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Combinatorial Commutative Algebra

Organised by
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Introduction by the Organisers

There has been very fruitful interaction between the fields of Combinatorics and Commutative Algebra since the 70's [1], [2], [3], [4]. Notably, there has been a surge in interest and research during the last 8 years: a great variety of new ideas and techniques were introduced, and substantial progress was made. Projects in this direction have been undertaken by both established mathematicians and graduate students or postodcs. Currently, the main centers for such research are Germany, Italy, Japan, and USA. The toolset from Commutative Algebra that helps to solve combinatorial problems ranges from Hilbert-series to local cohomology. On the other hand, Combinatorics enriches Commutative Algebra by supplying questions, methods and results that ask for a more general setting, a setting which in many cases has a ring theoretic framework.

The Oberwolfach workshop on "Combinatorial Commutative Algebra" was organized as an attempt to gather researchers from Combinatorics and Commutative Algebra in order to announce the latest developments, spread new problems, and spark further interaction. For that purpose only very few and only longer talks were scheduled, and they all reported on exciting recent developments. The talks covered a wide spectrum of topics ranging over f-vector theory, algebraic shifting, simplicial complexes, polytopes, Gröbner basis, free resolution, powers of ideals, Hilbert-Kunz functions, and related questions in Classical Algebraic Geometry.

The talks were confined to the morning session and the afternoons were kept free for research. Existing teams continued their collaboration and new teams were formed; conjectures announced during a lecture before lunch did not exist anymore at dinner time. Mathematics was on the move.

Gil Kalai envisaged in his talk yet another round of progress through the interaction of Commutative Algebra and Combinatorics. We hope that this conference has made a contribution for this vision to become true. The success of a conference is determined by many factors, one of them is the atmosphere at the conference location. The Oberwolfach staff created the perfect atmosphere and we are very grateful for their hospitality.

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Workshop: Combinatorial Commutative Algebra

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Abstracts

Kazhdan-Lusztig polynomials

Francesco Brenti

This has been an expository/survey talk about Kazhdan-Lusztig polynomials emphasizing their connections to the algebraic geometry of Schubert varieties and some recent developments.

1. Definitions

Let $[n] = \{1, \ldots, n\}$, $S_n = \{\sigma : [n] \to [n] : \ \sigma \text{ is a bijection}\}$,

$$S = \{(1,2), (2,3), \ldots, (n-1,n)\}\,, \text{ and } T = \{(i,j) : \ 1 \leq i < j \leq n\}\,,$$

(where (i,j) is the permutation that switches i and j and leaves everything else fixed). For $\sigma \in S_n$ let

$$\mathrm{inv}(\sigma) = |\{(i,j) \in [n]^2 : i < j, \ \sigma(i) > \sigma(j)\}|$$

(number *inversions* of σ, or *length* of σ, denoted $l(\sigma)$) and

$$D(\sigma) = \{(i, i+1) \in S : \ \sigma(i) > \sigma(i+1)\} (\Leftrightarrow l(\sigma(i,i+1)) < l(\sigma))$$

(*descent set* of σ).

There are two main combinatorial objects needed to define Kazhdan-Lusztig polynomials.

The *Bruhat graph* of S_n is the directed graph $B(S_n)$ having S_n as vertex set and where $u \to v$ if and only if there exist $(i,j) \in T$ such that $v = u(i,j)$ and $l(v) > l(u)$ (equivalently, such that $v = u(i,j)$, $i < j$ and $u(i) < u(j)$). Note that this digraph is always acyclic. The transitive closure of $B(S_n)$ is the *Bruhat order* of S_n, denoted by $\leq$.

Given $u, v \in S_n$ we let $[u,v] = \{a \in S_n : u \leq a \leq v\}$, $l(u,v) \stackrel{\text{def}}{=} l(v) - l(u)$, and write $u \lhd v$ if $|[u,v]| = 2$.

We are now in a position to define Kazhdan-Lusztig and R-polynomials. We begin with a "Theorem-Definition".

Theorem 1. *There is a unique family of polynomials* $\{R_{u,v}(q)\}_{u,v \in S_n} \subseteq \mathbb{Z}[q]$ *satisfying the following conditions:*

- **i):** $R_{u,v}(q) = 0$ *if* $u \not\leq v$;
- **ii):** $R_{u,v}(q) = 1$ *if* $u = v$;
- **iii):** *if* $s \in D(v)$ *then*

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q), & \text{if } s \in D(u), \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q), & \text{if } s \notin D(u). \end{cases} \tag{1}$$

A proof of the preceding theorem can be found in [11, §7.5]. The polynomials whose existence and uniqueness are guaranteed by Theorem 1 are called the *R-polynomials* of S_n. Theorem 1 can be used to compute the polynomials $\{R_{u,v}(q)\}_{u,v\in W}$, by induction on $l(v)$. It is not hard to show that $R_{u,v}(q)$ is a monic polynomial of degree $l(u,v)$, and that $R_{u,v}(0) = (-1)^{l(u,v)}$.

We now come to the definition of the Kazhdan-Lusztig polynomials. This is again a "Theorem-Definition".

Theorem 2. *There exists a unique family of polynomials* $\{P_{u,v}(q)\}_{u,v\in S_n} \subseteq \mathbb{Z}[q]$ *satisfying the following conditions:*

i): $P_{u,v}(q) = 0$ *if* $u \not\leq v$;
ii): $P_{u,v}(q) = 1$ *if* $u = v$;
iii): $deg(P_{u,v}(q)) < \frac{1}{2}l(u,v)$ *if* $u < v$;
iv):

$$q^{l(u,v)}\, P_{u,v}\left(\frac{1}{q}\right) = \sum_{u\leq a\leq v} R_{u,a}(q)\, P_{a,v}(q)$$

if $u \leq v$.

The preceding theorem was first proved in [12] and a proof of it can also be found in [11, §7.10]. The polynomials $\{P_{u,v}(q)\}_{u,v\in S_n}$ whose existence and uniqueness are guaranteed by Theorem 2 are called the *Kazhdan-Lusztig polynomials* of S_n. It can be shown that $P_{u,v}(0) = 1$ for all $u, v \in S_n$. So, in particular, $P_{u,v}(q) = 1$ if $l(u,v) \leq 2$.

Once the R-polynomials have been computed, then Theorem 2 can be used to compute recursively the polynomials $\{P_{u,v}(q)\}_{u,v\in S_n}$, by induction on $l(u,v)$.

2. Classical Results

Kazhdan-Lusztig polynomials (which can be defined for any Coxeter group) were introduced by Kazhdan and Lusztig in [12] in order to construct representations of the associated Hecke algebra, which is a deformation of the group algebra.

The Kazhdan-Lusztig polynomials have then found numerous and unexpected applications also in other areas of mathematics, including the representation theory of semisimple algebraic groups (see, e.g., [1] and the references cited there), the theory of Verma modules (see, e.g., [2], [5]), the algebraic geometry and topology of Schubert varieties (see, e.g., [13], [15], [3]), canonical bases ([7], [16]), and immanant inequalities ([10]).

Here are the main connections to Schubert varieties. For a permutation $v \in S_{n+1}$ let Ω_v be the Schubert cell indexed by v, and $\overline{\Omega}_v$ (Zariski closure) be the corresponding Schubert variety (we refer the reader to, e.g., [8], or [3] for the definition of, and further information about, Schubert cells and varieties). It is well known (and not hard to see) that $\overline{\Omega}_v = \biguplus_{u\leq v} \Omega_u$ so that $u \leq v$ if and only if $\overline{\Omega}_u \subseteq \overline{\Omega}_v$. Denote by $IH^*(\overline{\Omega}_v, \mathbf{C})_{\Omega_u}$ the (middle perversity) local intersection cohomology of $\overline{\Omega}_v$ at a (equivalently, any) point of Ω_u. This is a graded vector space, and we denote by $IH^i(\overline{\Omega}_v, \mathbf{C})_{\Omega_u}$ $(i \in \mathbf{N})$ its graded pieces (we refer the reader to, e.g., [9], or [14], for further information about intersection (co)homology). The following result was first proved by Kazhdan and Lusztig in [13, Theorem 4.3].

Theorem 3. *Let $u, v \in S_{n+1}$, $u \leq v$. Then*

$$P_{u,v}(q) = \sum_{i \geq 0} q^i \; dim_{\mathbf{C}}(IH^{2i}(\overline{\Omega}_v, \mathbf{C})_{\Omega_u}).$$

Note that it is known that $\dim_{\mathbf{C}}(IH^i(\overline{\Omega}_v, \mathbf{C})_{\Omega_u}) = 0$ if $i \equiv 1 \pmod 2$.

Theorem 3 implies that the coefficients of $P_{u,v}(q)$ are nonnegative for all $u, v \in S_n$ (something that Kazhdan and Lusztig had conjectured in [12]). No combinatorial, or commutative algebra, interpretation is known, in general, for these coefficients.

Here are two other connections between the Kazhdan-Lusztig polynomials and the algebraic geometry of Schubert varieties (see, e.g., [3]).

Theorem 4. *Let $u, v \in S_n$, $u \leq v$. Then $P_{u,v}(q) = 1$ if and only if $\overline{\Omega_v}$ is smooth at any point of Ω_u.*

Theorem 5. *Let $v \in S_n$. Then $P_{e,v}(q) = 1$ (where e is the identity) if and only if $\overline{\Omega_v}$ is smooth.*

What about the R-polynomials? Do they have any connections to geometry? The following result is a simple consequence of the main theorem in [6].

Theorem 6. *Let $\mathbf{F}$ be a finite field of order q and $u, v \in S_n$. Then $R_{u,v}(q) = |\Omega_v \cap \Omega_u^*|$ where Ω_v^* is the Shubert cell opposite to Ω_v.*

3. Recent Developments

Let P be (any) poset, and M be a complete matching of the Hasse diagram of P. For $x \in P$ denote by $M(x)$ the match of x.

Definition 7. *We say that M is a* special matching *if, for all $x, y \in P$, such that $M(x) \neq y$, we have that*

$$x \lhd y \Rightarrow M(x) \leq M(y).$$

Note that this implies, in particular, that if $x \lhd y$ and $M(x) \rhd x$ then $M(y) \rhd y$ and $M(y) \rhd M(x)$, and dually that if $x \lhd y$ and $M(y) \lhd y$ then $M(x) \lhd x$ and $M(x) \lhd M(y)$.

It is a surprising fact that special matchings can be used to compute the Kazhdan-Lusztig polynomials.

Theorem 8. *Let $v \in S_n$, and M be a special matching of $[e, v]$. Then*

$$R_{u,v}(q) = \begin{cases} R_{M(u),M(v)}(q), & \text{if } M(u) \lhd u, \\ qR_{M(u),M(v)}(q) + (q-1)R_{u,M(v)}(q), & \text{if } M(u) \rhd u, \end{cases} \tag{2}$$

for all $u \in [e, v]$. So the polynomials $R_{x,y}(q)_{x,y \in [e,v]}$ (and hence $P_{x,y}(q)_{x,y \in [e,v]}$, and hence the intersection homology of the Schubert variety $\overline{\Omega}_v$) depend only on $[e, v]$ as an abstract poset.

The preceding result has been recently proved in [4, Theorem 5.2].

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Initial algebras of determinantal rings, Cohen-Macaulay and Ulrich ideals

Winfried Bruns

Let K be a field and X an $m \times n$ matrix of indeterminates over K. Let $K[X]$ denote the polynomial ring generated by all the indeterminates X_{ij}. For a given positive integer $r \le \min\{m,n\}$ we consider the determinantal ideal $I_{r+1} = I_{r+1}(X)$ generated by all $r+1$ minors of X if $r < \min\{m,n\}$ and $I_{r+1} = (0)$ otherwise. Let $R_{r+1} = R_{r+1}(X)$ be the determinantal ring $K[X]/I_{r+1}$.

