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Algebraische Zahlentheorie

Organised by
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Abstract. The workshop on Algebraic Number Theory was attended by 53 participants, many of them young researchers. In 19 talks an overview of recent developments in Algebraic Number Theory and Arithmetic Algebraic Geometry was given.

Mathematics Subject Classification (2000): 11Sxx, 11Rxx, 11Mxx, 11Gxx, 14Fxx.

Introduction by the Organisers

The meeting was organized by Christopher Deninger (Münster), Peter Schneider (Münster) and Anthony Scholl (Cambridge). The subject of the conference was Algebraic Number Theory and Arithmetic Algebraic Geometry. It brought together internationally leading experts in these areas as well as a considerable number of young researchers. Correspondingly the talks were a mixture of lectures on recent important progress by experts and of reports about outstanding dissertations. This created a very stimulating atmosphere for discussions and exchange of ideas.

Workshop: Algebraische Zahlentheorie

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Abstracts

On the reduction modulo p of some crystalline representations

Laurent Berger
(joint work with C. Breuil)

Introduction

The goal of this talk was to recall part of Breuil's "mod p" and "continuous" p-adic Langlands correspondences (in the supersingular case), and to explain the proof of the conjecture relating the two. Note : most relevant articles are in the bibliographies of the three articles given as references, the first two of which contain the proofs of the two main results discussed in the lecture.

1. Objects in characteristic p

On the $\mathrm{GL}_2(\mathbf{Q}_p)$-side, the objects which we're interested in are the smooth irreducible $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ having a central character. Choose $r \in \{0, \cdots, p-1\}$ and let χ be a smooth character of $\mathbf{Q}_p^\times$. Define

$$\pi(r,\chi) := \left[\left(\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r \overline{\mathbf{F}}_p^2\right)/T\right] \otimes (\chi \circ \det),$$

where T is a Hecke operator defined by Barthel and Livné. The representations $\pi(r,\chi)$ are called supersingular, and any smooth irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ having a central character is either one-dimensional, a special series, a principal series or supersingular. Furthermore, the intertwinings between the $\pi(r,\chi)$'s are given by

$$\pi(r,\chi) = \pi(p-1-r, \chi\omega^r) = \pi(p-1-r, \chi\omega^r\mu_{-1}) = \pi(r, \chi\mu_{-1}),$$

where ω is the mod p cyclotomic character and μ_λ is the unramified character of $\mathbf{Q}_p^\times$ sending p to λ.

On the Galois side, it is easy to classify the irreducible 2-dimensional $\overline{\mathbf{F}}_p$-representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let ω_2 be Serre's fundamental character of level 2, and for $s \in \{1, \cdots, p\}$ let $\mathrm{ind}(\omega_2^s)$ denote the unique irreducible 2-dimensional $\overline{\mathbf{F}}_p$-representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ whose determinant is ω^s and whose restriction to inertia is $\omega_2^s \oplus \omega_2^{ps}$. If $r \in \{0, \cdots, p-1\}$ and if χ is a continuous character of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, let $\rho(r,\chi) := \mathrm{ind}(\omega_2^{r+1}) \otimes \chi$. Any irreducible 2-dimensional $\overline{\mathbf{F}}_p$-representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is isomorphic to some such $\rho(r,\chi)$ and the intertwinings between the $\rho(r,\chi)$'s are given by

$$\rho(r,\chi) = \rho(p-1-r, \chi\omega^r) = \rho(p-1-r, \chi\omega^r\mu_{-1}) = \rho(r, \chi\mu_{-1}).$$

It is therefore natural to define a correspondence : $\pi(r,\chi) \leftrightarrow \rho(r,\chi)$. Breuil has also defined a correspondence between special & principal series and reducible representations but this was not covered in the talk due to lack of time.

2. Objects in characteristic 0

Let E be a finite extension of $\mathbf{Q}_p$, choose $k \geq 2$ and $a_p \in E$ such that $\mathrm{val}(a_p) > 0$.

We start by defining the objects on the Galois side, they are 2-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let D_{k,a_p} be the filtered φ-module $D_{k,a_p} := Ee_1 \oplus Ee_2$ where

$$\begin{cases} \varphi(e_1) = p^{k-1}e_2 \\ \varphi(e_2) = -e_1 + a_p e_2 \end{cases} \quad \text{and} \quad \mathrm{Fil}^i D_{k,a_p} = \begin{cases} D_{k,a_p} & \text{if } i \leq 0, \\ E \cdot e_1 & \text{if } 1 \leq i \leq k-1, \\ 0 & \text{if } i \geq k. \end{cases}$$

We define V_{k,a_p} as the crystalline representation such that $\mathrm{D}_{\mathrm{cris}}(V_{k,a_p}^*) = D_{k,a_p}$ (note that then $V_{k,a_p}^* = V_{k,a_p}(1-k)$) so that V_{k,a_p} is a crystalline representation whose Hodge-Tate weights are 0 and $k-1$. Choose a $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-stable lattice T of V_{k,a_p} and set $\overline{V}_{k,a_p} := (\overline{\mathbf{F}}_p \otimes_{O_E} T)^{\mathrm{ss}}$. This representation does not depend on the choice of T by Brauer-Nesbitt.

We now move to the $\mathrm{GL}_2(\mathbf{Q}_p)$-side, and we define the following locally algebraic representation :

$$\Pi_{k,a_p} := \left(\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^{k-2}\overline{\mathbf{Q}}_p^2 \right) / (T - a_p).$$

Breuil has studied this representation and conjectured the existence of a $\overline{\mathbf{Z}}_p$-lattice. This is now a theorem (see [1] – and note that for simplicity, we assumed that φ is semi-simple on D_{k,a_p}) and we then set $\overline{\Pi}_{k,a_p} := [\overline{\mathbf{F}}_p \otimes_{\overline{\mathbf{Z}}_p} (\text{this lattice})]^{\mathrm{ss}}$. This does not depend on the choice of lattice by Brauer-Nesbitt. Breuil's conjecture regarding the compatibility of the correspondences $\Pi_{k,a_p} \leftrightarrow V_{k,a_p}$ and $\pi(r,\chi) \leftrightarrow \rho(r,\chi)$ is then the following :

Conjecture — We have $\overline{\Pi}_{k,a_p} = \pi(r,\chi)$ if and only if $\overline{V}_{k,a_p} = \rho(r,\chi)$.

Note that there is also a conjecture concerning the "reducible" cases. The above conjecture is now a theorem (see [2]); it was first proved for $k \leq p$ by Breuil, who also computed $\overline{\Pi}_{k,a_p}$ for $k \leq 2p+2$.

3. A new model for Π_{k,a_p}

We start by explaining the proof of the existence of a lattice in Π_{k,a_p}. The idea of the proof (based on a similar construction carried out by Colmez in the semi-stable case) is to construct a Banach representation $\Pi(V_{k,a_p})$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ with a map $\Pi_{k,a_p} \to \Pi(V_{k,a_p})$. One then uses the theory of (φ,Γ)-modules to prove that $\Pi(V_{k,a_p}) \neq 0$ and is irreducible (and admissible) which shows the existence of a lattice of Π_{k,a_p} (the map $\Pi_{k,a_p} \to \Pi(V_{k,a_p})$ being injective with dense image).

Let $\mathcal{O} := \{\sum_{i\in\mathbf{Z}} a_i X^i$, where $a_i \in O_E$ and $a_i \to 0$ as $i \to -\infty\}$. We endow this ring with a frobenius φ such that $\varphi(X) = (1+X)^p - 1$ and an action of $\Gamma := \mathbf{Z}_p^\times$ such that $[a] \cdot X = (1+X)^a - 1$. A (φ,Γ)-module is an $\mathcal{O}$-module $\mathbf{D}$ of finite type, endowed with a frobenius ϕ and an action of Γ which are continuous, semi-linear, commute with each other, and such that $\phi^*(\mathbf{D}) = \mathbf{D}$. Recall that

Fontaine constructed an equivalence of categories between the category of O_E-representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and the category of (ϕ,Γ)-modules. If $V = E\otimes_{O_E} T$ is an E-linear representation, we'll call $\mathbf{D}(V) := E\otimes_{O_E}\mathbf{D}(T)$ the associated (ϕ,Γ)-module.

One can show that if $\mathbf{D}$ is a (φ,Γ)-module and $y\in\mathbf{D}$, then there exist uniquely determined elements $y_0,\cdots,y_{p-1}$ such that $y=\sum_{i=0}^{p-1}(1+X)^i\varphi(y_i)$ and we define $\psi(y) := y_0$, which makes ψ into a Γ-equivariant left inverse of φ. Let V_{k,a_p} be the representation defined in the previous section and let $(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$ denote the set of sequences $(v_0,v_1,\cdots)$ such that $\psi(v_i)=v_{i-1}$ and such that the sequence $\{v_0,v_1,\cdots\}$ is bounded for the "weak topology" (which one can think of as the (p,X)-adic topology). We make $(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$ into a representation of the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ in the following way; if $v\in(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$, then :

$$\left(\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix}\cdot v\right)_n = x_0^{k-2}v_n \text{ where } x=p^{\mathrm{val}(x)}x_0;$$
$$\left(\begin{pmatrix} 1 & 0\\ 0 & p^j\end{pmatrix}\cdot v\right)_n = v_{n-j}=\psi^j(v_n);$$
$$\left(\begin{pmatrix} 1 & 0\\ 0 & a\end{pmatrix}\cdot v\right)_n = [a](v_n);$$
$$\left(\begin{pmatrix} 1 & z\\ 0 & 1\end{pmatrix}\cdot v\right)_n = \psi^m((1+X)^{p^{n+m}z}v_{n+m}),\ n+m\geq -\mathrm{val}(z).$$

The Banach space $\Pi(V_{k,a_p})$ alluded to above can be shown to be isomorphic to another Banach space of "class $\mathcal{C}^r$ functions" (for an r depending on k and a_p) on $\mathbf{P}^1$ satisfying a number of conditions, together with an action of $\mathrm{GL}_2(\mathbf{Q}_p)$. Using ideas of Colmez, we can interpret elements of $(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$ as distributions on $\mathbf{P}^1$, i.e. elements of the dual of this new Banach space and thus we have $\Pi(V_{k,a_p})^*\simeq(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$ as representations of the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. A classical result allows one to interpret Π_{k,a_p} as a space of locally polynomial functions on $\mathbf{P}^1$ satisfying similar conditions as those used to define $\Pi(V_{k,a_p})$ and this explains why Π_{k,a_p} injects into $\Pi(V_{k,a_p})$ (the image is dense because $\Pi(V_{k,a_p})$ is irreducible).

4. Proof of the conjecture

Using the isomorphism $\Pi(V_{k,a_p})^*\simeq(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$, one can show (see [3]) that $\overline{\Pi}_{k,a_p}^*\simeq(\varprojlim_\psi \mathbf{D}(\overline{V}_{k,a_p}))^b$ and this already proves one implication in the conjecture. Now suppose that $\overline{\Pi}_{k,a_p}=\pi(r,\chi)$ for some r,χ. By examination, we see that there exists $k'\in\{2,\cdots,p\}$ such that $\pi(r,\chi)=\overline{\Pi}_{k',0}$ and the conjecture in this case results from computations of Breuil. In particular, $\overline{V}_{k',0}\simeq\rho(r,\chi)$ on the one hand and $\overline{\Pi}_{k',0}^*\simeq(\varprojlim_\psi \mathbf{D}(\overline{V}_{k',0}))^b$ on the other hand so that $(\varprojlim_\psi \mathbf{D}(\overline{V}_{k,a_p}))^b\simeq(\varprojlim_\psi \mathbf{D}(\overline{V}_{k',0}))^b$ and using the theory of (φ,Γ)-modules it is not hard to show that $\overline{V}_{k,a_p} = \overline{V}_{k',0}$ so that $\overline{V}_{k,a_p} = \rho(r,\chi)$. This proves the other implication in the conjecture.

