tameness of the embedding $X^\tau \subset X$, which holds for smooth actions on manifolds or for spherical conjugation complexes.

References

[A] Atiyah M. K-Theory and reality. *Quat. J. of Math. Oxford* **17** (1966) 367–386.

[BGH] Biss D, Guillemin V. and Holm T. The mod 2 cohomology of fixed point sets of anti-symplectic involutions. Preprint `math.SG/0107151`. To appear in *Advances in Mathematics*

[DJ] Davis, M. & Januszkiewicz, T. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.* **62** (1991), 417–451

[HM] Hattori A. and Masuda M. Theory of multi-fans. em Osaka J. of Mathematics 40 (2003) 1–68

[HK] Hausmann J-C. and Knutson A. The cohomology ring of polygon spaces. Grassmannians. *Annales de l'Institut Fourier* (1998) 281-321.

[S] Schmid C. Cohomologie équivariante de certaines variétés hamiltoniennes et de leur partie réelle. *Thesis* (2001), University of Geneva.

Equivariant Schubert calculus

Matthieu Willems

1. Introduction

Let G be a connected complex semi-simple group, $B \subset G$ a Borel subgroup of G and $H \subset B$ a Cartan subgroup of B. We denote by T the maximal compact torus of H and by $X = G/B$ the flag variety associated to this data. The torus T acts on X by $t.(gB) = (tg)B$ for $t \in T$ and $g \in G$. The set of fixed points of the action of T on X can be identified with $W = N_G(H)/H$, the Weyl group of G, which is generated by r simple reflections s_i. We denote by $\alpha_i \in \mathfrak{h}^*$ the corresponding simple roots and by $\alpha_i^\vee \in \mathfrak{h}$ the corresponding simple coroots ($\mathfrak{h}$

is the Lie algebra of H). For a simple root α_i, we set $P_{\alpha_i} = Bs_iB \cup B$. The T-equivariant cohomology of X (with complex coefficients) is an algebra over the T-equivariant cohomology of a point, which can be identified with the symmetric algebra S of $\mathfrak{h}^*$. In fact, $H_T^*(X)$ is a free module over S with basis $\{\hat{\xi}^w\}_{w\in W}$ defined by the relations

$$\int_{\overline{X_v}} \hat{\xi}^w = \delta_{v,w},$$

where $\overline{X_v}$ is the closure of the Schubert cell $X_v = BvB/B \subset X$. One of the problems in the Schubert calculus is to calculate the polynomials $p_{u,v}^w \in S$ such that

$$\hat{\xi}^u\hat{\xi}^v = \sum_{w\in W} p_{u,v}^w \hat{\xi}^w.$$

2. Bott-Samelson varieties

We use Bott-Samelson varieties to give a method to compute these polynomials. Let $w = s_{\mu_1}\cdots s_{\mu_N}$ be a reduced decomposition of an element w of W. We denote by $\Gamma = \Gamma(\mu_1,\ldots,\mu_N)$ the Bott-Samelson variety associated to the sequence $\mu_1, \ldots, \mu_N$. It is the space of orbits of the action of B^N on $P_{\mu_1}\times\cdots\times P_{\mu_N}$ defined by

$$(g_1, g_2, \ldots, g_N)(b_1, b_2, \ldots, b_N) = (g_1b_1, b_1^{-1}g_2b_2, \ldots, b_{N-1}^{-1}g_Nb_N).$$

We will denote by $[g_1, g_2, \ldots, g_N]$ the class of $(g_1, g_2, \ldots, g_N)$ in Γ. The torus T acts on Γ by

$$t[g_1, g_2, \ldots, g_N] = [tg_1, g_2, \ldots, g_N].$$

We set $\mathcal{E} = (\mathbb{Z}/2\mathbb{Z})^N$. The set of fixed points of the action of T on Γ can be identified with $\mathcal{E}$ because $\Gamma^T \simeq \prod_{1\le i\le N} N_{P_{\mu_i}}(H)/H \simeq (\mathbb{Z}/2\mathbb{Z})^N$.