Determinantal ideals and rings are well-known objects and the study of these objects has many connections with algebraic geometry, invariant theory, representation theory and combinatorics. See Bruns and Vetter [BV] for a detailed discussion.

We develop an approach to determinantal rings via initial algebras. We cannot prove new structural results on the rings R_{r+1} in this way, but the combinatorial arguments involved are extremely simple.

The classical generic point is the embedding $\varphi : R_{r+1} \to K[Y, Z]$ where Y and Z are matrices of indeterminates of formats $m \times r$ and $r \times n$ respectively, and φ is induced by the substitution $X_{ij} \mapsto (YZ)_{ij}$. In other words, R_{r+1} is identified with the subalgebra of $K[Y, Z]$ generated by the entries of the product matrix YZ.

By the straightening law [DRS] the standard bitableaux form a K-basis of R_{r+1}. With respect to a suitable term order on $K[Y, Z]$, the initial monomials of the standard bitableaux (or rather their images under φ) have a very simple description and are pairwise different. Therefore these initial monomials span the initial algebra $\mathrm{in}(R_{r+1})$, and we can easily deduce that the latter is a normal semigroup ring. Using general results about toric deformations (see [CHV]) and the properties of normal semigroup rings [Ho], one obtains immediately that R_{r+1} is normal, Cohen-Macaulay [HE], with rational singularities in characteristic 0, and F-rational in characteristic p. (One should not mix up this approach with the construction of Gröbner bases for the determinantal ideals; see [BC].)

Toric deformations of determinantal rings have been constructed by Sturmfels [St] for the coordinate rings of Grassmannians (via initial algebras) and Gonciulea and Lakshmibai [GL] for the class of rings considered by us. The advantage of our approach, compared to that of [GL], is its simplicity.

Moreover, it allows us to determine the Cohen-Macaulay and Ulrich ideals of R_{r+1}. Suppose that $1 \le r < \min\{m, n\}$ and let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the ideal of R_{r+1} generated by the r-minors of the first r rows (resp. the first r columns) of the matrix X. The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of height one and hence they are divisorial, because R_{r+1} is a normal domain. The divisor class group $\mathrm{Cl}(R_{r+1})$ is isomorphic to $\mathbb{Z}$ and is generated by the class $[\mathfrak{p}] = -[\mathfrak{q}]$. (See [BV, Section 8].) The symbolic powers of $\mathfrak{p}$ and $\mathfrak{q}$ coincide with the ordinary ones. Therefore the ideals $\mathfrak{p}^k$ and $\mathfrak{q}^k$ represent all reflexive rank 1 modules.

The precise description of the monomial cone underlying $\mathrm{in}(R_{r+1})$ allows us to show that the initial ideals of $\mathfrak{p}^k$ (resp. $\mathfrak{q}^k$) are conic ideals in the sense of [BG] if $k \le m - r$ (resp. $k \le n - r$). It follows that the intial ideal and therefore $\mathfrak{p}^k$ (resp. $\mathfrak{q}^k$) itself is Cohen-Macaulay for $k \le m - r$ (resp. $k \le n - r$). Since the minimal number of generators of the remaining divisorial ideals exceeds the multiplicity of R_{r+1} (for example, see [Gh]), they cannot be Cohen-Macaulay. Comparing the minimal number of generators and the multiplicity, we prove that the powers $\mathfrak{p}^{m-r}$ and $\mathfrak{q}^{n-r}$ are even Ulrich ideals (see [BHU]).

These results were obtained in joint work with Tim Römer (Osnabrück) and Attila Wiebe (Essen), see [BRW].

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Regularity of ideals and their powers

Marc Chardin

We lectured on some parts of a recent work providing bounds for the Castelnuovo-Mumford regularity of homogeneous ideals and their powers in terms of degrees of generators.

A key point in these estimates is variants of a lemma due to Gruson, Lazarsfeld and Peskine in their joint work on regularity of reduced curves [3]. In its simplest form it says :

Lemma 1. *Let $F_\bullet$ be graded complex of free modules over a polynomial ring R, with $F_i = 0$ for $i < 0$ and $F_i = \oplus_j R[-j]^{\beta_{ij}}$. If $\dim H_i(F_\bullet) \leq i$ for $i \geq 1$, then*

$$\operatorname{reg}(H_0(F_\bullet)) \leq \max_i \{b_i((F_\bullet)\},$$

where $b_i(F_\bullet) := \max\{j \mid \beta_{ij} \neq 0\}$ if $F_i \neq 0$ and $-\infty$ else.

Notice that, by definition, the inequality above is an equality if $F_\bullet$ is acyclic.

More refined versions using the same kind of ideas give results on certain local cohomology modules of $H_0(C_\bullet)$ in terms of some cohomology modules of the C_i's whenever $C_\bullet$ is a graded complex of R-modules which is not too far from being acyclic. In a geometric context, this kind of results have been previously used by Ein and Lazarsfeld (see for instance [2]). For precise statments we refer to the first section of our preprint [1].

A first application concerns the regularity of Frobenius powers, and partially solves a conjecture of M. Katzmann motivated by the challenging problem of localization of tight closure.

Theorem 2. *Let S be a standard graded ring over a field of characteristic $p > 0$ and M a finitely generated graded S-module.*
Assume that $\dim(\mathrm{Sing}(S)\cap\mathrm{Supp}(M)) \leq 1$ *and set* $b_i^S(M) := \max\{j \mid \mathrm{Tor}_i^S(M,k)_j \neq 0\}$ *if* $\mathrm{Tor}_i^S(M,k) \neq 0$, *and* $b_i^S(M) := -\infty$ *else. Then, denoting by $\mathcal{F}$ the Frobenius functor, one has*

$$\mathrm{reg}(\mathcal{F}^e M) \leq \max_{0\leq i\leq j\leq \dim S}\{p^e b_i^S(M)+a_j(S)+j-i\} \leq \mathrm{reg}(S)+\max_{0\leq i\leq \dim S}\{p^e b_i^S(M)-i\}.$$

As a second example, we presented the following theorem on regularity of an intersection of projective schemes :

Theorem 3. *Let k be a field, $\mathcal{Z}_1,\dots,\mathcal{Z}_s$ be closed subschemes of a projective k-scheme $\mathcal{Z}$ of dimension d. Assume that* $\mathrm{reg}(\mathcal{Z}_1)\geq\cdots\geq\mathrm{reg}(\mathcal{Z}_s)$. *If the intersection of the $\mathcal{Z}_i$'s is of dimension at most 1, then*

$$\mathrm{reg}(\mathcal{Z}_1\cap\cdots\cap\mathcal{Z}_s) \leq \mathrm{reg}(\mathcal{Z}) + \sum_{i=1}^{\min\{d,s\}} \mathrm{reg}(\mathcal{Z}_i).$$

For the regularity of surfaces, we have for instance,

Theorem 4. *Let S be a standard graded Gorenstein ring, $\mathcal{Z} :=$* $\mathrm{Proj}(S)$, *$n :=$* $\dim\mathcal{Z}$ *and I be a graded S-ideal generated by forms of degrees $d_1\geq\cdots\geq d_s$. Set $X :=$* $\mathrm{Proj}(S/I)$ *and $\ell :=$* $\min\{s,n\}$. *Assume that* $\dim X = 2$, *the component of dimension two of X is a reduced surface $\mathcal{S}$ and $\mu(\mathcal{I}_{X,x}) \leq$* $\dim\mathcal{O}_{\mathcal{Z},x}$ *for $x\in\mathcal{S}$ except at most at finitely many points, then*

$$\mathrm{reg}(X) \leq \mathrm{reg}(S) + d_1 + d_2 + \cdots + d_\ell - \ell.$$

Without the hypothesis on the local number of generators, we are able to get a bound which is essentially twice the bound above.

Using the approximation complexes introduced by Simis and Vasconcelos as a candidate resolution for the symmetric powers of an ideal, we also provide the following result that estimates the regularity of powers in terms of the regularity of the ideal sheaf :

Theorem 5. *Let $I\subset R = k[X_0,\dots,X_n]$ be an ideal generated in degrees $d_1\geq\cdots\geq d_s$. Set $X :=$* $\mathrm{Proj}(R/I)\subseteq\mathbf{P}^n_k$, *$X^j :=$* $\mathrm{Proj}(R/I^j)$ *for $j\geq 2$ and $\ell :=$* $\min\{s,n\}$.
Assume that $\dim X\leq 3$, $\mu(I_\wp)\leq\dim R_\wp$ *for all* $\wp\in\mathrm{Supp}(R/I)$ *of codimension at most $n-2$, and further X is generically reduced if* $\dim X = 3$, *then*

$$\mathrm{reg}(X^j) \leq (j-1)d_1 + \max\{\mathrm{reg}(X), d_2+\cdots+d_\ell-\ell\}.$$

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Betti numbers and initial ideals

Aldo Conca

The goal of the talk is to present some results obtained in collaboration with J.Herzog and T.Hibi [3] and with S.Hosten and R.Thomas [4]. The main setting is the following. Let K be a field of characteristic 0 and $R = K[x_1, \dots, x_n]$ the polynomial ring equipped with the standard grading. Let I be an homogeneous ideal of R. The ideal I has a finite minimal graded R-free resolution.

$$0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to F_0 \to I \to 0.$$

The modules F_i are graded and R-free so that they are direct sums of rank 1 graded R-free. A rank 1 graded R-free module has the form $R(-j)$ for some integer j. Here $R(-j)$ denotes the graded module whose degree h component equals the degree $(h-j)$ component of R. By collecting terms we may write $F_i = \oplus_j R(-j)^{\beta_{ij}}$. The numbers β_{ij} are called (graded) Betti numbers of I and they are denoted by $\beta_{ij}(I)$ to stress their dependence on I. Concretely, $\beta_{ij}(I)$ is the number of minimal generators of degree j of the i-syzygy module of I. One knows that

$$\beta_{ij}(I) = \dim \operatorname{Tor}_i^R(I, K)_j$$

Two important invariants related to Betti numbers are the projective dimension:

$$\operatorname{proj}\dim(I) := \max\{i : \beta_{ij}(I) \neq 0 \text{ for some } j\}$$

and the Castelnuovo-Mumford regularity:

$$\operatorname{reg}(I) := \max\{j - i : \beta_{ij}(I) \neq 0\}$$

A standard way to attach to I a discrete object, a monomial ideal, is to take an initial ideal $\operatorname{in}_\tau(gI)$ with respect to term order τ after a change of homogeneous coordinates defined by an invertible matrix $g \in \mathrm{GL}_n(K)$. Set $J = \operatorname{in}_\tau(gI)$. It is well-known that the Betti numbers can only increase by passing from I to J and hence so do the regularity and the projective dimension.

$$\beta_{ij}(I) \leq \beta_{ij}(J) \text{ for all } i,j \quad \operatorname{proj}\dim(I) \leq \operatorname{proj}\dim(J) \quad \operatorname{reg}(I) \leq \operatorname{reg}(J) \quad (1)$$

For a given I and τ there is an Zariski open and not empty subset U of $\mathrm{GL}_n(K)$ such that $\operatorname{in}_\tau(gI) = \operatorname{in}_\tau(hI)$ if $g, h \in U$. This constant value is called the generic initial ideal of I with respect to τ and it is denoted by $\text{gin}_\tau(I)$. It just depends on I and on τ but not on the coordinates system. In generic coordinates and with respect to the degree reverse lexicographic order (rl for short) two of the three inequalities of (1) become equalities:

Theorem 1 ([2]). *Let* $J = \text{gin}_{\text{rl}}(I)$. *Then* $\text{proj dim}(I) = \text{proj dim}(J)$ *and* $\text{reg}(I) = \text{reg}(J)$.

In general, even when $J = \text{gin}_{\text{rl}}(I)$, the Betti numbers of I might be strictly smaller than those of J. But one has:

Theorem 2 ([1]). *Let* $J = \text{gin}_{\text{rl}}(I)$. *If* $\beta_{0j}(I) = \beta_{0j}(J)$ *for all* j *then* $\beta_{ij}(I) = \beta_{ij}(J)$ *for all* i, j.