References

[1] L. Berger, C. Breuil – *Représentations cristallines irréductibles de* $\mathrm{GL}_2(\mathbf{Q}_p)$, preprint.

[2] L. Berger, C. Breuil – *Sur la réduction des représentations cristallines de dimension* 2 *en poids moyens*, preprint.

[3] P. Colmez – *Une correspondance de Langlands locale p-adique pour les représentations semi-stables de dimension* 2, preprint.

On number fields with given ramification

Gaëtan Chenevier

Let p be a prime number and $\mathbb{Q}_{\{\infty,p\}}$ a maximal algebraic extension of $\mathbb{Q}$ which is unramified outside p and ∞. A well known problem in algebraic number theory is to determine whether the field embeddings $\mathbb{Q}_{\{\infty,p\}} \longrightarrow \overline{\mathbb{Q}}_p$ have a dense image, or equivalently if the associated (conjugated) group homomorphisms

$$\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\mathbb{Q}_{\{\infty,p\}}/\mathbb{Q})$$

are injective.

In this talk, I will explain how Langland's conjectures shed some light on this question, at least when the field $\mathbb{Q}_{\{\infty,p\}}$ is replaced by the "slightly" bigger $\mathbb{Q}_{\{\infty,p,l\}}$ (with obvious definition) where l is any prime number different from p. I will give also proof of this weaker form in many cases. The basic ideas are the following. We have to construct a lot of number fields in $\mathbb{Q}_{\{\infty,p,l\}}$, and then to be able to control their ramification at p. Following Grothendieck and al., such number fields can be constructed by looking at the galois actions on the $H^*_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)$ for each proper smooth $\mathbb{Q}$-scheme X having good reduction outside lp, and there properties at p should be readable on the bad reduction of models of X at p. Some examples of such X's are provided by the theory of PEL Shimura varieties. Better, there cohomology is conjecturally controlled by the cohomological automorphic forms of the group used to define them. In fact, Langlands conjectured the existence of a very precise $\mathbb{Q}$-motive attached to any algebraic automorphic form of a reductive $\mathbb{Q}$-group. We would like to use these motives in the following way. Let $K/\mathbb{Q}_p$ be a finite galois extension and let $\rho : \mathrm{Gal}(K/\mathbb{Q}_p) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be an injective representation. The local Langlands correspondence attaches to ρ a smooth irreducible representation $\pi(\rho)$ of $\mathrm{GL}_n(\mathbb{Q}_p)$. Assume now that we can find an algebraic automorphic representation Π of GL_n over $\mathbb{Q}$ which is unramified outside p and whose p-part Π_p is $\pi(\rho)$. If we look at the galois representation $\rho_l(\Pi)$ attached to an l-adic realization of the hypothetical motive of Π, then tautologically the subfield $\overline{\mathbb{Q}}$ which is fixed by the kernel of $\rho_l(\Pi)$ is unramified outside lp, and its p-adic completions are exactly K, what we wanted.

All we have to check is then the existence of Π as above, that is to construct algebraic automorphic forms with "prescribed everywhere" local components, which is somehow an analytic question. In fact, the problem as stated above may be as hard as the initial one, because the condition $\pi(\rho) = \Pi_p$ is very strong. A much more reasonable condition is to ask that the Weil-Deligne representations associated to $\pi(\rho)$ and Π_p coincide when restricted to the absolute inertia group of $\mathbb{Q}_p$. For instance, under this weaker assumption, Π could have any motivic weight and its Weil numbers are not specified. We would conclude again then by using the easy lemma that a galois (over $\mathbb{Q}_p$) subfield of $\overline{\mathbb{Q}_p}$ which generates the whole of $\overline{\mathbb{Q}_p}$ with $\mathbb{Q}_p^{\mathrm{ur}}$ is in fact already $\overline{\mathbb{Q}_p}$. Using Bushnell and Kutzko's theory of types, our problem takes then the following shape, which I state for a general group G instead of GL_n. Let G be a connected reductive group over $\mathbb{Q}$, J a compact open subgroup of the finite adèles of G and τ an irreducible, complex, smooth representation of J: can we find an discrete automorphic representation of G which is algebraic (or better cohomological), and whose restriction to J contains τ ? Even when G is a torus the answer is no in general (this is also a reason why we cannot solve simply the initial problem using class-field theory), and the general answer we could find is the following: assume that $G(\mathbb{R})$ as a holomorphic discrete series, and that the center of G has finite arithmetic groups, then the answer is yes. Although the first condition is not satisfied when $G = \mathrm{GL}_n$ and $n > 2$, for which we still don't know what should be, it is true for example for all symplectic groups. As a consequence, by modifying slightly the arguments sketched above, we can prove that standard conjectures in Langlands theory and in types theory imply that the field embeddings $\mathbb{Q}_{\{\infty,p,l\}} \longrightarrow \overline{\mathbb{Q}_p}$ has a dense image. Although we will say also something of the case $l = p$, we are less definite in this case.

More importantly, let us explain what we can show unconditionally. In the work of Harris and Taylor on the local Langlands conjecture for GL_n, they constructed the galois representations attached to some cohomological cuspidal automorphic representations of GL_n over a CM field, and they prove the local-global compatibility for them expected by Langlands. Such automorphic forms can be constructed by base change from unitary groups, which satisfy the conditions of the paragraph above. The main theorem we can prove is then the following:

Theorem: Let E be a CM field, F its maximal totally real subfield, $v = uu'$ a finite place of F which splits in E, l a prime number prime to v, and S the set of places of E dividing v and l. If E_S denotes a maximal algebraic extension of E which is unramified outside S, then any E-embedding $E_S \longrightarrow \overline{E_u}$ has a dense image.

It has the following consequences:

Corollary:

(a) Under the hypothesis of the theorem, the procardinal of $\mathrm{Gal}(E_S/E)$ is divisible by any integer.

(b) Assume that p and l are distinct prime numbers such that $l \equiv 3 \bmod 4$ and that p splits in $\mathbb{Q}(\sqrt{-l})$ (that is, some congruence). Then the field embeddings $\mathbb{Q}_{\{\infty,p,l\}} \longrightarrow \overline{\mathbb{Q}}_p$ have a dense image.

Non-commutative Iwasawa theory

John Coates

The lecture described two curious arithmetic phenomena which arise when one studies the main conjecture for an elliptic curve E over $\mathbb{Q}$ over the p-adic Lie extension of $\mathbb{Q}$ generated by the p-power roots of unity and the p-power roots of some fixed integer $m > 1$. These phenomena were suggested by the numerical data found by Vladimir and Tim Dokchister in support of the main conjecture (formulated by Fukaya, Kato, Sujatha, Venjakob and the author) in this case.

$\mathcal{L}$ invariants of p-adically uniformized varieties

Ehud de Shalit
(joint work with A. Besser)

This is a report on work in progress (with A. Besser) concerning $\mathcal{L}$-invariants of p-adically uniformized varieties. $\mathcal{L}$-invariants have been attached in the past to Mumford curves, or to modular forms with certain behavior at p. They are local invariants, and can be defined either from the p-adic cohomology of the curve (Fontaine-Mazur definition) or transcendentally, using p-adic integration (Coleman's approach). For some time, the equivalence of the two approaches was open, but in the 90's this question was settled by various authors. The $\mathcal{L}$-invariants appear in the study of special values of p-adic L-functions. Recently, Breuil showed how they come into play in phrasing a p-adic Langlands correspondence for $GL_2(\mathbb{Q}_p)$.

Our goal in this talk is to show how present knowledge of the cohomology of p-adically uniformized varieties (in particular, the monodromy-weight conjecture) allows us to attach "Fontaine-Mazur type" $\mathcal{L}$-invariants to them, and to speculate about a transcendental definition involving p-adic integration in higher dimensions.

0.1. **Fontaine-Mazur type $\mathcal{L}$-invariants.** Let K be a finite extension of $\mathbb{Q}_p$. Let $\mathfrak{X}$ be Drinfeld's p-adic symmetric domain of dimension d over K, Γ a discrete cocompact subgroup of $G = PGL_{d+1}(K)$, and X the smooth projective variety over K whose associated rigid analytic space is $\Gamma\backslash\mathfrak{X}$. Let Y be the special fiber of the semistable model of X obtained from the formal scheme $\widehat{\mathfrak{X}}$ upon dividing by Γ, and $Y^\times$ the log-scheme obtained when we endow it with its canonical log-structure (Y being a divisor with normal crossings on X). We fix m (only $m = d$ really matters), write K_0 for the maximal absolutely unramified subfield of K, and look at

$$D = H^m_{\log-crys}(Y^\times/K_0), \quad H = H^m_{dR}(X/K). \tag{1}$$

Then D comes equipped with a semi-linear bijective endomorphism (Frobenius) Φ and a nilpotent endomorphism (monodromy) N, satisfying the relation

$$N\Phi = p\Phi N. \tag{2}$$

The Hyodo-Kato isomorphism

$$\rho_\pi : K \otimes_{K_0} D \simeq H \tag{3}$$

depends on π. The pair (D, H), where H is endowed with the Hodge filtration, is the *filtered* (Φ, N)*-module* associated to X in degree m.

The *weight decomposition*

$$D = D^0 \oplus D^2 \oplus \cdots \oplus D^{2m} \tag{4}$$

involves in this case only even weights. Here D^i is the subspace of D on which a suitable power of Φ, which acts linearly, has eigenvalues which are Weil numbers of weight i. The weight filtration

$$F_W^i D = \sum_{j \le i} D^j \tag{5}$$

is transported to H via ρ_π, and is then independent of π. Up to indexing, it is the same as the *covering filtration* $F_\Gamma^\cdot$ which is obtained from the covering spectral sequence

$$E_2^{i,j} = H^i(\Gamma, H_{dR}^j(\mathfrak{X})) \Rightarrow H_{dR}^{i+j}(X/K). \tag{6}$$

More precisely, we have

$$F_W^{2m-2i} H = F_\Gamma^i H. \tag{7}$$

Since the covering spectral sequence degenerates at E_2, we have

$$gr_\Gamma^i H \simeq H^i(\Gamma, C_{har}^{m-i}). \tag{8}$$

Recall [dS1] that $H_{dR}^j(\mathfrak{X}) \simeq C_{har}^j$ (the space of harmonic j-cochains on the Bruhat Tits building $\mathcal{T}$ of G) via $\omega \mapsto c_\omega$, where $c_\omega(\sigma) = res_\sigma(\omega)$ for any oriented j-simplex σ.

We know the following facts.

