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Heterotic Strings, Derived Categories, and Stacks

Organised by
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November 13th – November 19th, 2005

Abstract. This workshop brought together both mathematicians and physicists interested in mathematical aspects of heterotic strings and physical aspects of derived categories and stacks. These three topics in mathematics and physics are all involved in modern approaches to and extensions of mirror symmetry, and much of the technical machinery in understanding their physics and mathematics overlap, so by bringing together experts in these areas we hope to help spur further developments.

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Introduction by the Organisers

The miniworkshop *Heterotic strings, derived categories, and stacks*, organised by Björn Andreas (Berlin), Emanuel Scheidegger (Vienna) and Eric Sharpe (Utah) was held November 13th–November 19th, 2005. This meeting was well attended with 14 participants with broad geographic representation. This workshop was a nice blend of researchers with various backgrounds in both mathematics and physics.

The three topics represent areas of mathematics and physics with significant technical overlap. Heterotic strings are types of string theories whose compactifications involve complex Kähler manifolds with holomorphic vector bundles, and most of the complications revolve around those vector bundles. Derived categories (of coherent sheaves) have an obvious mathematical link with holomorphic vector bundles, and appear physically in studies of D-brane/antibrane systems. Details of the physical model in which derived categories enter physics are also closely related to the details of the physical model in which stacks enter physics: in each case, only a distinguished subclass of presentations can be realized physically, and the nonuniqueness of presentations in that subclass is conjectured to be washed out by a physical process called renormalization group flow.

These topics also form elements of generalizations of a conjectured generalization of "mirror symmetry." Mirror symmetry is a symmetry exchanging pairs of complex Kähler manifolds with trivial canonical bundle. It has been of interest to algebraic geometers because it provides a new approach to enumerative geometry: (usually difficult) curve-counting questions were mapped to comparatively trivial questions about the mirror manifold. Mirror symmetry was originally developed for spaces, but recently has been extended to stacks. One of the conjectured generalizations of mirror symmetry, known as "(0,2) mirror symmetry," exchanges pairs consisting of complex Kähler manifolds with holomorphic vector bundles, and is an analogue of ordinary mirror symmetry for heterotic strings. Another generalization, known as "homological mirror symmetry," exchanges derived categories of coherent sheaves on one of the mirrors with a derived Fukaya category of the other. As the topics of this miniworkshop show up in these new areas of mirror symmetry, this miniworkshop could have instead been titled "New developments in mirror symmetry."

Since understanding these topics involves an interplay between mathematics and physics, for this miniworkshop we brought together a collection of both mathematicians and physicists.

B. Andreas, V. Braun, and E. Scheidegger spoke specifically on mathematical aspects of heterotic strings, and E. Sharpe gave an overview of a few current problems in heterotic strings. A. Tomasiello spoke on mirror symmetry in flux backgrounds, using ideas recently developed by Hitchin to extend mirror symmetry for type II strings. (The same ideas can also, it is thought, be used to solve certain technical problems in understanding heterotic strings in flux backgrounds, as discussed in E. Sharpe's talk.) D. Ploog spoke on general aspects of derived categories and Fourier-Mukai transforms, then U. Bruzzo and D. Hernandez Ruiperez gave a collection of talks on Fourier-Mukai transforms, relevant to both derived categories (encoding automorphisms thereof) and heterotic strings (encoding T-dualities). E. Macri spoke on pi-stability, a physical aspect of derived categories. K.-G. Schlesinger and C. Lazaroiu spoke on A_∞ and L_∞ algebras, as relevant to open and closed string field theory, and which play a role in the physical understanding of derived categories. Finally, E. Sharpe and P. Horja gave a collection of talks on physical aspects of stacks.

Workshop: Heterotic Strings, Derived Categories, and Stacks

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Fourier-Mukai transforms and finite groups

David Ploog

The first part of this talk will recapitulate the use of derived categories and Fourier-Mukai transforms in algebraic geometry. We put special emphasis on the notion of *D-equivalence*: two smooth projective varieties X and Y are D-equivalent if there is a triangulated equivalence $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ of their bounded derived categories. Note that by a result of Verdier (see [5]) the corresponding notion for the abelian category of coherent sheaves is not interesting: we have $\mathrm{Coh}(X) \cong \mathrm{Coh}(Y)$ as abelian categories if and only if $X \cong Y$.

In order to investigate D-equivalence, the derived analogue of classical correspondences is used: to an object $K \in \mathrm{D}^b(X \times Y)$ we associate the functor

$$\mathsf{FM}_K : \mathrm{D}^b(X) \to \mathrm{D}^b(Y), \quad F \mapsto Rp_{Y*}(P \otimes^L p_X^* F).$$

FM_K is called the *Fourier-Mukai transform* with *kernel* K. A theorem of Orlov [10] states that every fully faithful functor (in particular, every equivalence) $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is of Fourier-Mukai type, i.e. there is an object $K \in \mathrm{D}^b(X \times Y)$ (unique up to isomorphism) such that $F \cong \mathsf{FM}_K$. The following criterion is the work of many people (cf. [5]) : suppose $S \subset \mathrm{D}^b(X)$ is a *spanning class*, i.e. $S^\perp := \{F \in \mathrm{D}^b(X) \,:\, \mathrm{Hom}^i(s, F) = 0 \forall s \in S, i \in \mathbb{Z}\} = 0$ and ${}^\perp S = 0$, then

$$\begin{aligned} F \text{ is fully faithful } &\iff F \text{ is fully faithful on } S, \text{ i.e.} \\ &\qquad \mathrm{Hom}^i_{\mathrm{D}^b(X)}(s, s') \xrightarrow{\sim} \mathrm{Hom}^i_{\mathrm{D}^b(Y)}(F(s), F(s')) \ \forall s, s' \in S \\ F \text{ is an equivalence } &\iff \text{additionally } \dim(X) = \dim(Y) \text{ and} \\ &\qquad F(s \otimes \omega_X) \cong F(s) \otimes \omega_Y \ \forall s \in S \end{aligned}$$

Typical examples of spanning classes are $S = \{k(x) \ : \ x \in X\}$ (the set of all skyscraper sheaves) and $S = \{L^{\otimes n} \,:\, n \in \mathbb{Z}\}$ for an ample line bundle L on X.

As an application, this is enough to conclude that an abelian variety A and its dual $\hat{A}$ are always D-equivalent [9]: an appropriate kernel is given by the Poincaré bundle $\mathcal{P}$ in view of $\mathsf{FM}_{\mathcal{P}}(k(\alpha)) = \mathcal{P}|_{\{\alpha\} \times A} = \alpha$ and $\mathrm{Hom}_A(\alpha, \beta) = 0$ for $\alpha, \beta \in \hat{A}$ with $\alpha \ncong \beta$.

Some wellknown and useful facts about Fourier-Mukai transforms are:

- Due to Grothendieck-Riemann-Roch, FM is compatible with passage to cohomology [5]:

$$\begin{array}{ccc} \mathrm{D}^b(X) & \xrightarrow[x]{v} & \mathrm{H}^*(X, \mathbb{Q}) \\ {\scriptstyle \mathsf{FM}_K} \downarrow & & \downarrow {\scriptstyle \mathsf{FM}_K^{\mathrm{H}}} \\ \mathrm{D}^b(Y) & \xrightarrow[Y]{v} & \mathrm{H}^*(Y, \mathbb{Q}) \end{array}$$

where $v_X(\cdot) = \mathrm{ch}(\cdot)\sqrt{\mathrm{td}_X}$ is the Mukai vector and $\mathsf{FM}^{\mathrm{H}}_K$ is the correspondence given by the class $\mathrm{ch}(K)\sqrt{\mathrm{td}_{X\times Y}} \in \mathrm{H}^*(X \times Y, \mathbb{Q})$. For example, if A is a principally polarised abelian variety, then $\mathsf{FM}_{\mathcal{P}}$ is an autoequivalence of $\mathrm{D}^b(A)$ and $\mathsf{FM}^{\mathrm{H}}_{\mathcal{P}} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ since $\mathsf{FM}_{\mathcal{P}}(k(0)) = \mathcal{O}_A$ and $\mathsf{FM}_{\mathcal{P}}(\mathcal{O}_A) = k(0)[-1]$.

- Bondal and Orlov proved the following reconstruction theorem [1]: if X and Y are smooth projective D-equivalent varieties, and if ω_X is ample or anti-ample, then $X \cong Y$.
- Orlov has shown a derived analogoue of the Torelli theorem for K3 surfaces [10]: two K3 surfaces X and Y are D-equivalent if and only if their transcendental lattices coincide.
- Two abelian varieties A and B are D-equivalent if and only if $A\times\hat{A} \cong B\times\hat{B}$ are symplectically isomorphic (Polichshuk [13], Orlov [11]).

There seems to be a connection between D-equivalence and birational equivalence: Kawamata's conjecture [6] states in particular, that birational smooth projective varieties with trivial canonical class should be D-equivalent. Note that the case of an abelian variety and its dual shows that the reverse implication cannot hold. As evidenced by a simple blow-up, birational equivalence does in general not induce D-equivalence. Bridgeland has proved the conjecture for Calabi-Yau manifolds of dimension 3 [3]. Furthermore, the classification of birational isomorphisms of hyperkähler fourfolds implies the conjecture for this class of manifolds as well.

A famous use of derived categories, which stimulated the current activity of research quite a lot, is given by Kontsevich's mirror symmetry conjecture [7]. Since this is not of primary interest in heterotic string compactifications, we mention here another application: the derived McKay correspondence, as proved by Bridgeland, King and Reid [2]. This can be used to compare the automorphism group $\mathrm{Aut}\,(\mathrm{D}^G(X))$ of equivariant equivalences with the group $(\mathrm{Aut}\,(\mathrm{D}^b(X)))^G$ of autoequivalences commuting with all g^* [12]. Using this for the obvious permutation action of S^n on X^n together with $\mathrm{D}^{S_n}(X^n) \cong \mathrm{D}(S_n\text{-Hilb}(X^n))$ (BKR) and $S_n\text{-Hilb}(X^n) \cong \mathrm{Hilb}_n(X)$ (Haiman [4]), we show the following proposition [12]: D-equivalent K3 (or abelian) surfaces X_1 and X_2 have D-equivalent Hilbert schemes of points, i.e. $\mathrm{D}^b(\mathrm{Hilb}_n(X_1)) \cong \mathrm{D}^b(\mathrm{Hilb}_n(X_2))$. Furthermore, two birational Hilbert schemes H_1, H_2 of K3 surfaces X_1, X_2 must have isomorphic transcendental lattices T_{H_1}, T_{H_2}. Since $T_{H_i} = T_{X_i} \oplus \mathbb{Z}$, we deduce that $T_{X_1} \cong T_{X_2}$ and hence $\mathrm{D}^b(X_1) \cong \mathrm{D}^b(X_2)$ by Orlov's derived Torelli theorem. Using the above proposition, H_1 and H_2 must be D-equivalent, thus giving further evidence for Kawamata's conjecture. Note that there are examples by Markman [8] of non-birational Hilbert schemes with isomorphic transcendental lattices; these are still D-equivalent using the same arguments.

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Aspects of heterotic string compactifications

Björn Andreas

A compactification of the ten-dimensional heterotic string is given by a holomorphic, stable G-bundle V (with G some Lie group specified below) over a Calabi-Yau manifold X. The Calabi-Yau condition, the holomorphy and stability of V are a direct consequence of the required supersymmetry in the uncompactified space-time. We assume that the underlying ten-dimensional space M_{10} is decomposed as $M_{10} = M_4 \times X$ where M_4 (the uncompactified space-time) denotes the four-dimensional Minkowski space and X a six-dimensional compact space given by a Calabi-Yau threefold. To be more precise: supersymmetry requires that the connection A on V satisfies: $F_A^{2,0} = F_A^{0,2} = 0$, $F^{1,1} \wedge J^2 = 0$ (J denotes a Kähler form of X). It follows that the connection has to be a holomorphic connection on a holomorphic vector bundle and in addition to satisfy the Donaldson-Uhlenbeck-Yau equation that has a unique solution if and only if the vector bundle is polystable.