Generalizing Theorem 2, we have obtained:

Theorem 3 ([3]). *Let* $J = \text{gin}_{\text{rl}}(I)$. *If* $\beta_{ij}(I) = \beta_{ij}(J)$ *for all* j *and for a given* i *then* $\beta_{kj}(I) = \beta_{kj}(J)$ *for all* $k \geq i$ *and all* j.

We call this behavior "rigidity". It can be rephrased by saying that if the minimal free resolutions of I and J coincide (numerically) in position i then they coincide all the way back toward the end of the resolution. The same sort of rigidity is established also when J is replaced by any other gin of I or with the (unique) lex-segment ideal with the Hilbert function of I. It is easy to construct examples showing that the resolutions of I and J can be (numerically) different at the beginning and equal at the end. In other words, there is no rigidity toward the beginning of the resolution. Also, if J is an initial ideal is some special coordinates system then there is no rigidity at all. For example, toric and determinantal ideals arising from classical constructions are known to be minimally generated by Gröbner bases in their special coordinates systems. We call these ideals "classical". The family of classical ideals includes ideals defining Segre products of polynomial rings, ideals defining Veronese subrings of polynomial rings, ideal of minors of given size in generic or symmetric matrices of variables, ideal of Pfaffians of generic skew-symmetric matrices, ideals of Plücker relations, etc... For any such classical ideal I we know an explicit initial ideal J (called the classical initial ideal) which is Cohen-Macaulay and has as many generators as I, that is, $\beta_{0j}(I) = \beta_{0j}(J)$. But for most of the cases one has $\beta_{ij}(I) \neq \beta_{ij}(J)$ for some i (very often already $i = 1$) and some j. So we are led to ask:

Question 4. *Given a classical ideal* I *does there exist an initial ideal (in the given coordinates)* H *such that* $\beta_{ij}(I) = \beta_{ij}(H)$ *for all* i, j?

Such an ideal, if it exists, is clearly the best possible initial ideal of I, at least homologically speaking. For instance for maximal minors of generic matrices or submaximal minors of generic symmetric matrices already the classical initial ideals have the property required in Question 4. This is simply because they have minimal multiplicity with respect to their defining degree. For other "small" cases like submaximal minors of generic square matrices, the classical initial ideals does not have the correct resolution but one can find an initial ideal H as required in

Question 4. On the other hand, in most of the cases we think (and sometime we can prove) that such an H does not exist. For instance:

Example 5. *Let I be the ideal of 2-minors of a generic 4×4-matrix i.e. I defines the Segre embedding of $\mathbf{P}^3 \times \mathbf{P}^3$ in $\mathbf{P}^{15}$. With the help of the program TiGERS (Toric Gröbner bases Enumeration by Reverse Search) written by B.Huber and R.Thomas, one checks that all the quadratic initial ideal of I have at least 2 non-linear first syzygies. Since I has only linear first syzygies, it follows that there is no H has requested in Question 4.*

The next best thing that one could ask for is an initial ideal which has the correct number of generators and Cohen-Macalay type, i.e. head and tail of the resolution. This led us to:

Question 6. *Given a classical ideal I which is Gorenstein does there exist a Gorenstein initial ideal (in the given coordinates) H such that I and H have the same number of generators?*

Recent work of Reiner and Welker [5] on the Neggers-Stanley conjecture gives a positive answer to Question 6 for the the class of Gorenstein Hibi rings associated with posets of width at most 2. This class includes Segre embedding of $\mathbf{P}^n \times \mathbf{P}^n$ and it includes also the Grassmanninan $G(2, n)$ via a Sagbi deformation.

Our main result is the following:

Theorem 7 ([4]). *For the ideal defining Segre embedding $\mathbf{P}^n \times \mathbf{P}^n$ and for the ideal defining the Veronese embedding of $\mathbf{P}^n$ with forms of degree 2 and n odd we describe Gorenstein square-free initial ideals with explicit shellings of the associated simplicial complexes.*

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Instant Elimination, Powers of Ideals and an Oberwolfach Example

David Eisenbud

(joint work with Craig Huneke and Bernd Ulrich)

In our recent preprint entitled "The Regularity of Tor and Graded Betti Numbers" (www.arxiv.org/Math.AC/0405373) Craig Huneke, Bernd Ulrich and I studied a group of problems centered around powers (or products) of homogeneous ideals with linear presentations or linear relations in polynomial rings. In the first two sections of this report I will describe the motivating problem and our best theorem about it. The third section contains a more elementary conjecture and a result relating the powers of an ideal of finite colength to its resolution. In the last section we present a counter-example to a stronger conjecture, discovered by Mike Stillman at Oberwolfach in response to my talk.

The main technical result in our preprint is a bound for the Castelnuovo-Mumford regularity of the local cohomology of certain Tor modules. We will not present this bound here. However, the results below are all applications of it. The preprint gives many further applications as well.

1. Instant Elimination and the Cubic Surface

One of my favorite results from classical algebraic geometry states that any smooth cubic surface X in $\mathbf{P}^3$ (over an algebraically closed field K) is isomorphic to the blow-up of the projective plane $\mathbf{P}^2$ at 6 points. Conversely given a set $\Gamma \subset \mathbf{P}^2$ of 6 points such that Γ is not contained in a conic and no three points of Γ lie on a line, the blow-up of $\mathbf{P}^2$ at Γ is embedded as a smooth cubic surface X by the linear series of cubic plane curves passing through the 6 points. More recent, but still classical (possibly it was discovered by Room in [4]), is an explicit method of writing down the equation of X, given a basis for the space of cubic forms vanishing on Γ. I want to begin by reminding you of this construction, which was the motivation for the work by Huneke, Ulrich and myself that I will describe.

The points of Γ inmpose independent conditions on cubic forms, so the vector space of cubics through Γ has dimension 4; suppose that $F_i(x_0, x_1, x_2)$ $(i = 1, \ldots, 4)$ form a basis. By the Hilbert-Burch Theorem (see for example [2]), the ideal I_Γ of Γ is generated by the $n \times n$ minors of an $n \times n-1$ matrix M of forms of positive degree, which occurs as the presentation matrix of I. Since Γ is not contained in a conic, four of the minimal generators of I are cubic forms. We deduce at once that the number n must be 4 (so I_Γ is generated by cubics) and the entries of M must be linear forms.

Now the 3×4 matrix M of linear forms in the polynomial ring $S := K[x_0, x_1, x_2]$ may be regarded as a tensor in $K^3 \otimes K^4 \otimes S_1 = K^3 \otimes K^4 \otimes K^3$, where $S_1 = K^3$ denotes the degree 1 part of S. Let T be the homogeneous coordinate ring of $\mathbf{P}^3$, a polynomial ring in 4 variables $T = K[z_0, \ldots, z_3]$. If we identify T_1 with the middle tensor factor K^4, then such a tensor equally represents a 3×3 matrix N of linear forms over T.

Theorem 1 (Room). *The image of $\mathbf{P}^2$ under the linear series of cubics through Γ that are the minors of M is defined by the determinant of N.*

We would normally compute the ideal of X by eliminating the variables x_i from the four polynomials $z_i - F_i(x_0, x_1, x_2)$, and this is a relatively complicated calculation. Thus we think of theorem 1 as an "instant elimination" formula, and we ask when such instant elimination is possible.

Analyzing the proof of theorem 1, one finds that it rests on two independent propositions:

(1) The cokernel of N is a module whose annihilator is precisely the ideal of X.
(2) The annihilator of the cokernel of N is equal to its Fitting ideal (the ideal generated by the determinant of N.

The relationship between the Fitting ideal and the annihilator is an old subject, and Item (2) is a condition that has been studied elsewhere, for example in [1]. In the case of a square matrix N the equality follows as soon as the submaximal minors of N generate an ideal of grade at least 2, which covers the case of theorem 1.

2. Rees Algebra and Symmetric Algebra

On the other hand, condition (1) is rather mysterious. It belongs to the theory of Rees algebras, defined as follows. Let $I \subset S$ be any ideal. For simplicity, we will restrict ourselves to the case where S is a polynomial ring over K. The *Rees algebra* of I is the graded algebra $\mathcal{R}(I) := \oplus_d I^d = S[It] \subset S[t]$, where t is a new variable. When I is a homogeneous ideal, $\mathcal{R}(I)$ is a bigraded algebra with (i, j) component equal to $((I^j)_i)$. Thus for example when I is generated by cubics as above, $\mathcal{R}(I)$ has (i, j) component equal to 0 when $i < 3j$, and the space $\mathcal{R}(I)_{3j,j}$ is the space of minimal generators (of degree $3j$) of the ideal I^j.

We may write $\mathcal{R}(I)$ as a homomorphic image of the symmetric algebra $\mathrm{Sym}(I)$, say $\mathrm{Sym}(I)/\mathcal{B}$. Let $\mathcal{B}_t$ be the component of $\mathcal{B}$ in $\mathrm{Sym}_t(I)$, so that $\mathcal{B} = \oplus_{t \geq 2}\mathcal{B}_t$. It is easy to see that $\mathcal{B}_t$ is the torsion submodule of $\mathrm{Sym}_t(I)$.

The relations defining $\mathrm{Sym}(I)$ are easily derived from the relations defining I: if $G_1 \to G_0 \to I \to 0$ is a free presentation of I as an S-module, then $\mathrm{Sym}(I) = \mathrm{Sym}(G_0)/G_1 \mathrm{Sym}(G_0)$. That is, the defining ideal of $\mathrm{Sym}(I)$ in the polynomial ring $\mathrm{Sym}(G_0)$ is generated by the image of G_1, regarded as a space of forms that are linear in the variables corresponding to generators of G_0. Thus the difficult part of understanding $\mathcal{R}(I)$ is to understand $\mathcal{B}$.

Returning to the setup for instant elimination, suppose that I is generated by forms of a single degree d, which we think of as defining a rational map from the projective space $\mathrm{Proj}\, S$ to a projective space whose coordinates z_i correspond to the generators of I. Suppose further that the presentation matrix of I is a matrix of linear forms M, as in the example of the ideal of 6 points in the plane above. It is not hard to show that the matrix N derived from M by the process in the example is actually the presentation matrix of the module $\mathcal{N} := \oplus_j \mathrm{Sym}_j(I)_{dj+1}$ over the polynomial ring in variables corresponding to a set of generators of I. On the other hand, the module $\mathcal{N}' := \oplus_j (I^j)_{dj+1}$ has annihilator equal to the image of $\mathrm{Proj}(S)$ under the rational map defined by the generators of I. The module $\mathcal{N}'$ is naturally a homomorphic image of $\mathcal{N}$, and it is not hard to show that condition (1) of the "instant elimination process" is satisfied if and only if they are equal, $\mathrm{Sym}_j(I)_{dj+1} = I^j_{dj+1}$ for all j; that is, $(\mathcal{B}_j)_{dj+1} = 0$ for all j.

The simplest way in which to have $(\mathcal{B}_j)_{dj+1} = 0$ for all j is for $\mathcal{B} = 0$; in this case the ideal I is said to be of *linear type*. However, the ideal of 4 cubics in the example above is not of linear type, so this simple case doesn't even cover that classical example.

