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Theory of the Riemann Zeta and Allied Functions

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Introduction by the Organisers

This meeting, the second Oberwolfach workshop devoted to zeta functions, was attended by 42 participants representing 16 countries. The scientific program consisted of 32 talks of various lengths and a problem session. In addition, social activities were organised: a hike in the mountains and piano recitals by Peter Elliott and Valentin Blomer.

Since the times of Dirichlet and Riemann, zeta functions and Dirichlet series have played a central role in analytic number theory, and in recent times connections have been found with other areas of mathematics and its applications, including theoretical physics. The talks represented the various aspects of the theory of zeta functions. In particular, the following topics were discussed, among others:

- Connections of classical zeta functions with automorphic functions and spectral theory.
- Estimates of the size of zeta and L-functions, both at individual points and in mean value.
- Problems concerning the zeros of zeta functions (Riemann's hypothesis and other questions such as the Siegel zero and the distribution of zeros of Epstein's zeta functions).
- Applications of zeta and L-functions to arithmetic functions, and the duality between arithmetic and analysis.
- Random Matrix Theory, which shed new light light on mean value estimates and their consequences.
- Numerical calculations related to the zeros of Riemann's zeta function and other computational projects.

Workshop: Theory of the Riemann Zeta and Allied Functions

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Abstracts

Omega results for the Riemann zeta function and the error terms in the summatory functions of arithmetic functions

R. Balasubramanian

Let $f(s)=\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series, convergent in $\sigma>1$ and analytically continuable in $\sigma>-a$ with a finite number of poles. Here a is a positive real number or ∞. We assume that the poles are situated in $|t|\leqslant A$. We further suppose that $f(s)$ is of finite order, that is there exists a constant $B>0$, such that $|f(\sigma+it)|=O\big(|t|+1)^B\big)$ for $|t|\geqslant A$.

In this talk, we shall discuss the lower bound for $\int_T^{T+H}|f(\sigma+it)^2\,dt$ and some applications to arithmetic function.

We start with some preliminary observations. First note that $|\exp(\sin^2 z)|\leqslant \exp\big(-\exp(y/2)\big)$ where $z=x+iy$. The exponential decay of $\exp(\sin^2 z)$ immediately yields

Lemma 1. *If* $|x|<\pi/6$, $x\neq 0$, *then*

$$\int_{-\infty}^{\infty}\Big|\frac{\exp(\sin^2 z)}{z}\Big|\,dy \;=\; O\Big(\log\frac{1}{|x|}\Big)\,.$$

We shall employ an averaging technique which we explain. Fix T sufficiently large, $H\leqslant(\log\log T)^2\leqslant T$, $U=H^{7/8}$, $r=100$. Instead of considering a function $g(t)$, it becomes advantageous to consider $\widetilde{g}(t)=\int_0^u du_1\int_0^u du_2\cdots\int_0^u g(t-S)du_r$, where $S=u_1+u_2+\ldots+u_r$.

For Dirichlet series, which are functions of $s=\sigma+it$ the averaging is done with respect to the variable t.

Lemma 2. *Let* $a(n)$, $b(n)$ *be complex numbers which are* $o(n^{\varepsilon})$. *Let*

$$A(s)=\sum_{n\leqslant H} a_n n^{-s}\,,\; B(s)=\sum_{m\geqslant H+H^{1/4}} b_m m^{-s},\quad \textit{and}\;\; f(s)=\overline{A(1-s)}B(s)\,.$$

Then for any $\sigma>1$,

$$\widetilde{f}(\sigma+it)\;\ll\; H^{-10}\sum_{\substack{n\leqslant H\\ m\geqslant H+H^{1/4}}}\frac{|a_n||b_m|}{n^{1-\sigma}m^{\sigma}}\,.$$

Proof. Note that

$$f(\sigma+it-iS)=\sum_{\substack{n\leqslant H\\ m\geqslant H+H^{1/4}}}\frac{a_n b_m}{n^{1-\sigma}m^{\sigma}}\Big(\frac{n}{m}\Big)^{i(t-S)}$$

Integrating with respect to $u_1, u_2, \ldots, u_r$ we get

$$\widetilde{f}(\sigma+it) = \frac{1}{U^r}\sum_{n,m}\frac{a_n b_m}{n^{1-\sigma}m^{\sigma}}\left(\frac{1-\left(\frac{n}{m}\right)^{-iu}}{i\log\frac{n}{m}}\right)^r\left(\frac{n}{m}\right)^{it}$$
$$\ll \sum_{n,m}\frac{a_n b_m}{n^{1-\sigma}m^{\sigma}}\left(\frac{2}{U\log\frac{m}{n}}\right).$$

Since $U\log\frac{m}{n} \geqslant H^{7/8}\log\frac{H+H^{1/4}}{H} \geqslant \frac{1}{2}H^{1/8}$, we get the lemma.

Lemma 3. *With the above notation, assume $B(s)$ has an analytic continuation in $\sigma>0$. Then, for $T+U\leqslant t\leqslant T+H-U$,*

$$\int_{1/2}^{2}\widetilde{f}(\sigma+it)\,d\sigma \ll \frac{1}{U}\int_T^{T+H}\left|f(\tfrac{1}{2}+it)\right|dt.$$

Proof. We have

$$f(\sigma+it) = \frac{1}{2\pi i}\int_{\mathcal{D}} f(\sigma+it+w)\exp\left(\sin^2(w/10)\right)\frac{dw}{w}$$

where $\mathcal{D}$ is the rectangle $1/2-\sigma\pm iu,\ 2-\sigma\pm iU$. This gives

$$\widetilde{f}(\sigma+it) = \int \widetilde{f}(\sigma+it+w)\exp\Big(\sin^2(w/10)\Big)\frac{dw}{w}.$$

On the right vertical $\widetilde{f}$ is small by Lemma 2. On the horizontals, the integral is small because of $\exp\left(\sin^2(w/10)\right)$. On the left vertical putting $w=1/2-\sigma-it+is+iv$ we get the contribution to be

$$\leqslant \frac{1}{2\pi}\frac{1}{U^r}\int_0^U du_2\cdots\int_0^U du_r\int_T^{T+H}|f(\tfrac{1}{2}+iv)|dv$$
$$\times\int_{1/2}^{2}d\sigma\int\left|\frac{\exp\Big(\sin^2\left(\frac{1}{10}(\frac{1}{2}-\sigma-it+is-iv)\right)}{\frac{1}{10}\left(\frac{1}{2}-\sigma-it+is-iv\right)}\right|du_1.$$

Now by Lemma 1, the last integral is $O\Big(\log\left(\frac{\sigma-1/2}{10}\right)\Big)$ and

$$\int_{1/2}^{2}\log\left(\frac{\sigma-1/2}{10}\right) = O(1).$$

This completes the proof.

Lemma 4. *If I is an interval contained in $[T+U, T+H-U]$ then*

$$\int_I\widetilde{f}\left(\tfrac{1}{2}+it\right)dt \ll H^{-9}\sum_{n,m}\frac{|a_n||b_m|}{n^{1-\sigma}m^{\sigma}} + \frac{1}{U}\int_T^{T+H}|f(\tfrac{1}{2}+it)|\,dt.$$

Proof. Move the line of integration from $\sigma=1/2$ to $\sigma=2$ and by Lemma 3, the error is small. Since $\widetilde{f}(2+it)$ is small, the result follows.
We finally observe that, by passing to average, we do not lose much, at least for the positive function.

LEMMA 5. *If* $g(t)$ *is non-negative, then*

$$\int_T^{T+H} g(t)dt \geqslant \int_{T+101U}^{T+H} \widetilde{g}(t)\,dt \geqslant \int_{T+101U}^{T+H-101U} g(t)\,dt\,.$$

PROOF.

$$\begin{aligned}\int\limits_{T+101U}^{T+H} \widetilde{g}(t)\,dt &= \frac{1}{U^r}\int\limits_0^U du_1 \int\limits_0^U du_2 \cdots \int\limits_0^U du_r \int\limits_{T+101U}^{T+H} g(t-S)\,dt \\ &= \frac{1}{U^r}\int\limits_0^U du_1 \int\limits_0^U du_2 \cdots \int\limits_0^U du_r \int\limits_{T+101U-S}^{T+H-S} g(t)\,dt\,.\end{aligned}$$

Since

$$\int_{T+101U-S}^{T+H-S} g(t)dt \leqslant \int_T^{T+H} g(t)dt$$

the first inequality follows. The proof of the second inequality is similar.

We are now in a position to state the main result.

THEOREM. *We have*

$$\int_T^{T+H} |f\left(\tfrac{1}{2}+it\right)|^2 dt \geqslant \big(H+O(U)\big)\sum_{n\leqslant H}\frac{|a_n|^2}{n}\left(1+O\Big(\frac{1}{M}\Big)\right)+O\left(\sum |a_n|^2\right)$$

where $M^2 \displaystyle\sum_{H<n\leqslant H+H^{1/4}} \frac{|a_n|^2}{n} = \sum_{n\leqslant H}\frac{|a_n|^2}{n} =: L.$

PROOF. Let

$$\begin{aligned}F(t) &= f(\tfrac{1}{2}+it) - \sum_{n\leqslant H}\frac{a_n}{n^{1/2+it}} - \sum_{H<n\leqslant H+H^{1/4}}\frac{a_n}{n^{1/2+it}} \\ &= G_1(t)-G_2(t)-G_3(t)\,.\end{aligned}$$

We assume that the result is not true and get a contradiction. Then $\int_T^{T+H}|G_1(t)|^2dt = O(HL)$. Using the Montgomery-Vaughan inequality $\int_T^{T+H}|G_2(t)|^2dt \ll HL$ and $\int_T^{T+H}|G_3(t)|^2dt \ll HLM^{-2}$. Now $G_1(t)=F(t)+G_2(t)+G_3(t)$. Hence

$$\begin{aligned}\int \big|G_1(t)\big|^2 dt &= \int \big|F(t)\big|^2 dt + \int \big|G_2(t)\big|^2 dt + \int \big|G_3(t)\big|^2 \\ &\quad + 2\Re\bigg(\int_T^{T+H} G_2(t)\,\overline{G_3(t)}\,dt + \int_T^{T+H} F(t)\,\overline{G_2}(t)\,dt + \int_T^{T+H} F(t)\overline{G_3}(t)\,dt\bigg)\end{aligned}$$

$$\int_T^{T+H} \left|G_1(t)\right|^2 dt \geqslant \int_{T+101U}^{T+H} \left|\widetilde{G}_1(t)\right|^2 dt$$
$$\geqslant \int_{T+101U}^{T+H} \left(\left|\widetilde{G}_2(t)\right|^2 + 2\Re\overline{F(t)\widetilde{G}_2(t)} + 2\Re\overline{F(t)G_3(t)} + 2\Re G_2(t)\overline{G_3(t)}\right) dt\,.$$

Since $\int_T^{T+H} |G_3(t)|^2 dt \ll HSM^{-2}$, the terms involving $G_3(t)$ give a small error term by the Cauchy-Schwarz inequality.
Since $G_2(t)$ is $\sum_{n\leqslant H} a_n n^{-s}$ at $s = 1/2 + it$, and $F(t)$ is the analytic continuation of $\sum_{n>H+H^{1/4}} a_n n^{-s}$ at $s = 1/2 + it$, the integral $\int_{T+U}^{T+H-101U} G_2(t)\overline{F(t)}$ is of the form discussed in Lemma 4 and hence small. Again by Lemma 5,

$$\int\limits_{T+101U}^{T+H} |\widetilde{G}_2(t)|^2\, dt \geqslant \int\limits_{T+101U}^{T+H-101U} |G_2(t)|^2 dt = \sum_{n\leqslant H} \left(H - 204U + O(n)\right) \frac{|a_n|^2}{n}\,.$$

This completes the proof.

Applications to omega results for the error term in the summatory functions of arithmetic functions were also given.

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A continuity property connected with Nyman's criterion for the Riemann hypothesis

Michel Balazard

(joint work with Nicolas Jousse)

Let $\{u\}$ denote the fractional part of the real number u, and, for $u > 0$, $\alpha \geqslant 0$, define $g_\alpha(u) := \{\alpha/u\}$. Let $\mathcal{B}$ be the set of linear combinations of the g_α's, and $\mathcal{N}$ be the subset of those elements of $\mathcal{B}$ which vanish on $(1, +\infty)$. We observe that $\mathcal{B} \subset L^p(0, +\infty)$ for $1 < p \leqslant +\infty$, and that $\mathcal{N} \subset L^\infty(0, 1)$.

A striking result from Nyman's thesis (1950) is the following.

Theorem 1. (Nyman [2]) *The Riemann hypothesis is equivalent to $\mathcal{N}$ being dense in $L^2(0,1)$.*

It is easy to see that $\mathcal{N}$ is dense in $L^2(0,1)$ if, and only if the characteristic function χ of $(0,1)$ lies in the closure of $\mathcal{B}$ in $L^2(0,+\infty)$. Thus Nyman's theorem gives a reformulation of the Riemann hypothesis (RH) as an approximation problem in a Hilbert space. A further rephrasing of (RH) involves the quantity

$$\delta_n := \inf\Big\{\big\|\chi - \sum_{k=1}^{n} c_k g_{\alpha_k}\big\| : c_k \in \mathbb{C},\, 0 \leqslant \alpha_k \leqslant 1,\, k = 1,\ldots,n\Big\}, \tag{1}$$

the distance in $L^2(0,+\infty)$ between χ and the set of all n-terms linear combinations of the g_α's ; (RH) is plainly equivalent to $\delta_n = o(1)$, $n \to +\infty$.

The inequality $\delta_n \leqslant \delta_{n+1}$ is obvious. Is the sequence (δ_n) strictly decreasing? The answer is positive and was given by Nicolas Jousse in his thesis (2004).

Theorem 2. (Jousse [1]) *or every* $n \geqslant 1$, *one has* $\delta_{n+1} < \delta_n$.

It turns out that the main step in the proof of Theorem 2 consists in showing that the infimum in (1) is in fact a minimum. With this goal in mind, we denote by P_V the orthogonal projection on the closed subspace V of the (implicit) Hilbert space H. In the case $H = L^2(0,+\infty)$, one has

$$\delta_n = \inf\Big\{\big\|\chi - P_{\mathrm{Vect}(g_{\alpha_1},\ldots,g_{\alpha_n})}(\chi)\big\| : 0 \leqslant \alpha_1,\ldots,\alpha_n \leqslant 1\Big\},$$

so that a sufficient condition for this infimum to be a minimum is the continuity of the map

$$\begin{aligned} [0,+\infty[^n &\to H \\ (\alpha_1,\ldots,\alpha_n) &\mapsto P_{\mathrm{Vect}(g_{\alpha_1},\ldots,g_{\alpha_n})}(\chi)\,. \end{aligned}$$

This last question is an instance of a general problem studied by Jousse. We discuss now the simplest form of this problem, whereas the application to (δ_n) is handled by means of a slightly modified variant.

Let G be a locally compact abelian group, noted multiplicatively, with Haar measure μ. For $G \in H := L^2(G)$, and $\alpha \in G$, define $g_\alpha(x) := g(x\alpha^{-1})$, $x \in G$.

Definition. *The function* $g \in L^2(G)$ *is admissible if, for every* $y_0 \in G$, *and every positive integer* n, *the map*

$$\begin{aligned} G^n &\to H \\ (\alpha_1,\ldots,\alpha_n) &\mapsto P_{\mathrm{Vect}(g_{\alpha_1},\ldots,g_{\alpha_n})}(y_0)\,. \end{aligned}$$

is continuous.

The general problem is to obtain a characterization of admissible functions. It may be difficult to get a complete answer. Let us note that some very regular functions, such as non-zero continuously differentiable functions with compact support (in the case where G is the real line), are *not* admissible.

Jousse described a class of admissible functions. We begin with two definitions and then state his result.

Definition. *A measurable function* $f\colon g \to \mathbb{C}$ *is an exponential polynomial if the vector space* $\mathrm{Vect}(f_\alpha,\, \alpha \in \mathbb{C})$ *has finite dimension.*

In the case where $G = (]0,+\infty[,\times)$, the exponential polynomials are linear combinations of functions $x \mapsto x^\rho \log^k x$, $\rho \in \mathbb{C}$, $k \in \mathbb{N}$.

Definition. *A measurable function $f : g \to \mathbb{C}$ is countably simple if $f(G)$ is countable.*

Theorem 3. (Jousse [1]) *Assume G is σ-compact, metrizable and not compact. Let $f \in L^2(G)$ be such that $f = f' + f''$, where f' is an exponential polynomial, and f'' is countably simple. Then f is admissible.*

Observe that the error terms of analytic number theory are often sums (or differences) of an exponential polynomial on $]0,+\infty[$ and a countably simple function. In particular, this is the case for $t \mapsto \{1/t\}$.

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Non-vanishing of class group L-functions at the central point

Valentin Blomer

The question as to whether an L-function vanishes at a special point on the critical line has arisen in various contexts and is apparently a fundamental one. By now there are numerous results that many members – in some cases even a positive proportion – of a certain family of L-functions do not vanish at the central point. This is of interest in various aspects such as the Birch–Swinnerton–Dyer conjecture, the Siegel zero (see [4]) and the theory of modular forms of half-integral weight (see [6]). A large number of old and new results around this theme can be found in [5].

Here we consider L-functions attached to class group characters of an imaginary quadratic field. Let $K = \mathbb{Q}(\sqrt{-D})$ be the imaginary quadratic field of discriminant $-D$. We denote its class group by $\mathcal{C}$ and write $h = \#\mathcal{C}$. For each character $\chi \in \hat{\mathcal{C}}$ we have an L-function

$$L_K(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N\,\mathfrak{a})^{-s},$$

the summation being taken over all nonzero integral ideals $\mathfrak{a}$. For real characters $L_K(s,\chi)$ is the product of two Dirichlet L-functions, while for complex characters $L_K(s,\chi)$ comes from the cusp form $\sum_{\mathfrak{a}} \chi(\mathfrak{a}) e(zN\,\mathfrak{a})$ of weight 1 for $\Gamma_0(D)$ and character χ_D. We shall obtain the following result [1].

Theorem. *There is an absolute constant $c > 0$ such that*

$$\frac{1}{h}\#\{\chi \in \hat{\mathcal{C}} \mid L(1/2,\chi) \neq 0\} \geqslant c \prod_{p \mid D}\Big(1 - \frac{1}{p}\Big) \tag{1}$$

for sufficiently large D *.*

Since we appeal to Siegel's lower bound for $L(1,\chi_{-D})$, we do not know how large D must be chosen to ensure the validity of (1). The constant c is in principle computable, but we did not make any effort to do so because our method yields only a very small value for c, something around 10^{-6}. According to general conjectures on zeros of L-functions coming from random matrix theory we would expect that 100 percent of the $L_K(s,\chi)$ do not vanish at $s=1/2$. In fact, the present family of L-functions has been studied by Fouvry and Iwaniec [3] who showed under the Riemann hypothesis for the $L_K(s,\chi)$ that the distribution of low-lying zeros is governed by the symplectic group.

