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Heat Kernels, Stochastic Processes and Functional Inequalities

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Abstract. The conference brought together mathematicians belonging to several fields, essentially analysis, probability and geometry. One of the main unifying topics was certainly the study of heat kernels in various contexts: fractals, manifolds, domains of the Euclidean space, percolation clusters, infinite dimensional spaces, metric measure spaces.

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Introduction by the Organisers

The workshop *Heat kernels, stochastic processes and functional inequalities* was organized by Thierry Coulhon (Cergy), Bruno Franchi (Bologna), Takashi Kumagai (Kyoto) and Karl-Theodor Sturm (Bonn). It was held from November 27th to December 3nd. The meeting was attended by 56 participants from Australia, Austria, Canada, Finland, France, Germany, Italy, Japan, Poland, Switzerland, United Kingdom, and USA. This workshop was sponsored by the European Union, which allowed the invitation of 18 young people, who contributed positively to the atmosphere of the meeting.

The conference brought together mathematicians belonging to several fields, essentially analysis, probability and geometry. One of the main unifying topics was certainly the study of heat kernels in various contexts: fractals, manifolds, domains of the Euclidean space, percolation clusters, infinite dimensional spaces, metric measure spaces. Some related aspects of geometric analysis were also considered such as L^p-cohomology and mass transportation. There was a stimulating exchange between probabilistic and analytic points of view, together with a geometric emphasis in most of the problems. We had 5 one hour survey lectures and

21 thirty-five minutes talks. A lot of time was devoted to discussions and exchange of ideas.

Among the highlights were relations between mass transportation, generalized Ricci bounds and contraction properties, connections between heat kernel estimates and percolation clusters, non-linear aspects of diffusions, functional analytic approach to parabolic regularity, geometric and functional analytic aspects of infinite dimensional analysis.

This diversity of topics and mix of participants stimulated many extensive and fruitful discussions. It also helped initiate new collaborations, in particular for the younger researchers.

Heat Kernels, Stochastic Processes and Functional Inequalities

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Abstracts

The Kato square root problem: a review and new developments

Pascal Auscher

The following theorem has been proved in [2] in full generality.

Let $B = B(x)$ be an $n \times n$ ($n \geq 1$ is all this paper) matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda|\xi|^2 \leq \Re e\, B\xi \cdot \xi^* \text{ and } |B\xi \cdot \zeta^*| \leq \Lambda|\xi||\zeta|, \tag{1}$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \leq \Lambda < \infty$. Here, $u \cdot v = u_1v_1 + \cdots + u_nv_n$ and u^* is the complex conjugate of u so that $u \cdot v^*$ is the usual inner product in $\mathbb{C}^n$ and, therefore, $B\xi \cdot \zeta^* \equiv \sum_{j,k} b_{j,k}(x)\xi_k\, \overline{\zeta}_j$. We define a second order divergence form operator

$$Jf \equiv -\mathrm{div}(B\nabla f) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{i,j}\,\frac{\partial}{\partial x_j}\right), \tag{2}$$

which we interpret in the usual weak sense via a sesquilinear form, where ∇ is the gradient operator and div the divergence operator, its negative adjoint.

The accretivity condition (1) enables one to define a square root $J^{1/2} \equiv \sqrt{J}$, and a fundamental question is to determine whether one can solve the "Kato square root problem", i.e. establish the estimate

$$\|\sqrt{J}f\|_2 \sim \|\nabla f\|_2, \tag{3}$$

where $\sim$ is the equivalence in the sense of norms, with constants C depending only on n, λ and Λ, and $\|f\|_2 = (\int_{\mathbb{R}^n} |f(x)|_H^2\, dx)^{1/2}$ denotes the usual norm for functions on $\mathbb{R}^n$ valued in a Hilbert space H.

Theorem 1. *For any operator as above the domain of $\sqrt{J}$ coincides with the Sobolev space $W^{1,2}(\mathbb{R}^n)$ and $\|\sqrt{J}f\|_2 \sim \|\nabla f\|_2$.*

We first mention that an abstract Hilbert space formulation (where B, ∇ are replaced by abstract operators) was disproved by McIntosh, hence this is really a theorem about elliptic differential operators. In fact, harmonic analysis methods involving square function estimates, Carleson measures, maximal functions, variants of the $T(b)$ theorem and stopping-time arguments are at the heart of the proof. The case $n = 1$ was proved back to 1982 in a celebrated paper of Coifman, McIntosh and Meyer.

The estimate contained in the statement applies to time-evolution parabolic, hyperbolic or elliptic partial differential equations involving J as the main elliptic part and where the matrix B could also depend smoothly on time. We review such applications in this lecture.

Let us concentrate here on one aspect, namely elliptic problems. This estimate also contains the L^2 boundedness of a boundary Neumann to Dirichlet map for the following elliptic problem. Consider the elliptic problem

$$\begin{cases} \partial_t^2 u(x,t) - Ju(x,t) = 0 \text{ in } \mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^n \times (0,\infty)\} \\ u(x,0) = f(x) \text{ on } \mathbb{R}^n. \end{cases} \tag{4}$$

Then, a solution is given by the semigroup equation $u(x,t) = e^{-t\sqrt{J}}f(x)$ and the Neumann data equals $-\partial_t u(x,0) = \sqrt{J}f(x)$ so that the Neumann to Dirichlet map alluded to is $\nabla\sqrt{J}^{-1}$. Hence, one can solve the Neumann problem with data in L^2 with an estimate on ∇u and $\partial_t u$. Also this map is invertible, with inverse $-\sqrt{J}^{-1}\text{div}B$ acting on gradient fields, and one can solve the Dirichlet problem with Dirichlet data f having gradients in L^2 with an estimate on ∇u and $\partial_t u$ (this is called the regularity problem). This holds for this general class of equations with complex coefficients. Furthermore, one can impose the solution to satisfy some control such as quadratic estimates and then uniqueness can be discussed.

This elliptic problem can be embedded in a larger class of problems where we allow mixed second order derivatives

$$\begin{cases} Lu(x,t) = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ u(x,0) = f(x) \text{ on } \mathbb{R}^n. \end{cases} \tag{5}$$

where, now,

$$L = -\text{div}A\nabla \equiv -\sum_{i,j=1}^{n+1} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right)$$

(we use the notational convention that $t = x_{n+1}$ and $\partial/\partial_t = \partial/\partial_{x_{n+1}}$ and ∇, div now denote the full gradient and divergence operators on $\mathbb{R}^{n+1}$) where $A = A(x)$ is an $(n+1)\times(n+1)$ matrix of complex-valued L^∞ coefficients, defined on $\mathbb{R}^n$ (i.e., independent of the t variable), and satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \Re e\, A(x)\xi\cdot\xi^*, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda, \tag{6}$$

for some $\lambda > 0$, $\Lambda < \infty$, and for all $\xi \in \mathbb{C}^{n+1}$, $x \in \mathbb{R}^n$. The preceding case, we say that A is of Kato type, corresponds to $a_{n+1,n+1} = 1$ and the non-diagonal coefficients $a_{i,n+1} = a_{n+1,i} = 0$ if $i = 1,\ldots,n$. We refer to the forthcoming articles [1] for references on what follows as well for the statements which are new.

In this situation, there no longer is semigroup structure for solving. The construction of the Neumann to Dirichlet map can be made formally using limits of the layer potentials at the boundary. However, the study of boundedness and invertibility of the Neumann to Dirichlet map becomes much harder in general.

A trivial case is when A is constant, as this map and its inverse are bounded (on $L^2(\mathbb{R}^n)$) singular integrals of convolution type, that is Calderón-Zygmund operators. For example, if A is the identity, they are associated to the standard Riesz transforms on $\mathbb{R}^n$.

Another situation where things are known is the case where A has *real and symmetric* L^∞ coefficients. Then, Jerison and Kenig solved (I will not be precise in saying in which class) the Dirichlet problem with L^2 data by the method of harmonic measure. Later Kenig and Pipher solved (again, in some class) both the Neumann and regularity problems with datum in $L^2(\mathbb{R}^n)$ using the previous result of Jerison and Kenig, and Rellich identities. These identities furnish the *a priori* comparison between tangential and normal components of the gradient of any solution at the boundary, namely

$$\int_{\mathbb{R}^n} |\partial_\nu u(x,0)|^2\, dx \sim \int_{\mathbb{R}^n} |\nabla_T u(x,0)|^2\, dx, \tag{7}$$

where ∇_T is the tangential gradient, that is the gradient with respect to x and ∂_ν is the exterior conormal derivative, that is $-e_{n+1}\cdot A\nabla$ with e_{n+1} the unit upward vector along the x_{n+1} co-ordinate. Hence, one controls the full gradient of a solution at the boundary provided one controls either its conormal component (Neumann datum) or its tangential component (regularity datum), whence solvability. This method does not rely on boundedness of the Neumann to Dirichlet map. However, if this boundedeness could be proved, then invertibility follows directly from the Rellich estimate. Such a boundedness is far from trivial. To indicate why, let us mention that this class of L's includes pull-backs of the Laplace operator under a Lipschitz change of coordinate, and thus the class of Neumann to Dirichlet maps includes double layer potentials on Lipschitz graphs from potential theory, whose boundedness is due to Coifman, McIntosh and Meyer in the 1982 paper mentioned before. Then, Verchota understood in 1984 how to invert the Neumann to Dirichlet map incorporating the Rellich identities, so that he could use the method of layer potential to solve both the Neumann and regularity problems.

In the last 20 years some new technology has been developed to reprove the boundedness of the double layer potential, one of them is called the $T(b)$ theorem, a criterion to check L^2 boundedness for singular integrals (no longer of convolution type). For the purpose of solving the Kato problem and also some other geometric problems in complex function theory (the Painlevé problem), the $T(b)$ theorem has been adapted to various settings and more powerful and flexible versions are available nowadays. One of them can be proved and applied to obtain the

Theorem 2. *Let L be real and symmetric, then the Neumann to Dirichlet map is bounded and invertible.*

As Verchota did in his situation, we can reprove the results of Kenig and Pipher mentioned above by the method of layer potentials. But that is not all. We can also develop a perturbation theory by allowing the coefficients of A to vary in L^∞ norm. Another type of perturbation theory allowing the coefficients to depend in a minimally smooth way on the transverse variable $t = x_{n+1}$ was developed by R. Fefferman, Kenig, Pipher and then Kenig, Pipher.

Theorem 3. *Assume that $A(x)$ is either constant and complex, or real and symmetric with L^∞ coefficients or real of Kato type (see above) with L^∞ coefficients. Then there exists $\varepsilon > 0$ depending only on the ellipticity constants of A such that if $A'(x)$ is another complex coefficients matrix with*

$$\|A' - A\|_\infty \le \varepsilon \tag{8}$$

then the Neumann to Dirichlet map corresponding to $L' = -divA'\nabla$ *is bounded and invertible. Hence, the Neumann and Regularity problems for* L' *with* L^2 *data are solvable.*

In fact, this theorem is a consequence of a more general statement that would be too complicated to explain here. The proof of boundedness is done by a perturbation method comparing the layer potentials associated to A and A'. A remarkable point of this proof is that the Kato part of A', that is the matrix obtained by setting the non-diagonal term to 0 and replacing $a'_{n+1,n+1}$ by 1, plays an important role and in fact many of the estimates proved to establish Theorem 1, and some similar ones, are instrumental in the argument. The proof of invertibility for the perturbed Neumann to Dirichlet map is done via the method of continuity. This is here that we need to have handled complex coefficients so as to use tools of analytic functions such as the Cauchy formula.

To finish, we mention that in [1], we also develop another approach to obtain solvability of Neumann and Regularity problems (and more general transmission problems with Lipschitz interface). More precisely we imbed the elliptic problem in a first order systems (generalized Cauchy-Riemann equations) in some exterior algebra Λ. This first order system takes the form

$$\partial_t F + TF = 0 \quad \text{in } \mathbb{R}^{n+1}_+ \tag{9}$$

where F is Λ-valued and T is some bi-sectorial Dirac-type operator on $L^2(\mathbb{R}^n, \Lambda)$ that is independent of t. In this setting, there is no layer potential but there is an essential operator, $sgn(T) = T\sqrt{T^2}^{\,-1}$, and the goal is to prove that it is a bounded operator on $L^2(\mathbb{R}^n, \Lambda)$. The solvability of the BVP is equivalent to having specific numbers in the resolvent set of another related operator whose boundedness follows from that of $sgn(T)$. And openness of resolvent sets allows perturbation. This allows somehow more general perturbation results than the ones stated above.

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Heat kernels on percolation clusters

Martin Barlow

Let G be an infinite (connected) graph. For $x, y \in G$ let $\mu_{xy} = 1$ if $\{x, y\}$ is edge, and $\mu_{xy} = 0$ otherwise. Define the *vertex degree* $\mu(x) = \sum_y \mu_{xy}$. Assume $\mu(x) < \infty$ for all $x \in G$, and extend μ to a measure on G. Set $V(x, r) = \mu(B(x, r))$. We define a Dirichlet form by:

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y} (f(x) - f(y))^2 \mu_{xy}.$$

The continuous time simple random walk on G (CTSRW) on G is the process $Y = (Y_t, t \in [0, \infty), P^x, x \in G)$ associated with $(\mathcal{E}, L^2(G, \mu))$. This process waits at a point x for an exponential time with mean 1, then moves to $y \sim x$ with probability μ_{xy}/μ_x. The *heat kernel* on G is

$$q_t(x, y) = P^x(Y_t = y)\mu_y^{-1}.$$

Now consider bond percolation on $\mathbb{Z}^d$; write $\mathcal{C}(x)$ for the open cluster containing x, and let $\theta(p) = \mathbb{P}_p(|\mathcal{C}(x)| = \infty)$. Recall that in the supercritical regime ($p > p_c$) there exists a unique infinite cluster, which we denote by $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$. The CTSRW on the infinite percolation cluster $\mathcal{C}_\infty$ is just the CTSRW on the graph $(\mathcal{C}_\infty(\omega), E(\omega)|_{\mathcal{C}_\infty(\omega)})$; denote its heat kernel by $q_t^\omega(x, y)$.

In this talk I considered the heat kernel aspects of the following theorem

Theorem 4. [1] *Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exist r.v. $N_x(\omega)$ with $\mathbb{P}_p(N_x \geq n) \leq c \exp(-n^{\varepsilon_d})$ and (non-random) constants $c_i = c_i(d, p)$ such that the transition density of X satisfies, for $x, y \in \mathcal{C}_\infty(\omega)$:*

$$\frac{c_1}{t^{d/2}} e^{-c_2|x-y|^2/t} \leq q_t^\omega(x, y) \leq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t}, \qquad (GE)$$

provided

$$t \geq N_x(\omega), \text{ and } t \geq |x - y|.$$

Let Q be a (large) box in $\mathbb{Z}^d$ with side r, $\mathcal{C}(Q)$ be the largest connected cluster in Q, and Q' be a box with side $(9/10)r$ and the same centre as Q. We use four properties of $\mathcal{C}(Q)$; these all hold with probability at least $1 - e^{-cr^\delta}$.