The next simplest situation occurs when the regularity of $\mathcal{B}_j$ is jd, that is, when each $\mathcal{B}_j$ is a vector space concentrated in degree jd. This is exactly what happens in the example of the four cubics, above. The meaning of this condition is that the "extra" relations on the Rees algebra, beyond those of the symmetric algebra, are just those given by the K-linear relations among the monomials of degree j in the minimal generators of I that are the "obvious" generators of I^j. An obvious necessary condition, equivalent to the statement that each $\mathcal{B}_j$ has finite length, is that $Sym_j I = I^j$ locally on the punctured spectrum of S, that is, *I is of linear type on the punctured spectrum.*

Here is a general conjecture that covers the case of the ideal of 6 points:

Conjecture 2. *If $I \subset S = K[x_0, \ldots, x_r]$ has linear type on the punctured spectrum. If I is generated by forms of degree d and has linear free resolution, then (with notation above) $\mathcal{B}_j$ has regularity jd.*

When S/I has finite length, then I is automatically of linear type on the punctured spectrum. In this case we can prove the conjecture in a strengthened form:

Theorem 3. *Let $I \subset S$ be a homogeneous $\mathbf{m}$-primary ideal. Suppose that I is generated in degree d. If the resolution of I is linear for $\lceil n/2 \rceil$ linear steps, then $\mathcal{B}_t$ is concentrated in degree dt for every t; in particular, $(\mathcal{B}_j)_{dj+1} = 0$ for all j.*

3. Powers of Ideals

This is just the beginning of a whole series of conjectures about powers of ideals with linear presentation or linear resolution. Throughout we work with a homogeneous ideal I in a polynomial ring $S = K[x_1, \ldots, x_n]$. Let $\mathbf{m} = (x_1, \ldots, x_n)$ be the homogeneous maximal ideal. Suppose that I is generated in a single degree d. We say that I has linear resolution for t steps if the j-th syzygies of I are generated in degree $d + j$ for $j = 1, \ldots, t$ (so that the first t matrices in the minimal free resolution of I are matrices of linear forms.) For example, to say that I has linear resolution for 1 step just means that I is linearly presented. We say that I has linear resolution if this holds for $t = n - 1$, so that all the matrices in a minimal free resolution of I are matrices of linear forms. Here is a well-known (and elementary) result that sets the stage for our conjectures:

Proposition 4. *Suppose that I is a homogeneous $\mathbf{m}$-primary ideal generated in a single degree d. The ideal I has linear free resolution if and only if I is a power of $\mathbf{m}$.*

In this sense, an $\mathbf{m}$-primary ideal I ought to be "more like" a power of the maximal ideal if it has linear resolution for t steps for large t. Even $t = 1$ is enough to see some evidence:

Theorem 5. *Suppose that I is a homogeneous $\mathbf{m}$-primary ideal generated in a single degree. If I has linear presentation, then some power of I is equal to a power of the maximal ideal $\mathbf{m}$.*

Experimentation suggests that the following sharper version might be true:

Conjecture 6. *If I is a homogeneous $\mathbf{m}$-primary ideal of S generated in a single degree d, and I has linear free presentation, then $I^{n-1} = \mathbf{m}^{(n-1)d}$.*

For $n \leq 2$ a linearly presented ideal has linear resolution, so the conjecture follows at once from proposition 4. We can prove the conjecture in the first nontrivial case, $n = 3$, from the following more general version:

Theorem 7. *If I is an $\mathbf{m}$-primary homogeneous ideal of S generated in a single degree d, and I has linear resolution for t steps with $t \geq (n-1)/2$, then $I^2 = \mathbf{m}^{2d}$. In particular, if $n = 3$ and I has linear presentation then $I^2 = \mathbf{m}^{2d}$.*

4. An Oberwolfach Counterexample

A natural strengthening of conjecture 6 would be the statement that if $I \subset S$ is a homogeneous ideal generated by forms of degree d and having linear presentation, then I^j has linear resolution for j steps. Huneke, Ulrich and I had made this conjecture too, and I explained it in my talk. It provoked a lively discussion, and we speculated about how one might search among "randomly" chosen ideals with simple sets of generators to look for a counterexample. Since we could prove the result for monomial ideals, one would have to use at least monomials and binomials. Over the next couple of days Mike Stillman programmed Macaulay2 [3] to make such a search, and eventually found the following example, saving us lots of useless work.

Example 8. *Let $J \subset K[a, b, c, d, e, f, g]$ be the $\mathbf{m}$-primary ideal in 7 variables generated by the cubes of the 7 variables together with the two binomials*

$$d^2(b-e), \quad f(ce-g^2).$$

If $I = J \cap (a, b, c, d, e, f, g)^5$, then I has linear presentation but I^2 does not have linear resolution for two steps.

Note that the ideal J used to make the example is obtained from an ideal not involving the variable a by factoring out a^3. This suggests a way of analyzing the example—perhaps the subject of a future Oberwolfach presentation!

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Enriched homology and cohomology modules of simplicial complexes

Gunnar Fløystad

1. Definitions and basic properties. For a simplicial complex Δ on $[n] = \{1, \dots, n\}$ and a field k, one has the augmented oriented chain complex $\tilde{C}(\Delta; k)$ and the reduced homology groups, which depend only on the topological realization of Δ. The chain complex is defined by letting $\tilde{C}_i(\Delta; k)$ be the vector space $\oplus kF$ generated by the faces F of dimension i and the differential be defined by

$$F \mapsto \sum_{\dim F' = i-1} \varepsilon(F, F')F'$$

where $\varepsilon : \Delta \times \Delta \to \{-1, 0, 1\}$ is a suitable incidence function. The reduced homology groups $\tilde{H}_i(\Delta; k)$ are then the homology groups of this complex.

However the combinatorial structure makes the simplicial complex a richer object than its topological realization. We define enriched homology of the simplicial complex as follows. Let $S = k[x_1, \dots, x_n]$ be the polynomial ring. We get a complex $L(\Delta; k)$ of free S-modules be letting $L_i(\Delta; k)$ be the free S-modulee $\oplus SF$ generated by the faces F of dimension i and the differential be given by

$$F \mapsto \sum_{F = F' \cup \{l\}} \varepsilon(F, F')x_l F'.$$

We define the *symmetric homology modules* $HS_i(\Delta; k)$ (or just $HS_i(\Delta)$) as the homology modules of this complex. The justification for calling this enriched homology is that $\mathrm{rank}_S\ HS_i(\Delta)$ is equal to $\dim_k \tilde{H}_i(\Delta)$.

This definition is inspired by the theory of cellular resolutions [2] by attaching the monomial x_i to vertex i, and by Koszul duality between S and the exterior algebra E in n variables [4],[6] by applying the functor L defined there to the graded dual of the exterior face ring $k\{\Delta\}$ (a quotient of E).

If $R \subseteq [n]$ denote by Δ_R the restricted simplicial complex consisting of faces of Δ which are subsets of R. For $\mathbf{b}$ in $\mathbf{N}^n$, let R be the support of $\mathbf{b}$. Then $HS_i(\Delta)_{\mathbf{b}}$ is equal to $\tilde{H}_i(\Delta_R)$. Hence one sees that the symmetric homology module contains exactly the same information as the linear strands of the resolution of the Stanley-Reisner ring $k[\Delta]$, see [9, ch.2]. Our approach gives a different perspective to this set of data and new questions are natural to ask.

Let $L(\Delta; k)^\vee$ be the dual complex $\mathrm{Hom}_S(L(\Delta), S(-n))$ of $L(\Delta)$. Then we define the *symmetric cohomology modules* $HS^i(\Delta; k)$ (or just $HS^i(\Delta)$) to be the cohomolgy modules of this complex. Again $\mathrm{rank}_S\ HS^i(\Delta)$ is equal to $\dim_k \tilde{H}^i(\Delta)$. Furthermore $HS^i(\Delta)_{\mathbf{b}}$ is equal to $\tilde{H}^{i-r}(lk_\Delta R)$ where R is the complement of the support of $\mathbf{b}$ in $[n]$, $r = |R|$ and $lk_\Delta R$ is the link of R in Δ, see [9].

Now recall from [9] that Δ is a Cohen-Macaulay simplical complex iff $\tilde{H}_{i-r}(lk_\Delta R)$ vanishes for $i < \dim \Delta$ and all R. We get the following as a consequence.

Proposition 1. *Δ is Cohen-Macaulay iff the cohomolgy modules $HS^i(\Delta)$ vanish for $i < \dim \Delta$ iff $L(\Delta)^\vee$ is a resolution of $HS^{\dim \Delta}(\Delta)$.*

2. Cohen-Macaulay connectivity and Gorenstein* property. If Δ is a Cohen-Macaulay (CM) simplicial complex we then get a unique non-vanishing cohomology module $HS^{d-1}(\Delta)$ where $d-1 = \dim \Delta$. It is natural to ask in this case how properties of this modules reflects itself in the properties of Δ.

Theorem 2. *$HS^{d-1}(\Delta)$ is a rank one torsion free S-module iff Δ is Gorenstein*. In this case $HS^{d-1}(\Delta)^{\vee\vee} \cong S$, (provided all vertices are in Δ), and $HS^{d-1}(\Delta)$ identifies as the Stanley-Reisner ideal of the Alexander dual simplicial complex Δ^*.*

In [1], Baclawski introduced the notion of Δ being l-Cohen-Macaulay. This is defined to be that Δ_R is Cohen-Macaulay of the same dimension of Δ for all R such that $[n]\backslash R$ has cadinality $l-1$. As an example, if Δ is a graph then Δ is l-CM iff Δ is l-connected.

Theorem 3. *$HS^{d-1}(\Delta)$ can occur as an $l-1$'th syzygy module in an S-free resolution iff Δ is l-CM.*

In the theory of convex polytopes, Brasilinskys theorem [8], says that if P is a d-polytope, then the 1-skeleton i d-connected. In the simplicial case this has been proven more generally for homology spheres [7]. The following is a rather comprehensive generalization of this with a one-line proof, given the above theorem.

Corollary 4. *If Δ is l-CM then the codimension i skeleton is $(l+i)$-CM.*

Proof. $L(\Delta_{\leq d-i})^\vee$ is the truncation of $L(\Delta)^\vee$ in cohomological degrees $\leq d-i$. □

3. Vanishing of homology modules. We assume now that Δ contains all the vertices. From [6] one has the following, which is essentially also the main result in [3].

Theorem 5. *Only one homology module $HS_i(\Delta)$ is non-zero (except $HS_{-1}(\Delta) = k$) iff the Alexander dual Δ^* is Cohen-Macaulay.*

If Δ is also Cohen-Macaulay so both Δ and Δ^* are Cohen-Macaulay, we call Δ bi-Cohen-Macaulay. As an example, if Δ is a graph, then Δ is bi-CM iff Δ is a tree. When Δ is bi-CM we get now that there is only one non-vanishing cohomology module $HS^{d-1}(\Delta)$ and only one non-vanishing homology module $HS_{c-1}(\Delta)$ (with the abovemenitoned exception) where c is the maximum of $\{i \mid \text{all } i\text{-sets are in } \Delta\}$. This gives rather strong conditions on Δ and the f-vector will be determined by c, d, and n.

Now defined the *girth* of Δ to be the minimum of $\{i \mid HS_{d-1}(\Delta)_i \neq 0\}$ (or ∞ if $HS_{d-1}(\Delta) = 0$). If Δ is a graph this is the minimal length of a cycle in the graph.

Proposition 6. *Assume Δ is l-CM. Then its girth is $\leq n+2-l$ (for $l \geq 2$), and $d \leq n-l$ (except when Δ is the skeleton of a simplex).*

Now we put some strong conditions on Δ.

Theorem 7. *Let Δ be l-CM. The i) its girth is the maximal possible $n+2-l$ (or ∞ if $l=1$) and ii) only one $HS_i(\Delta)$ is non-zero for $i < \dim \Delta$ (except $HS_{-1}(\Delta) = k$) iff Δ_R is bi-CM of the same dimension as Δ for all R such that $[n]\backslash R$ has cardinality $l-1$.*

We call such Δ for *l-Cohen-Macaulay designs.* In this case the f-vector is determined by the invariants $c, d, n,$ and l. As examples, when $l = 1$ we have the bi-CM simplcial complexes, when $l = 2$ and $d = 2c$ cyclic polytopes of even dimensions are examples, and when d has the maximal possible value $n - l$ we get exactly the Alexander duals of Steiner systems $S(c, d, n)$.