For the proof of the theorem we compare different weighted averages over the set of L-functions for $\mathbb{Q}\big(\sqrt{-D}\big)$. We consider

$$\mathcal{L}_1 := \frac{1}{h}\sum_{\chi\in\hat{\mathcal{C}}}\Big(\sum_{q\in\mathcal{Q}}\lambda(q)\sum_{\mathfrak{q}:N\mathfrak{q}=q}\chi(\mathfrak{q})\Big)L_K(1/2,\chi)$$

and

$$\mathcal{L}_2 := \frac{1}{h}\sum_{\chi\in\hat{\mathcal{C}}}\Big(\sum_{q\in\mathcal{Q}}\lambda(q)\sum_{\mathfrak{q}:N\mathfrak{q}=q}\chi(\mathfrak{q})\Big)^2\big|L_K(1/2,\chi)\big|^2.$$

Here $\mathcal{Q}:=\{q\leqslant D^\eta:\,\mu^2(q)=1,(p\mid q\Rightarrow\chi_{-D}(p)=1)\}$ with a small parameter $0<\eta<1/4$, and the $\lambda(q)$'s are suitably chosen real numbers. By the Cauchy-Schwarz inequality we have

$$\#\{\chi\in\hat{\mathcal{C}}:\ L_K(1/2,\chi)\neq 0\}\geqslant h\mathcal{L}_1^2\mathcal{L}_2^{-1}\,.$$

The sum $\sum\lambda(q)\sum_{\mathfrak{q}}\chi(\mathfrak{q})$ is supposed to work as a mollifier and to smooth out irregularities in the behaviour of the $L_K(1/2,\chi)$ so that not too much is lost when we apply the Cauchy-Schwarz inequality.

It is a priori not clear whether the set $\mathcal{Q}$ contains elements other than 1, so the character sum may be trivial if the class number is extraordinarily small and there are only few or no small split primes. However, it turns out that the smaller the class number is, i.e. the less effective the mollifier works, the less variation exists, roughly speaking, in the values of $L_K(1/2,\chi)$. Thus our method works even in the improbable case of an exceptionally small class number, and we need not to appeal to any unproven hypothesis.

The mean $\mathcal{L}_1$ can be evaluated asymptotically relatively easily using the functional equation for the Epstein zeta function. The second moment is hard to evaluate, and the estimations rely ultimately on deep results from spectral theory of modular forms and equidistribution properties of Heegner points [2]. Finally we optimize the coefficients $\lambda(q)$ by minimizing the quadratic form $\mathcal{L}_2$ subject to the linear constraint coming from $\mathcal{L}_1$. We end up with an estimate of the type

$$\mathcal{L}_1^2\mathcal{L}_2^{-1}\gg\Big(\sum_{j=0}^{3}c_jL^j(1,\chi_{-D})(\log D)^{1-j}\Big)^{-1}\sum_{g\in\mathcal{Q}}\frac{\tau(g)}{g}$$

with certain positive constants c_j . Using some sieve ideas, this can be estimated by $\ll_\eta \prod_{p|D}(1-p^{-1})$ which completes the proof of the theorem.

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On the zeros of certain Epstein zeta functions

Enrico Bombieri

(joint work with Julia Mueller)

Let $Q(x,y)=ax^2+bxy+cy^2$ be a positive definite quadratic form with $a,b,c\in\mathbb{Z}$, $a>0$, and discriminant $D:=b^2-4ac$. The Epstein zeta function associated to Q is

$$E(s,Q):=\sum_{(m,n)\neq(0,0)}\frac{1}{Q(m,n)^s}.$$

If the class number is 1 then the Epstein zeta function coincides up a constant factor with the Dedekind zeta function of the quadratic field. In particular, it has an Euler product and it is expected to satisfy a corresponding Riemann Hypothesis. If the class number is 2 or more, the distribution of zeros of Epstein's zeta functions is also quite mysterious. In this case, Davenport and Heilbronn [2] have shown that $E(s,Q)$ has infinitely many zeros in the half-plane $\Re(s)>1$.

If the class number is 2, let $\zeta_K(s)$ and $L_K(s)$ be the Dedekind zeta function and the L-function associated to the non-trivial real character of the class group. Then the equation $E(s,Q_i)=0$, $i=1,2$, is equivalent to $\zeta_K(s)/L_K(s)=\mp 1$, and we can exploit the Euler product and Bohr almost periodicity. A similar idea occurs in the work of Gonek [3] in the study of the distribution of zeros on the line $\Re(s)=1/2$ of special instances of the Hurwitz zeta function $\zeta(s,a)$.

We consider only the simplest case with discriminant -20 and the two reduced quadratic forms $Q_1=m^2+5n^2$, $Q_2=2m^2+2mn+3n^2$. Then

$$\frac{\zeta_K(s)}{L_K(s)}=\left(\frac{1+2^{-s}}{1-2^{-s}}\right)\prod_{p\equiv 3,7\ (\mathrm{mod}\ 20)}\left(\frac{1+p^{-s}}{1-p^{-s}}\right)^2$$

and

$$E(s,Q_1) = \zeta_K(s) + L_K(s)\,, \quad E(s,Q_2) = \zeta_K(s) - L_K(s)\,.$$

Let $\sigma(Q)$ be the abscissa defining the largest half-plane free from zeros of $E(s,Q)$.

THEOREM 1. *The abscissa $\sigma(Q_1)$ is the unique solution $\sigma > 1$ of the equation*

$$\arctan(2^{-\sigma}) + 2 \sum_{p\equiv 3,7 \,(\mathrm{mod}\ 20)} \arctan(p^{-\sigma}) = \frac{\pi}{2}\,.$$

The abscissa $\sigma(Q_2)$ is the unique solution $\sigma > 1$ of the equation

$$\arctan(2^{-\sigma}) - 2 \sum_{p\equiv 3,7 \,(\mathrm{mod}\ 20)} \arctan(p^{-\sigma}) = 0\,.$$

We have

$$\sigma(Q_1) = 1.133906322092828621637158^+,$$
$$\sigma(Q_2) = 2.158504990900088015136407^+.$$

THEOREM 2. *For any fixed $\varepsilon > 0$ there are infinitely many zeros of $Z(s,Q_i)$ in the region*

$$\sigma > \sigma(Q_i) - \frac{1}{(\log(|t|+3))^{1-1/\sigma(Q_i)-\varepsilon}}\,.$$

Moreover, there are positive constants A, B such that

$$\sigma > \sigma(Q_i) - \frac{A}{(|t|+3)^B}$$

contains no zeros of $Z(s,Q_i)$.

The proofs use on a method of Bohr and Jessen [1] on sum of convex domains, a quantitative form of Kronecker's theorem on simultaneous approximations, Newton iteration method to locate zeros, and a lower bound for the approximation of the ratio of two logarithms by rational numbers obtained by Baker's method of linear forms in logarithms. We expect that the first statement of the second theorem is quite close to give the true zero-free region for $Z(s,Q_i)$.

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On the explicit spectral formula for the fourth moment of the Riemann zeta function

Roelof W. Bruggeman
(joint work with Yoichi Motohashi)

In Chapter 4 of [4], Motohashi gives an explicit formula for the fourth moment of the Riemann zeta function. I sketch a new proof, to appear in [2].

The explicit formula expresses

$$\int_{-\infty}^{\infty} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 g(t)\,dt$$

as a sum of three terms, some of which are seemingly unrelated to the Riemann zeta function. The test function g is holomorphic and quickly decreasing on a wide horizontal strip.

The first of the terms is called the trivial one:

$$-2\pi\Re\big((c_E-\log 2\pi)g\,(i/2)+g'\,(i/2)\,/2\big).$$

The *main term* is the value at $(1/2,1/2,1/2,1/2)$ of the sum of 13 meromorphic functions in $\mathbf{w}=(w_1,w_2,w_3,w_4)$, with a common structure. As an example, I give the third term:

$$\frac{\zeta(w_3+w_4)\zeta(w_1+w_2-1)\zeta(w_3-w_2+1)\zeta(w_1-w_4+1)}{\zeta(w_3+w_4-w_1-w_2+2)} \\ \times\Gamma(w_1+w_2-1)\int_{-\infty}^{\infty}\frac{\Gamma(1-w_1-it)}{\Gamma(w_2-it)}\,g(t)\,dt\,.$$

Conrey, Farmer, Keating, Rubinstein, Snaith, [3], have emphasized the common structure of terms 1–12 .

The *spectral term* is related to the decomposition of $L^2\,(\Gamma\backslash G)$ in terms of automorphic forms, where $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$ and $G=\mathrm{PSL}_2(\mathbb{R})$. Let $\{V\}$ be the complete orthogonal system of Hecke-invariant irreducible subspaces of the cuspidal subspace of $L^2\,(\Gamma\backslash G))$. Each V contributes to the spectral term the quantity $c_V H_V(1/2)^3\Theta(\nu_V;g)$, where

$$\Theta(\nu;g) = \int_0^\infty \widehat{g}\left(\frac{1}{2\pi}\log\left(1+\tfrac{1}{r}\right)\right)\Xi(r;\nu)\,\frac{dr}{\sqrt{r^2+r}}\,,$$

$$\Xi(r;\nu) = \int_{\mathbb{R}\setminus\{0\}} j_0(-u)j_\nu(u/r)\frac{du}{|u|^{3/2}}\,,$$

$$j_\nu(u) = \frac{\pi|u|}{\sin\pi\nu}\left(J_{-2\nu}^{(\operatorname{sign}u)}\left(4\pi|u|^{1/2}\right) - J_{2\nu}^{(\operatorname{sign}u)}\left(4\pi|u|^{1/2}\right)\right)\,,$$

$$J_{2\nu}^{(+)} = J_{2\nu}\,, \qquad J_{2\nu}^{(-)} = I_{2\nu}\,.$$

c_V is essentially the Fourier coefficient of V of order one, and H_V the finite part of the L-function of V. There is also an integral with an analogous structure, corresponding to the Eisenstein series.

The two known proofs use a meromorphic function of four variables $\mathbf{w} = (w_1, w_2, w_3, w_4)$:

$$J(\mathbf{w}) = \int_{-\infty}^{\infty} \zeta(w_1 - it)\zeta(w_2 + it)\zeta(w_3 + it)\zeta(w_4 - it)g(t)\,dt\,,$$

and break it up as a quadruple sum (for $\Re w_j > 1$):

$$\text{(1)} \qquad J(\mathbf{w};g) = \sum_{a,b,c,d\geqslant 1} a^{-w-1}b^{-w_2}c^{-w_3}d^{-w_4}\,\widehat{g}\left(\frac{\log(ad/bc)}{2\pi}\right)$$
$$= J_0(\mathbf{w};g) + J_1(\mathbf{w};g) + J_1(w_2,w_1,w_4,w_3)\,,$$

according to $ad = bc$, $ad > bc$, and $ad < bc$.

The original proof of Motohashi transforms J_1, and reduces it essentially to a double sum of Kloosterman sums. The sum formula of Kuznetsov, see e.g., Chap. 2 of [4], relates this to sums of products of Fourier coefficients of modular forms. The relation between $|\zeta|^4$ and modular forms looks accidental.

In the present proof

$$J_1(\mathbf{w};g) = \frac{1}{8}\sum_{a,b,c,d\in\mathbb{Z},\ ad>bc} f_\psi\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\,,$$

$$f_\psi\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = |a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4}\psi\left(\frac{ad}{bc}\right)\,,$$

with $\psi(x) + \psi(-x) = 2\widehat{g}\left(\frac{\log|x|}{2\pi}\right)$. We use the Hecke decomposition $\mathrm{M}_2(\mathbb{Z}) \cup \mathrm{GL}_2^+(\mathbb{R})$ to write this as a sum of Hecke operators applied to a Poincaré series on G:

$$J_1(\mathbf{w};g) = \frac{1}{4}\sum_{n\geqslant 1} n^{-z_1} T_n P f_\psi(1)\,,$$

$$T_n f(X) = n^{-1/2} \sum_{ad=n,\, b(d)} f\left(\tfrac{1}{\sqrt{n}}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) X\right) \qquad \text{(Hecke operator)},$$

$$P f(X) = \sum_{\gamma\in\Gamma} f(\gamma X) \qquad \text{(Poincaré series)},$$

with $z_1 = \frac{1}{2}(w_1+w_2+w_3+w_4-1)$. To make the Poincaré series convergent for $X\in G$, we replace f_ψ by $f_{\psi,\eta}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = f_\psi\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\eta\left(\frac{d}{c}\right)$, where η is a smooth function on $\mathbb{R}$, highly zero at 0, and $\pm\infty$, which we view as an approximation of 1. For technical reasons, we also add a factor $\tau(ad)$, with τ a smooth approximation of the characteristic function ι of $(-\infty,0)$. Now

$$J_1(\mathbf{w};g) = \lim_{\eta\uparrow 1}\lim_{\tau\uparrow\iota} \frac{1}{2}\sum_{n\geqslant 1} n^{-z_1} T_n P f_{\psi,\tau,\eta}(1). \tag{2}$$

After subtraction of a sum of Eisenstein series, we arrive at $P_0 f_{\psi,\tau,\eta}\in L^2(\Gamma\backslash G)$. We carry out the spectral expansion in a reasonable explicit way. After taking the limits of τ and η we arrive at a explicit formula for $\mathbf{w}$ in a suitable region. All terms allow meromorphic continuation to a neighborhood of the point $(1/2,1/2,1/2,1/2)$. Thus the explicit spectral formula is obtained.

In the contributions $\mathcal{M}_0(g),\ldots,\mathcal{M}_{12}(g)$ to the main term, the term $\mathcal{M}_0$ comes from J_0 in (1). The term $\mathcal{M}_2$ (in the notation of [4]) comes from the non-square integrable contribution to $Pf_{\psi,\tau,\eta}$. The terms $\mathcal{M}_1$, and $\mathcal{M}_3,\ldots,\mathcal{M}_6$ come from the continuous spectrum. The terms $\mathcal{M}_7,\ldots,\mathcal{M}_{12}$ arise similarly from $J_1(w_2,w_1,w_4,w_3)$. So most of the contributions to the “main term” also have a spectral nature. This proof does not explain all symmetries pointed out in [3]. However, in it the connection with automorphic forms arises naturally.

I do not think that the method extends to $\int_{-\infty}^{\infty}|\zeta(1/2+it)|^{2k} g(t)\,dt$ for $k\geqslant 3$. The algebraic group leaving invariant the form $v_1\cdots v_k - w_1\cdots w_k$ does not seem suitable for the present purpose. However, there may be a relation with the paper [1] of Beineke and Bump. Embed M_2 into GL_4 by

$$X \mapsto \left(\begin{array}{c|c} I & X \\ \hline 0 & I\end{array}\right).$$

Conjugation by $\left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2\end{array}\right)$, with $g_1,g_2\in\mathrm{GL}_2$, $\det g_1 = \det g_2$, gives the group Q of all linear transformation of M_2 that leave invariant the determinant form. We have used automorphic forms on the subgroup $\left\{\begin{array}{c|c} * & 0 \\ \hline 0 & I\end{array}\right\}$ of Q. The group Q is contained in the group

$$M = \left\{\left(\begin{array}{c|c} * & 0 \\ \hline 0 & *\end{array}\right)\right\}$$

that plays a role in the discussion by Beineke and Bump of the connection between $\int |\zeta|^4$ and Eisenstein series on GL_4.

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Automorphic summation formulae and moments of $\zeta(s)$

Daniel Bump[1]

(joint work with Jennifer Beineke[2])

We begin by recalling a parallel, pointed out by Beineke and Bump [1], between conjectural asymptotics for the moments of the Riemann zeta function and the constant terms of Eisenstein series. Let $\alpha = (\alpha_1, \dots, \alpha_{2n})$, $|\alpha_i|$ small,

$$Z(s,\alpha) = \zeta(s+\alpha_1)\cdots\zeta(s+\alpha_n)\zeta(1-s-\alpha_{n+1})\cdots\zeta(1-s-\alpha_{2n}).$$

We are interested in

$$\int_{-\infty}^{\infty} Z\left(\tfrac{1}{2}+it,\alpha\right) g(t)\,dt$$

with a suitably smooth test function $g(t)$ and the α_i near zero. Conrey, Farmer, Keating, Rubinstein and Snaith [3] conjectured that this should match

$$\int_{-\infty}^{\infty} M\left(\tfrac{1}{2}+it,\alpha\right) g(t)\,dt\,,$$

where M will now be describe. It is a sum of $\binom{2n}{n}$ terms indexed by $w \in \Xi$, the subset of the symmetric group S_{2n} consisting of permutations that satisfy

$$w(1) < \cdots < w(n), \qquad w(n+1) < \cdots < w(2n).$$

Let $R(\alpha) = A(\alpha)\,N(\alpha)$ where

$$N(\alpha) = \prod_{\substack{1\leqslant j\leqslant n\\ n+1\leqslant k\leqslant 2n}} \zeta(1+\alpha_j-\alpha_k).$$

We won't define the "arithmetic part" $A(\alpha)$. It is an Euler product, convergent if $\Re(\alpha_i)$ are small. We have

$$M\left(\tfrac{1}{2}+it,\alpha\right) = \left(\frac{t}{2\pi}\right)^{-\alpha_1-\ldots-\alpha_n+\alpha_{n+1}+\ldots+\alpha_{2n}} \times \sum_{w\in\Xi} R(w\alpha)\left(\frac{t}{2\pi}\right)^{-\alpha_{w(1)}-\ldots-\alpha_{w(n)}+\alpha_{w(n+1)}+\ldots+\alpha_{w(2n)}}.$$

To summarize, the $2n$-th moment

$$\int_{-\infty}^{\infty}\prod_{j=1}^{n}\zeta\left(\tfrac{1}{2}+\alpha_j\right)\zeta\left(\tfrac{1}{2}-\alpha_{n+j}\right)g(t)\,dt$$

is conjecturally a sum of $\binom{2n}{n}$ terms, each involving a product $N(w\alpha)$ of n^2 zeta functions.

Let $\mathbb{A}$ be the adele ring on $\mathbb{Q}$, and let E_α be the Eisenstein series on $\mathrm{GL}(2n,\mathbb{A})$ whose L-function is

$$\zeta(s+\alpha_1)\cdots\zeta(s+\alpha_n)\zeta(s+\alpha_{n+1})\cdots\zeta(s+\alpha_{2n}) = \chi(s+\alpha_{n+1})\cdots\chi(s+\alpha_{2n})Z(s,\alpha).$$

It is shown in [1] that the constant term

$$\int_{\mathrm{Mat}_n(\mathbb{Q})\backslash\mathrm{Mat}_n(\mathbb{A})} E\left(\begin{pmatrix} I_n & X \\ & I_n\end{pmatrix} g\right)dX$$

is a sum of $\binom{2n}{n}$ terms indexed by $w\in\Xi$, each of which is $N(w\alpha)$ times an Eisenstein series on $\mathrm{GL}_n\times\mathrm{GL}_n$. See [1] for a precise statement. Note however that the parallel between the Eisenstein series and the moment conjectures is very striking

- The L-function of the Eisenstein series matches the integrand in the $2n$-th moment;
- The constant term of the Eisenstein series, like M in the moment conjectures is a sum of $\binom{2n}{n}$ terms, each involving a product of n^2 zeta functions.