We define a T-invariant cell decomposition $\Gamma = \coprod_{\epsilon\in\mathcal{E}} \Gamma_\epsilon$, where Γ_ϵ is the set of classes $[g_1, g_2, \ldots, g_N]$ such that

$$\forall 1 \le i \le N, \begin{cases} g_i \in B & \text{if } \epsilon_i = 0, \\ g_i \notin B & \text{if } \epsilon_i = 1. \end{cases}$$

Since each Γ_ϵ is an even cell, the S-algebra $H_T^*(\Gamma)$ is a free S-module with a basis $\{\hat{\sigma}_\epsilon\}_{\epsilon\in\mathcal{E}}$ defined by the relations

$$\int_{\overline{\Gamma_{\epsilon'}}} \hat{\sigma}_\epsilon = \delta_{\epsilon',\epsilon}.$$

For $\epsilon \in \mathcal{E}$, we set $\pi_+(\epsilon) = \{1 \le i \le N, \epsilon_i = 1\}$, and for $1 \le j < i \le N$, we set $a_{j,i} = \mu_i(\mu_j^\vee)$. For $1 \le i \le N$, we denote by (i) the element of $\mathcal{E}$ defined by $(i)_j = \delta_{i,j}$.

Theorem 1. *We have the following relations:*

(1) $\hat{\sigma}_\epsilon = \prod_{i\in\pi_+(\epsilon)} \hat{\sigma}_{(i)}$

(2) $\hat{\sigma}_{(i)}^2 = \mu_i\hat{\sigma}_{(i)} - \sum_{j<i} a_{j,i}\hat{\sigma}_{(i)}\hat{\sigma}_{(j)}$.

3. Schubert calculus

We define a T-equivariant map $g\colon \Gamma \to X$ by

$$g([g_1, \dots, g_N]) = g_1 \times \cdots \times g_N \ [B].$$

For $\epsilon \in \mathcal{E}$, we set $l(\epsilon) = \mathrm{card}(\pi_+(\epsilon))$. The following theorem explains the link between the cohomology of Γ and the cohomology of X.

Theorem 2. *Let v be an element of W. We have:*

$$g^*(\hat{\xi}^v) = \sum_{\substack{\epsilon\in\mathcal{E},\, l(\epsilon)=l(v)\\ \text{and } g(\epsilon)=v}} \hat{\sigma}_\epsilon.$$

Using Theorems 1 and 2, we can give a method to compute the polynomials $p^w_{u,v}$ (see [6] for more details). It gives an expression in terms of simple roots and Cartan numbers. Unfortunately, it is not a positive formula (in the sense of Graham [4]). In [2] Haibao Duan gives similar formulas for the ordinary cohomology of X, and in [3] he uses his formulas to give a program for multipying Schubert classes in ordinary cohomology.

References

[1] Alberto Arabia, *Cohomologie T-équivariante de la variété de drapeaux d'un groupe de Kac-Moody*, Bulletin de la Société Mathématique de France **117** (1989), 129–165.

[2] Haibao Duan, *Multiplicative rule of Schubert classes*, Math.AG/0306227 (2003).

[3] Haibao Duan and Xuezhi Zhao, *A program for multiplying Schubert classes*, Math.AG/0309158 (2003).

[4] William Graham, *Positivity in equivariant Schubert calculus*, Duke Mathematical Journal **109** (2001), 599–614.

[5] Bernard Kostant and Shrawn Kumar, *The Nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Advances in Mathematics **68** (1986), 187–237.

[6] Matthieu Willems, *Cohomologie équivariante des tours de Bott et calcul de Schubert équivariant*, Journal de l'Institut de Mathématiques de Jussieu (to appear).

Actions on flag manifolds

Volker Hauschild

Let G be a compact connected Lie group. Consider a connected subgroup $U \subset G$ and the corresponding homogeneous space $X = G/U$. Then G acts on X in the standard way by left translation. It is a classical question in transformation group theory if this action represents the only way G can act on X. There are some examples which show that there are indeed homogeneous spaces of G which allow different G-actions, see e.g. [7]. As a rule for big subgroups U, however, one would expect that there are not many G-actions on G/U. This has been confirmed by the results of [2, 3, 4], [5], [6]. Call a homogeneous space G/U *standard* if every locally smooth action of G on G/U is conjugate to the standard action by left translation. For example, in the notes [3, 4] it has been shown that the flag manifold G/T is standard where $T \subset G$ is a maximal torus. In this talk I shall consider the case of the complex Grassmannians $G_{n+1,2} = SU(n+1)/S(U(n-1) \times U(2))$.