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Graphs, lattices, and monomial ideals

Jürgen Herzog

This is a survey on joint papers with Hibi and Zheng in which we study classes of monomial ideals associated with a graph or a meet-semilattice.

Let G be a finite graph with no loops and double edges. The edge ideal $I(G)$ of the graph G is the monomial ideal in $K[x_1, \dots, x_n]$ generated by the the set of monomials

$$\{x_i x_j \ \{i, j\} \in E(G)\}.$$

We say a graph G is Cohen-Macaulay (over K) if $S/I(G)$ is Cohen-Macaulay.

Theorem 1 (Herzog-Hibi). *Let G be a bipartite graph with vertex partition $V \cup V'$. Then the following conditions are equivalent:*

(a) G is a Cohen-Macaulay graph;

(b) $|V| = |V'|$ and the vertices $V = \{x_1, \dots, x_n\}$ and $V' = \{y_1, \dots, y_n\}$ can be labelled such that:

  (i) $\{x_i, y_i\}$ are edges for $i = 1, \dots, n$;

  (ii) if $\{x_i, y_j\}$ is an edge, then $i \leq j$;

  (iii) if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are edges, then $\{x_i, y_k\}$ is an edge.

For the proof of the implication (b) $\Rightarrow$ (a) we make a detour. Let $\mathcal{L}$ be a distributive lattice, P the poset of join irreducible elements of $\mathcal{L}$. The set $\mathcal{J}(P)$ of poset ideals with union as join, and intersection as meet is a distributive lattice.

The fundamental theorem of Birkhoff says that $\mathcal{L} \cong \mathcal{J}(P)$.

For any distributive lattice $\mathcal{L} = \mathcal{J}(P)$ we associate an ideal $H_{\mathcal{L}} \subset K[\{x_p, y_p\}_{p\in P}]$ as follows: for each poset ideal $I \subset P$ let

$$u_I = \prod_{p\in I} x_p \prod_{q\notin I} y_q.$$

Theorem 2. (a) $H_{\mathcal{L}}$ *has a linear resolution.*

(b) $H_{\mathcal{L}} = \bigcap_{p\leq q}(x_p, y_q)$.

Let Δ be the simplicial complex with $I_\Delta = H_{\mathcal{L}}$, $\mathcal{L} = \mathcal{J}(P)$, and let $\Delta^\vee$ be the Alexander dual of Δ. Then $I_{\Delta^\vee} = (\{x_p y_q\}_{p\leq p})$. Since $H_{\mathcal{L}}$ has a linear resolution, the Eagon-Reiner theorem implies that $S/I_{\Delta^\vee}$ is Cohen-Macaulay, and obviously it is the edge ideal of a bipartite graph, which we denote by $G(P)$.

Now we can prove the implication (b)$\Rightarrow$ (a) of Theorem 1: Let G be bipartite graph with vertex decomposition $V = \{x_p\}_{p\in P}$ and $V' = \{y_p\}_{p\in P}$, where $P = [n]$, and suppose G satisfies the conditions (b) of the theorem. On P we define a partial order $<$ by setting $p < q$ if $\{x_p, y_q\} \in E(G)$. Then $G = G(P)$, and hence G is Cohen-Macaulay.

We now extend the definition of the ideal $H_{\mathcal{L}}$ to more general lattices. Let $\mathcal{L}$ be a finite meet-semilattice and $\hat{0}$ its unique minimal element. In a finite meet-semilattice $\mathcal{L}$, each element of $\mathcal{L}$ is the join of elements of $\mathcal{L}$.

A finite meet-semilattice $\mathcal{L}$ is called *meet-distributive* if each interval $[x, y] = \{p \in \mathcal{L}\, x \leq p \leq y\}$ of $\mathcal{L}$ such that x is the meet of the lower neighbors of y in this interval is Boolean.

As in the case of a distributive lattice we introduce the squarefree monomial ideal $H_{\mathcal{L}}$ associated with a finite meet-semilattice $\mathcal{L}$. Let P be the set of join irreducible elements of $\mathcal{L}$. Let K be a field and $S = K[\{x_p, y_p\}_{p\in P}]$ the polynomial ring in $2|P|$ variables over K. We associate each element $p \in \mathcal{L}$ with the poset ideal $\ell(p) = \{q \in P \;:\; q \leq p\}$, and for each element $q \in \mathcal{L}$ we write $u_q = \prod_{p\in\ell(q)} x_p \prod_{p\in P\setminus\ell(q)} y_p$, and set $H_{\mathcal{L}} = (u_q)_{q\in\mathcal{L}}$.

Note that height$(H_{\mathcal{L}}) = 2$ if $\mathcal{L}$ is a lattice. In fact, $H_{\mathcal{L}} \subset (x_p, y_p)$ for any $p \in P$ while on the other hand $u_{\hat{0}} = \prod_{p\in P} y_p$ and $u_{\hat{1}} = \prod_{p\in P} x_p$ both belong to $H_{\mathcal{L}}$ and have no common factor.

We have the following algebraic characterization meet-distributive meet-semilattices

Theorem 3 (Herzog-Hibi-Zheng)**.** *Let $\mathcal{L}$ be an arbitrary finite meet-semilattice. The following conditions are equivalent:*

(i) *$\mathcal{L}$ is meet-distributive;*
(ii) *$H_{\mathcal{L}}$ has a linear resolution;*
(iii) *$H_{\mathcal{L}}$ has linear relations.*

On the other hand one can construct for *any* finite meet-semilattice $\mathcal{L}$ a finite free resolution of $H_{\mathcal{L}}$. This resolution is a *cellular resolution* in the sense of Bayer and Sturmfels.

Theorem 4 (Herzog-Hibi-Zheng). *Let $\mathcal{L}$ be finite meet-semilattice.*

(a) *There exists a finite multigraded free S-resolution $\mathbb{F}$ of $H_{\mathcal{L}}$ such that for each $i \geq 0$, the free module F_i has a basis with basis elements*

$$b(p; S)$$

where $p \in \mathcal{L}$ and S is a subset of the set of lower neighbors $N(p)$ of p with $|S| = i$.

(b) *The following conditions are equivalent:*
  (i) *the resolution constructed in (a) is minimal;*
  (ii) *$\mathcal{L}$ is meet-irredundant, i.e. for any $p \in \mathcal{L}$ and any proper subset $S \subset N(p)$ the meet $\bigwedge\{q\, q \in S\}$ is strictly greater than the meet $\bigwedge\{q\, q \in N(p)\}$.*

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Hilbert-Kunz Functions for Normal Rings

Craig Huneke

The talk reported on joint work with Moira McDermott and Paul Monsky on our recent result [HMM] about the Hilbert-Kunz function of normal rings. To describe the result and its background, we adopt the following terminology: $(R, \mathfrak{m})$ is a Noetherian local $\mathbb{Z}/p\mathbb{Z}$-algebra of Krull dimension d, and $I \subset R$ is $\mathfrak{m}$-primary. We let n be a varying non-negative integer, and let $q = p^n$. By $I^{[q]}$ we will denote the ideal generated by x^q, $x \in I$. If M is a finite R-module, $M/I^{[q]}M$ has finite length; we denote this length by $e_n(M, I)$, or more briefly by $e_n(M)$. We use $\lambda(-)$ to denote the length of an R-module. Our basic question is how does $e_n(M)$ depend on n? The results of [Mo1] show that $e_n(M) = \alpha q^d + O(q^{d-1})$ for some real α. Our main theorem strengthens this by proving:

Theorem 1. *Let $(R, \mathfrak{m}, k)$ be an excellent, local, normal ring of characteristic p with a perfect residue field and* $\dim R = d$. *Then $e_n(M) = \alpha q^d + \beta q^{d-1} + O(q^{d-2})$ for some α and β in $\mathbb{R}$.*

Moreover our results establish that $\beta(M) = 0$ whenever M is torsion-free and the class group of R is torsion.

One could hope that Theorem 1 could be generalized to prove that there exists a constant γ such that $e_n(M) = \alpha q^d + \beta q^{d-1} + \gamma q^{d-2} + O(q^{d-3})$ whenever R is non-singular in codimension two. However, this cannot be true. For example, if $R = \mathbb{Z}/5\mathbb{Z}[x_1, x_2, x_3, x_4]/(x_1^4 + \cdots + x_4^4)$, then with $I = (x_1, ..., x_4)$, $e_n(R) = \frac{168}{61}(5^{3n}) - \frac{107}{61}(3^n)$ by [HaMo]. Note that R is a 3-dimensional Gorenstein ring with isolated singularity. [1]

Our proof proceeds in a number of steps, each of which studies the rate of growth of the lengths of certain Tor modules. A first key step is:

Lemma 2. *If T is a finitely generated torsion R-module with* $\dim T = \ell$, *then* $\lambda(\mathrm{Tor}_1^R(R/I_n, T)) = O(q^\ell)$.

A consequence of this lemma is the next result, a rather surprising fact about the rate of growth of Tor modules of torsion-free modules. A priori one would expect these modules to growth as $O(q^{d-1})$.

Lemma 3. *If M is torsion-free,* $\lambda(\mathrm{Tor}_1^R(R/I_n, M)) = O(q^{d-2})$.

Definition 4. *Let $(R, \mathfrak{m}, k)$ be a local, normal ring of characteristic p. If M is torsion-free of rank r, $\delta_n(M) = e_n(M) - re_n(R)$.*

Our main results are summarized in the theorems and corollaries below.

Theorem 5. *Let $(R, \mathfrak{m}, k)$ be an excellent, local, normal ring of characteristic p with a perfect residue field. Let M be a torsion-free finite R-module. There is a real constant $\tau(M)$ such that $\delta_n(M) = \tau(M)q^{d-1} + O(q^{d-2})$.*

Let R be an integrally closed Noetherian domain. A Weil divisor on R is an element of the free abelian group on the height 1 primes of R. A principal Weil divisor is a divisor of the form $\sum_P \mathrm{ord}_P(f) \cdot P$ with $f \neq 0$ in the field of fractions of R. $C(R)$ is the quotient of the group of Weil divisors by the subgroup of principal divisors. Let M be a finite R-module. Then M admits a filtration with quotients (isomorphic to) R/P_i where each P_i is prime. Consider the Weil divisor $-\sum P_i$, the sum extending over those P_i that are of height 1. The image of this divisor in $C(R)$ is independent of the choice of filtration, and is denoted by c(M). c is additive on exact sequences and $c(R) = 0$. If P is a height 1 prime of R the exact sequence $0 \to P \to R \to R/P \to 0$ shows that $c(P) = P$.

Corollary 6. *Let $(R, \mathfrak{m}, k)$ be an excellent, local, normal ring of characteristic p with a perfect residue field. There is a homomorphism $\tau\colon C(R) \to \mathbb{R}, +$ with the following property. If M is torsion-free of rank r then $e_n(M) = re_n(R) + \tau q^{d-1} + O(q^{d-2})$ with $\tau = \tau(c(M))$.*

We remark that it is immediate from this corollary that τ is the zero map whenever the class group of R is torsion. In particular:

Remark 7. *Suppose that $(R, \mathfrak{m}, k)$ is a complete local normal two-dimensional ring, and k is the algebraic closure of the field with p elements. Then $C(R)$ is a torsion group.*

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[1] For additional work see [BuCh] for computations of the Hilbert-Kunz function for plane cubics, as well as [Mo2]-[Mo4] and [Te] for other concrete computations of the Hilbert-Kunz function, and [WY1]-[WY3], [BE] for work on minimal possible values for the Hilbert-Kunz multiplicity. Recent work of Brenner [Br1, Br2] and Trivedi [T] give even more information in the case R is a two-dimensional graded domain.

