Mathematisches Forschungsinstitut Oberwolfach

Report No. 21/2005

Mini-Workshop: Aspects of Ricci-Flow

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May 1st – May 7th, 2005

Abstract. The workshop studies Hamilton-Ricciflow of Riemannian metrics on 3-manifolds. The participants give detailed technical lectures on recent work of G. Perelman concerning a priori estimates and surgeries during the flow. The workshop was able to verify major sections of Perelman's work and identified points that need a more detailed exposition.

Mathematics Subject Classification (2000): 53C44.

Introduction by the Organisers

The motion of a Riemannian metric along its Ricci curvature,

$$\tfrac{d}{dt} g_{ij} = -2R_{ij}(g(t))$$

was proposed in 1982 by Richard S. Hamilton as a geometric version of the heat equation suitable for uniformizing and smoothing the geometry of a given initial Riemannian manifold (M^3, g_0). Hamilton's work has opened up the whole area of geometric evolution equations, leading to the discovery of new phenomena in these equations and to topological applications such as the classification of 3-manifolds of positive Ricci-curvature and certain 4-manifolds. Recent work of G. Perelman has indicated how to approach a proof of the Poincaré-conjecture and the Thurston Geometrization conjecture for 3-manifolds using Hamilton-Ricciflow.

The mini workshop has concentrated on a thorough technical investigation of that part of the work of Perelman that is related to Hamilton-Ricciflow with surgeries on a finite time interval. Together with further work of Perelman and also Colding-Minicozzi this part of Perelman's work implies a proof of the Poincaré conjecture when confirmed. The efforts of the workshop were greatly helped by previous work of other mathematicians on Perelman's work., e.g. the notes of B. Kleiner and J. Lott.

The workshop was able to confirm major sections in the two papers of G. Perelman including the entropy- and reduced volume estimates, the compactness properties of ancient solutions to the flow and the surgery construction. It was also able to reinterpret several arguments involving Alexandrov spaces from the viewpoint of smooth Differential Geometry.

When the workshop had to come to an end, its participants agreed that it would be very desirable to establish self-contained expositions of the following points:

(1) The boundedness of the curvature $\sup_{B_\rho(x)} R \leq c(\rho)$ in the proof of the approximation theorem I.12.1 of Perelman's first paper.
(2) The approximation of mini-max surfaces in the varifold distance for immersed surfaces in the paper of Colding-Minicozzi.
(3) The survival of the reduced volume estimate past surgeries.
(4) The uniform control of a fixed scale $\rho > 0$ past all surgeries on a finite time interval, below which the approximation theorem applies.

It seems that detailed self-contained expositions of (3) and in particular (4) require more effort than (1) and (2).

Mini-Workshop: Aspects of Ricci-Flow

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Abstracts

Uniform non-collapsing and an elliptic Harnack-type inequality

Huai-Dong Cao

During the mini-workshop, I first described Richard Hamilton's dimension reduction argument for the Ricci flow, see §21 and §22 in [1].

In another lecture, I discussed part of Perelman's work on the Ricci flow and gave a detailed presentation of how to prove uniform noncollapsing for ancient noncollapsing solutions as was indicated in §11.9 of [2].

I also showed how to obtain an elliptic Harnack type inequality for the scalar curvature. From the latter one easily obtains the important gradient estimates for the scalar curvature by combining with Shi's local derivative estimates. More precisely, we gave a detailed proof of the following:

Theorem *There exist positive constants κ_0, η and a positive function ω defined on $[0,\infty)$ with the following properties. Suppose we have a three-dimensional noncompact ancient κ-solution $(M, g_{ij}(t))$, $-\infty < t \leq 0$, for some $\kappa > 0$. Then:*

(i) the ancient κ-solution is either an ancient κ_0-solution, or a metric quotient of the round three-sphere;

(ii) for every $x, y \in M$ and $t \in (-\infty, 0]$, we have

$$R(x,t) \leq R(y,t)\omega(R(y,t)d_t^2(x,y)).$$

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[1] R. Hamilton, *The formation of singularities in the Ricci flow*, Collection: Surveys in differential geometry **Vol. II** (1995), 7–137.

[2] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, `arXiv:math.DG/0211159v1 11Nov2002`.

Entropy and lower volume ratio bounds

Klaus Ecker

In this talk we discussed Perelman's entropy formula for the Hamilton-Ricciflow. This formula features in chapters 3 and 4 of [1].

Perelman considers an integral expression which resembles the standard L^2-integral of a function f over the manifold with respect to the volume measure of the Ricci flow evolving metric at time t, augmented by the scalar curvature at that time and multiplied by a weight function involving f which is analogous to the standard Gaussian in Euclidean space. Moreover, there is a positive scaling factor τ which roughly corresponds to negative time or more precisely the time left to a fixed given time. Perelman's entropy is the infimum of this expression over all functions f which satisfy a normalisation condition akin to the requirement that the weight corresponds to a probability distribution on the manifold.

In the case of Euclidean space, one obtains this integral expression by writing all the integrals appearing in the standard logarithmic Sobolev inequality on one side of the inequality. The entropy in this simple case thus equals zero. In general, the entropy at time t can be estimated from below in terms of the constant in the logarithmic Sobolev inequality on our manifold at that time and the scaling parameter τ. In a way, a lower bound on the entropy corresponds to a lower bound arising in a Poincaré inequality, or in other words to a lower bound on the lowest eigenvalue of the Laplacian plus the scalar curvature.

An upper estimate for the entropy with scaling parameter equal to r^2 is given by the logarithm of the natural volume ratio of a geodesic ball of radius r and another term which is bounded if the norm of the Riemann tensor, scaled by r^2, is bounded on that ball. If both a lower and an upper bound hold for the entropy one thus obtains a positive lower bound on the volume ratios of geodesic balls.

It was one of Perelman's ingenious achievements to realize that the entropy for a solution of Hamilton-Ricciflow, considered as a function of t is non-increasing. This is established by a long but standard calculation. The achievement consisted in suggesting the correct monotone quantity. It is interesting to note that in Perelman's entropy formula he is taking a time derivative in a kind of Sobolev inequality at time t so in other words obtains control on how isoperimetric information is maintained by the Hamilton-Ricciflow.