1. 'Good volume':

$$c_1 r^d \leq \mu(\mathcal{C}(Q)) \leq c_2 r^d. \qquad (V_d)$$

2. 'Good distances': $|x - y|_1 \leq d_\omega(x, y) \leq c_2|x - y|_1$ for $x, y \in Q' \cap \mathcal{C}_\infty$.
3. 'Good isoperimetric inequality'. If $A \subset Q \cap \mathcal{C}_\infty$ with $\mu(A) \geq \frac{1}{2}\mu(\mathcal{C}(Q))$ then

$$\frac{\mu(A, \mathcal{C}(Q) - A)}{\mu(A)} \geq \frac{C}{r}. \qquad (1)$$

4. 'Good surface effects': $\mathcal{C}(Q)$ and $\mathcal{C}_\infty \cap Q$ do not differ by much. In particular, one has

$$Q' \cap \mathcal{C}(Q) = Q' \cap \mathcal{C}_\infty.$$

(1) implies a Poincaré inequality for $B = \mathcal{C}(Q)$: if $f : \mathcal{C}(Q) \to \mathbb{R}$ then

$$\int_{\mathcal{C}(Q)} (f - \overline{f})^2 d\mu \le cr^2 \sum_{x,y\in\mathcal{C}(Q)} (f(x) - f(y))^2 \mu_{xy}. \qquad (PI)$$

(As usual $\overline{f}$ is the real number which minimises the left hand side.)

A guide to the expected behaviour of Y on $\mathcal{C}_\infty$ is given by the following theorem of Delmotte, which translates into the graph setting work of Grigoryan [4] and Saloff-Coste [7] on manifolds.

Theorem 5. [3] *Let Γ be an infinite connected graph, with heat kernel $q_t(x, y)$. The following are equivalent:*
(a) Γ satisfies (VD) (volume doubling) and (PI) (Poincaré inequality).
(b) For $x, y \in G$, $t \ge d(x, y)$,

$$\frac{c_1}{t^{d/2}} e^{-c_2|x-y|^2/t} \le q_t(x, y) \le \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t}. \qquad (GE)$$

(c) Γ satisfies (PHI).

Proof. Of course the hardest part of the argument is to obtain (b) or (c) from (a). Delmotte proved that (a) $\mathbb{R}ightarrow$ (c) using Moser's argument.

Since $\mathcal{C}_\infty$ is random one can consider 'annealed' bounds (i.e. on $\mathbb{E}_p(q^\omega_t(x, y)|x, y \in \mathcal{C}_\infty)$ or 'quenched' bounds, on $q^\omega_t(x, y)$. The first quenched bounds were obtained in by Mathieu and Remy in [6]:

$$q^\omega_t(x, y) \le c_1 t^{-d/2}, \quad t \ge N_x(\omega), \qquad (UB1)$$

where $\mathbb{P}_p(N_x < \infty|x \in \mathcal{C}_\infty) = 1$.

Before considering how one can prove (UB1), it may be useful to see how not to do so. It is well known that the 'Nash inequality'

$$\mathcal{E}(f, f) \ge c||f||_2^{2+4/d}||f||_1^{-4/d}. \qquad (N)$$

is equivalent to:

$$q_t(x, x) \le Ct^{-d/2} \text{ for all } x, y \in G,\ t \ge 1. \qquad (GUB)$$

However (GUB) is false for $\mathcal{C}_\infty$, since $\mathcal{C}_\infty$ contains arbitrarily large 'bad' regions – for example one-sided strings of length n connected at one end to the rest of $\mathcal{C}_\infty$. Thus (N) (and related global Sobolev inequalities) must fail for $\mathcal{C}_\infty$, and one needs 'local' methods.

The basic idea is that if (VD) and (PI) hold for all large balls then one should also get (GE) for all large times t. To make this more precise we make the following definitions.

Definition.
1. A ball $B = ga(x, r)$ is *good* if (V_d) and (PI) hold for B.

2. Let $\alpha = 1/(11(2+d))$. The ball $B = B_\omega(x,R)$ is *very good* (VG) if $B_\omega(y,r)$ is good for all

$$B_\omega(y,r) \subset B \text{ and } R^\alpha \le r \le R.$$

3. $B = B_\omega(x,R)$ is *exceedingly good* (EG) if B is very good and satisfies a further, rather complicated, condition.

Theorem 6. *Let Γ be an infinite connected graph such that $V(x,r) \le C_0 r^d$ for all $x \in G$, $r \ge 1$.*
(a) Suppose for each $x \in G$ there exists $r_x < \infty$ such that $B(x,R)$ is VG for all $R \ge r_x$. Let $N_x = (r_x)^c$. Then

$$q_t(x,y) \le \frac{c_3}{t^{d/2}} e^{-c_4 |x-y|^2/t}, \quad t \ge N_x \vee d(x,y),$$

and

$$q_t^\omega(x,y) \ge \frac{c_1}{t^{d/2}} e^{-c_2 |x-y|^2/t}, \quad t \ge N_x \vee d(x,y)^3.$$

(b) If $B(x,R)$ is EG for all $R \ge r_x$ then the lower bound holds for $t \ge d(x,y) \vee N_x$.

One can prove that the hypotheses of Theorem 3(b) hold for $\mathcal{C}_\infty$ a.s., and so one derives Theorem 1.

Proof of Theorem 3. The basic idea is to work on an infinite connected graph Γ, and prove that if a ball $B(x_0,R)$ is good/VG/EG then we have the right upper and lower bounds on $q_t(x,y)$ for $x,y \in B(x_0,\frac{1}{2}R)$ and for suitable t. Ideally we would have $t \approx R^2$, but in fact the arguments needed $t \le R^2/\log R$.

Then, since for Theorem 3(a) we suppose that $B(x_0,R)$ is VG for all $R \ge R_x$ we obtain bounds on $q_t(x,y)$ for all large enough t.

Outline (for experts).
1. One can prove on-diagonal upper bounds on $q_t(x,x)$ directly from the Poincaré inequalities, using an argument of Kusuoka-Zhou – see [5].
2. The hardest part is usually to obtain off-diagonal upper bounds. Of the approaches available one cannot easily use Davies' method, as it relies on (global) Sobolev inequalities. An alternative might be to prove a Harnck inequality first, as in [3], but in [1] I used a method due to Bass and Nash – see [2].
3. For near diagonal lower bounds – i.e.

$$q_t(x,y) \ge c t^{-d/2} \quad \text{if } d(x,y) \le t^{1/2} \tag{2}$$

one can use the Fabes-Stroock method based on a weighted Poincaré inequality, which can be derived from (PI) by the method of [8].
4. Full lower bounds (i.e. if $t \ge d(x,y) \gg t^{1/2}$) require a chaining arguement which needs (2) in small balls. So 'very good' isn't good enough, and we need 'exceedingly good'.

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Symmetric Markov chains on Z^d with unbounded range

Richard Bass

(joint work with Takashi Kumagai)

Let X_n be a symmetric Markov chain on Z^d. We say that X_n has *bounded* range if there exists $K > 0$ such that $P(X_{n+1} = y \mid X_n = x) = 0$ whenever $|y - x| \geq K$. The range is *unbounded* if for every K there exists x and y (depending on K) with $|x - y| > K$ such that $P(X_{n+1} = y \mid X_n = x) > 0$. There is a great deal known about Markov chains on graphs when the chains have bounded range. The purpose of this talk is to discuss results for Markov chains on Z^d that have unbounded range.

Suppose C_{xy} is the conductance between x and y. We impose a condition on C_{xy} which essentially says that the C_{xy} satisfy a uniform second moment condition. Let Y_t be the continuous time Markov chain on Z^d determined by the C_{xy}, while X_n is the discrete time Markov chain determined by these conductances. The transition probabilities for the Markov chain X are defined by

$$P^x(X_1 = y) = \frac{C_{xy}}{\sum_z C_{xz}},$$

while the process Y_t is the Markov chain that has the same jumps as X but where the times between jumps are independent exponential random variables with parameter 1. When the second moment condition holds, together with two very mild regularity conditions, we obtain upper bounds on the transition probabilities of the form

$$P(Y_t = y \mid Y_0 = x) \leq ct^{-d/2}$$

and some corresponding lower bounds when x and y are not too far apart. Unlike the case of bounded range, reasonable universal bounds of Gaussian type need not hold when the range is unbounded. We also obtain bounds on the exit probabilities $P(\sup_{s\le t}|Y_s-x|>\lambda t^{1/2})$.

We say a uniform Harnack inequality holds for X if whenever h is nonnegative and harmonic for the Markov chain X in the ball $B(x_0,R)$ of radius $R>1$ about a point x_0, then

$$h(x)\le Ch(y),\qquad |x-x_0|,|y-x_0|<R/2,$$

where C is independent of R. Even when X_n is a random walk, i.e., the increments X_n-X_{n-1} form an independent identically distributed sequence, a uniform Harnack inequality need not hold. However, if we impose an additional strong assumption on the conductances, then we can prove such a Harnack inequality.

We prove that if we have Markov chains $X^{(n)}$ on Z^d satisfying the secnd moment condition uniformly in n, then the sequence of processes $X^{(n)}_t=X_{[nt]}/\sqrt{n}$ is tight in the space $D[0,\infty)$ of right continuous, left limit functions, and all subsequential limit points are continuous processes. Under an additional condition on the conductances (different than the one needed for the Harnack inequality), we then show that the $X^{(n)}_\cdot$ converge weakly as processes to the law of the diffusion corresponding to an elliptic operator

$$\mathcal{L}f(x)=\sum_{i,j=1}^d\frac{\partial}{\partial x_i}\Big(a_{ij}(\cdot)\frac{\partial f}{\partial x_j}(\cdot)\Big)(x)$$

in divergence form.

Ultracontractivity and embedding into L^∞

Alexander Bendikov

(joint work with Th. Coulhon, L. Saloff-Coste)

One of the classical uses of Sobolev embedding theorem is to show that an L^2 function on $\mathbb{R}^d$ having k derivatives in L^2 with $k>d/2$ is a bounded function. This has been generalized as follows. Let e^{-tA} be a semigroup of self-adjoint operators on $L^2(X,\mu)$. Assume that, for all $t\in(0,1)$,

$$\|e^{-tA}\|_{2\to\infty}=\sup_{\|g\|_2\le1}\|e^{-tA}g\|_\infty\le Ct^{-\nu/4}.$$

Then any function $f\in L^2(X,\mu)$ such that $A^kf\in L^2(X,\mu)$ for some $k>\nu/4$ (roughly speaking, this corresponds to $2k$ derivatives in L^2) must be a bounded function. See, e.g., [2, Théorème 1] and the references therein.

The aim of the present paper is to obtain results in this spirit when the semigroup e^{-tA} satisfies an ultracontractivity bound of the type

$$\|e^{-tA}\|_{2\to\infty}=\sup_{\|g\|_2\le1}\|e^{-tA}g\|_\infty\le e^{m(t)},\ \ t>0 \tag{1}$$

with a function m which tends to infinity at least as fast as $\log 1/t$ as t tends to 0. We call such a function m an ultracontractivity function for e^{-tA}.

More precisely, we would like to obtain equivalences between (1) and properties such as

$$g \in \bigcap_{0}^{\infty} \mathrm{Dom}(A^n) \text{ and } \limsup_{n\to\infty} \frac{\|A^n g\|_2^{1/n}}{\phi(n)} \le 1 \Longrightarrow g \in L^\infty(X,\mu) \tag{2}$$

where the function m in (1) and the function ϕ in (2) are related in some explicit way. We call a function ϕ such that (2) holds an embedding function for A.

We also relate these properties to Nash type inequalities and characterize those functions f on the real line such that $\|e^{-f(A)}\|_{2\to\infty} < \infty$. Similar questions were discussed in [1] which focussed on problems related to the long time behavior of the semigroup. In this paper, the focus is on the short time behavior.

In fact, the connection between Nash inequalities and ultracontractivity bounds on the one hand, embedding properties of the form (2) (or, to follow the terminology of [2], generalized Gagliardo-Nirenberg inequalities) and ultracontractivity bounds on the other hand have a different range of validity, as far as the behaviour of the function m is concerned. Nash inequalities are relevant when the explosion of m at 0 is not too fast, embedding properties are relevant when the explosion of m at 0 is fast enough. There is a common zone where both operate (see Theorem 7), and two exclusive zones where only one of them operates.

We also consider the case where X has finite measure and A has discrete spectrum together with L^∞ bounds on the eigenfunctions. In that case, one can obtain the connection with generalized Gagliardo-Nirenberg inequalities even when m does not belong to the favorable zone. This applies in particular to left-invariant Markov generators acting on locally compact metric groups.

Finally, we exhibit families of concrete examples, namely invariant diffusions on infinite dimensional tori and symmetric Lévy semigroups on the real line, which display the whole variety of behaviours we have been considering.

Definition 1. *Let M be a non-negative non-increasing function defined on $(0,+\infty)$ and such that $M(0_+) = \infty$. For non-negative x, set*

$$F(x) = F_M(x) = \inf_{t>0}\{tx + M(t)\} \tag{3}$$

and

$$\Phi(x) = \Phi_M(x) = \sup_{t>0}\left\{\frac{x}{t}e^{-M(t)/x}\right\}. \tag{4}$$

Definition 2. *Let M be a non-increasing non-negative function defined on $(0,+\infty)$ and such that $M(0_+) = \infty$. For any real x, set*

$$N(x) = \sup_{t>0}\{xt - tM(1/t)\}. \tag{5}$$

When $M \in \mathcal{C}^1$, set also

$$Q(x) = \begin{cases} -M' \circ M^{-1}(x) & \text{if } x \ge M(\infty) \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We can now state our main result.

Theorem 7. *Let M, F, Φ be as in the above definition. Assume that M is C^1, convex, and that there are constants $a, b \in (0, \infty)$ such that $aM(t) \leq -tM'(t) \leq bM(t)$ for all t small enough. Let $-A$ be the infinitesimal generator of a sub-Markovian semigroup on $L^2(X, \mu)$. Then the following properties are equivalent:*

(1) *There exists $c_1 \in (0, \infty)$ such that, for all $t > 0$, $\|e^{-tA}\|_{2\to\infty} \leq e^{c_1 M(t)}$.*
(2) *There exists $t_0 > 0$ such that, for all $t > t_0$, $\|e^{-tF(A)}\|_{2\to\infty} < \infty$.*
(3) *There exists $C_1 \in (0, \infty)$ such that for any function $f \in \bigcap_0^\infty \mathrm{Dom}(A^n)$ we have*

$$\limsup_{n\to\infty} \left\{ \frac{\|A^n f\|_2^{1/n}}{\Phi(n)} \right\} \leq C_1 \Longrightarrow f \in L^\infty(X, \mu).$$

(4) *There exists $C_2 \in (0, \infty)$ such that the Nash inequality*

$$\forall\, f \in \mathrm{Dom}(A) \text{ with } \|f\|_1 \leq 1, \quad \|f\|_2^2 \Phi\left(\log \|f\|_2\right) \leq C_2 \left(\langle Af, f\rangle + \|f\|_2^2\right)$$

is satisfied.

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On the Robin problem in fractal domains

Krzysztof Burdzy

(joint work with R. Bass, Z.-Q. Chen)

The Robin problem (also known as the "third" boundary problem) for a Euclidean domain $D \subset \mathbf{R}^d$ is to find a function u such that

$$\Delta u(x) = 0, \qquad x \in D, \tag{1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = cu, \qquad x \in \partial D, \tag{2}$$

with one or more side conditions, where $\mathbf{n}$ is the unit inward normal vector field on ∂D, $\partial u/\partial \mathbf{n}$ is the normal derivative of u in the distributional sense and $c > 0$ is a constant. See Gustafson and Abe [8] for the history of this problem.