The explanation for this mystery when $n=1$ involves the summation formula of Oppenheim [5]. It is explained in [1] how Oppenheim's summation formula can be deduced from the theory of Eisenstein series. On the other hand, it is known (see Matsumoto [4] and [1]) that the Oppenheim summation formula implies the conjectures of [3] (proved in 1927 by Ingham) for the second moment of zeta.

A generalization of the Oppenheim summation formula to GL_{2n} was obtained in Beineke and Bump [2]. This generalization involves a generalization of the classical divisor function also studied in Sato [6]. Let $L\subseteq\mathbb{Z}^n$ be a lattice, that is, a subgroup of finite index. Let $a\in\mathbb{C}$, and define

$$\tau_a(L) = \sum_{\substack{\text{lattice } L' \\ L\subseteq L'\subseteq\mathbb{Z}^n}}\left(\frac{[\mathbb{Z}^n:L']}{[L':L]}\right)^a.$$

We have

$$\sum_L \tau_a(L)\,[\mathbb{Z}^n : L]^{-s} = \prod_{k=0}^{n-1} \zeta(s+a-k)\zeta(s-a-k)\,.$$

Let $\mathrm{Mat}_n^*(\mathbb{Z}) = \mathrm{Mat}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{R})$. This is a parameter space for lattices: If $A \in \mathrm{Mat}_n^*(\mathbb{Z})$, let L_A be a row lattice, $A \to L_A$ is a bijection

$$\mathrm{GL}_n(\mathbb{Z}) \setminus \mathrm{Mat}_n^*(\mathbb{Z}) \to \left\{\text{Lattices } L \subseteq \mathbb{Z}^n\right\}.$$

If $A \in \mathrm{Mat}_n^*(\mathbb{Z})$, we will denote $\tau_a(A) = \tau_a(L_A)$. If $\Phi \in C_c^\infty(\mathrm{GL}_n(\mathbb{Z}) \setminus \mathrm{GL}_n(\mathbb{R}))$ the new summation formula expresses

$$\sum_{A \in \mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{Mat}_n^*(\mathbb{Z})} \tau_{n(s-1/2)}(A)\,\Phi(A)$$

in terms of a certain "Hankel transform" $\widetilde{\Phi}$ of the original function Φ. It is an integral of Φ against Bessel distribution on GL_n, arising from the study of the degenerate principal series representations of $\mathrm{GL}_{2n}(\mathbb{R})$. There are two main terms

$$\zeta(2ns)\,\zeta(2ns-1)\cdots\zeta(2ns-n+1) \int_{\mathrm{GL}_n(\mathbb{R}) \backslash \mathrm{Mat}_n(\mathbb{R})} |\det(g)|^{n(s-1/2)}\,\Phi(g)\,dg$$

$$+\,\zeta(2n-2ns)\,\zeta(2n-2ns+1)\cdots\zeta(n+1-2ns) \times \int_{\mathrm{GL}_n(\mathbb{R}) \backslash \mathrm{Mat}_n(\mathbb{R})} |\det(g)|^{n(1/2-s)}\,\Phi(g)\,dg\,.$$

There is also

$$\sum_{A \in \mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{Mat}_n^*(\mathbb{Z})} \tau_{n(s-1/2)}(A)\,\widetilde{\Phi}(A)\,.$$

If $n = 1$, this is all, and this is the Oppenheim summation formula. If $n \geqslant 2$, there are more terms, and the case $n = 2$ is made completely explicit in [2]. Although this formula will itself not be directly applicable to the moments of ζ, it is hoped that a variant will be so applicable. The proof depends on the theory of Eisenstein series on GL_{2n}.

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The ratios conjecture for the Riemann zeta function

Brian Conrey[1]
(joint work with Nina Snaith[2])

Let $\zeta(s)$ be the Riemann zeta function, let $s=1/2+it$, and let $\Re\gamma,\delta>0$ and $|\alpha|,|\beta|<1/4$. Then the ratios conjecture of Conrey, Farmer, and Zirnbauer [5] asserts that

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt = \int_0^T \Bigg(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)}\,A_\zeta(\alpha,\beta,\gamma,\delta)$$
$$+\Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta}\frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)}\,A_\zeta(-\beta,-\alpha,\gamma,\delta)\Bigg)\,dt+O\Big(T^{1/2+\varepsilon}\Big)$$

where

$$A_\zeta(\alpha,\beta,\gamma,\delta) = \prod_p \frac{\big(1-\frac{1}{p^{1+\gamma+\delta}}\big)\big(1-\frac{1}{p^{1+\beta+\gamma}}-\frac{1}{p^{1+\alpha+\delta}}+\frac{1}{p^{1+\gamma+\delta}}\big)}{\big(1-\frac{1}{p^{1+\beta+\gamma}}\big)\big(1-\frac{1}{p^{1+\alpha+\delta}}\big)}\,.$$

This conjecture is an analogue of the random matrix theorem (see [5] or [4])

$$\int_{U(N)} \frac{\Lambda_A(e^{-\alpha})\Lambda_{A^*}(e^{-\beta})}{\Lambda_A(e^{-\gamma})\Lambda_{A^*}(e^{-\delta})}\,dA = \frac{z(\alpha+\beta)z(\gamma+\delta)}{z(\alpha+\delta)z(\beta+\gamma)} + e^{-N(\alpha+\beta)}\frac{z(-\alpha-\beta)z(\gamma+\delta)}{z(-\beta+\delta)z(-\alpha+\gamma)}$$

where $z(x)=(1-e^{-x})^{-1}$. Here, $\Lambda_A(s)$ denotes the characteristic polynomial of an $N\times N$ unitary matrix A and is defined by

$$\Lambda_A(s)=\det(I-sA^*)=\prod_{n=1}^N(1-se^{-i\theta_n})\,;$$

A^* denotes the matrix which is the conjugate transpose of A, and the $e^{i\theta_n}$ are the eigenvalues of A.

More generally, [5] gives precise conjectures for averages of ratios of L-functions from a family, where the ratio may have any number of L-functions in the numerator or denominator.

We give some applications of this collection of ratios conjectures to various calculations of interest to number theorists. The first application is to the pair-correlation conjecture of Montgomery. We assume the Riemann Hypothesis, and let the non-trivial zeros of the Riemann zeta function be denoted by $1/2+i\gamma$. The pair-correlation conjecture is an assertion about the distribution of the differences $\gamma-\gamma'$ as γ and γ' range independently over the zeros of ζ in some interval. A convenient way to state the pair-correlation conjecture is through the use of a test function f. We assume that f is holomorphic in a strip of fixed width, say 1, around the real axis and rapidly decaying as the absolute value of the real part of the variable gets large; suppose also that f is even.

THEOREM 1. *Assuming the ratios conjecture,*

$$\sum_{0<\gamma,\gamma'\leqslant T} f(\gamma-\gamma') = \frac{1}{(2\pi)^2}\int_0^T \biggl(2\pi f(0)\log\frac{t}{2\pi} + \int_{-T}^{T} f(r)\Bigl(\log^2\frac{t}{2\pi} + 2\Bigl((\zeta'/\zeta)'(1+ir) + \Bigl(\frac{t}{2\pi}\Bigr)^{-ir}\zeta(1-ir)\zeta(1+ir)A(ir) - B(ir)\Bigr)\Bigr)\,dr\biggr)\,dt + O\bigl(T^{1/2+\varepsilon}\bigr)\,;$$

here the integral is to be regarded as a principal value near $r=0$,

$$A(\eta) = \prod_p \frac{\bigl(1-\frac{1}{p^{1+\eta}}\bigr)\bigl(1-\frac{2}{p}+\frac{1}{p^{1+\eta}}\bigr)}{\bigl(1-\frac{1}{p}\bigr)^2}\,,$$

and

$$B(\eta) = \sum_p \left(\frac{\log p}{\bigl(p^{1+\eta}-1\bigr)}\right)^2 .$$

This assertion is a much more precise version of the 'usual' pair-correlation conjecture:

$$\sum_{0<\gamma,\gamma'\leqslant T} f\Bigl(\frac{(\gamma-\gamma')\log T}{2\pi}\Bigr) \sim \frac{T\log T}{2\pi}\left(f(0) + \int_{-\infty}^{\infty} f(u)\Bigl(1-\Bigl(\frac{\sin\pi u}{\pi u}\Bigr)^2\Bigr)\,du\right)$$

for suitable test-functions. The formula above was originally derived by Bogomolny and Keating heuristically from the Hardy-Littlewood conjectures about the distribution of pairs of primes with a fixed difference. The ratios conjecture leads to the same result but with a much simpler calculation.

As a second example, we consider the second moment of $|\zeta'(\rho)|$ averaged over zeros $\rho = 1/2+i\gamma$ of $\zeta(s)$ with $0<\gamma<T$. Gonek [6] obtained the leading term for this, assuming only the Riemann Hypothesis.

THEOREM 2. *The ratios conjecture implies*

$$\begin{aligned}\sum_{\gamma<T} \bigl|\zeta'(\rho)\bigr|^2 = \frac{1}{2\pi}\int_0^T \biggl(&\frac{1}{12}\log^4\frac{t}{2\pi} + \frac{2\gamma}{3}\log^3\frac{t}{2\pi} + (\gamma^2-2\gamma_1)\log^2\frac{t}{2\pi} \\ &-\bigl(2\gamma^3+10\gamma\gamma_1+\gamma_2\bigr)\log\frac{t}{2\pi} + 2\gamma^4+12\gamma^2\gamma_1+14\gamma_1^2+8\gamma\gamma_2+\frac{10\gamma_3}{3}\biggr)\,dt \\ &+O(T^{1/2+\epsilon})\end{aligned}$$

where the γ_j *are the coefficients from the Laurent expansion of* $\zeta(1+s)$ *around* $s=0$:

$$\zeta(1+s) = \frac{1}{s} + \gamma - \gamma_1 s + \frac{\gamma_2}{2!}s^2 - \frac{\gamma_3}{3!}s^3\cdots .$$

It is possible that a similar result might be obtained by Gonek's method assuming only RH. The ratios conjectures similarly imply precise conjectures for the $2k$-th moment of $|\zeta'(\rho)|$ for any positive integer k. These agree with Hughes' conjectures [7] for the leading order terms of these moments.

As a third application, we compute a "mollified fourth moment" of $\zeta(1/2+it)$. Chris Hughes has recently proven an asymptotic formula for

$$\int_0^T \left|\zeta(1/2+it)\right|^4 \left|A(1/2+it)\right|^2 dt$$

where $A(s)=\sum_{n\leqslant y} a_n n^{-s}$ is an arbitrary Dirichlet polynomial and where $y=T^\theta$ with $\theta<5/27$. For applications to zeros of $\zeta(s)$ it is useful to know this asymptotic formula more explicitly in the case that $A(s)=M_2(s,P)$ is a mollifying polynomial

$$M_2(s,P)=\sum_{n\leqslant y}\mu_2(n)P\left(\frac{\log\frac{y}{n}}{\log y}\right)n^{-s}$$

where $y=T^\theta$ and μ_2 is defined by $1/\zeta(s)^2=\sum_{n=1}^\infty \mu_2(n)n^{-s}$.

Theorem 3. *Assuming the ratios conjecture, if $P(x)$ is a real polynomial for which $P^{(j)}(0)=0$ for $0\leqslant j\leqslant 4$, then for any $\theta>0$ we have*

$$\begin{aligned}
&\frac{1}{T}\int_0^T \left|\zeta(1/2+it)\right|^4 \left|M_2(1/2+it,P)\right|^2 dt\\
&\quad\sim \int_0^1 \frac{(1-\eta)^3}{6}\bigg(2P(\eta)P^{(4)}(\eta)+8P^{(3)}(\eta)P'(\eta)+\frac{4}{\theta}(P^{(4)}(\eta)P'(\eta)\\
&\qquad +3P''(\eta)^2)+\frac{16}{\theta}P^{(3)}(\eta)P''(\eta)+\frac{4}{\theta^2}P^{(4)}(\eta)P''(\eta)+\frac{4}{\theta^2}P^{(3)}(\eta)^2\\
&\qquad +\frac{4}{3\theta^3}P^{(4)}(\eta)P^{(3)}(\eta)+\frac{1}{12\theta^4}P^{(4)}(\eta)^2\bigg)\,d\eta\,.
\end{aligned}$$

We remark that this is a computation that can, in principle, be carried out unconditionally (for restricted θ) by Hughes' method; the ratios conjecture affords a relatively simple way to perform the computation, and also serves as a check.

There are a number of other applications in the forthcoming paper [3], including some applications to lower order terms in computations of the one-level density for zeros of different families of L-functions. Also, we mention that Tsz Ho Chan [2] has used the ratios conjectures to compute all of the lower order main terms in the second moment $\int_0^T S(t)^2\,dt$ where $S(t)=(1/\pi)\arg\zeta(1/2+it)$; these are all of the terms of the size $T/(\log^n T)$ for some n.

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An arithmetic characterisation of the logarithm

Peter D. T. A. Elliott

In 1946 P. Erdős [1], proved that a real-valued additive arithmetic function, monotonic on the positive integers, must be a constant multiple of a logarithm. The result of Erdős was reformulated by P. Turán, c.f. E. Wirsing [3], p.46, as a characterisation of the Riemann zeta function amongst those Dirichlet series that posses non-decreasing positive coefficients and an Euler product. In the lecture I presented an overview of the following sharpening of Erdős' result.

Theorem. *A real-valued additive arithmetic function is monotonic on all sufficiently large shifted primes, $p+1$, if and only if it has the form $A\log$ on the odd integers, whilst also being of the form $A\log + \text{constant}$ on the powers of 2.*

It follows from an old result of Hardy and Littlewood [2], that for some positive absolute constant c, any interval of length y contains at most $cy/\log y$ shifted primes. To this extent there are decidedly fewer primes than integers. Moreover, no initial bound upon the additive function is apparent.

To establish the theorem I employ the representation of integer powers by ratios of shifted primes, $(p+1)/(q+1)$, concentration function estimates for additive functions on shifted primes, and the fact that primes are well distributed in residue classes to moduli that are multiplicative perturbations of a high power of an integer. Necessary auxiliary results are obtained via harmonic analysis.

The theorem reinforces the notion that in some multiplicative sense the set of shifted primes contains almost as much information as the set of positive integers. What would an analogue of Turan's reformulation be?

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Irregularities in the distribution of imaginary parts of the Riemann zeta function and L-functions

Kevin Ford[1]

(joint work with Alexandru Zaharescu)

Fix a nonzero real number α. We study the sequence $\{\alpha\gamma\}$, where $\{y\}$ is the fractional part of y (the image of y in the torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}$) and γ runs over the imaginary parts of the zeros of $\zeta(s)$. Rademacher [9] showed that the sequence is uniformly distributed on RH, and later Hlawka [5] gave an unconditional proof. Hlawka further showed that the discrepancy function

$$D^*_\alpha(T) = \sup_{0\leqslant y\leqslant 1} \Big| \frac{1}{N(T)} \sum_{\substack{0<\gamma\leqslant T\\ \{\alpha\gamma\}\leqslant y}} 1 - y \Big|$$

satisfies $D^*_\alpha(T) \ll_\alpha 1/\log T$ on RH. Fujii [3] has shown unconditionally that $D^*_\alpha(T) \ll_\alpha (\log\log T)/(\log T)$.

We study the finer distribution of $\{\alpha\gamma\}$, uncovering an irregularity when α is a rational multiple of $(\log p)/(2\pi)$ for some prime p. In particular, our results imply that for such α, $D^*_\alpha(T) \gg_\alpha 1/\log T$. Central to these investigations is a formula of Landau [7]:

$$\sum_{0<\gamma\leqslant T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O_x(\log T) \qquad (x>1),$$

where $\rho=\beta+i\gamma$ are the nontrivial zeros of ζ, $\Lambda(x)$ is the von Mangoldt function for integral $x>1$ and $\Lambda(x)=0$ for non-integral $x>1$. On RH,

$$\sum_{0<\gamma\leqslant T} x^{i\gamma} = -\frac{T}{2\pi\sqrt{x}}\Lambda(x) + O_x(\log T) \qquad (x>1). \tag{1}$$

Based on this observation, Rademacher [9] asserted that if $\alpha = k(\log p)/(2\pi)$ where p is prime and k a positive integer, then there should be a "predominance of terms which fulfill $|\{\alpha\gamma\}-1/2|<1/4$".

To each α we associate a function g_α on $\mathbb{T}$, which is identically zero unless $\alpha = \frac{a}{q}\frac{\log p}{2\pi}$ with p prime, a and q positive integers with $(a,q)=1$. In this case

$$g_\alpha(t) = -\frac{(p^{a/2}\cos 2\pi qt - 1)\log p}{\pi(p^a - 2p^{a/2}\cos 2\pi qt + 1)}\,.$$

THEOREM 1. (cf. [1], Theorem 1) *Let* $h : \mathbb{T} \to \mathbb{C}$ *and suppose* $\alpha > 0$. *Then*

$$\frac{1}{T}\sum_{0<\gamma\leqslant T} h(\alpha\gamma) - \frac{N(T)}{T}\int_{\mathbb{T}} h \;\to\; \int_{\mathbb{T}} hg_\alpha \qquad (T \to \infty) \tag{2}$$

holds for all $h \in C^2(\mathbb{T})$. *On RH,* (2) *holds for all absolutely continuous functions* h *on* $\mathbb{T}$.

The main tools used to prove Theorem 1 are a version of Landau's formula with explicit dependence on x [4], zero-density estimates and results on the rate of convergence of Fourier series. When $\alpha = (\log p)/(2\pi)$, $g_\alpha(0) < 0$ and thus there is a shortage of zeros with $\{\alpha\gamma\}$ near 0.

We conjecture that (2) holds with h the characteristic function of $[0, y]$, the convergence uniform in y. One consequence is

$$D^*_\alpha(T) = (1 + o(1))\frac{\log p}{\pi q}\;\frac{\arcsin(p^{-a/2})}{\log T}$$

for $\alpha = \frac{a}{q}\frac{\log p}{2\pi}$, and $D^*_\alpha(T) = o(1/\log T)$ for α not of this form. Similar conjectures were made by Kaczorowski [6], concerning the quantity

$$\sup_{0\leqslant t\leqslant 1}\Bigg|\frac{1}{n!\,S_n}\sum_{\substack{\gamma>0\\ 0\leqslant\{\alpha\gamma\}<t}} e^{-\gamma}\gamma^n - t\Bigg|, \qquad S_n = \frac{1}{n!}\sum_{\gamma>0} e^{-\gamma}\gamma^n\,,$$

where γ runs over the imaginary parts of zeros of $\zeta(s)$ or of a Dirichlet L-function.

In the opposite direction, (2) cannot hold for all functions h which are continuous and differentiable on $\mathbb{T}$. This is a consequence of a property of general sequences, uniformly distributed or not.