In addition to X and V we have to specify a B-field on X of field strength H. In order to get an anomaly free theory, the Lie group G is fixed to be either $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ or one of their subgroups (the commutant of G in E_8 corresponds to the unbroken gauge-group observed in four dimensions) and H has to satisfy the identity $dH = \mathrm{tr} R \wedge R - \mathrm{Tr} F \wedge F$ where R and F are the associated curvature forms of the spin connection on X and the gauge connection on V. Also tr refers to the trace of the composite endomorphism of the tangent bundle to X

and Tr denotes the trace in the adjoint representation of G. For any closed four-dimensional submanifold X_4 of the ten-dimensional space-time M_{10}, the four form $\mathrm{tr}R \wedge R - \mathrm{Tr}F \wedge F$ must have trivial cohomology. Thus a necessary topological condition V has to satisfy is $\mathrm{ch}_2(TX) = \mathrm{ch}_2(V)$.

A physical interpretation of the third Chern-class can be given as a result of the decomposition of the ten-dimensional space-time into a four-dimensional flat Minkowski space and X. The decomposition of the corresponding ten-dimensional Dirac operator with values in V shows that massless four-dimensional fermions are in one to one correspondence with zero modes of the Dirac operator D_V on X. The index of D_V can be effectively computed using the Hirzebruch-Riemann-Roch theorem and is given by

$$\mathrm{index}(D_V) = \int_X \mathrm{Td}(X)\mathrm{ch}(V) = \frac{1}{2}\int_X c_3(V)\,,$$

equivalently, we can write the index as $\mathrm{index}(D) = \sum_{i=0}^{3}(-1)^k \dim H^k(X,V)$. For stable vector bundles we have $H^0(X,V) = H^3(X,V) = 0$ and so the index computes the net-number of fermion generations N_{gen} in the respective model.

Now it has been observed that the inclusion of background five-branes changes the anomaly constraint . Various five-brane solutions of the heterotic string equations of motion have been discussed in: the gauge five-brane, the symmetric five-brane and the neutral five-brane. It has been shown that the gauge and symmetric five-brane solution involve finite size instantons of an unbroken non-Abelian gauge group. In contrast, the neutral five-branes can be interpreted as zero size instantons of the $SO(32)$ heterotic string. The magnetic five-brane contributes a source term to the Bianchi identity for the three-form H, $dH = \mathrm{tr}R \wedge R - \mathrm{Tr}F \wedge F + n_5 \sum_{\mathrm{five-branes}} \delta_5^{(4)}$ and integration over a four-cycle in X gives the anomaly constraint $\mathrm{ch}_2(TX) - \mathrm{ch}_2(V) + [W] = 0$. The new term $\delta_5^{(4)}$ is a current that integrates to one in the direction transverse to a single five-brane whose class is denoted by $[W]$. The class $[W]$ is the Poincaré dual of an integer sum of all these sources and thus $[W]$ should be a integral class, representing a class in $H_2(X,\mathbb{Z})$. $[W]$ can be further specified taking into account that supersymmetry requires that five-branes are wrapped on holomorphic curves thus $[W]$ must correspond to the homology class of holomorphic curves. This fact constrains $[W]$ to be an algebraic class. Further, algebraic classes include negative classes, however, these lead to negative magnetic charges, which are un-physical, and so they have to be excluded. This constrains $[W]$ to be an effective class. Thus for a given Calabi-Yau threefold X the effectivity of $[W]$ constrains the choice of vector bundles V.

Example 1: Let $V = TX$ be the tangent bundle of X. Now, as X has $SU(3)$ holonomy it follows that the unbroken gauge-group in four dimensions is $E_6 \times E_8$ (here the second E_8 is often referred to as the "hidden sector". One problem of string compactifications based on TX is that they yield a rather large number of generations (for instance for X being given by the quintic hypersurface in $\mathbb{P}^4$, the index computation gives -100 generations).

Example 2: Three approaches to construct holomorphic vector bundles, with structure group the complexification $G_{\mathbb{C}}$ of a compact Lie group G, on elliptically fibered Calabi-Yau threefolds have been introduced in [1]. The parabolic bundle approach applies for any simple G. One considers deformations of certain minimally unstable G-bundles corresponding to special maximal parabolic subgroups of G. The spectral cover approach (or relative Fourier-Mukai transform) applies for $SU(n)$ and $Sp(n)$ bundles and can be essentially understood as a relative Fourier-Mukai transformation. The case of $U(n)$ bundles has been analyzed in [2]. To illustrate the idea, let $V \to X$ be a holomorphic vector bundle of rank n which is semistable and of degree zero on each fibre $\mathfrak{f}$ of $X \to B$, then its Fourier-Mukai transform $\mathrm{FM}^1(V)$ is a torsion sheaf of pure dimension two on X. The support of $\mathrm{FM}^1(V)$ is a surface $i\colon C \hookrightarrow X$ which is finite of degree n over B. Moreover, $\mathrm{FM}^1(V)$ is of rank one on C and if C is smooth, then $\mathrm{FM}^1(V) = i_*L$ is just the extension by zero of some line bundle $L \in Pic(C)$. Conversely given a sheaf $\mathcal{G} \to X$ of pure dimension two which is flat over B, then $\mathrm{FM}(\mathcal{G})$ is a vector bundle on X of rank equal to the degree of $\mathrm{supp}(\mathcal{G})$ over B. This correspondence between vector bundles on X and sheaves on X supported on finite covers of B is known as the spectral cover construction. The torsion sheaf $\mathcal{G}$ is called the spectral sheaf (or line bundle) and the surface $C = \mathrm{supp}(\mathcal{G})$ is then called the spectral cover. The third approach is called del Pezzo surface approach and applies for E_6, E_7 and E_8 bundles and uses the relation between subgroups of G and singularities of del Pezzo surfaces.

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Some open problems in perturbative heterotic string theory

Eric Sharpe

In this talk I outlined several problems or challenges in better understanding perturbative heterotic string compactifications.

(1) (0,2) mirror symmetry. One such challenge involves understanding (0,2) mirror symmetry. This is a conjectured generalization of ordinary mirror symmetry, in which instead of exchanging pairs of Calabi-Yau's, one instead exchanges pairs of Calabi-Yau's with holomorphic vector bundles. Ordinary mirror symmetry exchanges Hodge numbers of the Calabi-Yau's; (0,2) mirror symmetry exchanges sheaf cohomology. For example, if X_1 and X_2 are an ordinary mirror pair of Calabi-Yau threefolds, then $h^{1,1}(X_1) = h^{2,1}(X_2)$. If $(X_1, \mathcal{E}_1)$, $(X_2, \mathcal{E}_2)$ are a (0,2) mirror pair, then $h^1(X_1, \mathcal{E}_1) = h^1(X_2, \mathcal{E}_2)$. At the level of moduli, ordinary mirror symmetry exchanges complex and Kähler moduli, whereas (0,2) mirror symmetry mixes complex, Kähler, and bundle moduli.

In the special case that each $\mathcal{E}_i \cong TX_i$, (0,2) mirror symmetry should reduce to ordinary mirror symmetry.

Unfortunately relatively little is known about (0,2) mirror symmetry. Analogues of Greene-Plesser constructions have been worked out [1], numerical calculations of sheaf cohomology groups in large numbers of examples have been performed to check whether it is at least plausible [2], and more recently Hori-Vafa-Morrison-Plesser-type techniques [3, 4] have been applied to (0,2) mirrors [5]; more recently, heterotic analogues of the A and B model topological field theories ("holomorphic field theories," in a sense) have been studied [6] in order to work out the heterotic analogue of curve-counting. However, that does not actually amount to a lot – for example, there is currently no known general procedure for constructing heterotic (0,2) mirrors, unlike ordinary mirror symmetry, not even a meaningful conjecture.

In the special case of elliptically-fibered K3 surfaces, one might wonder whether (0,2) mirror symmetry could be expressed as some sort of Fourier-Mukai transform. After this talk was given, David Ploog studied this matter for the special case that the complex structures on the mirror K3's can be rotated into one another; his answer is presented in the appendix.

(2) Gauge bundles with other structure groups. Historically only heterotic string compactifications in which the gauge bundle has structure group $(S)U(n)$ have been considered. However, in principal, any principal G-bundle together with an embedding $G \rightarrow E_8$ could also be considered in an $E_8 \times E_8$ string. In general terms, the resulting worldsheet structure would be very interesting. Recall that the E_8 is built on the worldsheet from left-movers in NS and R sectors; more mathematically, the left-moving fermions couple to a Spin(16) bundle, and the GSO projection on each E_8 realizes a projection Spin(16) $\rightarrow$ Spin(16)/$\mathbf{Z}_2$. The group Spin(16)/$\mathbf{Z}_2$ naturally sits inside E_8 (whereas by contrast neither Spin(16) nor $SO(16)$ do the same).

Put another way, if we start with some principal G-bundle on spacetime together with an embedding of G into E_8, then to realize that on the worldsheet, we must first reduce the resulting E_8 bundle to a Spin(16)/$\mathbf{Z}_2$ bundle, then lift that Spin(16)/$\mathbf{Z}_2$ bundle to a Spin(16) bundle, and it is that resulting Spin(16) bundle to which the left-moving fermions couple. (When this can be done – there are obstructions. For reducing E_8 bundles to Spin(16)/$\mathbf{Z}_2$ bundles, there is an obstruction in degree 10, *i.e.* on 10-manifolds, but not on lower-dimensional manifolds [7]. Curiously, that same obstruction has appeared previously in the physics literature in [8], for completely different reasons. There is also an analogue of a Stiefel-Whitney class describing obstructions to lifting Spin)(16)/$\mathbf{Z}_2$ bundles to Spin(16) bundles.)

For $SU(n)$ bundles, as have been historically considered, this process is all very trivial, partly because the embedding $SU(n) \rightarrow SO(2n)$ factors through Spin($2n$). As a result, for $SU(n)$ bundles, one can identify the gauge bundle on spacetime with the bundle to which the left-moving fermions couple. More generally, however, one expects that those bundles will be different, and this story does not seem to have been explored.

(3) Sheaves and derived categories. In the paper [9] a few examples of (0,2) GLSM's describing non-locally-free (but torsion-free) sheaves on Calabi-Yau's. Their models were well-behaved and nonsingular, which begs the question, under what circumstances can locally-free sheaves be replaced with more general sheaves? These likely will occupy boundary components in a moduli space of heterotic conformal field theories, but it would be desirable to understand the structure of those boundary components, the rules for which sheaves are physically relevant to heterotic strings.

Another question one might ask, in the same vein, is whether derived categories are relevant for heterotic strings. Heterotic T-duality can sometimes be understood in terms of Fourier-Mukai transforms, which most naturally act on derived categories. However, unlike type II strings, where there is now a consistent picture of how derived categories enter physics (see *e.g.* [10]), it is not at all obvious at present how they could enter into the physics of heterotic strings.

(4) Non-Kähler compactifications and H flux. So far I have listed several challenges in understanding traditional heterotic compactifications on Calabi-Yau's, in which the H flux vanishes to zeroth order in α'. If one turns one H flux at leading order, then one works with complex non-Kähler manifolds with trivial canonical bundle [11].

One of the first questions one can ask is, under what circumstances are these theories well-defined. There is a no-go result that says the conditions for spacetime supersymmetry are incompatible with having a trivialization of the canonical bundle and a closed three-form H. One can evade these restrictions in a heterotic string by working away from large radius, with a gauge bundle distinct from the tangent bundle, then the anomaly cancellation condition says $dH \neq 0$. (Of course, there might still be a stronger version of the no-go theorem that applies in greater generality.) What happens on the worldsheet is more mysterious. Worldsheet supersymmetry is a weaker condition than spacetime supersymmetry; perhaps one has consistent worldsheet theories but only spacetime supersymmetry for certain fixed values of the Kähler moduli. Another option, suggested by experience with WZW models, is that the theories are nonunitary for most Kähler moduli, and hence not usually well-defined.

Assuming that the worldsheet theory is well-defined, one can ask what the massless spectrum should be. In heterotic compactifications with $H \neq 0$ at leading order, the BRST operator acts like $\overline{\partial}$, and the massless spectrum is given by sheaf cohomology. Naively, if I make H nonzero, the BRST operator is deformed to something of the form $\overline{\partial} + H\wedge$, which only appears to have a $\mathbf{Z}_2$ grading. Worldsheet chiral primary fields should be (bi)integrally graded, however. This problem has been considered in the context of type II string worldsheets (formally, ignoring the no-go results mentioned above), and there it was discovered that an alternate ("Clifford") grading [12] yielded cohomology of the BRST operator that was integrally-graded, as should be the case physically. Presumably there is an extension to the heterotic case, though this is not yet understood by the author.