Our interest in the Robin problem stems from some recent applications in physics, electrochemistry, heterogeneous catalysis and physiology; see [4], [5], [7], [11] and the references therein. Consider the mixed Dirichlet-Robin problem

$$\Delta u(x) = 0, \qquad x \in D \setminus B_*, \tag{3}$$

$$\frac{\partial u}{\partial \mathbf{n}} = cu, \qquad x \in \partial D, \tag{4}$$

together with the side condition

$$u(x) = 1, \qquad x \in \partial B_*, \tag{5}$$

where $B_* \subset D$ is a fixed closed ball with non-zero radius. The solution to (3)-(5) represents the steady state of a system in which some particles move randomly in $D \setminus B_*$ and cross a semi-permeable membrane ∂D. The other part of the boundary, ∂B_*, is a source of particles and can be controlled so that we can assume a condition of type (5). The constant c in (4) is a physical characteristic of the membrane ∂D. The constant c will play no role in our theorems so we take $c = 1$.

In some applied situations, it is desirable to have as much flux through the boundary as possible. The points of a man-made or natural membrane ∂D where there is no flux can be considered an inefficient use of material. Hence, it is interesting to know when the flux is non-negligible through all points of the membrane. In other words, we would like to know whether $\inf_{x \in \partial D} \partial u / \partial \mathbf{n}(x) > 0$. In view of the relation (4) between the flux $\partial u / \partial \mathbf{n}$ and the density u of particles and the maximum principle for the harmonic function u, this condition is equivalent to $\inf_{x \in D \setminus B_*} u(x) > 0$.

Definition 1.1. *We say that the whole surface of D is active if*

$$\inf_{x \in D \setminus B_*} u(x) > 0. \tag{6}$$

If it is not the case that the whole surface is active, we say part of the surface is nearly inactive.

In this paper we investigate the following problem.

Problem 1.2. *Give necessary and sufficient conditions of a geometric nature for the whole surface of D to be active.*

It is not difficult to show that the whole surface of a bounded Lipschitz domain is always active. We have posed Problem 1.2 in terms of u rather than $\partial u / \partial \mathbf{n}$ because we are interested in non-Lipschitz domains D; so there are some boundary points where $\mathbf{n}$ is not well-defined while the solution u is always well-defined, and, in fact, is smooth in $D \setminus B_*$. We do not have a complete solution to Problem 1.2, but we give a fairly explicit answer for some natural families of domains with fractal boundary.

We will approach Problem 1.2 using probabilistic methods. This agrees well with the motivating physical models. Suppose that X is reflecting Brownian motion in D, L is its local time on ∂D, and T_{B_*} is the hitting time of B_* by X. When D is a bounded C^3-smooth domain, it is known that (see [9] and [10]

$$u(x) = E_x \left[\exp\left(-\frac{1}{2} L_{T_{B_*}} \right) \right]. \tag{7}$$

This formula indicates that the third boundary problem (4) is more difficult to study from the probabilistic point of view than the corresponding Dirichlet and Neumann problems. This is because the Dirichlet problem corresponds to killed Brownian motion and killing on the boundary presents no technical problems. The Neumann boundary problem corresponds to reflecting Brownian motion. The construction of reflecting Brownian motion in an arbitrary domain D is a major technical challenge. Although this feat has been accomplished long time ago by Fukushima [6] on an abstract compactification, called the Martin-Kumarochi compactification, of D, many questions about the construction of reflecting Brownian motion on the Euclidean closure of a domain remain open (see [1]). Formula (7) shows that the Robin boundary problem (3)-(5) requires the construction and understanding of the local time. This is harder than constructing reflecting Brownian motion itself, because it is known that reflecting Brownian motion does not have a semimartingale decomposition in some domains. For some results in this area, see, e.g., DeBlassie and Toby [3]. For information on the eigenvalue problem for the Laplacian with Robin boundary conditions, see Smits [12], [13].

The following are some results proved in this paper.

- (i) The solution of Problem 1.2 for a class of domains with fractal boundaries.
- (ii) A semimartingale decomposition of reflecting Brownian motion in a class of fractal domains.
- (iii) A sharp estimate for the Green function with Neumann boundary conditions in long and thin domains.
- (iv) A new version of the Neumann boundary Harnack principle, stronger than the one in [2].

A simple example illustrating our main theorems is a cusp domain, defined for a fixed $\alpha > 1$ by

$$D = \Big\{ x = (x_1, x_2, \dots, x_d) : 0 < x_1 < 1 \text{ and } x_1^\alpha > (x_2^2 + \cdots + x_d^2)^{1/2} \Big\}.$$

Applying the main results of this paper, we can show that the whole boundary of D is active if $\alpha \in (1, 2)$, and part of ∂D is nearly inactive if $\alpha \geq 2$.

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Correlation inequalities of Brascamp-Lieb inequalities and the heat kernel

Eric Carlen

We discuss recent work with Elliott Lieb and Michael Loss, showing that a simple heat kernel argument can be used to prove correlation inequalities of Brascamp–Lieb type, and explain some new examples obtained this way.

1. Introduction

Consider an integral of a product of functions, such as

$$\int_X \prod_{j=1}^{N} f_j(\pi_j(x))\mathrm{d}\mu \tag{1}$$

where the π_j are certain "projections" from the measure space X to $\mathbb{R}$. For example when X is the unit sphere in $\mathbb{R}^N$, one might have $\pi_j(x) = \vec{e}_j \cdot x$, where $\vec{e}_j$ is the jth standard basis vector of $\mathbb{R}^N$.

When μ is a probability measure, the π_j, and hence the $f_j \circ \pi_j$ are random variables. In general, they will not be independent. This is the case in the example mentioned above when μ is the normalized probabilty measure on the unit sphere. Then of course, the integral of the product will not equal the product of the integrals. Indeed, it can even be the case that the integral of the product is infinite, due to correlations, while each individual integral is finite.

A Brascamp–Lieb inequality is one that relates an integral of a product, such as the one in (1), to a product of the L^p norms of the $f_j \circ \pi_j$, as in

$$\prod_{j=1}^{N} \left(\int_X |f_j(\pi_j(x))|^{p_j}\mathrm{d}\mu\right)^{1/p_j} . \tag{2}$$

The classical theorem of Brascamp and Lieb [2],[6] concerns the case in which X is $\mathbb{R}^M$, with $M < N$, $\mathrm{d}\mu$ is Lebesgue measure, and $\pi_j(x) = \vec{a}_j \cdot x$ where the $\{\vec{a}_1, \dots, \vec{a}_N\}$ span $\mathbb{R}^M$. Given L^p indices p_j with $1 \le p_j \le \infty$ for $j = 1, 2 \dots, N$, form the vector

$$\vec{p} = (1/p_1, 1/p_2, \dots, 1/p_N) \ , \tag{3}$$

and define
(4)
$$D(\vec{p}) = \sup\left\{ \frac{\int_{\mathbb{R}^M} \prod_{j=1}^N f_j(\vec{a}_j \cdot x)\mathrm{d}^N x}{\prod_{j=1}^N \|f_j\|_{p_j}} \ : \ f_j \in L^{p_j}(\mathbb{R}) \qquad j = 1, 2, \dots, N \right\} .$$

Next, let $\mathcal{G}$ denote the set of all centered Gaussian function functions on $\mathbb{R}$; i.e., those of the form $g(x) = e^{-(sx)^2/2}$ for some $s > 0$. Define $D_{\mathcal{G}}(\vec{p})$ by

(5) $$D_{\mathcal{G}}(\vec{p}) = \sup\left\{ \frac{\int_{\mathbb{R}^M} \prod_{j=1}^N g_j(\vec{a}_j \cdot x)\mathrm{d}^N x}{\prod_{j=1}^N \|g_j\|_{p_j}} \ : \ g_j \in \mathcal{G} \qquad j = 1, 2, \dots, N \right\} .$$

The classical theorem of Brascamp and Lieb states that $D(\vec{p}) = D_{\mathcal{G}}(\vec{p})$, and hence

(6) $$\int_{\mathbb{R}^M} \prod_{j=1}^N f_j(\vec{a}_j \cdot x)\mathrm{d}^N x \le D_{\mathcal{G}}(\vec{p}) \prod_{j=1}^N \|f_j\|_{p_j}$$

for all non negative $f_1, \dots, f_N$.

An important special case arises when $N = 3$, $M = 2$ and

$$\vec{a}_1 = \vec{e}_1 \qquad \vec{a}_2 = \vec{e}_1 - \vec{e}_2 \qquad \text{and} \qquad \vec{a}_3 = \vec{e}_2 \ .$$

Then, denoting f_1, f_2, f_3 by f, g, h respectively, (1) becomes

$$\int_{\mathbb{R}^2} f(x)g(x-y)h(y)\mathrm{d}x\mathrm{d}y \ .$$

In this case, it is quite easy to compute $D_{\mathcal{G}}(\vec{p})$, and thus one obtain the sharp constant in Young's inequality. (This important special case was also treated independently and simultaneously by Beckner.)

Several new examples of Brascamp–Lieb type correlation inequalities have recently been proven by Lieb, Loss and the author [3],[4]. The first of these holds when, as discussed in the introduction, the measure space X in (1) is the unit sphere S^{N-1} in $\mathbb{R}^N$ equipped with the uniform probability measure μ:

Theorem 8. *For all $N \ge 2$, given non–negative measurable functions $f_1, \dots, f_N$, on $[-1, 1]$,*

(7) $$\int_{\mathbb{S}^{N-1}} \left(\prod_{j=1}^N f_j \circ \pi_j \right) \mathrm{d}\mu \le \prod_{j=1}^N \|f_j \circ \pi_j\|_{L^p(\mathbb{S}^{N-1})} \ .$$

for all $p \ge 2$. Moreover, the L^2 norm is optimal in that for each $p < 2$, there exist functions f_j so that $\|f_j \circ \pi_j\|_{L^p(S^{N-1})} < \infty$ for each j, while the integral on the left side of (7) diverges. Finally, for every $p \ge 2$ and $N \ge 3$, there is equality in (7) if and only if some function f_j vanishes identically, or else each f_j is constant.

In the second of these, X is $\mathcal{S}^n$ denote the symmetric group on N letters; i.e., the group of all permutations σ of $\{1, \dots, N\}$. Let the (composition) product in $\mathcal{S}^n$ be denoted by juxtaposition, and for each $1 \le i, j \le N$ with $i \ne j$, let $\sigma_{i,j}$ be the pair permutation with $\sigma_{i,j}(i) = j$, $\sigma_{i,j}(j) = i$, $\sigma_{i,j}(k) = k$ for $k \neq i, j$. Let μ denote the uniform probability measure on $\mathcal{S}^n$ so that if g is any function on $\mathcal{S}^n$,

$$\int_{\mathcal{S}}^{n} g(\sigma)\mathrm{d}\mu = \frac{1}{N!}\sum_{\sigma\in\mathcal{S}^n} g(\sigma)\ . \tag{8}$$

Theorem 9. *For any $N \geq 2$ given functions $f_1, \ldots, f_N$, on $\{1, \ldots, N\}$,*

$$\int_{\mathcal{S}^n}\left(\prod_{j=1}^{N} f_j\circ\pi_j\right)\mathrm{d}\mu \leq \prod_{j=1}^{N}\|f_j\circ\pi_j\|_{L^p(\mathcal{S}^n)}\ . \tag{9}$$

there is equality in (9) if and only if at least one of the functions f_j is zero, or else each $f_j(k) = C_j e^{i(\alpha_j-\beta_k)}$ for some $\alpha_1, \ldots, \alpha_N$, $\beta_1, \ldots, \beta_N$, and $C_1, \ldots, C_N$

This can be reformulated as a theorem about permanents, if we identify functions f on $\{1, \ldots, N\}$ in the obvious way: Let F be an $N \times N$ complex matrix whose jth column is the vector $\vec{f_j}$ in $\mathbb{C}^N$. Let $|f_j|^2$ denote the sum of the absolute squares of the entries of $\vec{f_j}$. Hadamard's inequality for determinants [5] states that $|\det(F)| \leq \prod_{j=1}^N |\vec{f_j}|$. Then it is easy to reformulate the previous theorem as follows:

Theorem 10. *For any vectors $\vec{f_1}, \ldots, \vec{f_N}$ in $\mathbb{C}^N$ we have the inequality*

$$|\mathrm{perm}(F)| \leq \frac{N!}{N^{N/2}}\prod_{j=1}^{N}|\vec{f_j}|\ . \tag{10}$$

For $N > 2$, there is equality in (10) if and only if at least one of the vectors $\vec{f_j}$ is zero, or else F is a rank one matrix and, moreover, each of the vectors $\vec{f_j}$ is a constant modulus vector; i.e., its entries all have the same absolute value.

All of the theorems mentioned above, including the original paradigm, can be proved using a simple heat kernel interpolation. Barthe [1] had earlier given a proof of the original Brascamp–Lieb Theorem using an interpolation constructed by means of optimal mass transportation. This works very well on $\mathbb{R}^N$, but it is not clear how it could be adapted to work on S^{N-1} or on $\mathcal{S}^n$. On the other hand, the heat kernel proof is very simple, and works in essentially the same way in each of the cases mentioned above. We now biefly describe how this works.

For any non negative function f on $\{1, \ldots, N\}$ and any $1 \leq j \leq N$, and any $1 \leq p < \infty$, consider the function defined by $\left(e^{t\Delta}(f\circ\pi_j)^p\right)^{1/p}$. It is easy to see that the result is also of the form $g\circ\pi_j$. Moreover, the L^p norm is constant during the evolution. All one has to do is a calculation to check that the integral of the product increases with t. Finally, on S^{N-1} or on $\mathcal{S}^n$, $\left(e^{t\Delta}(f\circ\pi_j)^p\right)^{1/p}$ tends to a constatn, while on $\mathbb{R}^M$, it gets "more and more Gaussian". For the details, see the cited papers.

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L^p cohomology and the boundness of the Riesz transform

Gilles Carron
(joint work with Th. Coulhon, A. Hassell)

Let M be a complete Riemannian manifold with infinite measure. The Riesz transform T on M is the operator

$$f \to d\Delta^{-1/2} f,$$

where Δ is the positive Laplace operator on M. Thanks to the Green formula we have :

$$\forall f \in C_0^\infty(M),\ \|\Delta^{1/2} f\|_{L^2}^2 = \langle \Delta f, f\rangle = \|df\|_{L^2}^2$$

hence the Riesz transform is always a bounded map from $L^2(M)$ to $L^2(M; T^*M)$. It is of interest to figure out the range of p for which T extends to a bounded map $L^p(M) \to L^p(M; T^*M)$. Equivalently, we can ask whether

$$\| df \|_p \leq \|\Delta^{1/2} f\|_p \text{ for all } f \in C_c^\infty(M).$$

There is a lot of result in this direction. I will only mention few of such results :

i) On $\mathbb{R}^n$, for all $p > 1$, the Riesz transform is bounded on L^p.
ii) On manifold with non negative Ricci curvature, the Riesz transform is bounded on L^p for all $p > 1$ ([3]).
iii) On certain Cartan-Hadamard manifolds, the Riesz transform is bounded on L^p for all $p > 1$
([7]).
iv) On simply connected nilpotent Lie groups endowed with a left invariant metric, the Riesz transform is bounded on L^p for all $p > 1$ ([1]).
v)In [5], T. Coulhon and X.T. Duong have shown that if for some $C > 0$ and $\nu > 0$, (M, g) satisfies the the relative Faber-Krahn inequality :

$$\forall x \in M, R > 0 \text{ and } \Omega \subset B(x,R)\ \lambda_1(\Omega) \geq \frac{C}{R^2}\left(\frac{\mathrm{vol}\Omega}{\mathrm{vol}B(x,R)}\right)^{-2/\nu},$$

then for all $p \in]1,2]$ the Riesz transform is bounded on L^p.

