Mathematisches Forschungsinstitut Oberwolfach

Report No. 39/2005

Analysis and Geometric Singularities

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August 21st – August 27th, 2005

Abstract. This workshop focused on several of the main areas of current research concerning analysis on singular and noncompact spaces. Topics included harmonic analysis and Hodge theory on, and the theory of compactifications of, locally symmetric spaces, new topological techniques in index theory, nonlinear elliptic problems related to metrics with special geometry, and various more traditional problems in spectral geometry concerning estimation of eigenvalues and the spectral function.

Mathematics Subject Classification (2000): 58J50, 58J60, 58J20, 58C40, 53C25, 53C35 .

Introduction by the Organisers

This workshop was a successful and happy meeting of 23 researchers working on various aspects of partial differential equations on singular spaces and noncompact manifolds. This field encompasses such diverse topics as L^2 cohomology, harmonic analysis and spectral analysis on locally symmetric spaces, spectral geometry, and constructions of metrics with special geometry. All of these topics, and several others, were covered extensively at this meeting.

One noteworthy feature was a series of four expository survey talks by leading experts. These served to stimulate extensive discussion which carried over into all the other sessions. Lizhen Ji presented a view of the extensive compactification theory for locally symmetric spaces; Werner Müller gave a detailed survey of harmonic analysis on locally symmetric spaces, leading up to a presentation of his important new work in this area; Thomas Schick discussed the topological and C^* algebraic techniques now used extensively in index theory; finally, Michael Singer explained some of the analytic aspects of the burgeoning field of extremal Kähler metrics.

The rest of the talks complemented these surveys. For example, Leslie Saper presented an overview of his recent and ongoing work on the L^2 cohomology of locally symmetric spaces and the algebraic machinery of $\mathcal{L}$-modules he has developed for this goal. Gilles Carron talked about the L^2 cohomology of QALE spaces and Eugenie Hunsicker's talk explored further relationships between L^2 and intersection cohomology theory on manifolds with fibred boundaries, while Ulrich Bunke discussed his recent work on twisted K-theory. There were several talks on more traditional problems in spectral geometry, including one by Iosif Polterovich on new methods to obtain lower bounds for the spectral function of the Laplacian, one on estimates for the first positive eigenvalue of the Dirac operator by Bernd Ammann, Daniel Grieser's talk on the 'max flow min cut' method adapted from graph theory applied to the estimation of the Cheeger constant, and Sergiu Moroianu's talk on the disappearance of the essential spectrum for certain magnetic Laplacians. Talks focusing on metrics with special geometry included the one by Robin Graham on the Dirichlet-to-Neumann operator for Poincare-Einstein metrics, Hartmut Weiss' results on deformations of three-dimensional constant curvature cone metrics, and Frank Pacard's presentation of his construction, with Arezzo, of new constant scalar curvature Kähler metrics. Finally, Nader Yeganefar discussed his work on topological restrictions associated with quadratic curvature decay.

One of the things that stood out about this meeting was the spirited participation by a number of young researchers, many of whom are clearly poised to take their places amongst the next generation of leaders in this field. The quality of the talks was uniformly high, and their clarity certainly furthered the main goal, which was to further stimulate the discussion between the various groups of researchers at this meeting.

Several major results were announced and discussed here. These illustrate that this field is now reaching a certain maturity. A number of these problems have been outstanding for many years, but new ideas have finally allowed for more serious attacks on problems of a truly global nature, and on problems involving iterated singularities. In any case, the meeting certainly demonstrated the current vitality of this field.

Workshop: Analysis and Geometric Singularities

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Abstracts

The first Dirac eigenvalue in a conformal class

Bernd Ammann
(joint work with Emmanuel Humbert)

Let (M, g_0) be a compact n-dimensional Riemannian manifold equiped with a fixed spin structure that will not be mentioned explicitly in the notation, $n \geq 2$. Let $[g_0]$ be the set of all metrics conformal to g_0 having volume 1. For any metric $g = f^2 g_0 \in [g_0]$ we obtain a spinor bundle $\Sigma^g M$ and a Dirac operator $D^g : \Gamma(\Sigma^g M) \to \Gamma(\Sigma^g M)$. We identify $\Sigma^g M$ with $\Sigma^{g_0} M$ such that

$$D^g \phi = f^{-1} D^{g_0} \phi,$$
$$|\phi|_g = f^{(n-1)/2} |\phi|_{g_0}.$$

In the following the index g will be sometimes omitted if $g = g_0$.

We study the first positive eigenvalue of the Dirac operator as a function on $[g_0]$. At first, we sketch the proof that the first positive eigenvalue of the Dirac operator $\lambda_1^+(g)$ is not bounded from above. Then, we turn our attention to the infimum, denoed by

$$\lambda^+_{\min}(M, [g_0]) := \inf_{g \in [g_0]} \lambda_1^+(g).$$

This invariant satisfies $\lambda^+_{\min}(M, [g_0]) \leq \lambda^+_{\min}(S^n) \leq (n/2)\omega_n^{1/n}$ where the sphere will be always carry the standard Riemannian metric, and where ω_n denotes its volume [Hij86, Bär92, Amm03, GH05].

Then, we will show that $\lambda^+_{\min}(M, [g_0])$ is always positive, i.e. the first eigenvalue is uniformly bounded away from 0. This result is due to J. Lott if the Dirac operator is invertible [Lot86], for the general case see [Amm03]. In order to prove this result it is helpful to reformulate the problem as a variational problem. Namely we define the conformally invariant functional

$$\mathcal{F}(\phi) = \frac{\|D\phi\|^2_{L^{4/3}}}{\int \langle D\phi, \phi \rangle}.$$

One then shows that

$$\lambda^+_{\min}(M, [g_0]) := \inf \mathcal{F}(\phi)$$

where the infimum runs over all spinors of regularity $H_1^{4/3}$ with $\int \langle D\phi, \phi \rangle > 0$. The infimum of $\mathcal{F}$ is attained iff $\lambda_1^+(g)$ attains its infimum in a "generalized" metrics. Here a generalized metric is a metric of the form $f^2 g_0$, where f is a real function that may have zeros (see [Amm03a, Amm03b] for details). The main result of the first part of the talk is the following.

Theorem 1 ([Amm03b]). *Let $\alpha := 2/(n-1)$ if $n \geq 4$, and let $\alpha \in (0,1)$ if $n \in \{2, 3\}$. Assume that*

$$\lambda^+_{\min}(M, [g_0]) < \lambda^+_{\min}(S^n) \tag{1}$$

holds.

(A) *Then there is a spinor field* $\phi \in C^{2,\alpha}(\Sigma M) \cap C^\infty(\Sigma(M \setminus \phi^{-1}(0)))$ *on* (M, g_0) *minimizing* $\mathcal{F}$ *among all spinors with* $\int \langle D\phi, \phi\rangle > 0$. *In particular, it satisfies the Euler-Lagrange equation of* $\mathcal{F}$*:*

$$D_{g_0}\phi = \lambda^+_{\min} |\phi|^{2/(n-1)}\phi, \qquad \|\phi\|_{2n/(n-1)} = 1.$$

(B) *There is a generalized metric* g *conformal to* g_0 *with volume* 1 *such that*

$$\lambda_1^+(g) = \lambda^+_{\min}.$$

The metric has the form $g = |\phi|^{4/(n-1)} g_0$ *where* ϕ *is a spinor as in (A).*

(C) *If* $n = 2$, *then the metric* g *is smooth and the zero set of the conformal factor of* g, *denoted* $\mathcal{S}_g$, *is finite. Furthermore*

$$\#\mathcal{S}_g < \text{genus}(M).$$

In particular, if M *is diffeomorphic to a* 2*-torus, then there* g *is an ordinary metric.*

Roughly speaking, the inequality (1) avoids concentration of minimizing sequences for our functional.

Inequality (1) is strongly related to the Yamabe problem. The Yamabe problem [Yam60] is the problem to find a metric of constant scalar curvature in the given conformal class $[g_0]$. It has been solved affirmatively in [Tru68], [Aub76], [Sch84], see also the well-written overview article [LP87]. The most difficult step in the solution of the Yamabe problem is to show that any manifold not conformal to the standard sphere satisfies

$$Y(M, [g_0]) < Y(S^n) \tag{2}$$

It is a direct consequence of Hijazi's inequality [Hij86] that inequality (1) implies inequality (2). Hence, proving (1) for a given conformal spin manifold $(M.[g_0], \sigma)$ provides an alternative proof for the solvability of the Yamabe problem.

Furthermore, the solution of the Yamabe is structurally very similar to the proof of the above theorem.

Obviously, one wishes to determine all conformal manifolds $(M, [g_0])$ and all corresponding spin structures such that (1) holds. Unfortunately, this problem is widely open. In particular, it is not known whether (1) holds for all manifolds not conformal to S^n.

However, combining methods of [Aub76] and [BG92] with several original ideas and some involved calculations we obtain the following result.

Theorem 2 ([AHM03])**.** *If* $n \geq 7$ *and if* (M, g_0) *is not conformally flat, then (1) holds.*

A similar result exploiting information about the Green function of the Dirac operator has been proven in the conformally flat case in [AHM03].

In low dimensions inequality (1) can often be proved by applying Bär extrinsic estimate method [Bär98].

One motivation for studying the above minimization problem is that the Euler-Lagrange equations in the case $n = 2$ can be locally translated to constant mean curvature surfaces in $\mathbb{R}^3$ and S^3. The tool for this translation is the spinorial Weierstrass representation as explained in [KS96].

We obtain the following result [Amm03].

Assume that the Riemann surface (M, g) with spin structure σ carries a metric g such that the first positive eigenvalue of the Dirac operator is smaller than $2\sqrt{\pi/\mathrm{area}(M,g)}$. Then there is a periodic branched conformal constant mean curvature immersion F based on (M, g). The regular homotopy class of F is determined by the spin-structure σ. The indices of all branching points are even, and the sum of these indices is smaller than $2\mathrm{genus}(M)$. In particular, if M is a torus, there are no branching points.

For $n \geq 4$, the *geometric* interpretation of the Euler-Lagrange equations is unknown, however we have the following unpublished partial result. Suppose that (M, g_0) is an *analytic* 3-manifold, and ϕ a solution of the Euler-Lagrange equation. Then there is a (non-complete) 4-dimensional manifold (N, h) carrying a parallel spinor and an isometric embedding $(M \setminus \phi^{-1}(0), |\phi|^2 g_0)$ into (N, h) of constant mean curvature.

Note that 4-manifolds with a parallel spinor are Ricci-flat Kähler manifolds.

Further informations, all my publications and related material can be found on my webpage

`http://www.berndammann.de/publications.`

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Aspects of equivariant twisted K-theory of Lie groups

Ulrich Bunke

Let G be a compact semisimple Lie group which we consider as a G-space with respect to the conjugation action. Isomorphism classes of equivariant twists of G are classified by $H^3_G(G,\mathbb{Z})$. As a model for the twists one can think of G-equivariant bundles of algebras $\mathcal{H}$ over G with fibre K, the compact operators on a separable complex Hilbert space. The twisted equivariant K-theory is then defined as the equivariant K-theory of C^*-algebras $K_G(G,\mathcal{H}) := K^G(C(X,\mathcal{H}))$, where $C(X,\mathcal{H})$ denotes the G-algebra of sections of $\mathcal{H}$.

The graded group $K_G(G,\mathcal{H})$ is a very interesting object. Much of the motivation comes from physics, but we will not attempt to review the literature at this place. Freed-Hopkins-Teleman [2] calculated $K_G(G,\mathcal{H})$ and related it with the positive energy representations of G. Consider now the product $m : G\times G \to G$. We call the twist primitive if it satisfies

$$m^*\mathcal{H} \cong \mathtt{pr}_1^*\mathcal{H}\otimes \mathtt{pr}_2^*\mathcal{H}\ . \tag{1}$$

For a primitive twist and if m is K-oriented we can define a product

$$m_! : K_G(G,\mathcal{H})\times K_G(G,\mathcal{H}) \to K_G(G,\mathcal{H}).$$

It can be calculated explicitly and related to the fusion product of positive energy representations [2], [3].

In the paper [1] we reproduce the calculation of $K_G(G,\mathcal{H})$. While the approach of [2] is quite complicated and mixes analytic with topological arguments in [1] we only use the calculus of topological stacks and the formal properties of the K_G-theory functor.

In [1] we further adress the question how to produce twists which are primitive, and how to fix the isomorphism in (1) such that the product becomes additive. There is a way to associate twists to central extensions of the loop group LG of G. We show the following. Let V be a finite-dimensional unitary representation of G. The loop group LG then acts on $L^2(S^1,V)$. The latter Hilbert space is polarized. The action of LG lifts to a projective action on the correspnding Fock space and determines a central extension of LG. If $\pi_1(G)$ is finite and does not contain two-torsion then the twists obtained from unitary representations of G are primitive, and the isomorphism (1) can be choosen so that the product is associative. Our approach uses moduli spaces of flat connections on surfaces, in particular on the pair of pants surface.

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L^2 cohomology of QALE spaces

Gilles Carron

I will describe my result on the (reduced) L^2 cohomology of some crepant resolutions of $\mathbb{C}^n/G$ (where $G \subset \mathrm{SU}(n)$ is a finite subgroup of $\mathrm{SU}(n)$) endowed with a Quasi-Asymptotic-Locally-Euclidean Riemannian metric.

1. What are QALE spaces ? QALE stands for Quasi-Asymptotic-Locally-Euclidean, this notion has been introduced by D. Joyce in order to find new examples of complete Kaehler Einstein flat manifolds ([9]).

An Asymptotic-Locally-Euclidean or ALE space asymptotic to $\mathbb{R}^d/G$ (where $G \subset O(d)$ is a finite group acting freely on $\mathbb{S}^{d-1}$) is a Riemannian manifold (M^d, g) which outside a compact subset is diffeomorphic to $\left(\mathbb{R}^d \setminus \mathbb{B}\right)/G$ and such that within this diffeomorphism, the metric g is asymptotic to the euclidean metric. For instance $Y = T^*\mathbb{P}^1(\mathbb{C})$ is outside the zero cross section diffeomorphic to

$$]0, \infty[\times \mathrm{SO}(3) = \left(\mathbb{C}^2 \setminus \{0\}\right)/\{\pm \mathrm{Id}\},$$

and Y carries a ALE metric asymptotic to $\mathbb{C}^2/\{\pm\mathrm{Id}\}$; in fact Y also carries a Kaehler Einstein flat ALE metric metric known as the Eguchi-Hanson metric.

We now assume that the finite subgroup $G \subset \mathrm{SU}(n)$ satisfies the following hypothesis :

$$\mathbb{S}^{2n-1}/G \text{ has isolated singularities,} \tag{1}$$

that is to say these singularities are the quotient by G of a finite number of non intersecting sub-sphere of $\mathbb{S}^{2n-1}$. For instance it is always the case when $G \subset \mathrm{SU}(3)$ or $G \subset \mathrm{Sp}(2)$.

In that case

$$V = \{v \in \mathbb{C}^n, \exists g \in G \setminus \{\mathrm{Id}\}, gv = v\} = V_1 \cup ... \cup V_l.$$

is a union of linear subspace of $\mathbb{C}^n$ intersecting only at $\{0\}$ i.e. $i \neq j \Rightarrow V_i \cap V_j = \{0\}$.

Let V^ϵ be the ϵ neighborhood of V. Then the action of G is free on $\mathbb{C}^n \setminus V^\epsilon$ and also on $\widehat{\Sigma}_0 = C_1(\mathbb{S}^{2n-1} \setminus V^\epsilon)$ the part of the cone over the $\mathbb{S}^{2n-1} \setminus V^\epsilon$ $C_1(\mathbb{S}^{2n-1} \setminus V^\epsilon)$: the set of points of $\mathbb{C}^n$ with norm larger than 1 and staying on the line passing through $\mathbb{S}^{2n-1} \setminus V^\epsilon$. Let $\Sigma_0 = \widehat{\Sigma}_0/G$. Then $(\mathbb{C}^n \setminus \widehat{\Sigma}_0)/G$ is (for $\epsilon > 0$ small enough) a disjoint union of open sets which up to a finite cover, are

$$\{(w,v) \in V_i^\perp/A_i \times V_i,\ |v| \geq 1, |w| \leq \epsilon |v|\},$$

for some finite sub group $A_i \subset \mathrm{SU}\left(V_i^\perp\right)$. Moreover our assumption (1) implies that A_i acts freely on the unit sphere of $V_i^\perp$. Let Y_i be a ALE space asymptotic to $V_i^\perp/A_i$ and put

$$\Sigma_i = \{(y,v) \in Y_i \times V_i,\ |v| \geq 1, |y| \leq \epsilon |v|\},$$

where we denote by $|y|$ a smooth non negative function which coincides with the euclidean norm outside a compact set.