- Transversality of the Hodge and weight filtrations (Iovita-Spiess, see also [A-dS1])

$$H = F_\Gamma^i H \oplus F_{dR}^{m+1-i} H. \tag{9}$$

- Monodromy-weight conjecture ([A-dS2],[dS2], also Ito): Let

$$\nu = gr_\Gamma N : gr_\Gamma^i H \to gr_\Gamma^{i+1} H. \tag{10}$$

  Then ν is an isomorphism, except if $m = d$ is even, and $i = d/2 - 1$, when it is injective (with one-dimensional cokernel) or $i = d/2$, when it is surjective (with one-dimensional kernel). Call these two cases the *exceptional cases.*

It follows from transversality that

$$gr_\Gamma^i H \simeq H^i(\Gamma, C_{har}^{m-i}) = F_\Gamma^i H \cap F_{dR}^{m-i} H \simeq gr_{dR}^{m-i} H. \tag{11}$$

Now look at a subquotient of length 2 in the filtration

$$W = F_\Gamma^i / F_\Gamma^{i+2} = H/(F_{dR}^{m+1-i} \oplus F_\Gamma^{i+2}) = F_{dR}^{m-i-1}/F_{dR}^{m-i+1}. \tag{12}$$

$(0 \le i \le m-1)$. There are m such pieces to look at. We have a weight decomposition

$$W = W^0 \oplus W^2_\pi \tag{13}$$

where W^0 is of weight $2m-2i-2$ and W^2_π of weight $2m-2i$. Only W^2_π depends on π. We also have the "cross"

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & gr_\Gamma^{i+1} = H^{i+1}(\Gamma, C_{har}^{m-i-1}) & & & & \\
 & & & & \downarrow & \overset{\cong}{\searrow} & & & \\
0 & \rightarrow & gr_{dR}^{m-i} & \rightarrow & W & \rightarrow & gr_{dR}^{m-i-1} & \rightarrow & 0 \\
 & & & \overset{\cong}{\searrow} & \downarrow & & & & \\
 & & & & gr_\Gamma^{i} = H^{i}(\Gamma, C_{har}^{m-i}) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \tag{14}$$

where the diagonal arrows are isomorphisms by transversality. The monodromy operator induces $\nu : gr^i_\Gamma \to gr^{i+1}_\Gamma$.

Assume we are not in the exceptional case, so that ν is an isomorphism. Then $W^0 = gr^{i+1}_\Gamma$ and W^2_π projects isomorphically onto gr^i_Γ. Define

$$\lambda^{FM}_\pi : gr^i_\Gamma \to gr^{i+1}_\Gamma \tag{15}$$

by first lifting to gr^{m-i}_{dR}, then projecting along the weight decomposition onto W^0. Define

$$\mathcal{L}^{FM}_{\pi,i+1} = \lambda^{FM}_\pi \circ \nu^{-1} \in End(H^{i+1}(\Gamma, C^{m-i-1}_{har})). \tag{16}$$

These $\mathcal{L}$-invariants are defined for $1 \le i+1 \le m$. We have

$$\mathcal{L}^{FM}_{\pi',k} - \mathcal{L}^{FM}_{\pi,k} = -\log(\pi'/\pi)I. \tag{17}$$

They are therefore invertible for generic π.

[In the exceptional case note that we have a direct sum decomposition

$$H^{d/2}(\Gamma, C^{d/2}_{har}) = \mathrm{Im}(\nu) \oplus \ker(\nu)$$

with $\ker(\nu)$ one-dimensional. This should be an orthogonal direct sum with respect to cup product and Poincaré duality. Now in both the exceptional cases λ^{FM}_π is well defined as before. Assume first that $i+1 = d/2$. To complete the definition of $\mathcal{L}_{\pi,i+1}$ we have to show that $\mathrm{Im}(\lambda^{FM}_\pi) \subseteq \mathrm{Im}(\nu)$. Equality will hold for generic π. The $\mathcal{L}$-invariant will be an endomorphism of $\mathrm{Im}(\nu)$ only. In the other case, $i = d/2$, we have to show that $\ker(\lambda^{FM}_\pi) \supseteq \ker(\nu)$, again with equality for generic π.]

0.2. **$\mathcal{L}$-invariants via Coleman integration.** This section is largelay speculative.

We want to use p-adic integration to define an extension of G-modules

$$(18) \qquad 0 \to H^{j-1}(\mathfrak{X}) \to ?_\pi \to H^j(\mathfrak{X}) \to 0$$

depending on π, such that the connecting homomorphism

$$(19) \qquad \lambda_\pi^{Col} : H^i(\Gamma, H^j(\mathfrak{X})) \to H^{i+1}(\Gamma, H^{j-1}(\mathfrak{X}))$$

coincides with λ_π^{FM} as in the case of Mumford curves. We expect also the relation

$$(20) \qquad [?_{\pi'}] - [?_\pi] = -\log(\pi'/\pi)[\tilde{C}_{har}^{j-1}]$$

to hold in $Ext_G^1(C_{har}^j, C_{har}^{j-1})$, already before we take Γ-cohomology. The extension $[\tilde{C}_{har}^{j-1}]$ is the one defined in [A-dS2].

We believe that the extension $[?_\pi]$ should come from a modification of Besser's integration, which would cover the case of weakly formal schemes/dagger spaces with semistable reduction (or more generally, with log structure). Besser's original approach, via "finite polynomial cohomology", produces (for $\mathfrak{X}$ the analytic space associated to a smooth scheme rather than to a weakly formal scheme with log structure) an extension

$$(21) \qquad 0 \to H^{j-1}/F^n \to ? \to F^nH^j \to 0.$$

In the smooth case is does not depend on π, but in the semistable case we expect it to depend on π. Moreover, with our $\mathfrak{X}$, H^j is pure of weight $2j$ (whatever this means, e.g. for Frobenius structure), so choosing $n = 2j$ would produce the desired extension.

Once defined, weight considerations should yield the equality $\lambda_\pi^{Col} = \lambda_\pi^{FM}$ between the analytically defined map and the algebraically defined one. Viewed in a different optic, this would be a way to compute a subquotient of the Hyodo-Kato isomorphism via p-adic integration.

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A generalization of Dieudonné Theory

Jean-Marc Fontaine
(joint work with Uwe Jannsen)

Let k be a perfect field of char $p > 0$. Let $\mathcal{C}$ be the full sub-category of the category of schemes of f.t. over k which are l.c.i.. Let T be the class of quiet morphism, that is the smallest class of morphisms of $\mathcal{C}$ containing étale morphisms and ?extensions? of p-th roots. Let k_T be the corresponding topos (coverings are surjective families of quiet morphisms). k_T^{p-tor} = category of abelian sheaves of p-torsion. $W = W(k)$, σ = absolute Frobenius on W. $\mathfrak{U}_{\phi}^{\leq 0}(W, \varphi)$ = category of coeffective? finite gauges over k:

An object is a diagram of W-modules

$$\ldots \rightleftarrows_v^{\phi} M^{-r-1} \rightleftarrows_v^{\phi} M^{-r} \rightleftarrows_v^{\phi} \ldots \rightleftarrows_v^{\phi} M^{-1} \rightleftarrows_v^{\phi} M^0$$

of finite length with $\phi v = v\phi = p$ and $v : M^{-r} \to M^{-r-1}$ is an isomorphism for $r >> 0$, and $\varphi : M^0 \to M^{-\infty}(= M^{-r}$ for $r >> 0)$ is a $\sigma-$semilinear isomorphism.

We construct a functor of triangulated categories

$$RV : \mathcal{D}^b(\mathfrak{U}_{\phi}^{\leq 0}(W, \varphi))) \longrightarrow \mathcal{D}^b(k_T^{p-tor})$$

An object M of $\mathfrak{U}_{\phi}^{\leq 0}(W, \varphi)$ of level ≤ 1 (i.e. such that $v : M^{-r} \longrightarrow M^{-r-1}$ is an isomorphism for $r >> 1$) can be identified with a normal Dieudonné-module (i.e. $M = M^{-1}, F = \varphi\phi, V = v\varphi^{-1}$), $RV(M[0])$ is represented by a finite group scheme concentrated? in $d = 0$. Therefore this construction generalizes the classical Dieudonné theory. We expect that RV is fully faithful and have partial results.

On the Tamagawa Number Conjecture for motives attached to modular forms

Matthew Gealy

For any d-dimensional smooth proper variety X over Spec $\mathbb{Q}$ and any integer j, Beilinson [1] conjectures a relationship between a value of the L-function $L(h^d(X), s)$ attached to X and rational structures on certain cohomology groups. More specifically, Beilinson proposes two different $\mathbb{Q}$ lattices inside the plus part of the real singular cohomology of X , and conjectures that away from the center of symmetry $j = \frac{d+1}{2}$, the determinant of the change of basis matrix computes, up to $\mathbb{Q}^*$, the first non-zero coefficient in the Taylor expansion of $L(h^d(X), s)$ at $s = j$. The L-function here is defined by having Euler factors

$$L_p(h^d(X), s) = \det(1 - \mathrm{Frob}_p p^{-s} | H^d(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)^{I_p})$$

Analytic continuation and independence of l are already implicit in the conjecture, as are conjectures on the ranks of the motivic cohomology groups of X. As evidence Beilinson showed that the weak form of the conjecture holds when X is a closed modular curve and $j < 0$, where weak means one avoids those questions about motivic cohomology which are (still today) unknown. The argument was extended to higher modular varieties by Scholl [2]

Beilinson's conjecture only predicts the value of $L(h^d(X), s)$ up to $\mathbb{Q}^*$. Due to the work of Bloch-Kato [3], Fontaine-Perrin-Riou [4], and others, the conjecture was strengthened to predict the precise value. Namely, the original conjecture roughly involves comparing the Deligne realization of motivic cohomology classes to Betti cohomology classes with rational coefficients. For each prime l, one can also attempt to compare the l-adic realizations of said motivic classes to the l-adic elements derived from the Betti classes via classical-étale comparison. The Tamagawa Number Conjecture predicts that when bases are chosen so that the precise L-value is realized, then their l-adic avatars differ by the order of a generalized Shafarevich-Tate group and some Tamagawa numbers. A precise, modern statement, along with equivariant strengthenings, can be found in [5].

In the case where $X = \text{Spec } K$ is a cyclotomic field, the full equivariant conjecture is known for all j, by the work of many people, culminating in papers of Burns, Flach, and Greither (see [5]. When X is an abelian extension of a quadratic imaginary field, the Beilinson conjecture is due to Deninger , and under certain conditions the TNC is known by Johnson-Leung [6]. The only known non-critical case of TNC where X has positive dimension was the case of a CM elliptic curve over a quadratic imaginary field and j negative, due to Kings [7] in an appropriately weakened form. The proofs in these three cases use, respectively, cyclotomic units and the Iwasawa Main Conjecture, elliptic units and Rubin's Main Conjecture, and Eisenstein symbols and again Rubin's Main Conjecture.

We study the Tamagawa Number Conjecture when X is the motive attached to a new cuspidal eigenmodular form of weight ≥ 2, that is, X is the part of (the middle dimensional cohomology of) a modular variety, cut out by the associated eigensystem. Here the (weak) Beilinson conjecture is known by Beilinson and Scholl, as mentioned above. Explicit motivic cohomology elements are furnished by products of Eisenstein symbols. The method of Beilinson and Scholl was to use the Rankin-Selberg method to interpret the change of basis value as an automorphic integral, then to identify the terms at unramified primes. For a certain choice of Eisenstein symbols, we also calculate the contributions from bad primes, by going case by case according to the associated local representation. As a result, one obtains explicit bases whose comparison yields the exact L-value. For the same choice of Eisenstein symbols, we also use a lemma of Kings to show that their l-adic realizations give,up to normalization, elements used by Kato to construct his Euler system [3]. In addition to providing a more motivic description of these elements, this allows one to reduce the TNC in this case to an instance of Kato's Main Conjecture. This is done under some mild technical hypothesis on the cusp form, and, as in the work of Kings, the more serious assumption of the finiteness of a Shafarevich-Tate type group.