Appendix A. By D. Ploog

E. Sharpe posed the question whether there is a functorial way to map vector bundles $V \mapsto \varphi(V)$ on a (elliptic) K3 surface, fixing ranks and such that $H^1(X,V) \cong H^1(X,\varphi(V)^\vee)$. (More generally (0,2) mirror symmetry need not map the K3's in such a way that their complex structures can be related via hyperKähler transform, but, only the special case in which that assumption is made is analysed here.) An obvious answer would be $\varphi(V) = V^\vee$. However, this is contravariant. If we look for covariant mappings (i.e. Fourier-Mukai transforms) which are involutive, chances are slim: an equivalence $\mathsf{FM}_K : \mathrm{D}^b(X) \cong \mathrm{D}^b(X)$ yields a Hodge isometry $\mathsf{FM}_K^H : H^*(X) \cong H^*(X)$. By assumption we know that

$$\mathsf{FM}_K^H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $A : H^2(X) \cong H^2(X)$ is an isometry of the second cohomology. Now by the involutive property of φ all eigenvalues of A are 1 or -1. On general K3 surfaces, only the obvious matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \iota := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\mathrm{id} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are isometries. Excluding the identity, ι seems to deserve special attention. It has occured to several people, that the image of the group homomorphism $\mathrm{Aut}(\mathrm{D}^b(X)) \to \mathrm{Aut}(\tilde{H}(X))$ (the latter denoting the group of all integral Hodge isometries of $H^*(X)$) has index at most 2, see [15], [17], [18]. While it is not known at present whether ι actually is in the image, popular belief indicates that it is not.

Evidence for this is given by rephrasing the question in terms of orientation, where ι generates the subgroup of orientation reversing isometries (see [16], and [14] for a related effect on the real 4-fold underlying K3 surfaces). Coming from another angle, Bridgeland has used stability conditions on $\mathrm{D}^b(X)$ to put the conjecture in a natural setting [13].

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A heterotic standard model

Volker Braun

For many years physicists have been trying to understand how string theory can give rise to realistic low-energy physics. While string theory itself is basically unique, there is a wide range of possible compactifications, which almost always yield vastly different 4-dimensional physics than our universe. Recently, we have discovered [1, 2, 3] a compactification of the $E_8 \times E_8$ heterotic string with 3 generations of quarks and leptons, 0 anti-generations, and no color triplets. The only difference of our matter spectrum to the minimal supersymmetric Standard Model (MSSM) is the existence of a second pair of Higgs–Higgs conjugate particles.

Since then, we managed to refine our construction and found a model that yields the precise MSSM matter content (that is, with a single Higgs–Higgs conjugate pair). There are no anti-generations or vector-like pairs. Moreover, Yukawa couplings are allowed by the elliptic fibrations, in contrast to our previous model [4]. Finally, there are 3 complex structure moduli, 3 Kähler moduli, and 13 vector bundle moduli.

A nice way to embed the Standard Model gauge group with an additional $U(1)_{B-L}$ into the E_8 of the heterotic string is by first picking a maximal regular $SU(4) \times Spin(10)$ subgroup, and second a $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup inside the $Spin(10)$. If one turns on the corresponding $SU(4)$ instanton and $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson line, then the effect is to break the gauge group to the commutant of $SU(4) \times \mathbb{Z}_3 \times \mathbb{Z}_3$ inside E_8, which is precisely

$$SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}. \tag{1}$$

Moreover, the matter content of the Standard Model fits nicely into the fermions of the heterotic string, which transform in the **248** of E_8. Viewed as $SU(4) \times Spin(10)$

representation, it branches as

$$\mathbf{248} \to \big(\mathbf{1},\mathbf{45}\big) \oplus \big(\mathbf{15},\mathbf{1}\big) \oplus \big(\mathbf{4},\mathbf{16}\big) \oplus \big(\bar{\mathbf{4}},\bar{\mathbf{16}}\big) \oplus \big(\mathbf{6},\mathbf{10}\big). \tag{2}$$

The first two summands are $Spin(10)$ gauginos and vector bundle moduli, and the last three are matter fields. The latter three representations of $Spin(10)$ split further under the $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson line as

$$\begin{aligned} \mathbf{16} =\, & \chi_1\chi_2^2\big(\mathbf{3},\mathbf{2},1,1\big) \oplus \chi_2^2\big(\mathbf{1},\mathbf{1},6,3\big) \oplus \chi_1^2\chi_2^2\big(\bar{\mathbf{3}},\mathbf{1},-4,-1\big) \oplus \chi_1^2\big(\bar{\mathbf{3}},\mathbf{1},2,-1\big)\oplus \\ & \oplus \big(\mathbf{1},\bar{\mathbf{2}},-3,-3\big) \oplus \chi_2\big(\mathbf{1},\mathbf{1},0,3\big), \\ \mathbf{10} =\, & \chi_2\big(\mathbf{1},\mathbf{2},3,0\big) \oplus \chi_1\chi_2\big(\mathbf{3},\mathbf{1},-2,-2\big) \oplus \chi_2^2\big(\mathbf{1},\bar{\mathbf{2}},-3,0\big) \oplus \chi_1^2\chi_2^2\big(\bar{\mathbf{3}},\mathbf{1},2,2\big), \end{aligned} \tag{3}$$

where χ_1, χ_2 are the two basic characters of $\mathbb{Z}_3 \times \mathbb{Z}_3$. While every summand in the decomposition of the **16** corresponds to one of the quarks and leptons that actually occur in nature, the decomposition of the **10** yields Higgs fields and color triplets. The latter simply do not exist. The need to project them out while retaining Higgs fields is known as the doublet-triplet splitting problem.

These group theory considerations suggest the following. First, recall that to specify a geometric compactification of heterotic string theory one has to specify a Calabi-Yau threefold together with a stable, holomorphic vector bundle on it. Since we require $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines, we need a Calabi-Yau manifold with

$$\pi_1\big(X\big) = \mathbb{Z}_3 \times \mathbb{Z}_3. \tag{4}$$

We constructed [6] such a manifold using a fiber product $\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2$ of two dP_9 surfaces B_1 and B_2. For special dP_9 surfaces, the fiber product allows for a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action. The quotient

$$X = \widetilde{X}\big/(\mathbb{Z}_3 \times \mathbb{Z}_3) \tag{5}$$

is then the requisite Calabi-Yau manifold. Furthermore, we found an equivariant[1] vector bundle $\widetilde{V}$ on $\widetilde{X}$ with cohomology groups

$$\begin{aligned} H^i\Big(\widetilde{X},\widetilde{V}\Big) &= \begin{cases} 0 & i=3 \\ 0 & i=2 \\ 3R[\mathbb{Z}_3\times\mathbb{Z}_3] & i=1 \\ 0 & i=0 \end{cases}, \\ H^i\Big(\widetilde{X},\wedge^2\widetilde{V}\Big) &= \begin{cases} 0 & i=3 \\ \chi_2 \oplus \chi_2^2 \oplus \chi_1\chi_2^2 \oplus \chi_1^2\chi_2 & i=2 \\ \chi_2 \oplus \chi_2^2 \oplus \chi_1\chi_2^2 \oplus \chi_1^2\chi_2 & i=1 \\ 0 & i=0 \end{cases}. \end{aligned} \tag{6}$$

The low-energy spectrum of heterotic string theory compactified on $\big(\widetilde{X},\widetilde{V}\big)$ is determined by the invariant cohomology after tensoring with the extra character coming from the Wilson line.

[1]The category of vector bundles on X is identical to the category of equivariant vector bundles on $\widetilde{X}$. For technical reasons we work in the latter context.

This solve the doublet-triplet splitting problem as follows. The relevant dimensions are

$$\begin{aligned}\# \text{ of Higgs } &= \dim_{\mathbb{C}} \Big[\chi_2^1 H^1\big(\widetilde{X}, \wedge^2 \widetilde{V}\big)\Big]^{\mathbb{Z}_3\times\mathbb{Z}_3} = 1 \\ \# \text{ of triplets } &= \dim_{\mathbb{C}} \Big[\chi_1^2\chi_2^2 H^1\big(\widetilde{X}, \wedge^2 \widetilde{V}\big)\Big]^{\mathbb{Z}_3\times\mathbb{Z}_3} = 0.\end{aligned} \tag{7}$$

and the same for the conjugate Higgs and conjugate color triplets. Hence, there is precisely one Higgs–Higgs conjugate pair, and zero color triplets. Furthermore, $H^1\big(\widetilde{X}, \widetilde{V}\big)$ is three times the sum of all $\mathbb{Z}_3 \times \mathbb{Z}_3$ representations, yielding three whole families of quarks and leptons. Finally, $H^1\big(\widetilde{X}, \widetilde{V}^\vee\big) = 0$ tells us that there are no anti-generations.

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Fourier-Mukai transforms on singular varieties

Daniel Hernández Ruipérez

(joint work with A.C. López Martín, F. Sancho de Salas)

In the last years derived categories and their equivalences have been of great important in physics and in birational geometry. On the one hand, they have been used in string theory because homological mirror symmetry predicted that objects in the derived category of a Calabi-Yau threefold can be taught as D-branes of B-type and equivalences of derived categories (which are integral functors in the smooth case by Orlov reconstruction theorem [15]) should mirror monodromies on the special Lagrangian side. Participants of this mini-workshop have contributed to give evidences of this conjecture [2, 9], recent surveys of the subject are [1, 3]. On the other side, there are many results supporting the belief that derived categories are important invariants of algebraic varieties and that both them and their equivalences are most suitable in birational geometry, particularly the minimal program model in higher dimensions. One can mention a result due to Bondal and Orlov [5] which says that if X is a smooth projective variety whose canonical divisor is either ample or anti-ample (X is of general type or Fano), then X can be reconstructed out of its derived category. It is known however that there exist non-isomorphic (even non-birational) smooth varieties with equivalent derived categories, they are called Mukai partners, and examples for abelian varieties and K3 surfaces were provided by Mukai [14] and Orlov [15]. After them, the study of Mukai partners has been contemplated by many people [7] [10]. Kawamata [10] proved that if X and Y are smooth projective varieties with equivalent derived categories and X is of general type, then X and Y are K-equivalent, i.e., there exist birational morphisms $f\colon Z \to X$, $g\colon Z \to Y$ such that $f^*K_X \sim g^*K_Y$. Other important contributions are owed to Bridgeland [6], who proved that two crepant resolutions of a projective threefold with terminal singularities have equivalent derived categories; therefore, two birational Calabi-Yau threefolds have equivalent derived categories. This has been conjectured to hold true in higher dimensions by Kawamata.

However, very little attention has been paid so far to singular varieties in the Fourier-Mukai literature, probably because the fundamental results about integral functors do not easily generalise to the singular situation. We would like to mention two of the most important.

One is the aforementioned Orlov's reconstruction theorem which has been generalised by Kawamata [12] to the smooth stack associated to a normal projective variety with only quotient singularities; therefore D-equivalence also implies K-equivalence for those varieties. Bridgeland result about flopping contractions [6] has been generalised by to quasi-projective varieties with only Gorenstein terminal singularities by Van de Bergh [16] and Chen [8]. Finally, Kawamata [11] has obtained analogous results for some $\mathbb{Q}$-Gorenstein threefolds using algebraic stacks.

Other is Bondal and Orlov's characterisation of those integral functors between the derived categories of two smooth varieties that are fully faithful [4]. We present here a generalisation to projective Gorenstein singularities of any kind [13]. The precise statement is this:

Theorem *Let X and Y be projective Gorenstein schemes over an algebraically closed field of characteristic zero, and let $\mathcal{K}^\bullet$ be an object in $D^b(X \times Y)$ of finite projective dimension over X and over Y. Assume also that X is integral. Then the integral functor $\Phi^{\mathcal{K}^\bullet}\colon D^b(X) \to D^b(Y)$ is fully faithful if and only if the kernel $\mathcal{K}^\bullet$ is strongly simple over X.*

It is important to notice that we first had to adapt to the singular case the usual definition of strong simplicity. The role played in the definition by the structure sheaves $\mathcal{O}_x$ of the points is now played by the structure sheaves of the zero-cycles of finite homological dimension supported on points (if x is a singular point $\mathcal{O}_x$ has not finite homological dimension) and this implies a lot of technical complications. Bridgeland's criterion that characterizes when a fully faithful integral functor is an equivalence is also valid in the Gorenstein case. Moreover, since for a Gorenstein variety one has a more natural spanning class given by the structure sheaves of zero cycles of finite homological dimension supported on points, one also proves the following alternative result.