Theorem 1. *The complex flag manifold $SU(5)/S(U(3) \times U(2))$ is standard.*

In the following we suppose $X = SU(n+1)/S(U(n-1) \times U(2))$ for general n and only at the end of the proof we specialize to $n = 4$. Observe that the induced action of the maximal torus T on X has a nonempty fixed space by the standard equality of the Euler numbers: $e(X) = e(X^T)$. Let $p \in X^T$ and let G_p be the corresponding isotropy group. By dimension reasons and the classification of the maximal rank subgroups of Lie groups the connected component G_p^0 must be conjugate to one of the following subgroups:

$$SU(n+1),\ S(U(n) \times U(1)),\ S(U(n-1) \times U(2)).$$

The first case would mean that $G = SU(n+1)$ has a fixed point on X, a possibility excluded by Theorem 4.6 in the note [1].

The third case means that the orbit $G(p)$ has the same dimension as X itself and so must coincide with the full manifold. Since X is simply connected, it follows that G_p is connected and the action must be transitive with isotropy group $S(U(n-1) \times U(2))$. Therefore if we are able to eliminate the second case, our theorem is proved.

Let σ_p be the slice at p. Then G_p acts on σ_p via a linear representation whose principal isotropy group must be positive-dimensional. Now we have $\dim \sigma_p = 2n - 4$. But the least-dimensional nontrivial real representation of $SU(n)$ is the realification of the standard complex representation, which has real dimension $2n$. It follows that the subgroup $SU(n)$ of G_p^0 acts trivially on σ_p, and consequently the principal isotropy group of the whole action must be a finite extension of $S(U(n) \times U(1))$ and then all isotropy groups are of maximal rank or it must be a finite extension of $SU(n)$. The second case can be excluded in the following way: One can verify that under the hypothesis of the theorem there are no nontrivial homomorphisms $H^*(X;\mathbb{Q}) \to H^*(\mathbb{C}\mathrm{P}^n; Q)$. Using this fact and some simple theory of characteristic classes one sees that the principal isotropy group must be conjugate to $SU(n)$. In this case one can show that $H^*(X;\mathbb{Q})$ must be a flat module over $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Q})$, which is not possible. We conclude that every isotropy group must be of maximal rank. Let

$$X^{(T)} = \big\{ x \in X \bigm| gTg^{-1} \subset G_x \text{ for some } g \in G \big\}.$$

It follows that $X = X^{(T)}$. If X^T is the fixed set of the induced T-action, then the Weyl group $WG = NT/T$ acts on X^T in a natural way. Moreover one can easily verify that in this situation the subset X_0^T of those fixed set components of T which intersect the principal orbit type $X_{(H)}$ nontrivially must be identical to X^T. It follows that the inclusion $X^T \subset X$ induces an isomorphism

$$X^T/WG \cong X^{(T)}/G \cong X/G.$$

Since X is connected, WG must act transitively on the set of connected components of X^T. Let $F_0 \subset X^T$ be a connected component of X^T and let $W_0 \subset WG$

be the stabilizer of F_0. Now the full G-action on X is completely determined by the induced W_0-action on F_0. For the equivariant cohomology (with rational coefficients) of the G-action it is not difficult obtain the following isomorphism of graded $H^*(B_G)$-algebras:

$$H^*_G(X) \cong (H^*(B_T) \otimes H^*(F_0))^{W_0} .$$

Projecting modulo the ideal generated by the elements of positive degree in $H^*(F_0)^{W_0}$ induces a surjection of $H^*(B_G)$-algebras