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Arithmetic Cohomology and Duality

Thomas Geisser

In the first half we discuss how to combine ideas of Lichtenbaum [5] and Voevodsky [6] to construct a cohomology theory $H^i_c(X_{\mathrm{ar}},\mathcal{F})$ for varieties X over a finite field $\mathbb{F}_q$. If one takes the motivic complex as coefficients is, then one expecte the following to hold:

- The cohomology groups are finitely generated abelian groups.
- The cohomology groups form an integral model of l-adic cohomology.
- $\mathrm{ord}_{s=n}\zeta(X,s)=\sum_i(-1)^i i\,\mathrm{rank}\,H^i_c(X_{\mathrm{ar}},\mathbb{Z}(n))=:\rho_n$.
- For $s\to n$, up to a power of q

$$\zeta(X,s)\sim(1-q^{n-s})^{\rho}_n\chi(H^*_c(X_{\mathrm{ar}},\mathbb{Z}(n)),e).$$

Here $\chi(H^*_c(X_{\mathrm{ar}},\mathbb{Z}(n)),e)$ is the Euler characteristic of the complex

$$\to H^{i-1}_c(X_{\mathrm{ar}},\mathbb{Z}(n))\xrightarrow{\cup e}H^i_c(X_{\mathrm{ar}},\mathbb{Z}(n))\xrightarrow{\cup e}H^{i+1}_c(X_{\mathrm{ar}},\mathbb{Z}(n))\to,$$

and e is a generator of $H^1((\mathbb{F}_q)_{\mathrm{ar}},\mathbb{Z})$. The statements above are true for smooth and proper X if and only if Tate's conjecture holds over finite fields, and if rational and numerical equivalence agree up to torsion [3][4]. For arbitrary varieties one also has to assume resolution of singularites.
In the second half of the talk we discuss Bloch's cycle complex $\mathbb{D}$ of dimension 0, (in degree $-i$ it consists of cycles of dimesnion i on $X\times\Delta^i$ meeting faces properly). We define extension groups $\mathrm{Ext}^i(\mathcal{F},\mathbb{D})$, construct a pairing

$$R\,\mathrm{Hom}(\mathcal{F},\mathbb{D})\to R\,\mathrm{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_{\mathrm{ar}},\mathcal{F}),\mathbb{Z}).$$

and show the following Theorem:

Assume resolution of singularites up to dimension d and that $CH_0(S,i)$ is torsion for $i>0$ and S smooth and projective of dimension at most d. Then the *pairing is a quasi-isomorphism of complexes with finitely generated cohomology groups for every X of dimension at most d and $\mathbb{Z}$-construcible $\mathcal{F}$.*

In particular, one gets perfect pairings

$$\mathrm{Ext}^{-i}(\mathcal{F},\mathbb{D})/tor \times H^i_c(X_{\mathrm{ar}},\mathcal{F})/tor \to \mathbb{Z}$$
$$\mathrm{Ext}^{-i}(\mathcal{F},\mathbb{D})_{tor} \times H^{i+1}_c(X_{\mathrm{ar}},\mathcal{F})_{tor} \to \mathbb{Q}/\mathbb{Z}$$

This extends results of Deninger [1] (curves), Grothendieck ($\mathcal{F}$ torsion prime to q) and Moser-Milne ($\mathcal{F}$ torsion of q-power order). The condition that $CH_0(X,i)$ is torsion for $i>0$ and X smooth and projective is a special case of Parshin's conjecture $K_i(X)$ is torsion for $i>0$, and follows for example if either of the following conjectures holds (by [2] and an argument of Jannsen/ Soule, respectively)
1) The cycle map is an isomorphism (Tate), and rational and numerical equivalence agree up to torsion (Beilinson).
2) Every smooth and projective scheme over $\mathbb{F}_q$ is finite dimensional in the sense of Kimura.

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Logarithmic Differential Forms on Drinfel'd Symmetric Spaces

Elmar Grosse-Kloenne

Let p be a prime numer and let K be a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k. Fix $d\in\mathbb{N}$ and let $\mathfrak{X}$ denote the Drinfel'd symmetric space of dimension d over K, i.e. the complement in projective d-space $\mathbb{P}^d_K$ of all K-rational hyperplanes. This is a K-rigid spaces acted on by the group $\mathrm{GL}_{d+1}(K)$. Let $\Gamma\subset\mathrm{GL}_{d+1}(K)$ be a cocompact discrete (torsionfree and sufficiently small) subgroup and let M be a K-rational representation of $\mathrm{GL}_{d+1}(K)$. The de Rham complex $\underline{M}\otimes_K\Omega^\bullet_{\mathfrak{X}}$ on $\mathfrak{X}$ with coefficients in M descends to a de Rham complex $\mathcal{M}\otimes_K\Omega^\bullet_{\mathfrak{X}_\Gamma}$ on the quotient $\mathfrak{X}_\Gamma=\Gamma\backslash\mathfrak{X}$ which is (the analytification of) a projective K-scheme. The stupid filtration (i.e. by degree) of $\mathcal{M}\otimes_K\Omega^\bullet_{\mathfrak{X}_\Gamma}$ endows its cohomology $H^d(\mathfrak{X}_\Gamma,\mathcal{M}\otimes_K\Omega^\bullet_{\mathfrak{X}_\Gamma})$ with a Hodge filtration. On the other hand, the spectral sequence which expresses $H^d(\mathfrak{X}_\Gamma,\mathcal{M}\otimes_K\Omega^\bullet_{\mathfrak{X}_\Gamma})$ in terms of the cohomology groups $H^*(\mathfrak{X},\underline{M}\otimes_K\Omega^\bullet_{\mathfrak{X}})$ endows $H^d(\mathfrak{X}_\Gamma,\mathcal{M}\otimes_K\Omega^\bullet_{\mathfrak{X}_\Gamma})$ with another filtration, the covering filtration. According to a conjecture of Schneider [6], these filtrations are oppposite to each other. In the case of constant coefficients $M = K$ this conjecture had been proved first by Iovita and Spiess [5]; later proofs were given by Alon and de Shalit [1] as well as the speaker [2].

In this talk a proof was indicated in the cases $M = K$, $M = K^{d+1}$ (the standard representation) and $M = (K^{d+1})^*$ (see [3], [4]). The method is to construct certain equivariant sheaves of logarithmic (in particular closed) differential forms (with coefficients in M) inside $\underline{M} \otimes_K \Omega^\bullet_{\mathfrak{X}}$ and then to show that the corresponding descended sheaves on $\mathfrak{X}_\Gamma$ generate the spaces $H^d(\mathfrak{X}_\Gamma, \mathcal{M} \otimes_K \Omega^\bullet_{\mathfrak{X}_\Gamma})$. It was explained how the choice of equivariant $\mathcal{O}_K$-lattices in all K-vector space sheaves involved — there existence follows from a construction of equivariant integral structures in the constant sheaves $\underline{M}$ on $\mathfrak{X}$ for *any* K-rational representation M of $\mathrm{GL}_{d+1}(K)$ — reduces the verification of the above generation statement to statements on the de Rham cohomology (with coefficients) of certain smooth projective k-schemes.

In the last part of the talk it was explained how the above conjecture and a variant of it (equally to be found in [6]) translates into a conjecture on certain filtered (ϕ, N)-modules in the sense of Fontaine, if $\mathfrak{X}_\Gamma$ is regarded as the completion (at p) of a suitable global Shimura variety. The basic observation here is that the above covering filtration on $H^d(\mathfrak{X}_\Gamma, \mathcal{M} \otimes_K \Omega^\bullet_{\mathfrak{X}_\Gamma})$ can then be interpreted as a slope filtration coming from a certain natural Frobenius structure on $\mathcal{M} \otimes_K \Omega^\bullet_{\mathfrak{X}_\Gamma}$, as explained in [2]. Moreover some speculations on the monodromy operator which appears in this context were added.

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On a Conjecture of Rapoport–Zink

Urs Hartl

Let $K_0 = W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$ and denote by σ the absolute Frobenius of F. Let D be a K_0-vector space of dimension n equipped with a σ-linear automorphism ϕ. Fontaine's (weakly) admissible filtrations of (D, ϕ) with fixed Hodge–Tate weights are parametrized by a period domain $\mathcal{F}^{wa}$ which is a rigid analytic subspace of a partial flag variety over K_0. Rapoport–Zink [4, p. 29] have conjectured the existence of an étale morphism $\mathcal{F}^a \to \mathcal{F}^{wa}$ of rigid analytic spaces which is bijective on points, and a representation of the étale fundamental group [2]

$$\rho : \pi_1(\mathcal{F}^a, \bar{x}) \;\; \rightarrow \;\; \mathrm{GL}_n(\mathbb{Q}_p)$$

($\bar{x}$ is a geometric base point of $\mathcal{F}^a$.), such that for the underlying K-valued point x of $\bar{x}$ the composite of ρ with $\mathrm{Gal}(\overline{K}, K) = \pi_1(x, \bar{x}) \rightarrow \pi_1(\mathcal{F}^a, \bar{x})$ is the crystalline Galois representation obtained by applying Fontaine's functor to the filtered module corresponding to x.

In our talk we propose a candidate for both $\mathcal{F}^a$ and ρ. Namely by recent work of Berger [1] (the filtered module corresponding to) a point of $\mathcal{F}^{wa}$ is admissible, if the ϕ-module $\mathcal{M}$ over the Robba ring which it defines [1, §II], is isoclinic of slope zero. Due to Kedlaya's classification [3, Thm. 4.5.7] of ϕ-modules over the "algebraic closure" of the Robba ring, the later holds iff $\mathcal{M}$ contains no ϕ-submodule of negative slope. We therefore propose that, in terms of the associated Berkovich spaces, $\mathcal{F}^a$ is the subspace of $\mathcal{F}^{wa}$ consisting of those Berkovich points for which there is no such ϕ-submodule of $\mathcal{M}$. We prove that $\mathcal{F}^a \subset \mathcal{F}^{wa}$ is an open immersion of Berkovich spaces. By the theorem of Colmez-Fontaine, which is reproved in [1], it is surjective on the level of classical rigid analytic points (which are rational over a finite extension of K_0). We give an example showing that it need not be surjective on general Berkovich points. We conjecture that the ϕ-invariants $\mathcal{M}^{\phi=1}$ form a local system of $\mathbb{Q}_p$-vector spaces on $\mathcal{F}^a$; cf. [2]. This would give rise to the representation ρ.

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Hasse invariants for some unitary Shimura varieties

Tetsushi Ito

It is well-known that the Hasse invariant, which is a modular form mod p of weight $p-1$, plays an important role in the study of the geometry of modular curves in characteristic $p > 0$. The aim of this talk is to give a generalization of the Hasse invariant for some unitary Shimura varieties by using an idea of Ekedahl-Oort stratification.

First, we briefly recall the definition of the classical Hasse invariant. Let $\mathcal{E} \rightarrow X$ be a family of elliptic curves over an algebraic variety X over an algebraically closed field k of characteristic $p > 0$. Let $\omega := e^*\Omega^1_{\mathcal{E}/X}$ be the Hodge bundle on X, where $e\colon X \rightarrow \mathcal{E}$ denotes the zero section. Let $G := \ker\left(F\colon \mathcal{E} \rightarrow \mathcal{E}^{(p)}\right)$ be the kernel of the relative Frobenius morphism. Then, G is a finite flat group scheme of order p on X. Since the Verschiebung $V\colon G^{(p)} \rightarrow G$ induces a map $V\colon (\mathrm{Lie}\, G)^{(p)} \rightarrow \mathrm{Lie}\, G$

on the Lie algebra, and ω is isomorphic to the dual $(\mathrm{Lie}\, G)^\vee$ of $\mathrm{Lie}\, G$, we have a section

$$\text{“}V\text{”} \in \mathrm{Hom}\big((\mathrm{Lie}\, G)^{(p)}, \mathrm{Lie}\, G\big) \cong \mathrm{Hom}\big(\omega^{\otimes(-p)}, \omega^{\otimes(-1)}\big) \cong H^0\big(X, \omega^{\otimes(p-1)}\big),$$

which is nothing but the classical Hasse invariant. It is easy to see that the Hasse invariant is invertible on the ordinary locus and vanishes on the supersingular locus. By a theorem of Igusa, for the universal elliptic curve over a modular curve $\mathcal{E} \to X(N)$ with $(N,p) = 1$, $N \geq 3$, the Hasse invariant has a simple zero at each supersingular point of $X(N)$. By a theorem of Deuring, the number of supersingular points on the j-line is equal to the class number of a definite quaternion algebra B over $\mathbb{Q}$ of discriminant p. Using the Hasse invariant, Igusa calculated the number of supersingular points on the λ-line and the j-line, and obtained an algebraic proof of Eichler's class number formula for B ([3]).