Theorem *Let X, Y and $\mathcal{K}^\bullet$ be as in the previous theorem with Y connected. A fully faithful integral functor $\Phi^{\mathcal{K}^\bullet}\colon D^b(X) \to D^b(Y)$ is an equivalence of categories if and only if for every point $x \in X$ there exists a zero cycle Z_x of finite homological dimension supported on x such that $\Phi^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_x}) \simeq \Phi^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_x}) \otimes \omega_Y$.*

We can the extend to the Gorenstein case some geometric consequences of the existence of Fourier-Mukai functors which are analogous to certain well-known properties of smooth schemes.

We also prove a new result that characterises when a relative integral functor is fully faithful or an equivalence, and generalises [8, Prop. 6.2]. This result, together with the characterisation of Fourier-Mukai functors in the absolute Gorenstein case gives a criterion to ascertain when a relative integral functor between the derived categories of the total spaces of two Gorenstein fibrations is an equivalence. We expect that this theorem could be applied to very general situations. As a first application we give here a very simple and short proof of the (known) invertibility result for elliptic fibrations:

Theorem *Let S be an algebraic scheme over an algebraically closed field of characteristic zero, $X \to S$ an elliptic fibration with integral fibres and a section, $\hat{X} \to S$ the dual fibration and $\mathcal{P}$ the relative Poincaré sheaf on $X \times_S \hat{X}$. The relative integral functor*

$$\Phi^{\mathcal{P}}\colon D^b(X) \to D^b(\hat{X}$$

is an equivalence of categories.

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String compactifications on stacks

Eric Sharpe
(joint work with T. Pantev)

In this talk I outlined some recent work on understanding string compactifications on stacks. Stacks have much of the basic structure that one ordinarily needs to make sense of string compactifications, *e.g.* metrics, spinors, *etc.* So, can one make sense of strings propagating on stacks? On all stacks, or only some? Are there new conformal field theories?

The basic idea described in [1, 2] is that every[1] (smooth Deligne-Mumford) stack has a presentation of the form $[X/G]$ for G some group, not necessarily finite, and not necessarily effectively-acting. To such a presentation, one can associate a "G-gauged sigma model on X," which is the technical physical description of a physical theory describing a string propagating G-equivariantly on X. Unfortunately, such presentations are not unique, and the physical theories one associates to different presentations of the same stack can be very different. For example, $[\mathbf{C}^2/\mathbf{Z}_2]$ (where the $\mathbf{Z}_2$ acts by flipping signs of coordinates, leaving the origin as a fixed point) is the same stack as $[X/\mathbf{C}^\times]$, where

$$X = \frac{\mathbf{C}^2 \times \mathbf{C}^\times}{\mathbf{Z}_2}$$

In X, the $\mathbf{Z}_2$ acts on the $\mathbf{C}^2$ as in $[\mathbf{C}^2/\mathbf{Z}_2]$, and simultaneously on the $\mathbf{C}^\times$ as a rotation, so that the action on the product is free. If we follow the procedure above, then to $[\mathbf{C}^2/\mathbf{Z}_2]$ we associate a $\mathbf{Z}_2$ gauged sigma model on $\mathbf{C}^2$, *i.e.* a $\mathbf{Z}_2$ string orbifold, which turns out to be a special kind of quantum field theory which possesses a conformal invariance property. To the presentation $[X/\mathbf{C}^\times]$, on the other hand, we associate a $U(1)$-gauged[2] nonlinear sigma model on X. This theory is not conformally invariant, unlike the physical theory we associated to $[\mathbf{C}^2/\mathbf{Z}_2]$.

On the face of it, we have a contradiction: different presentations of the same stack define different physical theories, so our assignment of physical theories to stacks does not seem to be well-defined. However, there is a workaround, involving

[1] There are some pathological exceptions to this rule. By 'every', we really mean, every stack that is likely to be relevant for physics.

[2] In a supersymmetric theory, gauging a reductive algebraic group is realized by gauging a compact Lie group; the noncompact directions in the group are taken care of by "D-terms," which break conformal invariance classically.

a physical process called "renormalization group flow." Given one physical theory, the renormalization group (actually a semigroup) constructs other theories.

In particular, we conjecture that after applying the renormalization group, the different physical theories associated to different presentations of a single stack, all evolve to the same physical theory. Unfortunately, there is no known way to directly verify such a claim; no one knows how to explicitly follow renormalization group flow in nontrivial examples.

This general setup is closely analogous to the manner in which derived categories enter physics (see *e.g.* [3]). Given a single isomorphism class of objects in a derived category, there will be many presentations, but only some of those presentations can be associated to physical D-brane/antibrane systems (just as for stacks, it is only known how to associate presentations of the form $[X/G]$ to physics). However, the class of physically-realizable presentations is sufficiently broad that every isomorphism class of objects has a physically-realizable presentation. For derived categories, physically-realizable presentations include complexes of locally-free sheaves, and every bounded complex of coherent sheaves is quasi-isomorphic to at least one complex of locally-free sheaves. (Similarly, in stacks, (almost) every stack admits at least one presentation of the form $[X/G]$.) However, physically-realizable presentations are not unique, and the different physical theories one gets are very different. One conjectures that under renormalization group flow, that potential presentation-dependence washes away, but as it is impossible to explicitly follow renormalization group flow, one must perform indirect tests, preferably as many as possible.

In the case of stacks, unfortunately, the obvious indirect tests tend to fail, which makes one worry that perhaps the renormalization group does not respect stacks, and sends different presentations of the same stack to different theories. One of the first things to check is whether the mathematical deformations of a stack match the physical deformations of the corresponding theory. For strings on spaces, these match, so one would expect a matching for stacks also. Unfortunately, they do not match, even in very simple cases. For example, the stack $[\mathbf{C}^2/\mathbf{Z}_2]$ is rigid, whereas the corresponding physical theory admits deformations.

How should such a result be interpreted? Perhaps there is an alternate notion of deformation theory for stacks, in which a matching is restored, or perhaps physics depends upon the choice of presentation and the conjectured behavior of the renormalization group is wrong, in which case stacks are simply not a useful concept for physics. Understanding this problem, in one form or another, has been one of the driving forces for several years now behind our efforts to understand whether the notion of strings on stacks can be made consistent.

Other problems arise elsewhere, for example in massless spectra. The mathematically natural manner to define a cohomology theory of stacks is in terms of the associated inertia stack. Physically, that would mean that the massless spectrum of a theory on a stack should be calculated by cohomology of the inertia stack. However, in the physics community, the only cases in which massless spectra have previously been calculated are for global quotients by finite effectively-acting groups. For global quotients by finite noneffectively-acting groups, the calculation is doable but subtle and was not thought through in the physics community prior to [1, 2]. For global quotients by nonfinite groups, as appear in *e.g.* the alternate presentation of $[\mathbf{C}^2/\mathbf{Z}_2]$, there is no known explicit calculation of the massless spectrum, and moreover, because the theories are not conformally invariant, one can argue that an explicit calculation may be impossible.

In particular, we have a physics prediction: if the massless spectrum of a string on a stack is given by a cohomology of the inertia stack, then we have implicitly made a prediction for massless spectra of gauged sigma models in cases where no physical calculation existed previously.

On the other hand, this is another place where the general procedure described above could fail, and indeed that is what seems to happen. For gerbes, *i.e.* gaugings by noneffectively-acting groups, the massless spectrum predicted by the inertia stack contains multiple dimension zero operators, a violation of "cluster decomposition," one of the fundamental axioms upon which quantum field theory is based. Perhaps the notion of strings on stacks is flawed, or perhaps strings can only be sensibly compactified on *some* stacks, but not others.

These issues formed the basis for the work in [1, 2], and we now believe we understand how these issues are resolved. For example, for banded abelian G-gerbes, we believe that a string propagating on the gerbe is indistinguishable from a string propagating on copies of the underlying space, together with flat B fields. In particular, for strings propagating on disconnected target spaces, cluster decomposition also breaks, but in the mildest possible way, so that the physical theory is still well-defined. (This is very nearly the only mechanism by which violating cluster decomposition is acceptable, however.) Mathematically, this makes a prediction for Gromov-Witten invariants of gerbes, namely that they are computable in terms of Gromov-Witten invariants of the underlying space.

Understanding physically how and when it is consistent to compactify strings on stacks also gives rise to physical calculations of quantum cohomology rings, through one-loop effective actions and other gauge-theoretic methods, as well as through Toda duals.

Much of [1, 2] is also devoted to understanding mirror symmetry for stacks. The mirrors turn out to possess some slightly exotic features – for example, Landau-Ginzburg-point mirrors to stacks often feature discrete-valued physical fields, a characteristic which plays an important role in understanding how to generalize Batyrev's old prescription for mirrors of hypersurfaces in toric varieties, to toric stacks. As an easy special case, consider Toda duals. The Toda dual (a Landau-Ginzburg) model to a projective space $\mathbf{P}^{N-1}$ is described by a "superpotential"

$$W = \exp(-Y_1) + \cdots + \exp(-Y_{N-1}) + \exp(Y_1 + \cdots + Y_{N-1})$$

The Toda dual to a $\mathbf{Z}_k$ gerbe over $\mathbf{P}^{N-1}$ with characteristic class $-n \mod k$ is given by

$$W = \exp(-Y_1) + \cdots + \exp(-Y_{N-1}) + \Upsilon^n \exp(Y_1 + \cdots + Y_{N-1})$$

where Υ is a discrete-valued field, taking values in kth roots of unity. Mirror symmetry has taken the noneffective gauging, which appears physically through nonperturbative effects, and turned those nonperturbative effects into a purely perturbative phenomenon, namely a discrete-valued field.

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Mirror symmetry and generalized complex geometry

Alessandro Tomasiello
(joint work with W.–y. Chuang, S. Fidanza, M. Graña, S. Kachru, R. Minasian, M. Petrini)

Mirror symmetry is a stunning phenomenon arising in string theory compactifications on Calabi–Yau manifolds. In this talk I want to propose evidence that it might also hold in a more general context. It will consists of three parts, loosely based respectively on [1, 2, 3].

The simplest such a generalization, the one we will consider for most of the talk, consists of pairs of a complex manifold and a symplectic manifold. This was first suggested years ago in [4]. In the last part of this talk I will indeed show mirror pairs of string theory vacua of this type. (These examples are not yet enough by themselves to prove a general phenomenon, and for this reason we will present different evidence in the first two parts.) In a looser sense, this mirror symmetry might also be valid more generally for manifolds whose direct sum of the tangent and cotangent bundle, $T \oplus T^*$, admits a reduction to SU(3)×SU(3) (this condition is also necessary, but not sufficient, for the existence of $N = 1$ vacua, as we will see below). This is clearly only a topological condition; clearly such manifolds will not in general be vacua, and hence they can constitute mirror pairs only in the sense that they give rise to equal four–dimensional effective theories.

1. The first piece of evidence comes from the classification of $N = 1$ vacua of type II supergravity [1]. By vacua we mean solutions of the form $\text{Minkowski}_{3,1}$ (or $\text{AdS}_{3,1}$) $\times$ a six dimensional manifold M_6. We allow a priori all the supergravity fields: ϕ (the dilaton), H (the curvature of the B–field) and F_k (the RR curvatures; k is even for the IIA theory, and odd for the IIB theory), besides of course the metric g.

There are two types of conditions that the manifold and the fields on it have to satisfy: algebraic and differential. The algebraic conditions are topological, and say that M_6 needs to have an SU(3)×SU(3) reduction on $T \oplus T^*$. This is equivalent to the existence of a pair $(\Phi_+, \Phi_-) \in \Lambda^{\text{even}}T^*, \Lambda^{\text{odd}}T^*$ with special properties: *i)* each Φ has to be a pure spinor when viewed as a spinor for the Clifford algebra built on $T \oplus T^*$; *ii)* $\Phi_\pm$ have to satisfy a certain compatibility condition.