$$H^*_G(X) \to H^*(B_T)^{W_0}.$$

This in turn induces a surjective $\mathbb{Q}$-algebra homomorphism

$$H^*(X) \to H^*(G/T)^{W_0}.$$

We observe that W_0 must be the the Weyl group of $SU(n)$ and therefore

$$H^*(G/T)^{W_0} \cong H^*(\mathbb{C}\mathrm{P}^n).$$

We therefore have a nontrivial homomorphism

$$H^*(X) \to H^*(\mathbb{C}\mathrm{P}^n).$$

But it can be shown that for $G_{5,2} = SU(5)/S(U(3) \times U(2))$ every graded homomorphism

$$h\colon H^*(G_{5,2}) \to H^*(\mathbb{C}\mathrm{P}^4)$$

of graded $\mathbb{Q}$-algebras must be trivial in the sense that the image of h is in $H^0(\mathbb{C}\mathrm{P}^4)$. This excludes our case and the theorem is proved.

References

[1] Allday, C., Hauschild, V., Puppe, V.: *A non-fixed point theorem for Hamiltonian Lie group actions*, Trans. Amer. Math. Soc. **354**, 2971–2982 (2002)

[2] Hauschild, V.: *Über den Symmetriegrad der rational azyklischen homogenen Räume*, Manuscripta Math. **32**, 365–379 (1980)

[3] Hauschild, V.: *Actions of compact Lie groups on homogeneous spaces*, Math. Z. **189** 475–486 (1985)

[4] Hauschild, V.: *The Euler characteristic as an obstruction to compact Lie group actions*, Trans. Amer. Math. Soc. **298**, 549–578 (1986)

[5] Hsiang, W. Y.: Cohomology theory of topological transformation groups, *Ergeb. Math. Grenzgeb.* **85**, Springer-Verlag, Berlin-Heidelberg-New York 1975

[6] Hsiang, W. Y., Tomter, P.: *Transformation groups on complex Stiefel manifolds*, Acta math. **152**, 107–126 (1986)

[7] Kreck, M., Stolz, S.: *Nonconnected moduli spaces of positive sectional curvature metrics*, J. Amer. Math. Soc. **6**, 825–850 (1993)

Morse interpolation for Hamiltonian GKM spaces

Catalin Zara

(joint work with Victor Guillemin)

A Hamiltonian GKM space is a compact symplectic manifold (M^{2m}, ω) with a Hamiltonian action of a compact torus T^n of dimension $n \geq 2$, such that (1) the fixed point set M^T is finite and (2) for each fixed point $p \in M^T$, the weights $\alpha_{p,1}, \ldots, \alpha_{p,m}$ of the (complex) representation of T on the tangent space T_pM are pairwise linearly independent.

An equivariant cohomology class $f \in H_T^*(M;\mathbb{C})$ is determined by its restrictions to fixed points, and the first condition above implies that $H_T^*(M)$ is a subring of the ring $Maps(M^T, \mathbb{S}(\mathfrak{t}^*))$, where $\mathfrak{t}^*$ is the dual of the Lie algebra $\mathfrak{t}$ of T and $\mathbb{S}(\mathfrak{t}^*)$ is the symmetric algebra. The second condition allows us to construct a simple combinatorial object that encodes the conditions which have to be satisfied by an element $f \in H_T^*(M) \subset Maps(M^T, \mathbb{S}(\mathfrak{t}^*))$ ([GZ1]). This combinatorial object is a regular graph $\Gamma = (V, E)$, with m edges meeting at each vertex, and with a labeling $\alpha\colon E \to \mathfrak{t}^*$ of the oriented edges by weights of T. The vertices of the graph correspond to the fixed points $p \in M^T$, the edges correspond to non-trivial connected components of the sets of points fixed by codimension one subtori $T' \subset T$, and the labels of the oriented edges with initial vertex p are the weights $\alpha_{p,1}, \ldots, \alpha_{p,m}$. From the pair (Γ, α) one constructs a graded ring

$$H_\alpha(\Gamma) = \left\{ f\colon V \to \mathbb{S}(\mathfrak{t}^*) : f(p) \equiv f(q) \mod \alpha_e \text{ for every edge } e = (p,q) \text{ of } \Gamma \right\}$$

and $H_T^*(M) \simeq H_\alpha(\Gamma)$ as graded rings ([GKM]). Both graded rings are free $\mathbb{S}(\mathfrak{t}^*)$-modules, and our goal is:

Construct a canonical basis of $H_T^*(M)$ as elements of $H_\alpha(\Gamma)$.