An immediate consequence of the monotonicity of the entropy and the upper and lower bounds discussed above is the fact that on any finite time solution of Hamilton-Ricciflow one obtains a lower bound on the volume ratios in balls of radius less than a constant proportional to the square root of the existence time of the solution. This lower bound depends only on the initial metric. Considering this information on sequences of rescaled solutions of the flow allows one to take longer and longer time intervals and therefore obtains a lower volume ratio estimate for all radii on smooth limiting solutions. Such an estimate does not hold on certain eternal solutions of the Hamilton-Ricciflow such as the product of the two-dimensional cigar soliton with the real line. This solution admits sequences of geodesic balls with radii r_k tending to infinity for which the volume grows like r_k^2 rather than r_k^3 as it should for a 3-manifold. Therefore, such solutions cannot occur as rescaling limits of finite time solutions of the flow.

The latter information has been a vital step in proceeding with Hamilton's programme for the use of Hamilton-Ricciflow in trying to settle the geometrization programme. Moreover, lower volume ratio bounds easily imply lower bounds on the injectivity radius, which is a crucial condition for the applicability of Hamilton's compactness theorem for solutions of the Ricciflow.

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The surgery construction for the Hamilton-Ricciflow

Gerhard Huisken

In his paper [1] concerning Ricciflow with surgery R. Hamilton develops a detailed quantitative concept of necks that are sufficiently close to a cylinder. The lecture describes the surgery in the 3-dimensional case in the context of Perelman's paper on Ricciflow with surgery [2]. In particular it is shown that for a given class of initial data the neck-parameters and surgery parameters can be chosen in such a way that surgery preserves the pinching estimate of Hamilton-Ivey for 3-dimensional Hamilton-Ricciflow.

References

[1] R. Hamilton, *Four-manifolds with positive isotropic curvature* Comm. anal. Geom **7** (1997), 1–92.

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Properties of reduced distance and volume

Tom Ilmanen

In the lecture, we summarized the basic properties of the reduced distance $\ell(x,t) = \ell_{q,T}(x,t)$ and reduced volume

$$\tilde{V}(t) = \tilde{V}_{q,T}(t) := (4\pi(T-t))^{-n/2} \int e^{-\ell(x,t)} dvol_t(x)$$

on a Ricci flow $(M, (g(t))_{0\leq t\leq T})$, see [1, chapter 7]. In particular, we reviewed the three weak inequalities satisfied by the reduced distance, and the monotonicity of the reduced volume, and addressed the limiting properties of these quantities at the initial time t=0 and the final time $t \to T$.

Particular attention was paid to the lower bound for $\tilde{V}_{q,T}(0)$ in terms of the initial geometry. We checked that it takes the form

$$\tilde{V}_{q,T}(0) \geq c(K, i, T)$$

where $K := \max_x |Rm_{g(0)}(x)|_{g(0)}$ and $i := \min_x \mathrm{inj}_{g(0)}(x)$.

There are two proofs of the Bishop-Gromov monotonicity formula, the first by Jacobi fields, the second by $(n-1)$-dimensional volume elements that are transverse to a geodesic (i.e. the Riccati equation). Perelman's derivation of the monotonicity of reduced volume is like the former. It would be interesting to see a proof analogous to the latter.

In [1, Chap. 11.4] it is shown that the case $0 < \mathcal{R} < \infty$ is impossible, where $\mathcal{R} := \limsup_{x\to\infty} R(x)d(x,p)^2$ is the asymptotic scalar curvature ratio of an ancient solution with positive curvature operator. Perelman's proof uses a blowdown of the solution in the sense of Alexandrov spaces. Another method uses smooth estimates for triangles and a smooth limit.

There is a third way to prove this, based on the Bishop-Gromov monotonicity formula. The key is the well-known fact that a smooth blowdown of a manifold of nonnegative Ricci curvature and Euclidean volume growth is a smooth cone. For a Ricci flow with positive curvature operator, this is then impossible by the splitting result from Hamilton's strong maximum principle for tensors.

We present this well-known fact as a sequence of two lemmas.

Let M be a Riemannian manifold with $Rc \geq 0$, $p \in M$, $B_r = B_r(p)$, $\theta(r) = \theta(B_r) := |\partial B_r|/r^n$, $s(x) := d(x,p)$. Note that $\theta(r)$ is monotone nonincreasing. Let h, H, and A be the metric, mean curvature, and second fundamental form of ∂B_r.

Lemma 1. *For every $\theta_0 > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that if*

$$\theta(r) - \theta(3r) \leq \delta, \qquad \theta(3r) \geq \theta_0,$$

then

$$|Rc(\nu,\nu)| + \left|A - \frac{H}{n-1}h\right|^2 + \left|H - \frac{n-1}{s(x)}\right|^2 \leq \frac{\varepsilon}{r^2} \tag{1}$$

in $B_{3r} \setminus B_r \setminus Z_0$, where Z_0 is some set of measure less than εr^n. Furthermore, there is $Y_0 \subseteq \partial B_r$ such that $|Y_0| \leq \varepsilon r^{n-1}$ and every $x \in \partial B_r \setminus Y_0$ is on a minimizing geodesic from p to a point in ∂B_{3r}.

The elementary proof of this lemma involves tracing through the quantities in the Bishop-Gromov monotonicity formula.

Now let (M_i, g_i, p_i) be a sequence of pointed Riemannian manifolds, $B^i_r := B^{g_i}_r(p_i)$, $r_i \to 0$, $\Omega_i := B^i_{1/r_i} \setminus B^i_{r_i}$, and $d(q_i, p_i) = 1$. Suppose that

$$(\Omega_i, g_i, q_i) \to (\Omega, g, q)$$

as pointed manifolds in the sense of Cheeger, that is, (Ω, g) is a Riemannian manifold (not complete in this case) and there are diffeomorphisms

$$\phi_i : U_i \subseteq \Omega_i \xrightarrow{\cong} V_i \subseteq \Omega$$

such that $\phi_i(q_i) = q$, the U_i exhaust the Ω_i (in a certain sense), V_i exhausts Ω, and

$$(\phi_i)_*(g_i) \to g \quad \text{in } C^\infty,$$

on compact subsets of Ω.