THEOREM 2. ([1], Theorem 7) *Let* $a_1, a_2, \ldots$ *be an arbitrary sequence of numbers in* $\mathbb{T}$, *let* t *be a point in* $\mathbb{T}$, *and let* $f(x)$ *be a function decreasing monotonically to* 0 *arbitrarily slowly. Then there is a function* h, *continuous and differentiable on* $\mathbb{T}$, *and which is* $C^\infty(\mathbb{T}\backslash\{t\})$, *so that for infinitely many positive integers* n,

$$\Bigg|\,\frac{1}{n}\sum_{j=1}^{n} h(a_j) \;-\; \int_{\mathbb{T}} h\,\Bigg| \;\geqslant\; f(n)\,.$$

In a second paper [2], we consider generalizations to other L-functions F in the Selberg class. We assume that for some $A > 0$ and $B \geqslant 0$,

$$N_F(\sigma, T) \ll T^{1-A(\sigma-1/2)}\log^B T\,, \tag{3}$$

where $N_F(\sigma, T)$ is the number of zeros $\beta + i\gamma$ of F with $0 \leqslant \gamma \leqslant T$ and $\beta \geqslant \sigma$. Such zero density estimates are known for the Riemann zeta function, Dirichlet

L-functions, and certain L-functions attached to holomorphic cusp forms (all with $B = 1$). Define $g_{F,\alpha}$ to be identically zero unless $\alpha = \frac{a}{q}\frac{\log p}{2\pi}$, in which case

$$g_{F,\alpha}(t) = -\frac{1}{\pi}\Re\sum_{m=1}^{\infty}\frac{\Lambda_F(p^{am})}{p^{am/2}}e^{2\pi iqmt}.$$

Here $\Lambda_F(n)$ are the coefficients in the Dirichlet series of $-F'(s)/F(s)$.

Theorem 3. *Assume* (3) *holds. Let* $\alpha > 0$ *and* $h : \mathbb{T} \to \mathbb{C}$. *Then*

$$\frac{1}{T}\sum_{0<\gamma\leqslant T} h(\alpha\gamma) - \frac{N(T)}{T}\int_{\mathbb{T}} h \;\to\; \int_{\mathbb{T}} hg_{F,\alpha} \qquad (T\to\infty) \tag{4}$$

holds for all $h \in C^2(\mathbb{T})$. *On RH for the function* F, (4) *holds for all absolutely continuous functions* h *on* $\mathbb{T}$.

In particular, if F is a Dirichlet L-function attached to a Dirichlet character χ, $\alpha = \frac{\log p}{2\pi}$ and $\chi(p) = e^{2\pi i\xi}$, then $g_{F,\alpha}(t)$ has a minimum at $t = 1-\xi$. Consequently, there is a shortage of zeros of F with $\{\alpha\gamma\}$ near $1-\xi$.

We also estimate the discrepancy function $D^*_{F,\alpha}(T)$ using the Erdős-Turán inequality and the moment method of Selberg and Fujii (see [3]).

Theorem 4. *Suppose* (3) *holds and* $g_{F,\alpha}$ *is not identically zero. Then*

$$D^*_{F,\alpha} \gg_{F,\alpha} \frac{1}{\log T}.$$

Theorem 5. *Fix* $\alpha > 0$ *and assume* (3) *holds. Then*

$$D^*_{F,\alpha}(T) \ll_{F,\alpha} \Big(\frac{\log\log T}{\log T}\Big)^{2/3}.$$

If (3) *holds with* $B = 1$, *then*

$$D^*_{F,\alpha}(T) \ll_{F,\alpha} \frac{\log\log T + \sqrt{Q\log\log T}}{\log T}, \qquad Q = \sum_{p\leqslant T}\frac{|a_F(p)|^2}{p},$$

where $a_F(n)$ *are the coefficients of the Dirichlet series for* $F(s)$. *If RH is true for* F, *then*

$$D^*_{F,\alpha}(T) \ll_{F,\alpha} \frac{1}{\log T}.$$

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Discriminants of cubic fields

Étienne Fouvry

(joint work with Karim Belabas)

Let p always denote a prime number and let $\mathcal{K}$ be the set of the cubic extensions K of $\mathbb{Q}$, satisfying $K \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. We denote by $\widetilde{K}$ the orbit of K under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and $\widetilde{\mathcal{K}}$ be the set of these orbits. The aim of [2] is to study the distribution of $\operatorname{disc} \widetilde{K}$ in the arithmetic progressions. For $X < Y$ real numbers and q integer we define

$$N(X, Y; q) = \operatorname{card} \left\{ \widetilde{K} \in \widetilde{\mathcal{K}} : \, X < \operatorname{disc} \widetilde{K} < Y, \ q \mid \operatorname{disc} \widetilde{K} \right\}.$$

We prove

Theorem 1. *For every positive ε, we have*

$$N(0, X; q) \sim \frac{1}{12\,\zeta(3)} \prod_{p \mid q} \Big(\frac{p^2+p}{p^2+p+1} \Big) \cdot \frac{X}{q}$$

and

$$N(-X, 0; q) \sim \frac{1}{4\,\zeta(3)} \prod_{p \mid q} \Big(\frac{p^2+p}{p^2+p+1} \Big) \cdot \frac{X}{q},$$

when $X \to +\infty$, uniformly for q a squarefree integer, satisfying $q \leqslant X^{1/24-\varepsilon}$.

Theorem 2. *There exists a positive absolute constant* C_0, *such that, for every* $X \geqslant 2$ *and every squarefree* $q \leqslant X^{1/4}$, *we have the inequality*

$$N(-X, X; q) \leqslant C_0 \prod_{p \mid q} \Big(3 + \frac{1}{p}\Big) \cdot \frac{X}{q}\,.$$

When q *is a prime, we have the inequality*

$$N(-X, X; q) \leqslant C_0' \cdot \frac{X}{q}\,,$$

for some positive C_0', *for any* $X \geqslant 2$, *and* $q \leqslant X^{15/58} (\log X)^{-6/29}$.

Theorem 3. *There exist two absolute constants* $c_1 > 0$ *and* c_2, *such that, for any* $X \geqslant 2$, *and for any integer* q *not divisible by* 16 *and not divisible by the square of an odd prime, we have the lower bound*

$$N(0, X; q),\ N(-X, 0; q) \ \geqslant\ c_1 \frac{\varphi(q)}{q} \cdot \frac{X}{q} - c_2 \Big(\frac{X}{q}\Big)^{1/2} \prod_{p \mid q} \big(1 + p^{-1/2}\big)\,.$$

In particular, there exists X_0 *and* $c_3 > 0$ *such that, for any integer* q *as above, satisfying the inequality* $q \leqslant X \exp\big(-\sqrt{\log X}\big)$, *we have the lower bound*

$$N(0, X; q),\ N(-X, 0, q) \ \geqslant\ c_3 \frac{\varphi(q)}{q} \cdot \frac{X}{q}\,,$$

for $X \geqslant X_0$.

The cornerstone of these results is the famous paper of Davenport and Heilbronn [3], where Theorem 1 is proved in the particular case $q = 1$. Note that Theorem 1 is the first result dealing with the distribution of cubic discriminants in arithmetic progressions, with a large uniformity over the modulus (however note the result ([1], Théorème 7.1) where such a question is studied for cubic discriminants which are also discriminants of quadratic fields).

Following the theory of Delone–Faddeev [4] and Davenport–Heilbronn, we construct a function Φ which, to each $\widetilde{K}$, associates a class (modulo the action of $\mathrm{GL}(2, \mathbb{Z})$) of cubic forms

$$F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

with integers coefficients. This class of cubic form $F = \Phi(\widetilde{K})$ has the same discriminant

$$\Delta(a, b, c, d) = b^2c^2 + 18abcd - 27a^2d^2 - 4b^3d - 4c^3a$$

as $\widetilde{K}$. The problem is now to ensure that Φ is bijective, by finding a fundamental domain. Let $\mathcal{V}$ be the set of points $(a, b, c, d) \in \mathbb{Z}^4$ satisfying the inequalities

$$a \geqslant 1,\ \ |bc - 9ad| \leqslant b^2 - 3ac \leqslant c^2 - 3bd\,, \tag{1}$$

and some local conditions U_p, for each prime p (U_p is too long to be defined here). Then Davenport and Heilbronn proved that, roughly speaking, Φ can be seen as a bijection between the set $\widetilde{\mathcal{K}}^+$ of class of cubic fields with positive discriminants and the set $\mathcal{V}$ modulo the identification $(a,b,c,d) \sim (a,-b,c,-d)$. The same type of result is true for negative cubic discriminants. Since Φ preserves the discriminant, we see that, with a negligible error, $N(0,X;q)$ is equal to

$$\frac{1}{2} \cdot \mathrm{card}\,\Big\{(a,b,c,d) \in \mathcal{W}(X) : \ (a,b,c,d) \in U_p \text{ (for every } p),\ q \mid \Delta(a,b,c,d)\Big\},$$

where $\mathcal{W}(X)$ is defined by (1) and by the inequality $0 < \Delta(a,b,c,d) < X$. We are led to count integers points in a volume $\mathcal{W}(X)$ in $\mathbb{Z}^4$, satisfying congruences conditions. As in [3], we dissect $\mathcal{W}(X)$ into a cusp (which is treated trivially) and, as in [1], into hypercubes of dimension 4, where we study the polynomial congruence $\Delta(a,b,c,d) \equiv 0$ modulo q, eventually with the help of exponential sums. Note that the local conditions U_p are responsible of the rather small domain of uniformity in q.

Theorem 2 is the analogue of the Brun–Titchmarsh theorem for primes in arithmetic progressions. Its proof is based on the geometric property of $\mathcal{W}(X)$ to contain rather long segments in c.

The proof of Theorem 3 is independent of Davenport–Heilbronn theory. The key tool is a result due to Mayer ([5], Theorem 1.1), consequence of Hasse's theory. It gives a formula for $\sum_{f' \mid f} m(\Delta f'^2)$, where Δ is a discriminant of a quadratic field, f an integer and $m(\mathfrak{d})$ is the number of elements of $\widetilde{\mathcal{K}}$ with discriminant equal to $\mathfrak{d}$. This formula introduces rather delicate algebraic quantities. However, we deduce a useful lower bound which leads to the question of counting squarefree integers in arithmetic progressions.

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On a Divisor Sum

John B. Friedlander[1]

(joint work with Henryk Iwaniec[2])

We are interested in the divisor sums

$$\sum_{n \leqslant x} a_n \, \tau_k(n)$$

for polynomial sequences a_n. In addition to being of interest on their own these sums arise as a tool in the study of the distribution of primes, for example by means of Linnik's identity [3]:

For each integer $n > 1$,

$$\frac{\Lambda(n)}{\log n} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} t_j(n)$$

where $t_j(n)$ denotes the number of ways of writing n as the ordered product of j integers each being strictly greater than one. These in turn can easily be expressed in terms of the usual divisor functions $\tau_k(n)$. Thus, success with the above divisor sums for a given sequence serves as a measure of our progress toward the goal of an asymptotic formula for counting the primes in that sequence.

Previously in [2] we proved the asymptotic formula for the distribution of prime values of $a^4 + b^2$. The methods suggested that it should become possible, once sufficiently strong bounds were available, to do the same for the prime values of $a^6 + b^2$, which would also then have a nice application to the theory of elliptic curves. It is thus important to study the above divisor sums for this sequence. Here the cases $k = 1$ and $k = 2$ are relatively straight–forward and indeed are already known for the much sparser sequence $1 + b^2$; see for example [1] which has the sharpest results to date on this.

In this talk the main result gives the asymptotic formula for the much more difficult case of τ_3. We have

$$\sum_{\substack{a^6+b^2 \leqslant x \\ (a,b)=1}} \tau_3(a^6 + b^2) = \kappa P x^{2/3} (\log x)^2 + O\left(x^{2/3} (\log x)^{15/8}\right) .$$

Here the constant P is a rather complicated looking Euler product and κ is the integral which counts asymptotically the number of integer points (a, b) in the region under consideration.

It would be possible to evaluate the corresponding sum where the co–primality condition on a with b is dropped but it would be technically more complicated and in any case is irrelevant for the problem of counting primes.

We described a number of subsidiary results used in the derivation of the main theorem, thus suggesting the flow of the proof. We believe these to be of independent interest. In addition we mentioned some other applications of this circle of ideas.

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Counting primes, groups, and manifolds

Dorian Goldfeld[1]

(joint work with Alexander Lubotzky, Nikolay Nikolov and László Pyber)

Let Γ denote the modular group $\mathrm{SL}_2(\mathbb{Z})$ and $C_n(\Gamma)$ the number of congruence subgroups of Γ of index at most n. We prove that

$$\lim_{n\to\infty} \frac{\log C_n(\Gamma)}{(\log n)^2/\log\log n} = \frac{3-2\sqrt{2}}{4}.$$

The proof of this asymptotic formula is achieved by first obtaining a lower bound by counting primes satisfying certain congruence conditions and then obtaining an upper bound by counting subgroups of an abelian group. Remarkably, although the two counting methods are totally different, the same answer is achieved.

The counting methods used in the above result are capable of vast generalization. Consider an absolutely simple, connected, simply connected algebraic group G defined over a number field k and let $\Gamma = G(\mathcal{O}_S)$ (here S denotes a finite set of places, $\mathcal{O}_S$ denotes the ring of S-integers of k). Let h denote denote the Coxeter number of the root system associated to G, and define

$$\gamma(G) = \frac{\left(\sqrt{h(h+2)}-h\right)^2}{4h^2}.$$

The following theorem is proved in [1].

Theorem 1. *Let G, Γ and $\gamma(G)$ be as defined above. Assuming GRH we have*

$$\lim_{n\to\infty} \frac{\log C_n(\Gamma)}{(\log n)^2/\log\log n} = \gamma(G),$$

and moreover, this result is unconditional if G is of inner type (e.g. G splits) and k is either an abelian extension of $\mathbb{Q}$ or a Galois extension of degree less than 42.

The GRH in Theorem 1 refers to the generalized Riemann hypothesis for Artin-Hecke L-functions of number fields. The cases where the theorem can be proved unconditionally make use of the Bombieri-Vinogradov theorem (Riemann hypothesis on the average [2] over arithmetic progressions) and its generalization to number fields [3]

If one considers the simplest possible situation where the counting techniques used in the above theorems apply, then one is led to the following extremal problem in multiplicative number theory.

For $n \to \infty$, let

$$M_1(n) = \max\left\{ \prod_{1 \leqslant i,j \leqslant t} \gcd(a_i, a_j) :\ 0 < t, a_1 < a_2 < \ldots < a_t \in \mathbb{Z},\ \prod_{i=1}^{t} a_i \leqslant n \right\},$$

$$M_2(n) = \max\left\{ \prod_{p,p' \in \mathcal{P}} \gcd(p-1, p'-1) : \mathcal{P} = \text{set of distinct primes where} \prod_{p \in \mathcal{P}} p \leqslant n \right\}.$$

One can prove the following theorem which can be considered as a baby version of Theorem 1 above.

Theorem 2. *Let* $\lambda(n) = (\log n)^2 / \log\log n$. *Then*

$$\liminf_{n\to\infty} \frac{\log M_1(n)}{\lambda(n)} = \limsup_{n\to\infty} \frac{\log M_2(n)}{\lambda(n)} = \frac{1}{4}.$$

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Approximating prime tuples

Daniel Goldston[1]
(joint work with Cem Yalçın Yıldırım)

The twin prime conjecture is a special case of a more general conjecture for prime tuples, namely that the tuple $(n+h_1, n+h_2, \ldots, n+h_k)$ will have primes in every component for infinitely many n, provided the set of shifts $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ does not exclude this possibility by filling every residue class for some prime p. Hardy and Littlewood [3] made the quantitative conjecture that, letting

$$\Lambda(n;\mathcal{H}) \;=\; \Lambda(n+h_1)\Lambda(n+h_2)\cdots\Lambda(n+h_k)$$

with Λ the von Mangoldt function,

$$\sum_{n\leqslant N} \Lambda(n;\mathcal{H}) \;=\; N\big(\mathfrak{S}(\mathcal{H}) + o(1)\big), \quad \text{as} \quad N \to \infty,$$

where

$$\mathfrak{S}(\mathcal{H}) \;=\; \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right),$$

and $\nu_p(\mathcal{H})$ is the number of distinct residue classes modulo p the elements of $\mathcal{H}$ occupy. This conjecture lies very deep.

For a number of years we have been working on approximations for primes and prime tuples which may be applied to the problem of finding small gaps between primes. In [1, 2], based on the formula

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{1}{d} \,,$$

we used the approximation

$$\Lambda_R(n) = \sum_{\substack{d|n \\ d \leqslant R}} \mu(d) \log \frac{R}{d}$$

to define the tuple approximation

$$\Lambda_R(n;\mathcal{H}) = \Lambda_R(n+h_1)\Lambda_R(n+h_2)\cdots\Lambda_R(n+h_k)\,.$$

With this approximation we were able to obtain the result that there are infinitely often two prime numbers within 1/4 of the average spacing between primes.

We have recently found a better approximation. Let

$$P_{\mathcal{H}}(n) = (n+h_1)(n+h_2)\cdots(n+h_k).$$

If the tuple $(n+h_1, n+h_2, \ldots, n+h_k)$ is a prime tuple, then $P_{\mathcal{H}}(n)$ will have k prime factors, which we can detect with the generalized von Mangoldt function

$$\Lambda_k(n) = \sum_{d|n} \mu(d) \left(\log \frac{n}{d}\right)^k$$

which is zero if n has more than k distinct prime factors. Our prime tuple detecting function is

$$\frac{1}{k!}\,\Lambda_k\big(P_{\mathcal{H}}(n)\big)\,,$$

where the normalization by $1/k!$ simplifies the statement of our results, and now we take our new prime tuple approximation to be

$$\Lambda_R(n;\mathcal{H}) = \frac{1}{k!} \sum_{\substack{d|P_{\mathcal{H}}(n) \\ d \leqslant R}} \mu(d) \left(\log \frac{R}{d}\right)^k.$$

Preliminary results indicate this approximation improves the previous results on small gaps between primes, and also simplifies the proofs. There is also the possibility that when utilizing the Elliott–Halberstam conjecture one obtains for the first time primes closer than any small fraction of the average spacing.

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Work in progress on exponential sums and the Riemann zeta function

Martin N. Huxley

In the Bombieri-Iwaniec method for estimating exponential sums

$$\sum_{m=M+1}^{2M} e\left(t\,F\left(\tfrac{m}{M}\right)\right),$$

symmetry prevents cancellation. The symmetries appear as affine maps

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} G \\ H \end{pmatrix}$$

with integer coefficients A, B, C, D, G, and H, preserving the integer lattice, acting on

$$X = TF\left(\tfrac{x}{M}\right), \qquad Y = \frac{T}{2M} F'\left(\tfrac{x}{M}\right).$$

For fixed A, B, C, D, the symmetries are parametrised by integer points close to a certain plane curve, the resonance curve. The resonance curves can now be regarded as translations of plane sections of a certain seven-dimensional manifold in ten-dimensional space. The resonance curve usually has a cusp, so the manifold is not uniformly smooth.