They also indicated that for $p > n$, the Riesz transform is not bounded on L^p on a connected sum of two Euclidean space.

Our result is the following :

Theorem *Let M be a complete Riemannian manifold of dimension $n \geq 3$ which is the union of a compact part and a finite number of Euclidean ends. Then the Riesz transform is bounded from $L^p(M)$ to $L^p(M; T^*M)$ for $1 < p < n$, and is unbounded on L^p for all other values of p if the number of ends is at least two.*

Let's describe a key point in the proof in the case of a connected sum of two Euclidean space $M = \mathbb{R}^n \# \mathbb{R}^n$. Such manifold is topologically the product $\mathbb{R}\times\mathbb{S}^{n-1}$ endowed with the metric

$$(dr)^2 + (1 + r^2)(d\theta)^2$$

where $(d\theta)^2$ is the standard round metric on the sphere $\mathbb{S}^{n-1}$ and r the radial variable. One of our argument is a precise description of the behavior for large x and y of the Schwarz kernel of the operator :

$$\Delta^{-1/2} = \frac{2}{\pi}\int_0^\infty (\Delta + k^2)^{-1}\, dk.$$

If $P(x, y)$ is the kernel of this operator, we show that when $x \in M$ and $y = (r, \theta)$ then for $r \to \pm\infty$

$$P(x,(r,\theta)) \simeq \frac{u_\pm(x)}{r^{n-1}}$$

where $u_\pm$ is the harmonic function such that

$$\lim_{r\to\pm\infty} u_\pm(x) = 1$$

$$\lim_{r\to\mp\infty} u_\pm(x) = 0$$

At infinity, we also get

$$|du_\pm(r,\omega)| = O(r^{1-n}).$$

With this estimate, we can show that the Riesz transform can be bounded only when $p < n$. To get the full result, we obtain sharper estimates by using the scattering calculus introduced by R. Melrose ([8]).

In ([2]), authors have remarked that if the Riesz transform is bounded on $L^{p/(p-1)}$ and on L^p then the operator $P = d\Delta^{-1}d^*$ extend to a bounded operator on $L^p(T^*M)$. On $L^2(T^*M)$, this operator P is the orthogonal projection on the closure of the space $dC_0^\infty(M)$. But on (M, g), the space of L^2 differential one-forms admits the Hodge decomposition

$$L^2(T^*M) = \mathcal{H}^1(M) \oplus \overline{dC_0^\infty(M)} \oplus \overline{d^*C_0^\infty(\Lambda^2T^*M)},$$

where $\mathcal{H}^1(M) = \{\alpha \in L^2(T^*M), d\alpha = 0 = d^*\alpha\}$ (see [6]). Let us recall now the definition of reduced L^p-cohomology: for $p \geq 1$, the first space of reduced L^p cohomology of (M, g) is

$$H^1_p(M) = \frac{\{\alpha \in L^p(T^*M), d\alpha = 0\}}{\overline{dC_0^\infty(M)}}\ ,$$

where we take the closure in L^p. On L^2, we have

$$\{\alpha \in L^2(T^*M), d\alpha = 0\} = \mathcal{H}^1(M) \oplus \overline{dC_0^\infty(M)},$$

hence the first space of reduced L^2 cohomology can be identified with $\mathcal{H}^1(M)$.

We assume now that the manifold (M, g) satisfies the following conditions

$$\left(\int_M |f|^{\frac{2\nu}{\nu-2}} d\mathrm{vol}\right)^{1-2/\nu} \leq C \int_M |df|^2 d\mathrm{vol},\ \forall f \in C_0^\infty(M),$$

And that at one point $x_0 \in M$, we have a control on the growth of geodesic balls centered at x_0 :

$$\mathrm{vol}\, B(x_0, r) \leq C r^\nu, \forall r \geq 1.$$

Proposition *Under these two hypotheses, if the Riesz transform is bounded in L^p for some $p > 2$, then*

$$H_p^1(M) = \{\alpha \in L^p(M; T^*M) \mid d^*\alpha = d\alpha = 0\}.$$

If moreover the Ricci curvature of M is bounded from below then there is a natural map

$$H_2^1(M) \to H_p^1(M)$$

which is injective.

Corollary *If M has at least two ends, then the Riesz transform is not bounded on L^p for any $p \geq \nu$.*

Let's us describe two examples :

First example: The manifold $M = \mathbb{R}^n \# \mathbb{R}^n = \mathbb{R} \times \mathbb{S}^{n-1}$ endowed with a metric $(dr)^2 + (1 + r^2)(d\theta)^2$. Then it is easy to show by direct computation that

$$\{\alpha \in L^p(M; T^*M) \mid d^*\alpha = d\alpha = 0\} = \mathbb{R}\frac{dr}{(1 + r^2)^{\frac{n-1}{2}}}.$$

Where as

$$H_p^1(M) = \begin{cases} \mathbb{R} & \text{if} p < n \\ \{0\} & \text{if } p \geq n \end{cases}$$

Hence the map $H_2^1(M) \to H_p^1(M)$ is injective only for $p < n$. The proposition gives in this case the right range of p's where the Riesz transform is bounded on L^p.

Second example: Let M be a connected sum of several (say $l \geq 2$) copies of a simply connected nilpotent Lie group N endowed with a left invariant metric, let ν be the homogeneous dimension of N; that is

$$\nu = \lim_{R \to \infty} \frac{\log \mathrm{vol} B(o, R)}{\log R},$$

Then M satisfies the Sobolev inequality, the upper bound on the volume growth of geodesic balls and more over it's clear that its Ricci curvature is bounded from below. We known that the Riesz transform is bounded on L^p for $p \in]1, 2]$ and not bounded on L^p for any $p \geq \nu$. Moreover we can compute the L^p cohomology of M:

$$H^1_p(M) = \begin{cases} H^1_c(M) \simeq \mathbb{R}^{l-1} & \text{if } p < \nu \\ \{0\} & \text{if } p > \nu \end{cases}$$

Hence the map

$$H^1_2(M) \to H^1_p(M)$$

is injective when $p < \nu$. It is then tempting to propose the following conjecture :

On such a manifold M , is the Riesz transform bounded on any $p \in]1, \nu[$?

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Boundary Processes of Markov Processes

Zhen-Qing Chen

(joint work with Masatoshi Fukushima, Jiangang Ying)

Time change is one of the most basic and very useful transformations for Markov processes, which has been studied by many authors. However a precise characterization of the time-changed process of a Markov process X on a state space E by a Revuz measure whose quasi-support F is a proper subset of E has only been started very recently. The time-changed process has F as its state space so it can be regarded as the trace process of X on F.

The following is a prototype of the problem we discussed in the talk. Suppose X is a Lévy process in $\mathbf{R}^n$ that is the sum of a Brownian motion in $\mathbf{R}^n$ and an independent spherically symmetric α-symmetric stable process in $\mathbf{R}^n$, where $n \geq 1$ and $\alpha \in (0, 2)$. Denote by $B(x, r)$ the open ball in $\mathbf{R}^n$ centered at $x \in \mathbf{R}^n$ with radius r. Its Euclidean closure is denoted by $\overline{B(x, r)}$. Let $F = \overline{B(0, 1)} \cup \partial B(x_0, 1)$, where $x_0 \in \mathbf{R}^n$ with $|x_0| = 3$. What is the trace process of X on the closed set

F? More precisely, let $\mu(dx) := 1_{\overline{B(0,1)}}(x)\,dx + \sigma_{\partial B(x_0,1)}$, where $\sigma_{\partial B(x_0,1)}$ denotes the Lebesgue surface measure of $\partial B(x_0,1)$. It is easy to see that μ is a smooth measure of X and it uniquely determines a positive continuous additive functional $A^\mu = \{A_t^\mu,\, t \geq 0\}$ of X having μ as its Revuz measure. Define its inverse

$$\tau_t := \inf\{s > 0 :\, A_s^\mu > t\} \qquad \text{for } t \geq 0.$$

Then the time changed process $Y_t := X_{\tau_t}$ is a symmetric Markov process on F, which can be regarded as the trace process of X on F. So the more precise question is

Question: Can we characterize the time changed process Y?

In fact, one can ask such questions for a general irreducible Markov process X that has a weak dual process on a general state space E which not only can have discontinuous sample paths but also can have killings inside E or have finite lifetime, and for any *quasi-closed* subset F of E.

When X is a general m-symmetric Markov process, we have obtained a complete characterization of the time changed process Y in terms of its Burling-Deny decomposition in [1]. In particular, the characterization of Lévy system of Y is obtained in terms of Feller measure and supplement Feller measure. These measures are intrinsic quantities for the part process of X killed upon leaving $E \setminus F$.

We also studied time changes for non-symmetric Markov processes. Let $(X, \widehat{X})$ be a pair of Borel standard processes on a Lusin space E that are in weak duality with respect to some σ-finite measure m that has full support on E. Let F be a finely closed subset of E. In [2], we have obtained the characterization of a Lévy system of the time changed process of X by a positive continuous additive functional (PCAF in abbreviation) of X having support F, under the assumption that every m-semipolar set of X is m-polar for X. The characterization of the Lévy system is again in terms of Feller measures, which are intrinsic quantities for the part process of X killed upon leaving $E \setminus F$. Along the way, various relations between the entrance law, exit system, Feller measures and the distribution of the starting and ending point of excursions of X away from F are studied. We also show that the time changed process of X is a special standard process having a weak dual and that the μ-semipolar set of Y is μ-polar for Y, where μ is the Revuz measure for the PCAF used in the time change.

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These work extend the previous results obtained by LeJan in [4]-[5] and by Fukushima, He and Ying in [3].

This talk is based on joint work with Masatoshi Fukushima and Jiangang Ying [1] and [2].

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Limit theorems for geometric approximations to Wiener measure

Bruce Driver

A typical path integral is an **informal** expression of the form

$$\frac{1}{Z}\int_{\mathcal{F}} f(\sigma)e^{-E(\sigma)}\mathcal{D}\sigma \tag{1}$$

where $\mathcal{F}$ is a space of maps from one manifold to another, f is a real valued function on $\mathcal{F}$, $E(\sigma)$ is the energy of the map σ, $\mathcal{D}\sigma$ is "Lebesgue measure" and Z is a normalization constant. The use of path integrals for "quantizing" classical mechanical systems (whose classical energy is E) started with Feynman in [3] with very early beginnings being traced back to Dirac [2]. We present three rigorous interpretations to the path integral in (1) in the simplest non-trivial case, namely when $\mathcal{F} = \{\sigma \in C([0,1] \to M) : \sigma(0) = o\}$ where M is a compact, d – dimensional Riemannian manifold and o is a fixed point in M. This is done by first replacing $\mathcal{F}$ by an increasing sequence $\{\mathcal{F}_n\}$ of approximating finite dimensional "submanifolds" consisting of piecewise geodesic paths. The formal measure $\mathcal{D}\sigma$ on $\mathcal{F}$ is then replaced by the Riemannian volume measure on $\mathcal{F}_n$ relative to three different natural Riemannian metrics. The limit of these three approximating measures is shown to exist as a measure on $\mathcal{F}$. Each of the limiting measure are absolutely continuous relative to the Wiener measure (ν) on $\mathcal{F}$ and their (distinct) Radon-Nikodym derivatives are explicitly computed.

In more detail, let $\mathcal{P}_n := \{s_i := \frac{i}{n} : i = 0, 1, 2, \ldots,\}$ be a uniform partition of $[0,1]$,

$$\mathcal{F}_n = \{\sigma \in \mathcal{F} : \nabla\sigma'(s)/ds = 0 \text{ off } \mathcal{P}_n\}.$$

For $\sigma \in \mathcal{F}_n$, and $X, Y \in T_\sigma\mathcal{F}_n$, define:

(1) the H^1–**Metric on** $\mathcal{F}_n$ by

$$G^1(X,Y) := \int_0^1 \left\langle \frac{\nabla X(s)}{ds}, \frac{\nabla Y(s)}{ds} \right\rangle ds,$$

(2) the **Riemann sum approximation** to the H^1–**Metric on** $\mathcal{F}_n$ by

$$G^1_n(X,Y) := \sum_{i=1}^n \left\langle \frac{\nabla X(s_{i-1}+)}{ds}, \frac{\nabla Y(s_{i-1}+)}{ds} \right\rangle \frac{1}{n},$$

and

(3) the **Riemann sum approximation** to the L^2–**Metric on** $\mathcal{F}$ by

$$G^0_n(X,Y) := \sum_{i=1}^n \langle X(s_i), Y(s_i)\rangle \frac{1}{n}.$$

Further define three measure ν_n^0, ν_n^1, and ν_n on $\mathcal{F}_n$ as:

$$d\nu_n^0 := \frac{1}{Z_n^0} e^{-\frac{1}{2}E_M} \cdot d\mathrm{Vol}_{G_n^0},$$
$$d\nu_n^1 := \frac{1}{Z_n^1} e^{-\frac{1}{2}E_M} \cdot d\mathrm{Vol}_{G_n^1},$$

and

$$d\nu_n := \frac{1}{Z_n^1} e^{-\frac{1}{2}E_M} \cdot d\mathrm{Vol}_{G^1|_{T\mathcal{F}_n}}$$

where

$$E_M(\sigma) := \int_0^1 |\sigma'(s)|_g^2 \, ds,$$

is the energy functional, and Z_n^0 and Z_n^1 are normalization constants given by

$$Z_n^0 := \left(2\pi/n^2\right)^{nd/2} \text{ and } Z_n^1 := (2\pi)^{dn/2}.$$

In each of the three theorems below f is assumed to be a bounded and continuous (in the sup-norm topology) real valued function on $\mathcal{F}$.

Theorem 11 (Andersson and Driver [1]). *The sequence of measures* $\left\{\nu_n^1\right\}_{n=1}^{\infty}$ *satisfy,*

$$\lim_{n\to\infty} \int_{\mathcal{F}_n} f(\sigma) d\nu_n^1(\sigma) = \int_{\mathcal{F}} f(\sigma) d\nu(\sigma)$$

Theorem 12 (Andersson and Driver [1]). *Let* $\mathrm{Scal}\,(m)$ *be the scalar curvature of* M *at a point* $m \in M$. *Then*

$$\lim_{n\to\infty} \int_{\mathcal{F}_n} f(\sigma) d\nu_n^0(\sigma) = \int_{\mathcal{F}} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))ds} d\nu(\sigma).$$

For each $m \in M$, let R_m be the curvature tensor at $m \in M$ and define $\Gamma_m \in \mathrm{End}\,(T_m M)$ by

$$\Gamma_m = \sum_{i,j=1}^{d} \Big(R_m\left(e_i, R_m(e_i, \cdot)e_j\right) e_j + R_m\left(e_i, R_m(e_j, \cdot)e_i\right) e_j + R_m\left(e_i, R_m(e_j, \cdot)e_j\right) e_i \Big)$$

where $\{e_i\}_{i=1,2,\ldots,d}$ is any orthonormal basis for $T_m(M)$. For $\sigma \in \mathcal{F}$, let $u\,(s) := //_s\,(\sigma)$ denote stochastic parallel translation along σ and define a linear operator on $\mathrm{L}^2\left([0,1]\,; T_o M\right)$ by

$$\left(K_\sigma f\right)(t) := \int_0^1 t \wedge s \left[u\,(s)^{-1}\,\Gamma_{\sigma(s)} u\,(s)\right] f\,(s)\, ds.$$

Theorem 13 (Adrian Lim [4]). *Suppose the sectional curvatures of M are bounded below by 0 and above by $1/(2 \cdot \dim(M))$, then*

$$\lim_{n\to\infty} \int_{\mathcal{F}_n} f(\sigma)\, d\nu_n(\sigma) = \int_{\mathcal{F}} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))\, ds} \cdot \det\left(I + \frac{1}{12} K_\sigma\right) d\nu(\sigma).$$

We refer the reader to the references in [1] and [4] for more background and related references to these three theorems.