Lemma 2. *Assume that $Rc_{g_i} \geq 0$ and $\theta(B^i_{1/r_i}) - \theta(B^i_{r_i}) \to 0$. Then Ω has a smooth cone structure, that is,*

$$\Omega = (0,\infty) \times (\Sigma, g_\Sigma), \qquad g = ds^2 + s^2 g_\Sigma,$$

for some smooth Riemannian $(n-1)$-manifold (Σ, g_Σ).

Note that the situation of the lemma would arise (using Cheeger's compactness theorem and Hamilton's curvature estimates) for the blowdown of an ancient Ricci flow of nonnegative curvature operator and quadratic curvature decay.

Lemma 1 implies Lemma 2. If everything were known to converge smoothly, then it would simply be a matter of integrating up the quantites that vanish in

(1). But one must take care (and use the minimizing property of the geodesics in the second part of Lemma 1) because the limit of distance functions $s(x) := d_{g_i}(\phi_i^{-1}(x), p_i)$ and its level-sets are not known a-priori to be C^2.

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An examination of surgery strategies

Dan Knopf

In geometric evolution equations such as mean curvature flow and Ricci flow, a solution that becomes singular (in the sense that its curvature becomes infinite at some finite time) will typically asymptotically approach a standard model as the singularity time is reached. This behavior is analogous to (but is more complicated than) that observed for semilinear parabolic equations of the form $u_t = \Delta u + u^p$. (See for instance [3, 4].) If these standard singularity models are understood well enough, a natural way of resolving developing singularities is to perform geometric-topological surgeries at discrete singularity times $t_1 < t_2 < \cdots$. In each such surgery, an (almost) standard neighborhood of the developing singularity is removed and replaced in such a way that curvature control is regained, at least for a short time. (See [5], [6, 7], [9, 10], and [8].) Specifically, a surgery program assigns to every initial manifold (M^n, g_0) a discrete sequence of times $0 = t_0^- < t_0^+ = t_1^- < t_1^+ < \cdots$ together with a family M_k^n of smooth manifolds and metrics $g(t)$ such that

- i each $(M_k, g(t))$ is a smooth geometric evolution for $t_k^- \leq t < t_k^+$;
- ii each $(M_{k+1}, g(t_{k+1}^-))$ is obtained from $(M_k, \lim_{t \nearrow t_k^+} g(t))$ by replacing part of M_k by a (possibly empty) geometrically standard piece; and
- iii each replacement is triggered by parameters depending only on the initial data, and reduces the curvature below a fixed level also depending only on the initial data.

For Ricci flow in dimension three, the most natural mechanism of singularity formation and conjecturally the most common is the *neck*: a region that resembles a quotient of the shrinking cylinder $ds^2 + \rho^2 g_{S^2}$. (See below for a precise definition.) In the mathematical literature, one finds two main strategies for detecting and performing surgery on necks. The first was pioneered by Hamilton [5] and is applied to mean curvature flow by Huisken–Sinestrari [7]. The second was introduced by Perelman [9, 10]. In what follows, we briefly review and contrast some basic elements of their approaches. Almost everything below is taken from [5], [9, 10], and the Kleiner–Lott notes [8].

A motivating example. Consider $[-L, L] \times S^2$ with metric $g = ds^2 + r^2 g_{S^2}$, where g_{S^2} is a standard round metric. Suppose a singularity forms at the point $\{0\} \times S^2$ at time $T < \infty$. (This happens for an open set of rotationally symmetric metrics on S^3; see [1].) Define $\sigma = \frac{s}{\sqrt{T-t}}$ and $\tau = \log \frac{1}{T-t}$. Let $\rho(t) = \sqrt{2(T-t)}$ denote the radius of the standard self-similar cylinder solution. Then one has the precise asymptotics $\frac{r}{\rho} \approx 1 + \frac{\sigma^2 - 2}{8\tau}$ for $\frac{|\sigma|}{\sqrt{\tau}} = \mathcal{O}(1)$ and $\frac{r}{\rho} \approx 1 + \frac{\sigma^2}{8(\tau + \log \frac{1}{\sigma^2})}$ for $e^{-\delta\tau}|\sigma| = \mathcal{O}(1)$. (See [2].)

Topological and geometric necks. For $a < b$, define the cylinder $C^3 = [a, b] \times S^2 \subset \mathbb{R}^4$. A *topological neck* in a manifold M^3 is a local diffeomorphism $N : C^3 \looparrowright M^3$. Let G denote the Riemannian metric on M^3; let $g = N^*(G)$ denote the pullback metric on C^3; and let $\bar{g}$ denote the induced (product) metric on C^3, normalized to have scalar curvature $\bar{R} = 1$. For $x \in [a, b]$, the *mean radius* of $S^2(x) := \{x\} \times S^2$ is defined as

$$r(x) = \sqrt{\frac{\mathrm{Area}(S^2(x), g)}{8\pi}},$$

and the area-normalized pullback metric is $\hat{g} = r^{-2} g$.

For $\varepsilon > 0$, $k \geq 1$, and $L > 0$, one says a topological neck N is *geometrically* (ε, k, L)*-cylindrical* if

I $|\hat{g} - \bar{g}|_{\bar{g}} \leq \varepsilon$;
II $\sum_{j=1}^{k} \left|\bar{\nabla}^j \hat{g}\right|_{\bar{g}} \leq \varepsilon$;
III $\sum_{j=1}^{k} \left|\frac{\partial^j}{\partial x^j} \log(r(x))\right| \leq \varepsilon$; and
IV $b - a \geq L$.

Perelman's formulation. An (ε, ρ)*-neck* in the sense of Perelman is a metric ball $B_t := B_{g(t)}(x, \frac{\rho}{\varepsilon})$ such that $(B_t, \rho^{-2} g(t))$ is an $(\varepsilon, \lceil\varepsilon\rceil + 1, 2\varepsilon^{-1})$ cylindrical neck. A *strong* (ε, ρ)*-neck* in the sense of Perelman is a family $\{B_s = B_s(x, \frac{\rho}{\varepsilon}) : t - \rho^2 \leq s \leq t\}$ of balls such that each $\left(B_s, \frac{1}{(1+t-s)\rho^2} g(s)\right)$ is an $(\varepsilon, \lceil\varepsilon\rceil + 1, 2\varepsilon^{-1})$ cylindrical neck. An (ε, ρ)*-cap* is the union of an (ε, ρ)-neck and either B^3 or else $\mathbb{R} \times_{\mathbb{Z}_2} S^2 \approx \mathbb{RP}^3 \backslash \bar{B}^3$.