Let us introduce a class of unitary Shimura varieties studied by Kottwitz and Harris-Taylor ([2]). Fix a prime number p. Let B be a central division algebra over an imaginary quadratic field E such that $\dim_E B = n^2$, and p splits as $p = v \cdot v^c$ in E, where v^c denotes the complex conjugate of v. Moreover, let $*$ be a positive involution of second kind on B such that $B_v = B \otimes_E E_v$ is isomorphic to the matrix algebra $M_n(E_v)$ over E_v. Let V be a free B-module of rank 1, and $\langle , \rangle \colon V \times V \to \mathbb{Q}$ a $*$-hermitian nondegenerate alternating form. We define an algebraic group G over $\mathbb{Q}$ by

$$G(R) = \{(g, \lambda) \in \mathrm{End}_B(V \otimes_{\mathbb{Q}} R) \times R^\times \mid \langle gv, gw \rangle = \lambda \langle v, w \rangle \ \forall v, w \in V\}$$

for a $\mathbb{Q}$-algebra R. We assume $G(\mathbb{R}) \cong GU(1, n-1)$.

Let $K_p \subset G(\mathbb{Q}_p)$ be a maximal compact subgroup, $K^{p,\infty} \subset G(\mathbb{A}_{\mathbb{Q}}^{p,\infty})$ a sufficiently small open compact subgroup, and $K_\infty \subset G(\mathbb{R})$ a subgroup which is maximal compact modulo center. Then, we get a Shimura variety Sh defined over E which is a moduli space of PEL type satisfying

$$Sh(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}}) / K_p K^{p,\infty} K_\infty.$$

Recently, the geometry and cohomology of Shimura varieties of this type is extensively studied by Kottwitz, Harris-Taylor and many other people. By choosing an order $\mathcal{O}_B$ of B which is stable by $*$ and maximal at p, we get an integral model $Sh_{\mathcal{O}_{E,v}}$ over $\mathcal{O}_{E,v}$, which is defined as a moduli space of polarized abelian varieties of dimension n^2 with an action of $\mathcal{O}_B$ and a level structure satisfying certain conditions. Then, $Sh_{\mathcal{O}_{E,v}}$ is a projective smooth scheme of relative dimension $n-1$ over $\mathcal{O}_{E,v}$.

Let $k := \overline{k(v)}$ be an algebraic closure of the residue field $k(v)$ of $\mathcal{O}_{E,v}$. We define a stratification of $X := Sh_{\mathcal{O}_{E,v}} \otimes_{\mathcal{O}_{E,v}} k$ as follows. We decompose the p-divisible group $\mathcal{A}[p^\infty]$ of the universal abelian variety $\mathcal{A}$ over X according to the action of $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_{B,v} \times \mathcal{O}_{B,v^c}$ as $\mathcal{A}[p^\infty] = \mathcal{A}[v^\infty] \oplus \mathcal{A}[(v^c)^\infty]$. Then, we get a 1-dimensional Barsotti-Tate group $\mathcal{G} = \mathrm{diag}(1, 0, \ldots, 0) \cdot \mathcal{A}[v^\infty]$. By Serre-Tate theory and Morita equivalence, the formal spectrum of the complete local ring $\mathcal{O}_{X,s}^\wedge$ of X at a closed point $s \in X$ is isomorphic to the universal deformation space of $\mathcal{G}_s$ in characteristic $p > 0$. Then, we get a stratification $X = X^{(0)} \amalg \cdots \amalg X^{(n-1)}$, where $X^{(h)}$ denotes the locus where the rank of the étale part of $\mathcal{G}$ is equal to h. Let $X^{[h]} := X^{(0)} \amalg \cdots \amalg X^{(h)}$ be the locus where the rank of the étale part of $\mathcal{G}$ is equal to or less than h. Then, $X^{[h]}$ coincides with the Zariski closure of $X^{(h)}$ in X, and deformation theory tells us that $X^{(h)}$ and $X^{[h]}$ are smooth varieties of dimension h over k, and $X^{[h]}$ is projective over k (cf. [2]).

Let $\omega := \det\bigl(e^*\Omega^1_{\mathcal{A}/X}\bigr)$ be the Hodge bundle, where $e\colon X \to \mathcal{A}$ is the zero section, and $\mathcal{L} := (\operatorname{Lie}\mathcal{G})^\vee$ the dual of the Lie algebra $\operatorname{Lie}\mathcal{G}$ of $\mathcal{G}$. Since ω is an ample line bundle on X and ω is isomorphic to some power of $\mathcal{L}$, we see that $\mathcal{L}$ is an ample line bundle on X.

Theorem. For each h, $0 \le h \le n-1$, we have a section ("generalized Hasse invariant")

$$H_h \in H^0\bigl(X^{[h]}, \mathcal{L}^{\otimes(p^{n-h}-1)}\bigr),$$

which is invertible on the open stratum $X^{(h)} \subset X^{[h]}$, and has a simple zero along the "boundary" $X^{[h-1]} = X^{[h]} \backslash X^{(h)}$ if $h \ge 1$. For $h = n-1$, $X^{(n-1)} \subset X$ is the ordinary locus, and H_{n-1} coincides with the classical Hasse invariant.

The idea of the proof is as follows (see also [4],[2]). We consider the following exact sequence on the closed stratum $X^{[h]}$

$$0 \longrightarrow \ker\bigl(F^{n-h}\colon \mathcal{G}[p] \to \mathcal{G}[p]^{(p^{n-h})}\bigr) \longrightarrow \mathcal{G}[p] \longrightarrow G' \longrightarrow 0,$$

where $\ker F^{n-h}$ is of order p^{n-h}, and the cokernel G' is of order p^h. It is easy to see that G' is étale on the open stratum $X^{(h)} \subset X^{[h]}$, and is not étale on $X^{[h-1]} = X^{[h]} \backslash X^{(h)}$. We put $G_i := \ker F^i / \ker F^{i-1}$. Each G_i is a finite flat group scheme of order p on $X^{[h]}$, and the following Frobenius morphisms

$$G_{n-h}^{(p)} \xrightarrow{F} G_{n-h-1}^{(p^2)} \xrightarrow{F} \cdots \xrightarrow{F} G_1^{(p^{n-h})}$$

are isomorphisms. Moreover, the Verschiebung $V\colon G_{n-h}^{(p)} \to G_1$ is an isomorphism on $X^{(h)}$, and is zero on $X^{[h-1]}$. Therefore, we get a map

$$\text{“}V \circ (F^{-1})^{n-h-1}\text{”}\colon (\operatorname{Lie} G_1)^{(p^{n-h})} \longrightarrow \operatorname{Lie} G_1,$$

and a section

$$H_h := \text{“}V \circ (F^{-1})^{n-h-1}\text{”} \in H^0\bigl(X^{[h]}, \mathcal{L}^{\otimes(p^{n-h}-1)}\bigr),$$

which is invertible on $X^{(h)}$, and vanishes on $X^{[h-1]}$. Recall that $\operatorname{Lie} G_1 = \operatorname{Lie}\mathcal{G} = \mathcal{L}^\vee$ since $\mathcal{G}$ is a 1-dimensional Barsotti-Tate group. To calculate the vanishing order of H_h along $X^{[h-1]}$, we may use Zink's theory of displays (cf. [5]).

As a corollary, we see that the closed stratum $X^{[h-1]}$ is an ample divisor of $X^{[h]}$, and $X^{(h)}$ is an affine variety over k. Note that the general theory of Ekedahl-Oort stratification (cf. [4]) only tells us that each $X^{(h)}$ is quasi-affine (i.e. $X^{(h)}$ is an open subvariety of an affine variety). The affineness of $X^{(h)}$ has the following interesting consequences.

(1) For $\ell \ne p$ and $\Lambda = \mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{Z}/\ell^r\mathbb{Z}$, the étale cohomology $H^i\bigl(X^{(h)}, \Lambda\bigr)$ vanishes for $i > h = \dim X^{(h)}$.

(2) By the weak Lefschetz theorem, the number of connected components of $X^{[h]}$ are the same for $1 \leq h \leq n-1$. Therefore, we have $\pi_0\big(X^{[h]}\big) = \pi_0(X) = \pi_0(Sh)$ for $1 \leq h \leq n-1$.

Since the Hodge bundle ω is isomorphic to some power of $\mathcal{L}$, we can prove a formula relating the number of p-rank zero points $\#X^{(0)}$ on X to the degree of ω. This should be considered as a higher dimensional analogue of a classical theorem of Deuring and Igusa on the number of supersingular points on the j-line ([3], see also [1]).

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A Stickelberger index for the Tate-Shafarevich group

Hendrik Kasten

In this work, we obtain an algebraic description of the critical value of the L–series of rational elliptic curves of analytical rank 0 over an arbitrary real abelian number field. We do this by constructing a matching Stickelberger-ideal and computating its index. By inserting the result into the formula of the conjecture of Birch and Swinnerton-Dyer, we can describe the order of the Tate-Shafarevich group conjecturally in purely algebraic terms. For a more detailed exposition of this idea compare [2].

Let K be an arbitrary abelian number field, $G = \mathrm{Gal}(K/\mathbb{Q})$ the Galois group of K over $\mathbb{Q}$, and $\hat{G}$ the character group of G. In this work, we will always identify the characters $\chi \in \hat{G}$ with their respective primitive Dirichlet characters via Kronecker-Weber. Further let E be an elliptic curve defined over $\mathbb{Q}$. We are looking for an algebraic description of the critical L–value $L(E/K^+, 1)$, where K^+ is the maximal real subfield of K. The L–series of E over K is the product of the twists by the characters $\chi \in \hat{G}$ of the L-series of E over $\mathbb{Q}$. As those are modular by [1], we can apply a theorem of Manin and Birch (cf. [3], Theorem 4.2) on those curves E, whose conductor N_E is sufficiently coprime to the conductor of K. We introduce our notation of the modular symbol of an arbitrary $r \in \mathbb{Q}$ as follows:

$$\int\limits_{\gamma_r} f(q)\frac{dq}{q} =: [r]_f =: [r]_f^+\,\Omega^+ + [r]_f^-\,\Omega^-,$$

where γ_r denotes the geodesic in $X_0(N_E)$ from 0 to r, Ω^+ and Ω^- are complex periods independent of r, and the coefficients $[r]_f^+$ and $[r]_f^-$ are rational numbers. By explicit calculation we can prove the following proposition:

Proposition 1. *Let $\chi \neq \mathbf{1}$ be a Dirichlet character with conductor f_χ and f a Hecke eigenform belonging to an elliptic curve E. Then for any multiple m of f_χ with $\gcd(N_E, \frac{m}{f_\chi}) = 1$ the formula*

$$\sum_{c \bmod m}^{*} \chi(c) \left[\frac{c}{m}\right]_f = \mathfrak{C}_m(\chi) \cdot \sum_{b \bmod f_\chi}^{*} \chi(b) \left[\frac{b}{f_\chi}\right]_f$$

holds with $\mathfrak{C}_m(\chi)$ in $\mathbb{Z}[\chi]$.