Definition : *We say that a Riemannian manifold (M,g) is QALE asymptotic to $\mathbb{C}^n/G$ if there is a compact set K such that outside K : $M \setminus K = \Sigma_0 \cup U$ where U is the disjoint union of open subsets which are (up to finite cover) diffeomorphic to a Σ_i. Moreover we require that on Σ_0 the metric is asymptotic to the euclidean metric and that when pull back to Σ_i the metric g is asymptotic to the product metric of the ALE metric on Y_i and of the euclidean one on $V_i \setminus \mathbb{B}_i$.*

2. What is L^2 cohomology ? Let (X,g) be a Riemannian manifold. The reduced L^2 cohomology space of (X,g) are defined as follow : Let $Z_2^k(X)$ be the kernel of unbounded operator d acting on $L^2(\Lambda^k T^*X)$, or equivalently

$$Z_2^k(X) = \{\alpha \in L^2(\Lambda^k T^*X),\ d\alpha = 0\},$$

where the equation $d\alpha = 0$ has to be understood in the distribution sense i.e. $\alpha \in Z_2^k(X)$ if and only if

$$\forall \beta \in C_0^\infty(\Lambda^{k+1}T^*X),\ \int_X \langle \alpha, \delta\beta \rangle = 0\ .$$

That is to say $Z_2^k(X) = \left(\delta C_0^\infty(\Lambda^{k+1}T^*X)\right)^\perp$. Let $B_2^k(X)$ be the L^2-closure of the space $\{d\beta, \beta \in L^2(\Lambda^{k-1}T^*X), d\beta \in L^2(\Lambda^k T^*X)\}$, then the reduced L^2 cohomology spaces of (X,g) are

$$\bar{H}_2^k(X) = Z_2^k(X)/B_2^k(X).$$

These spaces depend only of the topology of the L^2 spaces of differential forms, hence there are quasi isometric invariant of (X,g). That is to say if g_1 and g_2 are two Riemannian metrics such that for a $C \geq 1$ we have

$$C^{-1} g_1 \leq g_2 \leq C g_1,$$

then $\bar{H}_2^k(X, g_1) \simeq \bar{H}_2^k(X, g_2)$. When (X,g) is a complete Riemannian manifold then the space $L^2(\Lambda^k T^*X)$ has the following Hodge-DeRham-Kodaira orthogonal decomposition

$$L^2(\Lambda^k T^*X) = \mathcal{H}^k(X) \oplus \overline{dC_0^\infty(\Lambda^{k-1}T^*X)} \oplus \overline{\delta C_0^\infty(\Lambda^{k+1}T^*X)},$$

where the closure is taken with respect to the L^2 topology and where we note

$$\mathcal{H}^k(X) = \{\alpha \in L^2(\Lambda^k T^*X),\ d\alpha = \delta\alpha = 0\},$$

the space of harmonic L^2 k-forms; moreover we also have

$$Z_2^k(X) = \mathcal{H}^k(X) \oplus \overline{dC_0^\infty(\Lambda^{k-1}T^*X)} \text{ and } B_2^k(X) = \overline{dC_0^\infty(\Lambda^{k-1}T^*X)}$$

hence we have the identification

$$\mathcal{H}^k(X) \simeq Z_2^k(X)/\overline{dC_0^\infty(\Lambda^{k-1}T^*X)}.$$

3. Our result : Theorem: *If $X \to \mathbb{C}^n/G$ is a crepant resolution endowed with a QALE metric asymptotic to $\mathbb{C}^n/G$ then*

$$\bar{H}_2^k(X) \simeq \operatorname{Im}(H_c^k(X) \to H^k(X)).$$

The crepancy condition insures that $K_X = 0$ and it is a natural restriction to X carrying a Kaehler Einstein flat QALE metric; in fact, D. Joyce has shown that if X carries moreover a Kaehler QALE metric then its Kaehler class contains a unique Kaehler Einstein flat QALE metric (theorem 9.3.3 of [9]).

The cohomology of crepant resolution [1] is well known : according to deep result of Y. Ito & M. Reid ($n = 3$), V. Batyref and J. Denef & F. Loeser ([7, 1, 5]) we can compute $\dim H^k(X)$ only in terms of G : let $X \xrightarrow{\pi} \mathbb{C}^n/G$ a crepant resolution. If $g \in G$ we note $\operatorname{age}(g) = \theta_1 + \theta_2 + ... + \theta_n$ where the eigenvalues of g are $(e^{i2\pi\theta_1}, e^{i2\pi\theta_2}, ..., e^{i2\pi\theta_n})$ with $\theta_i \in [0, 1[$, as $\det g = 1$ we get $\operatorname{age}(g) \in \mathbb{N} \cap [0, n[$; it is also clear that $\operatorname{age}(g)$ only depends of the conjugacy class of g $[g] \in \mathcal{C}(G)$.

The cohomology of X is $\{0\}$ in odd degree and in even degree we get :

$$\dim H^{2k}(X) = \operatorname{card}\{[g] \in \mathcal{C}(G),\ \operatorname{age}(g) = k\}.$$

A corollary of this result and of our theorem is the following

Corollary: *Let $G \subset \mathrm{SU}(3)$ or $G \subset \mathrm{Sp}(2)$ be a finite subgroup and assume that $X \to \mathbb{C}^n/G$ is a crepant resolution endowed with a QALE metric asymptotic to $\mathbb{C}^n/G$, then:*

$$\begin{aligned}\chi_{L^2}(X) &= \sum_{k=0}^{2n}(-1)^k \dim \mathcal{H}^k(X) = \sum_{l=0}^{n} \dim \mathcal{H}^{2l}(X) \\ &= \operatorname{card}\big\{[g] \in \mathcal{C}(G), \ker(g - \mathrm{Id}) = \{0\}\big\}.\end{aligned}$$

4. About the proof : The description of the geometry of QALE space make tempting to use exact sequence of Mayer -Vietoris type to compute the spaces of reduced L^2 cohomology of X. But unlike the L^2 cohomology defined by

$$H_2^k(X) = Z_2^k(X)/\{d\beta, \beta \in L^2(\Lambda^{k-1}T^*X), d\beta \in L^2(\Lambda^k T^*X)\},$$

the spaces of reduced L^2 cohomology are not cohomology space that is exact sequence doesn't hold if the range of $d : \{\beta \in L^2(\Lambda^{k-1}T^*X), d\beta \in L^2(\Lambda^k T^*X)\} \to L^2(\Lambda^k T^*X)$ is not closed.

To avoid these difficulties we introduce the following notion :

[1] $G \subset \mathrm{SU}(n)$ is now general i.e.it does not necessary satisfies (1)

Definition : *Let (X,g) be a Riemannian manifold, we say that the range of d is almost closed in degree k and with respect to $w \in L^\infty(X)$, $w>0$ if when we introduce :*

$$\mathcal{C}_w^{k-1}(X) = \{\alpha \in L^2(\Lambda^{k-1}T^*X, w d\mathrm{vol}_g), d\alpha \in L^2(\Lambda^k T^*X, d\mathrm{vol}_g)\}$$

then $d\mathcal{C}_w^{k-1}(X) = \overline{\mathcal{C}_1^{k-1}(X)} = B_2^k(X)$. Some classical examples:

- *If X is an ALE space of real dimension d then the range of d is almost closed in all degree with respect to $w = (1+d(o,x))^{-2}$ if $d \geq 3$ and with respect to $w = \big[(1+d(o,x))\log(2+d(o,x))\big]^{-2}$ if $d=2$.*
- *If $X = C_1(B)$ is the part of the cone over a compact manifold B of dimension $(d-1)$ (perhaps with a boundary) which is at distance larger than 1 of the vertex then the range of d is almost closed in degree k with respect to $w = (1+d(o,x))^{-2}$ if $k \neq d/2$ or if $k = d/2$ and the cohomology space of B vanish in degree $(k-1)$ ($b_{k-1}(V)=0$) and with respect to $w = \big[(1+d(o,x))\log(2+d(o,x))\big]^{-2}$ if $k=d/2$ and $b_{k-1}(V) \neq 0$.*

The following proposition is straightforward :

Proposition : *Let $X = U \cup V$, we assume that the range of d is almost closed in degree k and with respect to w on X, U, V and $U \cap V$, we moreover assume that*

$$\{0\} \to \mathcal{C}_w^{k-1}(X) \xrightarrow{r^*} \mathcal{C}_w^{k-1}(U) \oplus \mathcal{C}_\mu^{k-1}(V) \xrightarrow{\delta} \mathcal{C}_w^{k-1}(U\cap V) \to \{0\}$$

Let $\bar{H}_{2,w}^{k-1}$ be the reduced $L^2(w d\mathrm{vol}_g)$ cohomology that is to say :

$$\bar{H}_{2,w}^{k-1}(X) = \frac{\{\alpha \in L^2(\Lambda^k T^*X, w d\mathrm{vol}_g),\ d\alpha = 0\}}{\overline{\{d\beta, \beta \in L^2(\Lambda^{k-1}T^*X, w d\mathrm{vol}_g), d\beta \in L^2(\Lambda^k T^*X, w d\mathrm{vol}_g)\}}}.$$

Then the following sequence is exact :

$$\bar{H}_{2,w}^{k-1}(U) \oplus \bar{H}_{2,w}^{k-1}(V) \xrightarrow{\delta} \bar{H}_{2,w}^{k-1}(U\cap V) \xrightarrow{b} \bar{H}_2^k(X) \xrightarrow{r^*} \bar{H}_2^k(U) \oplus \bar{H}_\mu^k(V) \xrightarrow{\delta} \bar{H}_2^k(U\cap V).$$

The scheme of the proof is first to compute the reduced L^2 and $L^2(w d\mathrm{vol}_g)$ cohomology of the Σ_i and prove that the range of d is almost closed, then secondly to used an avatar of this proposition to compute the reduced L^2 cohomology of a neighborhood of infinity and finally to used again this exact sequence and the result about the cohomology of crepant resolutions to deduce our theorem.

I will only briefly describe how we can compute in degree $k<n$ the reduced L^2 cohomology of the Σ_i and how we can show that the range of d is almost closed on Σ_i. Recall that

$$\Sigma_i = \{(y,v) \in Y_i \times V_i,\ |v| \geq 1, |y| \leq \epsilon|v|\}$$

where Y_i is an ALE space asymptotic to $V_i^\perp/A_i$, let $m_i = \dim_{\mathbb{C}} V_i^\perp$ and $n_i = dim_{\mathbb{C}} V_i$.

First, the reduced L^2 cohomology of Y_i is well known ([3],[12]):

$$\bar{H}_2^k(Y_i) \simeq \begin{cases} H_c^k(Y_i) & \text{if } k \leq 2m_i - 1 \\ H^k(Y_i) & \text{if } k \geq 1 \end{cases}$$

In particular, the natural map $\bar{H}_2^k(Y_i) \to H^k(Y_i)$ is injective. Let be $K \subset Y_i$ a compact subset with smooth boundary on which Y_i retracts. Then thanks to the Kunneth formula, the natural map

$$\bar{H}_2^k(Y_i \times (V_i \setminus \mathbb{B})) \simeq \bigoplus_{p+q=k} \bar{H}_2^p(Y_i) \otimes \bar{H}_2^p((V_i \setminus \mathbb{B}))$$
$$\to \bar{H}_2^k(K \times (V_i \setminus \mathbb{B})) \simeq \bigoplus_{p+q=k} \bar{H}_2^p(Y_i) \otimes \bar{H}_2^p((V_i \setminus \mathbb{B}))$$

is also an injective map.

Hence the map $\bar{H}_2^k(Y_i \times (V_i \setminus \mathbb{B})) \to \bar{H}_2^k(\Sigma_i)$ is also injective. We are now going to prove that this natural map is an isomorphism for $k < n = m_i + n_i$: and that the range of d is almost closed in these degree with respect to $w = [\rho_2 \log(\rho_2 + 1)]^{-2}$ where ρ_2 depends only on the second variable and is the radial function on $(V_i \setminus \mathbb{B})$.

It is also true that if $\phi \in B_2^k(Y_i \times (V_i \setminus \mathbb{B}))$ then there is $u, v \in L^2_{loc}(\Lambda^{k-1}T^*(Y_i \times (V_i \setminus \mathbb{B}))$, with $du, dv \in L^2$ and also $u/(\rho_1 \log(\rho_1 + 1)) \in L^2$ and $v/(\rho_2 \log(\rho_2 + 1)) \in L^2$ where ρ_1 depends only on the first variable, is always larger than 2 and equals to the radial function on $(V_i^\perp \setminus \mathbb{B})/A_i \subset Y_i$.

Let $\alpha \in B_2^k(\Sigma_i)$, first note that $\partial\Sigma_i$ has a neighborhood inside the interior of $Y_i \times (V_i \setminus \mathbb{B})$ which is quasi-isometric to $\Omega = C_1(I \times \mathbb{S}^{2m_i-1} \times \mathbb{S}^{2n_i-1})$ where $I \subset \mathbb{R}$ is an open interval. Hence the restriction of α to Ω satisfies that

$$i_\Omega^* \alpha \in B_2^k(\Omega).$$

But on Ω, the range of d is almost closed in degree $k < n$ with respect to the weight $1/r^2$ where r is the radial function. On Ω the radial function is comparable to ρ_2, hence if $w' = \rho_2^{-2}$ then there is $\psi \in \mathcal{C}_{w'}^{k-1}(\Omega)$ such that

$$i_\Omega^* \alpha = d\psi.$$

Now we can easily extend this ψ to a $\bar{\psi} \in \mathcal{C}_{w'}^{k-1}(\Sigma_i)$, but now if we extend $\alpha - d\bar{\psi}$ by zero we get a closed L^2 form on $Y_i \times (V_i \setminus \mathbb{B})$ whose reduced L^2 cohomology class $c \in \bar{H}_2^k(Y_i \times (V_i \setminus \mathbb{B}))$ is map to the L^2 cohomology class of α in $\bar{H}_2^k(\Sigma_i)$. But $\alpha \in B_2^k(\Sigma_i)$ and this map $\bar{H}_2^k(Y_i \times (V_i \setminus \mathbb{B})) \to \bar{H}_2^k(\Sigma_i)$ is injective, so we can write

$$\alpha - d\bar{\psi} = du + dv$$

where $u/(\rho_1 \log \rho_1) \in L^2$ and $v/(\rho_2 \log \rho_2) \in L^2$ but on Σ_i we also get $i_{\Sigma_i}^* u/(\rho_2 \log \rho_2) \in L^2$, hence if we put $\beta = \bar{\psi} + i_{\Sigma_i}^* u + i_{\Sigma_i}^* v$ we have $\beta \in \mathcal{C}_w^{k-1}(\Sigma_i)$. Hence $B_2^k(\Sigma_i) \subset d\mathcal{C}_w^{k-1}(\Sigma_i)$, the reverse inclusion follows from an argument borrowed to P. Li ([10] and also [4, 6, 11, 8]).

The reduced L^2 cohomology of Ω vanishes in degree $k < n$: $Z_2^k(\Omega) = B_2^k(\Omega)$; hence the same argument starting from an element $\alpha \in Z_2^k(\Sigma_i)$ will provided a $\alpha - d\bar{\psi}$ which when extends by zero lies in $Z_2^k(Y_i \times (V_i \setminus \mathbb{B}))$ hence the L^2 reduced

L^2 cohomology class of α lies in the image of the map $\bar{H}_2^k(Y_i \times (V_i \setminus \mathbb{B})) \to \bar{H}_2^k(\Sigma_i)$; so this map is also surjective.