Normal necks. One says a topological neck N is *normal* if

A for all $x \in [a, b]$, $S^2(x)$ is a constant mean curvature surface in (C^3, g);
B for all $x \in [a, b]$, the map id : $(S^2(g), \bar{g}) \to (S^2(x), g)$ is harmonic with center of mass $(x, \vec{0})$;
C for all $x \in [a, b]$ and all Killing vector fields $\bar{V}$ of $\bar{g}|_{S^2(x)}$, one has

$$\int_{S^2(x)} \bar{g}(\bar{V}, \nu)\, dA = 0,$$

where ν is the g-unit normal to $S^2(x)$; and
D for all $[\alpha, \beta] \subseteq [a, b]$, $\mathrm{Vol}([\alpha, \beta] \times S^2, g) = 8\pi \int_\alpha^\beta r(x)^3\, dx$.

One says N is a *maximal normal* (ε, k, L)*-neck* if for any (ε, k, L)-neck N_0 such that $N = N_0 \circ F$, the map $F : \mathrm{dom}(N) \to \mathrm{dom}(N_0)$ is onto. Normal necks enjoy good existence, uniqueness, and maximal extension properties. (See [5] for precise statements.) In particular, one has the following important maximality result.

Theorem 1 (Hamilton)**.** *For all $L > 0$, there exist $\varepsilon > 0$ and $k \geq 1$ such that any normal (ε, k, L)-neck is contained in a maximal normal (ε, k, L)-neck, unless $M^3 \approx (\mathbb{R} \times S^2)/\Gamma$.*

Neck detection. Let N be a topological neck and let $P \in C^3$. One says the curvature at P is (ε, k, L)*-cylindrical* if $|\operatorname{Rm} - \overline{\operatorname{Rm}}|_{\bar g} \leq \varepsilon$ and $\sum_{j=1}^{k} |\bar\nabla^j \operatorname{Rm}|_{\bar g} \leq \varepsilon$ in $B_{\bar g}(P, L)$. This property makes it possible to detect necks, as follows.

Theorem 2 (Hamilton)**.** *For every (ε, k, L), there exists (ε', k', L') such that if the curvature at a point P of a topological neck is (ε', k', L')-cylindrical, then P lies at the center of a geometric (ε, k, L)-neck.*

Canonical neighborhoods. In order to perform surgery, one must know that every region of sufficiently high curvature has a standard form. Let $\varphi : [1, \infty) \to (0, \infty)$ be monotone with $\lim_{R\to\infty} \varphi(R) = 0$ and let $r : [0, \infty) \to (0, \infty)$ be monotone decreasing. Perelman says a Ricci flow surgery program is φ*-pinched* if for all (x, t) with $R(x, t) \geq 1$, one has

$$\frac{1}{R} \operatorname{Rm} \geq -\varphi(R)\, g \mathbin{\bar\wedge} g.$$

Perelman says a Ricci flow surgery program satisfies the *Canonical Neighborhood Assumption* if every (x, t) with $R(x, t) \geq r^{-2}$ belongs to a parabolic neighborhood $\{B_{g(s)}(x, t) : t - \varepsilon^2 r^2 \leq s \leq t\}$ whose properties match those of neighborhoods in ancient κ-solutions (singularity models), as set forth in the following.

Theorem 3 (Perelman)**.** *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exist $c_1(\varepsilon) \geq \frac{30}{\varepsilon}$ and $c_2(\varepsilon) \geq 3$ such that for each (x, t) in any ancient κ-solution, there exist r and B with*

$$r \in \left[\frac{1}{c_1 \sqrt{R(x,t)}}, \frac{c_1}{\sqrt{R(x,t)}} \right]$$

and $B_{g(t)}(x, r) \subseteq B \subseteq B_{g(t)}(x, 2r)$ such that either

(1) B is the final time slice of a strong $(\varepsilon, \varepsilon r)$-neck;
(2) B is an $(\varepsilon, \varepsilon r)$-cap;
(3) B is a closed manifold diffeomorphic to S^3 or $\mathbb{RP}^3$; or
(4) B is a quotient of the round sphere.

Moreover, at time t, one has

a $|\nabla R| < \eta R^{3/2}$ and $|R_t| < \eta R^2$ everywhere;
b $\frac{1}{c_2} R(x, t) \leq R \leq c_2 R(x, t)$ in B;
c $\mathrm{Vol}(B) \geq \frac{1}{c_2} (R(x, t))^{-3/2}$ in cases (1)–(3); and
d $\mathrm{sect} \geq \frac{1}{c_2} R(x, t)$ in case (3).

The critical issue in Perelman's surgery program is to show that the Canonical Neighborhood Assumption continues to hold after the first surgery time.

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The compactness of the set of κ-solutions

Bernhard List

The content of the talk was the following compactness theorem of Perelman [1, §11.7] which is used to prove a gradient estimate for the scalar curvature, uniform on the set of κ-solutions.

Here a κ-solution is defined to be an ancient, complete, non-flat, κ-noncollapsed solution to the Ricci Flow with bounded sectional curvature and positive curvature operator.

The theorem is a crucial tool in Perelman's paper, since the further study of regions with high curvature near a singular point is done by comparing a general solution with an approximate κ-solution. The precise statement is

Theorem *The set of pointed κ-solutions is compact modulo scaling, i.e. from every sequence of such solutions and base points $(M_k, x_k, g_k(t))$ satisfying $R(x_k, 0) = 1$ we can extract a smooth converging subsequence, whose limit is also a pointed κ-solution.*

Since the original proof is very short, a more detailed exposition following the lines of Perelman's proof was given, including a careful inspection of the crucial point-picking arguments and the constructions of the collapsing balls.