Fix a circle $S \subset T$ such that $M^S = M^T$ and let $\xi \in \mathfrak{t}$ be the infinitesimal generator of S. Then the ξ-component $\phi^\xi\colon M \to \mathbb{R}$ of the moment map $\phi\colon M \to \mathfrak{t}^*$ is a Morse function on M. In the combinatorial setting, ξ induces a partial ordering on the graph Γ. We say that an oriented edge e is ascending if $\alpha_e(\xi) > 0$ and is descending if $\alpha_e(\xi) < 0$; for two vertices p and q, we say that $p \prec q$ if there exists an ascending path from p to q.

For each $p \in V$, there exists a class $\tau_p \in H_\alpha(\Gamma)$ such that: (1) τ_p is supported on the flow-up $\mathcal{F}_p = \{q \in V : p \preceq q\}$, (2) $\tau_{p,q} (:= \tau_p(q))$ is homogeneous, of the same degree for all $q \in \mathcal{F}_p$, and (3) $\tau_{p,p}$ is the product of weights associated to descending edges with initial vertex p. A collection $\{\tau_p\}_{p \in V}$ of such classes is a basis of $H_\alpha(\Gamma)$ over $\mathbb{S}(\mathfrak{t}^*)$. There might be several classes τ_p satisfying the conditions above, but one can use local index maps similar to the ones defined in [GK] to select a canonical class.

For $q \in V$, let $e_1, \ldots, e_k$ be the descending edges from q, let $q_1, \ldots, q_k$ be the other vertices of these edges, and let $\alpha_1, \ldots, \alpha_k$ be the weights associated to these edges. The local index map $I_q\colon H_\alpha(\Gamma) \to \mathbb{S}(\mathfrak{t}^*)$ is defined by

$$I_q(f) = \frac{f(q)}{\prod \alpha_i} + \sum_{i=1}^{k} \frac{\rho_i(f(q_i))}{(-\alpha_i)\prod_{j \neq i} \rho_i(\alpha_j)} ,$$

where $\rho_i\colon \mathbb{S}(\mathfrak{t}^*) \to \mathbb{S}(\mathfrak{t}^*)$ is the ring morphism determined by

$$\rho_i(\beta) = \beta - \frac{\beta(\xi)}{\alpha_i(\xi)}\alpha_i \qquad \text{for all } \beta \in \mathfrak{t}^* .$$

In this form, it is quite hard to see that $I_q(f) \in \mathbb{S}(\mathfrak{t}^*)$; however, this becomes obvious from the following description of the local index: $f(q)$ is a solution of the system of congruences $\{g \equiv f(q_i) \mod \alpha_i$, for all $i = 1, \ldots, k\}$. Such a solution is defined only up to an element of $(\prod \alpha_i)\mathbb{S}(\mathfrak{t}^*)$, and $(\prod \alpha_i)I_q(f)$ is the element that corresponds to the solution $f(q)$. Moreover, the formula for $I_q(f)$ is the localization formula ([AB], [BV]) for computing an integral over a symplectic cut of the stable manifold at q: the first term corresponds to the fixed point q, and the sum corresponds to fixed points in a weighted projective space (see [GK] for the similar construction in equivariant K-theory). Using local index maps, one can define canonical classes τ_p:

$$\text{For every } p \in V, \text{ there exists a unique class } \tau_p \in H_\alpha(\Gamma)$$
$$\text{such that } I_q(\tau_p) = \delta_{p,q} \text{ for all } q \in V.$$

The homogeneity and support conditions follow from these conditions.