In the special case $F(x) = -\log x$, corresponding to partial sums of the Riemann zeta function in the normalisation $\zeta\left(\frac{1}{2} + 2\pi i T\right)$, the manifold is algebraic. We can state some explicit formulas. We start with the magic matrix, an integer matrix with

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1, \quad A > 0, \quad B < 0, \quad C > 0, \quad D < 0.$$

Now we choose two more integer matrices of determinant one with

$$\begin{pmatrix} f' & e' \\ s' & r' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} f & e \\ s & r \end{pmatrix},$$

where

$$r,\, s,\, r',\, s' > 0,$$

$$\frac{e}{r} \leqslant \frac{-BC + \sqrt{-BC}}{AC} \leqslant \frac{f}{s},$$

$$\frac{e'}{r'} \leqslant \frac{-BC - \sqrt{-BC}}{-CD} \leqslant \frac{f'}{s'}.$$

This data determines the resonance curve, and in particular, the position of its cusp. We want a bound on how often the cusp of the resonance curve is close to an integer point. We have

$$\frac{r'}{r} = \frac{(1+\Delta)\big(\sqrt{-BC}+1\big)}{A}, \qquad \frac{s'}{s} = \frac{(1+E)\big(\sqrt{-BC}+1\big)}{A},$$

where Δ and E are small. Modulo a shift by an integer vector, the cusp occurs at a point (y,z) where

$$y = 3\,\Delta r\sqrt{\frac{2T(-B)(\sqrt{-BC}+1)}{A\sqrt{-BC}}}\,,$$

$$z = 3\,Es\sqrt{\frac{2T(-B)(\sqrt{-BC}+1)}{A\sqrt{-BC}}}\,.$$

An obvious difficulty is that T is a very large parameter, and the typical sizes for the matrix entries are small powers of T, around $T^{1/7}$. The coordinates of the resonance curve have order of magnitude $\sqrt{T}$, very large, but smaller than T. The resonance curves belong to the dignified arithmetic large-scale structure, not to the volatile short-scale structure which can be modelled by random matrices. This enables us to prove short interval means of the type

$$\int_{T-U}^{T+U} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt = O\left(T^{89/285}U^{5/19}\log^2 T\right)$$

for $U = O\left(T^{1/2004}\right)$; the exponents are by way of illustration using the results of “Area, Lattice Points and Exponential Sums”, Oxford 1996, not the latest form of the method, to appear in Proc. London Math. Soc. in 2004/5.

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The Riemann zeta function and the divisor problem

Aleksandar Ivić

Let $d(n)$ denote the number of divisors of n, and let

$$\Delta(x) \;=\; \sum_{n\leqslant x} d(n) - x(\log x + 2\gamma - 1)$$

denote the error term in the Dirichlet divisor problem, and

$$E(T) \;=\; \int_0^T \left|\zeta(\tfrac12+it)\right|^2 dt - T\left(\log\left(\frac{T}{2\pi}\right) + 2\gamma - 1\right)$$

is the error term in the mean square formula for $|\zeta(\frac12+it)|$, where $\gamma = 0.577215\ldots$ is Euler's constant. In view of F.V. Atkinson's explicit formula [1] for $E(T)$, the divisor analogue of $E(T)$ is the function

$$\begin{aligned}\Delta^*(x) &:= \; -\Delta(x) + 2\Delta(2x) - \tfrac12\Delta(4x)\\ &= \tfrac12\sum_{n\leqslant 4x}(-1)^n d(n) - x(\log x + 2\gamma - 1)\,.\end{aligned}$$

M. Jutila [6] proved that

$$\int_0^T \left(E^*(t)\right)^2 dt \;\ll\; T^{4/3}\log^3 T\,, \tag{1}$$

$$E^*(t) :=\; E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right).$$

We present a proof (see [4] for details) of the bound

$$\int_0^T \left(E^*(t)\right)^4 dt \;\ll_\varepsilon\; T^{16/9+\varepsilon}, \tag{2}$$

using (among other things) the recent result of Robert-Sargos [7] that the number of integers $N < n_1, n_2, n_3, n_4 \leqslant 2N$ such that $\left|\sqrt{n_1}+\sqrt{n_2}-\sqrt{n_3}-\sqrt{n_4}\right| < \delta\sqrt{N}$ is $\ll_\varepsilon N^\varepsilon(N^4\delta + N^2)$ $(\delta > 0)$. It is indicated that, by similar methods, one can also prove the new result

$$\int_0^T \left|E^*(t)\right|^5 dt \;\ll_\varepsilon\; T^{2+\varepsilon}. \tag{3}$$

Note that (1) and (3) yield, by Hölder's inequality, the bound (2). We also prove that

$$\sum_{r=1}^R\left(\int_{t_r-G}^{t_r+G}\left|\zeta(\tfrac12+it)\right|^2 dt\right)^4 \ll_\varepsilon T^{2+\varepsilon}G^{-2} + RG^4T^\varepsilon,$$

provided that $T^{1/5+\varepsilon} \leqslant G \ll T, T < t_1 < \cdots < t_R \leqslant 2T, t_{r+1} - t_r \geqslant 5G$ $(r = 1,\cdots,R-1)$ (see [3] for similar results). This bound yields a new proof of Heath-Brown's result [2] that $\int_0^T \left|\zeta(\frac12+it)\right|^{12} dt \ll_\varepsilon T^{2+\varepsilon}$. Power moments of

$$J_k(t,G) = \frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty}|\zeta(\tfrac12+it+iu)|^{2k}\mathrm{e}^{-(u/G)^2}\,du \qquad (t \asymp T,\; T^\varepsilon \leqslant G \ll T),$$

where k is a natural number, are investigated. The results that are obtained are used to show how bounds for $\int_0^T|\zeta(\frac12+it)|^{2k}\,dt$ may be obtained. In particular, by employing the foregoing method, it is proved that $\int_T^{2T} J_1^m(t,G)\,dt \ll_\varepsilon T^{1+\varepsilon}$ for $T^\varepsilon \leqslant G \leqslant T$ if $m = 1, 2$; for $T^{1/7+\varepsilon} \leqslant G \leqslant T$ if $m = 3$, and for $T^{1/5+\varepsilon} \leqslant G \leqslant T$ if $m = 4$.

Finally, we present the recent results obtained jointly with P. Sargos [5] on higher moments of $\Delta(x)$:

$$\int_1^X \Delta^3(x)\,dx \;=\; BX^{7/4} + O_\varepsilon\big(X^{7/5+\varepsilon}\big)\,,$$
$$\int_1^X \Delta^4(x)\,dx \;=\; CX^2 + O_\varepsilon\big(X^{23/12+\varepsilon}\big)\,,$$

where $B, C > 0$, improving on the exponents 45/23 and 47/28, respectively, of K.-M. Tsang [8].

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A uniform bound for Hecke L-functions

Matti Jutila[1] and Yoichi Motohashi

Let

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s}$$

be the Hecke L-function attached to the j-th Maass form ψ_j . Thus the coefficients $t_j(n)$ are the corresponding eigenvalues of the Hecke operators $T(n)$. The form ψ_j is also an eigenfunction of the hyperbolic Laplacian, and we write the eigenvalue as $1/4 + \kappa_j^2$. In this talk, a proof of the uniform estimate

$$H_j(\tfrac{1}{2} + it) \ll (t + \kappa_j)^{1/3+\varepsilon}, \quad t \geqslant 0\,,$$

is outlined. This is not entirely new; in fact, this was shown for $t \gg \kappa_j^3$ by T. Meurman [4] and for $t = 0$ by A. Ivić [1], and we settled earlier the case $0 \leqslant t \ll \kappa_j^{2/3}$ (see [3]). The whole assertion follows immediately from the following theorems (where $\alpha_j = |\rho_j(1)|^2/\cosh(\pi\kappa_j)$ in the standard notation of spectral theory; see e. g. [5]).

Theorem 1. *Let K be large and*

$$G = (K+t)^{4/3}K^{-1+\varepsilon}, \quad 0 \leqslant t \ll K^{3/2-\varepsilon}.$$

Then

$$\sum_{K\leqslant\kappa_j\leqslant K+G} \alpha_j|H_j(\tfrac{1}{2}+it)|^4 \ll GK^{1+\varepsilon}.$$

Theorem 2.

$$\sum_{K\leqslant\kappa_j\leqslant K+G} \alpha_j|H_j(\tfrac{1}{2}+it)|^2 \ll (GK+t^{2/3})^{1+\varepsilon}, \quad t\geqslant 0,\ 1\leqslant G\leqslant K.$$

In the proof of Theorem 1, the original spectral sum is first transformed into an arithmetic form involving Kloosterman sums by the Bruggeman–Kuznetsov sum formula, and after a Voronoi transformation we end up with the additive divisor problem. This leads back to spectral theory by an identity due to the second named speaker, and the new spectral sum is estimated either by the spectral large sieve or by the following "hybrid" mean value estimate.

Theorem 3.

$$\sum_{K\leqslant\kappa_j\leqslant 2K} \alpha_j H_j^2(\tfrac{1}{2})|H_j(\tfrac{1}{2}+it)|^2 \ll (K^2+t^{4/3})^{1+\varepsilon}.$$

The proof is analogous to that of Theorem 1 except that, in connection with the additive divisor problem, the contribution of the holomorphic cusp forms is comparable with that of the Maass forms, a new phenomenon in applications of the spectral theory.

The argument of the proof of Theorem 2 (see [2]) is somewhat different from those sketched above, for the additive divisor problem plays now no role. Instead, we need some other devices such as estimates for exponential sums.

The proof of Theorem 1 applies, with minor modifications, to the estimation of the Rankin zeta function

$$L(s,\psi\otimes\psi_j) = \zeta(2s)\sum_{n=1}^{\infty} t_\psi(n)t_j(n)n^{-s},$$

where ψ is a fixed cusp form, holomorphic or real analytic. We get a spectral mean square estimate which implies the bound

$$L(\tfrac{1}{2}+it,\psi\otimes\psi_j) \ll \kappa_j^{2/3+\varepsilon}, \quad 0\leqslant t\ll\kappa_j^{2/3}.$$

An analogous but weaker "subconvexity estimate" has been obtained by P. Sarnak [6], for fixed t.

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A refined nonvanishing theorem for automorphic L-functions on $\mathbf{GL}(\boldsymbol{n}, \boldsymbol{A_Q})$

Wenzhi Luo[1]

Let F be a number field, π an irreducible cuspidal automorphic representation of $\mathrm{GL}(n, \boldsymbol{A}_F)$, let S be a finite set of places of F, and let $\beta \in \mathbb{C}$. One asks whether there always exist infinitely many primitive ray class characters ψ of F such that ψ is unramified at the places in S and the twisted standard L-function $L(\beta, \pi \otimes \psi) \neq 0$. When $n = 1$ or $n = 2$ the answer to this question is affirmative, in view of the works of Goldfeld–Hoffstein–Patterson [3] and Rohrlich [9] respectively, see also [2] and [8]. For $n \geqslant 3$, Barthel and Ramakrishnan [1] proved the same nonvanishing result under the condition $\Re(\beta) \notin [1/n,\ 1-1/n]$.

In the current work, we show that if the base field $F = \mathbb{Q}$, then the answer to the above question is affirmative for $n = 3$. Moreover for $n \geqslant 4$, the same nonvanishing theorem is true under the condition $\Re(\beta) \notin [2/n,\ 1-2/n]$.

Our improvement results from estimating the second moment of the dual sums in the approximate functional equations, and it is inspired by the work [4]. This type of nonvanishing theorems are closely related to the Selberg eigenvalue conjecture and the generalized Ramanujan conjecture, see [5], [6] and [7].

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The sum of the Möbius function

Helmut Maier
(joint work with Hugh L. Montgomery[1])

Let
$$M(x) = \sum_{n\leqslant x} \mu(n)\,.$$

Several authors have given estimates for $M(x)$ on assumption of the Riemann Hypothesis using the method of complex integration.

In 1912 Littlewood [2] proved that $M(x) \ll x^{1/2+\varepsilon}$. Landau [1] and Titchmarsh [4] then replaced ε by a function $\varepsilon(x)$. Titchmarsh in 1927 showed that $\varepsilon(x) = 1/\log\log x$ is admissible. The present authors improve on this by showing that $\varepsilon(x) = 1/(\log x)^{22/61}$ is admissible.

The improvement is due to the choice for the path of integration. In all cases the crucial estimate is that of
$$\int_{\mathcal{C}} \frac{x^s}{s\,\zeta(s)}\,ds$$
where $\mathcal{C}$ is a curve close to the critical line.

The choice of the earlier authors for $\mathcal{C}$ was that of a simple curve resembling a straight line. The present authors choose a piecewise linear path. The distance of the vertical pieces from the critical line is large in regions where $|\zeta'(s)/\zeta(s)|$ assumes large values and small otherwise. The frequency of the occurrence of large values is determined by a method inspired by work of Selberg [3].

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A Burgess-like bound for twisted L-functions

Philippe Michel

(joint work with Valentin Blomer and Gergely Harcos)

Here we discuss a subconvexity problem for L-functions of a modular form twisted by a character: namely we prove the following

Theorem. *Let f be an arbitrary modular form (either holomorphic or Maass) of level D and some nebentypus, and let χ be a primitive character of conductor q. Then for $\Re s = 1/2$ and any $\varepsilon > 0$*

$$L(f \otimes \chi, s) \ll_{\varepsilon,f} q^{1/2-(1-2\theta)/8}.$$

Here θ denotes any approximation towards the Ramanujan–Petersson conjecture (the current best value of θ is $7/64$) and the dependancy is at most polynomial in the parameters of f.

The method of proof builds on the original amplification method of Duke–Friedlander–Iwaniec, the Vonoroi summation formulae, Jutila's variant of the δ-symbol method and on the large sieve inequalities of Deshouillers–Iwaniec.

Changes of sign of $\psi(x) - x$

Hugh L. Montgomery[1]

(joint work with Ulrike M. A. Vorhauer[2])

We assume the Riemann Hypothesis (RH) throughout. It is classical that there is an absolute constant $C > 1$ such that $\psi(x) - x$ changes sign in every interval $[x, Cx]$ for $x \geqslant 1$. Since $\psi(x) < x$ for $1 \leqslant x < 19$, it is clear that $C \geqslant 19$, and it is likely that one could show that $C = 19$ works for all $x \geqslant 1$. Our interest is in the limit of constants C that work for all $x \geqslant x_0(C)$.

First we consider the classical argument. Put

$$f_m(y) = -\sum_{\rho} \frac{e^{i\gamma y}}{\rho(i\gamma)^m}$$

where ρ runs over all the trivial non-trivial zeros of $\zeta(s)$. From the explicit formula for $\psi(x)$ we know that

$$\frac{\psi(e^y)-e^y}{e^{y/2}} = f_0(y) + o(1)\,.$$

Let $0 < \gamma_1 \leqslant \gamma_2 \leqslant \cdots$ denote the ordinates of the zeros of the zeta function in increasing order, and set $\rho_j = 1/2 + i\gamma_j$. We find that

$$\frac{1}{|\rho_1|\gamma_1^m} < \sum_{j=2}^{\infty} \frac{1}{|\rho_j|\gamma_j^m}$$

for $m = 1, 2$ but that

$$\frac{1}{|\rho_1|\gamma_1^3} > \sum_{j=2}^{\infty} \frac{1}{|\rho_j|\gamma_j^3}\,.$$

Let θ be determined, $0 \leqslant \theta < 2\pi/\gamma_1$, so that

$$\frac{e^{i\gamma_1\theta}}{\rho_1(i\gamma_1)^3} < 0\,,$$

and put $y_r = r\pi/\gamma_1 + \theta$. Thus $f_3(y_{2r}) > 0$ and $f_3(y_{2r+1}) < 0$, and moreover the quantities $|f_3(y_r)|$ are bounded away from 0. Hence $f_3(y_{r+3}) - 3f_3(y_{r+2}) + 3f_3(y_{r+1}) - f_3(y_r)$ is large in one sign, and $f_3(y_{r+4}) - 3f_3(y_{r+3}) + 3f_3(y_{r+2}) - f_3(y_{r+1})$ is large in the opposite sign. It follows that $f_0(y)$ takes both (relatively) large positive and negative values in an interval of the form $[y_r, y_{r+4}]$. Since any interval $[y, y+5\pi/\gamma_1]$ contains a subinterval of this form, it follows that $f_0(y)$ takes large positive and negative values in any interval of length at least $5\pi/\gamma_1 = 1.1113\ldots$. Consequently, any $C > \exp(5\pi/\gamma_1) = 3.038\ldots$ suffices, for all sufficiently large x.

We propose now a new method by which the above bound can be improved. Let $k(y)$ be an even, nonnegative function of $L^1(\mathbb{R})$ with support contained in the interval $[-\alpha, \alpha]$ for some $\alpha > 0$. The sum $f_0(y)$ has a logarithmic singularity at $y = 0$, but for $y > 0$ it is uniformly convergent except in the neighborhood of the logarithm of a prime power, where it is boundedly convergent. Thus we may integrate term-by-term to see that

$$\int_{-\alpha}^{\alpha} k(y) f_0(Y+y)\,dy = -\sum_{\rho} \frac{e^{i\gamma Y}}{\rho}\widehat{k}\Big(\frac{-\gamma}{2\pi}\Big)\,. \tag{1}$$

We want k to be sufficiently smooth so that the right hand side is absolutely convergent, and even more, we want

$$\Big|\frac{\widehat{k}(\gamma_1/(2\pi))}{\rho_1}\Big| > \sum_{j=2}^{\infty} \Big|\frac{\widehat{k}(\gamma_j/(2\pi))}{\rho_j}\Big|\,. \tag{2}$$

We now redefine θ to fit the new situation: We choose θ, $0 \leqslant \theta < 2\pi/\gamma_1$, so that

$$\frac{e^{i\gamma_1\theta}}{\rho_1}\,\widehat{k}(-\gamma_1/(2\pi)) < 0\,.$$

We take $y_r = \pi r/\gamma_1 + \theta$ as before, but now with this new θ. From (1) and (2) it follows that $f_0(y)$ takes a (relatively) large value of one sign in an interval $[y_r - \alpha, y_r + \alpha]$, and a value of the opposite sign in $[y_{r+1} - \alpha, y_{r+1} + \alpha]$. Hence $f_0(y)$ takes positive and negative values, bounded away from 0, in any interval of the form $[y_r - \alpha, y_{r+1} + \alpha]$. Since any interval $[y, y + 2\pi/\gamma_1 + 2\alpha]$ contains such an interval, it follows that f_0 changes sign in any interval of length $> 2\pi/\gamma_1 + 2\alpha$.

We have not yet tried to optimize the choice of the kernel k, but even the simple choice $k(y) = \max(0, 1 - |y|)$ allows us to reduce the constant $1.1113\ldots$ to a value near 0.75. It is possible that with a better choice of the kernel, one might achieve a value nearer 0.5.

Concerning the limitations of our approach we make the following observation: Our argument would apply equally well to the function

$$f(y) = \frac{\cos\gamma_1 y}{|\rho_1|} - \frac{\cos\gamma_2 y}{|\rho_2|} - \frac{\cos\gamma_3 y}{|\rho_3|} + \frac{\cos\gamma_4 y}{|\rho_4|} + \frac{\cos\gamma_5 y}{|\rho_5|} - \frac{\cos\gamma_6 y}{|\rho_6|}.$$

Since this function is positive for $-0.197 \leqslant y \leqslant 0.197$, it follows that our method is incapable of reaching the value 0.394.

The function $f_0(y)$ becomes increasingly erratic as y increases, and it is unclear to us whether one would expect the gaps between sign changes to tend to 0 or not. Some good heuristics on this point would be valuable.