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The canonical neighbourhood assumption

Jan Metzger

The key to the surgery procedure for the three dimensional Ricci flow as it is proposed in Perelman [1],[2] is the so called canonical neighborhood assumption. This assumption, which is actually a property, guarantees that the Ricci flow, as it goes singular, approaches the special geometry of a long neck. The knowledge of this fact then allows the construction of a surgery procedure adapted to this geometry.

In Theorem 12.1 of his first paper, Perelman establishes this assumption in the case that at the time of interest, no surgeries were made for some time.

In the program of constructing a solution to Ricci flow with surgery, this theorem allows one to get off the ground and perform surgery at the first singular time.

In addition the proof of the theorem contains important ideas, that are used later in the program. A modified version of this proof is used in Chapter 5 of [2] to establish the canonical neighborhood assumption in the presence of surgeries in the past.

The full statement of Theorem 12.1 from [1] is:

Theorem *Let $\varepsilon > 0$, $\kappa > 0$ and a decreasing function $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\lim_{s\to\infty}\phi(s) = 0$ be given. Then there exists $r_0 = r_0(\epsilon, \kappa, \phi)$ with the following property.*

Let $(M, g(t))$ be a family of Riemannian manifolds satisfying the Ricci flow equation

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}$$

for $t \in [0, T]$ and the following assumptions:

(1) (M,g(t)) is κ-noncollapsed on scales $< r_0$, i.e. for all $(x, t) \in M \times [0, T]$ and $0 < r < r_0$, whenever $|Rm| \leq r^{-2}$ on $B(x, r) \times [t - r^2, t]$ the volume of this ball with respect to $g(t)$ satisfies $r^{-n}\mathrm{Vol}(B(x, r)) \geq \kappa$.

(2) (M,g(t)) is ϕ-almost nonnegatively curved, i.e.

$$\mathrm{Rm}(x, t) \geq -\phi(R(x, t))R(x, t)$$

for each (x, t) with $R(x, t) \geq 1$.

Then if $(x_0, t_0) \in M \times [0, T]$ is a point with $t_0 \geq 1$ and $Q := R(x_0, t_0) \geq r_0^{-2}$, then in the parabolic neighborhood

$$B_{t_0}(x_0, (\varepsilon Q)^{-1/2}) \times [t_0 - (\varepsilon Q)^{-1}, t_0]$$

after rescaling, the metric $g(t)$ is close to a κ-solution. Here $\mathrm{Rm}(x, t)$ denotes the Riemannian curvature operator of $g(t)$ at $x \in M$, R denotes the scalar curvature and $B_t(x, r)$ denotes the metric ball around x with radius r measured with respect to the metric $g(t)$.

A κ-solution to the Ricci flow is a solution on the time interval $(-\infty, 0]$, which is complete, non-flat, κ-noncollapsed, has nonnegative curvature operator and bounded curvature on every time slice.

Such solutions are deeply analyzed in Chapter 11 of [1] and portions of them provide the canonical neighborhoods from the introduction.

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Finite extinction time—report on a paper of Colding and Minicozzi

Leon Simon

The finite extinction time result of Colding and Minicozzi [2], like the alternative finite extinction time result originally proposed by Perelman, is designed to ensure that if we start with a counterexample $M = M^3$ to the Poincaré conjecture (we stick to this case in the present discussion, although Colding and Minicozzi's paper applies somewhat more generally—indeed to any $M = M^3$ with $\pi_3(M) \neq 0$) then the Hamilton Ricci flow with surgeries can exist for at most a fixed time $T < \infty$ depending only on the initial data $(M, g(0))$.

This effectively means, since each Hamilton surgery takes place at a fixed scale and results in a fixed reduction in the volume of M each time it is performed, that only finitely many surgeries would be required before an immediate contradiction is obtained (modulo checking the appropriate facts about the discreteness of surgery times and preservation of various bounds after each surgery), thus significantly simplifying the Ricci flow approach to the Poincaré conjecture as distinct from the Ricci flow approach proposed to settle the full Thurston geometrization conjecture.

The general approach to the proof of finite extinction time (both by Perelman and Colding-Minicozzi) is the following:

Suppose that there is a bounded function $w : [0, T) \to (0, \infty)$ (not necessarily continuous) such that

$$\frac{\bar{d}w(t)}{dt} \leq -\theta - \frac{\alpha}{t+\beta} w(t) \tag{1}$$

for all $t \in [0, T)$, where θ, β are given positive constants and $\alpha \in (0, 1)$ is also a given constant, and where $\frac{\bar{d}f(t)}{dt} = \limsup_{h \downarrow 0} \frac{f(t+h)-f(t)}{h}$ for any f. Then we must have

$$T \leq \Big(\frac{1-\alpha}{\theta} \beta^{-\alpha} w(0) + \beta^{1-\alpha}\Big)^{1/(1-\alpha)} - \beta. \tag{2}$$

This is proved in the obvious way, by observing that (1) implies that the function $(t+\beta)^{-\alpha} w(t) + \frac{\theta}{1-\alpha}(t+\beta)^{1-\alpha}$ is decreasing on $[0, T)$. Furthermore (since we want to apply the above principle in a setting where t is the time variable of a Hamilton Ricci flow $(M, g(t))$ in the case when surgeries are allowed) we should note that we will still be able to prove the inequality (2) if there are $0 = t_0 < t_1 < t_2 < \cdots < t_Q < T = t_{Q+1}$ such that (1) holds on each interval $[t_{j-1}, t_j)$, $j = 1, \ldots, Q+1$, provided that

$$\limsup_{t \uparrow t_j} w(t) \le w(t_j), \quad j = 1, \ldots, Q. \tag{3}$$

The idea is to find a suitable function $w(t)$ on $(M, g(t))$, where $(M, g(t))$ is a Hamilton Ricci flow with surgeries, so that the above general principle can be applied. A candidate for such a function is the "width" function w of $(M, g(t))$, which we proceed to define. Assume that our 3-dimensional manifold M is a homotopy 3-sphere which is not diffeomorphic to S^3, so that in particular $\pi_3(M) \approx \mathbb{Z}$, and let

$$w(t) = \inf_{\gamma \in \Gamma_{p,q,\lambda}(M)} \max_{s \in [0,1]} \mathcal{E}_{g(t)}(\gamma_s). \tag{4}$$