It can be shown that such a pair (Φ_+, Φ_-) determines a metric g. The second set of conditions, the differential one, is actually expressed more conveniently in terms of this pair, rather than in terms of g. Schematically, the conditions read $(d+H\wedge)\Phi_+ = 0$, $(d+H\wedge)\Phi_- = F$ for IIA, and $(d+H\wedge)\Phi_+ = F$, $(d+H\wedge)\Phi_- = 0$ for IIB. Here F is a formal sum of all F_k, that hence $\in \Lambda^{\rm even}T^*$ for IIA, and $\in \Lambda^{\rm odd}T^*$ for IIB.

One sees that the equations are symmetric under an exchange $\Phi_+ \leftrightarrow \Phi_-$. This feature was far from obvious from the formulation of the IIA and IIB theories, and it already seems to suggest a mirror pairing. The conditions can be interpreted in terms of the *generalized complex geometry* of Hitchin [5]; for example, they imply that the manifold is, in his notation, generalized Calabi–Yau. One important particular case is when the SU(3)×SU(3) structure on $T \oplus T^*$ comes from an SU(3) structure on T. In that case, the two pure spinors have the form $\Phi_+ = e^{iJ}$, $\Phi_- = \Omega$. It follows that supersymmetric vacua are symplectic in IIA, and complex in IIB.

2. We can also forget temporarily about the condition that M_6 support a supersymmetric vacuum, and ask how T–duality acts (when it makes sense to apply it), in a SYZ–like approach, on the Φ's; even more importantly, on their exterior derivatives $d\Phi_\pm$. So far, to classify the differential type of a manifold of SU(3) structure, the exterior derivatives $d\Omega$ and dJ have been used, appropriately decomposed in SU(3) representations. For example, one function one gets this way is $\Omega \llcorner dJ$. Collectively, such quantities are known as *intrinsic torsion.*

The original computations are performed in [2]; it turns out that they can be cast in a neater form using again generalized complex geometry. One can define a new set of intrinsic torsions, which transform under three T–dualities with some simple exchanges. Or, more insightfully, the transformation can be viewed as a rotation of a modified Hodge diamond. One of these improved functions reads $(\Phi_- \wedge (d + H\wedge)\Phi_+)_{\rm top}$.

As a particular consequence of these exchanges, we find again that complex and symplectic manifolds should be mirror.

3. We have seen how the conditions for supersymmetric vacua in type II supergravity lead to elegant mathematical conditions. Unfortunately, explicit examples are hard to come by so far, for a variety of reasons, among which the need of introducing negative sources, for example *orientifolds*. Supergravity is however only an approximation to string theory. There are corrections signaling the stringy nature of gravity; these can help avoiding orientifolds, but are not described (so far) by equations as elegant as the ones for supergravity. In [3], we work around this problem by considering extremal transitions. There exist indeed transitions from Calabi–Yau manifolds to manifolds which are only complex, or only symplectic. We are able to show that some of these provide vacua for IIB or IIA respectively. These also come in mirror pairs. For example, one such pair has $(b_2, b_3/2) = (243, 3)$ or $(2, 244)$ respectively.

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Toric stacks and mirror symmetry

R. Paul Horja
(joint work with Lev A. Borisov)

Toric varieties have traditionally provided an indispensable testing ground for many of the predictions made in the field of mirror symmetry. Recently, Borisov, Chen and Smith [BCS] gave a combinatorial definition of toric smooth Deligne–Mumford (DM) stacks and computed their orbifold (Chen–Ruan) cohomology.

In the first part of this talk, I present some of the technical aspects of the definition of reduced toric DM stacks as quotient stacks $\mathbb{P}_\Sigma = [Z/G]$. The construction is similar in spirit with the one given by Cox and Katz for simplicial toric varieties. It uses the combinatorics of the stacky fan $(\Sigma, \{v_i\}_{1\leq i\leq n})$, where Σ is a usual simplicial fan in the lattice $N \cong \mathbb{Z}^d$ and v_i are vectors generating the one dimensional cones of the fan Σ, one for each cone. The vectors v_i are not necessarily the minimal integer generators of the one dimensional cones. Moreover, according to a result of Vistoli, the category of coherent sheaves on $[Z/G]$ is equivalent to that of G-linearized coherent sheaves on Z, see [V, Example 7.21]. Using this description, in the paper [BH1], we gave a description of the Grothendieck group of a smooth toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$. In the reduced case, $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ is generated by the classes R_i of the invertible sheaves $\mathcal{L}_i$ which correspond to the one-dimensional cones of the fan Σ. These sheaves generalize the sheaves $\mathcal{O}(D_i)$ for codimension one strata D_i in a smooth toric variety.

Theorem [BH1]. *Let B be the quotient of the ring $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ by the ideal generated by the relations*

- $\prod_{i=1}^n x_i^{f(v_i)} = 1$ *for any linear function* $f : N \to \mathbb{Z}$,
- $\prod_{i\in I}(1 - x_i) = 0$ *for any set* $I \subseteq [1, \ldots, n]$ *such that* $v_i, i \in I$ *are not contained in any cone of* Σ.

Then the map $\rho : B \to K_0(\mathbb{P}_{\mathbf{\Sigma}})$ which sends x_i to R_i is an isomorphism.

In the second part of the talk, I briefly describe some applications of the construction of toric stacks to homological mirror symmetry, following our work [BH2]. To a set of vectors $\mathcal{A} := \{v_1, \ldots, v_n\}$ contained in an affine hyperplane in N located at unit distance from the origin, Gelfand, Kapranov and Zelevinski [GKZ] associated a holonomic system of partial differential equations which turned out to be of crucial importance in mirror symmetry.

For each stacky fan Σ supported on the cone $\mathbb{R}_{\geq 0}\Delta$ and induced by a regular triangulation of the polytope Δ, we construct a mirror symmetry map defined on the dual of the Grothendieck ring of the toric DM stack $\mathbb{P}_\Sigma$

$$MS_\Sigma : (K_0(\mathbb{P}_\Sigma, \mathbb{C}))^\vee \to \mathcal{S}ol(U_\Sigma)$$

that produces GKZ solutions (periods) which are analytic in the complex domain U_Σ in $\mathbb{C}^n$ associated to the fan Σ. Moreover, if Σ_+ and Σ_- are two fan structures induced by two regular triangulations of the polytope Δ that correspond to two vertices joined by an edge of the secondary polytope determined by $\mathcal{A}$, the associated toric DM stacks $\mathbb{P}_{\Sigma_+}$ and $\mathbb{P}_{\Sigma_-}$ are birationally equivalent, and, according to Bondal and Orlov [BO] and Kawamata [K], their bounded derived categories of coherent sheaves are equivalent. The equivalence is given by a Fourier-Mukai functor $FM : D^b(\mathbb{P}_{\Sigma_-}) \to D^b(\mathbb{P}_{\Sigma_+})$.

The main result of [BH2, theorem 5.4] shows that the mirror symmetry maps $MS_{\Sigma_\pm}$ corresponding to the two birationally equivalent DM stacks are compatible with the K-theoretic action of the Fourier-Mukai functor $FM^\vee : (K_0(\mathbb{P}_{\Sigma_+}, \mathbb{C}))^\vee \to (K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C}))^\vee$, and to the mirror analytic continuation procedure. Namely, we have that

$$MB \circ MS_{\Sigma_+} = MS_{\Sigma_-} \circ FM^\vee,$$

where MB is the analytic continuation operator whose definition involves Mellin-Barnes integrals. At the end of the talk, I outline further applications of our results and some related speculations in the field of mirror symmetry.

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Topological strings on K3 fibrations

Emanuel Scheidegger

In this talk we considered the problem of computing part of the holomorphic free energy of the topological string for a particular class of K3 fibrations. For a previous presentation of related results, see [1].

We recall that given a holomorphic map $\phi : \Sigma_g \to X$, from a Riemann surface Σ_g into a smooth Calabi–Yau threefold X, the Gromov–Witten invariant of the class $\beta = [\phi(\Sigma_g)] \in H_2(X, \mathbb{Z})$ is defined as

$$N_\beta^{(g)} = \int_{[\overline{\mathcal{M}_g(\beta,X)}]^{\mathrm{virt}}} 1. \tag{1}$$

Since the virtual fundamental class of the moduli space of stable maps $\overline{\mathcal{M}_g(\beta, X)}$ is a stack, this invariant is a rational number. In counting problems we would prefer to have integer invariants, in particular the integer instanton numbers in physics in general do not agree with the Gromov–Witten invariants.

There are integer invariants which contain the same information as the Gromow–Witten invariants, though they are difficult to define. We start by reexpanding the generating function for the Gromow–Witten invariants, *i.e.* the holomorphic free energy of the topological string [2], as follows:

$$\begin{aligned}\mathcal{F}(t,\lambda) &= \sum_{g=0}^{\infty} \sum_{\beta\in H_2(X,\mathbb{Z})} N_\beta^{(g)} q^\beta \lambda^{2g-2} \\ &= \frac{c(t)}{\lambda^2} + l(t) + \sum_{g=0}^{\infty} \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{m=1}^{\infty} n_\beta^{(g)} \frac{1}{m} \left(2 \sin \frac{m\lambda}{2}\right)^{2g-2} q^{\beta m},\end{aligned} \tag{2}$$

with $q^\beta = \exp(i\langle\beta, \omega(t)\rangle)$ and t are coordinates on the Kähler moduli space $\mathcal{M}(X, \omega)$. The equality of these two expansions relies on a physics argument by Gopakumar and Vafa [3] and and has not been proven mathematically. Therefore this expansion should be viewed as an indirect definition of the Gopakumar–Vafa invariants $n_\beta^{(g)}$. A physicist's way of defining these invariants directly is to say that they are the number of BPS states with right spin r of an M2–brane wrapping a curve $C \in X$ such that $[C] = \beta$. From this, it is clear that $n_\beta^{(g)} \in \mathbb{Z}$. This can be reexpressed in mathematical terms as follows. Consider the moduli space $\mathcal{M}_C$ of complex structure deformations of embeddings of the curve C into X. Next, consider the space $\widehat{\mathcal{M}}_C$ which is a fibration over $\mathcal{M}_C$ with fibers $J(C)$, the Jacobian of the curve C. Note that since we also deform the complex structure of C, the curve can become singular, and the fibration is non–trivial. For both $\mathcal{M}_C$ and $\widehat{\mathcal{M}}_C$ a mathematical definition is in general an open problem. Nevertheless, we further assume that there exists an $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ Lefshetz action on $H^*(\widehat{\mathcal{M}}_C)$ such that $\mathrm{SU}(2)_L$ acts on the fiber, $H^*(J(C))$, and $\mathrm{SU}(2)_R$ acts on the base, $H^*(\mathcal{M}_C)$. The claim is then that $H^*(\widehat{\mathcal{M}}_C) = \bigoplus_{r\geq 0} \left(\left(\frac{1}{2}\right)_L \oplus 2(0)_L\right)^r \otimes R_r(\beta)$, where $R_r(\beta)$ is the $\mathrm{SU}(2)_R$ representation on $H^*(\mathcal{M}_C)$. Using this decomposition, the Gopakumar–Vafa invariants could be defined as $n_\beta^{(r)} = \mathrm{tr}_{R_r(\beta)}(-1)^{2H_R}$, where H_R is the Casimir operator of $\mathrm{SU}(2)_R$. Using $\sum_{r=0}^{\infty} C_g(r)\lambda^{2r} = \left(\frac{\sin(\lambda/2)}{\lambda/2}\right)^{2g-2}$ we can relate the Gromov–Witten invariants to the Gopakumar–Vafa invariants as follows: $N_\beta^{(g)} = \sum_{r=0}^{g} C_r(g-r)n_\beta^{(r)}$. Finally, we again emphasize that all these statements are conjectures based on physics arguments and impressive enumerative evidence.