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Dirichlet-to-Neumann map for Poincaré-Einstein metrics

C. Robin Graham

Let X be the interior of a compact $(n+1)$-dimensional manifold with boundary $\overline{X}$ and let $[g]$ be a conformal class of Riemannian metrics on ∂X. A Poincaré-Einstein metric with conformal infinity $[g]$ is a Riemannian metric g_+ on X such that $\mathrm{Ric}(g_+) = -ng_+$ and such that $x^2 g_+$ extends at least continuously as a metric to $\overline{X}$ satisfying $x^2 g_+|_{T\partial X} \in [g]$, where x is a defining function for ∂X. A motivating example is the hyperbolic Poincaré metric $h = 4(1-|y|^2)^{-2} \sum (dy^i)^2$ on the ball B^{n+1}, with conformal infinity the conformal class of the usual metric $g^{(0)}$ on the sphere $\mathbb{S}^n$. It was shown in [GL] that if $n \geq 3$ and g is a metric on $\mathbb{S}^n$ which is sufficiently close to $g^{(0)}$, then there is a Poincaré-Einstein metric g_+ near h with conformal infinity $[g]$. Such a metric g_+ is unique up to a diffeomorphism of $\overline{B^{n+1}}$ restricting to the identity on $\mathbb{S}^n$.

If g_+ is a Poincaré-Einstein metric, then a choice of representative metric g in the conformal infinity of g_+ induces for some $\epsilon > 0$ an identification of a neighborhood of ∂X with $\partial X \times [0, \epsilon) \subset \partial X \times \mathbb{R}$ such that in this identification, g_+ takes the geodesic normal form $g_+ = x^{-2}(dx^2 + g_x)$. Here g_x is a 1-parameter family of metrics on ∂X satisfying $g_0 = g$ and x denotes the coordinate in $\mathbb{R}$. If g is C^∞ and n is odd, a boundary regularity theorem asserts that $g_x \in C^\infty(\partial X \times [0,\epsilon))$ (see [A1], [CDLS], [A3], [H]). Moreover, the Taylor expansion of g_x is even to order n: there are ϕ_x, $\psi_x \in C^\infty(\partial X \times [0,\epsilon))$ whose Taylor expansions at $x=0$ contain only even terms such that $g_x = \phi_x + x^n\psi_x$. Of course $g_0 = \phi_0 = g$. One has $\mathrm{tr}_g\psi_0 = 0$ and $\mathrm{div}_g\psi_0 = 0$. It is the case that ψ_0 is locally formally undetermined subject only to these two conditions and the full Taylor expansion of g_x is formally determined in terms of g and ψ_0. The term ψ_0 plays the role of Neumann data for this problem. In the AdS/CFT corresondence, ψ_0 corresponds to the stress-energy tensor of the boundary conformal field theory. For the hyperbolic metric h, one can calculate that $\psi_0 = 0$.

If n is even and g is C^∞, it is shown in [CDLS] that g_x has an expansion involving $\log x$. One can define analogous Neumann data in this case as well, but we assume throughout the rest of this note that n is odd.

Define the Dirichlet-Neumann relation of X to be the set of pairs (g,ψ) such that there is a Poincaré-Einstein metric g_+ on X with conformal infinity $[g]$ and for which ψ is the ψ_0 determined by g. In general, existence and uniqueness fail for Poincaré-Einstein metrics with prescribed conformal infinity. However, by the result of [GL] mentioned above, near $(g^{(0)},0)$ the Dirichlet-Neumann relation can be written as the graph of a well-defined Dirichlet-to-Neumann map.

Let $\mathcal{M}^\infty$ denote the space of C^∞ metrics on $\mathbb{S}^n$ and $\mathcal{M}^\infty_0 \subset \mathcal{M}^\infty$ a neighborhood of $g^{(0)}$. Define the Dirichlet-to-Neumann map $\mathcal{N} : \mathcal{M}^\infty_0 \to C^\infty(S^2T^*\mathbb{S}^n)$ as follows. If $g \in \mathcal{M}^\infty_0$, let g_+ be a Poincaré-Einstein metric near h with conformal infinity $[g]$. Write g_+ in the geodesic normal form determined by g and define $\mathcal{N}(g) = \psi_0$. One can show that $\mathcal{N}$ satisfies the following equivariance properties with respect to diffeomorphisms and conformal changes:

$$\begin{aligned} \mathcal{N}(\Phi^*g) &= \Phi^*\mathcal{N}(g)\,, \quad \Phi \in \mathrm{Diff}(\mathbb{S}^n) \\ \mathcal{N}(\Omega^2 g) &= \Omega^{2-n}\mathcal{N}(g)\,, \quad 0 < \Omega \in C^\infty(\mathbb{S}^n). \end{aligned} \tag{1}$$

The first result below identifies the linearization $d\mathcal{N}_{g^{(0)}}$. For $n \geq 4$, let $\mathcal{W}(g)$ denote the Weyl tensor of the metric g, so that $\mathcal{W} : \mathcal{M}^\infty \to C^\infty(\otimes^4 T^*\mathbb{S}^n)$. Let $W = d\mathcal{W}_{g^{(0)}}$ and let W^* denote the adjoint of W with respect to the L^2 inner product induced by $g^{(0)}$. For $n = 3$, let $\mathcal{C} : \mathcal{M}^\infty \to C^\infty(S^2T^*\mathbb{S}^n)$ denote the Cotton-York tensor of g, normalized by $\mathcal{C}_{ij} = 2\mu_i{}^{kl}\nabla_k P_{jl}$, where $P_{jl} = R_{jl} - \frac{R}{4}g_{jl}$ and μ_{ikl} is the volume form, and set $C = d\mathcal{C}_{g^{(0)}}$. The operator $C : C^\infty(S^2T^*\mathbb{S}^n) \to C^\infty(S^2T^*\mathbb{S}^n)$ is self-adjoint and we set $|C| = \sqrt{C^2}$. Let ∇ denote the covariant derivative and $\Delta = \nabla^*\nabla$ the rough Laplacian with respect to $g^{(0)}$, acting on $C^\infty(S^2T^*\mathbb{S}^n)$. We remark that W^*W commutes with Δ: both are $O(n+1)$-equivariant and they can be simultaneously diagonalized.

Theorem 1. *The linearization of $\mathcal{N}$ at $g^{(0)}$ is given by the following.*

$$d\mathcal{N}_{g^{(0)}} = \begin{cases} a\,W^*W(\Delta + c_1)\dots(\Delta + c_m)\sqrt{\Delta + c_{m+1}} & n \geq 5 \\ \frac{1}{3}\,|C| & n = 3\,, \end{cases}$$

where $m = (n-5)/2$, $a \neq 0$, and $c_i > 0$ for $1 \leq i \leq m+1$.

Theorem 1 is a consequence of the equivariance properties (1). The actions in (1) determine an action on $\mathcal{M}^\infty$ of the semidirect product of the positive C^∞ functions with Diff($\mathbb{S}^n$). The identity component of the isotropy group of $g^{(0)}$ under this action can be identified with the identity component $O_e(n+1,1)$ of the conformal group. Linearizing (1) shows that $d\mathcal{N}_{g^{(0)}}$ is equivariant with respect to two actions of $O_e(n+1,1)$ on $C^\infty(S^2T^*\mathbb{S}^n)$. These actions are principal series representations of $O_e(n+1,1)$ and $d\mathcal{N}_{g^{(0)}}$ is therefore an intertwining operator between two specific principal series representations. A result of [BÓØ] shows that for $n \geq 5$, the space of such intertwining operators is one-dimensional and identifies the spectral decomposition of the intertwining operator. Some computation shows that the operator identified in Theorem 1 is the operator with the prescribed spectrum. For $n = 3$, the space of intertwining operators is 2-dimensional, spanned by C and $|C|$. Consideration of the behavior under orientation reversal shows that $d\mathcal{N}_{g^{(0)}}$ must be a multiple of $|C|$ and evaluation on an example determines the constant.

The properties (1) completely describe the behavior of $\mathcal{N}$ under diffeomorphism and conformal change. This behavior is degenerate: $\mathcal{N}$ collapses the orbit of $g^{(0)}$ to 0. The next result, proved via the implicit function theorem, shows that $\mathcal{N}$ is well-behaved in the transverse directions. Let $\mathcal{M}^{k,\alpha}$ denote the space of $C^{k,\alpha}$ metrics on $\mathbb{S}^n$. Define $\mathcal{T} \subset C^{k,\alpha}(S^2T^*\mathbb{S}^n)$ to be the space of $C^{k,\alpha}$ trace-free, divergence-free tensors with respect to $g^{(0)}$ and let $\mathcal{S}$ be a smooth submanifold of $\mathcal{M}^{k,\alpha}$ near $g^{(0)}$ containing $g^{(0)}$ and tangent to $\mathcal{T}$ at $g^{(0)}$. For example, one choice for $\mathcal{S}$ is the intersection of $g^{(0)} + \mathcal{T}$ with a neighborhood of $g^{(0)}$.

Theorem 2. *For $k > n$, $\mathcal{N}$ extends to a neighborhood of $g^{(0)}$ in $\mathcal{M}^{k,\alpha}$ and $\mathcal{N}|_{\mathcal{S}} : \mathcal{S} \to C^{k-n,\alpha}(S^2T^*\mathbb{S}^n)$ is a smooth embedding of Banach manifolds.*

Next we describe an application to the LeBrun positive frequency conjecture. First we have the following local unique continuation theorem at infinity for $\pm$ self-dual Poincaré-Einstein metrics.

Theorem 3. *Let X be the interior of a 4-dimensional manifold with boundary $\overline{X}$ and let $U \subset \overline{X}$ be open and connected with $U \cap \partial X \neq \emptyset$. Let g_+ be a Poincaré-Einstein metric on $U \cap X$ with a C^k compactification, where k is fixed but sufficiently large. If g is a metric in the conformal infinity of g_+, then g_+ is $\pm$ self-dual if and only if $3\psi_0 = \pm\mathcal{C}(g)$ on $U \cap \partial X$.*

Theorem 3 is proved by a formal power series analysis of the self-duality equations mentioned in [FG] together with a unique continuation theorem of Mazzeo. Anderson ([A2]) has observed that a globally defined nondegenerate self-dual g_+ satisfies $3\psi_0 = \mathcal{C}(g)$ by consideration of the signature and Gauss-Bonnet formulae.

Let $\mathcal{M}_\pm$ denote the space of metrics g on $\mathbb{S}^n$ near $g^{(0)}$ such that $[g]$ is the conformal infinity of a $\pm$ self-dual Poincaré-Einstein metric. An immediate consequence of Theorem 3 is the following characterization in terms of the Dirichlet-to-Neumann map:

Theorem 4. $\mathcal{M}_\pm = \{g : 3\mathcal{N}(g) = \pm\mathcal{C}(g)\}$

The positive frequency conjecture states that if $\mathcal{S}$ is as above, then $\mathcal{M}_\pm \cap \mathcal{S}$ are smooth submanifolds of $\mathcal{S}$ and $T_{g_{(0)}}\mathcal{S} = T_{g_{(0)}}(\mathcal{M}_+ \cap \mathcal{S}) \oplus T_{g_{(0)}}(\mathcal{M}_- \cap \mathcal{S})$. This was proved by Biquard ([B1]) by deforming the associated twistor spaces. A different proof can be given based on Theorems 1 and 4 by applying the implicit function theorem in a manner similar to that used by Biquard in [B2] for the analogous problem in the asymptotically complex hyperbolic case.

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Cheeger's inequality revisited

Daniel Grieser

In this talk, I presented the ideas and results from the preprint 'The first eigenvalue of the Laplacian, isoperimetric constants, and the Max Flow Min Cut Theorem', arXiv.org:math.DG/0506243.

Cheeger's inequality gives a lower bound for the first eigenvalue of the Dirichlet Laplacian on a compact Riemannian manifold Ω with boundary (assumed Lipschitz),

$$\lambda_\Omega = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}, \tag{1}$$

in terms of 'Cheeger's constant'

$$h_\Omega = \inf_{S \subset \Omega} \frac{|\partial S|}{|S|}. \tag{2}$$

Cheeger proved [1]

$$\lambda_\Omega \geq h_\Omega^2/4. \tag{3}$$

We consider the problem of estimating h_Ω from below. First, it is an immediate consequence of Green's formula that, for any number h and vector field V on Ω satisfying the pointwise estimates

$$|V| \leq 1 \tag{4}$$
$$\operatorname{div} V \geq h, \tag{5}$$

one has $h_\Omega \geq h$. This simple fact seems to be little known in the geometric analysis community.

It is a remarkable fact that this estimate is sharp:

Theorem 1. *We have*

$$h_\Omega = \sup\{h : \exists V \text{ satisfying } (4), (5)\},$$

where the supremum is taken over smooth vector fields V on Ω.

A maximizer exists of regularity $V \in L^\infty, \operatorname{div} V \in L^2$.

This theorem may be regarded as a continuous version of the classical Max Flow Min Cut Theorem for networks. In the case of Euclidean domains Ω, it was first proved by Strang [5] in two dimensions and by Nozawa [3] in general. Their proofs carry over immediately to the case of Riemannian manifolds. Essentially, the supremum in the theorem is regarded as a convex optimization problem. By standard theory, it has a dual problem of the same value, which turns out to be

$$\text{Minimize } \frac{\int_\Omega |\nabla\phi|}{\int_\Omega \phi}, \quad \text{subject to } \phi \geq 0, \phi_{\partial\Omega=0}. \tag{6}$$

Now the Cavalieri principle and the coarea formula easily imply that this infimum doesn't change if ϕ is restricted to be a characteristic function of a subset $S \subset \Omega$, in which case the numerator has to be interpreted as $|\partial S|$. Therefore, this mimimum is simply h_Ω, and this proves the theorem.

The characterization of Cheeger's constant in terms of vector fields also gives a new approach to the inequality

$$\lambda_\Omega \geq \frac{1}{4\rho_\Omega^2}, \tag{7}$$

(where ρ_Ω is the inradius and Ω is assumed to be a plane simply connected domain). This inequality is usually attributed to Osserman [4], but in fact it was first proved by E. Makai [2].

Problem: In the example of a square the Cheeger constant and a minimizing subset S can be found explicitly (round off the corners by quarter circles, optimize over their radius). However, there does not seem to be a simple formula for the optimal vector field in the theorem. What is this vector field?

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Interpolating L^2 Hodge and Signature Theorems

Eugénie Hunsicker

1. Introduction

One approach to studying L^2 Hodge and Signature theorems is to consider a manifold, M, which is the interior of a manifold with corners $\overline{M}$, and to consider metrics on M, generally singular on $\overline{M}$, but with specific nice regularity near the boundary. If $\overline{M}$ has boundary Y, where Y is the total space of a fibre bundle over a compact manifold, B, and with fibre F, then M can be endowed with various metrics which reflect this boundary structure. Two incomplete classes of metrics for which L^2 Hodge and Signature theorems have been obtained are metrics which are quasi-isometric near the boundary to the product metric on $(0,1) \times Y$ and metrics which are quasi-isometric near the boundary to the bundle over B whose fibres are truncated metric cones on the boundary fibres, F. The Hodge theorem in this case can be expressed in terms of an extended definition of intersection cohomology, and the signature results differ by a topological invariant of the boundary fibration, called τ and originally introduced by Dai. This paper concerns a family of metrics on M which interpolates in a natural geometric way between these two extremal metrics. The L^2 Hodge and Signature theorems obtained for the metrics in this family interpolate in a natural topological way between the results in the extremal cases.

2. Main results

Over the past thirty years or so, there have been efforts to extend the Hodge and Signature theorems for compact complete smooth manifolds to compact manifolds with singularity and to noncompact manifolds. One approach has been to consider metrics which have a specified regularity near the singularity or at infinity. For example, consider a manifold $\overline{M}$ with boundary $\partial\overline{M} = Y$ where $Y \xrightarrow{\phi} B$ is a fibre bundle with f-dimensional fibre F. For a value $c \geq 0$, we may endow $M = \overline{M} - \partial\overline{M}$ with a metric g_c which is quasi-isometric near the boundary to one of the form

$$ds_c^2 = dr^2 + r^{2c}\tilde{h} + \phi^* ds_B^2$$

where h is a two form which restricts to a metric on each fibre of Y. Let X be the compactification of M obtained by collapsing the fibres of the boundary fibration. In the case that $c = 0$, this gives a metric which is smooth to the boundary and gives the region near the boundary the structure of a finite cylinder. In the case that $c = 1$, this metric gives a neighborhood of the boundary the structure of a truncated cone bundle over B. If in addition, F is odd dimensional, then the exterior derivative, d and thus the Laplacian have unique closed extensions in L^2. In other cases, generally d has various closed extensions, and one must be specified in an L^2 Hodge or signature theorem. The minimal extension is given by the graph closure and the maximal extension has domain which includes any form α for which the weak derivative$d\alpha$ is L^2. Both extensions give cohomology groups, denoted $H^*_{min}(M, g_c)$ and $H^*_{max}(M, g_c)$, respectively. In addition, we get two extensions of the Hodge Laplacian, $D_{rel} = d_{min} + d^*_{min}$ and $D_{abs} = d_{max} + d^*_{max}$, each of which defines a Hodge cohomology, or space of L^2 harmonic forms. We denote these by $\mathcal{H}^*_{\mathrm{rel}}(M, g_c)$ and $\mathcal{H}^*_{\mathrm{abs}}(M, g_c)$, respectively.