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Intertwined diffusions

David Elworthy
(joint work with Y. LeJan, X.-M. Li-Hairer)

A) Intertwined diffusion generators. Consider a smooth surjective map $p : N \to M$ between manifolds N and M with smooth diffusion generators $\mathcal{A}$ and $\mathcal{B}$ on the manifolds M and N respectively, intertwined by p. This means that for any smooth $f : M \to \mathbb{R}$ we have

$$\mathcal{B}(f \circ p) = \mathcal{A}(f) \circ p.$$

They have symbols $\sigma^{\mathcal{A}}$ and $\sigma^{\mathcal{B}}$ related by the commutative diagram

$$\begin{array}{ccc} T_u^* N & \xrightarrow{\sigma_u^{\mathcal{B}}} & T_u N \\ {\scriptstyle (T_u p)^*} \uparrow & & \downarrow {\scriptstyle T_u p} \\ T_{p(u)}^* M & \xrightarrow[\sigma_{p(u)}^{\mathcal{A}}]{} & T_{p(u)} M \end{array}$$

where $T_u p$ denotes the derivative map of p at the point u, a linear map between the relevant tangent spaces, and $(T_u p)^*$ is its adjoint acting on cotangent spaces. Recall that the (principal) *symbol* of a second order differential operator such as $\mathcal{A}$ is defined by

$$df_x(\sigma_x^{\mathcal{A}}(dg_x)) = \frac{1}{2}\big(\mathcal{A}(fg)(x) - f(x)\mathcal{A}(g)(x) - g(x)\mathcal{A}(f)(x)\big)$$

for $x \in M$ and smooth functions f and g on M. The symbols of our diffusion generators determine symmetric, positive semi-definite, bilinear forms on the cotangent spaces. Let E_x be the image of $\sigma_x^{\mathcal{A}}$ and let $\langle -, - \rangle$ be the inner product induced on it by $\sigma_x^{\mathcal{A}}$. In the elliptic case E_x the symbol is positive definite and E_x is the whole of the tangent space T_xM, furnishing M with a Riemannian metric.

There are obvious questions which arise from this set up. One is on the relationships between the operators $\mathcal{A}$ and $\mathcal{B}$, for example is there a nice decomposition of $\mathcal{B}$? are there spectral relationships? Others concern the diffusion processes generated by the operators, for example the existence of a skew product decomposition of $\mathcal{B}$-diffusions, or the filtering problem of finding the conditional law of the $\mathcal{B}$-diffusion given its projection on M by p. We have not considered relations between the spectra of $\mathcal{A}$ and $\mathcal{B}$, for this in the special cases when M and N are Riemannian manifolds and the map p is a Riemannian submersion with $\mathcal{A}$ and $\mathcal{B}$ the corresponding Laplace -Beltrami operators, see [1]. However the other questions are discussed in [4], with an earlier version for the case when $p : N \to M$ is a principal bundle, and so in particular M is the quotient of N by a group action in [3]. Most of our results require a "cohesiveness" condition on $\mathcal{A}$, described below.

Earlier results have mainly concerned the cases where $\mathcal{B}$ is the Laplacian : see [2] for a skew product decomposition when p is Riemannian with totally geodesic fibres, with an application to a factorisation theorem for harmonic maps; there is an extension of this by Liao in [7] to more general Riemannian submersions; for a discussion of skew -product decompositions of Brownian motions, with many examples, see Pauwels& Rogers [9], or [10].

B) Semi-Connections. The key step in our discussions is the construction of a *semi-connection* associated to our intertwining. For this we need the symbol of $\mathcal{A}$ to have constant rank, q say. this means that if E denotes the union of all the subspaces $E_x, x \in M$ then E is a subbundle of the tangent bundle TM.

By a (non-linear) *semi-connection on* $p : N \to M$ *over* E we will mean a smooth horizontal lift map $\mathcal{H}$ giving for each $u \in N$ a linear mapping $\mathcal{H}_u : E_{p(u)} \to T_uN$ which is a right inverse to the derivative $T_up : T_uN \to T_{p(u)}M$ of p at u. For such a semi-connection let H_u denote the image of $\mathcal{H}_u$; this is the *horizontal subspace* at u. Let F_u be the sum of H_u with the *vertical subspace* $KerT_up$ and $\Pi_u : F_u \to H_u$ the projection. When $E = TM$ we have a *(non-linear) connection*[8].

If we assume that $\sigma^{\mathcal{A}}$ has constant rank and so has image a subbundle E we obtain a semi-connection over E characterised by the requirement

$$\mathcal{H}_u \circ \sigma_{p(u)}^{\mathcal{A}} = \sigma^{\mathcal{B}}(T_up)^*.$$

A semi-connection $\mathcal{H}$ over E determines a covariant differention $\nabla^{\mathcal{H}}$ in the E-directions acting on smooth sections $f : M \to N$ of p. For this note that the derivative T_xf at a point $x = p(u)$ of such a section maps E_x to $F_{f(x)}$. Then, by definition, $\nabla_v^{\mathcal{H}} := T_xf(v) - \Pi_{f(x)}T_xf(v) \in KerT_{f(x)}p$, for all $v \in E_x$. Also any curve σ in M with $\dot{\sigma}(t) \in E_{\sigma(t)}$ for all t has a unique maximal horizontal lift $\tilde{\sigma}$ with $\sigma(0)$ any given point above $\sigma(0)$ and $\dot{\tilde{\sigma}}(t) \in H_{\tilde{\sigma}(t)}$ for all t for which it is defined.

Using Stratonovich differentials there is the corresponding result for continuous semi-martingales in M.

To lift an $\mathcal{A}$-diffusion by this "corresponding result" we need to assume that $\mathcal{A}$ is *cohesive* or *along* E. There are various possible definitions of this; one is that $\mathcal{A}$ has a smooth Hormander form representation

$$\mathcal{A} = 1/2 \sum_j \mathcal{L}_{X^j} \mathcal{L}_{X^j} + \mathcal{L}_A$$

where $\mathcal{L}_A$ denotes Lie differentiation along the vector field A, and $X^j, j = 1, ..., m$ and A are vector fields taking values in E.

Given this cohesiveness there is a canonical decomposition of $\mathcal{B}$ into the sum of a "horizontal " and a "vertical" diffusion generator. If $p : N \to M$ is a principal bundle and $\mathcal{B}$ is equivariant then so is this decomposition and there is a skew product representation of $\mathcal{B}$-diffusions. In this case the semi-connection is equivariant, and is sometimes called a "partial connection" or "connection over E".

C.Stochastic flows and diffusion of tensors. This theory is applied to stochastic flows to obtain a skew product decomposition of the flow (given a cohesive generator), and a representation of the conditioned flow given the one point motion from a chosen point of M. Also a stochastic flow for $\mathcal{A}$ determines a diffusion of sections of tensor bundles giving a semi-group on tensor fields generated by the right hand side of our Hormander form for $\mathcal{A}$ using the usual extension of lie differentiation to tensor fields. Our techniques give an expression for the 'filtering out of the redundant noise' from these diffusions, generalising results in [6] and [5].

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One point extensions of general Markov processes

Masatoshi Fukushima

The 'boundary problem of Markov processes' was one of the popular subjects in probability theory in the 1960's ([1],[2],[3],[4], [5], [6]). Later developements in the theory of Markov processes, especially the excursion theory based on exit systems [8] and the time change theory based on Dirichlet forms [9] were rooted in the study of the boundary problem. In a series of joint works with H.Tanaka, Z.-Q.Chen and J.Ying ([11],[13],[14], [15]), I have been reconsidering those issues left open in the boundary problem by using the well developed current theory of Markov processes.

In [11] and [15], we are concerned with the one point extensions of general Markov processes originated in the seminal work of Itô [6] supplemented by Meyer [7]. Itô only gave an analysis and left open the synthesis, namely, construction problem starting with the minimal process. A motivation of Itô's work was in his study with McKean [1] on the Brownian motion on the half line, while FT was motivated by the extension of the absorbing Brownian motion on the multi-dimensional bounded domain to its one point compactification presented in [4] in terms of the associated Sobolev space.

I shall talk about [15] which extends [11] from the symmetric diffusion process to the pair of standard processes in weak duality.

More specifically, Let a be a non-isolated point of a topological space E. Suppose we are given standard processes X^0 and $\widehat{X}^0$ on $E_0 = E \setminus \{a\}$ in weak duality with respect to a σ-finite measure m on E_0 which are of no killing inside E_0 but approachable to a. We first show that their right process extensions X and $\widehat{X}$ to E admitting no sojourn at a and keeping the weak duality are uniquely determined by the approaching probabilities of X^0, $\widehat{X}^0$ and m up to a non-negative constant δ_0 representing the killing rate of X at a. We then construct, starting from X^0, such X by piecing together returning excursions around a and a possible non-returning excursion including the instant killing. This extends [11] which treats the case where X^0, X are m-symmetric diffusions and X admits no sojourn nor killing at a. While the Dirichlet form theory plays a role in [11], the recent studies of exit systems in [12] and [14] replace the role. As typical examples, the cases where X^0 is the censored symmetric stable process studied in [10] and a non-symmetric diffusion process on an Euclidean domain are presented.

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Quasi-invariance of the Wiener Measure on the Loop Space of a Riemannian Manifold

Elton Hsu

Let M be a compact Riemannian manifold and $P_x(M)$ the space of continuous functions $\gamma : [0,1] \to M$ such that $\gamma(0) = x$. The standard filtration of Borel σ-fields is denoted by $\{\mathscr{B}_s,\ 0 \le s \le 1\}$. Let $\mathbb{P}_x$ be the law of Brownian motion on M from $x \in M$ and $\mathbb{P}_{xy}$ be the law of Brownian bridge on M from x to y with time length $[0,1]$. We know that these two measures are mutually equivalent on $\mathscr{B}_s$ for all $s < 1$. The measure $\mathbb{P}_{xy}$ is concentrated on $P_{xy}(M) = \{\gamma \in P_x(M) :\ \gamma(1) = y\}$.

Introduce the Cameron-Martin norm

$$|h|_{\mathscr{H}} = \sqrt{\int_0^1 |\dot{h}_s|^2 ds}$$

and the Cameron-Martin space

$$\mathscr{H}_0 = \left\{ h \in P(\mathbb{R}^n) :\ h(0) = h(1) = 0 \quad \text{and} \quad |h|_{\mathscr{H}} < \infty \right\}.$$

Consider the vector field D_h defined on $P(M)$:

$$D_h(\gamma)_s = U(\gamma)_s h_s,$$

where $U(\gamma)$ is the horizontal lift of γ starting from a fixed frame u_x at x. It has been shown ([1] and [2]) that the vector field D_h generates a flow $\{\zeta^t,\ t \in \mathbb{R}\}$ on $P_{xy}(M)$ and the Wiener measure $\mathbb{P}_{xy}$ is quasi-invariant under the flow. More precisely, the shifted Wiener measure $P^t_{xy} = P_{xy} \circ \zeta^{-t}$ is mutually absolutely continuous with $\mathbb{P}_{xy}$. In this note, we outline a proof of this result which is simpler than the previous approaches. Details will be found in the forthcoming paper [3] coauthored with Fuzhou Gong.

For $s < 1$, the two measures $\mathbb{P}^t_{xy}$ and $\mathbb{P}_{xy}$ are equivalent on $\mathscr{B}_s$. The Radon-Nikodym derivative

$$l_s = \left.\frac{d\mathbb{P}^{-t}_{xy}}{d\mathbb{P}_{xy}}\right|_{\mathscr{B}_s}, \quad 0 \le s < 1$$

is an exponential martingale. Hence, it must have the form $l_s = e^{z_s}$, where

$$z_s = \int_0^s \langle \beta_\tau, db_\tau \rangle - \frac{1}{2}\int_0^s |\beta_\tau|^2 d\tau.$$

Here b_s is a euclidean Brownian motion (the martingale part of the stochastic development w_s of γ_s. The process β_s can be expressed as

$$\beta_s = \left(O^t_s\right)^* c(1-s, u^t_s) - c(1-s, u_s) + \left(O^t_s\right)^* A^t_s,$$

where

$$c(1-s, u_s) = u_s^{-1} \nabla \ln p_M(1-s, \gamma_s, y)$$

with $p_M(\cdot,\cdot,\cdot)$ being the heat kernel of M and u^t_s being the horizontal lift of $\gamma^t_s = \zeta(\gamma)^t_s$ starting from a fixed frame u_0. The processes $\{O^t_s,\ A^t_s\}$ are given in terms of the flow ζ^t as follows. Let $J : P_0(\mathbb{R}^n) \to P_x(M)$ be the Itô map (stochastic development) and $w^t = J^{-1}\gamma^t : P_x(M) \to P_0(\mathbb{R}^n)$ the image of the flow in the flat path space $P_0(\mathbb{R}^n)$. It can be uniquely written in the form

$$dw^t_s = O^t_s\, dw_s + A^t_s ds,$$

where $O^t_s \in O(n)$, the orthogonal group. The flow equation for the vector field D_h is equivalent to a flow equation for w^t (or equivalently, for $\{O^t,\ A^t\}$):

$$O^t = I - \int_0^t K(w^\lambda) O^\lambda d\lambda, \quad A^t = O^t \int_0^t \left(O^\lambda\right)^{-1} \left\{\dot{h} - \frac{1}{2}\mathrm{Ric}_{u^\lambda} h\right\} d\lambda.$$

Here Ric_u is the scalarized Ricci transform at a frame $u \in \mathscr{O}(M)$ and $K_h(w^t)_s \in o(n)$ (antisymmetric matrices). We do not need to $K(w^t)_s$ explicitly save for the fact that $K(w^t)_s$ is the vertical component of $d\left(u^t_s\right)/dt$.