Here the notation is as follows: p, q are any pair of points (not necessarily distinct) in M, $\Gamma_{p,q,\lambda}(M)$ is the set of C^1 maps $\gamma : S^3_* \to M$ which are homotopic to a fixed continuous $\lambda : S^3_* \to M$ which generates $\pi_3(M)$, where S^3_* is the piecewise C^1 homeomorph of the standard S^3 defined by $S^3_* = (\{0\} \times B^3) \cup ([0,1] \times S^2) \cup (\{1\} \times B^3)$, with $B^3 = \{x \in \mathbb{R}^3 : |x| \le 1\}$, and where $\gamma|\{0\} \times B^3 \equiv p$, $\gamma|\{1\} \times B^3 \equiv q$, $\gamma_s(\omega) = \gamma(s, \omega)$, $(s, \omega) \in (0,1) \times S^2$, γ_0, γ_1 are the constant maps from S^2 into M taking values p, q respectively (thus $\{\gamma_s\}_{s \in [0,1]}$ is a continuous one-parameter family of C^1 maps $S^2 \to M$), and $\mathcal{E}_{g(t)}$ is the energy functional defined for C^1 maps of S^2 into $(M, g(t))$ by

$$\mathcal{E}_{g(t)}(\varphi) = \frac{1}{2} \int_{S^2} |d\varphi_{|_\omega}|^2_{g(t)}\, d\omega = \frac{1}{2} \int_{S^2} \left(|d\varphi_{|_\omega}(\tau_1)|^2_{g(t)} + |d\varphi_{|_\omega}(\tau_2)|^2_{g(t)} \right) d\omega,$$

where τ_1, τ_2 are any orthonormal basis for $T_\omega(S^2)$, $\omega \in S^2$. For maps $\gamma \in \Gamma_{p,q,\lambda}(M)$, the corresponding families $\{\gamma_s\}_{s \in [0,1]} \in C^1([0,1]; C^1(S^2; M))$ are referred to as "sweep-outs" of M. The area functional $\mathcal{A}_{g(t)}$ is

$$\mathcal{A}_{g(t)}(\varphi) = \int_{S^2} |d\varphi_{|_\omega}(\tau_1) \wedge d\varphi_{|_\omega}(\tau_2)|_{g(t)}\, d\omega,$$

and it is a standard fact that

$$\mathcal{A}_{g(t)}(\varphi) \le \mathcal{E}_{g(t)}(\varphi) \text{ with equality } \iff \varphi \text{ is conformal} \tag{5}$$

in the sense that $|d\varphi_{|_\omega}(\tau_1)| \equiv |d\varphi_{|_\omega}(\tau_2)|$ and $\langle d\varphi_{|_\omega}(\tau_1), d\varphi_{|_\omega}(\tau_2)\rangle \equiv 0$. It can then be checked that the above width function can equivalently be defined in terms of the area functional, because

$$\inf_{\gamma \in \Gamma_{p,q,\lambda}(M)} \max_{s \in [0,1]} \mathcal{E}_{g(t)}(\gamma_s) = \inf_{\gamma \in \Gamma_{p,q,\lambda}(M)} \max_{s \in [0,1]} \mathcal{A}_{g(t)}(\gamma_s). \tag{6}$$

Colding and Minicozzi in [2] prove that in fact the function w in (4) satisfies an inequality of the form (1) with $\theta = 4\pi$, $\alpha = \frac{3}{4}$, and β any number in $(0, \frac{-1}{\min_M R_{|t=0}}]$ if $\min_M R_{|t=0} < 0$ and with any $\beta > 0$ if $\min_M R_{|t=0} \ge 0$. They establish this by using known results about the "min-max" in (4)—in particular the facts that for each given $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ and a sequence $\gamma^j \in \Gamma_{p,q,\lambda}(M)$

(also depending on ϵ) such that

min-max (i): $\max_{s\in[0,1]} \mathcal{E}_{g(t)}(\gamma^j_s) \to w(t)$

min-max (ii): $d_{\mathrm{var}}(\cup_i \Sigma^j_{is}, \gamma^i_s) < \epsilon \ \ \forall s \in [0,1]$ such that $\mathcal{E}_{g(t)}(\gamma^j_s) > w(t) - \delta$

(finite union over i), where each Σ^j_{is} is a branched conformal minimal immersion and d_{var} is the varifold metric defined by

$$d_{\mathrm{var}}(\Sigma_1, \Sigma_2) = \sup_{|f|\le 1,\, \mathrm{Lip} f\le 1} \Big| \int_{\Sigma_1} f(x, \nu_{\Sigma_1}) - \int_{\Sigma_2} f(x, \nu_{\Sigma_2}) \Big| \tag{7}$$

for any pair of branched immersed surfaces Σ_1, Σ_2 in M, where ν_{Σ_j} is a continuous unit normal for Σ_j and where the functions f referred to in the sup are Lipschitz maps $f : \cup_{x\in M}\{x\} \times (T_xM)^\perp \to \mathbb{R}$.

The properties min-max(i), (ii) are justified in the paper [2] by appealing to a general theory developed by Jost [3] (which ensures a property like min-max(ii) with respect to weak convergence in $W^{1,2}$ for some s), and the proof that the paths γ^j_s can be modified to ensure the implication in min-max(ii) holds with respect to the metric d_{var} for <u>all</u> s such that $\mathcal{E}_{g(t)}(\gamma^j_s) > w(t) - \delta$ is justified by appealing to [1]. However the reference [3] is restricted to a classical setting and does not explicitly discuss approximation in the varifold metric of min-max(ii) (although it seems clear that this could be done by using the harmonic replacement property), while the paper [1] actually refers to a minimax setting which is a little different than that mentioned above; in view of the importance of the results under consideration, it would perhaps be desirable to have a self-contained justification of min-max(i),(ii).