Hodge and signature theorems in the case where $c = 1$ have been proved in [7] and [12]. The Hodge theorems in the case where $c = 0$ were proved in [1], and a signature theorem for this cases follows from this Hodge theorem and the signature theorem in [2] for complete manifolds with cylindrical ends. The Hodge theorems can all be understood in terms of intersection cohomology, either middle perversity intersection cohomology of X (in the cone case) or the extremal perversities (not originally considered in the work of Goresky and MacPherson) which correspond to relative and absolute cohomology of M. The other possible intersection cohomology groups on X naturally interpolate between these topologically defined groups, and it is natural to ask if this topological interpolation is related to any geometric interpolation between the metrics. In addition, the signature results in the cases where $c = 1$ and $c = 0$ differ by an invariant, called the τ-invariant, introduced by Dai in [8], which has a natural decomposition, and this suggests that signature theorems over the interpolating family of metrics may have some expression in terms of that decomposition.

In fact, both the Hodge and Signature theorems interpolate in a satisfying manner:

Theorem 1. *Let (M, g_c) and X be as above. Then*

$$H^*_{min}(M, g_c) \cong \mathcal{H}^*_{rel}(M, g_c) \cong \begin{cases} IH^*_{\underline{\mathfrak{m}}+[[1+\frac{1}{2c}]]}(X, B) & f \text{ is even} \\ IH^*_{\underline{\mathfrak{m}}+[[\frac{1}{2}+\frac{1}{2c}]]}(X, B) & f \text{ is odd} \end{cases}$$

and

$$H^*_{max}(M, g_c) \cong \mathcal{H}^*_{abs}(M, g_c) \cong \begin{cases} IH^*_{\overline{\mathfrak{m}}-[[1+\frac{1}{2c}]]}(X, B) & f \text{ is even} \\ IH^*_{\overline{\mathfrak{m}}-[[\frac{1}{2}+\frac{1}{2c}]]}(X, B) & f \text{ is odd} \end{cases},$$

where $[[x]]$ denotes the greatest integer strictly less than x and $\underline{\mathfrak{m}}$ and $\overline{\mathfrak{m}}$ are the two middle perversities for X.

Here we use the notation $IH^*_{\mathfrak{p}}(X,B)$ instead of $IH^*_{\mathfrak{p}}(X)$ in order to indicate a slightly more general definition of intersection cohomology than is standard. In particular, it allows us to include the case where the boundary fibration fibre F is trivial, so X is our original manifold with boundary, $\overline{M}$.

Note that when $c = 1$, we get $[[\frac{1}{2} + \frac{1}{2c}]] = 0$, so this result reduces to the result for manifolds with edges in [12], and in the case that the fibre is even dimensional, it reduces to Cheeger's result in [6]. In the extended definition of intersection cohomology , for c sufficiently close to 0, these spaces become relative and absolute cohomologies of M, respectively, thus reducing to the known results for manifolds with boundary. As c goes from 0 to 1, the intersection cohomology groups isomorphic to the maximal cohomology interpolate between middle perversity and absolute cohomology, while the intersection cohomology groups isomorphic to the minimal cohomology interpolate between middle perversity and relative perversity.

Using this Hodge theorem, we can also obtain a signature theorem for the manifolds (M, g_c) through a signature theorem for intersection cohomology. If p and q are dual perversities with $p \geq q$, then the intersection pairing between their intersection cohomology groups descends to a nondegenerate pairing on $\mathrm{Im}\left(IH^{n/2}_p(X,B) \longrightarrow IH^{n/2}_q(X,B)\right)$. Call its signature $\sigma_p(X)$. The τ-invariant defined by Dai in [8] is given by a sum $\tau = \sum_{i=2}^{\infty} \tau_i$, where τ_i is the signature of a form defined on the E_i term of the Leray-Serre spectral sequence for the boundary fibration of M. We obtain the interpolating result:

Theorem 2. *If $p = \underline{\mathfrak{m}} + k$, then the signature of the intersection form on these spaces is given by:*

$$\sigma_p(X) = sgn\, Im\left(H^*(M, \partial M) \longrightarrow H^*(M))\right) + \sum_{i=2+2k}^{\infty} \tau_i.$$

Thus as the metric becomes less and less cylindrical and more and more conical, the signature theorem picks up more and more of the τ_i terms, until, when the metric is close to conical, the signature includes all of τ.

The proof of the Hodge result follows by soft analytic techniques using the general theory of Hilbert complexes as described in [4] and a local calculation as in Cheeger's paper on manifolds with conical singularities, [6]. The signature result follows from the Hodge result in a purely topological manner. The first thing we need is a theorem that says the signature can be calculated on M and near the singular stratum in X independently. This is accomplished by a version of Novikov additivity:

Theorem 3. *If X is a pseudomanifold with a single compact smooth singular stratum and if $Y \subset X$ is a compact codimension 1 submanifold such that $X = Z \bigcup_Y Z'$ where $Z \subset\subset X^{reg}$, then*

$$\sigma_p(X) = \hat{\sigma}(Z) + \hat{\sigma}_p(Z').$$

The proof of this theorem is similar to the proof of the original Novikov additivity result for complete compact manifolds. The last part of the proof is calculation of the signature of the neighborhood of the singular stratum in X, which has the form of a bundle over B whose fibres are truncated cones over F. This is accomplished using a modification of the technique used in [7] which involves analyzing the spectral sequence of the boundary fibration.

An interesting consequence of the Novikov additivity result is that although the space on which the intersection pairing is defined is a priori global, that is, it does not satisfy the standard Mayer-Vietoris sequence, the signature of the intersection pairing on this space can be calculated by cutting off the end and considering it separately. This together with results by Carron in, for example, [5] suggest that cut and paste techniques may be applicable in Hodge results which involve spaces of this form, as in [10], as well. It would be interesting to examine this. It seems likely, as well, that a much more general Novikov additivity result should be true.

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Compactifications of symmetric and locally symmetric spaces

Lizhen Ji

In this talk, we gave a survey of compactifications of both symmetric and locally symmetric spaces, and together with related compactifications of Teichmüller spaces and moduli spaces of Riemann surfaces.

The topics of talk consisted of the following: (1). List of compactifications and relations between them. (2). Motivations and original constructions of compactifications. (3). Uniform construction of all compactifications (4). Teichmüller spaces, mapping class groups and moduli spaces. (6). Applications to the integral Novikov conjectures for arithmetic and S-arithmetic groups.

§1. Definitions. A Riemannian manifold M is called *locally symmetric* if for any point $x \in M$, the local geodesic symmetry is a local isometry. M is called *symmetric* if it is locally symmetric and each local geodesic symmetry extends to a global isometry. They are special Riemannian manifolds: the covariant derivative of the sectional curvature is equal to 0. The definitions here tie up with Lie group theory immediately.

§2. Examples of symmetric spaces.

$\mathcal{P}_2$, the space of positive definite binary quadratic forms of determinant 1. Then $SL(2,\mathbb{R})$ acts on $\mathcal{P}_2$ by $g \cdot A = g^t A g$. Hence $\mathcal{P}_2 = SL(2,\mathbb{R})/SO(2)$, and it can also be identified with the upper half plane $\mathbf{H} = \{x + iy \mid y > 0, x \in R\}$.

A generalization of $\mathcal{P}_2$ is given by $\mathcal{P}_n$, space of positive definite quadratic forms in n variables of determinant 1, $\mathcal{P}_n = SL(n,\mathbb{R})/SO(n)$, $n \geq 3$. Another is given by the Siegel upper half space $\mathcal{H}^n = \{X + iY \mid X, Y > 0\}$, where X, Y are real symmetric $n \times n$ matrices. When $n = 1$, it reduces to $\mathbf{H}$. It is known that $\mathcal{H}^n = Sp(n,\mathbb{R})/U(n)$, $n \geq 2$. Another example is $SL(2,\mathbb{C})/SU(2)$, the real hyperbolic space of dimension 3, also the space of Hermitian positive definite forms in 2 variables of determinant 1. $\mathcal{P}_n$ and $\mathcal{H}^n$ are symmetric spaces of higher rank. Clearly, they are noncompact, since positive definite quadratic forms can degenerate and become only semi-definite.

Another type of generalization is given by **Teichmüller spaces**. Let Σ_g be a closed surface of genus $g \geq 0$. The equivalence classes of all marked complex structures (or hyperbolic metrics) on Σ_g, $f : \Sigma_g \to S$, f a homotopy equivalence, S a Riemann surface, forms the Teichmullar space T_g. If we forget the marking, we get the moduli space of complex structures on Σ_g. When $g = 1$, $T_g = \mathbf{H}$. Therefore for $g \geq 2$, T_g is a generalization of $\mathbf{H}$. Though T_g is not a symmetric space, it enjoys some properties similar to those of symmetric spaces of noncompact type.

§3. Examples of locally symmetric spaces. Two positive definite quadratic forms $A, B \in \mathcal{P}_n$ are equivalent if there exists $\gamma \in SL(n,\mathbb{Z})$ such that $\gamma^t A \gamma = B$. This equivalence is important in number theory, since equivalent integral quadratic forms represent the same set of integers. Existence of such integral representation and number of them are important and basic problems. The set of equivalence classes in $\mathcal{P}_n$ is $SL(n,\mathbb{Z})\backslash\mathcal{P}_n$, a locally symmetric space. When $n = 2$, it is the moduli space of elliptic curves. The moduli space of abelian varieties of dimension n with the principal polarization is given by $Sp(n,\mathbb{Z})\backslash\mathcal{H}^n = Sp(n,\mathbb{Z})\backslash Sp(n,\mathbb{R})/U(n)$. All these locally symmetric spaces are noncompact since elliptic curves, abelian varieties, and quadratic forms can degenerate. Many other moduli spaces are also given or related to locally symmetric spaces.

The general set-up for locally symmetric spaces. Let $\mathbf{G} \subset GL(n, C)$ be a semisimple linear algebraic group defined over $\mathbb{Q}$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup, i.e., Γ is commensurable with $\mathbf{G} \cap GL(n, \mathbb{Z})$. We can assume that Γ is torsion free by taking a subgroup of finite index. $G = \mathbf{G}(\mathbb{R})$ the real locus, $K \subset G$ a maiximal compact subgroup, $X = G/K$ symmetric space of noncompact type when given a G-invariant metric. Γ acts properly on X, and the quotient $\Gamma\backslash X$ is a locally symmetric space of finite volume. Assume that $\Gamma\backslash X$ is noncompact, i.e., the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive, equivalently, Γ does not contain some nontrivial unipotent elements.

§4. Mapping class groups. Let Σ_g be a closed orientable surface of genus g, and $\mathrm{Diff}^+(\Sigma)$ all orientation preserving diffeomorphisms, $\mathrm{Diff}^0(\Sigma)$ all diffeomorphism isotropic to the identity map. Then $Mod_g = \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}^0(\Sigma)$ is the mapping class group and acts on the Teichmuller space T_g. The quotient $Mod_g\backslash T_g$ is the moduli space of complex structures on Σ_g. When $g = 1$, $Mod_g = SL(2, \mathbb{Z})$. So the moduli space $Mod_g\backslash T_g$ is similar to a locally symmetric space.

Problems. Compactifying the spaces X and $\Gamma\backslash X$, and relate their compactifications. We also want to compactify the Teichmüller space T_g and the moduli space $Mod_g\backslash T_g$.

§5. Why compactifications? The general answers are: 1. It is easier to deal with compact spaces. 2. Useful to understand how objects degenerate. 3. Sometimes almost necessary.

More specifically, (a) For harmonic analysis on a symmetric space X, for example, harmonic functions on the unit disc $D = SU(1,1)/U(1) = \mathbf{H}$, the boundary values/behaviors of eigenfunctions are important, for example, the Poisson integral formula. To generalize results on D to X, we need to start with some boundary of X, and hence compactifications of X. To understand better behaviors of functions at infinity of noncompact manifolds, use more refined compactifications. (b) Understand geometry at infinity, i.e., asymptotic geometry of the spaces. (c) Behaviors at infinity of automorphic forms on locally symmetric spaces, to pass from compactifications X to compactifications $\Gamma\backslash X$.

§6. List of compactifications of symmetric spaces. (1) Satake compactifications $\overline{X}^S$, finitely many of them, the most common one being the maximal Satake $\overline{X}^S_{\max}$, (2) Furstenberg compactifications $\overline{X}^F$, isomorphic to the Satake compactifications. (3) Geodesic compactification $X \cup X(\infty)$, (4) Martin compactification $X \cup \partial_\lambda X$, $\lambda \leq \lambda_0$. (5) Karpelevic compactification $\overline{X}^K$. There are also other constructions, using the space of subgroups, space of continuous functions in the Gromov compactification. Relations between them can be described. See [GJT] and [BJ]. In the above compactifications of X, X is open and dense. There are also compactifications of X which are closed manifolds and contain X as an open but not dense subset. For example, the Oshima compactification $\overline{X}^O$, the Oshima-Sekiguchi compactification $\overline{X}^{OS}$, and the real locus of the wonderful compactification $\overline{\mathbf{X}_c}^W$ of $\mathbf{X}_c$, the complexification of X. When $X = SL(n, R)/SO(n)$, $\mathbf{X}_c = SL(n, \mathbb{C})/SO(n, \mathbb{C})$,

§7. Compactifications of locally symmetric spaces. (1) Satake compactifications $\overline{\Gamma\backslash X}^{S}$, (2) Baily-Borel compactification $\overline{\Gamma\backslash X}^{BB}$, (3) toroidal compactifications $\overline{\Gamma\backslash X}^{tor}_{\Sigma}$, infinitely many in general, depending on cone decomposition Σ, (4) Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$, (5) Reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$, (6) Geodesic compactification $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$, (7) Tits compactification $\overline{\Gamma\backslash X}^{T}$, (8) Gromov compactification $\overline{\Gamma\backslash X}^{G}$, (9) subgroup compactification $\overline{\Gamma\backslash X}^{sb}$. There are also compactifications of $\Gamma\backslash G$. Relations between them can be determined. See [BJ].

The Teichmüller space T_g admits many different compactifications: (1) Teichmüller compactification, (2) Thurston compactification, (3) Bers compactification, (4) Weil-Patterson completion or the visual compactification as a CAT(0)-space. The moduli space $Mod_g\backslash T_g$ also has two compactifications: Mumford-Deligne compactification by stable curves, a projective variety: Borel-Serre type compactification, a manifold with corners, outlined by Harvey, completed by Ivanov.

§8. History of compactifications.

Symmetric spaces. (1) X is simply connected and nonpositively curved. Geodesics naturally lead to the geodesic compactification $X \cup X(\infty)$, where $X(\infty)$ is the set of equivalence classes of geodesics. If X is a proper CAT(0)-space, the same construction gives a compactification of X. A CAT(0)-space is a geodesic length space such that every triangle is thinner than a corresponding triangle in $\mathbf{R}^2$ (of the same side lengths) Teichmuller space T_g with the Weil-Patterson metric is a CAT(0)-space, but not a proper space. Bruhat-Tits buildings are also CAT(0)-spaces. (2) The modern theory of compactifications of X started with Satake, followed by Furstenberg, Moore, Kotanyi, Herman, Karpelevic, and also by Oshima et al in another direction.