The mutual absolute continuity of $\mathbb{P}^{-t}_{xy}$ and $\mathbb{P}_{xy}$ on $\mathscr{B}_1$ is equivalent to the uniform integrability of the exponential martingale l_s. Writing $l_1 = e^{z_s} e^{z_1 - z_s}$ and using Jensen's inequality we have

$$l_s^{-1} \mathbb{E}_{xy}\left[l_1 | \mathscr{B}_s\right] = \mathbb{E}_{xy}\left[e^{z_1 - z_s} \middle| \mathscr{B}_s\right] \ge e^{\mathbb{E}_{xy}[z_1 - z_s | \mathscr{B}_s]}.$$

Suppose that we can find a constant C independent of $0 \le s < 1$ such that

$$\mathbb{E}_{xy}\left[z_1 - z_s | \mathscr{B}_s\right] \ge -C. \tag{1}$$

Then $0 \le l_s \le e^C \, \mathbb{E}_{xy} \{l_1 | \mathscr{B}_s\}$, which immediately implies the uniform integrability of $\{l_s, \; 0 \le s < 1\}$. Therefore it is enough to prove (1), or

$$\mathbb{E}_{xy} \left[\int_s^1 |\beta^t_\tau|^2 d\tau \middle| \mathscr{B}_s \right] \le C. \tag{2}$$

the process β_s has three terms. The last term is easy to deal with because

$$\left| O^{-t*} A_s^{-t} \right| \le C \, |t| \left\{ |h_s| + |\dot{h}_s| \right\}.$$

The difference which forms the first two terms is the integral from 0 to t of the derivative of $O_s^{t*} c(1-s, u_s^t)$ (with respect to t). From the flow equation we have

$$\frac{d}{dt} O_s^{t*} = O_s^{t*} K(w^t)_s.$$

Here we have used the antisymmetry of $K(w^t)_s$. As for the derivative du^t/dt, its horizontal component is $u_s^t h_s$ by the definition of horizontal lift, and its vertical component is $K(w^t)_s$, as we have pointed out before. Therefore the derivative of $c(1-s, u_s^t)$ with respect to t is equal to

$$\left\langle \left(u_s^t\right)^{-1} \nabla^2 \ln p_M(1-s, \gamma_s^t, y), h_s \right\rangle - K(w^t)_s \, c(1-s, u_s^t),$$

the first term on the right side being just the matrix multiplication. Now, in the derivative of $O_s^{t*} c(1-s, u_s^t)$ with respect to t, the two terms involving $K(w^t)_s$ cancel each other and we obtain simply

$$\frac{d}{dt} \left\{ O_s^{t*} c(1-s, u_s^t) \right\} = O_s^{t*} \left\langle \left(u_s^t\right)^{-1} \nabla^2 \ln p_M(1-s, \gamma_s^t, y), h_s \right\rangle.$$

This shows that the inequality (2) we wanted to prove is implied by

$$\mathbb{E}_{xy} \left\{ \int_s^1 |\nabla^2 \ln p_M(1-s, \gamma_s^t, y)|^2 |h_\tau|^2 \, d\tau \right\} \le C.$$

Using the estimate

$$\left| \nabla^k \ln p_M(s, z_1, z_2) \right| \le C_k \left[\frac{1}{\sqrt{s}} + \frac{d_M(z_1, z_2)}{s} \right]^k.$$

we see that the left side is bounded by

$$\mathbb{E}_{xy} \int_0^1 \left[\frac{d_M(\gamma_s^t, y)^4}{(1-s)^4} + \frac{1}{(1-s)^2} \right] |h_s|^2 \, ds.$$

We have the bound

$$\mathbb{E}_{xy}, d_M(\gamma_s^t, y)^4 \le C \left(1 + |h|_{\mathscr{H}}^4 |t|^4 \right) (1-s)^2.$$

It follows that there is a constant C (depending on t and $|h|_{\mathscr{H}}$) such that

$$\mathbb{E}_{xy} \int_0^1 \left[\frac{d_M(\gamma_s^t, y)^4}{(1-s)^4} + \frac{1}{(1-s)^2} \right] |h_s|^2 \, ds \le C \int_0^1 \left| \frac{h_s}{1-s} \right|^2 ds \le 4C |h|_{\mathscr{H}}.$$

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Jump processes, $\mathcal{L}$-harmonic functions, continuity estimates

Moritz Kassmann

(joint work with R. Husseini and M. Barlow/R. Bass/Z.-Q. Chen)

Given a family of Lévy measures $\mu(x, dh)$ we study the following set of questions:

- When can one construct corrsponding jump processes in $\mathbb{R}^d$?
- What can one say about heat kernels ?
- Are harmonic functions with respect to theses processes regular ?
- Does the generator extend to the generator of a Feller semi-group ?

This program is not new at all, it is rather natural to raise the questions above and there are several publications on this subject. One way to summarize existing results is: As long as $x \mapsto \mu(x, dh)$ is smooth and $A \mapsto \mu(x, A)$ is asymptotically rotational invariant and non-degenerate, one can construct corresponding Feller processes and harmonic functions are smooth.

In our presentation we concentrated on cases where these two assumptions may fail. First, we explained what can happen in general situations. Assume $\mu(x, dh) = n(x, h)\, dh$ and

$$\frac{c_0}{|h|^{d+\alpha}} \leq n(x,h) \leq \frac{c_1}{|h|^{d+\beta}}, \qquad n(x,h) = n(x+h, -h) \tag{1}$$

where d is the space dimension and $\alpha, \beta \in (0,2)$, $\alpha < \beta$ are two real numbers. With the help of Dirichlet form techniques one can associate a strong Markov process to $\mu(x, dh)$. Quite astonishing is the following result which we presented:

Theorem 14. [2] *Under the above assumption (1) there are discontinuous $\mathcal{L}$-harmonic functions and the martingale problem for*

$$\mathcal{L}u(x) = \text{p. v.} \int \big(u(x+h) - u(x)\big) n(x,h)\, dh$$

is not well-posed.

In [1] we concentrate on continuity a-priori estimates for harmonic functions. Hölder regularity of functions being harmonic w.r.t to diffusions or jump processes has been an important object of studies for many years. The standard method used to prove Hölder regularity in the general case goes back to Krylov and Safonov (1979). At the heart of these techniques are exit time estimates of the following type: There exists a $c > 0$ such that, given any starting point x, a small ball $B(x, r)$ and any measurable set $A \subset B(x, r)$ not too small relative to the ball, the probability of hitting A before leaving $B(x,r)$ the first time is bounded from below by c.

Nonetheless often a much weaker uniform control on the modulus of continuity of harmonic functions is sufficient. In the talk we presented continuity estimates for a class of pure jump processes where the above exit time estimates do not necessarily hold. In fact the probability of hitting A before leaving $B(x,r)$ may tend logarithmicly to 0 for $r \to 0$, as can be illustrated by an example.

As an application a certain kind of uniqueness of the martingale problem is established in [1]. Combining the results of [1] and [2] the problem of well-posedness of the martingale problem for non-local operators is now much better understood than it was before.

Preprints with references to other papers can be found at the web page:
`http://www.iam.uni-bonn.de/~kassmann`

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Resistance forms and heat kernel estimates

Jun Kigami

The notion of resistance form was introduced to characterize a space where every pair of points has a finite resistance. Let $(\mathcal{E},\mathcal{F})$ be a resistance form on X. Under mild assumptions, for any nonempty compact subset B of X,we have the B-Green function $g_B : X \times X \to [0,\infty)$ which satisfies (GB1) and (GB2):

(GF1) $g_B(x,y) = g_B(y,x)$ and $g_B(x,y) = 0$ for any $y \in B$.

(GF2) Define $g_B^x(y) = g_B(x,y)$, then $g_B^x \in \mathcal{F}$ and

$$\mathcal{E}(g_B^x, u) = u(x)$$

for any $u \in \mathcal{F}$ with $u|_B \equiv 0$.

If $(\mathcal{E},\mathcal{F})$ is a local regular Dirichlet form on $L^2(X,\mu)$ as well, we may associate a diffusion process and have

$$E_x(\tau_B) = \int_X g_B(x,y)\mu(dy),$$

where $\tau_B = \inf\{t|X_t \in B\}$.

Applying those facts, we will sutdy an asymptotic behavior of the associated heat kernel. In particular, desired heat kernel estimates are the sub-Gaussian upper estimate (1) and the near diagonal lower sub-Gaussian estimate (2) below.

The upper sub-Gaussian estimate:

$$p(t,x,y) \le \frac{c_1}{V(x,t^{1/\beta})} \exp\Big(- c_2 \Big(\frac{d(x,y)^\beta}{t}\Big)^{\frac{1}{\beta-1}} \Big), \tag{1}$$

where $V(x, r) = \mu(\{y | d(x, y) < r\})$.

The near diagonal lower sub-Gaussian estimate:

$$\frac{c}{V(x, t^{1/\beta})} \leq p(t, x, y) \tag{2}$$

if $d(x, y) \leq \epsilon$.

We will try to find a "good distance" d under which we have (1) and (2)

Theorem Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. Assume that $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(X, \mu)$. Let $p(t, x, y)$ be the heat kernel associated with $(\mathcal{E}, \mathcal{F})$ and μ. Then the following two conditions (A) and (B) are equivalent.
(A) μ has the volume doubling property with respect to the resistance metric associated with the resistance form $(\mathcal{E}, \mathcal{F})$.
(B) There exists a distance d and $\beta > 1$ such that d is quasisymmtric to the resistance metric and (1) and (2) hold.

A logarithmic Sobolev form of the Li-Yau inequality

M. Ledoux
(joint work with D. Bakry)

This is a summary of a paper to appear in Revista Mat. Iberoamericana.

We present a finite dimensional version of the logarithmic Sobolev inequality for heat kernel measures of non-negatively curved diffusion operators that contains and improves upon the Li-Yau parabolic inequality in Riemannian manifolds with non-negative Ricci curvature. This new inequality is of interest already in Euclidean space for the standard Gaussian measure, and provides a finite dimensional extension of the classical logarithmic Sobolev inequality. The result may also be seen as an extended version of the semigroup commutation properties under curvature conditions, and the functional inequality is actually equivalent to the curvature-dimension condition. The proof relies on the classical Bakry-Emery semigroup argument which is improved by the dimension hypothesis together with a suitable integration of the resulting differential inequality. Partial results under arbitrary curvature conditions are also discussed, although the optimal form seems difficult to obtain. The main result may be applied, as time goes to infinity, to reach, in this setting, optimal Euclidean logarithmic Sobolev inequalities for the invariant measure. Exponential Laplace differential inequalities through the Herbst argument furthermore yield diameter bounds and dimensional estimates on the heat kernel volume of balls.

On the range of random walks on percolation clusters

Pierre Mathieu

Let

$$N(t) = \#\{X(s)\,;\,0 \leq s \leq t\}$$

denote the **range** up to time t of a nearest neighbor path in the lattice $\mathbb{Z}^d$, say $(X(s), s \geq 0)$.

When $(X(t), t \geq 0)$ performs a **simple symmetric random walk** on $\mathbb{Z}^d$, the asymptotics of the Laplace transform of $N(t)$ are given by the:

Theorem 15. *(M.D. Donsker, S.R.S. Varadhan,* [3]*)*
Let E denote the expectation with respect to the law of the simple symmetric random walk on $\mathbb{Z}^d$. Let $\alpha \in]0,1[$. As t tends to $+\infty$,

$$t^{-d/(2+d)} \log E(\alpha^{N(t)})$$

converges to some limit: $c_d(\alpha) \in]-\infty, 0[$.

This abstract reports on our attempt to obtain a similar bound for the symmetric random walk on a percolation cluster.

Random walks on percolation clusters

Consider super critical Bernoulli bond percolation in $\mathbb{Z}^d$: let $\mathbb{E}_d$ be the set of edges of the grid $\mathbb{Z}^d$. Let Ω be the set of sub-graphs of $(\mathbb{Z}^d, \mathbb{E}_d)$. We identify Ω with the product space $\{0,1\}^{\mathbb{E}_d}$ and define Q to be the product Bernouilli(p) measure on Ω.
We assume that $p > p_c$, where p_c is the critical probability for the appearance of an infinite connected component in ω, see [5]. Thus Q.a.s. the graph ω has a unique infinite connected component, the so-called **infinite cluster**, that we will denote with $\mathcal{C}(\omega)$. Finally define

$$Q_0(.) = Q(.|0 \in \mathcal{C}(\omega))\,.$$

For a given $\omega \in \Omega$ such that $0 \in \mathcal{C}(\omega)$, let P^ω be the law of the **simple symmetric random walk** on $\mathcal{C}(\omega)$ started at point 0. We use the notation E^ω for the expectation with respect to P^ω.

Theorem 16. *(C. Rau, in preparation)*
Let $\alpha \in]0,1[$. There exist constants $C_1 = C_1(\alpha, p, d)$ and $C_2 = C_2(\alpha, p, d)$ such that Q_0 almost surely, for large enough times t one has

$$e^{-C_1 t^{-d/(2+d)}} \leq E^\omega(\alpha^{N(t)}) \leq e^{-C_2 t^{-d/(2+d)}}\,.$$

Comparing with Theorem 15, one notices that the order of magnitude of the Laplace transform of $N(t)$ is the same as for the random walk on the full grid $\mathbb{Z}^d$.

We do not know wether $t^{-d/(2+d)} \log E^\omega(\alpha^{N(t)})$ actually has a limit or not.

Some indication on the proof: The lower bound is easy to prove. Below, we only discuss the upper bound when $\alpha = 1/2$.

Following [4], we use the interpretation of the Laplace transform of $N(t)$ as a return probability for a Markov chain on the **wreath product** $\mathcal{C}(\omega) \wr \{0,1\}$, say $E^\omega[2^{-N(t)} 1_{X(t)=0}] = P^\omega[Z(t) = 0]$. To get an upper bound on $P^\omega[Z(t) = 0]$, we first obtain a sharp control on the isoperimetric profile of the cluster $\mathcal{C}(\omega)$ in the spirit of [6], and then deduce isoperimetric bounds on the wreath product as in [4]. Using general results on the link between isoperimetric inequalities, Nash inequalities and return probabilities as in [2], we can conclude the proof.

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Sharp decay rates for the fastest conservative diffusions

Robert J. McCann

In many diffusive settings, initial disturbances will gradually disappear and all but their crudest features — such as size and location — will eventually be forgotten. Quantifying the rate at which this information is lost is sometimes a question of central interest. Joint works with Yong Jung Kim (KAIST) and Dejan Slepcev (UCLA) address this issue for the conservative nonlinearities in a model problem known as the fast diffusion equation

$$u_t = \Delta(u^m), \qquad (n-2)_+/n < m < 1 \quad u, t \geq 0, \quad x \in \mathbf{R}^n,$$

which governs the decay of any integrable, compactly supported initial density towards a characteristically spreading self-similar profile. For other values of the parameter m, this equation has been used to model heat transport, population spreading, fluid seepage, curvature flows, and avalanches in sandpiles.

For the fastest conservative nonlinearities $m \leq n/(n+2)$, we develop a potential theoretic comparison technique with Kim which establishes the sharp conjectured power law rate of decay $1/t$ uniformly in relative error, and in weaker norms such as $L^1(\mathbf{R}^n)$. For nonlinearities $m \geq (n-1)/n$ we attain nearly the same $L^1(\mathbf{R}^n)$ rate as a second order asymptotic after centering the mass of the solution, using an entropy dissipation approach with Slepcev. This leaves a gap in dimensions $n \geq 3$.

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Besov Spaces on Fractals and Conformal Geometry

Hervé Pajot

The main goal of this talk is to explain how to define classical tools of analysis (like Besov spaces or Poincaré inequalities) in singular spaces and how to use them to study some problems in conformal geometry.

Let (X, d, μ) be a metric measure space. Assume that μ is Q-regular, that is there exists $C > 0$ such that

$$C^{-1}R^Q \leq \mu(B(x,R)) \leq CR^Q$$

whenever $x \in X$ and $R \in (0, \mathrm{diam} X)$. Note that Q is the Hausdorff dimension of (X, d). We will say that d is Q-regular if the Hausdorff measure of d is Q-regular.

For any $p \geq 1$, the Besov p-norm of the function $u : X \to \mathbb{R}$ is given by

$$||u||_p = \left(\int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{2Q}} d\mu(x) d\mu(y) \right)^{1/p}.$$

Given two functions u and v, we write $u \sim v$ if $u - v$ is constant almost everywhere (with respect to μ). For $p \geq 1$, we define the Besov space $B_p(d)$ by

$$B_p(d) = \{u : X \to \mathbb{R} \text{ in } L^p(\mu); ||u||_p < +\infty\}/\sim.$$

In the Euclidean case of $\mathbb{R}^n$, $B_p(d_{\mathrm{eucl}})$ is the classical Besov space $B^{n/p}_{p,p}(\mathbb{R}^n)$ (if $n/p < 1$).