We close with a remark concerning computation. Odlyzko has placed the first 10^4 ordinates γ_j, to 9-digit accuracy, on his web page. Of course many more zeros beyond that are known to lie on the critical line. To supplement this, it is useful to note that certain sums over zeros can be computed to high accuracy, without having to compute the zeros. For example,

$$\left(\frac{\xi'}{\xi}\right)'(s) = -\sum_\rho \frac{1}{(s-\rho)^2}. \tag{3}$$

Since $\xi(s)$ and its derivatives are easily computed by the Euler–Maclaurin sum formula, we find that

$$\sum_\rho \frac{1}{(\rho - 1/2)^2} = -0.04620998\ldots.$$

By repeatedly differentiating (3) we can similarly evaluate any sum of the form $\sum_\rho (\rho - 1/2)^{-2m}$, $m = 1, 2, 3, \ldots$. Moreover, this evaluation is unconditional.

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On a property of the multiplicative order of a (mod p)

Leo Murata[1]
(joint work with Koji Chinen and Carl Pomerance)

For a fixed natural number $a \geqslant 2$, let $D_a(p)$ denote the residual order of a in $(\mathbb{Z}/p\mathbb{Z})^*$. $\mathbb{P}$ denotes the set of all prime numbers, and $\mathbb{N}$ denotes the set of all natural numbers. It is known that the map D_a fluctuates quite irregularly, and at the same time, this funtion is (almost) surjective from $\mathbb{P}$ to $\mathbb{N}$. In order to study the property of the function D_a more closely, we consider two types of sets:

$$Q_a(x;k,\ell) = \{p \leqslant x\colon\ p \in \mathbb{P},\ D_a(p) \equiv \ell \,(\mathrm{mod}\ k)\}, \quad 0 \leqslant \ell < k \in \mathbb{N},$$

$$M_2(x) = \{p \leqslant x\colon\ p \in \mathbb{P},\ D_2(p) \in \mathbb{P}\}.$$

The natural density of $Q_a(x;k,\ell)$ (joint work with K. Chinen)

First we can prove the existence of the natural density of $Q_a(x;k,j)$ for general residue classes:

Theorem 1. ([1]–[4]) *We assume the Generalized Riemann Hypothesis (GRH), and assume a is not a perfect b-th power with $b \geqslant 2$. Then, for any residue class $j\,(\mathrm{mod}\ k)$, the set $Q_a(x;k,j)$ has the natural density $\Delta_a(k,j)$, and the values of $\Delta_a(k,j)$ are effectively computable.*

Moreover, we can prove some number theoretical properties of $\Delta_a(k,\ell)$ as a number theoretical function of k and ℓ.

Theorem 2. ([2]) *(equi-distribution property) We assume GRH.*
(I) *If q is an odd prime and $r \geqslant 2$, then for an arbitrary j, we have*

$$\Delta_a(q^r,j) = \frac{1}{q}\,\Delta_a(q^{r-1},j).$$

(II) *If $q = 2$ and $r \geqslant 4$, then for any j, we have the same relation.*

It seems an interesting phenomenon that, for the remaining cases — when r is "very small" — we actually find some irregularity. Here we remark that $\Delta_a(k,\ell)$ does not have "multiplicativity", so it seems difficult to obtain a explicit formula for a general value of $\Delta_a(k,j)$.

An estimate for $\#M_2(x)$ (joint work with C. Pomerance)

Here we take $a = 2$. On the cardinality of the set $M_2(x)$, Pomerance already proved

Theorem 3. ([6]) *We have unconditionaly*

$$\#M_2(x) \ll \pi(x)\frac{\log\log\log x}{\log\log x},$$

and under GRH,

$$\#M_2(x) \ll \pi(x)\frac{\log\log x}{\log x}\,.$$

We can improve the latter estimate as follows:

Theorem 4. ([5]) *We assume GRH. Then we have*

$$\#M_2(x) \ll \pi(x)\frac{1}{\log x}\,.$$

Here we remark that, this estimate seems to be *best possible*. In fact, let us consider the set

$$L(x) = \left\{ p \leqslant x\colon\ \frac{p-1}{2} \ \text{ is also prime, } \ p \equiv 7 \,(\mathrm{mod}\ 8) \right\}.$$

Then, it is easy to see that $L(x) \subset M_2(x)$, and it is (not yet proved but) conjectured that

$$\#L(x) \sim C\ \pi(x)\,\frac{1}{\log x}$$

with a strictly positive constant C, which gives a lower bound of $\#M_2(x)$.

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Pair correlation and the Chebotarev Density Theorem

V. Kumar Murty[1]

(joint work with M. Ram Murty)

In this recent joint work we study a non-Abelian pair correlation hypothesis and its implication for the distribution of primes.

Let K/F be a finite Galois extension of number fields with group G. For each character χ of G, we have the associated Artin L-function $L(s,\chi)$. It is defined by an absolutely convergent Euler product for $\Re(s) > 1$ and has a continuation as a meromorphic function of s. Artin's holomorphy conjecture (AC) asserts that

$L(s,\chi)$ is entire except possibly for a pole at $s = 1$ of order $< \chi, 1_G >$ the multiplicity of the trivial character in χ.

Let us set

$$d_\chi \;=\; \chi(1)[F:\mathbb{Q}]$$

and

$$A_\chi \;=\; d_F^{\chi(1)\mathbf{N}\mathfrak{f}_\chi}$$

the Artin conductor of χ. Let us set

$$w(u) = \frac{4}{4+u^2}\,.$$

Assuming the GRH (that is, the Riemann Hypothesis for all Dedekind zeta functions), define

$$P_T(Y,\chi) \;=\; \sum_{-T\leqslant\gamma_1,\gamma_2\leqslant T} w(\gamma_1-\gamma_2)\exp\big(2\pi i(\gamma_1-\gamma_2)Y\big)\,.$$

The pair correlation hypothesis (PC) in this context is the statement that for

$$0 < Y \leqslant A d_\chi \log T,$$

we have the bound

$$P_T(Y,\chi) \;\ll_A\; T(\log A_\chi + d_\chi \log T)\,.$$

We apply this hypothesis to obtaining a sharper error term for the Chebotarev Density Theorem. Let $C \subseteq G$ be a conjugacy set (i.e. a union of conjugacy classes) of G. Define

$$\pi_C(x) \;=\; \#\big\{\mathfrak{p} \text{ prime of } F:\ \mathbb{N}_{F/\mathbb{Q}}\mathfrak{p} \leqslant x,\ (\mathfrak{p}, K/F) \subseteq C\,\big\}\,.$$

Then the Chebotarev Density Theorem asserts that

$$\pi_C(x) \;\sim\; \frac{|C|}{|G|}\,\pi_F(x)$$

where $\pi_F(x)$ denotes the number of prime ideals of F of norm less than or equal to x. This can be made effective with an explicit error term that depends on various parameters of the fields (and of course on x). In applications, the theorem is often applied to a family of fields and therefore, it is necessary that the implied constants in any error estimate be absolute and that attention be paid to making the dependence of the error term on field constants as optimal as possible.

Assuming AC, GRH and PC, we show that

$$\Big|\pi_C(x) \;-\; \frac{|C|}{|G|}\,\pi_F(x)\Big| \;\ll\; n_F^{1/2}|C|^{1/2}\Big(\frac{|G^\#|}{|G|}\Big)^{1/4} x^{1/2}\log M(K/F)\,x$$

where $G^\#$ denotes the number of conjugacy classes of G and $M(K/F)$ is a "harmless" term depending on the primes of F that are ramified in K.

Observe that the quotient $|G|/|G^\#|$ is the average size of a conjugacy class. Observe also that if G is Abelian, this quotient is 1.

This result has many applications, but we shall indicate two. First, we discuss Artin's primitive root conjecture. For an integer $a \neq \pm 1$, and not a square, denote by $N_a(x)$ the number of primes $p \leqslant x$ for which a is a primitive root modulo p. Hooley showed that assuming the GRH,

$$N_a(x) = c(a)\frac{x}{\log x} + O\Big(x\frac{\log\log x}{(\log x)^2}\Big).$$

Using the GRH and PC (in this case, AC is known), we show that

$$N_a(x) = c(a)\,\mathrm{Li}\,x + O\big(x^{10/11}(\log x)^2(\log a)\big).$$

Thus, we save a power of x in the error term.

Second, we consider one of the Lang–Trotter problems. Let f be a holomorphic cusp form of weight $k \geqslant 2$, level N that is a normalized eigenform for the Hecke operators. Assume that the Fourier coefficients of f are rational integers and denote by $a_f(n)$ the n-th Fourier coefficient. Set

$$\pi_{f,a}(x) = \#\big\{p \leqslant x\colon\ a_f(p) = a\big\}.$$

We prove that for $a \neq 0$, we have

$$\pi_{f,a}(x) \ll x^{3/4}(\log Nx)^{1/2}.$$

This improves on all earlier results. We also get a result for $a = 0$.

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The Selberg Class: Linear and non-linear twists

Alberto Perelli[1]

(joint work with Jerzy Kaczorowski[2])

Selberg [10] introduced the following axiomatic class of L-functions, now called the Selberg class $\mathcal{S}$. A function $F(s)$ belongs to $\mathcal{S}$ if

(i) *(ordinary Dirichlet series)* $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is absolutely convergent for $\sigma > 1$;

(ii) *(analytic continuation)* $(s-1)^m F(s)$ is an entire function of finite order for some integer $m \geqslant 0$;

(iii) *(functional equation)* $F(s)$ satisfies a functional equation of type $\Phi(s) = \omega\overline{\Phi}(1-s)$, where $\overline{f}(s) = \overline{f(\overline{s})}$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $r \geqslant 0$, $Q > 0$, $\lambda_j > 0$, $\Re\mu_j \geqslant 0$, and $|\omega| = 1$;

(iv) *(Ramanujan conjecture)* $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$;

(v) *(Euler product)* $\log F(s) = \sum_{n=1}^{\infty} b(n) n^{-s}$ with $b(n) = 0$ unless $n = p^m$ with $m \geqslant 1$, and $b(n) \ll n^{\vartheta}$ for some $\vartheta < 1/2$.

The first three axioms are more of analytic nature, and we denote by $\mathcal{S}^\sharp$ the extended Selberg class of the not identically vanishing functions satisfying axioms *(i) - (iii)*.

Here are few examples of functions (conjecturally) in $\mathcal{S}$: the Dirichlet L-functions $L(s,\chi)$, the Hecke L-functions $L_K(s,\chi)$ and suitably normalized functions $L_f(s)$ associated with modular forms are in $\mathcal{S}$. The Artin L-functions (assuming the Artin conjecture) and the automorphic L-functions (assuming the Ramanujan conjecture) are also in $\mathcal{S}$.

The Structure of $\mathcal{S}$

One of the main problems in the Selberg class theory is: what does $\mathcal{S}$ contain? The Main Conjecture asserts that

$\mathcal{S}$ is equal to the class of automorphic L-functions.

If true, this conjecture is very deep since it morally implies the Langlands' conjectures. Define the *degree* of $F \in \mathcal{S}$ by

$$d_F = 2\sum_{j=1}^{r} \lambda_j \,,$$

and let $\mathcal{S}_d = \{F \in \mathcal{S} : d_F = d\}$. For example $d_\zeta = 1$, $d_{L(\cdot,\chi)} = 1$, $d_{L_f} = 2$, $d_{\zeta_K} = [K:Q]$. Then the Main Conjecture splits as

CONJECTURE 1. (General converse theorem) *For* $d \in \mathbb{N}$

$\mathcal{S}_d$ is equal to the automorphic L-functions of degree d.

CONJECTURE 2. (Degree conjecture) *For* $d \notin \mathbb{N}$

$$\mathcal{S}_d = \emptyset \,.$$

Remark. Conjecture 2 is expected to hold for $\mathcal{S}^\sharp$ as well, but is definitely false if "ordinary Dirichlet series" in axiom *(i)* is replaced by "general Dirichlet series".

Let $D(\lambda,\mu,Q,\omega)$ be the vector space of general Dirichlet series satisfying *(ii)* and *(iii)*.

THEOREM 1. ([8]) *$D(\lambda,\mu,Q,\omega)$ has an uncountable basis.*

Conjectures 1 and 2 are true for $0 \leqslant d < 5/3$. More precisely: $\mathcal{S}_0 = \{1\}$ and $\mathcal{S}_d = \emptyset$ for $0 < d < 1$ (Richert [9], Bochner [1], Conrey-Ghosh [3]; other proofs have been given more recently). $\mathcal{S}_1 = \{L(s+i\theta,\chi)\}$ with χ a primitive Dirichlet character and $\theta \in \mathbb{R}$ (Kaczorowski–Perelli [5]); and $\mathcal{S}_d = \emptyset$ for $1 < d < 5/3$ (Kaczorowski-Perelli [6]).

Linear twists

The main tool for $d \geqslant 1$ are the linear twists

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n\alpha)$$

where $e(x) = e^{2\pi i x}$. In order to study their analytic properties, let $N, \alpha > 0$ and $K \in \mathbb{N}$. By the Mellin transform and the functional equation we get

$$F_N(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}\, e(-n\alpha)\, e^{-n/N}$$
$$= R_N(s,\alpha) + \omega Q^{1-2s} \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{1-s}} H_K\Big(\frac{n}{Q^2(\frac{1}{N}+2\pi i\alpha)}, s\Big)$$

where

$$H_K(z,s) = \frac{1}{2\pi i} \int_{(-K-\frac{1}{2})} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s)+\overline{\mu}_j-\lambda_j w)}{\Gamma(\lambda_j s+\mu_j+\lambda_j w)} \Gamma(w)\, z^w\, dw$$

are hypergeometric functions. For s fixed, such functions were studied by Braaksma [2]. Their behaviour depends on the value of

$$\mu = 2\sum_{j=1}^{r} \lambda_j - 1 = d_F - 1\,;$$

$\mu = 0$ ($d_F = 1$) is a simpler, while $\mu > 0$ ($d_F > 1$) is more complicated due to the presence of the "exponential part". Hence, development of a two-variables theory is required; since $N \to \infty$, the main interest is for $H_K(-iy, s)$ with $y = n/(2\pi Q^2\alpha)$. Define the conductor q_F and the shift θ_F by

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j} \quad \text{and} \quad \theta_F = \Im\Big(2\sum_{j=1}^{r}(\mu_j - \tfrac{1}{2})\Big)\,.$$

Moreover, the critical value is defined by $n_\alpha = q_F d_F^{-d_F} \alpha^{d_F}$, and we write $a(n_\alpha) = 0$ if $n_\alpha \notin \mathbb{N}$. Further, let

$$\kappa = \frac{1}{d-1}\,, \qquad A = (d-1)\, q_F^{-\kappa}\,, \qquad s^* = \kappa(s + \frac{d}{2} - 1 + i\theta_F)\,.$$

Then the properties of the linear twists for $1 \leqslant d < 2$ are summarized by

Theorem 2. ([5]) *Let $F \in \mathcal{S}_1^\sharp$ and $\alpha > 0$. Then $F(s,\alpha)$ is entire if $a(n_\alpha) = 0$, while if $a(n_\alpha) \neq 0$ then $F(s,\alpha)$ has at most simple poles at $s_k = 1 - k - i\theta_F$ $(k = 0, 1, \ldots)$ with non-vanishing residue at $s = s_0$.*

Theorem 3. ([6]) *Let $1 < d < 2$, $F \in \mathcal{S}_d^\sharp$, and $\alpha > 0$. Then*

$$F(s,\alpha) \;=\; e^{as+b} \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{s^*}}\, e\big(A(n/\alpha)^\kappa\big) + G(s,\alpha)$$

where a, b are suitable constants, and $G(s,\alpha)$ is holomorphic for $\sigma^* > \sigma_a(F) - \kappa$.

Note that $\sigma^* > \sigma$ for $\sigma > 1/2$ and $1 < d < 2$. This is important, and immediately proves the non-existence of polar functions in the range $1 < d < 2$.

Non-linear twists

For $F \in \mathcal{S}_d^\sharp$ with $d > 0$ consider the non-linear twists

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}\, e\big(-n^{1/d}\alpha\big) \qquad (\alpha > 0)\,.$$

It turns out that Theorem 2 is a special case of the following general result for non-linear twists.

THEOREM 4. ([7]) *Let $d > 0$, $F \in \mathcal{S}_d^\sharp$ and $\alpha > 0$. Then $F(s,\alpha)$ is entire if $a(n_\alpha) = 0$, while if $a(n_\alpha) \neq 0$ then $F(s,\alpha)$ has at most simple poles at $s_k = \frac{d+1}{2d} - \frac{k}{d} - i\frac{\theta_F}{d}$ $(k = 0, 1, ...)$, with non-vanishing residue at $s = s_0$.*

Bounds on vertical strips, uniform for $F(s)$ in suitable families $\mathcal{F}$ (roughly, bounded degree and μ-coefficients) can also be obtained. We conclude with an application of Theorem 4; see [7] for other applications.

For $\phi(u)$ smooth with compact support and $F \in \mathcal{S}_d^\sharp$ consider the non-linear exponential sum

$$S_F(x,\alpha) = \sum_{n=1}^{\infty} a(n) e\big(-n^{1/d}\alpha\big)\phi(n/x)\,.$$

Then an asymptotic expansion of type

$$S_F(x,\alpha) = \sum_k c_k(F,\alpha)\, x^{s_k} + O(x^{-A})$$

can be obtained. This extends and improves results by Iwaniec-Luo-Sarnak [4] for GL_2 L-functions, obtained by a different method. Uniform versions are also obtainable.

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On a sum involving derivative of $\zeta(s)$ over simple zeros

Ayyadurai Sankaranarayanan
(joint work with Maubariz Garaev)

For any positive integer m let $\zeta^{(m)}(s)$ denote the m^{th} derivative of $\zeta(s)$, $N^{(1)}(T)$ and $N(T)$ denote the number of simple and the total number of zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ in the rectangle $0\leqslant\beta\leqslant 1, 0<\gamma<T$ respectively. The Riemann Hypothesis (RH) asserts that all the non-trivial complex zeros of $\zeta(s)$ are on the critical line $\Re s=1/2$.

On p. 374 of Titchmarsh's book [12] revised by D. R. Heath-Brown, it is proved that the series

$$\sum\left|\rho\zeta^{(1)}(\rho)\right|^{-1} \tag{1}$$

diverges, assuming RH, and that all the zeros of $\zeta(s)$ are simple. In [7], S. M. Gonek studies the asymptotic formula for the quantity

$$\sum_{0<\gamma<T}\zeta^{(\mu)}\left(\rho+i\alpha L^{-1}\right)\zeta^{(\nu)}\left(1-\rho-i\alpha L^{-1}\right)$$

(with $L=\frac{1}{2\pi}\log\frac{T}{2\pi}$, $|\alpha|\leqslant L/2$). Assuming RH, one deduces from his asymptotic formula

$$\sum_{0<\gamma<T}\left|\zeta^{(\mu)}(\frac{1}{2}+i\gamma)\right|^2\sim N(T)\Big(\frac{\mu}{\mu+1}\Big)^2\bigg(\frac{1}{T}\int_0^T\left|\zeta^{(\mu)}(\frac{1}{2}+it)\right|^2dt\bigg). \tag{2}$$

One can combine (2) for $\mu=1$ with Heath-Brown's estimate [8] (see also [4] and [9])

$$N^{(1)}(T)\geqslant\frac{1}{3}N(T)$$

or even with the better estimate of Conrey [3]

$$N^{(1)}(T)\geqslant\frac{2}{5}N(T),$$

assuming RH, to obtain the estimate

$$\sum_{|\gamma|\leqslant T}^{\star}\left|\rho\zeta^{(1)}(\rho)\right|^{-1}\gg(\log T)^{1/2} \tag{3}$$

where the star means that the summation is taken over the simple zeros only. Estimate (3) was improved in [5, 6] to the unconditional result

$$\sum_{|\gamma|\leqslant T}^{\star}\left|\rho\zeta^{(1)}(\rho)\right|^{-1}\gg(\log T)^{3/4}.$$

From the arguments of [6] it follows that

$$\sum_{|\gamma|\leqslant T}^{\star} \left|\zeta^{(1)}(\rho)\right|^{-1} \gg T(\log T)^{-1/4}.$$

We discussed two methods of proving the following result:

Theorem. *If all the zeros of $\zeta(s)$ are simple, then the estimate*

$$\sum_{|\gamma|\leqslant T} \left|\zeta^{(1)}(\rho)\right|^{-1} \gg T \tag{4}$$

holds and we do not need the Riemann Hypothesis to uphold the above inequality. Here the sum runs over all the zeros of the Riemann zeta function.