In the case where X admits a K3 fibration $\pi : X \to B$ with generic fiber $Y \cong \mathrm{K3}$, a subset of these invariants can be computed explicitly as we will show in the remainder of the text. We restrict the classes β to the fiber, *i.e.* we define $\mathcal{F}_{\mathrm{K3}}(t,\lambda)$ as in (2), but the sum now runs only over $\beta \in H_2(Y,\mathbb{Z})$. Consider first the trivial fibration, $X = \mathrm{K3} \times T^2$. For C a curve in the class β in the K3 with $C^2 = 2g-2$ a formula for $\mathcal{F}_{\mathrm{K3}}(t,\lambda)$ was given in [4]. It is based on the model $\widehat{\mathcal{M}}_C$ of the moduli space of M2–branes, which leads to the Hilbert scheme of points on K3 [5]

$$\mathcal{H}(\lambda,t) = \left(\tfrac{1}{2}\sin(\tfrac{\lambda}{2})\right)^2 \prod_{n\geq 1} \frac{1}{(1-e^{i\lambda}q^n)^2(1-q^n)^{20}(1-e^{-i\lambda}q^n)^2} \, .$$

We have extended this argument to regular K3 fibrations in [6] and found that the topological free energy for the fiber classes can be expressed as

$$\mathcal{F}_{\mathrm{K3}}(\lambda,t) = \frac{\Theta(q)}{q}\mathcal{H}(\lambda,t) \tag{3}$$

where $\Theta(q)$ is a modular form determined from the topological properties of both the fiber and the fibration. This formula is clearly inspired by the results of heterotic–type II duality [7], [8]. $\Theta(q)$ is related to an automorphic form of the classical duality group $\mathrm{SO}(2, h^{1,1}(X)-1, \mathbb{Z})$ by the Borcherds lifting.

In the last part we explain how the modular form $\Theta(q)$ can explicitly be determined in a particular class of K3 fibrations [9]. For this purpose, we first review some general facts about the topology of K3 fibrations. For simplicity, we assume that $H_2(X,\mathbb{Z}) = i_*\mathrm{Pic}(Y) \oplus \mathbb{C}\cdot[B]$, where $\mathrm{Pic}(Y)$ is the Picard lattice of the fiber and $[B]$ is the class of the base. First, consider the properties of the fiber only. One topological invariant is the Picard lattice $\mathrm{Pic}(Y)$, its rank ρ and its intersection form I which is always even. Later, we will only be interested in the simple situation with $\rho = 1$. Note that then $I = \langle 2n\rangle = \{e \in \mathbb{Z} | e^2 = 2n\}$, and furthermore $h^{1,1}(X) = 2$. Next, we look at the global properties of the fibration. Again, an important topological characteristic is its intersection form. If we take $h \in i_*\mathrm{Pic}(Y)$ and its dual $H \in H_4(X,\mathbb{Z})$, as well as the class of the fiber $L = [i(Y)] \in H_4(X,\mathbb{Z})$ and its dual $l = [B] \in H_2(X,\mathbb{Z})$, then, since $L^2 = 0$, we have $L^3 = 0$, and $HL^2 = 0$. Furthermore, $H^2L = 2n$ and $H^3 = p$. Finally, a basic invariant is of course the Euler characteristic $\chi(X) = \chi(Y)\chi(B) + \chi(\text{singular fibers})$. For the singular fibers, a classification similar to the one for singular elliptic fibers would be helpful. However, there are no results known in the mathematics literature. Therefore we further restrict ourselves to K3 fibrations obtained as complete intersections in toric varieties. Using the methods described in [6], we find 29 examples with $h^{1,1} = 2$. The analysis of the singular fibers can be explicitly performed and yields

$$
\begin{array}{llll}
 & n=1 & \chi(X) = -28(1+2m) & 1 \le m \le 4 \\
 & n=2 & \chi(X) = -28(2+m) & 1 \le m \le 5 \\
 & n=3 & \chi(X) = -4(23+m) & 1 \le m \le 14 \\
(4) & n=4 & \chi(X) = -112 &
\end{array}
$$

With this result in our hands we proceed to determine the modular form $\Theta(q)$. First, we observe that (3) can be rewritten as

$$
\mathcal{F}_{\mathrm{K3}}(\lambda, t) = -\frac{\Theta(q)}{4\pi^2\Delta(q)} \sum_{g=0}^{\infty} S_g(G_2, \tfrac{1}{2}G_4, \ldots, \tfrac{1}{g}G_{2g}) \left(\frac{\lambda}{2\pi}\right)^{2g-2} . \tag{5}
$$

where S_g is the Schur function for the unnormalized Eisenstein series $G_{2k}(q)$ and $\Delta(q) = \eta(q)^{24}$. We notice that the relevant modular form is actually $f_{\mathrm{K3}}(q) = \frac{\Theta(q)}{\Delta(q)}$. In order to give a modular form, we need to know its weight k and the modular group under which it transforms. It turns out that the weight of f_{K3} is related to ρ by $k = -\left(1 + \frac{\rho}{2}\right)$, *i.e.* $k = -\frac{3}{2}$ for $\rho = 1$. Therefore $\Theta(q)$ has to be a modular form of half integral weight which is only defined for the congruence subgroup $\Gamma_0(4N)$. Due to $\Delta(q)$ in the denominator $f_{\mathrm{K3}}(q)$ is a nearly holomorphic form of half integral weight. There exists a particular basis for the space $M^!_{*+\frac{1}{2}}(\Gamma_0(4N)$ of such forms. For $l = 1, 2, 3, 4, 6, 7$ there exists a pair $(\{f_d(q)\}, \{g_D(q)\})$ of modular forms of weight $(\frac{3}{2} - l, l + \frac{1}{2})$ with the property that $f_d(q) = q^{-d} + O(q)$ and $g_D(q) = q^{-D} + O(1)$. Then

$$
\{f_d(q) | -d = \Box \mod 4N, d \ge 0\} \qquad \{g_D(q) | D = \Box \mod 4N, D > 0\} \tag{6}
$$

form two bases of $M^!_{*+\frac{1}{2}}(\Gamma_0(4N))$. The final result is that in terms of the basis $f_d(q)$ for $l = 3$ we can show that

$$
f_{\mathrm{K3}}(q) = f_{4n}(q) - m f_{-n^2 \mod 4n}(q) \tag{7}
$$

where n and m are determined by (4).

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Some problems from the deformation theory of vertex algebras

Karl-Georg Schlesinger

Kodaira-Spencer theory describes the holomorphic deformations of a complex manifold. Concretely, a complex structure on a smooth manifold can be defined by giving a $\overline{\partial}$-operator. A holomorphic deformation then means that we deform $\overline{\partial}$ into a new operator

$$\overline{\partial} + A^i \partial_i$$

which is supposed to act as antiholomorphic derivation, again, on the deformed manifold. Here, A is a $(1,0)$-vector field with values in $(0,1)$-forms. The condition on $\overline{\partial} + A^i \partial_i$ to define a new complex structure translates into the Kodaira-Spencer equation

$$\overline{\partial} A + \frac{1}{2} [A, A] = 0$$

where the bracket means the bracket on vector fields and wedging.

In [BCOV] it was shown that the Kodaira-Spencer equation on a complex Calabi-Yau 3-fold X can be derived from an action principle and that the perturbative treatment of the resulting field theory agrees - on a perturbative level - with the string field theory of the B-model on X. To achieve this result it is essential that one can find a special gauge fixing - the Tian gauge - for the Kodaira-Spencer equation which makes the theory calculationally accessible. In tree level approximation the perturbative treatment of the field theory leads to an iterative solution of the classical Kodaira-Spencer equation. Including loop corrections, one finds an anomaly - holomorphic anomaly - which forces one to include fields of non-classical ghost number. Mathematically speaking this means that one has to pass from the moduli space of classical Kodaira-Spencer theory, given by the first sheaf cohomology $H^1(TX)$ of the holomorphic tangent bundle of X, to the extended moduli space ([Kon], [Wit]) given by the total Hochschild complex of the structure sheaf of X. The classical moduli space is included since

$$HH^n(\mathcal{O}(X)) \simeq \oplus_{p+q=n} H^p \left(\bigwedge^q TX \right)$$

Extended moduli space is conjectured to be a moduli space of (triangulated) A_∞-categories ([Kon]).

From a more physical perspective, one can understand the appearance of A_∞-structures as arising form the BRST-complex ([HM]): For the open topological string the three point correlation functions C_{ijk} can be used as structure constants to define a product

$$\phi_i \bullet \phi_j = C_{ijk}\phi_k$$

on the fields ϕ_i. Using the Ward identities, this can be shown to be an A_∞-product. For the closed topological string, one can, in addition, define an L_∞-bracket

$$[\phi_i, \phi_j] = F_{ijk}\phi_k$$

where the F_{ijk} are the three point correlation functions with one of the fields replaced by its first BRST-descendant. Together the A_∞- and L_∞-structures combine into a G_∞-structure (homotopy Gerstenhaber structure).

For the open topological string, studying the corresponding string field theory from this perspective means studying the deformation theory of the A_∞-structure. This reproduces the Hochschild complex. For the closed topological string, the corresponding deformation theory of the G_∞-structure was shown in [HM] to lead to three different, mutually incompatible, deformation complexes. Only one of them is related to deformations by the fields of the topological string while the other two are supposed to be related to turning on background fields. Especially, one of them is supposed to be related to fivebrane backgrounds. The incompatibility of the tree complexes shows that an approach modelled after Hochschild cohomology seems to have limits in the presence of background fields.

This motivates to look for a different approach. Since the structures involved directly relate to the operator product expansion, we look for a deformation theory of vertex algebras. Since it is known that background fields need the introduction of vertex operators with non-integer exponents in the series, we have to allow for the deformations to be quantum vertex algebras in the sense of [Bor]. We use the abstract approach to vertex algebras in the sense of the chiral algebras of [BD] for this. The essential product structures of a chiral algebra are then, roughly speaking, given by an inner cocommutative Hopf algebra H in a monoidal category $\mathcal{C}$. We present the deformation theory for this part of the data of a chiral algebra. In the general deformation theory, we allow the deformation of the Hopf algebra structure of H, as well, as the tensor product and the composition of $\mathcal{C}$. Under a physically plausible assumption, we can reformulate this as the deformation problem of a pair of monoidal categories $\mathcal{M}$ and $\mathcal{C}$, together with a forgetful functor

$$\mathcal{M} \to \mathcal{C}$$

We have shown in [GS] that the deformation theory is described by a system of coupled differential equations, generalizing the Kodaira-Spencer equation, and that this system can formally be derived, again, from an action principle.

We finish by some open questions: Can one find an analogue of the Tian gauge? This would allow for a perturbative treatment of the deformation theory of vertex algebras. What are the physical degrees of freedom behind the action we have found (the analogue of the B-model in Kodaira-Spencer theory)? On the deformation complex, there is additional structure beyond a (homotopy) Gerstenhaber algebra given by three ingredients: A second differential, a curvature tensor, and a representation of a vertex algebra. We explain that this is a first indication that the deformation theory might, indeed, be related to the degrees of freedom of a fivebrane.

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Heisenberg-Clifford superalgebra and instantons

U. Bruzzo

Instanton moduli spaces. Let $\mathcal{M}_0^{reg}(r,n)$ be the moduli space of framed instantons on $\mathbb{R}^4$ of rank r and instanton charge n (taken modulo gauge transformations fixing the framing). It is a smooth complex affine variety of complex dimension $2rn$. It is not compact. A "partial compactification" is obtained by adding the so-called *ideal instantons*. An ideal instanton may regarded as a collection of m points x_i in $\mathbb{R}^4$ (with $0 < m \leq k$) and a framed instanton (∇, ϕ) on $\mathbb{R}^4 - \{x_1, \ldots, x_m\}$ of instanton charge $k-m$, such that the measure associated with the curvature of the ASD connection ∇ approaches the Dirac delta concentrated at x_i when $x \to x_i$. In this way one gets a moduli space $\mathcal{M}_0(r,n)$ which is singular at the points corresponding to the ideal instantons. Resolving the singularities one obtains a space $\mathcal{M}(r,n)$ which is a quasi-projective smooth variety, and may be regarded as a moduli space parametrizing geometric objects, namely, torsion-free coherent sheaves $\mathcal{E}$ on the complex projective plane $\mathbb{P}^2$, which are locally free in a neighbourhood of a fixed line $\ell_\infty \subset \mathbb{P}^2$, and are equipped with an isomorphism $\Phi\colon \mathcal{E}_{|\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$.