Finally, our main result:

Theorem 8. *Let $(R, \mathfrak{m}, k)$ be an excellent, local, normal ring of characteristic p with a perfect residue field and $\dim R = d$. Let M be finitely generated R-module. Then there exists $\alpha(M), \beta(M) \in \mathbb{R}$ such that $e_n(M) = \alpha(M)q^d + \beta(M)q^{d-1} + O(q^{d-2})$.*

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The multigraded Poincaré series and the Golod-property

Michael Jöllenbeck

Let k be a field of arbitrary characteristic, $S := k[x_1, \ldots, x_n]$ the polynomial ring in n commuting indeterminants with the natural multigrading $\deg(x_i) := e_i \in \mathbb{N}^n$, and $A := S/\mathfrak{a}$ the quotient algebra, where $\mathfrak{a} \subset S$ is a monomial ideal with minimal generating system $G(\mathfrak{a}) := \{m_1, \ldots, m_l\}$. We are interested in the multigraded Poincaré series of k as an A-modul, given by $P_A(\underline{x}, t) := \sum_{\substack{i \geq 0 \\ \alpha \in \mathbb{N}^n}} \dim_k(\operatorname{Tor}_i^A(k,k)_\alpha)\, t^i \underline{x}^\alpha$. If the Taylor resolution of $\mathfrak{a}$ as an S-modul is minimal, Charalambous and Reeves gave in [CR] an explicit description of P_A, namely

$$P_A(\underline{x},t) := \frac{\prod_{i=1}^{n}(1+t\,x_i)}{1+\sum_{I\subset G(\mathfrak{a})}(-1)^{cl(I)}m_I\,t^{cl(I)+|I|}}.$$

where $m_I := \mathrm{lcm}(m \in I)$, $cl(I) := |I/\sim|$ and $\sim$ is the transitive hull of the relation $m \sim n$ iff $\gcd(m,n) \neq 1$.

In general, they conjectured that the Poincaré series has a similar form. Particularly, they claimed that only the summation index of the sum in the denomiantor will change, but they did , however, not provide its specific structure.

Now, using an algebraic version of the discrete Morse theory (see [F1],[F2]), independently found by Emil Sköldberg ([Sk]), and Volkmar Welker and the author ([JW]), the general summation index can be discribed. Also, a proof is given in certain special cases. Furthermore, we present an idea how to prove the conjecture in the general setting. Finally, we obtain some interesting corollaries if A is assumed to be Golod. Then, for instance, the conjecture implies that a monomial ring is Golod if and only if the product on the Koszul homology (i.e. the first Massey operation) vanishes.

The main idea is to minimize the Taylor resolution of $\mathfrak{a}$ with a special sequence of acyclic matchings $\mathcal{M} := \mathcal{M}_1,\ldots,\mathcal{M}_r$, called the standard-matching, which always exists. It preserves the product on the Taylor complex in the sense that the resulting Morse complex is minimal and, if tensored with k, is isomorphic as an algebra to the Koszul homology.

Writing $I \notin \mathcal{M}_i$ if $I \subset G(\mathfrak{a})$ is not matched by the i-th acyclic matching, the conjecture of Charalambous and Reeve can be reformulated as follows.

Conjecture 1. *If $\mathcal{M}$ is a standard-matching on the Taylor resolution, then P_A is given by*

$$P_A(\underline{x},t) := \frac{\prod_{i=1}^{n}(1+t\,x_i)}{1+\sum_{\substack{I\subset G(\mathfrak{a})\\ I\notin\mathcal{M}_1}}(-1)^{cl(I)}m_I\,t^{cl(I)+|I|}}.$$

We are able to prove Conjecture 1 if

- the Taylor resolution of $\mathfrak{a}$ is minimal (theorem of [CR]),
- $A = S/\mathfrak{a}$ is Koszul, such that $\mathfrak{a}$ admits a quadratic Gröbner basis,
- the Koszul homology is an M-ring (see [Fr]), and either there exists a homomorphism $s : H_\bullet(K^A) \to K^A$, such that $\pi \circ s = \mathrm{id}$, or $\mathfrak{a}$ has a minimal resolution as S-modul, which carries the structure of a differential graded algebra.

For the general case, we give an outlining proof idea.

The algebra A is Golod, by definition, iff all Massey-operations on the Koszul homology vanish. Also, Golod proved that A is Golod iff the Poincaré series P_A

is given by

$$P_A(\underline{x},t) := \frac{\prod_{i=1}^{n}(1+t\,x_i)}{1 - \sum_{i\geq 0,\alpha\in\mathbb{N}^n} \dim_k(\mathrm{Tor}_i^S(A,k)_\alpha)\underline{x}^\alpha\, t^i}.$$

Now, Conjecture 1 implies that A is Golod iff the product, i.e. the first Massey-operation, on the Koszul homology vanishes.

Finally, we discuss some criteria for algebras to be Golod.

For instance, it is known that A is Golod if $\mathfrak{a}$ is componentwise linear (see [HRW]). This result can be generalized to

Theorem 2. *Let $\mathfrak{a}$ be generated by monomials with degree l.*

(i) *If $\dim_k(Tor_{i,i+j}) = 0$ for all $j \geq 2(l-1)$, then $A = S/\mathfrak{a}$ is Golod.*

(ii) *If A is Golod, then $\dim_k(Tor_{i,i+j}) = 0$ for all $j \geq i(l-2)+2$.*

In particular: If A is Koszul, then A is Golod iff $\mathfrak{a}$ has linear minimal resolution.

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Combinatorial Expectations from Commutative Algebra

Gil Kalai

Commutative algebra turned out to be a powerful tool to study enumerative and extremal combinatorial problems. It is especially useful in connections with graded combinatorial objects such as polytopes, simplicial complexes and arrangements of hyperplanes. In various situations, commutative algebra not only provide answers to basic questions but put the questions in what appears to be their right context. In this summary I will describe some problems and developments in this line of research.

Some very successful rings for combinatorial applications are the Stanly-Reisner ring (the face ring) of a simplicial complex and the Orlik-Solomon algebra associated to arrangements. The Cohen- Macaulay property turned out to be extremely important for combinatorics and so are also notions related to generic initial ideals, Gröbner bases, algebraic shifting, etc.

Towards a Homological VC-dimension. The questions discussed here are part of a long term project with Roy Meshulam. Discussions with Jiri Matousek and Tomas Kaiser were also very helpful.

Let K be a simplicial complex. We denote by $f_i(K)$ the number of i-dimensional faces of K and by $b_i(K)$ the ith Betti number of K. $b(K)$ denotes the sum of the Betti numbers of K.

Definition 1. A family of simplicial complexes is *strongly hereditary* if it is closed under induced subcomplexes and links.

(An induced subcomplex of a simplicial complex K is obtained by taking a subset U of the vertices and all faces contained in U.)

Definition 2. A strongly hereditary family of simplicial complexes has *HVC-dimension* d with constant A if for every complex K in the family with n vertices

$$b(K) < A \cdot n^d.$$

Conjecture 3. *If K has HVC dimension d then so is its algebraic shifting (GIN).*

Recent works of Bayer, Charalambous ,Popescu, Aramova,Herzog and Hibi [7, 2, 3, 4] give the ultimate generalization for what we know on face-numbers/Hilbert polynomials/Generic initial ideals/Algebraic shifting for Cohen-Macaulay like complexes. These works give a very good description of invariants that are preserved under algebraic shifting. I hope that the methods used by these authors can be used to settle Conjecture 1.

An example of hereditary family of simplicial complexes of HVC-dimension d is the class of d-Leray complexes:

Definition 4. A simplicial complex K is *d-Leray* if for every induced subcomplex K',

$$H_i(K') = 0,$$

for every $i \geq d$.

It follows from commutative algebra considerations (and as a very special case of the results mentioned above) that the property of being d-Leray is preserved under shifting. We describe now an important combinatorial consequence.

Definition 5. A class $\mathcal{K}$ of simplicial complexes satisfy the *fractional Helly property* of order d if for every $t > 0$ there is $s = s(t) > 0$ such that for every $K \in \mathcal{K}$, with $f_0(K) = n$

$$f_d(K) \geq t \cdot \binom{n}{d+1}$$

implies that

$$\dim K \geq s \cdot d.$$

The fact that d-Leray complexes are preserved under algebraic shifting implies that they satisfy a fractional Helly property of order d. Fractional Helly theorems have further important combinatorial applications, e.g. concerning the connection between covering numbers and fractional covering numbers, see [1]. See also [17] for much background on Helly-type theorems and also for the original notion of VC-dimension and its combinatorial significance.

Conjecture 6. *The class of simplicial complexes K with HVC-dimension d, with constant A , satisfies a fractional Helly theorem of order d.*

Conjecture 7. *Let A be a fixed positive constant. Consider families M of sets in R^d such that if L is the intersection of m sets in M then $b(L) \leq A \cdot (m^d)$. Then the nerve of M has HVC-dimension d.*

Conjecture 8. *In conjectures 1,2,3,4 we can replace $b(K)$- the sum of Betti numbers, by $\chi(K)$ - the Euler characteristic of K.*

(If we based HVC-dimension on the Euler characteristic rather than the sum of Betti numbers it becomes a purely combinatorial object).

Are all spheres Lefschetz? One of the most outstanding conjectures in commutative combinatorial algebra (see also [21]) asserts that all Gorenstein* face rings are Lefschetz. A Gorenstein* face rings are the face rings of homology sphere K (in the widest sense of the word). The Lefschetz property is a profound property of the face ring: Suppose that K is a $(d-1)$-dimensional Gorenstein* complex and let $R(K)$ be its face ring. Let $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ a generic system of parameters which are linear forms. Let

$$H(K) = \oplus_{i=0}^{d} H_i(H) = R(K)/ < \vartheta_1, \vartheta_2, \ldots, \vartheta_d >,$$

and let $h_i = \dim H_i(K)$. It follows from the fact that K is Gorenstein* that $h_i = h_{d-i}$. The conjecture is that for an additional generic linear form ω,

$$\omega^{d-2i} : H_i(K) \rightarrow H_{d-i}(K)$$

(the map is by multiplication) is an isomorphism.

When K is the boundary complex of a simplicial (rational) polytope the hard Lefschetz theorem for the associated toric variety implies that K is Lefschetz. (This is the reason for this name.) The conjecture that all Gorenstein* complexes are Lefschetz would imply the "g-conjecture" for spheres, namely a complete description of face numbers of simplicial spheres. Once proved, various far-reaching extensions of this problem are waiting to be attacked but at present it is open even for $d = 5$.

Face rings for polyhedral complexes - a Massey answer? A very early connection between commutative algebra and combinatorics is the characterization by Macaulay of f-vectors of order ideal of monomials (= Hilbert functions of standard polynomial rings). A similar characterization by Kruskal-Katona of f vectors of simplicial complexes provided a complete description of Hilbert functions of exterior face rings. It turns out that numerical conditions given by Macaulay and Kruskal-Katona theorems extend to much more general combinatorial objects such as polyhedral complexes (and much beyond them), see [22, 19] but a notion of a face ring is missing.

Perhaps we need a structure which is weaker than a ring (and yet had the same consequences on the face numbers/Hilbert functions). One example that come to mind is Massey product in algebraic topology [16] (there are other "products" in algebraic topology which are similar.). You have a graded vector space $\oplus H^i$ and the product of two elements $x \in H^j$ and $y \in H^k$ is defined in H^{j+k} only modulo some subspace of H^{j+k}.

Problem 9. *Define an abstract notion of a graded "ring" (with ambiguous product) that is strong enough to imply Macaulay's inequalities (and Kruskal-Katona inequalities in the exterior or square free cases).*

(Massey-like ?) product on toric intersection homology. The Lefschetz property of boundary complexes of simplicial polytopes extend in a beautiful complicated way to non simplicial polytopes using intersection homology of toric varieties. For some of the involved combinatorics see [21, 10]. Recently, a direct construction of graded modules which apply for arbitrary polytopes was achieved by Barthel, Brasselet, Fieseler and Kaup [5] and by Bressler and Lunts [8] and a direct proof for the Lefschetz property which apply even for nonrational polytopes was given by Karu [14]. The "combinatorial" definition of the intersection homology module has strong commutative algebra ingredients. (So far, these definition relies in a strong way on the geometry of the polytope and an extension for polyhedral spheres is not known.)