If we denote the Gauß sum of a character χ of G by $\tau(\chi)$, and its conductor by f_χ, then we get for a non–trivial character χ and $\gcd(N_E, m) = 1$:

$$\mathfrak{C}_m(\chi) \cdot L(E/\mathbb{Q}, \chi, 1) = -\frac{\tau(\chi)\ \Omega^{\mathrm{sgn}(\chi)}}{f_\chi} \cdot \sum_{x \bmod m}^{*} \bar{\chi}(-x) \left[\frac{x}{m}\right]_f^{\mathrm{sgn}(\chi)} .$$

Now we want to define our Stickelberger-ideal. Therefore, let $N := \mathbb{Q}(\zeta_n)$ for any $n \in \mathbb{N}$ be the cyclotomic field generated by a primitive n-th root of unity ζ_n and $G_N := \mathrm{Gal}(N/\mathbb{Q})$ its Galois group over $\mathbb{Q}$. Let further e^+ denote the idempotent $\frac{1}{2}(1+c)$ in the group ring $\mathbb{Q}[H]$ of any Galois group H over $\mathbb{Q}$ of an abelian number field, where c is the complex conjugation. We define Stickelberger elements

$$\theta_N^+(a) = \left(1 - \frac{s_N(G_N)}{|G_N|}\right) \sum_{x \bmod n}^{*} \left[\frac{ax}{n}\right]_f^+ e^+(x, N)^{-1} + s_N(G_N) \cdot [i\infty]_f^+ \, e^+$$

in $e^+\mathbb{Q}[G_N]$ for each N and each integer a and study their images

$$\hat{\theta}_N^+(a) = (\mathrm{cor}_{K_N}^K \circ \mathrm{res}_{K_N}^N)(\theta_N^+(a))$$

in $e^+\mathbb{Q}[G]$. Let S' denote the abelian subgroup generated in $e^+\mathbb{Q}[G]$ by all those elements and S the intersection of S' and $e^+\mathbb{Z}[G]$.

Making use of the generalised index, that Sinnott introduced in [4], we compute

$$(\star) \qquad \left(e^+\mathbb{Z}[G] : S\right) = \left(e^+\mathbb{Z}[G] : S'\right) \cdot (S' : S) .$$

Write m for the conductor of the number field K. The first index on the right hand side is dealt with by the following proposition:

Proposition 2. *If $L(E/K^+, 1) \neq 0$ and $\gcd(N_E, m) = 1$, up to primes in $\Sigma_K := \{p \text{ prime } \mid p \text{ divides } |G|\}$, and up to sign we have*

$$\left(e^+\mathbb{Z}[G] : S'\right) = \frac{|D_{K^+}|^{1/2}}{(\Omega^+)^{|G^+|}} \cdot L(E/K^+, 1).$$

Idea of Proof: First we need some notation. Let $R' := \mathbb{Z}[\zeta_{|G|}]$ and $R := R'[\frac{1}{|G|}]$. For all characters χ of G let $e_\chi = \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$ be the corresponding idempotent in $\mathbb{C}[G]$. We consider

$$\omega := \sum_{\chi\in\hat{G}} \left(-\frac{f_\chi}{\tau(\chi)\Omega^{\mathrm{sgn}(\chi)}} \cdot L(E/\mathbb{Q},\chi,1) \right) e_\chi.$$

Now the proposition follows from

Lemma 3. *Let* $\gcd(N_E, m) = 1$. *Then the following holds*

(a) *As an* R*-module,* $e_\chi(S' \otimes R)$ *is generated by* $e_\chi\omega$.
(b) *As an* $R[G]$*-module,* $(S' \otimes R)$ *is generated by the Stickelberger elements* $\hat{\theta}^+_{F_\chi}(1)$ *for all characters* χ *of* G*, where* $F_\chi = \mathbb{Q}(\zeta_{f_\chi})$.

Under assumption of a conjecture of Stevens ([5], Conjecture III), the second index on the right hand side of ($\star$) can be broken down into a product of powers of finitely many primes.

Proposition 4. *If Stevens' conjecture holds and* $L(E/K^+, 1) \neq 0$*, the index* $(S' : S)$ *is defined and divisible only by the primes in* $\Sigma := \Sigma_E \cup \Sigma'_K$. *Here,* $\Sigma'_K := \{p \text{ prime } \mid p \text{ divides } \varphi(m)m\}$ *and* $\Sigma_E \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}$.

Remark *If* E *is semistable and without complex multiplication over* $\bar{\mathbb{Q}}$*, then* Σ_E *lies in* $\{2, 3, 5, 7\}$.

Now we can sum up all those results in the following theorem:

Theorem 5. *Let* E *be an arbitrary elliptic curve over* $\mathbb{Q}$ *and* K *an arbitrary abelian number field. Assume, the critical* L*–value* $L(E/K^+, 1)$ *is different from zero and Stevens' conjecture holds. If the conductor* N_E *of* E *and the conductor* m *of* K *are coprime, the following equation holds up to primes in* Σ *and up to sign:*

$$\left[e^+\mathbb{Z}[G] : S\right] = \frac{|D_{K^+}|^{1/2}}{(\Omega^+)^{|G^+|}} \cdot L(E/K^+, 1).$$

Comparing this result with the predictions of the conjecture of Birch and Swinnerton-Dyer, we get

Corollary 6. *Under the requirements of Theorem 5 and assuming the conjecture of Birch and Swinnerton-Dyer for elliptic curves of analytic rank zero over* K*, the following equation holds up to primes in* Σ *and up to sign:*

$$|Ш(K^+, E)| = \left(\frac{\Omega^+}{\Omega_{\mathbb{Q}}}\right)^{|G^+|} \cdot \frac{[e^+\mathbb{Z}[G] : S]}{\prod_{\substack{\nu \text{ non-} \\ \text{arch.}}} \left[E(K^+_\nu) : E_0(K^+_\nu)\right]}$$

At this, the ratio of the periods is a rational number which can be taken to be 1 *by appropriate choice of the period lattice.*

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Specialization of monodromy of p-adic differential equations

Kiran S. Kedlaya

An old result of Borel [7, Lemma 4.5] states that a vector bundle with connection on a punctured unit disc, equipped with a polarized variation of Hodge structure, has quasi-unipotent monodromy transformation; that is, if one performs parallel transport on the space of local horizontal sections in a neighborhood of some point, going around a loop around the puncture effects a linear transformation whose eigenvalues are roots of unity.

The p-adic local monodromy theorem (so named because it fulfills a role in p-adic cohomology analogous to that of Grothendieck's ℓ-adic monodromy theorem in étale cohomology) of André [1], Mebkhout [6], and the speaker [2] makes a similar assertion in p-adic (rigid) analytic geometry. To fix notation, let K be a complete discretely valued field of characteristic 0, whose residue field k is algebraically closed of characteristic $p > 0$. Let $\mathcal{E}$ be a vector bundle equipped with a connection and compatible Frobenius structure on a rigid analytic open annulus over K of outer radius 1 and "unspecified" inner radius. That is, one is really working in the direct limit category as the inner radius tends to 1, or equivalently, one is really considering a finite projective (actually necessarily free) module M over the Robba ring

$$\mathcal{R} = \left\{ \sum_{n \in \mathbb{Z}} c_n t^n (c_n \in K) : \text{for some } \rho, \lim_{n \to \pm\infty} |c_n| \eta^n = 0 \text{ for all } \eta \in (\rho, 1) \right\},$$

equipped with a connection $\nabla : M \to M \otimes \Omega^{1,\mathrm{cont}}_{\mathcal{R}/K} = M \otimes dt$ and a Frobenius isomorphism $F : \sigma^* M \to M$ of modules with connection, for some $\sigma : \mathcal{R} \to \mathcal{R}$ lifting a power of the absolute Frobenius on $k((t))$. (More precisely, the subring $\mathcal{R}^{\mathrm{int}}$ of $\mathcal{R}$, consisting of series $\sum c_n t^n$ with all $c_n \in \mathfrak{o}_K$, is a discrete valuation ring with residue field $k((t))$; the map σ should preserve $\mathcal{R}^{\mathrm{int}}$ and act on its field by some power of Frobenius. For instance, given a q-power Frobenius lift σ_K on K, one could take σ to be given by $\sum c_n t^n \mapsto \sum c_n^{\sigma_K} t^{qn}$.)

The p-adic local monodromy theorem then asserts that $\mathcal{E}$ is necessarily quasi-unipotent. Here quasi-unipotence means that there is an exhaustive filtration of $\mathcal{E}$ (by subbundles stable under the Frobenius and connection actions) whose successive quotients can be trivialized as connection modules after pulling back along a finite étale covering of the annulus. Moreover, this covering can be taken to be one induced by a finite étale covering of $\operatorname{Spec} k((t))$, via the fact that the subring $\mathcal{R}^{\text{int}}$ of $\mathcal{R}$, consisting of series $\sum c_n t^n$ with all $c_n \in \mathfrak{o}_K$, is a henselian discrete valuation ring with residue field $k((t))$. (Namely, one then lifts the extension from $k((t))$ to $\mathcal{R}^{\text{int}}$ and then tensors up to $\mathcal{R}$.)

In particular, one can construct a semisimplified monodromy representation

$$\rho_{\mathcal{E}} : \pi_1(\operatorname{Spec} k((t)), \overline{x}) \cong \operatorname{Gal}(k((t))^{\text{sep}}/k((t))) \to \operatorname{GL}_n(K)$$

(for $\overline{x}$ a geometric point) by taking the natural action on horizontal sections. (If k is not algebraically closed, one obtains something similar but the representation must be taken with values in the maximal unramified extension of K, and is semilinear rather than linear.) One can also construct a unipotent representation and get an equivalence between Frobenius-connection modules and a certain representation category; see [5].

In this lecture, we speculate on what sort of monodromy representation one expects in a relative setting, at least for the semisimple part (the unipotent part seems much subtler). Specifically, let P be a smooth formal scheme over $\mathfrak{o}_K$, let Q be a formal disc bundle over P, and let Q^* be the complement of the zero section. (Locally, this means that $P = \operatorname{Spf} A$, $Q = \operatorname{Spf} A[[t]]$, and $R = \operatorname{Spf} \widehat{A((t))}$.) Take a Frobenius-connection module $\mathcal{E}$ on a relative annulus within the generic fibre Q_K. (For technical reasons, it seems necessary to assume that one has an "absolute" connection, i.e., one relative to K rather than to P_K. That means one must include an integrability hypothesis.) We then propose to exhibit a corresponding monodromy representation

$$\rho_{\mathcal{E}} : \pi_1(Q_k^*, \overline{x}) \to \operatorname{GL}_n(K)$$

(where Q_k^* is the special fibre, i.e., locally $\operatorname{Spec} \overline{A}((t))$) with the following compatibilities:

- The formation of $\rho_{\mathcal{E}}$ commutes with arbitrary base change (to another smooth formal scheme over $\mathfrak{o}_K$).
- The formation of $\rho_{\mathcal{E}}$ commutes with base change of the field K (to another complete discretely valued field with algebraically closed residue field).
- If $P = \mathfrak{o}_K$, then $\rho_{\mathcal{E}}$ coincides with the semisimplified monodromy representation constructed above.

For a given $\mathcal{E}$, one can construct $\rho_{\mathcal{E}}$ after replacing P by a strict localization, thanks to the "relative p-adic local monodromy theorem" [3, Theorem 5.1.3]. However, avoidance of the strict localization turns out to be closely related to the problem of "semistable reduction for overconvergent F-isocrystals" as investigated, for instance, in [4]. Indeed, we expect that:

- existence and uniqueness of $\rho_{\mathcal{E}}$ follow from a form of semistable reduction (existence of a canonical logarithmic extension after a suitable finite cover and blowup) for $\mathcal{E}$;

- semistable reduction for $\mathcal{E}$ follows from the existence of $\rho_{\mathcal{E}}$;
- the existence of $\rho_{\mathcal{E}}$ in general can be reduced to the case where $\dim P_k = 1$.