Method 1. There are two central ideas: The first is to use Perron's formula with small values of x in contrast to the usual usage of it with large values of x. The second idea is the crucial application of a theorem of K. Ramachandra and A. Sankaranarayanan (see [10]) which is an unconditional variant of a conditional result of J. E. Littlewood.

Method 2. There is a direct approach to this problem suggested by H. L. Montgomery. This method is quite simple and elegant through which we can avoid completely the usage of Perron's formula in this situation.

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Siegel zeros and number fields

Harold M. Stark

A CM field K is a totally complex quadratic extension of a totally real field. Improvements in the Brauer–Siegel theorem [1], [2] in the 1970's led to the conjecture that there are only finitely many CM fields of any fixed class-number and also proofs of this conjecture under GRH, thus no Siegel zeros, under Artin's conjecture on entire L-functions, and unconditionally when $K/\mathbb{Q}$ is normed. In 2000, it was shown that $h(K) \to \infty$ under the modified generalized Riemann hypothesis (MGRH): If $\zeta_K(\beta+i\gamma)=0$ then either $\beta=1/2$ or $\gamma=0$. Thus, MGRH allows Siegel zeros to exist. This means the obstruction lies in the possible existence of complex zeros of $\zeta_K(s)$ with $\beta>1/2$ or $\gamma\neq 0$.

The question naturally arises as to where the zeros are and how many there are. On investigating this, I have come across a very interesting series of related harmonic analysis questions of which I give two here. In both cases we are dealing with a generalized trigonometric polynomial

$$T_N(x)=\sum_{n=1}^{N} a_n\cos(\theta_n x)$$

where $\theta_1,\dots,\theta_N$ are real and $a_1,\dots,a_N$ are positive real numbers.

Question 1 (The unrestricted case). Suppose M is a large positive real number and that $T_N(x)\leqslant 0$ for $0<X_1\leqslant x\leqslant MX_1$. Find a lower bound for N in terms of M.

Question 2 (The restricted case). We now assume each a_n is in the range $0<a_n\leqslant 1$ and that $T_N(x)\leqslant -1/2$ for $0<X_1\leqslant x\leqslant MX_1$. Again, find a lower bound for N in terms of M.

We have essentially the best possible answer for Question 2:

$N>M/L$ where L is a power of $\log M$, and examples exist with $N<ML$.

For Question 1, the best I can prove so far is $N>M^{1/2}/L$.

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Joint universality of *L*-functions

Jörn Steuding

In 1975 Voronin [4] proved a remarkable analytical property of the Riemann zeta function. Roughly speaking, Voronin's universality theorem states that any non-vanishing analytic function can be approximated uniformly by certain purely imaginary shifts of the zeta function $\zeta(s)$ in the critical strip. More precisely, let $0 < r < 1/4$ and suppose that $g(s)$ is a non-vanishing continuous function on the disc $|s| \leqslant r$ which is analytic in the interior. Then, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \max_{|s|\leqslant r} \left|\zeta\left(s + 3/4 + i\tau\right) - g(s)\right| < \varepsilon\right\} > 0 .$$

Meanwhile, it is known that there exists a rich zoo of Dirichlet series having some universality property. Recently, Steuding [3] proved universality for a subclass of the Selberg class. This subclass $\widetilde{\mathcal{S}}$ differs from the Selberg class in two additional axioms. A Dirichlet series

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

from the Selberg class lies in $\widetilde{\mathcal{S}}$ if it satisfies the following two axioms:

- **Polynomial Euler product**: For $1 \leqslant j \leqslant m$ and each prime p there exist complex numbers $\alpha_j(p)$ such that

$$\mathcal{L}(s) = \prod_p \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} .$$

- **Mean-square:** There exists a positive constant κ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leqslant x} |a(p)|^2 \;=\; \kappa .$$

The class $\widetilde{\mathcal{S}}$ contains the Riemann zeta function, Dirichlet L-functions, Dedekind zeta functions, Hecke L-functions, L-functions associated with newforms, and Rankin-Selberg L-functions If one is willing to accept some widely believed conjectures, then a large class of functions belongs to $\widetilde{\mathcal{S}}$.

Voronin [5] also obtained joint universality for Dirichlet L-functions, that is simultaneous uniform approximation by a family of L-functions associated with non-equivalent characters; the non-equivalence of the characters assures a certain independence of the related L-functions, and this independence is necessary for joint universality. Recently, Laurinčikas & Matsumoto [2] proved a joint universality theorem for L-functions associated with newforms twisted by characters.

It is natural to ask for joint universality in the Selberg class. However, all known jointly universal families are given by (multiplicative or additive) twists of a single universal Dirichlet series by characters. In some sense, Selberg's Conjecture B states that primitive functions form an orthonormal system in the Selberg class. As proved by Bombieri & Hejhal [1], this implies the statistical independence of primitive functions. There is some hope that this can be used as substitute for the independence induced by non-equivalent characters in order to prove joint universality for distinct primitive L-functions from the Selberg class.

For $1 \leqslant j \leqslant m$, assume that the L-functions

$$\mathcal{L}_j(s) = \sum_{n=1}^{\infty} \frac{a_{\mathcal{L}_j}(n)}{n^s}$$

from $\widetilde{\mathcal{S}}$ satisfy the orthogonality condition

$$\sum_{p \leqslant x} \frac{a_{\mathcal{L}_j}(p)\overline{a_{\mathcal{L}_k}(p)}}{p} = \delta_{jk}\kappa_j \log\log x + O(1)\,, \tag{1}$$

where κ_j is a positive constant depending on $\mathcal{L}_j$, and $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise. This condition is known to hold for several families of L-functions in $\widetilde{\mathcal{S}}$, for example for Dirichlet L-functions associated with pairwise non-equivalent characters; it is expected to hold for any two distinct primitive L-functions from the Selberg class (Selberg's Conjecture B). Moreover, (1) is closely related to the axiom on the mean square in the definition of $\widetilde{\mathcal{S}}$.

Conjecture. Suppose that $\mathcal{L}_1, \ldots, \mathcal{L}_m$ are elements of $\widetilde{\mathcal{S}}$ satisfying condition (1). For $1 \leqslant j \leqslant m$ let $g_j(s)$ be a continuous function on $\mathcal{K}_j$ which is non-vanishing in the interior. Here $\mathcal{K}_j$ is a compact subset of the strip

$$\mathcal{D} := \{s\colon \max\{1/2, 1 - 1/d\} < \Re s < 1\}$$

with connected complement, and d is the maximum of the degrees of the $\mathcal{L}_j$ (a quantity determined by the functional equation for $\mathcal{L}_j$). Then, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T]:\ \max_{1\leqslant j\leqslant m} \max_{s\in\mathcal{K}_j} \big|\mathcal{L}_j(s+i\tau) - g_j(s)\big| < \varepsilon\Big\} > 0\,.$$

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The error term in the mean square formula for $\zeta(1/2 + it)$

Kai Man Tsang

(joint work with Yuk-Kam Lau)

Let $\Delta(x)$ (for $x \geqslant 1$) be the error term in the dirichlet divisor problem, that is,

$$\Delta(x) = \sum_{n\leqslant x} d(n) - x\log x - (2\gamma - 1)x\,.$$

Let

$$E(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt - T\log\frac{T}{2\pi} - (2\gamma-1)T$$

be the error term in the mean square of $\zeta(1/2+it)$. These are well-known error terms in analytic number theory and a lot of researches have been devoted to them. In 1992, Heath-Brown [1] found a new approach to show that both $x^{-1/2}\Delta(x^2)$ and $t^{-1/2}E(t^2)$ possess limiting distributions. His method, indeed, applies to functions $F(t)$ which satisfy the following hypothesis.

Hypothesis (H). *There exists a sequence* $a_1(t), a_2(t), \ldots$ *of continuous real-valued functions of period* 1 *such that*

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \min\left\{1, \left|F(t) - \sum_{n\leqslant N} a_n(\gamma_n t)\right|\right\}dt = 0 .$$

Here $\gamma_1, \gamma_2, \cdots$ *are constants which are linearly independent over the rationals.*

In view of the Voronoi formula for $\Delta(x)$ and the Atkinson formula for $E(T)$, both $\Delta(x)$ and $E(T)$ satisfy this hypothesis.

Theorem 1. (Heath-Brown, [1]) *If* $F(t)$ *satisfies the above Hypothesis* (H) *with the* $a_n(t)$*'s satisfying the additional conditions:*

(*i*) $\displaystyle\int_0^1 a_n(t)dt = 0 \qquad$ *for* $n = 1, 2, 3, \ldots$;

(*ii*) $\displaystyle\sum_{n=1}^{\infty}\int_0^1 a_n(t)^2 dt < \infty$;

(*iii*) $\displaystyle\max_{0\leqslant t\leqslant 1}|a_n(t)| \ll n^{1-\mu}$ *and* $\displaystyle\lim_{n\to\infty} n^{\mu}\int_0^1 a_n(t)^2 dt = \infty$ *for some constant* $\mu > 1$,

then there exists a function $f(\alpha)$ *for which*

$$\frac{1}{T}\,\mathrm{meas}\left\{t\in[0,T] : F(t)\in I\right\} \to \int_I f(\alpha)\,d\alpha$$

on each interval I *as* $T\to\infty$. *Furthermore, the probability density function* $f(\alpha)$ *satisfies*

$$\frac{d^k}{d\alpha^k} f(\alpha) \ll_{A,k} \left(1+|\alpha|\right)^{-A}$$

for any constant A *and* $k = 0, 1, 2, \ldots$.

Theorem 2. [2] *Under the same assumptions in the above theorem, we have, for any natural number* k,

$$\int_{-\infty}^{\infty}\alpha^k f(\alpha)\,d\alpha = \sum_{1\leqslant r\leqslant k}\ \sum_{\substack{\ell_1+\cdots+\ell_r=k\\ \ell_1,\cdots,\ell_r\geqslant 1}} \frac{k!}{\ell_1!\ell_2!\cdots\ell_r!}\sum_{1\leqslant n_1<n_2<\cdots<n_r}\prod_{i=1}^{r}\int_0^1 a_{n_i}(t)^{\ell_i}dt.$$

The innermost sum here converges absolutely.

Using Theorem 2, we show that the density function $f(\alpha)$ for the error term $\Delta_3(x)$ in the Pilz divisor problem satisfies

$$\int_{-\infty}^{\infty} \alpha^3 f(\alpha)\, d\alpha \neq 0\,.$$

Hence $f(\alpha)$ is asymmetric.

Instead of the k-th moments, one can consider more generally the mean value

$$\int_0^X \underbrace{F(\alpha x)F(\beta x)F(\gamma x)\cdots}_{k \text{ copies}}\, dx\,,$$

where $\alpha, \beta, \gamma, \cdots$ are fixed positive constants. The simplest case is $k = 2$, and we have the following theorem.

Theorem 3. *We have*

$$D_\Delta(\alpha) \;=\; \lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \Delta(\alpha x)\Delta(x)\, dx \;=\; \frac{\alpha^{1/4}}{6\pi^2} \sum_{\substack{n,m\geqslant 1\\ \alpha n = m}} \frac{d(n)\,d(m)}{(nm)^{3/4}}\,.$$

$$D_E(\alpha) \;=\; \lim_{T\to\infty} \frac{1}{T^{3/2}} \int_0^T E(\alpha t)\, E(t)\, dt \;=\; \left(\frac{2}{\pi}\right)^{1/2} \frac{\alpha^{1/4}}{3} \sum_{\substack{n,m\geqslant 1\\ \alpha n = m}} \frac{(-1)^{n+m} d(n)\,d(m)}{(nm)^{3/4}}\,.$$

Clearly, $D_\Delta(\alpha) = 0$ if α is irrational, and $D_\Delta(\alpha)$ is positive when α is a positive rational number. The same conclusion holds for $D_E(\alpha)$.

It is more interesting when we take a step further to consider mean values of triple products

$$\int_0^X F(\alpha x)\, F(\beta x)\, F(x)\, dx\,,$$

where F is Δ and E respectively.

Theorem 4. *For any $\alpha, \beta > 0$, we have*

$$\lim_{X\to\infty} X^{-7/4} \int_1^X \Delta(\alpha x)\, \Delta(\beta x)\, \Delta(x)\, dx \;=\; \frac{(\alpha\beta)^{1/4}}{28\pi^3} \sum_{\substack{\sqrt{\alpha n}\pm\sqrt{\beta m}\pm\sqrt{k}=0\\ n,m,k\geqslant 1}} \frac{d(n)\,d(m)\,d(k)}{(nmk)^{3/4}}\,,$$

which is positive or zero, according as the equations $\sqrt{\alpha n} \pm \sqrt{\beta m} \pm \sqrt{k} = 0$ have solutions in natural numbers $n, m,$ and k or not.

Theorem 5. *For any $\alpha, \beta > 0$ we have*

$$\lim_{T\to\infty} T^{-7/4} \int_0^T E(\alpha t)\, E(\beta t)\, E(t)\, dt$$
$$= \frac{1}{7\sqrt{2}} \Big(\frac{2}{\pi}\Big)^{3/4} (\alpha\beta)^{1/4} \sum_{\substack{\sqrt{\alpha n}\pm\sqrt{\beta m}\pm\sqrt{k}=0\\ n,m,k\geqslant 1}} (-1)^{n+m+k}\, \frac{d(n)\,d(m)\,d(k)}{(nmk)^{3/4}}\,.$$

If we denote the above sum on the right hand side by $C(\alpha,\beta)$ *, then* $C(1,1)>0$ *. However, for any integer* $h\geqslant 5$ *,* $C\big(2^{2/3},(1+2^{1/3})^2 2^h\big)<0$ *. Furthermore, there exist real numbers* θ *, arbitrarily close to* 1*, for which* $C(\theta,\theta)<0$ *.*

Remark. Comparing Theorems 4 and 5, we see a marked distinction between the behaviour of $\Delta(x)$ and $E(t)$. This distinction originates from the extra factor of $(-1)^n$ in each term of the Atkinson formula for $E(t)$.

Lattice points in circles

Let

$$P_{\boldsymbol{\alpha}}(x)=\sum_{\substack{|\boldsymbol{n}-\boldsymbol{\alpha}|\leqslant\sqrt{x}\\ \boldsymbol{n}\in\mathbb{Z}^2}}1\;-\;\pi x\,,$$

the error term in the counting of lattice points in the circle $\{\boldsymbol{z}:|\boldsymbol{z}-\boldsymbol{\alpha}|\leqslant\sqrt{x}\}$ whose center is at the fixed point $\boldsymbol{\alpha}$. Then, similar to the case for $\Delta(x)$, we have

$$P_{\boldsymbol{\alpha}}(x)=-\frac{x^{1/4}}{\pi}\sum_{m\leqslant X}r_{\boldsymbol{\alpha}}(m)m^{-3/4}\cos\big(2\pi\sqrt{mx}+\tfrac{\pi}{4}\big)+O\big(X^{\varepsilon}\big)$$

for $X<x\leqslant 4X$, where for $\boldsymbol{\alpha}=(a,b)$

$$r_{\boldsymbol{\alpha}}(m)=\sum_{\substack{u,v\in\mathbb{Z}\\ u^2+v^2=m}}\cos(2\pi au)\cos(2\pi bv)\,.$$

Hence $r_{\mathbf{0}}(m)=r(m)$, the number of ways of writing m as the sum of two squares. For $\boldsymbol{\alpha}=(1/2,1/2)$, $r_{\boldsymbol{\alpha}}(m)=(-1)^m r(m)$. Similar to Theorems 4 and 5 above, we show that $P_{\mathbf{0}}(x)$ and $P_{\boldsymbol{\alpha}}(x)$ for $\boldsymbol{\alpha}=(1/2,1/2)$ behave quite differently in certain mean values of triple products.

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A variance for k-free numbers in arithmetic progressions

Robert C. Vaughan[1]

Let $k\geqslant 2$ and μ_k denote the characteristic function of the k-free numbers, and define

$$Q_k(x;q,a)=\sum_{\substack{n\leqslant x\\ n\equiv a(\bmod q)}}\mu_k(n)\,,\quad Q_k(x)=Q_k(x;1,0)$$

$$f(q,a)=\sum_{\substack{m=1\\(m^k,q)|a}}^{\infty}\frac{\mu(m)(m^k,q)}{m^kq}\,,$$

$$V(q)=\sum_{a=1}^{q}\big(Q_k(x;q,a)-xf(q,a)\big)^2,\quad V(x,Q)=\sum_{q\leqslant Q}V(q)\,.$$

As promised on page 796 of Vaughan [5], we return to the study of $V(x,Q)$. It is a straightforward calculation to show that

$$f(q,a)=\lim_{x\to\infty}x^{-1}Q_k(x;q,a)\,,$$

and the variance $V(x,Q)$ has been studied by several authors, mostly in the special case of squarefree numbers, $k=2$. For general $k\geqslant 2$ in Brüdern, *et al* [1] (Lemma 2.2), it is shown that

$$V(x,Q)\ll\begin{cases}x^{\frac{2}{k}+\varepsilon}Q^{2-\frac{2}{k}}, & \text{when } 1\leqslant Q\leqslant x,\\ Q^2\log(2Q), & \text{when } Q>x,\end{cases}\tag{1}$$

it being obvious from (1) above that the $g_k(q,a)$ featuring in (2.6) of that paper satisfy

$$\zeta(k)^{-1}g_k(q,a)=\lim_{x\to\infty}x^{-1}Q_k(x;q,a)=f(q,a).$$

Theorem 1.2 of Vaughan [5] concerns the distribution of general sequences. In the special case $a_n=\mu_k(n)$ it is known that it is possible to take $\Psi(x)=x^{1-1/k}$ and so, when $\sqrt{x}\log(2x)<Q\leqslant x$,

$$V(x,Q)=x^2\mathcal{F}(x/Q)+O\left(x^{\frac{3}{2}}\log x+x^{\frac{4}{3}+\frac{2}{3k}}(\log x)^{\frac{4}{3}}\right)\tag{2}$$

where

$$\mathcal{F}(y)\sim c_ky^{\frac{1}{k}-2}\text{ as } x\to\infty\tag{3}$$

and c_k is a positive constant.