The "g-conjecture" that we mentioned before relies on the Lefschetz property and also on the ring structure of the face ring (or the cohomology of the associated toric variety). There is some evidence (see [10], [6]) that the Macaulay-type inequalities which follow from the ring structure continue to apply for the for the general case. However, no ring structure on intersection homology that can imply these relations is known or even expected. Perhaps one should look for some weaker form of product so Problem 5 can be relevant also here.

Other topics.

Clique complexes and the Charney - Davis conjecture. Clique complexes are defined as follows. We start with a graph G and form the simplicial complex whose faces are sets of vertices of G which form a complete graph in G. (Other related constructions where you replace the word "complete" by another property like "bipartite", "perfect", etc) are also of interest. There are important results and many problems concerning such complexes. Conjectures 1,2 and 4 above are of much interest for the special case of clique complexes.

The Charney-Davis conjecture is a beautiful combinatorial problem concerning face numbers of simplicial spheres which are also clique (flag) complexes. See [20] and references cited there. It can be regarded as a strong form of inequalities derived from the Lefschetz property. Commutative-algebraic approaches are proposed in [21], [20].

Khovanskii's Upper bound theorem. Khovanskii [15] extended the classic upper bound theorem (UBT) to describe an upper bound theorem for simplicial d-polytopes which are obtained as a section of a polytope Q. (The classical UBT is the case where Q is a simplex. The answer depends on the face numbers of Q) Khovanslii's motivation was to show that certain hyperbolic reflection groups do not exist in high dimensions.

Problem 10. *Extend Khovanskii's theorem:*

- *a) to the context of simplicial spheres (you have to find what is the analog of "section"),*
- *b) to give a complete description of the face numbers of P in terms of those of Q*
- *c) for general polytopes.*

Braden (unpublished) had some results on part c).

Missing faces and Betti numbers. A remarkable result by Migliore and Nagel [18] describes the maximum values of Betti numbers for Stanley-Reisner rings of simplicial polytopes (and Lefschetz spheres.) The extremal cases are the Billera-Lee polytopes [9] constructed for proving the sufficiency part of the g-conjecture.

Using a formula by Hochester for these Betti numbers this result has a concrete combinatorial statement which I expect will have many combinatorial applications. It appears to include as a special case an upper bound for the number of i-dimensional missing faces of a simplicial polytope with a prescribed f-vector.

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Non-generic initial ideals and face numbers of simplicial complexes

Isabella Novik

One of the central problems in geometric combinatorics is to characterize or at least to obtain significant necessary conditions on the face numbers of different classes of simplicial complexes. Among the earliest results in this quest was the Kruskal–Katona theorem [5], [6] that characterized the face numbers of all simplicial complexes. Since then many powerful tools and techniques have been developed, among them are the theory of Stanley-Reisner rings (see [10]), one of whose first applications was the characterization (due to Stanley [8]) of the face numbers of all Cohen-Macaulay complexes, and the theory of algebraic shifting (introduced by Kalai and closely related to the notion of reverse lexicographic generic initial ideals) that culminated in the characterization of the face numbers of all simplicial complexes with prescribed topological Betti numbers (due to Björner–Kalai [4]).

Recently algebraic shifting was extensively studied from the algebraic point of view by Aramova, Herzog, Hibi, and others. Its usefulness in attacking problems related to face numbers of simplicial complexes (or more generally, to the Hilbert function of homogeneous ideals) is explained by the fact that while reverse lexicographic generic initial ideal $\mathrm{Gin}(I)$ of a homogeneous ideal I has a much simpler combinatorial structure than I, it shares with I many combinatorial and algebraic invariants such as its Hilbert function, regularity, homological dimension, and more generally extremal Betti numbers (the latter result is due to Bayer-Charalambous-Popescu [2] and Aramova-Herzog [1]).

This technique has, however, one disadvantage: the resulting ideal is too simple, that is, any additional structure the original ideal (simplicial complex) may have been equipped with, such as symmetry, balancedness, etc. is completely destroyed when passing to its Gin, making Generic initial ideals inappropriate for studying face numbers of such complexes. This motivates a new approach: exploring the behavior of a special (only partially generic) initial ideal appropriate for the complex at hand. In [7] we develop and use this technique to strengthen Stanley's characterization of the face numbers of Cohen-Macaulay simplicial complexes (e.g., simplicial spheres) to certain necessary conditions on the face numbers of Cohen-Macaulay complexes with a proper $\mathbb{Z}/p\mathbb{Z}$-action We then generalize those conditions further for the class of Buchsbaum complexes (e.g., simplicial manifolds) with a proper $\mathbb{Z}/p\mathbb{Z}$-action. (The latter result is similar in spirit to the necessity portion of the theorem of Björner-Kalai on face numbers and Betti numbers of simplicial complexes.) As applications of this theorem we establish:

- a new version of the Upper Bound Theorem for centrally symmetric manifolds;
- a single generalization that covers both Kühnel's conjecture on the Euler characteristic of even-dimensional manifolds and Sparla's analog of this conjecture for centrally symmetric manifolds for all $2k$-manifolds on $n \geq 6k+3$ vertices.

Additional results (this is still a work in progress, parts of it are joint with Eric Babson) include

- A new and simpler proof of the characterization of the flag f-numbers of Cohen-Macaulay balanced complexes due to Stanley [9] (necessity) and Björner-Frankl-Stanley [3] (sufficiency), as well as a generalization of the necessity portion of this result to conditions on the flag f-numbers and Betti numbers of balanced Buchsbaum complexes.
- Certain necessary conditions on the face numbers and Betti numbers of general simplicial complexes with a proper $\mathbb{Z}/p\mathbb{Z}$-action.

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Prestable ideals and Sagbi bases

Hidefumi Ohsugi

This is a joint work with Takayuki Hibi. In order to find a reasonable class of squarefree monomial ideals I for which the toric ideal of the Rees algebra of I has a quadratic Gröbner basis, the concept of prestable ideals will be introduced. Prestable ideals arising from finite pure posets together with their application to Sagbi bases will be discussed.

Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$ and let $I \subset R$ be an ideal which is generated by monomials $u_1, \ldots, u_m$ with $\deg u_1 = \cdots = \deg u_m$. The Rees algebra of I is the subalgebra $\mathcal{R}(I) = K[x_1, \ldots, x_n, u_1t, \ldots, u_mt]$ of $R[t]$. Let $A = K[x_1, \ldots, x_n, y_1, \ldots, y_m] = R[y_1, \ldots, y_m]$ denote the polynomial ring over K and define the surjective homomorphism $\pi : A \to \mathcal{R}(I)$ by setting $\pi(x_i) = x_i$ and $\pi(y_j) = u_jt$. The toric ideal $J_{\mathcal{R}(I)}$ of $\mathcal{R}(I)$ is the kernel of π. Blum [1] proved that if $\mathcal{R}(I)$ is Koszul, then all powers of I have linear resolutions. Thus in particular if $J_{\mathcal{R}(I)}$ has a quadratic Gröbner basis, then all powers of I have linear resolutions. However, the existence of a quadratic Gröbner basis of $J_{\mathcal{R}(I)}$ is a rather strong condition which guarantees that all powers of I have linear resolutions. In [4] a much weaker condition, called the x-condition, for $J_{\mathcal{R}(I)}$ is introduced and it is proved that if $J_{\mathcal{R}(I)}$ satisfies the x-condition, then all powers of I have linear resolutions.

Recently, we introduced a new class of monomial ideals, the class of prestable ideals, which contains the stable ideals [2]. If I is prestable, then $J_{\mathcal{R}(I)}$ satisfies the x-condition and all powers of I have linear resolutions. We then discuss a class of prestable squarefree monomial ideals I arising from finite pure posets (partially ordered sets) such that $J_{\mathcal{R}(I)}$ has a quadratic Gröbner basis. As one of the applications of such prestable ideals coming from finite pure posets, Sagbi bases of the algebras studied in [3] will be determined.

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The geometry of 2-regular algebraic sets

Sorin Popescu
(joint work with D. Eisenbud, M. Green, K. Hulek)

This work was motivated by the classical results of Del Pezza (1886) and Bertini (1907) describing subvarieties of $\mathbb{P}^r$ of minimal degree, that is $X \subset \mathbb{P}^r$ such that $\deg(X) = \operatorname{codim}(X) + 1$.

Algebraically, these are characterized by the fact that their homogeneous ideal is 2-regular in the sense of Mumford and Castelnuovo. Geometrically, they are quadric hypersurfaces (codim = 1), the Vernonese surface in $\mathbb{P}^5$, scrolls

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d)) \xrightarrow{\mathcal{O}(1)} \mathbb{P}^N$$

and cones over all these.

We describe completely 2-regular ideals which are radical. More precisely, we show that the following assertions are equivalent for an algebraic set $X \subset \mathbb{P}^r$:

a) X is 2-regular.
b) X is *small* (i.e. every zero-dimensional linear section of X is 2-regular in its span, or, equivalently, consists of a scheme in linearly general position).
c) Each irreducible component X_i of X is a variety of minimal degree in its span, and there is an ordering of the components $X_1, \ldots, X_n$ of X such that for all i

$$(X_1 \cup \cdots \cup X_{i-1}) \cap X_i = \operatorname{span}(X_1 \cup \cdots \cup X_{i-1}) \cap \operatorname{span}(X_i)\,.$$

We also describe the combinatorics of all orderings of components of X that satisfy c) above.

Bounds for multiplicities

Tim Römer

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring with n variables over a field K equipped with the standard grading by setting $\deg(x_i) = 1$. Let $I \subset S$ be a graded ideal and $R = S/I$. Consider the minimal graded free resolution of R:

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta^S_{p,j}(R)} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta^S_{1,j}(R)} \to S \to 0$$

where we denote with $\beta^S_{i,j}(R) = \dim_K \operatorname{Tor}^S_i(R,K)_j$ the graded Betti numbers of R and $p = \operatorname{proj}\dim(R)$ is the projective dimension of R. Let $e(R)$ denote the multiplicity of R and for $1 \leq i \leq p$ define $M_i = \max\{j \in \mathbb{Z} \colon \beta^S_{i,j}(R) \neq 0\}$ and $m_i = \min\{j \in \mathbb{Z} \colon \beta^S_{i,j}(R) \neq 0\}$. Based on results of Huneke and Miller in [8], Huneke and Srinivasan conjectured that if R is Cohen-Macaulay then

$$(1) \qquad (\prod_{i=1}^{p} m_i)/p! \leq e(R) \leq (\prod_{i=1}^{p} M_i)/p!.$$

Herzog and Srinivasan proved this conjecture in [6] for several types of ideals: complete intersections, perfect ideals with quasipure resolutions (i.e. $m_i(R) \geq M_{i-1}(R)$ for all i), perfect ideals of codimension 2, codimension 3 Gorenstein ideals generated by 5 elements (the upper bound holds for all codimension 3 Gorenstein ideals), codimension 3 Gorenstein monomial ideals with at least one generator of smallest possible degree (relative to the number of generators), perfect stable ideals (in the sense of Eliahou and Kervaire [2]), perfect squarefree strongly stable ideals (in the sense of Aramova, Herzog and Hibi [1]). See also [7] for related results.

The lower bound fails to hold in general if R is not Cohen-Macaulay (see [6] for a detailed discussion). Let $c = \operatorname{codim}(R)$. Then Herzog and Srinivasan conjectured in this case the following inequality:

$$(2) \qquad e(R) \leq (\prod_{i=1}^{c} M_i)/c!.$$

Since the codimension is less or equal to the projective dimension and for all i we have that $M_i \geq i$, the inequality in Conjecture (2) is stronger than the corresponding one in Conjecture (1).