Heisenberg-Clifford superalgebra. We want to define operators $q_i[u]$, where i is an integer, and u is a homology class (with compact support) in $\mathbb{R}^4$. This will act as a linear map

$$q_i[u] : H_\bullet(\mathcal{M}(r,n),\mathbb{Q}) \to H_\bullet(\mathcal{M}(r,n+i),\mathbb{Q})$$

(the integer r will be kept fixed during the whole treatment). When $i \geq 0$ we consider the cartesian product $\mathcal{M}(r,n) \times \mathcal{M}(r,n+i) \times X$ with projections

$$X \xleftarrow{p_1} \mathcal{M}(r,n) \times \mathcal{M}(r,n+1) \times X \xrightarrow{p_2} \mathcal{M}(r,n) \times \mathcal{M}(r,n+i)\,.$$

We define the closed subscheme $\mathcal{M}_r^{[n,i]}$ of $\mathcal{M}(r,n)\times\mathcal{M}(r,n+i)\times X$ whose elements are the triples $(\mathcal{E},\mathcal{E}',x)$ such that the sheaves $\mathcal{E}$, $\mathcal{E}'$ fit into an exact sequence $0\to\mathcal{E}\to\mathcal{E}'\to A_x\to 0$, where A_x is a skyscraper sheaf concentrated at the point x. We define $\omega(u)\in H_\bullet(\mathcal{M}(r,n)\times M(r,n+i),\mathbb{Q})$ by letting $\omega(u)=p_{2*}\big(p_1^*u\cap[\mathcal{M}_r^{[n,i]}]\big)$, and define the linear map $q_i[u]$ by letting

$$q_i[u](\alpha)=\pi_{2*}\big(\pi_1^*\alpha\cap\omega(u)\big)$$

where π_1, π_2 are the projections of $\mathcal{M}(r,n)\times\mathcal{M}(r,n+i)$ onto the factors. When i is negative, the operator $q_i[u]$ is defined by replacing the product $\mathcal{M}(r,n)\times\mathcal{M}(r,n+i)$ with the product $\mathcal{M}(r,n+i)\times\mathcal{M}(r,n)$ and proceeding as above.

We define among these operators the graded commutator

$$\big[q_i[u],q_j[v]\big]=q_i[u]\circ q_j[v]-(-1)^{\deg(u)\cdot\deg(v)}q_j[v]\circ q_i[u]\,.$$

Theorem. *The operators $q_i[u]$ verify the commutation relations*

$$\big[q_i[u],q_j[v]\big]=(-1)^{ri-1}ri\;\langle u,v\rangle\;\delta_{i+j,0}\cdot Id\,, \tag{1}$$

where $\langle u,v\rangle$ is the intersection product of the homology classes u, v.

This result has been proved in the case $r=1$ by Nakajima [6] and Grojnowski [5].

The case $r=1$. In this case the moduli space $\mathcal{M}(1,n)$ reduces to the Hilbert scheme $\mathcal{H}_n=(\mathbb{C}^2)^{[n]}$ parametrizing 0-dimensional subschemes of length n of the space $\mathbb{C}^2$. The commutation relations (1) may be proved as follows. Define the closed subschemes $\mathcal{M}_n$ and $\mathcal{M}_n(p)$ of $\mathcal{H}_n$ (where p is a point in $\mathbb{C}^2$) as follows:

$$\mathcal{M}_n=\{Z\in\mathcal{H}_n/\ Z\text{ is topologically supported at one point }\},$$

$$\mathcal{M}_n(p)=\{Z\in\mathcal{H}_n/\ Z\text{ is topologically supported at }p\}.$$

Briançon has shown [2] that $\mathcal{M}_n$ and $\mathcal{M}_n(p)$ are irreducible projective varieties, with $\dim(\mathcal{M}_n)=n+1$ and $\dim\mathcal{M}_n(p)=n-1$. Moreover, Ellingsrud and Strømme [4] have computed the intersection product of these subschemes of $\mathcal{H}_n$, obtaining

$$[\mathcal{M}_n]\cap[\mathcal{M}_n(p)]=(-1)^{n-1}n\,.$$

This computes the constants in the commutation relations (1), since $q_n[\mathrm{pt}]\mathbb{I}=[\mathcal{M}_n(p)]$ and $q_n[X]\mathbb{I}=[\mathcal{M}_n]$, where $\mathbb{I}$ is the generator of $H_\bullet(\emptyset)$, and X is the fundamental class in the homology of $\mathbb{R}^4$ with compact support.

The constants may also be computed by noting that

$$[\mathcal{M}_n]\cap[\mathcal{M}_n(p)]=s_{n-1}(\mathcal{M}_n(p))$$

where $s_{n-1}(\mathcal{M}_n(p))$ is the top Segre class of the scheme $\mathcal{M}_n(p)$ [7].

The instanton case. For $r>1$ one introduces the subschemes of $\mathcal{M}(r,n)$

$$\mathrm{Quot}(r,n)=\big\{\mathcal{O}_X^{\oplus r}\to A\to 0\big\},\qquad \mathrm{Quot}_p(r,n)=\big\{\mathcal{O}_X^{\oplus r}\to A_p\to 0\big\}\,,$$

where A is a rank zero sheaf whose topological support is a point, and A_p is a rank zero sheaf whose topological support is a fixed point p. The sets $\mathrm{Quot}(r,n)$ and $\mathrm{Quot}_p(r,n)$ are irreducible projective varieties of dimension $rn+1$ and $rn-1$, respectively [1, 3]. Again the intersection product $[\mathrm{Quot}(r,n)]\cdot\mathrm{Quot}_p[(r,n)]$

computes the constants in the commutation relations (1). Moreover, also in this case the one has the identification

$$[\mathrm{Quot}(r,n)] \cdot \mathrm{Quot}_p[(r,n)] = s_{2n-1}(\mathrm{Quot}_p(r,n))$$

The idea is to compute this Segre classe using a Bott formula for the equivariant cohomology of the moduli space $\mathcal{M}(r,n)$ with respect to a naturally defined action of $\mathbb{C}^*$ [8].

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Nahm vs. Fourier-Mukai

U. Bruzzo

Other talks in this workshop have dealt with the Fourier-Mukai transform in a very general setting, i.e., as a functor

$$\Phi^{\mathcal{K}^\bullet}_{X\to Y} \colon D^-(X) \to D^-(Y)$$

where X and Y are smooth projective varieties, D^- denotes the derived category of complexes of coherent sheaves bounded from the right, and $\mathcal{K}^\bullet$ is a complex in $D^-(X \times Y)$. The functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is defined as

$$\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{E}^\bullet) = \mathbf{R}\pi_{Y*}(\pi_X^*(\mathcal{E}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)$$

where π_X, π_Y are the projections onto the components of $X \times Y$.

Under suitable conditions, the Fourier-Mukai transform may be recast into a differential-geometric setting, and then it may be shown to coincide, in a proper sense, with a generalization of the so-called *Nahm transform*, introduced by Nahm in 1983 [9] (see also [6, 10]). This correspondence was noted for the first time in [7] in the case of Abelian surfaces.

Let X, Y be compact Kähler manifolds, E a holomorphic vector bundle on X and Q a holomorphic vector bundle on $X \times Y$. Let us denote by $\mathcal{E}$ and $\mathcal{Q}$ the sheaves of holomorphic sections of the corresponding bundles. One can define the Fourier-Mukai transform of the sheaf $\mathcal{E}$ as the sheaf on Y

$$\hat{\mathcal{E}} = \Phi^{\mathcal{Q}}_{X\to Y}(\mathcal{E})\,. \tag{1}$$

Let us see how one defines the Nahm transform in this setting. We assume that X has trivial canonical bundle. One considers the relative Dolbeault complex $\Omega^{0,\bullet}_{X\times Y/Y}$ of the projection $\pi_Y\colon X\times Y\to Y$. Moreover, after equipping E and Q with hermitian metrics, and introducing the corresponding Chern connections on both bundles, one twists the Dirac operator obtained from the complex $\Omega^{0,\bullet}_{X\times Y/Y}$ with the Chern connection on $\pi_X^*E\otimes Q$. In this way one gets a family of (twisted) Dirac operators $\mathcal{D}=\{D_y\}_{y\in Y}$.

Theorem. *Assume that* $\ker(\mathcal{D})=0$. *Then the index vector bundle* $\hat{E}=\operatorname{ind}(\mathcal{D})$ *has a holomorphic structure such that the sheaf of holomorphic sections of* $\hat{E}$ *is isomorphic to the Fourier-Mukai transform (1).*

For a proof of this result the reader may refer to [1, 4, 5].

The bundle $\hat{E}$ carries a naturally defined hermitian metric $\hat{h}$ since the fibre $\hat{E}_y$ at a point $y\in Y$ is the kernel of the map

$$D_y^*\colon \Gamma(E\otimes S_-\otimes Q_y)\to\Gamma(E\otimes S_+\otimes Q_y)$$

where $Q_y=Q_{X\times\{y\}}$ and

$$S_+=\bigoplus_{k\,\text{even}}\Omega^{0,k}_X\,,\qquad S_-=\bigoplus_{k\,\text{odd}}\Omega^{0,k}_X\,.$$

Let h be the hermitian metric on E.

Definition. *The pair* $(\hat{E},\hat{h})$ *is the generalized Nahm transform of the pair* (E,h).

Let $\hat{\nabla}$ be the Chern connection induced by the hermitian metric $\hat{h}$ and by the holomorphic structure of $\hat{E}$. In his paper [9] Nahm deals with the case when X is a complex torus (of complex dimension 2), Y the dual torus, and Q the Poincaré bundle on $X\times Y$. Nahm shows that whenever the connection ∇ is an instanton, then the transformed connection $\hat{\nabla}$ is an instanton as well. This result may be generalized as follows.

Definition. *Let* Z *be a hyperkähler manifold, and* F *a vector bundle on* Z. *A connection on* F *is said to be a quaternionic instanton if its curvature is of type (1,1) with respect to every complex structure in the hyperkähler structure of* Z.

With reference to the previous construction, assume that both X and Y are hyperkähler manifolds, and that the Chern connections on E and Q are quaternionic instantons.

Theorem. *The Chern connection* $\hat{\nabla}$ *on* $\hat{E}$ *is a quaternionic instanton on* Y.

This result parallels the fact that under suitable conditions the Fourier-Mukai transform preserves the condition of stability [8, 2, 3].

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Non-commutative geometry and D-Brane systems

Calin Lazaroiu

I present a noncommutative-geometric description of the semiclassical dynamics of finite topological D-brane systems. Starting from the formulation in terms of A_∞ categories, I show that such systems can be described by the noncommutative symplectic supergeometry of $\mathbf{Z}_2$-graded quivers, and give a synthetic formulation of the boundary part of the generalized WDVV equations. In particular, a faithful generating function for integrated correlators on the disk can be constructed as a linear combination of quiver necklaces, i.e. a function on the noncommutative symplectic superspace defined by the quiver's path algebra. This point of view allows one to construct extended moduli spaces of topological D-brane systems as non-commutative algebraic 'superschemes'. They arise by imposing further relations on a $\mathbf{Z}_2$-graded version of the quiver's preprojective algebra, and passing to the subalgebra preserved by a natural group of symmetries.

Stability conditions on non-compact Calabi-Yau varieties

Emanuele Macri

1. Stability conditions on triangulated categories

Stability conditions were introduced by Bridgeland ([8, 9]) in 2003, relying on physical ideas due to Douglas ([14]), Aspinwall and others in the context of the study of D-branes in string theory (a nice review can be found in [2]). A stability condition on a triangulated category $\mathcal{T}$ is given by abstracting the usual properties of μ-stability for sheaves on projective varieties; more precisely, one introduces the notion of slope, using a group homomorphism from the Grothendieck group $K(\mathcal{T})$ to $\mathbb{C}$ (called central charge), and of semistable object and then requires that a stability condition is compatible with the shift functor and has the Harder-Narasimhan property (i.e., every nonzero object has a filtration in semistable ones).