Recall that an homeomorphism $f : (Z_1, d_1) \to (Z_2, d_2)$ is quasisymmetric (or shortly QS) if there exists an increasing homeomorphism $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d_1(x,y) \leq t d_1(x,z) \Longrightarrow d_2(f(x), f(y)) \leq \phi(t) d_2(f(x), f(z))$$

whenever x, y, $z \in Z_1$, $t \in (0, +\infty)$. The conformal gauge of the metric space (X, d) is defined by

$$\mathcal{J}(X,d) = \{\delta \text{ distance on } X; Id : (X,d) \to (X,\delta) \text{ is QS}\}.$$

The conformal dimension of (X, d) is

$$\mathrm{Cdim}(X,d) = \inf\{\mathrm{Hdim}(X,\delta); \delta \in \mathcal{J}(X,d)\}.$$

In [1], first groups of l_p-cohomology associated to the conformal gauge $\mathcal{J}(X,d)$ are construted (if (X,d) is also compact and uniformly perfect). They are denoted by $l_pH^1(\mathcal{J}(x,d))$. The critical exponent of l_p cohomology is given by

$$p(\mathcal{J}(X,d)) = \inf\{p \in [1,+\infty); l_pH^1(\mathcal{J}(x,d)) \neq \{0\}\}.$$

It turns out that $l_pH^1(\mathcal{J}(x,d))$ and $B_p(d)$ are isomorphic as Banach spaces. Hence,

$$p(\mathcal{J}(X,d)) = \inf\{p \in [1,+\infty), B_p(d) \neq \{0\}\}.$$

The conformal dimension and the critical exponent of l_p cohomology are invariant under QS homeomorphisms. Are they equal in general ?

Theorem [1]. *Let (X,d) be a compact, uniformly perfect metric space. Assume that there exists $\delta \in \mathcal{J}(x,d)$ such that δ is Q_0-regular and admits a $(1,Q_0)$-Poincaré inequality (see the definition below) for some $Q_0 > 1$. Then,*

$$Cdim(X,d) = p(\mathcal{J}(X,d)).$$

In fact, by a result of Tyson [3], $Q_0 = \mathrm{Cdim}(X,d)$. Observe that Lipschitz functions are in $B_p(d)$ for $p > Q_0$ and hence $p(\mathcal{J}(X,d) \leq Q_0$. To get the reverse inequality, we prove that any function in $B_p(d)$ with $p < Q_0$ is constant almost everywhere by using the Poincaré inequality.

Let $p \geq 1$. We say that (X,d,μ) supports a $(1,p)$ Poincaré inequality if there exists a constant $C_p > 0$ such that

$$\frac{1}{\mu_X(B)}\int_B |u-u_B|d\mu_X \leq C_p \mathrm{diam} B \left(\frac{1}{\mu_X(B)}\int_B \rho^p d\mu_X\right)^{\frac{1}{p}}$$

whenever
- B is a open ball in X,
- $u : X \to \mathbb{R}$ is continuous and bounded in B,
- $\rho : X \to \overline{\mathbb{R}^+}$ is an upper gradient of u in B,

and where u_B denotes the mean value of u on B. Recall that $\rho : X \to \overline{\mathbb{R}^+}$ is an upper gradient of u in B if for any $x \in X$, any $y \in X$, any rectifiable curve γ in X connecting x to y, we have $|u(x) - u(y)| \leq \int_\gamma \rho$. We say that d admits a $(1,p)$-Poincaré if (X,d,μ) (where μ is the Hausdorff mesure of d) supports a $(1,p)$-Poincaré inequality in the previous sense.

What happens in fractals, for instance for the Sierpinski carpet (SC) and the Sierpinski gasket (SG) ?

$$\mathrm{Hdim}(SC) = \log(3)/\log(2) \qquad \mathrm{Cdim}(SC) = 1 \qquad p(SC) = ???,$$

$$\mathrm{Hdim}(SG) = \log(8)/\log(3) \qquad \mathrm{Cdim}(SC) = ??? \qquad p(SC) = ???.$$

In fact, the conformal dimension of other pcf sets in the sense of Kigami (like the Sierpinski gasket) is also one (see [4]. Note that, by standard arguments, we get $1 + \log(2)/\log(3) \leq \mathrm{Hdim}(SC) < \log(8)/\log(3)$. A good general reference concerning conformal dimension and related topics is the recent book of Heinonen [2].

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Stochastic Generalized Porous Media Equations

M. Röckner

(joint work with J.-G. Ren, F.-Y. Wang)

Recall that for Δ, the Laplace operator on $\mathbb{R}^d$, the following differential equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \Delta\big(|X_t|^r \ \mathrm{sign}\, X_t\big)$$

is called the porous medium equation if $r > 1$ and the fast diffusion equation if $r \in (0,1)$. A special feature of this equation is that the solution decays algebraically fast in t when $r > 1$ and decays to zero at finite time when $r \in (0,1)$. For historical remarks and recent progress on this equation, we refer to [2, 1, 11] and the references therein.

In recent years, the stochastic version of the porous medium equation has been studied intensively, see [8] for the existence, uniqueness and long-time behavior of some stochastic generalized porous media equations with finite reference measures, [12] for the stochastic porous media equation on $\mathbb{R}^d$ where the reference (Lebesgue) measure is infinite and [3, 7] for the analysis of the corresponding Kolmogorov equations. See also [19] for large deviation results for a class of generalized porous media equations.

The basic motivation of this work is to study, for example, the following stochastic porous media and fast diffusion equation:

$$\mathrm{d}X_t = L\Big(\sum_{i=1}^{l} \alpha_i |X_t|^{r_i} \ \mathrm{sign}\, X_t\Big) + B(t, X_t)\ \mathrm{d}W_t\ , \tag{1}$$

where L is a negative definite self-adjoint operator on $L^2(\mathbf{m})$ for some σ-finite measure $\mathbf{m}$, W_t is the Brownian motion on a reference Hilbert space, and B a properly defined linear operator from this Hilbert space to the state space of X_t. Here, $l \in \mathbb{N}$ and $r_i, \alpha_i > 0$, $i = 1, \ldots, l$, are fixed numbers. If $r_i < 1$ and $r_j > 1$ for some $i, j \leq l$, we call this equation the stochastic porous media and fast diffusion equation. As a consequence of our main result, this equation has a unique solution as soon as B satisfies a Lipschitz type condition.

To solve such an equation, we first establish a general result for "monotone" stochastic equations, which extends the corresponding one of Krylov and Rozovskii ([13, Theorems II.2.1 and II.2.2]). As a direct application, we are able to solve (1) with $\sum_{i=1}^{l} \alpha_i |X_t|^{r_i} \operatorname{sign}(X_t)$ replaced by $\Psi(t, X_t)$ for a more general time–dependent function Ψ comparable with an N-function. The Orlicz space (cf. [16]) determined by N plays an essential role in our analysis.

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Variational convergence over metric spaces

Takashi Shioya
(joint work with K. Kuwae)

The talk is on the joint work [7, 8] with K. Kuwae.

Let $M_i \to M$ and $Y_i \to Y$ $(i = 1, 2, 3, \dots)$ be two pointed Gromov-Hausdorff convergent sequences of proper metric spaces, where '*proper*' means that any bounded subset is relatively compact, and let us give measures on M_i which converge to a measure on M with respect to the measured Gromov-Hausdorff topology. We are interested in the convergence and asymptotic behavior of maps $u_i : M_i \to Y_i$ and also energy functionals E_i defined on a family of maps from $M_i \to Y_i$. Note that there are several attempts to define natural energy functionals on the mapping space from M to Y by the measured metric structure of M and the metric structure of Y. We introduce a natural definition of the L^p-convergence of $u_i : M_i \to Y_i$ to $u : M \to Y$, $p \geq 1$, and establish a general theory for energy functionals E_i by extending the theory of variational convergences, mainly studied by Mosco [9]. Mosco introduced the asymptotic compactness of energy functionals $\{E_i\}$, which is a generalization of the Rellich compactness. The asymptotic compactness is useful to obtain the convergence of energy minimizers, i.e., harmonic maps, and also to investigate spectral properties in the linear case. Under a uniform bound of Poincaré constants and some property of the metric on M, we prove the asymptotic compactness of $\{E_i\}$. We focus on a Γ-convergence with the asymptotic compactness, say *compact convergence*. If $\{E_i\}$ is asymptotically compact, it has a compact convergent subsequence. We prove that the compact convergence is equivalent to the Gromov-Hausdorff convergence of the energy-sublevel sets, which is a geometric interpretation of compact convergence. This is connected with Gromov's study of spectral concentration, Section $3\frac{1}{2}.57$ of [1]. We also prove that the compact convergence of E_i is equivalent to a convergence of associated resolvents, provided that Y_i are all $CAT(0)$-spaces and E_i are lower semi-continuous convex functionals. Such the resolvent was defined by Jost [2] using the Moreau-Yosida approximation. As applications of our theory, we study the approximating energy functional and its spectral property. We also obtain the compactness of energy functionals if M_i are Riemannian manifolds with a lower bound of Ricci curvature. In the case where E_i are symmetric quadratic forms on L^2-spaces, there are some applications of our theory to a homogenization problem, [11], and convergence of Dirichlet forms, [3, 4, 10], including finite-dimensional approximation problems, [5, 6].

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The principal eigenvalue for time-changed processes and applications

Masayoshi Takeda

1. Gaugeability for Feynman-Kac Functionals

Chen [1] and Takeda [3] showed that the gaugeability, that is, the integrability of Feynman-Kac functionals, or the subcriticality of Schrödinger operators is equivalent to that the principal eigenvalue of time changed process is greater than one (Theorem 17 below). The objective of my talk is to give two applications of this result.

Let M be a non-compact, complete Riemannian manifold and m the volume. Let $\mathbb{M} = (\mathbb{P}_x, B_t)$ be the Brownian motion on M. For a domain $D \subset M$, let $\mathbb{M}^D$ be the absorbing Brownian motion on D. We assume that $\mathbb{M}^D$ is transient. Chen [1] generalized the notion of *Green-tightness* for general transient Markov processes and introduced a new Kato class. We denote by $\mathcal{S}^D_\infty$ this class associated with $\mathbb{M}^D$. For $\mu \in \mathcal{S}^D_\infty$, define

$$\lambda(\mu; D) = \inf\left\{\frac{1}{2}\int_D (\nabla u, \nabla v)\, dm :\ u \in C_0^\infty(D),\ \int_D u^2(x)\mu(dx) = 1\right\}.$$

Then $\lambda(\mu; D)$ is the principal eigenvalue of the time-changed process of $\mathbb{M}^D$ by A^μ_t. Here A^μ_t is the positive continuous additive functional in the Revus correspondence with μ. Let $p^{\mu,D}_t(x, y)$ be the integral kernel of *Feynman-Kac semigroup*:

$$p^{\mu,D}_t f(x) := \mathbb{E}_x[\exp(A^\mu_t) f(B_t); t < \tau_D] = \int_D p^{\mu,D}_t(x, y) f(y) dy$$

($\tau_D = \inf\{t > 0 : B_t \notin D\}$) and $G^{\mu,D}(x,y) := \int_0^\infty p_t^{\mu,D}(x,y)dt$.@

Theorem 17. ([1],[3]) *Let $\mu \in \mathcal{S}_\infty^D$. Then the following statements are equivalent:*

(1) (i)] (**gaugeability**) $\sup_{x\in D} \mathbb{E}_x[e^{A^\mu_{\tau_D}}] < \infty$;

(ii) (**subcriticality**) $G^{\mu,D}(x,y) < \infty$ *for* $x, y \in D$, $x \neq y$;

(iii) $\lambda(\mu; D) > 1$.

2. Applications of Theorem 17

(i) Let $\bar{\mathbb{B}} = (\bar{\mathbb{P}}_x, \bar{B}_t)$ be a branching Brownian motion with **branching rate** k and **branching mechanism** $\{p_n(x)\}_{n\geq 2}$: $\bar{\mathbb{P}}_x[T > t|\sigma(B)] = \exp(-A_t^k)$ (T is the **first splitting time**), $\sum_{n=2}^\infty p_n(x) = 1$. Set $Q(x) = \sum_{n\geq 2} np_n(x)$ and $\mu(dx) = (Q(x)-1)k(dx)$. We assume that $\sup_{x\in M} Q(x) < \infty$.

Theorem 18. ([4]) *Let K be a closed set with* $\mathrm{Cap}(K) > 0$ *and let* $\mu \in \mathcal{S}_\infty^D$, $D = M \setminus K$. *Then*

$$\lambda(\mu; D) > 1 \Longleftrightarrow \sup_{x\in D} \bar{\mathbb{E}}_x[N_K] < \infty.$$

Here N_K is the number of branches hitting K.

R.Z. Khas'minskii gave a sufficient condition for gaugeability known as Khas'minskii lemma, and applied it for showing the finiteness of the expectation of N_K. To prove Theorem 18, we show that for $\mu \in \mathcal{S}_\infty^D$

$$\sup_{x\in D} \mathbb{E}_x[\exp(A^\mu_{\tau_D})] < \infty \Longleftrightarrow \sup_{x\in D} \mathbb{E}_x[\exp(A^\mu_{\tau_D}); \tau_D < \infty] < \infty.$$

Then the identity

$$\bar{\mathbb{E}}_x[N_K] = \mathbb{E}_x[\exp(A^\mu_{\tau_D}); \tau_D < \infty]$$

leads us to Theorem 18.

(ii) Let $d(x,y)$ be the distance. Let $p(t,x,y)$ be the heat kernel. Suppose that it satisfies the Gaussian lower and upper bounds (**Li-Yau estimate**): for any $x, y \in M$ and $t > 0$,

$$\frac{C_1 \exp\left(-c_1 \frac{d^2(x,y)}{t}\right)}{m(B(x,\sqrt{t}))} \leq p(t,x,y) \leq \frac{C_2 \exp\left(-c_2 \frac{d^2(x,y)}{t}\right)}{m(B(x,\sqrt{t}))}$$

(C_1, c_1, C_2, c_2 are positive constants. $B(x,r) = \{y \in M : d(x,y) < r\}$). Let $p^\mu(t,x,y)$ be the heat kernel of the Schrödinger operator, $\frac{1}{2}\Delta + \mu$.

Theorem 19. ([5]) *Let $\mu \in \mathcal{S}_\infty^M$. Then $p^\mu(t,x,y)$ satisfies the Li-Yau estimate if and only if $\lambda(\mu; M) > 1$.*

Example: Let $M = \mathbb{R}^d$ ($d \geq 3$) and $\mu = \sigma_r$ the surface measure of $\{|x| = r\}$. Then

$$\lambda(\sigma_r; \mathbb{R}^d) = \frac{d-2}{2r}.$$

Hence $p^{\sigma_r}(t,x,y)$ satisfies the Gaussian estimate, if and only if $r < \frac{d-2}{2}$.

Remark 1. Let $d = 3$ and $\lambda(\mu; \mathbb{R}^3) = 1$, for example, $\mu = \sigma_{1/2}$. Then

$$p^{\mu}(t,x,y) \simeq \frac{C_1}{t^{3/2}}\left(1+\frac{\sqrt{t}}{< x >}\right)\left(1+\frac{\sqrt{t}}{< y >}\right)\exp\left(-c_1\frac{|x-y|^2}{t}\right)$$

($< x >:= 1+|x|$) (cf. [2, Theorem 10.15]).

Remark 2. Let $\mu = \mu^+ - \mu^-$ be a signed measures such that $\mu^+ \in \mathcal{S}_\infty^M$ and $\sup_{x\in M} G\mu^-(x) < \infty$. Then the Li-Yau estimate of the heat kernel $p^\mu(t,x,y)$ is equivalent to

$$\inf\{\mathcal{E}(u,u) + \int_M u^2 d\mu^- : u \in C_0^\infty(M), \int_M u^2 d\mu^+ = 1\} > 1.$$

For $\mu = -\mu^-$, the result above is shown in [2, Theorem 10.5].