With min-max(i),(ii) available it is then relatively straightforward to prove the claimed inequality (1) with the relevant values of α, β, θ; specifically

$$\frac{\bar{d}w(t)}{dt} \le -4\pi - \frac{3/4}{t+\beta} w(t) \tag{8}$$

where β is arbitrary $\in \big(0, \frac{-1}{\min_M R_{|t=0}}\big]$ if $\min_M R_{|t=0} < 0$ and with $\beta > 0$ completely arbitrary if $\min_M R_{|t=0} \ge 0$.

To prove this Colding-Minicozzi first recall the evolution equation $R_t = \Delta R + 2|Ric|^2 \ge \Delta R + \frac{2}{3}R^2$ which via a standard comparison/maximum principle argument (comparing with solutions of the ODE $\varphi' = \frac{2}{3}\varphi^2$ with $\varphi(0) = \min_M R_{|t=0}$ in case $\min_M R_{|t=0} \le 0$ and just using the normal parabolic maximum principle for $R_t \ge \Delta R$ in case $\min_M R_{|t=0} \ge 0$) gives the lower bound

$$R_{|t} \ge \frac{-3}{2(t+\beta)}, \tag{9}$$

where β is arbitrary $\in \big(0, \frac{-1}{\min_M R_{|t=0}}\big]$ in case $\min_M R_{|t=0} < 0$ and $\beta > 0$ is completely arbitrary in case $\min_M R_{|t=0} \ge 0$.

Then the identity $\frac{d}{dt}\mathcal{A}_{g(t)}(\Sigma) = -\int_\Sigma (R - Ric(\nu_\Sigma, \nu_\Sigma))$ is used, together with the min-max(i),(ii) above at a specific time τ to give a sequence of sweep outs $\gamma^j_s(\tau)$, and the Gauss-Bonnet theorem on Σ^j_{is}, to verify directly that, if s is such that $\mathcal{E}_{g(\tau)}(\gamma^j_s) \ge w(\tau) - \delta$, then

$$\frac{d}{dt}_{|t=\tau}\mathcal{A}_{g(t)}(\gamma^j_s(\tau)) \le -4\pi - \frac{1}{2}\int_{\gamma^j_s(\tau)} R + C_1\epsilon \le -4\pi + \frac{1}{2}\mathcal{A}_{g(\tau)}(\gamma^j_s(\tau))(-\min_M R_{|t=\tau}) + C_2\epsilon,$$

where C_1, C_2 depend on $|Ric_{|t=\tau}|_{C^1}$ but do not depend on j or s, and the sweep-out $\{\gamma^j_s\}_{s\in[0,1]}$ does not depend on t (the sweep-out is selected so that min-max(i),(ii) hold at the fixed time $t=\tau$). Then by the lower bound (9)

$$\frac{d}{dt}\mathcal{A}_{g(t)}(\gamma^j_s) \le -4\pi + \frac{3/4}{t+\beta}w(t) + C_3\epsilon.$$

By a Taylor approximation to order 2 we then have

$$h^{-1}\big(\mathcal{A}_{g(\tau+h)}(\gamma^j_s) - \mathcal{A}_{g(\tau)}(\gamma^j_s)\big) \le -4\pi + \frac{3/4}{t+\beta}w(t) + C_4\epsilon + C_5 h$$

for sufficiently small $h>0$ (independent of τ, j), and, since $\mathcal{A}_{g(\tau)}(\gamma^j_s) \le \mathcal{E}_{g(\tau)}(\gamma^j_s) \le \max_{s\in[0,1]}\mathcal{E}_{g(\tau)}(\gamma^j_s) \to w(\tau)$, we conclude, after appealing to (6) and then letting $j\to\infty$, that

$$h^{-1}(w(\tau+h) - w(\tau)) \le -4\pi + \frac{3/4}{t+\beta}w(t) + C_4\epsilon + C_5 h,$$

and hence (8) is established by taking $\limsup_{h\downarrow 0}$, and then letting $\epsilon \downarrow 0$.

Colding and Minicozzi make no explicit discussion of the fact that the width cannot jump up after a surgery, but after a remark of Tom Ilmanen at the workshop this seems quite clear (at least in the context of counterexamples to the Poincaré conjecture) from the following argument: Assume without loss of generality that M is a prime homotopy 3-sphere which is a counterexample to the Poincaré conjecture; then after a standard Hamilton surgery we get two components, one which is a standard S^3 and the other, $\widetilde{M}$ say, which is again a prime homotopy 3-sphere which is a counterexample to the Poincaré conjecture. In fact the surgery is by definition such that there is a diffeomorph Σ of S^2 contained in M such that $M\setminus\Sigma$ has two components $M_\pm$, with $\overline{M}_+$ a fake 3-ball and $\overline{M}_-$ a standard 3-ball, and $\widetilde{M}$ is the image of M under a distance decreasing transformation $\Psi : M \to \widetilde{M}$ which is C^1 and such there is a point $x_0 \in \widetilde{M}$ with $\Psi|M_+$ a diffeomorphism onto $\widetilde{M}\setminus\{x_0\}$ and $\Psi(M_-) = \{x_0\}$, so (since M_- is a standard 3-ball) Ψ gives a homotopy equivalence $M \approx \widetilde{M}$. Therefore $\Gamma_{p,q,\lambda}(M)$ is mapped bijectively to $\Gamma_{\widetilde{p},\widetilde{q},\widetilde{\lambda}}(\widetilde{M})$, where $\widetilde{p} = \Psi(p)$, $\widetilde{q} = \Psi(q)$, $\widetilde{\lambda} = \Psi\circ\lambda : S^3_* \to \widetilde{M}$ is a generator for $\pi_3(\widetilde{M})$, and $\gamma \in \Gamma_{p,q,\lambda}(M)$ maps to $\widetilde{\gamma} = \Psi\circ\gamma \in \Gamma_{\widetilde{p},\widetilde{q},\widetilde{\lambda}}(\widetilde{M})$, which trivially has $\mathcal{E}_{\widetilde{g}}(\widetilde{\gamma}_s) \le \mathcal{E}_g(\gamma_s)$ for each s because Ψ is distance decreasing. Thus w computed on $\widetilde{M}$ at times after the surgery has starting value which is $\le$ the value of w computed on M at the time of surgery.

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A priori estimates for the asymptotic scalar curvature ratio on κ-noncollapsed, positively curved solutions to the Ricci flow.