Herzog and Srinivasan proved this conjecture in the cases of stable ideals, squarefree strongly stable ideals and ideals with a d-linear resolution, i.e. $\beta^S_{i,i+j}(I) = 0$ for $j \neq d$. Furthermore Gold [4] established Conjecture (2) in the case of codimension 2 lattice ideals. This conjecture is also known to be true for so-called **a**-stable ideals by Gasharov, Hibi and Peeva [3] which generalizes the stable and squarefree stable case.

We show that Conjecture (2) is valid for codimension 2 ideals (see [9] for details). This generalizes the cases of perfect codimension 2 ideals of Herzog and Srinivasan and codimension 2 lattice ideals of Gold.

For $d \geq 0$ let $I_{\langle d \rangle} \subseteq I$ be the ideal which is generated by all elements of degree d in I. Recall from [5] that an ideal $I \subset S$ is called *componentwise linear* if for all $d \geq 0$ the ideal $I_{\langle d \rangle}$ has a d-linear resolution. We show that the upper bound for the multiplicity holds for componentwise linear ideals which generalizes some of the known cases since for example stable and squarefree stable ideals are componentwise linear. We prove that **a**-stable ideals are componentwise linear and can also deduce the conjecture in this case.

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Resolutions of some ideals generated by powers of linear forms

Henry K. Schenck

Let $R = \mathbb{K}[x_1, x_2, x_3]$ be a polynomial ring, and let $\varphi(r) = l + k(3 - r) > 0$, $k, l \in \mathbb{N}$ (the positive integers). For $char(\mathbb{K}) = 0$ or $char(\mathbb{K}) > \sum_{i=1}^{3} \varphi(i) - 2$, we consider the following families of ideals:

$$I_\varphi = \langle x_1^{\varphi(1)}, x_2^{\varphi(1)}, x_3^{\varphi(1)}, (x_1x_2)^{\varphi(2)}, (x_1x_3)^{\varphi(2)}, (x_2x_3)^{\varphi(2)}, (x_1x_2x_3)^{\varphi(3)}\rangle$$

$$J_\varphi = \langle x_1^{\varphi(1)}, x_2^{\varphi(1)}, x_3^{\varphi(1)}, (x_1+x_2)^{2\varphi(2)}, (x_1+x_3)^{2\varphi(2)}, (x_2+x_3)^{2\varphi(2)}, (x_1+x_2+x_3)^{3\varphi(3)}\rangle.$$

It is easy to use the Taylor resolution to obtain the Hilbert series of R/I_φ. In [7] Postnikov and Shapiro conjectured that the Hilbert series of the two families are equal (actually, that the Hilbert series are equal for analogous families of ideals, in any number of variables).

In [9], the inverse systems approach of Macaulay (see [1]) is used to translate questions about the Hilbert series of R/J_φ into questions about ideals of fatpoints. In the three variable case, we can apply results of Harbourne on rational surfaces with $K^2 > 0$ and Riemann-Roch to show that the Hilbert series of R/J_φ is

$$\frac{1 - 3t^{\varphi(1)} - 3t^{2\varphi(2)} - t^{3\varphi(3)} + 6t^{\varphi(1)+\varphi(2)} + 6t^{2\varphi(2)+\varphi(3)} - 6t^{\varphi(1)+\varphi(2)+\varphi(3)}}{(1-t)^3};$$

and conjecture that the minimal free resolution of J_φ is given by:

$$0 \longrightarrow R^6\Big(-\sum_{i=1}^{3}\varphi(i)\Big) \longrightarrow \begin{matrix} R^6(-2\varphi(2)-\varphi(3)) \\ \oplus \\ R^6(-\varphi(1)-\varphi(2)) \end{matrix} \longrightarrow \begin{matrix} R^3(-\varphi(1)) \\ \oplus \\ R^3(-2\varphi(2)) \\ \oplus \\ R(-3\varphi(3)) \end{matrix} \longrightarrow J_\varphi \longrightarrow 0.$$

Let G be a graph on vertices $\{0,\dots,n\}$, with edge $\{i,j\}$ having weight a_{ij}. For each $I \subseteq \{1,\dots,n\}$ set $d_I(i) = \sum_{j\notin I} a_{ij}$, $m_I = \prod_{i\in I} x_i^{d_I(i)}$; I_G is the ideal generated by the set of monomials m_I. In a similar fashion, set $D_i = \sum_{i\in I} d_I(i)$ and $p_I = (\sum_{i\in I} x_i)^{D_I}$; J_G is the ideal generated by the set of the p_I. Postnikov and Shapiro prove the equality of Hilbert series for R/I_G and R/J_G. For the complete graph G on $\{0,1,2,3\}$ with edge weights $a_{i,j} = k$ if $i,j > 0$ and $a_{0,i} = l$, we find that $I_G = I_\varphi$ and $J_G = J_\varphi$. Postnikov and Shapiro generalize the conjecture about the minimal free resolution of R/J_φ: for two related families of ideals I and J, if the Hilbert series agree, then so do the betti numbers of the minimal free resolutions. In [9], families I and J defined by "almost linear degree functions" are studied; while methods of [8] do not yield the Hilbert series, the approach using algebraic geometry does give the Hilbert series; and these families also yield a counterexample to Conjecture 6.10 of [8]. However, the counterexample is *not* obtained from a graph G, so the conjecture may be true if restricted to ideals of the form J_G.

The minimal free resolution of I_G is described in [8] for a *saturated digraph G* - the resolution is given by the Scarf complex. So for this restricted class, the question is if J_G has a "Scarf type" resolution.

Lemma 1. *The free resolution of* $L = \langle x_1^{\varphi(1)}, x_2^{\varphi(1)}, (x_1+x_2)^{2\varphi(2)}\rangle$ *is given by*

$$0 \longrightarrow R^2(-\varphi(1)-\varphi(2)) \longrightarrow R^2(-\varphi(1)) \oplus R(-2\varphi(2)) \longrightarrow L \longrightarrow 0.$$

Proof. Theorem 2.7 of [4] gives the minimal free resolution for any ideal generated by powers of bivariate linear forms. □

This gives one way to show that Conjecture 6.10 of [8] is true for graphs on $\{0,1,2\}$, since for a graph with edge weights $\{0,1\} = a, \{0,2\} = b, \{1,2\} = c$ we have an obvious resolution for I_G:

$$0 \longrightarrow R^2(-a-b-c) \xrightarrow{\begin{bmatrix} y^b & 0 \\ 0 & x^a \\ x^c & y^c \end{bmatrix}} \begin{matrix} R(-a-c) \\ \oplus \\ R(-b-c) \\ \oplus \\ R(-a-b) \end{matrix} \xrightarrow{\begin{bmatrix} x^{a+c} & y^{b+c} & -x^a y^b \end{bmatrix}} I \longrightarrow 0$$

On the other hand, in two variables, if the Hilbert series agree, then the resolutions must also agree, since the ideals will each have three generators and Hilbert-Burch resolutions.

The point of the lemma is that it does give some intuition for the first syzygies of the ideals $J_\varphi \subseteq \mathbb{K}[x_1,x_2,x_3]$. In particular, if

$$2l + 3k = \varphi(1) + \varphi(2) \le 2\varphi(2) + \varphi(3) = 3l + 2k,$$

then first syzygies of degree $\varphi(1)+\varphi(2)$ are the first syzygies of minimal degree (the Hilbert function has maximal growth to that point), so cannot cancel out. So the lemma identifies six of the twelve first syzygies, as long as $k \le l$ (it is easy to see the syzygies are independent).

The monomial ideals I_φ fall into the class identified in [8] as having Scarf resolution, so the question is if the ideals J_φ could have Scarf type resolutions. It may be possible to use results of Yuzvinsky's, which are an analog of the LCM lattice for monomial ideals given by Gasharov, Peeva, and Welker in [2]; some preliminary computations indicate that it is promising. Of course, it is necessary to choose different generators for J_φ, because the Taylor complex with the given generators is never a resolution, so the results of [10] do not apply.

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Syzygies of Projective Toric Varieties

Greg Smith

(joint work with Henry K. Schenck)

This lecture examined the relationship between the Minkowski sum of lattice polytopes and the syzygies of the associated toric ideal. A lattice polytope Δ gives rise to a toric ideal I_Δ in a polynomial ring S. If Δ has dimension n, then Ewald and Wessels [4] prove that the ring $S/I_{(n-1)\Delta}$ is normal. Bruns, Gubeladze and Trung [1] show that the ring $S/I_{n\Delta}$ is Koszul which implies that it is defined by quadratic relations. Using some tools from algebraic geometry, we extend these "lower order" results to higher syzygies.

The polytope Δ corresponds to an ample line bundle L on a toric variety X such that the lattice points in Δ give a basis for $H^0(X, L)$. In particular, we can view the polynomial ring S as $\operatorname{Sym} H^0(X, L)$ and the quotient S/I_Δ as $\bigoplus_{j\geq 0} H^0(X, L^{\otimes j})$. If $F_\bullet$ is the minimal graded free resolution of S/I_Δ, then Green and Lazarsfeld [7] say that L satisfies the property (N_p) provided $F_0 = S$ and $F_i = \bigoplus S(-i-1)$ for $1 \leq i \leq p$. Explicit conditions certifying that a line bundle satisfies the property

(N_p) are known for curves [6], smooth varieties [2], normal surfaces [5] and abelian varieties [11]. What are the analogous conditions for toric varieties?

To answer this question, we use multigraded Castelnuovo-Mumford regularity as defined by Maclagan and Smith [9]. Fix a finite collection $B_1, \dots, B_\ell$ of globally generated line bundles on X. For $\boldsymbol{u} \in \mathbb{Z}^\ell$, we simply write $B^{\boldsymbol{u}}$ for $B_1^{\otimes u_1} \otimes \dots \otimes B_\ell^{\otimes u_\ell}$. Assume that there exists a $\boldsymbol{u} \in \mathbb{N}^\ell$ such that $B^{\boldsymbol{u}}$ is an ample line bundle on X. Given a coherent $\mathscr{O}_X$-module $\mathscr{F}$ and a line bundle A on X, we say that $\mathscr{F}$ is A-regular (with respect to $B_1, \dots, B_\ell$) if $H^i(X, \mathscr{F} \otimes A \otimes B^{-\boldsymbol{u}}) = 0$ for all $i > 0$ and all $\boldsymbol{u} \in \mathbb{N}^\ell$ satisfying $|\boldsymbol{u}| = u_1 + \cdots + u_\ell = i$. If $\boldsymbol{\mathcal{B}}$ is the semigroup $\{B^{\boldsymbol{u}} : \boldsymbol{u} \in \mathbb{N}^\ell\} \subset \operatorname{Pic}(X)$, then the main result is:

Theorem 1. *Let $\boldsymbol{m}$, $\boldsymbol{w} \in \mathbb{N}^\ell$ such that $B^{\boldsymbol{m}}, B^{\boldsymbol{w}} \in \bigcap_{j=1}^{\ell} (B_j + \boldsymbol{\mathcal{B}})$. If the line bundle $B^{\boldsymbol{m}}$ is $\mathscr{O}_X$-regular, then $B^{\boldsymbol{m}+p\boldsymbol{w}}$ satisfies property (N_{p+1}).*

As an immediate corollary, we see that $S/I_{(n-1+p)\Delta}$ satisfies property (N_p) which generalizes the motivating lower order results. Hering [8] also establishes this corollary and Ogata [10] shows that $S/I_{(n-2+p)\Delta}$ satisfies property (N_p) when $n := \dim(\Delta) \geq 3$.

The techniques used to prove Theorem 1 yield stronger results when X is a product of projective spaces. In particular, this addresses a question raised in Eisenbud, Green, Hulek and Popescu [3]. Nevertheless, the following questions remain unanswered:

Question 2. *Does every ample line bundle on a smooth toric variety satisfy property (N_1)?*

Question 3. *If $n > 2$ then does $\mathscr{O}_{\mathbb{P}^n}(d)$ satisfy property (N_{3d-3})?*

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