The main result of Bridgeland's paper [8] is that these stability conditions can be described via a kind of moduli space of stabilities (it is better to say a parameters space: we will be more clear about this below). This moduli space becomes a (possibly infinite-dimensional) manifold, which is called stability manifold and denoted by $\mathrm{Stab}\,\mathcal{T}$, if a technical condition (local finiteness) is assumed. This condition allows our stability conditions to have also finite Jordan-Hölder filtrations in stable objects. For each connected component Σ of $\mathrm{Stab}\,(\mathcal{T})$ there is a linear subspace $V(\Sigma)$ of $(K(\mathcal{T})\otimes\mathbb{C})^*$ (with a well-defined topology) and a natural local homeomorphism $\mathcal{Z}$ from Σ to $V(\Sigma)$ which identify the tangent spaces to Σ with $V(\Sigma)$. In almost all examples we will deal with in the sequel, $K(\mathcal{T})$ will be finite dimensional. In general, referring to stability manifolds, one often considers only finite-dimensional submanifolds of it (for example, one asks factorizations of the central charge through the Chern character map, but, it is not clear what is the natural target space onto which this Chern character should map $K(\mathcal{T})$). There are no known examples where $V(\Sigma)$ is different from $(K(\mathcal{T})\otimes\mathbb{C})^*$ or where $\mathrm{Stab}\,(\mathcal{T})$ is neither connected nor contractible.

We want to point out some remarks and some mathematical motivations for this definition. First of all the key point is that, if X is a smooth projective Calabi-Yau 3-fold, then $\mathrm{Stab}\,(X) := \mathrm{Stab}\,D(X)$ (where $D(X) := D^b(Coh(X))$), modulo the natural action of the exact autoequivalences $\mathrm{Aut}\,eq(D(X))$, should be thought as an approximation of the "assumed" stringy Kähler moduli space, or better some extended version of it ([12]). Understanding the relations between these two spaces is the main conjecture and one of our goals is to try to study some evidences for it (for example to find a Frobenius manifold structure on $\mathrm{Stab}\,(X)$, or better some weaker version, which probably has something to do with the Cecotti-Vafa-Dubrovin structures described by Hertling in [16]). Unfortunately, up to now, no example is known of stability conditions on smooth projective Calabi-Yau 3-folds. But some hints on the validity of this conjecture can be searched in other ways. One way is to lower dimensions and to see what happens for elliptic curves and K3 and abelian surfaces. In these cases one can get an almost complete description of the stability manifold which effectively provides some evidences for this conjecture. Moreover, it is nice to see how stability conditions are related to non-commutative 2-tori (in the example of elliptic curves) or to the conjecture of describing the group of exact autoequivalences of the derived categories of K3 surfaces[1] ([22], [23], [9]).

Another way is to pass to the local case, which will be our main topic of interest. Before doing so, we will present other remarks on stability conditions. A further good point about $\mathrm{Stab}\,(\mathcal{T})$ is that it encodes the notion of "continuous" family of t-structures on $\mathcal{T}$. Here comes out the problem we observed before: this notion of continuity does not behave well, at least in the non Calabi-Yau case.

[1]Note that we always have an action of $\mathrm{Aut}\,eq(\mathcal{T})$ on $\mathrm{Stab}\,(\mathcal{T})$. Hence, stability manifolds can be useful in studying the structure of the autoequivalences group on every triangulated category.

It is easy to see that, for example, some discrete invariants of the abelian hearts, which is natural to assume they remain constant in "families" (like Hochschild cohomology, which has to do with deformations and so with stability conditions), are not preserved (an easy example is Stab$(\mathbb{P}^1)$, (3.1)). Probably this kind of problems boils down to the fact that the definition of stability condition should be rewritten in the context of DG-categories, which is a very interesting topic for future investigation.

Another application of stability conditions is to the construction of moduli spaces of complexes. There are several examples that show that the notion of moduli spaces of complexes is necessary in birational algebraic geometry (a good one is Bridgeland's construction of the flop of a Calabi-Yau threefold as a moduli space of perverse point sheaves [7]). Hence, stability conditions naturally arise in this context. For example, it is possible to reinterpret Bridgeland's construction for the local model of a flop, the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, in the language of stability conditions. In general, the notion of moduli spaces of stable complexes (with respect to a fixed stability condition) is not so precise as a question up to now, but some work has been done in this direction ([17], [19], [1]).

Finally, we want to notice the connections between stability conditions and representation theory of finite-dimensional algebras ([10, 11], [3], [20]): in most known examples where Stab$(\mathcal{T})$ can be described, this is due to the equivalence of $\mathcal{T}$ with the bounded derived category $D(A)$ of finitely generated (right) modules over an associative algebra A. Moreover, on $D(A)$, there is a natural open set of Stab(A) = Stab$(D(A))$ in which Bridgeland-Douglas stability reduces to King θ-stability ([18]).

2. Some examples of stability manifolds of non-compact Calabi-Yau varieties

2.1. Stab $(\mathcal{O}_{\mathbb{P}^1}(-2))$. A very interesting object to study in the context of stability conditions is the stability manifold of a non-compact Calabi-Yau manifold X. We try to give here and in the subsequent paragraph some hints on how this is related to other parts of derived category theory, to mirror symmetry and so to the main conjecture on the relations with the stringy Kähler moduli space. The idea is to consider non-compact Calabi-Yau varieties which contains a Fano variety S whose derived category is generated by an exceptional collection of sheaves (for example, the total space of the canonical bundle ω_S on a del Pezzo surface S). In this case it easy to construct stability conditions on the derived category $D_0(X)$ of complexes with support contained on S. Stability conditions on X are related, up to autoequivalences of $D_0(X)$, to stability conditions on S, and there it is enough to define the values of the central charge on the simple modules over the algebra associated to a strong exceptional collection ([6], [20], [10]).[2] As we have said in the previous section, stability conditions do not behave so well for non Calabi-Yau

[2]The precise relation is that exceptional objects on S correspond to spherical ones on X and modules over the path-algebra of a quiver correspond to nilpotent modules over the path-algebra of the completed quiver ([10]).

manifolds. In this examples this fact becomes evident, since the "degenerate" stability conditions on S (i.e. stability conditions associated to unfaithful bounded t-structures) become "non-degenerate" on passing to X.

The first example we encounter is $S = \mathbb{P}^1$, $X = \mathcal{O}_{\mathbb{P}^1}(-2)$ ([21], [20], [10, 11]). We have that $D(\mathbb{P}^1)$ is generated by $\mathcal{O}(k)[1]$ and $\mathcal{O}(k+1)$, for k fixed integer. Hence, one can get easily hearts of bounded t-structures on $D(\mathbb{P}^1)$ by considering the extension-closed subcategories $\mathcal{A}_{k,p}$ generated by $\mathcal{O}(k)[p+1]$ and $\mathcal{O}(k+1)$, for $p \geq 0$. Then $\mathcal{A}_{k,0} \cong$ mod-A, where A is the path-algebra associated to the Kronecker quiver ($D(\mathbb{P}^1) \cong D^b(\text{mod-}A)$) and for $p > 0$ is $Vect_{\mathbb{C}} \oplus Vect_{\mathbb{C}}$ and so it is unfaithful (i.e. $D(\mathbb{P}^1) \ncong D^b(Vect_{\mathbb{C}} \oplus Vect_{\mathbb{C}})$). Using these t-structures one easily defines stability conditions on $D(\mathbb{P}^1)$ and proves that the stability conditions that arise in this way are the only one, up to the action of the universal cover $\widetilde{GL^+(2,\mathbb{R})}$ of $GL^+(2,\mathbb{R})$. In particular, $\text{Stab}\,(\mathbb{P}^1) \cong \mathbb{C}$. Passing to $\text{Stab}\,(X)$ "eliminates" the degenerate t-structures. It can be proved that $D_0(X) \cong D^b(\mathcal{A})$, for every heart of t-structures associated to stability conditions in the connected component Σ of $\text{Stab}\,(X)$ containing the stability conditions induced by those of $\mathbb{P}^1$.

We now want (in order to relate $\text{Stab}\,(X)$ to the stringy Kähler moduli space) to construct an associative product on the tangent spaces in the points of Σ. Our guess is the following. Take a stability condition $\sigma \in \Sigma$ and let $\mathcal{A}_\sigma$ be the heart of the t-structure associated to it. First of all there exists a Chern character from $K(D_0(X)) \otimes \mathbb{C}$ with values in the Hochschild homology. In this case this map is an isomorphism of vector spaces. Since X is Calabi-Yau, $\mathrm{HH}_\bullet(\mathcal{A}_\sigma)^* \cong \mathrm{HH}^\bullet(\mathcal{A}_\sigma)$. This identifies in a natural way the tangent space to σ to $\mathrm{HH}^\bullet(\mathcal{A}_\sigma)$. Now, the Hochschild cohomology has a natural product. The point is that all the operation we have made are compatible at the level of DG-categories. Therefore, we get that the ring structures of the tangent spaces are compatible. This should give a product structure on $T\Sigma$. It seems plausible that this product is related to Gromov-Witten invariants on X. Probably an argument of Caldararu (see also [5] for related subject) should imply that Hochschild cohomology, in this non-compact case, is isomorphic, as a ring, to orbifold cohomology ([13], [24]) of the orbifold $[\mathbb{C}^2/\mathbb{Z}_2]$ whose crepant resolution is X.[3]

This product should define a sort of weak Frobenius structure on $\text{Stab}\,X$ ([16], [4]). When restricted to an appropriate submanifold Γ of $\text{Stab}\,(X)$ this should coincide with the picture that Bridgeland argued in [12].

Now, all this structure should generalize to other Calabi-Yau manifolds. In the next section we will concentrate on $\text{Stab}\,(\mathcal{O}_{\mathbb{P}^2}(-3))$. Here we still consider surfaces. Natural examples to consider are all the local crepant resolutions Y_G of kleinian singularities $\mathbb{C}^2/G$, where $G\mathbb{Z}SL(2,\mathbb{C})$ is a finite group. In this case again it is simple to give a precise description of a connected component of $\text{Stab}\,(Y_G)$ ([11]). But computations with Hochschild (co)homology become more complicated (one has to compute the Hochschild cohomology of an A_∞-algebra, instead of an

[3]Note that the Hochschild cohomology is also isomorphic to the Hochschild cohomology of the stack $[\mathbb{P}^1/X^*]$ (the stack associated to the trivial action of the dual of X on $\mathbb{P}^1$) via the equivalence between the corresponding abelian categories of coherent sheaves.

algebra, as in the case of X). But the general features should remain the same. Moreover, restricting as before to a particular submanifold Γ_G of Stab(Y_G), the product should induce a Frobenius structure. It seems an interesting problem to see the relations between the Frobenius manifold Γ_G and the one associated to G, arising from Dubrovin's almost duality ([15]).

2.2. **Stab$(\mathcal{O}_{\mathbb{P}^2}(-3))$.** A more interesting and difficult example of non-compact Calabi-Yau manifold is $S = \mathbb{P}^2$, $X = \mathcal{O}_{\mathbb{P}^2}(-3)$. Our hope is that our techniques of the previous paragraph could be extended to this case. Some problems arise. First of all the description of Stab$(\mathbb{P}^2)$ is more complicated ([20]). We have, as in the case of $\mathbb{P}^1$, that, given a complete strong exceptional collection on $D(\mathbb{P}^2)$, we can define a stability condition by looking at the t-structure induced by the equivalence $D(\mathbb{P}^2) \cong D(\text{mod-}A)$, where A is the path-algebra of the quiver (with relations) associated to the exceptional collection. But this time the combinatorics of the intersections of the open subsets of Stab$(\mathbb{P}^2)$ associated to such t-structures is more complicated and essentially involves the fact of knowing the action of the braid group on three generators on exceptional collections. We only have, up to now, a description of an open subset of a connected component of Stab$(\mathbb{P}^2)$. It is probably true that this open subset covers the full connected component. As before, going to X, eliminates degenerate stabilities and transforms mutations in functors (in the so called twist functors of [25]). Now, the description of the tangent spaces in term of Hochschild cohomology probably works the same (up to technical computations problems). But now the induced product structure should be related (as Bridgeland conjectured in [12]) to the quantum cohomology of $\mathbb{P}^2$, and this relation has to be clarified.

Similar arguments could be done for S a del Pezzo surface, X the total space of its canonical bundle. A partial description of Stab(S) is given in [20]. Here it is worth nothing that in $D(S)$ there are more unfaithful t-structures: the ones associated to exceptional collections which are not strong. The interesting thing is that, going to X, these degenerations probably disappear and, like in the case of $\mathcal{O}_{\mathbb{P}^2}(-3)$, one only has stability conditions associated to algebras.

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