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Let $(M, g(t))_{t\in(-\infty,1]}$ be a κ-noncollapsed solution to Ricci-flow, with non-negative curvature operator. Let A be the asymptotic scalar curvature, and assume $0 < A < \infty$. Fix a base point $p \in M$. We can then find a sequence of points $x_k \in M$ so that

$$\begin{aligned} d^2_{t=0}(x_k, p) &\to \infty \text{ as } k \to \infty, \\ R(x_k, 0)d^2_{t=0}(x_k, p) &\to A \text{ as } k \to \infty. \end{aligned} \tag{1}$$

This implies that $R(x_k, 0) \to \infty$ as $k \to \infty$. We set

$$g_k(\cdot, t) := R(x_k, 0)g(\cdot, \frac{t}{R(x_k, 0)}).$$

Notation: $R_k(x, t) := R(g_k(t))(x)$ and ${}^k d_t(x, y) := dist(g_k(t))(x, y)$. Then we have scaled so that $R_k(x_k, t) = 1$, and furthermore ${}^k d_0(x_k, p) \le 2A$. In particular, the Harnack inequality gives us that

$$R_k(x, t) \le \frac{2A}{{}^k d^2_0(x, p)},$$

for all ${}^k d^2_0(x, p) \ge b > 0$. Defining the sets:

$$\Omega_k := \{x \in M | b < {}^k d_0(x, p) < B, \},$$

we see that Ω_k is non-empty, since it contains x_k, and using the convergence theorems of Cheeger-Anderson (see [6], Section on convergence of Riemannian Manifolds) and Hamilton (see [3]) we obtain (in view of the fact that our solutions are κ- noncollapsed) a convergent subsequence to a smooth open Riemannian manifold Ω, and a solution to the Ricci flow $(\Omega, \tilde{g}(t)_{t\in(-1,0]})$. Returning to the original manifolds we see that $(M, {}^k g(0), p) = (M, \lambda_k g(\cdot, 0), p)$, where $\lambda_k \to 0$. A theorem of Gromov in [1, pages 58-59] gives: Such a sequence of metric spaces converges in the sense of Gromov-Hausdorff to a Tits cone C.

In our case this cone is positively curved and non-flat as $R_k(x_k, 0) = 1$. This would mean that $(\Omega, g(0))$ is a piece of a non-flat cone, which would contradict the maximum principle for the evolving curvature operater (see [4]). One open issue is to relate the two notions of convergence: Gromov-Hausdorff and Cheeger/Hamilton. It would be interesting to see if we could prove that such an asymptotic scalar curvature is impossible, without having to resort to the Gromov-Hausdorff convergence and Tits cones.

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Gradient estimates in Ricci flow and in mean curvature flow

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A fundamental step in Perelman's analysis of the singularities of 3-dimensional Ricci flow is an estimate of the scalar curvature. This follows combining Theorem 12.2 in [2] and the statement §1.5 in [3]. The former theorem is the well-known *canonical neighbourhood* property, saying that for any solution on a closed three-manifold and any $\varepsilon > 0$ there exists $r_0 > 0$ such that any point where $R > r_0^{-2}$ has a parabolic neighbourhood which is ε-close to an ancient κ-solution. The latter property, which was basically proved in §11 of [2], says that at any point of any ancient κ-solution we have the estimates

$$|\nabla R|^2 \leq \eta R^3 \qquad |R_t| \leq \eta R^2,$$

for some universal constant η not depending even on κ.

Rather than giving a full analysis of the ingenious proof of these estimates, in this short note we will point out the analogies and the differences with a gradient estimate for the mean curvature flow contained in the forthcoming paper [1].

Theorem *Let $\mathcal{M}_t$, $t \in [0, T[$ be smooth closed n-dimensional surfaces immersed in $\mathbb{R}^{n+1}$ evolving by mean curvature, with $n \geq 3$. Suppose that $\mathcal{M}_0$ is two-convex (i.e. the sum of the two smallest principal curvatures is positive everywhere). Then there exist $c, C > 0$ such that the inequalities*

$$|\nabla A|^2 \leq c|A|^4 + C, \qquad |\partial_t A| \leq c|A|^3 + C$$

hold everywhere on $\mathcal{M}_t$ for all $t \in [0, T[$. Here A denotes the second fundamental form. The constant C depends on the initial data $\mathcal{M}_0$ while c only depends on the dimension n.

Although the statements are very similar, the proof of the result is completely different in the two cases. The result for the mean curvature flow is proved using the maximum principle. The subtle part of the proof consists of the choice of the test function to which the maximum principle will be applied. The possibility of defining the test function relies on other estimates obtained previously, which use the two–convexity assumption. The proof also uses in an essential way the sharp inequality $|\nabla H|^2 \leq \frac{n+2}{3}|\nabla A|^2$, where H is the mean curvature.

Perelman's argument for the Ricci flow uses quite different arguments. As we have seen, he first analyzes the ancient κ-solutions, and then he shows that regions with sufficiently high curvature of an arbitrary solution are close to κ-solutions. The noncollapsing property plays a central role throughout the analysis, as well as other tools, like the behaviour at infinity of complete manifolds of positive curvature and the Toponogov splitting theorem. The central part of the argument is a compactness theorem for κ-solutions, which gives the gradient estimates as a corollary. The canonical neighbourhood property is then derived by a delicate contradiction argument using techniques from the theory of Alexandrov spaces.

It is remarkable that in both cases the coefficient of the leading term of the gradient estimate only depends on the dimension and not on the data. For the mean curvature flow, this term can be explicitly evaluated by looking at the evolution equation to which the maximum principle is applied. In the case of the Ricci flow, the coefficient could depend a priori on the noncollapsing constant κ. However, Perelman also proves that κ cannot be smaller than some fixed κ_0, except for the quotients of the sphere S^3. Thus, the constant η in the inequality can be chosen independently of κ.

In Perelman's proof many statements are obtained by contradiction arguments. In some respect it would be desirable to have more direct methods, like in a proof by maximum priciple, in order to have a more explicit dependence of the constants; this might also possibly simplify the analysis of the surgeries afterwards. However, finding an alternative approach to these estimates for the Ricci flow seems at the present time quite a difficult task.

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