Locally symmetric spaces. (1) Compactifications of modular curves $\Gamma\backslash\mathbf{H}$, $\Gamma \subset SL(2,\mathbb{Z})$, was known in the 19th century. (2) Siegel initiated the study of automorphic forms in multi-variables, found a fundamental domain of the Siegel modular group $Sp(n,\mathbb{Z})$ in $\mathcal{H}^n$, defined a compactification of this fundamental domain. (3) Satake proposed a procedure to compactify $Sp(n,Z)\backslash\mathcal{H}^n$ as a complex analytic V-manifold, and defined such a compactification when $n=2$. (4) Satake defined a compactification of $Sp(n,Z)\backslash\mathcal{H}^n$ by adding lower dimensional spaces of the same type, and showed it to be a normal complex space. (5) Baily showed that the Satake compactification of $Sp(n,Z)\backslash\mathcal{H}^n$ is a normal projective variety. (6) Satake started the general theory of compactifications of locally symmetric spaces using compactifications of symmetric spaces. Satake compactifications are partially ordered. (7) When X is Hermitian, it can be realized as a bounded symmetric domain. The corresponding Satake compactification of X (and $\Gamma\backslash X$) is a minimal compactification. Baily-Borel showed that it is a normal projective variety. (8) Ash-Mumford-Rapoport-Tai constructed toroidal compactifications to resolve the singularities of the Baily-Borel compactification. They depend on some combinatorial data (ployhderal cone decompositions, needed for torus embeddings). There are in general infinitely many. (10) Borel-Serre compactification, and the reductive Borel-Serre comactification by Zucker. (11) geodesic and Tits compactifications.

§9. Specific reasons for compactifications

Martin compactification. Let Δ be the Laplace operator of X. Consider $C_\lambda(X) = \{u \in C^\infty(X) \mid \Delta u = \lambda u, u > 0\}$. If nonempty, it is a convex cone. A problem in potential theory is to find the set of extremal elements and parametrize them in terms of some "geometric boundary" This problem is solved by the Martin compactification $X \cup \partial_\lambda X$, defined in terms of asymptotics of Green function. Its boundary points parametrize a set of generators of the cone $C_\lambda(X)$, and a subset, called the minimal Martin boundary, parametrizes the extremal ones. A natural problem is to identify the Martin compactification with other more geometrical compactifications, solved in [GJT] using harmonic analysis on symmetric spaces. Recent work of Mazzeo-Vasy used methods from many body scattering theory.

Oshima compactification. Let $\mathcal{D}(X)$ be the set of all invariant differential operators, which includes the Laplace operator Δ. If $\mathbf{H}$, $\mathcal{D}(X)$ is a polynomial algebra generated by Δ. Consider the joint eigenspace of $\mathcal{D}(X)$, $\varepsilon_\chi(X) = \{u \in C^\infty(X) \mid Du = \chi(D)u, D \in \mathcal{D}(X)\}$. The Helgason conjecture says that functions in $\varepsilon_\chi(X)$ are the Poisson transform of hyperfunctions on the Poisson boundary of X. The motivation is as follows. Due to the G-invariance of the operators, G acts on $\varepsilon_\chi(X)$. This gives a representation of G. Understanding this representation is related to spectral decomposition, or decomposition of the regular representation of G on X into irreducible representations. The Oshima compactification $\overline{X}^O$ was motivated by this conjecture. $\overline{X}^O$ is a closed real analytic manifold, containing the union of finitely many X as an open subset. Real analytic structure is needed for applying theory of regular singularities in order to get boundary values (as hyperfunctions) The Poisson boundary is contained in the closure of each copy of X, and the invariant differential operators on X have regular singularities along the Poisson boundary.

Baily-Borel compactification. Assume that X is Hermitian symmetric, i.e., has an invariant complex structure. For example, $X = \mathcal{H}^n$. $\Gamma\backslash X$ Hermitian locally symmetric space. Let $\mathcal{M}(\Gamma\backslash X)$ be the field of meromorphic functions. A problem proposed by Siegel is to determine the transcendental degree of $\mathcal{M}(\Gamma\backslash X)$. This is related to the growth of dimension of (holomorphic) modular forms as the weight increases.

If $\Gamma\backslash X$ is a projective variety, by GAGA, every meromorphic function is rational (or algebraic), and hence the trans. degree of $\mathcal{M}(\Gamma\backslash X) = \dim \Gamma\backslash X$. If $\Gamma\backslash X$ is compact, it is a projective variety. Assume $\Gamma\backslash X$ is noncompact. If it admits a compactification $\overline{\Gamma\backslash X}$ as a normal projective variety such that the boundary is of codimension at least 2, then the Riemann extension theorem implies that every meromorphic function on $\Gamma\backslash X$ extends to a meromorphically on the compactification. So $\mathcal{M}(\Gamma\backslash X) = \mathcal{M}(\overline{\Gamma\backslash X})$ is the field of rational functions on $\overline{\Gamma\backslash X}$, and tran. degree of $\mathcal{M}(\Gamma\backslash X) = \dim \Gamma\backslash X$. $\overline{\Gamma\backslash X}^{BB}$ has the desired properties. It is one of the minimal Satake compactifications. Piateski-Shapiro constructed a similar compactification when X is a classical symmetric domain and introduced important notions of Siegel domains of 3 kinds, important for toroidal compactifications below.

Borel-Serre compactification. Recall that X is contractible. If Γ is torsion free, then $\pi_1(\Gamma\backslash X) = \Gamma$, $\pi_i(\Gamma\backslash X) = \{1\}, i \geq 2$. Hence $\Gamma\backslash X$ is a $K(\Gamma, 1)$-space, or the classifying space $B\Gamma$ of Γ. An important problem in topology is to determine when exists a finite $B\Gamma$-space. This implies many finiteness properties such as finite presentation. If $\Gamma\backslash X$ is compact, then it admits a finite triangulation, $\Gamma\backslash X$ is a finite $B\Gamma$-space. If $\Gamma\backslash X$ is noncompact, the existence of a finite $B\Gamma$-space is not obvious.

If $\Gamma\backslash X$ has a compactification $\overline{\Gamma\backslash X}$ which is a homotopic to $\Gamma\backslash X$, and has a finite triangulation, then $\overline{\Gamma\backslash X}$ is the desired classifying space. The Borel-Serre compactification $\Gamma\backslash\overline{X}^{BS}$ is a manifold with corners and satisfies these two conditions. When $\Gamma\backslash X$ is a surface, $\overline{\Gamma\backslash X}^{BS}$ is obtained by adding a circle to each cusp end. As suggested by the notation, it is the quotient by Γ of a partial compactification $\overline{X}^{BS}$. $\overline{X}^{BS}$ is equal to the universal covering $E\Gamma$ of the classifying space $B\Gamma = K(\Gamma, 1)$.

Such an explicit model of $E\Gamma$ and compactifications of $E\Gamma$ is important in studying cohomology groups of Γ and other related invariants. It is also important in global and asymptotic geometry and other properties of Γ, for example, the (integral) Novikov conjectures for Γ in (algebraic) topology and C^*-algebras.

The Borel-Serre compactification of the Teichmüller space T_g and the moduli space are motivated by similar problems, and many similarities between Mod_g and arithmetic groups.

Reductive Borel-Serre compactification. To study L^2-cohomology of $\Gamma\backslash X$, the Borel-Serre compactification is too large since it does not support partitions of unity. Consider $X = \mathbf{H}$, and a cusp neighborhood of $\Gamma\backslash\mathbf{H}$. In a model of this cusp in terms of the strip in $\mathbf{H}$, the coordinates $z = x + iy$, dx is not square integrable, in particular not bounded. So to get partition of unity, need to make the partition function independent of x near the cusp. This is the same as collpasing the boundary circles of $\overline{\Gamma\backslash\mathbf{H}}^{BS}$ into points. In general, to solve this problem, Zucker blowed down nilmanifolds in the boundary of the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$. This collapsed compactification is the reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$, which admits partition of unity.

Toroidal compactifications. $\overline{\Gamma\backslash X}^{BB}$ is usually singular. In fact, expect the case of Riemann surfaces, its boundary points have $\mathbb{C}$ codim at least 2. Hironaka's theorem says that the singularities can be solved. But the resolutions are not canonical and has no group structures. Mumford et al constructed more canonical, explicit resolutions, i.e., compactifications $\overline{\Gamma\backslash X}^{tor}_{\Sigma}$, smooth projective varieties. Consider

$\Gamma\backslash\mathbb{H}$, one cusp neighborhood. Identify it with a neighborhood of the puncture in punctured disk $D^\times$. Filling in the puncture gives a smooth compactification of the cusp. $D^\times$ is contained in $\mathbb{C}^\times$, a torus over $\mathbb{C}$, $\mathbb{C}^\times = \mathbb{R}\times\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}/\mathbb{Z}$ is real torus. A general torus is $\Lambda\backslash\mathbb{C}^n \cong (\mathbb{C}^\times)^n$, where Λ is a lattice in $\mathbb{R}^n$. There are many ways to compactify tori, given by the theory of toric varieties, described in terms of fans. To understand $\overline{\Gamma\backslash X}^{tor}_\Sigma$, Σ some collections of compatible fans, a key point is to see that parts of infinity of $\Gamma\backslash X$ are contained in $\mathbb{C}$-torus bundles. For this purpose, Siegel domains of the three kinds are essential.

Wonderful compactification of symmetric varieties. $\mathbb{C}^\times$ is a commutative algebraic group. Its compactifications are toric varieties, crucial in toroidal compactifications above. A natural question is to compactify non-abelian algebraic groups $\mathbf{G}$ over $\mathbb{C}^\times$. If $\mathbf{G}$ is reductive, it is a symmetric variety via the identification $\mathbf{G} = \mathbf{G}\times\mathbf{G}/Diag(\mathbf{G})$ More general symmetric varieties $\mathbf{G}/\mathbf{H}$. Wonderful compactification is similar to the maximal Satake compactification. Important in enumerative algebraic geometry and other areas.

§10. Original constructions of compactifications.

Satake compactifications $\overline{X}^S_\tau$ of X are constructed in two steps: (1) Compactify the special symmetric space $\mathcal{P}_n = SL(n,\mathbb{C})/SU(n)$, the space of positive definite Hermitian matrices of determinant 1, a minimal Satake compactification $\overline{\mathcal{P}_n}^S$ (2) Embed X into $\mathcal{P}_n$ as a totally geodesic submanifold of $\mathcal{P}_n$ and take closure in $\overline{\mathcal{P}_n}^S$. Different embeddings in $\mathcal{P}_n$ give different Satake compactifications. Furstenberg compactifications are obtained by embedding into the space of probabulity measures on faithful Furstenberg boundaries.

Satake compactifications of locally symmetric spaces are constructed in 4 steps: (1). Decompose boundary of $\overline{X}^S_\tau$ into boundary components X_P, P real parabolic subgroups. (2). Choose "rational" boundary components to get a partial compactification $X\cup\coprod' X_P$ (3). Retopologize $X\cup\coprod' X_P$ (4). Show that Γ acts continuously on $X\cup\coprod' X_P$ with compact quotient, a desired compactification of $\Gamma\backslash X$

This construction of compactifications of $\Gamma\backslash X$ really depends on compactifications of X. On the other hand, the Borel-Serre compactification can be constructed directly in 3 steps: (1). For every $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$, define a corner $X(\mathbf{P})$, $X(\mathbf{P})\supset X$. (2). Form a partial compactification $\overline{X}^{BS}$ from these corners $\overline{X}^{BS} = \cup X(\mathbf{P})$. (3). Show that Γ acts properly on $\overline{X}^{BS}$ with a compact quotient.

This method can be modified and applied to construct almost all compactifications of $\Gamma\backslash X$ and X in [BJ]. Briefly, instead of corners, we assign a boundary component $e(\mathbf{P})$ for each $\mathbf{P}$, which are disjoint from each other. This uniform construction allows one to see clearly relations between different compactifications.

§11. Alternative Borel-Serre compactification and applications. Another approach to the$\overline{\Gamma\backslash X}^{BS}$ is to construct a compact submanifold homotopic to $\Gamma\backslash X$. This is related to the precise reduction theory. This idea can also be applied to the Teichmüller space T_g and the moduli space of Riemann surfaces. In some applications to integral Novikov conjectures for arithmetic groups Γ and more general S-arithmetic groups, we need a good compactification of $E\Gamma$, and hence of $\overline{X}^{BS}$. This alternative construction might be helpful for this purpose. The idea also applies to integral Novikov conjectures for the mapping class groups Mod_g.

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On the essential spectrum of magnetic Schrödinger operators

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(joint work with Sylvain Golénia)

We analyze the spectrum of magnetic Schrödinger operators, and also of the Laplacians on differential forms, on certain Riemannian manifolds X homeomorphic to the interior of a compact manifold with boundary $\overline{X}$. Let $M := \partial\overline{X}$ be the boundary of $\overline{X}$ and $x : \overline{X} \to [0,\infty)$ a boundary-defining function. On a tubular neighborhood $[0,\varepsilon) \times M$ of the boundary, the metric is assumed to take the form

$$g_p = x^{2p} a \cdot \left(\frac{dx}{x^2} + \alpha(x)\right)^2 + h(x) \tag{1}$$

for some $p > 0$. The function a, the family of 1-forms α, and the family of symmetric 2-tensors h, are all smooth down to $x = 0$.

Interesting particular cases of such metrics are obtained when $\alpha \equiv 0$, $a \equiv 1$, and h is constant in x. Then for h flat and $p = 1$, the ends are complete hyperbolic of finite volume, while for $p > 1$ we get the so called metric horns, incomplete and again of finite volume. As $p \searrow 0$, we approach an asymptotically cylindrical metric. Conical singularities are however *not* of the above type. If $a(0) \equiv 1$ and $\alpha(0) \equiv 0$ we call the metric *exact*. We can reduce ourselves to this case if $\alpha(0)$ is exact as a 1-form.

Our first results concern the Laplacian $\Delta_{k,p}$ on square-integrable sections of $\Lambda^k(X)$ with respect to g_p. If the Betti numbers h^k and h^{k-1} of M vanish, then $\Delta_{k,p}$ is essentially self-adjoint (even in the incomplete case $p > 1$), has purely discrete spectrum, and its eigenvalue counting function grows like

$$N_{k,p}(\lambda) \approx \begin{cases} C_1\lambda^{n/2} & \text{for } 1/n < p < \infty, \\ C_2\lambda^{n/2}\log\lambda & \text{for } p = 1/n, \\ C_3\lambda^{1/2p} & \text{for } 0 < p < 1/n. \end{cases} \tag{2}$$

If however one of these Betti numbers is non-zero, then assuming that g_p is exact, we show that the essential spectrum equals $[0,\infty)$ for $0<p<1$, $[c,\infty)$ for $p=1$ with an explicit c, and $\emptyset$ for $p>0$ for every self-adjoint extension. In this last case however we do not have a Weyl law. To determine the essential spectrum we use a comparison principle from [1] to reduce to the case of conformal cylinders.

The second operator studied is the magnetic Laplacian. Let B be a smooth exact real 2-form on X, A a primitive of B (i.e., $dA = B$), and let

$$\Delta_A := (d+iA)^*(d+iA)$$

be the so-called magnetic Laplacian, which physically is supposed to govern the evolution of a particle in the magnetic field B with the gauge A. If $H^1(X,\mathbb{R}) = 0$ then Δ_A is independent on the choice of A up to unitary equivalence, in particular its spectrum is determined only by B.

Assume for simplicity that B has compact support and also $H^1(X,\mathbb{R}) = 0$. We call B *integral* if its relative cohomology class belongs to the image of $2\pi H^2_c(X,\mathbb{Z})$. We prove that when B is non-integral, the operator Δ_A has purely discrete spectrum and the generalized Weyl law (2) holds. Else, if g_p is exact, then Δ_A has the same essential spectrum as the free Laplacian on functions. Note that for the existence of non-integral B it is necessary that $H^1(M) \neq 0$. In dimension 3, for orientable X this is incompatible with the assumption that $H^1(X) = 0$.

We add now to our operators an electric potential V of the form $x^{-2p}v$ for some smooth v on $\overline{X}$. If the restriction of v to M is strictly positive then the spectrum becomes purely discrete and (2) holds. Even more, for the magnetic Schrödinger Laplacian this facts hold if $v_{|M}$ is non-negative and not identically 0. In this way we get Schrödinger operators with purely discrete spectrum and semi-bounded below for a potential which may tend to $-\infty$ in some directions.

Note that for X a spin manifold, the spectrum of the Dirac operator of (X, g_p) was studied in [3].

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Spectral theory on locally symmetric spaces

Werner Müller

Spectral theory on locally symmetric spaces of finite volume is closely related with the theory of automorphic forms which in turn is immediately related with the theory of numbers. In this talk we give a survey of some of the recent developments in this area.