For the particular case of k-free numbers the part of the hypothesis of that theorem expressed by (1.12) in [5], can be replaced by a more precise statement of the form

$$\sum_{q>y}g(q)=c_k^*y^{\frac{1}{k}-1}+O\left(y^{\theta_k}\right)$$

where $\theta_k<\frac{1}{k}-1$ and this leads to the conclusion (2) with (3) replaced by

$$\mathcal{F}(y)=c_ky^{\frac{1}{k}-2}+O\left(y^{\frac{1}{k}-2-\delta_k}\right)$$

where $\delta_k>0$.

In the special case of squarefree numbers, following earlier work of Orr [3] & [4], Warlimont [6] and Croft [2], Warlimont [7] has shown that when $1\leqslant Q\leqslant x$,

$$V(x,Q)=x^2\mathcal{F}(x/Q)+O\left(x^{\frac{3}{2}}(\log x)^{\frac{7}{2}}\right)$$

where

$$\mathcal{F}(y) = c_k y^{-3/2} + O\left(y^{-\frac{7}{4}} \exp\big(- \widetilde{c}_k (\log y)^{1/5}\big)\right) .$$

In all of the methods used hitherto to establish the asymptotic formula for $V(x,Q)$, even in the special case $k=2$, there is a natural limit to the method which forces the error term to be at least as large as $x^{1+1/k}\log x$ (and when $k>2$ there is no published literature with the exponent of x as small as $1+1/k$). In this memoir it is shown how this limitation can be overcome.

THEOREM. *There are positive constants c_k^*, $\widetilde{c}_k$ such that, whenever $Q \leqslant x$,*

$$\begin{aligned} V(x,Q) &= c_k x^{1/k} Q^{2-1/k} + O\left(x^{\frac{1}{2k}} Q^{2-\frac{1}{2k}} \exp\left(- c_k^* \frac{(\log 2x/Q)^{3/5}}{(\log\log 3x/Q)^{1/5}}\right)\right) \\ &\quad + O\left(x^{1+1/k} \exp\left(- \frac{\widetilde{c}_k (\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right) \end{aligned} \tag{4}$$

where

$$c_k = \frac{2C_1 k^2\big(- \zeta(1/k-1)\big)}{\zeta(k)^2(2k-1)(k-1)}$$

and

$$C_1 = \prod_p \left(1 + \frac{(p-1)^2 \sum_{j=1}^{k-1} p^{(2-\frac{1}{k})j} - 2p^{2k} + p^{2k-1} + 2p^{k+1} - p}{p^2(p^k-1)^2}\right). \tag{5}$$

The approximation (4) is superior to (1) when $Q \gg x^{1/2}$, and the main term in (4) dominates the error terms when x/Q is large and Q is large compared with

$$Q_1 = x^{\frac{k}{2k-1}} \exp\left(- \frac{k}{2k-1} \widetilde{c}_k (\log x)^{3/5} (\log\log x)^{-1/5}\right). \tag{6}$$

One curiosity is that improvements in both error terms would follow from a better zero-free region for the Riemann zeta function alone.

The core of the proof depends heavily on Brüdern *et al* [1] where excellent bounds for

$$\sum_{n\leqslant x} \mu_k(n) e(n\alpha),$$

and related expressions, are obtained.

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Analogues for results of Deshouillers and Iwaniec

Nigel Watt

Let $\Gamma = \Gamma_0(q)$, a Hecke congruence subgroup. Take $\mathfrak{B} = \mathfrak{B}^{(q)} = \{u_1(z), u_2(z), \ldots\}$ to be an orthonormal basis for the space that is spanned by the non-holomorphic cusp forms for Γ with Petersson inner-product. Each $u_j(z)$ is an eigenfunction of the hyperbolic Laplacian operator: $\Delta u_j = \lambda_j u_j$, where $\lambda_j = 1/4 + \kappa_j^2$ and, as shown by Kim and Sarnak in [3], either $\kappa_j \in \mathbb{R}$ or $0 < i\kappa_j \leqslant 7/64 = \vartheta$ (say).

In 1982 Deshouillers and Iwaniec established a "large-sieve" upper-bound for

$$\mathfrak{S}_{\mathfrak{a},q,K}(\boldsymbol{b}, N) = \sum_{|\kappa_j| \leqslant K}^{(q)} \frac{1}{\cosh(\pi\kappa_j)} \Big| \sum_{N/2 < n \leqslant N} b_n \rho_{j\mathfrak{a}}(n) \Big|^2,$$

where $K, N \geqslant 1$, $\mathfrak{a}$ is a cusp for Γ, $\boldsymbol{b} = (b_n)$ is an arbitrary complex sequence, and $\rho_{j\mathfrak{a}}(n)$ is the n-th Fourier coefficient of $u_j(z)$ at the cusp $\mathfrak{a}$ (see [1], Theorem 2). More recently, in the course of working on mean values of Dirichlet's L-function, Glyn Harman, Kam Wong and I have needed upper bounds for

$$\sum_{|\kappa_j| \leqslant K}^{(q)} \frac{1}{\cosh(\pi\kappa_j)} \Big| \sum_{N/2 < n \leqslant N} b_n \rho_{j\infty}(Dn) \Big|^2 = \mathfrak{S}_{\infty,q,K}(\boldsymbol{b}^{\langle D \rangle}, DN),$$

where $D \in \mathbb{N}$ can be large and where, for $n \in \mathbb{N}$, $b_n^{\langle D \rangle} = b_{n/D}$, if $n \equiv 0 \pmod{D}$, and 0 otherwise. Given that $\mathfrak{B}$ is chosen so that each $u_j(z)$ is an eigenfunction of all Hecke operators T_d with $(d,q) = 1$, the multiplicative nature of $\rho_{j\infty}(Dn)$ permits exploitation of Kim and Sarnak's bound, $|\tau_j(d)| \leqslant d^{\vartheta}\tau(d)$ (from [3]). This leads to a proof that, if $\varepsilon > 0$ then, for $(q,D) = 1$ and $K, N \geqslant 1$, one has

$$\mathfrak{S}_{\infty,q,K}(\boldsymbol{b}^{\langle D \rangle}, DN) \ll_{\varepsilon} D^{2\vartheta}\tau^4(D) \left(K^2 + q^{-1}N^{1+\varepsilon}\right) \sum_{N/2 < n \leqslant N} |b_n|^2.$$

This bound helps us to show (in [2], Theorem 1) that, for $T \geqslant D^{3/5}$, one has

$$\frac{1}{\phi(D)T} \sum_{\chi \bmod D} \int_{-T}^{T} \left|L\left(\tfrac{1}{2}+it,\chi\right)\right|^4 \left|\sum_{m\leqslant M} a_m \chi(m) m^{-it}\right|^2 dt$$
$$\ll_\varepsilon \left(1 + D^\vartheta (DT)^{-1/2} M^2\right) T^\varepsilon M \max_{m\leqslant M} |a_m|^2 .$$

The bound for $\mathfrak{S}_{\infty,q,K}(\boldsymbol{b}^{\langle D\rangle}, DN)$ can be improved "on average" (over the level q), provided that suitable factorisations of D exist.

Theorem. *If no prime factor of D is greater than D^ε, then*

$$\sum_{q\leqslant Q} \mathfrak{S}_{\infty,q,K}(\boldsymbol{b}^{\langle D\rangle}, DN)$$
$$\ll_\varepsilon \left(Q + D^{2\vartheta}N + (DN)^{2\vartheta}\left(\min\left(Q,\sqrt{DN}\right)\right)^{1-4\vartheta}\right)\left(D^2QN\right)^\varepsilon K^2 \sum_{N/2<n\leqslant N} |b_n|^2 .$$

The idea of averaging over the level was introduced in Deshouillers and Iwaniec's paper [1]. The above theorem is essentially an analogue of one of their results, ([1], Theorem 6). For example, where Deshouillers and Iwaniec use the bound

$$Y^{2i\kappa_j} \leqslant Y^{2\vartheta} \qquad (Y \geqslant 1),$$

with $\vartheta = 1/4$ (Selberg's bound $\lambda_j \geqslant 3/16$ being the best available when [1] was written), I use instead the analogous bound

$$\tau_j^2(d) \leqslant d^{2\vartheta}\tau^2(d) \ll_\varepsilon d^{2\vartheta+\varepsilon} \qquad ((d,q) = 1).$$

I also use the Bruggeman–Kuznetsov and Kuznetsov summation formulae in a way that parallels their use in [1].

It is hoped to apply the above theorem (and a stronger result that applies in special cases) in future work on a mean-value of sums involving Dirichlet characters. This would be part of some proposed joint work with Glyn Harman, aimed at adding to what is known regarding the abundance of Carmichael numbers.

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An application of triple correlations of a divisor sum to the distribution of primes

Cem Yalçin Yildirim
(joint work with Daniel A. Goldston[1])

We use the truncated divisor sums

$$\lambda_R(n) = \sum_{r \leqslant R} \frac{\mu^2(r)}{\phi(r)} \sum_{d|(n,r)} d\mu(d)$$

to study the distribution of primes in short intervals. Goldston (unpublished) showed that these divisor sums arise from the local approximations used in the application of the circle method to the twin primes problem. Furthermore, Goldston observed that among sums of the kind

$$\sum_{\substack{d|n \\ d\leqslant R}} a(R,r) \quad \text{with} \quad a(R,1)=1,\ a(R,r)\in\mathbb{R}\,,$$

$\lambda_R(n)$ is the best approximation to von Mangoldt's prime counting function $\Lambda(n)$ in an L^2-sense. This involves a minimization which was solved in a more general setting by Selberg [8] for his upper bound sieve. As can be seen from our results, on average these divisor sums tend to behave similarly to $\Lambda(n)$ does or is conjectured to do.

The pure and mixed (with one factor of $\Lambda(n)$) correlations we are interested in evaluating are

$$\mathcal{S}_k(N,\boldsymbol{j},\boldsymbol{a}) = \sum_{n=1}^{N} \lambda_R(n+j_1)^{a_1}\lambda_R(n+j_2)^{a_2}\cdots\lambda_R(n+j_r)^{a_r}$$

and

$$\widetilde{\mathcal{S}}_k(N,\boldsymbol{j},\boldsymbol{a}) = \sum_{n=1}^{N} \lambda_R(n+j_1)^{a_1}\lambda_R(n+j_2)^{a_2}\cdots\lambda_R(n+j_{r-1})^{a_{r-1}}\Lambda(n+j_r)\,.$$

Here $\boldsymbol{j} = (j_1,j_2,\ldots,j_r)$, $\boldsymbol{a} = (a_1,a_2,\ldots,a_r)$, the j_i's are distinct integers, $a_i \geqslant 1$, and $\sum_{i=1}^{r} a_i = k$. In the case of the mixed correlations we assume that $r\geqslant 2$ and take $a_r = 1$.

For $k=1$ and $k=2$ these correlations have been evaluated before ([1], [6]). We calculate the cases with $k=3$, the triple correlations. Our method consists of evaluating the relevant sums straightforwardly. In the case of the mixed correlations an important ingredient is the Bombieri–Vinogradov theorem, and this brings a further restriction on the size of R. We obtain for $1\leqslant k\leqslant 3$, when $\max_i |j_i| \leqslant N^{1-\varepsilon}$ and $R \gg N^{\varepsilon}$,

$$\mathcal{S}_k(N,\boldsymbol{j},\boldsymbol{a}) = \big(\mathcal{C}_k(\boldsymbol{a})\mathfrak{S}(\boldsymbol{j}) + o(1)\big)N(\log R)^{k-r} + O\big(R^k\big)\,,$$

where $\mathcal{C}_k(\boldsymbol{a})$ has the values $\mathcal{C}_1((1))=1$, $\mathcal{C}_2((2))=1$, $\mathcal{C}_2((1,1))=1$, $\mathcal{C}_3((3))=3/4$, $\mathcal{C}_3((2,1))=1$, $\mathcal{C}_3((1,1,1))=1$. For $2\leqslant k\leqslant 3$, when $\max_i |j_i| \leqslant N^{1/(k-1)-\varepsilon}$

and $N^{\varepsilon} \ll R \ll N^{1/(2(k-1))-\varepsilon}$,

$$\widetilde{\mathcal{S}}_k(N,\boldsymbol{j},\boldsymbol{a}) = \big(\mathfrak{S}(\boldsymbol{j}) + o(1)\big) N (\log R)^{k-r}.$$

Here

$$\mathfrak{S}(\boldsymbol{j}) = \prod_{p\,\text{prime}} \Big(1-\frac{1}{p}\Big)^{-r} \Big(1-\frac{\nu_p(\boldsymbol{j})}{p}\Big),$$

and $\nu_p(\boldsymbol{j})$ is the number of distinct residue classes modulo p that the j_i's occupy. Recall that the Hardy–Littlewood prime r-tuple conjecture [4] states

$$\sum_{n=1}^{N} \Lambda(n+j_1)\Lambda(n+j_2)\cdots\Lambda(n+j_r) \sim \mathfrak{S}(\boldsymbol{j})\, N$$

when $\mathfrak{S}(\boldsymbol{j}) \neq 0$. For $r=1$ this reduces to the Prime Number Theorem which is the only settled case.

The above results agree with those obtained in our former work [3] which used the truncated divisor sums

$$\Lambda_R(n) = \sum_{\substack{d|n \\ d\leqslant R}} \mu(d) \log\Big(\frac{R}{d}\Big),$$

and therefore as a corollary one has as in [3]

$$\liminf_{n\to\infty} \Big(\frac{p_{n+r}-p_n}{\log p_n}\Big) \leqslant r - \frac{\sqrt{r}}{2}.$$

In fact, our results for the correlations are more detailed and with better error terms than their statement above for which a concise form covering all the cases was adopted. If the Generalized Riemann Hypothesis is assumed, so that Hooley's estimate [5] can be used instead of the Bombieri–Vinogradov theorem, the results obtained with $\lambda_R(n)$'s also permit us to obtain the following $\Omega_{\pm}$-result for the variation in the error term in the prime number theorem. As usual we write $\psi(N) = \sum_{n\leqslant N}\Lambda(n)$. For any arbitrarily small but fixed $\eta > 0$, and for sufficiently large N, with $\log^{14} N \leqslant h \leqslant N^{1/7-\varepsilon}$ and writing $h = N^{\alpha}$, there exist n_1, $n_2 \in [N+1, 2N]$ such that

$$\psi(n_1+h) - \psi(n_1) - h > \Big(\frac{\sqrt{1-5\alpha}}{2} - \eta\Big)(h\log N)^{1/2}$$

$$\psi(n_2+h) - \psi(n_2) - h < -\Big(\frac{\sqrt{1-5\alpha}}{2} - \eta\Big)(h\log N)^{1/2}.$$

It should be noted that an important ingredient in achieving this result is a recent theorem of Montgomery and Soundararajan [7] which makes it possible to evaluate the sum

$$\sum_{\substack{1\leqslant j_1,j_2,j_3\leqslant h \\ \text{distinct}}} \mathfrak{S}\big((j_1,j_2,j_3)\big)$$

with a good error term.

This is a new development in the sense that formerly our knowledge under the Generalized Riemann Hypothesis was restricted to results for the absolute value of this variation. The strongest of such results were attained in [2] in the more general case of primes in an arithmetic progression which yielded as a special case

$$\max_{N<n\leqslant 2N} |\psi(n+h)-\psi(n)-h| \gg_\varepsilon (h\log N)^{1/2}$$

for $1\leqslant h\leqslant x^{1/3-\varepsilon}$. In fact the general case was also obtained by using the correlations of $\lambda_R(n)$'s. There only the first and second level correlations were employed, nevertheless in the more general case of $n\in[N+1,2N]$ running through an arithmetic progression $n\equiv a(\bmod q)$.

Since the $\lambda_R(n)$'s are related to the Selberg sieve, our results for the correlations may turn out to be of use in other problems as well.

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The Mean values of L-functions

Qiao Zhang

Let f be an automorphic form over GL_n, then a central problem in analytic number theory is to estimate the mean values

$$\sum_{\substack{|D|\leqslant x\\ D\ \mathrm{fund.\,disc.}}} L\left(\tfrac12, f\otimes\chi_D\right) \tag{1}$$

and

$$\int_0^T \left|L\left(\tfrac12+it,f\right)\right|^2 dt\,. \tag{2}$$

In some special cases, this problem has been extensively studied with fruitful results and important arithmetic applications.

In this talk, we demonstrate how to attack these mean values through a different approach. With complex Tauberian theorems in mind, we consider the (discrete and continuous) Dirichlet series

$$Z_f\left(\tfrac{1}{2}, w\right) = \sum_{\substack{D=-\infty \\ D \text{ fund. disc.}}}^{\infty} L\left(\frac{1}{2}, f \otimes \chi_D\right) |D|^{-w} \qquad (\Re w \gg 1),$$

$$Z_f(w) = \int_1^{\infty} \left|L\left(\tfrac{1}{2}+it, f\right)\right|^2 t^{-w}\, dt \qquad (\Re w \gg 1)$$

and study their analytic properties, in particular their analytic continuations beyond $\Re w = 1$ and polar behaviours at the point $w = 1$.

For the integral mean value (2), our goal is to express $Z_f(w)$ (asymptotically) as an inner product of f with a certain kernel function, by directly exploring the symmetries satisfied by f itself, so that it suffices to study the analytic properties of this kernel function alone. This realizes and generalizes the ideas suggested by Good in his 1984 paper. A distinctive characteristic of this approach is that we have avoided delicate discussions of the arithmetic nature of those Fourier coefficients, such as estimates of the generalized "additive divisor problems" $\sum_{n \leqslant x} \lambda_f(n)\lambda_f(n+r)$. Instead, we reduce the problem to more routine analytic manipulations. Hence this approach is applicable to cases where our knowledge of the arithmetic nature of f is rather limited. This is the main motivation of the present work. As an example of this approach, asymptotic formulas for mean squares of modular L-functions over Hecke congruence subgroups have been obtained. During this process, a theory of nonholomorphic Poincaré series is developed.

For the discrete mean value (1), we observe that the double Dirichlet series

$$Z_f(s, w) = \sum_{\substack{D=-\infty \\ D \text{ fund. disc.}}}^{\infty} L(s, f \otimes \chi_D)\, |D|^{-w} \qquad (\Re s > 1, \Re w > 1)$$

enjoys two functional equations, one coming from that of the Dirichlet coefficient $L(s, f \otimes \chi_D)$ and the other coming from the quadratic reciprocity law. This fact enables us to analyze the analytic properties of $Z_f(s, w)$ itself. In particular, we study the cubic moment of quadratic Dirichlet L-functions. With some technical assumptions, we show that

$$\sum_{\substack{|D| \leqslant x \\ D \text{ fund. disc.}}} L\left(\tfrac{1}{2}, \chi_D\right)^3 = x R_3(\log x) + b x^{3/4} + O\big(x^{1/2+\varepsilon}\big),$$

where R_3 is some polynomial of degree 6 and b is a computable nonzero constant. The appearance of the "exceptional main term" $x^{3/4}$ in this asymptotic formula is very surprising, and suggests a much finer structure than once expected for the value distribution of automorphic L-functions with quadratic twists.

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