This would allow the semistable reduction conidtion to be checked by doing so only on two-dimensional ambient spaces, where a valuation-theoretic approach looks promising.

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Modularity in intermediate weight

Mark Kisin

Let $p > 2$ be a prime, S a finite set of primes of $\mathbb{Q}$ containing p, and $G_{\mathbb{Q},S}$ the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S. Fix a decomposition group at $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q},S}$ at p. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. The main point of the talk was to explain the following

Theorem 7. *Let $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation, and denote by $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O})$ the residual representation. Suppose that*

(1) *$\rho|_{G_{\mathbb{Q}_p}}$ is crystalline with Hodge-Tate weights 0 and $k-1$, where $2 \leq k \leq 2p-1$.*

(2) *$\bar{\rho}$ is modular, $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ has only scalar endomorphisms, and $\bar{\rho}|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible.*

Then ρ is modular.

The proof of the theorem uses the refinement of the Taylor-Wiles patching argument developed in [Ki]. One considers a suitable crystalline deformation ring $R_{0,k-1}$ of the local representation $\bar{\rho}|_{G_{\mathbb{Q}_p}}$. It is not hard to show that $R_{0,k-1}[1/p]$ is formally smooth of dimension 1, and the key point is to show that it is a domain.

To establish this we use some recent results of Berger-Breuil [BB] which were exposed by Berger during the conference. Namely, Berger-Breuil compute the reductions of crystalline representations with Hodge-Tate weights in the interval $[0, 2p-2]$. These computations suggest that the generic fibre of $R_{0,k-1}$ is either a disc or an annulus. Using Wach lattices [Be], one can construct a map from one of $W(\mathbb{F})[\![X]\!][1/p]$ or $W(\mathbb{F})[\![X,Y]\!]/(XY-p)[1/p]$ to $R_{0,k-1}[1/p]$. The results of Berger-Breuil imply that this map induced a bijection on closed points and an isomorphism of the residue fields of corresponding points. A result of Gabber implies that any such map is an isomorphism.

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The Euler system of Heegner points

Jan Nekovář

Let E be an elliptic curve over $\mathbf{Q}$ of conductor N and K an imaginary quadratic field in which all primes dividing N split. Fix a modular parametrisation $\pi : X_0(N) \longrightarrow E$ of E and an ideal $I \subset O_K$ such that $NO_K = I\overline{I}$. The cyclic isogeny $[\mathbf{C}/O_K \longrightarrow \mathbf{C}/I^{-1}]$ represents a point $x \in X_0(N)(H)$ defined over the Hilbert class field H of K. Define $y = \mathrm{Tr}_{H/K}(\pi(x)) \in E(K)$.

V.A. Kolyvagin [3] proved that, if the point y is not torsion, then the quotient group $E(K)/\mathbf{Z}y$ and the Tate-Šafarevič group of E over K are both finite.

Kolyvagin's result was subsequently generalised in several directions ([1], [4], [5], [6], [2]).

In our lecture we explained the following result.

Theorem 8. *Let X be a Shimura curve over a totally real number field F, A an F-simple quotient of the Jacobian of X corresponding to a Hilbert modular form f over F with trivial central character, x a CM point on X by a totally imaginary quadratic extension K of F and α a character of the Galois group $G(K(x)/K)$. Assume that f does not have CM by any quadratic extension of F contained in the fixed field of* $\mathrm{Ker}(\alpha)$. *If the α-component of the image of x in A is not torsion, then it generates, up to a finite group, the α-component of $A(K(x))$, and the α-component of the Tate-Šafarevič group of A over $K(x)$ is finite.*

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On the variation of Tate-Shafarevich groups of elliptic curves over hyperelliptic curves

Mihran Papikian

Let E be an elliptic curve over the function field $F := \mathbb{F}_q(t)$ of $\mathbb{P}^1_{\mathbb{F}_q}$, where q is a power of an odd prime p. Assume E has conductor $(\mathfrak{p}) \cdot \infty$, where ∞ is the place corresponding to $1/t$ and $\mathfrak{p} \in A := \mathbb{F}_q[t]$ is a prime. Further assume that the reduction of E at ∞ is split multiplicative. In this situation it is known that E is a quotient of the Drinfeld Jacobian variety $J := J_0(\mathfrak{p})$; see [1], [2].

Let $S := \{x_1, x_2, \dots, x_n\}$ be the set of isomorphism classes of super-singular Drinfeld modules over $\overline{\mathbb{F}}_\mathfrak{p}$, where we write $\mathbb{F}_\mathfrak{p} := A/(\mathfrak{p})$. It is known that $n = \dim(J) + 1$. For each $x_i \in S$ we let ϕ_i denote a super-singular Drinfeld module representing the isomorphism class corresponding to x_i. Let $\mathcal{M}$ denote the free $\mathbb{Z}$-module on the set S, and $\deg : \mathcal{M} \to \mathbb{Z}$ denote the $\mathbb{Z}$-linear map obtained by sending each $x_i \in S$ to $1 \in \mathbb{Z}$, and let $\mathcal{M}^0$ denote the kernel of deg. Define a symmetric, bilinear, $\mathbb{Z}$-valued pairing on $\mathcal{M}$ by the formula

$$\langle x_i, x_j \rangle = \frac{1}{q-1} \#\mathrm{Isom}(\phi_i, \phi_j). \tag{1}$$

In particular, $\langle x_i, x_j \rangle = 0$ for $i \neq j$. It is known that $\mathrm{Aut}(\phi_i) \cong \mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$ (the latter case can occur only when $\deg \mathfrak{p}$ is odd), so $\langle x_i, x_i \rangle = 1$ or $(q+1)$.

Denote by $\mathcal{E}$ the Néron model of E over $\mathbb{P}^1_{\mathbb{F}_q}$. Let $\mathcal{E}^0$ be the relative connected component of the identity of $\mathcal{E}$, i.e., the largest open subgroup-scheme of $\mathcal{E}$ in which all fibres are connected. Similarly, denote by $\mathcal{J}$ and $\mathcal{J}^0$ the Néron model of J and its relative connected component of the identity. It is known that the closed fibre $\mathcal{J}^0_{\overline{\mathbb{F}}_\mathfrak{p}}$ is a torus and the character group $\mathrm{Hom}_{\overline{\mathbb{F}}_\mathfrak{p}}(\mathcal{J}^0_{\overline{\mathbb{F}}_\mathfrak{p}}, \mathbb{G}_{m,\overline{\mathbb{F}}_\mathfrak{p}})$ is canonically isomorphic to $\mathcal{M}^0$. Moreover, the pairing in (1) restricted to $\mathcal{M}^0$ is Grothendieck's monodromy pairing discussed in [4]. The character group Υ of $\mathcal{E}^0_{\overline{\mathbb{F}}_\mathfrak{p}}$ is isomorphic to $\mathbb{Z}$. Let $\pi : J \to E$ be a non-trivial homomorphism. Let ρ be a generator of Υ. There results a functorial homomorphism between the character groups $\pi^* : \Upsilon \to \mathcal{M}^0$. Let $H_E := \pi^*(\rho) \in \mathcal{M}^0$. Note that H_E, up to a non-zero scalar multiple, is independent of π.

Now let $\mathfrak{d}$ be an irreducible polynomial in A of odd degree. Let $K = F(\sqrt{\mathfrak{d}})$. The field K is the function field of a hyperelliptic curve over $\mathbb{F}_q$. The extension K/F is ramified only at $(\mathfrak{d})$ and ∞. Let $\mathcal{O}$ be the integral closure of A in K. If we assume that the ideal $(\mathfrak{p})$ remains prime in $\mathcal{O}$ then the endomorphism rings of some super-singular Drinfeld modules ϕ_i contain $\mathcal{O}$ as a subring. There results an action of $\mathrm{Pic}(\mathcal{O})$ on a subset of S, and one produces from this action an element $H_K \in \mathcal{M}$.

Denote by $E_K := E \otimes_F K$ the base change of E to K. The first result which I discussed in my talk is the following theorem:

Theorem 1. *$L(E_K, 1) = 0$ if and only if $\langle H_E, H_K \rangle = 0$.*

Remark. It would be very interesting to have some cohomological explanation for this results, as our proof is indirect and follows the strategy in [3].

Now assume that E is optimal. Fix $\pi : J \to E$ to be the quotient map having connected and smooth kernel; such π exists since E is optimal. The next main result which I discussed during the talk gives a formula for the order of the Tate-Shafarevich group $\text{Ш}(E/K)$ in terms of $\langle H_E, H_K \rangle$ and $\omega := \deg(\Omega^1_{\mathcal{E}/\mathbb{P}^1}|_O)$, where $\Omega^1_{\mathcal{E}/\mathbb{P}^1}|_O$ is the pullback of $\Omega^1_{\mathcal{E}/\mathbb{P}^1}$ along the relative zero section:

Theorem 2. *If $\langle H_E, H_K \rangle \neq 0$ then*

$$\#\text{Ш}(E/K) = \left(\langle H_E, H_K \rangle \cdot q^{(\omega - 1)} \right)^2 .$$

In particular, this theorem says that the variation of $\#\text{Ш}(E/K)$ over different K depends only on the relative position of H_K and H_E in $\mathcal{M}$, and if $\omega > 1$ then for any K the Tate-Shafarevich group $\text{Ш}(E/K)$ has p-torsion. The formula in the theorem can also be used to compute $\#\text{Ш}(E/K)$.

To prove Theorem 1 we prove a more general result which gives a formula for the special values of L-functions of Drinfeld cusp forms at the center of the critical strip. This formula is the function field analogue of the result of Gross over $\mathbb{Q}$ [3]. As a preliminary step we prove the analogue of Eichler's theorem over F, that is, we show that a certain set of explicit theta series arising from quaternion algebras over F spans the whole space of Drinfeld automorphic forms. (This last result might be of some independent interest.) Theorem 2 is a consequence of the analogue of Gross' formula, the theorems of Tate [7] and Milne [5], and some results on the rational torsion of optimal elliptic curves [6].

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The characteristic class and the ramification of an ℓ-adic etale sheaf.

Takeshi Saito
(joint work with A. Abbes and K. Kato)

For an ℓ-adic etale sheaf on a variety, its characteristic class is already defined in SGA5 implicitly. If the variety is proper, it computes the Euler number, by the Lefschetz-Verdier trace formula.

For a smooth sheaf $\mathcal{F}$ on a smooth variety U, its Swan class $\mathrm{Sw}\mathcal{F}$ is defined using the ramification on the boundary as an element of $CH_0(X \setminus U)_{\mathbb{Q}}$ where X is a compactification. As a generalization of the Grothedieck-Ogg-Shafarevich formula, it also computes the Euler number. The main ingredient of the proof is a Lefschetz trace formula for open variety.

We expect to have a relation

$$C(j_!\mathcal{F}) - \mathrm{rank}\mathcal{F}C(j_!\Lambda) = \mathrm{cl}\ \mathrm{Sw}(\mathcal{F})$$

between the characteristic class and the Swan class. Two results on the relation are discussed. One is for an arbitrary rank but works under a certain technical assumption. The other is in the rank 1 case. It consists of establishing a relation with another invariant defined by Kato. The proof of the relation with the characteristic class is based on the observation that one can kill the ramification by the blow-up at the ramification locus in the diagonal in the self product.

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Sheaves of Analytic Families of Overconvergent p-adic Modular Forms

Glenn Stevens

From the work of H. Hida [3] (in the ordinary case) and R. Coleman [2] (in general) one knows that every modular eigenform of finite slope "belongs to a p-adic analytic family". More recently, I developed a theory of "overconvergent modular symbols", which associates to any analytic family of modular eigenforms forms an analytic family of "modular eigensymbols." These families of modular symbols interpolate the classical modular symbols attached, via the classical Eichler-Shimura theorem, to the classical modular forms in the families. Is there an extension of the Eichler-Shimura theorem that would directly relate the entire family of modular forms to the corresponding family of modular symbols?