The general set up is as follows. Let $S = G/K$ be a globally Riemannian symmetric space of the noncompact type defined by a real semisimple Lie group G and a maximal compact subgroup $K \subset G$. The standard example is $S = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ - the space of positive definite symmetric $n \times n$-matrices of determinant 1. Furthermore let $\Gamma \subset G$ be a lattice which means a discrete subgroup with finite co-volume (e.g. $\mathrm{vol}(\Gamma\backslash G) < \infty$ with respect to any Haar measure). A typical example is $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ or a principal congruence subgroup $\Gamma(N) \subset \mathrm{SL}(n,\mathbb{Z})$. Then $X_\Gamma := \Gamma\backslash S = \Gamma\backslash G/K$ is a locally symmetric space of finite volume. Let $\mathcal{D}(S)$ denote the ring of invariant differential operators on S, i.e., the ring of differential operators $D\colon C^\infty(S) \to C^\infty(S)$ which satisfy $DR_g = R_g D$, $g \in G$, where R_g is defined by $R_g f(x) = f(gx)$. It is well known that $\mathcal{D}(S)$ is a finitely generated commutative algebra. The minimal number of generators of $\mathcal{D}(S)$ equals the rank r of S and there exists a set of r formally selfadjoint generators. The Laplacian Δ of S belongs to $\mathcal{D}(S)$. The basic problem is:

Problem: Study the joint spectral resolution of $\mathcal{D}(S)$ acting on X_Γ.

This means, in particular, to study the joint eigenfunctions of $\mathcal{D}(S)$ i.e., $\phi \in C^\infty(X_\Gamma)$ satisfying $D\phi = \lambda_D \phi$, $D \in \mathcal{D}(S)$. If X_Γ is noncompact one has to impose growth conditions at infinity. There are both square-integrable and generalized eigenfunctions (which are not square-integrable).

There is a distinguished class of eigenfunctions called *cusp forms*. They are defined as follows. Let P be a parabolic subgroup of G and let N_P be the unipotent radical of P. If $N_P \cap \Gamma\backslash N_P$ is compact, P is called Γ-cuspidal. A function $\phi \in C^\infty(X_\Gamma)$ is called a cusp form, if ϕ is a simultaneous eigenfunction of $\mathcal{D}(S)$ which is slowly increasing and satisfies

$$\int_{N_P \cap \Gamma\backslash N_P} \phi(nx)\, dn = 0$$

for all proper Γ-cuspidal parabolic subgroups $P \subset G$. It follows that every cups form ϕ is rapidly decreasing and, hence belongs to $L^2(X_\Gamma)$. Cusp forms are the building blocks of the spectral theory. Langlands' theory of Eisenstein series [4] provides a decomposition

$$L^2(X_\Gamma) = L^2_d(X_\Gamma) \oplus L^2_c(X_\Gamma)$$

into invariant subspaces, where $L^2_d(X_\Gamma)$ is the subspace spanned by the square-integrable simultaneous eigenfunctions of $\mathcal{D}(S)$ and $L^2_c(X_\Gamma)$ corresponds to the continuous spectrum. The latter subspace is spanned by wave packets defined by Eisenstein series. Let $L^2_{\mathrm{cusp}}(X_\Gamma)$ be the subspace of $L^2(X_\Gamma)$ spanned by the cusp forms. Then $L^2_{\mathrm{cusp}}(X_\Gamma) \subset L^2_d(X_\Gamma)$ and the orthogonal complement $L^2_{\mathrm{res}}(X_\Gamma)$ of $L^2_{\mathrm{cusp}}(X_\Gamma)$ in $L^2_d(X_\Gamma)$ is spanned by iterated residues of Eisenstein series.

Remark. The whole theory can be stated more conveniently using representation theory. Let R be the right regular representation of G in the Hilbert space $L^2(\Gamma\backslash G)$. Then one of the main problems of spectral theory is to study the spectral decomposition of the unitary representation $(R, L^2(\Gamma\backslash G))$. This is the relation with the theory of automorphic forms.

Some of the basic problems of spectral theory of automorphic forms are:

Basic problems.

(1) Existence and construction of cusp forms
(2) Location and structure of the spectrum
(3) Functoriality principle

1) Existence of cusp forms and Weyl's law

An important problem in the theory of automorphic forms is the existence and construction of cusp forms. The interesting lattices are arithmetic subgroups of G for which the quotient $X_\Gamma = \Gamma\backslash S$ is noncompact. In this case it is not at all clear that there exists any cusp form. In fact the results of Phillips and Sarnak [8] suggest that the existence of many cusp forms is a nongeneric property and may be restricted to arithmetic groups.

A convenient way of counting the number of cusp forms for a given lattice Γ is to use the eigenvalues of the Laplace operator Δ. Regarded as unbounded operator in $L^2(X_\Gamma)$ with domain $C_c^\infty(X_\Gamma)$, Δ is essentially selfadjoint and its restriction to $L^2_{\text{cusp}}(X_\Gamma)$ decomposes discretely. Let $0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues, counted with multiplicity, of Δ in $L^2_{\text{cusp}}(X_\Gamma)$. Let $N^\Gamma_{\text{cusp}}(\lambda) = \#\{j \mid \lambda_j \le \lambda\}$ be the counting function of the cuspidal spectrum. In [1] Donnelly proved that

$$\limsup_{\lambda\to\infty} \frac{N^\Gamma_{\text{cusp}}(\lambda)}{\lambda^{d/2}} \le \frac{\operatorname{vol}(X_\Gamma)}{(4\pi)^{d/2}\Gamma(d/2+1)}, \tag{1}$$

where $d = \dim S$. A lattice Γ for which equality holds in (1) is called essentially cuspidal by Sarnak [10]. For such a Γ cusp forms exist in abundance. Sarnak [10] has conjectured that (G,Γ) is essentially cuspidal iff Γ is arithmetic. In particular, congruence subgroups are expected to be essentially cuspidal. The conjecture has been verified in the following cases: For congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ by Selberg [12], for congruence subgroups of a real rank one group by Reznikov [9], for Hilbert modular groups by Efrat [2], for $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ by S. Miller [6], for congruence subgroups of $\mathrm{SL}(n,\mathbb{Z})$, $n \ge 2$, by the author [7] and for congruence subgroups of $G = \mathcal{G}(\mathbb{R})$, where $\mathcal{G}$ is a split adjoint semisimple algebraic group, by Lindenstrauss and Venkatesh [5]. All these results establish Weyl's law for the cuspidal spectrum in its weak form, that is, without estimation of the remainder term. For principal congruence subgroups Γ of $\mathrm{SL}(2,\mathbb{Z})$ a much more precise result is known [13], namely

$$N^\Gamma_{\text{cusp}}(\lambda) = \frac{\operatorname{area}(\Gamma\backslash H)}{(4\pi)}\lambda + a\sqrt{\lambda}\log\lambda + b\sqrt{\lambda} + O\left(\frac{\sqrt{\lambda}}{\log\lambda}\right)$$

as $\lambda \to \infty$ with certain constants a and b which are determined by the surface. We announce the following result:

Theorem. *Let $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, $n \ge 2$, $d = \dim X$ and let Γ be a principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$. Then*

$$N^\Gamma_{\text{cusp}}(\lambda) = \frac{\operatorname{vol}(\Gamma\backslash X)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^{d/2} + R(\lambda)$$

and $R(\lambda)$ *satisfies*

$$R(\lambda) = \begin{cases} O(\sqrt{\lambda}\log\lambda), & n = 2, \\ O(\lambda^{(d-1)/2}), & n > 2. \end{cases} \tag{2}$$

as $\lambda \to \infty$.

2) Tempered spectrum and the Ramanujan–Selberg conjecture

Let $\Gamma(N) \subset \mathrm{SL}(2,\mathbb{Z})$ be the principal congruence subgroup of level N. One of the basic conjectures concerning the spectrum of the Laplacian on $X(N) = \Gamma(N)\backslash H$ is the *Selberg* conjecture which states that the first positive eigenvalue $\lambda_1(X(N))$ of the Laplacian on $X(N)$ satisfies

$$\lambda_1(X(N)) \geq 1/4 \tag{3}$$

for all N. An affirmative answer to this conjecture has important consequences in analytic number theory. The best known approximation to (3) is $\lambda_1(X(N)) \geq 975/4096 = 0.238...$ which is due to Kim and Sarnak [3, Appendix 2].

The conjecture can be reformulated representation theoretically. It holds if and only if every irreducible unitary representation of $\mathrm{SL}(2,\mathbb{R})$ which occurs in the space of cusps forms $L^2_{\mathrm{cusp}}(\Gamma(N)\backslash\mathrm{SL}(2,\mathbb{R}))$ is tempered (tempered means that every matrix coefficient of π is in $L^{2+\epsilon}$ for any $\epsilon > 0$). This is the archimedean *Ramanujan* conjecture. There is a corresponding conjecture for each finite place.

There are generalizations of the Ramanujan conjectures for other groups [11]. Assume that $\mathcal{G}$ is a semisimple linear algebraic group defined over $\mathbb{Q}$. Let $G = \mathcal{G}(\mathbb{R})$ and let Γ be a congruence subgroup of $\mathcal{G}(\mathbb{Q})$. Let $\widehat{G}_{\mathrm{unit}}$ denote the unitary dual of G equipped with the Fell topology and let $\widehat{G}_{\mathrm{aut}}$ be the closure of the set of all $\pi \in \widehat{G}_{\mathrm{unit}}$ which occur in the spectral decomposition of the regular representation of G in $L^2(\Gamma\backslash G)$ as Γ varies over all congruence subgroups. Furthermore let $\widehat{G}_{\mathrm{temp}}$ be the set of all $\pi \in \widehat{G}_{\mathrm{unit}}$ which occur in $L^2(G)$. Then one has

$$\{1\} \cup \widehat{G}_{\mathrm{temp}} \subset \widehat{G}_{\mathrm{aut}} \subset \widehat{G}_{\mathrm{unit}}. \tag{4}$$

The general Ramanujan problem is to determine the set $\widehat{G}_{\mathrm{aut}}$.

Now let $G = \mathrm{SL}(n)$. Then the Ramanujan conjecture for $\mathrm{SL}(n)$ is that the left inclusion in (4) is an equality. Actually, it suffices to show that every $\pi \in \widehat{G}_{\mathrm{unit}}$ which occurs in the space cusp forms for some congruence subgroup is tempered. This is definitely false for general G.

We consider the spherical part of the spectrum. Let $\widehat{G}^1_{\mathrm{temp}}$ be the set of all $\pi \in \widehat{G}_{\mathrm{temp}}$ whose restriction to $\mathrm{SO}(n)$ contains the trivial representation. Given $\pi \in \widehat{G}^1_{\mathrm{temp}}$, let λ_π be the Casimir eigenvalue of π and let $m_\Gamma(\pi)$ be the multiplicity with which π occurs in the space of cusp forms $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$. Finally, for $\lambda \geq 0$ let $\widehat{G}^1_{\mathrm{temp}}(\lambda)$ be the set of all $\pi \in \widehat{G}^1_{\mathrm{temp}}$ such that $|\lambda_\pi| \leq \lambda$. Then we have

Theorem 2. *Let* Γ *be a congruence subgroup of* $\mathrm{SL}(n, \mathbb{Z})$. Then

$$\lim_{\lambda \to \infty} \frac{\sum_{\pi \in \widehat{G}^1_{\mathrm{temp}}(\lambda)} m_\Gamma(\pi)}{N^\Gamma_{\mathrm{cusp}}(\lambda)} = 1.$$

In other words, the tempered spectrum has full density. For $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ this was proved by S. Miller [6].

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Blowing up Kähler manifolds of constant scalar curvature

Frank Pacard

(joint work with C. Arezzo)

We report some recent construction concerning Kähler metrics of constant scalar curvature on blow ups at points of compact manifolds which already carry Kähler constant scalar curvature metrics.

1. Introduction

Assume that (M,ω_0) is a constant scalar curvature compact Kähler manifold of complex dimension $m \geq 2$. Given m distinct points $p_1,\ldots,p_k \in M$, we define $\mathrm{Bl}_{p_1,\ldots,p_k}M$ to be the blow up of M at the points $p_1,\ldots,p_k$. The question we address is whether $\mathrm{Bl}_{p_1,\ldots,p_k}M$ can be endowed with a constant scalar curvature Kähler form. The uniqueness of these metrics has been recently proved by Chen-Tian [4] (and previously by Donaldson and Chen in some special cases).

It is well known that, if we perturb the Kähler form ω_0 into

$$\omega = \omega_0 + i\,\partial\,\bar\partial\varphi$$

then the scalar curvature of ω can be expanded into powers of φ as

$$\mathbf{s}\,(\omega) = \mathbf{s}\,(\omega_0) - \frac{1}{2}\,\Delta_0^2\,\varphi - \mathrm{Ric}_0\cdot\nabla^2\,\varphi + Q_0(\varphi),$$

where Q_0 collects all the nonlinear terms and where all operators in the right hand side of this identity are computed with respect to the metric g_0 induced by the Kähler form ω_0. We define the elliptic fourth order operator

$$\mathbb{L} := \frac{1}{2}\,\Delta_0^2 + \mathrm{Ric}_0\cdot\nabla^2,$$

It turns out that

$$\mathbb{L} = 2\,(\bar\partial\partial^{\#})^{\star}\,\bar\partial\partial^{\#}, \tag{1}$$

where $\partial^{\#}\varphi$ denotes the $(1,0)$-part of the g_0-gradient of φ. Using this result, the key observation of LeBrun-Simanca [7] is that the elements $\mathrm{Ker}\,\mathbb{L}$ are in one to one correspondence with holomorphic vector fields, namely $\partial^{\#}\varphi$, which vanish somewhere on M.

2. Main results

In the case where the manifold has no nontrivial holomorphic vector field which vanish somewhere (this condition is for example fulfilled when the group of automorphisms of M is discrete) we obtain the:

Theorem 1. *Assume that (M,ω_0) is a constant scalar curvature compact Kähler manifold without nontrivial holomorphic vector field vanishing somewhere. Then, $\tilde M$, the blow up of M at finitely many points has a constant scalar curvature Kähler form $\tilde\omega$. In addition, if the scalar curvature of ω_0 is not zero then the scalar curvatures of $\tilde\omega$ and of ω_0 have the same signs.*

In the case of 0-scalar curvature metrics, we also have the :

Theorem 2. *Assume that (M,ω_0) is a 0-scalar curvature compact Kähler manifold without nontrivial vanishing holomorphic vector field. Then, the blow up of M at finitely many points has a 0-scalar curvature Kähler form provided the first Chern class of M is non zero.*

This last result compliments, in any dimension, previous constructions which have been obtained in complex dimension $n = 2$ and for 0-scalar curvature metrics, by Kim-LeBrun-Pontecorvo [5], LeBrun-Singer [8] and Rollin-Singer [9], [10].

In the case where M has nontrivial vanishing holomorphic vector fields $X_1, \ldots, X_d$, with $d \geq 1$ (this condition in particular implies that M has a nontrivial continuous automorphisms group), our main result states that the blow up of M at sufficiently many points can be endowed with a constant scalar curvature Kähler metric provided these points are carefully chosen. To be more precise, we assume that the kernel of $\mathbb{L}_M$ is not trivial and we denote by $\xi_0 \equiv 1, \xi_1, \ldots, \xi_d$ the linearly independent functions which span this kernel and, without loss of generality, we assume that the functions $\xi_1, \ldots, \xi_d$ have mean 0. Observe that d either denotes the dimension of the space of vanishing holomorphic vector fields on M or the dimension of the nontrivial kernel of $\mathbb{L}$. We define the matrix

$$\mathfrak{M}(p_1, \ldots, p_k) := \begin{pmatrix} \xi_1(p_1) & \ldots & \xi_1(p_k) \\ \vdots & & \vdots \\ \xi_d(p_1) & \ldots & \xi_d(p_k) \end{pmatrix} \tag{2}$$

Our main result reads :

Theorem 3. *Assume that (M, ω_0) is a constant scalar curvature compact Kähler manifold. Let us assume that $k \geq 1$ and $p_1, \ldots, p_k$ are chosen so that*

$$\text{Rank}\, \mathfrak{M} = d \qquad \text{and} \qquad \text{Ker}\, \mathfrak{M} \cap K^k_+ \neq \emptyset, \tag{3}$$

where K^k_+ is the cone of vectors with positive entries in $\mathbb{R}^k$. Then, $\tilde{M}$, the blow up of M at the points $p_1, \ldots, p_k$ has a constant scalar curvature Kähler form.

The first condition Rank $\mathfrak{M} = d$ is easily seen to be generic (and open) in the sense that, the set of the points $(p_1, \ldots, p_k) \in M^k$ for which the condition os fulfilled is open and dense in M^k provided $k \geq d$.