The combinatorial construction of $\tau_{p,q}$ is an iterated Lagrange interpolation process, and $\tau_{p,q}$ is a sum of contributions of ascending paths from p to q. The contribution $E(\gamma)$ of an ascending path γ is a rational expression on $\mathfrak{t}$, depending on ξ (see formula 4.9 in [GZ2]). By sending the coordinates of ξ to 0 one at a time, most of these contributions become zero. We call a path relevant if the resulting contribution $E'(\gamma)$ is not zero, and we denote the set of relevant paths from p to q by $\Omega_{p,q}^{rel}$. Then

$$\tau_{p,q} = \sum_{\gamma \in \Omega_{p,q}^{rel}} E'(\gamma)\,,$$

and this formula appears to be a "path integral" formula, via localization.

If M is the Grassmannian of k-dimensional complex planes in $\mathbb{C}^n$, with the T^n-action induced from a linear action on $\mathbb{C}^n$, then the graph Γ is the Johnson graph: the vertices are k-element subsets of $\{1, \ldots, n\}$ and the edges join vertices p and q if $\#(p \cap q) = k - 1$, i.e. if $q = p - \{i\} \cup \{j\}$ for some $i \in p$ and $j \notin p$. For a suitable choice of ξ, the edge $p \xrightarrow{(i,j)} q$ is ascending if and only if $i < j$, and a path

$$p = p_0 \xrightarrow{(i_1,j_1)} p_1 \xrightarrow{(i_2,j_2)} \cdots \xrightarrow{(i_m,j_m)} p_m = q$$

is relevant if and only if $i_1 > i_2 > \cdots > i_m$. To each relevant path from p to q we attach a permutation $v \in S_k$ ([Za]), and the space $\Omega_{p,q}^{rel}$ is parametrized by $\{v \in S_k \;:\; v \preceq w_{p,q}\}$, where $w_{p,q}$ is a (231)-avoiding permutation. Then the relevant paths from p to q correspond to fixed points for an action of T^k over a smooth Schubert variety, and $\tau_{p,q}$ is, via the localization formula, the integral of an equivariant form, hence the term "path integral."

For the flag manifold of complete flags in $\mathbb{C}^n$, the vertices of Γ correspond to permutations $w \in S_n$, and two vertices p and q are joined by an edge if and only if they differ by a transposition. If $v \preceq w$, then the relevant paths from v to w are the ascending paths

$$v = v_0 \to \tau_{i_1j_1}v = v_1 \to \tau_{i_2j_2}v_1 = v_2 \to \cdots \to w$$

for which $j_1 \leq j_2 \leq \dots$. Let w_0 be the longest element in S_n and $u = (u_1, u_2, \dots, u_m)$ be a particular reduced word for vw_0. The relevant paths from v to w correspond bijectively to subwords (not necessarily reduced) u' of u which are words for ww_0, in such a way that the number of edges of a path is the same as the number of deleted letters for the corresponding subword (compare with [Ku, Lemma 3.5]).

References

[AB] Atiyah, M. F. and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.

[BV] Berline, N. and M. Vergne, *Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante*, C.R. Acad. Sci.Sér I Math **295** (1982), no. 9, 539–541.

[GK] Guillemin, V. and M. Kogan, *Morse theory on Hamiltonian G-spaces and equivariant K-theory*, math.SG/0309312.

[GKM] Goresky, M., R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83.

[GZ1] Guillemin, V. and C. Zara, *Equivariant de Rham theory and graphs*, Asian J. Math. **3**, (1999) no. 1, 49–76.

[GZ2] Guillemin, V. and C. Zara, *Combinatorial Formulas for Products of Thom Classes*, Proceedings of the Geometry, Mechanics, and Dynamics Workshop, Toronto, August 7-12, 2002, Springer-Verlag, NY.

[Ku] Kumar, S.,*The nil Hecke ring and singularity of Schubert varieties*, Invent. Math. **123** (1996), 471–506.

[TW] Tolman, S. and J. Weitsman, *On the cohomology rings of Hamiltonian T-spaces*, Northern California Symplectic Geometry Seminar, 251-258, Amer. Math. Soc. Transl. Ser. 2, **196**, Amer. Math. Soc., Providence, RI, 1999.

[Za] Zara, C., *Parking functions, stack-sortable permutations, and spaces of paths in the Johnson graph*, Electron. J. Combin. **9** (2) (2003) #R11.

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