Adrian Iovita and I have introduced a new and, we believe, more promising approach, based on a mixture of cohomology of coherent sheaves and étale cohomology of local systems. On the modular curve $X_0 := X(N,p)$ (level $\Gamma_1(N) \times \Gamma_0(p)$-structure), we define a big sheaf of $\mathcal{O}$-modules $\mathcal{H}^1(\mathcal{D})$ as well as a family of sheaves of $\mathcal{O}$-modules $\mathcal{H}^1(\mathcal{D}_k)$, $k \in \mathcal{X}$, together with specialization morphisms

$$\mathcal{H}^1(\mathcal{D}) \longrightarrow \mathcal{H}^1(\mathcal{D}_k),\ (k \in \mathcal{X}).$$

These sheaves are constructed locally as galois cohomology groups for the geometric fundamental group with coefficients in relative (over the modular curve) locally analytic distribution spaces. Iovita and I call $\mathcal{H}^1(\mathcal{D})$ the sheaf of analytic families of overconvergent p-adic modular forms and prove the following theorem in the spirit of Eichler-Shimura.

Theorem. *Let $v \in \mathbb{Q}^+$ with $v < \frac{p}{p+1}$. Let $k \in \mathbb{Z}$ and let $M_{k+2}(v) := \underline{\omega}^{k+2}(Z(v))$ be Katz's space of v-overconvergent modular forms. Then there is a natural map*

$$H^0\big(Z(v),\ \mathcal{H}^1(\mathcal{D}_k(1))\big) \longrightarrow M_{k+2}(v).$$

Moreover, for any rational number $h \geq 0$, both of these spaces have slope $\leq h$ decompositions and the above map induces an isomorphism

$$H^0\big(Z(v),\ \mathcal{H}^1(\mathcal{D}_k(1))\big)^{(\leq h)} \cong M_{k+2}(v)^{(\leq h)}.$$

The proof is based on p-adic Hodge theory as developed by Tate [4] over local fields and later generalized by Faltings [2].

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Fine Selmer groups of CM elliptic curves

Ramadorai Sujatha

(joint work with J. Coates)

The fine Selmer group of an elliptic curve over a strongly admissible p-adic Lie extension was defined and studied in [1]. In this talk, we study the special case when the elliptic curve has complex multiplication.

Let $\mathcal{K}$ be an imaginary quadratic field and $\mathcal{O}_{\mathcal{K}}$ be the ring of integers of $\mathcal{K}$. We consider an elliptic curve E which is defined over $\mathcal{K}$, and which has complex multiplication by $\mathcal{K}$ in the sense that the ring of endomorphisms of E is isomorphic to $\mathcal{O}_{\mathcal{K}}$. We write E_p and E_{p^∞} for the group of all p-division points and p-power division points in $E(\overline{\mathcal{K}})$, respectively. Define

(1) $$F = \mathcal{K}(E_p),\ F_\infty = \mathcal{K}(E_{p^\infty}),\ G = \mathrm{Gal}(F_\infty/F),\ \mathcal{G} = \mathrm{Gal}(F_\infty/\mathcal{K}).$$

Recall that given a profinite group G, the Iwasawa algebra of G, denoted by $\Lambda(G)$, is defined as the completed group algebra

$$\Lambda(G) = \varprojlim_{N} \mathbb{Z}_p[G/N].$$

Here N varies over open normal subgroups of G and the inverse limit is taken with respect to the natural maps.

If $\mathcal{L}$ is any field with $\mathcal{K} \subset \mathcal{L} \subset F_\infty$, we write $K(\mathcal{L})$ for the maximal unramified abelian p-extension of $\mathcal{L}$ in which every prime of $\mathcal{L}$ above p splits completely. Put

$$Y(\mathcal{L}) = G(K(\mathcal{L})/\mathcal{L}), \tag{2}$$

and endow $Y(\mathcal{L})$ with the natural action of $G(\mathcal{L}/\mathcal{K})$ via inner automorphisms. This action extends to an action of the whole Iwasawa algebra $\Lambda(G(\mathcal{L}/\mathcal{K}))$, and it is well known that $Y(\mathcal{L})$ is always finitely generated and torsion as a module over $\Lambda(G(\mathcal{L}/\mathcal{K}))$.

Conjecture: *The $\Lambda(G)$-module $Y(F_\infty)$ is always pseudo-null.*

Recall that a finitely generated module over a commutative Noetherian ring R of Krull dimension d is *pseudo − null* if the module has dimension strictly less than $d-1$. We discuss certain arithmetic criteria for the above conjecture to hold.

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Compatibility of local and global Langlands correspondences

Teruyoshi Yoshida

(joint work with Richard Taylor)

Let L be a number field (finite over $\mathbb{Q}$), n a positive integer, ℓ a fixed prime and $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ a fixed field isomorphism. We denote the absolute Galois group of L by $G_L = \mathrm{Gal}(\overline{L}/L)$. Conjectural global Langlands correspondence predicts a correspondence between (A) algebraic automorphic representation Π of $GL_n(\mathbb{A}_L)$ and (B) n-dimensional ℓ-adic Galois representation $R : G_L \to GL_n(\overline{\mathbb{Q}}_\ell)$ which is (1) almost everywhere unramified and (2) de Rham at ℓ. The cuspidals in (A) should correspond to irreducibles in (B). We are interested in cuspidal Π, and we denote the (conjectural) Galois representation attached to Π by $R = R_{\ell,\iota}(\Pi)$. Up to semisimplification, it is characterized by the property that the eigenvalues of Frobenius Frob_v equal the Satake parameters of Π_v for almost all places of v. One of the most general results in the direction $\Pi \longmapsto R$ is the one obtained by Kottwitz [K], Clozel [C] and Harris-Taylor [HT], which constructs the semisimple representation $R_{\ell,\iota}(\Pi)$ when L is an imaginary CM-field and Π is cuspidal, satisfying (1) Π is conjugate self-dual, (2) Π is regular algebraic and (3) there is a finite place x of L where Π_x is (essentially) square-integrable. We show the compatibility of this correspondence with the local Langlands correspondence at all places outside ℓ; we need to fix notations for the local Langlands correspondence.

Let K be a finite extension of $\mathbb{Q}_p$ and n a positive integer. We denote the maximal unramified extension of K by K^{ur}, the (geometric) Frobenius by Frob, and the Weil group by $W_K = \{\sigma \in G_K \mid \sigma|_{K^{ur}} \in \mathrm{Frob}^{\mathbb{Z}}\}$. The local Langlands correspondence rec (proved by Harris-Taylor [HT] and Henniart [He]) gives the correspondence from (A′) irreducible admissible representations (over $\mathbb{C}$) of $GL_n(K)$ to (B′) n-dimensional F-semisimple Weil-Deligne representations (over $\mathbb{C}$) of W_K. Recall (see Tate [Ta]) that a Weil-Deligne representation is a pair (r, N) of a finite dimensional representation $r : W_K \to GL(V)$ and an $N \in \mathrm{End}(V)$ such that $r(\sigma)N = \chi(\sigma)Nr(\sigma)$ for all $\sigma \in W_K$, where $\chi : W_K \to \mathbb{Q}^\times$ is the composite of the local reciprocity map $W_K \to W_K^{ab} \cong K^\times$ (sending lifts of Frob to uniformizers) and the normalized absolute value $|\,|_K : K^\times \to \mathbb{Q}^\times$. We can define the F-semisimplification $r^{F\text{-ss}}$ of r, and write $(r, N)^{F\text{-ss}} = (r^{F\text{-ss}}, N)$ and $(r, N)^{\mathrm{ss}} = (r^{F\text{-ss}}, 0)$. The cuspidal (resp. square integrable, tempered) representations in (A′) correspond to irreducible (resp. indecomposable, pure) representations in (B′). For the definition of pure Weil-Deligne representations, see [TY]. For a prime ℓ and an ℓ-adic Galois representation ρ of G_K (assume de Rham when $\ell = p$), denote the associated Weil-Deligne representation (over $\overline{\mathbb{Q}}_\ell$) of W_K by $\mathrm{WD}(\rho)$ (for $\ell \neq p$ use quasi-unipotence of Grothendieck; for $\ell = p$ use Berger's theorem and Fontaine's functor D_{pst}).

Theorem A. (Harris-Taylor [HT], Taylor-Yoshida [TY]) Let L be a CM-field, ℓ a prime and fix $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Let Π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ satisfying the three conditions above, and $R_{\ell,\iota}(\Pi)$ be the associated ℓ-adic representation of G_L. Then, for all finite place v of L not dividing ℓ:

$$\iota \mathrm{WD}(R_{\ell,\iota}(\Pi)|_{G_{L_v}})^{F\text{-ss}} \cong \mathrm{rec}\big(\Pi_v^\vee \cdot |\det|_K^{\frac{1-n}{2}}\big)$$

as Weil-Deligne representations over $\mathbb{C}$ of W_{L_v}.

Some remarks on the theorem:

(1) As we assumed the existence of x where Π_x is square-integrable, we obtain the irreducibility of $R_{\ell,\iota}(\Pi)$ if x does not divide ℓ. This is because $R_{\ell,\iota}(\Pi)$ is semisimple by definition, and the theorem tells us that its restriction at x is indecomposable.

(2) The equality of $(\,)^{\mathrm{ss}}$ of both sides in the theorem was one of the main results of Harris-Taylor ([HT], Introduction, Theorem C). The new result in [TY] is the determination of the monodromy operator N.

(3) In [TY], the theorem for v dividing ℓ is shown to follow from the functoriality of the p-adic weight spectral sequence of Mokrane [M].

We sketch the idea of proof of the theorem. First we note that the temperedness of Π_v is shown in [HT] (Introduction, Theorem C). Hence, by (Theorem A)$^{\mathrm{ss}}$, it suffices to prove that the left hand side is pure (in particular, this follows from the Weight-Monodromy Conjecture). Using the global base change, we reduce to the case when Π_v has a fixed vector by the Iwahori subgroup $\mathrm{Iw}_n = \{g \in GL_n(\mathcal{O}_{L,v}) \mid g \bmod v \text{ is upper triangular }\}$. We descend Π to an automorphic representation π of a unitary group G which locally at v looks like GL_n and at infinity looks like $U(1, n-1) \times U(0, n)^{[L:\mathbb{Q}]/2-1}$. Then we realise $R_{\ell,\iota}(\Pi)$ in the cohomology of a Shimura variety X associated to G with Iwahori level structure at v. More precisely (assume the infinitesimal character to be trivial for simplicity), the representation $R_{\ell,\iota}(\Pi)$ appears inside the semisimplification of the π^p-isotypic component of $H^{n-1}(X, \overline{\mathbb{Q}}_\ell)$. We show that X has strictly semistable reduction at v with a nice moduli-theoretic definition of the strata of the special fiber (the reduction of X at v), and use the results of [HT] to compute the cohomology of these (smooth, projective) strata as a virtual $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_v^{\mathbb{Z}}$-module. This description and the temperedness of Π_v shows that the π^p-isotypic component of the cohomology of any strata is concentrated in the middle degree. This implies that the π^p-isotypic component of the Rapoport-Zink weight spectral sequence ([RZ], [S]) degenerates at E_1, which shows that $\mathrm{WD}\big(H^{n-1}(X, \overline{\mathbb{Q}}_\ell)[\pi^p]|_{G_{L_v}}\big)$ is pure.

In the special case that Π_v is a twist of a Steinberg representation and Π_∞ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I]. After we had posted the first version of this paper, Boyer [B] has announced an alternative proof with presumably stronger results.

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