The second sufficient condition on the existence of points for which $\text{Ker}\, \mathfrak{M} \cap K^k_+ \neq \emptyset$ is more subtle. We are only able to prove that there exists $k_0 \geq d+1$ such that, for all $k \geq k_0$ the set of points $(p_1, \ldots, p_k) \in M^k$ for which both conditions in (3) are fulfilled is a nonempty and open subset of M^k. By opposition to the first condition, it is easy to convince oneself that this condition does not hold for generic choice of the points.

Collecting these results, we see that there exists $k_0 \geq d + 1$ and, for all $k \geq k_0$ there exist a nonempty open subset $U \subset M^k$ such that, for all $(p_1, \ldots, p_k) \in U$ there exist a constant scalar curvature Kähler form on the blow up of M at $p_1, \ldots, p_k$.

These results are obtained using a connected sum of the Kähler form ω_0 at each p_ℓ with a 0-scalar curvature Kähler form η which is defined on $N := \text{Bl}_0\, \mathbb{C}^n$, the blow up of $\mathbb{C}^n$ at the origin. When $n = 2$ this model (N, η) has been known for long time and it is usually referred to in the literature as Burns metric and in higher dimensions a similar model has been produced by Simanca [11]. In fact, we can be more precise and show that, in each result, there exists $\tilde{\omega}_\varepsilon$, for $\varepsilon \in (0, \varepsilon_0)$, a one parameter family of Kähler forms on $\mathrm{Bl}_{p_1,\ldots,p_k} M$ which converges in $\mathcal{C}^\infty$ norm as ε tends to 0, to the Kähler form ω_0 away from the points p_ℓ. Moreover, the sequence of Kähler forms $\varepsilon^{-2}\,\tilde{\omega}_\varepsilon$ converges in $\mathcal{C}^\infty$ norm to the Kähler form η on N the blow up at the origin of $\mathbb{C}^n$.

Example : We take $M = \mathbb{P}^n$, which we regard as the quotient of the unit sphere in $\mathbb{C}^{n+1}$ with complex coordinates $(z_1, \ldots, z_{n+1})$ by the standard S^1-action given by the restriction of scalar multiplication. We have :

Theorem 4. *When $M = \mathbb{P}^n$, then $d = n^2 + 2n$ and $k_0 \leq 2n\,(n+1)$.*

Applying Theorem 3, we get the existence of constant scalar curvature Kähler forms on the blow up of $\mathbb{P}^n$ at k points for $k \geq k_0$ which belong to some nonempty open set of M^k. One can make use of the symmetries of $\mathbb{P}^n$, and construct constant scalar curvature Kähler forms on the blow up of $\mathbb{P}^n$ at $p_1, \ldots, p_k$ with $k \leq d$. However this construction will not hold anymore for the choice of the points in some open set of $(\mathbb{P}^n)^k$. We get the :

Theorem 5. *The blow up of $\mathbb{P}^n$ at*

$$p_1 = [1, 0, \ldots, 0], \qquad \ldots, \qquad p_{n+1} = [0, \ldots, 0, 1]$$

carries a constant scalar curvature Kähler metric.

The above result is optimal in the number of points because, for $k \leq n$, $\mathrm{Bl}_{p_1,\ldots,p_k} \mathbb{P}^n$ does not satisfy the Matsushima-Lichnerowicz obstruction. Observe also that $\mathrm{Bl}_{p_1,\ldots,p_{n+1}} \mathbb{P}^n$ still has vanishing holomorphic vector fields.

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The spectral function and the remainder in local Weyl's law: View from below

Iosif Polterovich
(joint work with Dmitry Jakobson)

Let X be a compact Riemannian manifold of dimension $n \geq 2$. Consider the Laplacian Δ on X with the eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ and the corresponding orthonormal basis $\{\phi_i\}$ of eigenfunctions: $\Delta\phi_i = \lambda_i\phi_i$.Given $x, y \in X$, let

$$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x)\phi_i(y)$$

be the spectral function of the Laplacian. As $\lambda \to \infty$, it satisfies

$$N_{x,y}(\lambda) = O(\lambda^{n-1}), \qquad x \neq y.$$

On the diagonal $x = y$, the asymptotics of the spectral function (denoted simply by $N_x(\lambda)$) is given by the local Weyl's law:

$$N_x(\lambda) = \frac{\sigma_n}{(2\pi)^n}\lambda^n + R_x(\lambda), \qquad R_x(\lambda) = O(\lambda^{n-1}),$$

where σ_n is the volume of the unit ball in $\mathbb{R}^n$. The above–mentioned upper bounds on $N_{x,y}(\lambda)$ and $R_x(\lambda)$ are sharp. However, in many cases, particularly for non-positively curved manifolds, these bounds can be improved and a lot of research is devoted to this subject. For instance, the remainder $R_x(\lambda)$ on a flat square 2-torus coincides with the error term in the Gauss's circle problem. Estimating this term is an important question in number theory. In the talk we present some recent results obtained in [1, 2] concerning *lower* bounds on $N_{x,y}(\lambda)$ and $R_x(\lambda)$. We write $f_1(\lambda) = \Omega(f_2(\lambda))$ for an arbitrary function f_1 and a positive function f_2, if $\limsup_{\lambda\to\infty} |f_1(\lambda)|/f_2(\lambda) > 0$. We prove

Theorem 1. *(i) Let $x, y \in X$ be two points that are not conjugate along any shortest geodesic joining them. Then*

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).$$

(ii) Let $n > 3$ and let $x \in X$ be any point where the scalar curvature does not vanish. Then

$$R_x(\lambda) = \Omega(\lambda^{n-3}).$$

(iii) Assume X has no conjugate points. Then

$$R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}}).$$

In dimension two, part (*iii*) of this theorem can be viewed as a generalization of the classical Hardy-Landau lower bound for the remainder in the Gauss's circle problem. Assume now that X is negatively curved, i.e. for any pair of directions ξ, η the sectional curvature $K(\xi,\eta)$ satisfies

$$-K_1^2 \le K(\xi,\eta) \le -K_2^2.$$

In this case, Theorem 1 can be strengthened using methods of thermodynamic formalism.

Theorem 2. *On an n-dimensional compact negatively curved manifold*

$$(I) \qquad N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-H/2)}{h} - \delta}\right), \ \ \forall\, \delta > 0, \ x \neq y,$$

and

$$(II) \qquad R_x(\lambda) = \begin{cases} \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-H/2)}{h} - \delta}\right) \ \ \forall\, \delta > 0, & n \le 5; \\ \Omega(\lambda^{n-3}), \quad n \ge 6. \end{cases}$$

Here H is the Sinai-Ruelle-Bowen potential, P is the topological pressure and h is the topological entropy of the geodesic flow. One can show that

$$\frac{P\,(-H/2)}{h} \ge \frac{K_2}{2K_1} > 0.$$

In dimension two, estimate (II) improves an unpublished result of A. Karnaukh [3], which served as a starting point for our research. In [1, 2] we develop and extend the approach of [3]. It is instructive to compare (II) with a “folklore” conjecture that on a generic negatively curved surface the *global* remainder $R(\lambda) = \int_X R_x(\lambda)$ in the Weyl's law satisfies $R(\lambda) = O(\lambda^{\epsilon})$ for any $\epsilon > 0$. Such a conjecture looks plausible (at least on average) in view of some numerical results in quantum chaos. The gap between (II) and the predicted global bound is quite intriguing. It indicates that substantial cancellations occur when $R_x(\lambda)$ is integrated over the surface. It would be interesting to understand the nature of this effect.

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Index theory, C^*-algebras and applications; a survey

Thomas Schick
(joint work with Bernhard Hanke, Paolo Piazza)

Index theory has been one of the central and connecting themes in the interplay of analysis, geometry and topology.

We start with the classical Atiyah-Singer index theorem [1] and, via the Lichnerowicz formula, its application to the positive scalar curvature.

The main point of the talk is to show that the *natural* interpretation of indices of elliptic operators provides them as elements in K-theory groups of C^*-algebras. An instance of this is the family index theorem [2], a different direction is the Mishchenko-Fomenko index theorem [9], which deals with elliptic operators acting on C^*-algebraic bundles. This theorem has been used by Rosenberg [11] to obtain strong obstructions to positive scalar curvature: if a closed spin manifold admits a metric of positive scalar curvature, then the generalized index of the Dirac operator in the K-theory of the C^*-algebra of the fundamental group vanishes. This result still follows from the Lichnerowicz formula, because in the context of C^*-algebras one can apply positivity to conclude invertibility of the operator in question and therefore vanishing of its index.

The Gromov-Lawson-Rosenberg conjecture states that this index is the only obstruction to positive scalar curvature. This has been disproved by the speaker in [13]. The *stable Gromov-Lawson-Rosenberg conjecture*, introduced by Stephan Stolz, asserts that vanishing of the index implies at least that the product of the given manifold with certain high dimensional model manifolds admit a metric with positive scalar curvature.

The methods of index and K-theory have been abstracted in particular by Kasparov [7, 6]; the idea is to define K and KK-groups directly as equivalence classes of elliptic operators. This is therefore the most natural domain for the definition of an index. In [12] a proof can be found that the theory of Mishchenko-Fomenko fits into the KK-framework.

The K-theory of a C^*-algebra is easy to define, but hard to calculate explicitly. In the case of the group C^*-algebras, the Baum-Connes conjecture states that the assembly map

$$A\colon K_*(BG) \to K^*(C^*_{red}G)$$

is for a torsion-free discrete group G an isomorphism. $K_*(BG)$ is the K-homology of the classifying space of G. This group is by Baum-Douglas described as a set of equivalence classes of elliptic operators. The map A is then given by the index of this operator. $C^*_{red}G$ is the (reduced) C^*-algebra of the group G

If G contains torsion, the left hand side has to be replaced by the equivariant K-homology of the universal space $E(G, Fin)$ for proper G-actions. This can be defined in terms of equivariant homotopy theory, an approach by Davis and Lück in [3]. Alternatively, one can use KK-theory for its definition. It is the goal of the speaker, together with Paul Baum and Nigel Higson, to extend the geometric description of Baum and Douglas to the equivariant setting and even further to from discrete to totally disconnected groups).

The Baum-Connes is known to be true for large classes of important groups, e.g. for Gromov hyperbolic groups, or for amenable groups. Most of these results are obtained using methods of KK-theory.

We list only the following applications of the Baum-Connes conjecture.

(1) If G is torsion free and A is surjective for G, then the spectrum of every self-adjoint element of $C^*_{red}G$ is connected. It follows that the group algebra $\mathbb{C}G$ contains no projections except for 0 and 1.
(2) If A is rationally injective for G, then the Novikov conjecture about homotopy invariance of higher signatures is true for manifolds with fundamental group G, a result due to Mishchenko .
(3) If A is rationally injective, then the stable Gromov-Lawson-Rosenberg conjecture for closed spin manifolds with fundamental group G holds. This is proved by Stolz, compare [14], and uses the homotopy theoretic description of the Baum-Connes map.

We want to mention one further application of the Baum-Connes conjecture developed recently by Paolo Piazza and the speaker. This also uses extensions of (C^*-algebraic) index theory to singular spaces, in this cases to Atiyah-Patodi-Singer type index theorems for manifolds with boundary. The result is the following:

Theorem 1. *If G is torsion-free and the Baum-Connes conjecture is true for G, and if M is a closed spin manifold with positive scalar curvature, then the higher rho-invariants of the Dirac operator of M vanish identically.*

This result is of particular importance because these invariants are, if the fundamental group contains torsion, essentially the only ones which can distinguish different concordance classes of metrics with positive scalar curvature. Accordingly, no invariants are known to distinguish such invariants for manifolds with torsion-free fundamental group.

We quickly want to indicate the basic idea of the proof of the above theorem. Since M has positive scalar curvature, the K-theoretic index invariant of the Dirac operator vanishes. This implies, by injectivity of the Baum-Connes map, that the corresponding K-homology element vanishes. Now one can use classical algebraic topology to conclude that the manifold we started with is bordant to a special manifold with positive scalar curvature, for which it is easy to show that the rho invariant in question vanishes. Now one can apply suitable generalizations of the Atiyah-Patodi-Singer index theorem to this bordism. It relates the index of the Dirac operator on the bordism to the two rho-invariants of the boundary pieces. Since both boundary pieces have positive scalar curvature, the Dirac operator defines an honest Fredholm index problem and therefore has K-theoretic index. Now surjectivity of the assembly map implies that this index is equal to the index of a closed manifold. Because we study rho invariants (i.e. relative indices) this means that the index contribution in the index formula is zero. Putting everything together we obtain that the rho invariant we want to compute is equal to the rho invariant of the second boundary piece of the bordism, i.e. is equal to zero.

A prominent way to obtain numerical invariants is given by the use of traces defined on the algebra to be considered (or on a holomorphically closed subalgebra). The general theory for the use of such traces and for the development of corresponding index theorems is explained in [12]. It is applied to the question of positive scalar curvature in [5]. The main result of [5] says that an enlargeable spin manifold (in the sense of Gromov and Lawson [4] admits a natural construction of a C^*-algebra with a trace such that the index of the Dirac operator twisted with this trace is easy to calculate. This can then be used to prove that in this situation the Rosenberg index mentioned above is non-zero. This puts the results of [4] into the perspective of the general index theoretic philosophy of this talk: on the one hand, it shows that the information used in [4] to obtain obstructions to positive scalar curvature is contained in the K-theoretic information described above. On the other hand, we get —using the ideas of Gromov and Lawson— information about non-trivial elements in K-theory groups. We plan to further expand the philosophy that a given geometric situation will almost automatically determine the right C^*-algebra to use to obtain calculable index information, which can then be used to obtain information about the more mysterious K-theory of group C^*-algebras. We hope in particular to refine Mathai's [8] approach to the Novikov conjecture for low degree cohomology classes to a statement about non-vanishing of relevant elements in the K-theory of the (maximal) group C^*-algebra.

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L^2-harmonic forms on locally symmetric spaces

Leslie Saper

Let $D = G/K$ be a symmetric space of non-compact type, where G is the group of real points of a semisimple algebraic group defined over $\mathbb{Q}$ and K is a maximal compact subgroup. Let $\Gamma \subset G$ be an arithmetic subgroup and let $G \to \mathrm{GL}(E)$ be a finite dimensional complex representation. We consider the locally symmetric space $X = \Gamma\backslash D$ and the flat vector bundle $\mathbb{E} = D \times_\Gamma E$ on X. We give X the Riemannian metric induced from a G-invariant metric on D and we let $\mathbb{E}$ have the fiber metric induced from an admissible inner product on E. We wish to give a topological interpretation of the space $\mathcal{H}_{(2)}(X;\mathbb{E})$ of L^2-harmonic forms on X with coefficients in $\mathbb{E}$. This space plays an important role in differential geometry, harmonic analysis and, via Langlands's program, number theory.

In work in progress we prove the following

Theorem. *Assume that G has no $\mathbb{Q}$-simple factors with $\mathbb{Q}$-root system of type D_n, E_n, or F_4. If $E^* \cong \overline{E}$ then*

$$\mathcal{H}_{(2)}(X;\mathbb{E}) \cong \mathrm{Range}\big(I_mH(\widehat{X};\mathbb{E}) \to I_nH(\widehat{X};\mathbb{E})\big) \ .$$

Here $\widehat{X}$ is the reductive Borel-Serre compactification of X which was introduced by Zucker [9, (4.1)]. The topological invariants $I_mH(\widehat{X};\mathbb{E})$ and $I_nH(\widehat{X};\mathbb{E})$ are the two middle-perversity intersection cohomology groups of $\widehat{X}$ introduced by Goresky and MacPherson [2], [3]. We expect that the restriction on the $\mathbb{Q}$-root system of G can be removed.

The theorem follows from known results in one important case: where X admits a Satake compactification X^* with all real boundary components equal-rank. (This case includes the situation where X is a Hermitian locally symmetric space and X^* is the Baily-Borel-Satake compactification.) When X is equal-rank, Borel and Casselman [1] have proved that the L^2-cohomology $H_{(2)}(X;\mathbb{E})$ is finite dimensional; in particular, Range d is closed, where d denotes the unbounded operator on L^2-forms given by exterior differentiation, and $\mathcal{H}_{(2)}(X;\mathbb{E}) \cong H_{(2)}(X;\mathbb{E})$. On the other hand, the Zucker/Borel conjecture (proved in the Hermitian case independently by Looijenga [4] and Saper and Stern [8] and in general by Saper and Stern, see [7]) states that $H_{(2)}(X;\mathbb{E}) \cong IH(X^*;\mathbb{E})$. Finally the Rapoport/Goresky-MacPherson conjecture (proved by Saper in [5]) shows that $IH(X^*;\mathbb{E}) \cong I_mH(\widehat{X};\mathbb{E}) \cong I_nH(\widehat{X};\mathbb{E})$. These three results imply our theorem in this case. Note that the isomorphisms of the Zucker/Borel conjecture and the Rapoport/Goresky-MacPherson conjecture actually hold locally over X^*.

In the general case however Range d may not be closed in which case $H_{(2)}(X;\mathbb{E})$ is infinite-dimensional and is not represented by L^2-harmonic forms. Furthermore it does not seem possible to use a local argument on a Satake compactification X^* in general, but instead a more global argument on $\widehat{X}$ is needed.

In the talk we described our approach to prove the theorem in the general case based on our theory of $\mathfrak{L}$-modules; an exposition of this theory may be found in [6].

Details will appear elsewhere.

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Extremal Kähler metrics

Michael A. Singer

1. Introduction

The notion of 'extremal Kähler metric' is due to Calabi: the setting is as follows. M^m is a compact complex manifold and $\Omega \in H^2(M, \mathbb{R})$ is a given *positive* class, so that Ω can be represented by a positive form. Locally this means

$$\omega = \frac{i}{2} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta \tag{1.1}$$

where $(g_{\alpha\overline{\beta}}(z))$ is a positive-definite hermitian matrix at each point z. With (M, Ω) fixed, one then seeks critical points of the functional

$$\Omega \ni \omega \mapsto \int_M s(\omega)^2 \frac{\omega^m}{m!}; \tag{1.2}$$

here $s(\omega)$ is the scalar curvature of ω. The Euler-Lagrange equation states that $\nabla^{1,0}s$ should be a holomorphic vector field. Of course, if ω is Yamabe–Kähler (YK), so that $s(\omega)$ is constant, or *a fortiori* if it is Einstein–Kähler (EK), then ω is extremal.

The basic questions in the subject are those of existence and uniqueness: given (M, Ω) does there exist an extremal Kähler representative in Ω; and if so, is it unique?

If we fix $\omega_0 \in \Omega$, then we can write any other representative in Ω in the form $\omega = \omega_0 + i\partial\overline{\partial}\varphi$, where φ is a smooth real function. The Ricci forms are related by

$$\rho(\omega) = \rho(\omega_0) - i\partial\overline{\partial}\log(\omega^m/\omega_0^m) = \rho(\omega_0) - i\partial\overline{\partial}\log(1 + i\partial\overline{\partial}\varphi), \tag{1.3}$$

a fully nonlinear fourth-order expression in φ. Thus the condition that ω be YK is a fully nonlinear fourth-order elliptic equation for φ. There is a good deformation theory for YK and general extremal Kähler metrics due to LeBrun and Simanca [15, 14]. However, subtle obstructions to the existence of such metrics show that one cannot hope for a general existence theorem.

2. Obstructions, Examples

2.1. **Obstructions.** Let $\mathfrak{A}(M)$ denote the group of holomorphic automorphisms of M, $\mathfrak{A}_0(M)$ the connected component of the identity in $\mathfrak{A}(M)$. Lichnerowicz showed that $\mathfrak{A}_0(M)$ is reductive if M admits YK metrics in any Kähler class [2] and Calabi [5] gave an analogous (but technically more complicated) result for extremal Kähler metrics. Futaki introduced a remarkable invariant $F_\Omega(X) \in \mathbb{R}$, where X is a holomorphic vector field and proved that it must vanish if Ω contains a YK representative [2].

It is now known that there are more subtle obstructions, present even if $\mathfrak{A}(M) = 1$.

2.2. **The Einstein–Kähler case.** If $c_1(M)$ is positive, negative, or zero, then one can take $\Omega = \pm c_1(M)$ and any extremal Kähler metric in this class is automatically EK. The EK equation is a complex Monge–Ampère equation and the problem was completely solved if $c_1(M) \leq 0$ by Yau [20] in 1978. The case $c_1(M) > 0$ is much harder and, like the general YK or extremal case, is not fully understood; see, however [8, 16, 17, 19, 18]. The EK case is technically simpler because the PDE can be reduced to an equation of Monge–Ampère type; in particular this is a second-order nonlinear equation for φ.

2.3. **Variations on a theme of Calabi.** On $\mathbb{C}^m$, equipped with a fixed hermitian inner product, we can consider

$$\omega = \frac{i}{2}\partial\overline{\partial}f(t), \; t = \log|z| \tag{2.1}$$

so t is the logarithm of the euclidean distance-function. If $f(t) = e^{2t}$ we get the standard flat Kähler structure, while if $f(t) = t$ the resulting form is degenerate, but descends to give the Fubini–Study metric on $P(\mathbb{C}^m)$. Calabi also found an extremal Kähler metric with non-constant scalar curvature on the blow-up of a point in $P(\mathbb{C}^{m+1})$ by this method. This construction has now been substantially generalized, cf. [13, 1].

2.4. **Resolution of singularities and blowing up.** Another source of examples comes from recent work of Arezzo and Pacard (see Pacard's contribution in this chapter). Such results remain to be fully exploited in complex dimensions ≥ 3. This is because we lack examples of ALE scalar-flat Kähler metrics on (Kähler) resolutions of singularities $N \to \mathbb{C}^m/G$ in general, if $m \geq 3$. The existence of such metrics seems to me to be an important open problem in this subject. (For cyclic singularities in dimension 2, there are such metrics [6].)

3. Symplectic viewpoint

An alternative framework for studying the YK problem has been proposed by Donaldson [9, 10]. Instead of fixing the complex structure and Kähler class, one fixes the symplectic structure ω and allows the compatible complex structure J to vary. If X is the space of all such complex structures, then the group K of exact symplectomorphisms acts on X. Moreover, X is formally an infinite-dimensional Kähler manifold and the action of K is Hamiltonian, with moment map

$$\langle \mu(J), f \rangle = \int_M [s(J) - c] f \frac{\omega^m}{m!} \tag{3.1}$$

where c is any constant. Here the lie algebra $\mathfrak{k}$ of K has been identified with the Poisson algebra of smooth functions on M so that this formula identifies μ as a map into $\mathfrak{k}^*$. Also, we have written $s(J)$ for the scalar curvature of the metric g deteremined by ω and J.

From this point of view, *the problem of finding a YK metric is now equivalent to finding a zero of the moment map* μ.

This is important because there is a standard package of ideas and methods that are available for studying zeros of moment maps, at least in finite dimensions. In particular the reductivity of $\mathfrak{A}_0(M)$ if M admits a YK metric follows without computation from general facts about moment maps. This point of view has led to proofs of uniqueness of YK metrics for projective varieties with $\mathfrak{A}(M)$ discrete [11]. A proof of uniqueness for general extremal Kähler metrics has recently been announced by Chen and Tian [7].

The general picture now is that YK metrics should exist if and only if (M, Ω) is K-stable. This stability is essentially an algebro-geometric condition on (M, Ω), involving 1-parameter degenerations of M. The biggest challenge today is to relate this condition to the PDE for YK metrics. We note that important progress has been made by Donaldson in the case of *toric* varieties [12].

4. Further reading

The following bibliography is not comprehensive, but will enable the interested reader to orient him/herself in the subject. One could start with the relevant chapters of [2], and then move on to the survey articles [4] and [3].

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Quadratic curvature decay and asymptotic cones

Nader Yeganefar

Let M be a complete noncompact Riemmannian manifold. In this note, we are interested in the following problem: find geometric conditions which imply that M has finite topological type, i.e. is homeomorphic to the interior of a compact manifold with boundary. In dealing with such a problem, one usually expects that to get finite topological type, it is enough to impose geometric constraints only at infinity. For example, it is known that if M is flat or has nonnegative sectional curvature, then it has finite topological type; moreover this conclusion also holds even if the curvature conditions hold only outside some compact subset (see [2] and [3]). We would like to generalize these results by further relaxing the geometric assumptions. (For more details and references, the reader should consult [7].)

We consider here the class of manifolds with quadratic curvature decay. More precisely, M is said to have (lower) quadratic curvature decay if for some point m_0 and some constant $C > 0$, the sectional curvatures K at every point $m \in M$ satisfy

$$K \geq -C/d_M(m_0, m)^2, \tag{0.1}$$

where d_M denotes the distance function. It is easy to see that having quadratic curvature decay does not depend on the choice of m_0 and is a scale invariant property. A manifold with quadratic curvature decay is in some sense nonnegatively curved at infinity, and one could expect it to have finite topological type. However, by a result of Gromov, on any noncompact manifold there is a complete metric of quadratic curvature decay (see [5]) so that condition (0.1) does not impose anything on the topology. In contrast to this result, let us mention that if (0.1) holds with the exponent 2 replaced by $2 + \varepsilon$ for some $\epsilon > 0$, then Abresch proved that M has finite topological type [1].

In view of the above mentioned result of Gromov, we need to find additional assumptions that restrict the topology of a manifold M carrying a metric of quadratic curvature decay. Our main result here provides a link between the topology of M and its asymptotic cones. Before stating this result, let us first say a few words about asymptotic cones. If (M, m_0, d_M) is a pointed metric space and if $\{R_i\}$ is a sequence of positive numbers going to infinity, we can consider the sequence of rescaled pointed metric spaces $\{(M, m_0, d_M/R_i)\}$. If this sequence is precompact in the pointed Gromov-Hausdorff topology, then any of its limit points is called an asymptotic cone. Intuitively, an asymptotic cone has to reflect the large scale metric behaviour of M. Moreover, an asymptotic cone may not be unique (i.e may depend on the converging subsequence of the original sequence) and may even not be a metric cone. Now, even if $\{(M, m_0, d_M/R_i)\}$ is not precompact, there is a contruction using "ultrafilters" and "ultralimits" which allows us to get from this sequence a pointed metric space denoted by $C_{\omega,\{R_i\}}(M, m_0)$, where ω is a non-principal ultrafilter, see [4] for more details. We call this space also asymptotic cone, and it indeed generalizes the first definition given above. Recall finally that a metric space M is said to have a pole at some point m_0 if for any point m there exists a ray (i.e. a minimizing geodesic on $[0, \infty)$) starting at m_0 and passing through m.

Our result is the following: *Let M be complete noncompact manifold with lower quadratic curvature decay. If M has infinite topological type, then there is a sequence of positive numbers $\{R_i\}$ diverging to infinity such that $C_{\omega,\{R_i\}}(M, m_0)$ does not have a pole at its basepoint.*

A similar theorem is obtained by Petrunin and Tuschmann [6] in a more restrictive situation.

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Rigidity of cone-3-manifolds

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(joint work with Joan Porti in part)

A cone-manifold of curvature $\kappa \in \{-1, 0, 1\}$ is a metric space X, which is homeomorphic to a manifold and whose local geometry is modelled on the κ-cone over a cone-manifold of curvature $+1$ and one dimension lower. The set of points whose link (i.e. the cross-section of the model cone) is isometric to the standard round sphere is called the smooth part of the cone-manifold, its complement is called the singular locus and is usually denoted by Σ. In the following we will be concerned with 2- and 3-dimensional cone-manifolds only.

In two dimensions the singular locus consists of isolated points. The link of each cone point is isometric to the circle of a certain length, which we will refer to as the cone angle associated to that point. In three dimensions the singular locus is an embedded geodesic graph. The cone-angle associated to an edge will be the cone-angle of a transverse disk.

If cone-angles are $\leq 2\pi$, these spaces satisfy a lower curvature bound in the triangle comparison sense. If cone-angles are $\leq \pi$, the geometry is even more restricted, for example the Dirichlet-polyhedron will be convex and the valency of a vertex of the singular locus (in the 3-dimensional case) will be at most 3.

The concept of cone-3-manifold is a natural generalization of the concept of geometric 3-orbifold, where the cone-angles are of the form $2\pi/n$, $n \in \mathbb{Z}, n \geq 2$. Cone-3-manifolds play a significant role in the proof of the Orbifold Theorem, which has recently been accomplished by M. Boileau, B. Leeb and J. Porti, cf. [1]. The Orbifold Theorem states that a similar geometric decomposition as conjectured for 3-manifolds holds true for 3-orbifolds with non-empty singular locus. It was announced by W. Thurston around 1982.

In this talk I discussed the following global rigidity results for hyperbolic and spherical cone-3-manifolds, cf. [8]:

Theorem 1. *Let X, X' be hyperbolic cone-3-manifolds with cone-angles $\leq \pi$. If there exists a homeomorphism of pairs $(X, \Sigma) \cong (X', \Sigma')$ such that the cone-angles around corresponding edges coincide, then X and X' are isometric.*

Theorem 1 generalizes the global rigidity result of S. Kojima, cf. [4], which states the same rigidity property under the additional assumption that the singular locus is a link, i.e. a union of embedded circles. In the statement of Theorem 2, following [5], we call a cone-3-manifold X *Seifert fibered*, if X is Seifert fibered in the usual sense and if in addition Σ is a union of fibers.

Theorem 2. *Let X, X' be spherical cone-3-manifolds with cone-angles $\leq \pi$, which are not Seifert fibered. If there exists a homeomorphism of pairs $(X, \Sigma) \cong (X', \Sigma')$ such that the cone-angles around corresponding edges coincide, then X and X' are isometric.*

Furthermore I discussed the following result concerning Euclidean cone-manifold structures, which is joint work with J. Porti, cf. [6]:

Theorem 3. *Let X be a Euclidean cone-3-manifold with cone-angles $\leq \pi$ which is not almost product. Then for all multiangles $(\alpha_{,1}, \ldots, \alpha_N) \in (0, \pi)^N$ there exists a unique cone-manifold structure of curvature $\kappa \in \{-1, 01\}$ on X. Moreover, the set of Euclidean multiangles $E \subset (0, \pi)^N$ is a properly embedded hypersurfaces which splits $(0, \pi)^N$ into two components S and H, corresponding to multiangles of spherical and hyperbolic cone-manifold structures respectively.*

In the above statement we call a Euclidean cone-3-manifold (not) *almost product* if it is (not) the quotient of $E^2 \times S^1$ by a finite group of isometries respecting the product structure, where E^2 is a 2-dimensional Euclidean cone-manifold.

I indicated the proof of Theorem 1: We follow the same strategy as Kojima in [4], namely we construct a continuous path of hyperbolic cone-manifold structures on X with singular locus Σ and decreasing cone angles, which starts at the given structure and terminates at a complete hyperbolic structure of finite volume, possibly with totally geodesic boundary consisting of thrice-punctured spheres if vertices are present. This reflects the fact that the geometry of links of vertices has to change from spherical through horospherical to hyperspherical as we decrease cone angles. Here the link of a vertex is called horospherical if it is the horospherical cross-section of a singular cusp (i.e. a Euclidean turnover) and hyperspherical if it is a totally geodesic boundary component (i.e. a hyperbolic turnover).

To be able to carry out this strategy, we need to know that we can always deform cone-angles in an essentially unique way, furthermore we need to rule out degenerations as we decrease cone-angles. The first issue is handled by a local rigidity theorem for hyperbolic cone-manifold structures with singular cusps and totally geodesic boundary as above. This generalizes the local rigidity theorem for compact hyperbolic cone-3-manifolds in [7] and replaces the use of the local rigidity theorem of C.D. Hodgson and S.P. Kerckhoff, cf. [3], in Kojima's proof. The proof uses analytic techniques developed by J. Brüning and R. Seeley in [2]. As in [7], the assumption that cone angles are $\leq \pi$ is essential.

To rule out degenerations, we use the geometric results of [1], which enter the proof of the Orbifold Theorem. The description of thin parts of hyperbolic cone-3-manifolds with lower diameter bound and cone-angles $\leq \alpha < \pi$, which is achieved in [1], allows us to conclude that at most very mild degenerations can occur, namely tubes around short closed (smooth or singular) geodesics opening into rank-2 cusps. These in turn can be ruled out by a hyperbolic Dehn-surgery theorem in the setting of cone-manifolds.

If we are now given hyperbolic cone-manifolds X and X' as in Theorem 1, we can consider the corresponding paths of cone-manifold structures terminating at the complete structures. By Mostow-Prasad rigidity, these will be isometric. Using local rigidity along the paths, we conclude that X and X' are isometric.

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