Remark 3. Theorem 19 can be extended to the symmetric α-stable process ($d > \alpha$). Let $\mu \in \mathcal{S}_\infty^{\mathbb{R}^d}$ with $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^{\alpha-d} d\mu(x)d\mu(y) < \infty$. Then the heat kernel of $-\frac{1}{2}(-\Delta)^{\alpha/2} + \mu$ satisfies

$$C_1\left(\frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \le p^\mu(t,x,y) \le C_2\left(\frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),$$

if and only if

$$\lambda(\mu) = \inf\{\mathcal{E}^\alpha(u,u) : \ u \in C_0^\infty(\mathbb{R}^d), \ \int_{\mathbb{R}^d} u^2 d\mu = 1\} > 1.$$

Here

$$\mathcal{E}^{(\alpha)}(u,u) = K(\alpha,d)\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}dxdy.$$

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Brownian motion on Jordan curves and stochastic calculus of variations on the diffeomorphism group of the circle

Anton Thalmaier
(joint work with Hélène Airault and Paul Malliavin)

We report on recent progress of constructing unitarizing probability measures for the representation of the Virasoro algebra $\mathcal{V}$, using methods of Stochastic Analysis.

Denote by $\mathrm{Diff}(S^1)$ the group of orientation preserving diffeomorphisms of the circle S^1 and by $\mathrm{diff}(S^1)$ its Lie algebra. With the identification $\mathrm{diff}(S^1) \cong C^\infty(S^1;\mathbb{R})$ the Lie bracket takes the form $[u,v] = u\dot{v} - \dot{u}v$. We are interested in central extensions of $\mathrm{Diff}(S^1)$ and $\mathrm{diff}(S^1)$, the *Virasoro group*, resp., *Virasoro algebra.* In explicit terms the Virasoro algebra is given (modulo equivalence) by

$$\mathcal{V} \equiv \mathcal{V}_c = \mathbb{R} \oplus \mathrm{diff}(S^1)$$

with the Lie bracket $[s\kappa + u, t\kappa + v] = \frac{c}{12}\,\omega(u,v)\kappa + [u,v]$ (κ *central element*, $c > 0$ *central charge*), where ω an antisymmetric bilinear form satisfying the *cocycle condition* $\omega([u,v],w)+\omega([v,w],u)+\omega([w,u],v) = 0$. Such forms have been classified by Gelfand-Fuks and are modulo the choice of constants given by the fundamental cocycle (1.2.1) [2]. There are two well-known representations of $\mathcal{V}$:

- on the space $\mathcal{U}^\infty$ of univalent functions on the disk
- on the Sato Grassmannian via an embedding of $\mathrm{Diff}(S^1)$ into $\mathrm{Sp}(\infty)$.

For both the diffeomorphism group $\mathrm{Diff}(S^1)$ of the circle S^1 is a key object [2].

Brownian motion on $\mathrm{Diff}(S^1)$**.** Denote by $\mathscr{H} \subset \mathrm{Diff}(S^1)$ the restrictions to S^1 of the Poincaré group of homographic transformations. The Lie algebra of $\mathscr{H}$ is $\mathfrak{su}(1,1)$ and generated by the vector fields $\cos\theta, \sin\theta, 1$.

Theorem 20 ([3])**.** *There exists a unique semi-Hilbertian metric on* $\mathrm{diff}(S^1)$ *which is invariant under the adjoint action of* $\mathfrak{su}(1,1)$*.*

Let $\mathrm{Diff}(S^1)/S^1$ be the diffeomorphisms of S^1 which fix a point on the circle and let $\mathrm{diff}_0(S^1) = \{u \in \mathrm{diff}(S^1) : \int_{S^1} u(\theta)\,d\theta = 0\}$. Developing $u \in \mathrm{diff}_0(S^1)$ in terms of a Fourier series $u(\theta) = \sum_{k\geq 1}(a_k \cos k\theta + b_k \sin k\theta)$, let $\mathcal{J}u(\theta) = \sum_{k\geq 1}(-a_k \sin k\theta + b_k \cos k\theta)$. Then $\mathcal{J}^2 = -\,\mathrm{id}$ and $\mathcal{J}$ defines an *almost complex structure* on $\mathrm{diff}_0(S^1)$ which is integrable. The *Kähler metric* on $\mathrm{diff}_0(S^1)$ is

$$\|u\|^2 := \omega(u, \mathcal{J}u) = \sum_{k\geq 1} \alpha_k^2\,(a_k^2 + b_k^2), \quad \alpha_k = (k^3 - k)^{1/2}.$$

The Lie subalgebra of $\mathrm{diff}(S^1)$ generated by 1, sin, cos is $\mathfrak{su}(1,1)$, and

$$\mathrm{diff}(S^1)/\mathfrak{su}(1,1) = \Big\{\theta \mapsto \sum_{k\geq 2}(a_k \cos k\theta + b_k \sin k\theta)\Big\} \mathrel{\widehat{=}} H^{3/2}(S^1).$$

Brownian motion on the *universal Teichmüller space* $\mathrm{Diff}(S^1)/\,\mathrm{SU}(1,1)$, with respect to the $H^{3/2}$ metric, is constructed by exponentiating BM on $\mathrm{diff}(S^1)$, see

[10], [5], [8]. Let $x_k(t)$ be an independent sequence of scalar Brownian motions, and let

$$e_{2k+1} := \alpha_k^{-1}\sin k\theta, \quad e_{2k} := \alpha_k^{-1}\cos k\theta, \quad \alpha_k = \sqrt{k^3-k}, \quad k>1.$$

Brownian motion on $\mathrm{Diff}(S^1)$ is *formally* a solution of the Stratonovich SDE

$$(*) \qquad dg_t = \sum_{k>1}\Big(e_{2k}(g_t)\circ dx_{2k}(t) + e_{2k+1}(g_t)\circ dx_{2k+1}(t)\Big).$$

Solving $(*)$ via *Abel regularization*, we fix $\rho>0$ and replace α_k by $\sqrt{k^3-k}/\rho^k$. For $\rho<1$, the theory of stochastic flow of diffeomorphisms guarantees a unique solution g^ρ which takes values in the smooth diffeomorphisms of S^1.

Theorem 21 (**Limiting case** $\rho=1$)**.** *As $\rho\nearrow 1$, the limit $g_t(\theta)$ of $g_t^\rho(\theta)$ exists uniformly in θ and defines a solution of $(*)$, but $(\theta\mapsto g_t^\rho(\theta))\in \mathrm{Homeo}(S^1)$ only.*

Brownian motion on Jordan curves [4]**.** Let $\mathcal{J}=\{\Gamma\subset\mathbb{C}:\Gamma \text{ Jordan curve}\}$. To $\Gamma\in\mathcal{J}$ there is a continuous injective parametrization $\varphi: S^1\to\mathbb{C}$ such that $\varphi(S^1)=\Gamma$. *Holomorphic parametrizations* are constructed as follows. Each Jordan curve Γ splits the complex plane into two simply connected domains D_Γ^+, D_Γ^-. Let D be the open unit disc in $\mathbb{C}$. By the Riemann mapping theorem, there exist biholomorphic mappings $F^+\colon D\to D_\Gamma^+$ and $F^-\colon D\to D_\Gamma^-$ (unique modulo $\mathrm{SU}(1,1)$) which by Caratheodory extend to homeomorphisms $F^+\colon \bar D\to\bar D_\Gamma^+$, $F^-\colon\bar D\to\bar D_\Gamma^-$. Then $F^\pm|S^1$ parametrize Γ, and $g_\Gamma := (F^+)^{-1}\circ F^-|S^1$ defines an orientation preserving homeomorphism of S^1. Now let $\mathcal{J}^\infty := C^\infty$ Jordan curves .

Theorem 22 (Beurling-Ahlfors "*conformal welding*")**.** *The mapping $\Gamma\mapsto g_\Gamma$ from $\mathcal{J}^\infty$ to $\mathrm{Diff}(S^1)$ is surjective and induces a canonical isomorphism*

$$\mathcal{J}^\infty \cong \mathrm{SU}(1,1)\backslash \mathrm{Diff}(S^1)/\,\mathrm{SU}(1,1).$$

To construct BM on Jordan curves from BM "on $\mathrm{Diff}(S^1)$" we have to factorize g_t as $g_t=(F_t^+)^{-1}\circ F_t^-|S^1$. By conformal welding we first associate to the regularized process $t\mapsto {}^\rho g_t$ on $\mathrm{Diff}(S^1)$ a process $t\mapsto {}^\rho\Gamma_t$ with values in $\mathcal{J}^\infty$.

Theorem 23 (Brownian motion on Jordan curves [4])**.** *As $\rho\nearrow 1$, the regularized process $t\mapsto {}^\rho\Gamma_t$ converges to a diffusion on the space of continuous Jordan curves.*

Let $\mathcal{U}^+$ denote the space of univalent functions on D which are continuous and injective on $\bar D$, resp. $\mathcal{U}^-$ the equivalent space of functions, univalent on $\{z\in\mathbb{C}:|z|>1\}$, continuous and injective on $\{z\in\mathbb{C}:|z|\ge 1\}$.

Theorem 24 (Stochastic Sewing Theorem)**.** *There exist processes f_t^+, f_t^-, taking values in $\mathcal{U}^+$, resp. in $\mathcal{U}^-$, such that $g_t=(f_t^+)^{-1}\circ f_t^-|S^1$.*

Brownian motion on univalent functions. Denote by $\mathcal{U}^\infty$ the space of univalent functions $f\in C^\infty(\bar D;\mathbb{C})$, normalized by $f(0)=0$, $f'(0)=1$. To $f\in\mathcal{U}^\infty$ we associate the C^∞ Jordan curve $\Gamma=f(\partial D)$ around the origin 0. The Riemann mapping theorem provides a biholomorphic map $h_f\colon\{z\in\mathbb{C}:|z|>1\}\xrightarrow{\sim} D_\Gamma^-$ with $h_f(\infty)=\infty$, which extends to a diffeomorphism $h_f\colon\{z\in\mathbb{C}:|z|\ge 1\}\xrightarrow{\sim}\bar D_\Gamma^-$.

By Kirillov (1982), the mapping $\mathcal{U}^\infty \ni f \mapsto g_f := (f^{-1} \circ h_f)|S^1 \in \mathrm{Diff}(S^1)$ induces an isomorphism $\mathcal{U}^\infty \cong \mathrm{Diff}(S^1)/S^1$. In particular, $\mathcal{U}^\infty$ is a homogeneous space under the left action of $\mathrm{Diff}(S^1)$.

Theorem 25 ([4]). *As $\rho \nearrow 1$, the regularized process with values in $\mathcal{U}^\infty$ constructed from the motion $t \mapsto {}^\rho g_t$ on $\mathrm{Diff}(S^1)$ converges to a diffusion $t \mapsto f_t$ taking values in the univalent functions on D which are continuous and injective on $\bar{D}$.*

Unitarizing measures. We read $u \in \mathrm{diff}(S^1)$ as right-invariant vector field $\hat{u}$ on $\mathrm{Diff}(S^1)$, and consider the *Neretin differential form* Ω on $\mathrm{Diff}(S^1)$, see (2.2.5) [2].

Theorem 26. *Let $\mathscr{M} = \mathcal{U}^\infty = \mathrm{Diff}(S^1)/S^1$ and denote by $\mathcal{H}(\mathscr{M})$ the space of holomorphic functionals on $\mathscr{M}$. For any $c > 1$, we get a representation ρ of $\mathcal{V}_c$ on $\mathcal{H}(\mathscr{M})$ by defining $\rho(u)\phi = \langle \hat{u}, d\phi \rangle + c\,\langle \hat{u}, \Omega \rangle$ and $\rho(\kappa)\phi = i\phi\, c/12$.*

There exists a function K ("*Kähler potential*") on $\mathrm{Diff}(S^1)$, invariant under $\mathrm{SU}(1,1)$, such that $\partial\bar{\partial}K = \omega$ and $3\,\bar{\partial}K = \Omega$. Unitarizing measures should *formally* be of the type "$\mu_\gamma = c_0 \exp(-c\,K)\,d\mathrm{vol}$" where $d\mathrm{vol} \equiv$ "volume measure" of the infinite dimensional Kähler manifold $\Delta_{\mathscr{M}_1} := \mathrm{Diff}(S^1)/\Gamma$ with $\Gamma = \mathrm{SU}(1,1)$

Consider the horizontal Laplacian $\Delta^H = \frac{1}{2}\sum(\partial^r_{e_k})^2$ on $\mathrm{Diff}(S^1)$ and the Laplace Beltrami operator $\Delta_{\mathscr{M}_1}$ to the Kähler metric on $\mathscr{M}_1$. Let π be the projection from $\mathrm{Diff}(S^1)$ to $\mathscr{M}_1$. Then $\Delta^H(\phi \circ \pi) = (\Delta_{\mathscr{M}_1}\phi) \circ \pi$. By solving an appropriate SDE, a diffusion on $\mathscr{M}_1$ is constructed to the operator $\Delta_{\mathscr{M}_1} - c\,\nabla K \cdot \nabla$ where $\nabla K \cdot \nabla = \sum_k (\partial^r_{e_k} K)\,\partial^r_{e_k}$. The vector field ∇K is called *unitarizing drift*.

Theorem 27. *The Ornstein-Uhlenbeck process associated to $\Delta_{\mathscr{M}_1} - c\,\nabla K \cdot \nabla$ exists on the group of Hölder homeomorphisms.*

Theorem 28 (Airault-Malliavin-Thalmaier [3]). *An unitarizing measure must be supported by $\mathrm{Homeo}(S^1)/\Gamma$ and will be an invariant measure for the Ornstein-Uhlenbeck diffusion to $\Delta_{\mathscr{M}_1} - c\,\nabla K \cdot \nabla$, where c is the central charge.*

Theorem 29 (Bowick-Rajeev [6], Airault [1], Gordina-Lescot [9]).

$$\mathrm{Ricci}(\mathrm{Diff}(S^1)/\Gamma) = -13/6 \times \text{identity}.$$

Theorem 30. *For $\gamma > 13/6$ an invariant measure exists and is unique.*

The proof uses the method of "*confinement under curvature positivity*" [7].

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Optimal transport, Ricci curvature and synthetic geometry

Cedric Villani

In this survey talk, I reported on a recent direction of research mixing analysis, geometry and probability theory. I first illustrated the power of monotone changes of variables with an elementary proof of optimal Sobolev inequalities (joint works with Cordero-Erausquin, Nazaret and Maggi). On a Riemannian manifold, there is a "canonical" construction of monotone maps based on optimal transport theory with cost equal to the square of the geodesic distance. This leads to the interplay between optimal transport and Riemannian geometry, in particular Ricci curvature lower bounds (Otto and I; Cordero-Erausquin, McCann and Schmuckenschläger). These ideas were recently used by Lott and I on one hand, Sturm on the other hand, to start the development of a synthetic theory of Ricci curvature lower bounds on metric-measure length spaces: curvature-dimension bounds $\mathrm{CD}(K, N)$ are defined in terms of the behavior of certain nonlinear convex functions of the density along geodesics of optimal transport.

Much more information can be found in the following references:

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Local Poincare Inequality by Transportation

Max-K von Renesse

The sharp geometric properties of optimal mass transportation [2], [8] can be used to develop a theory of singular metric measure spaces with generalized lower Ricci curvature bounds [7], [3].

The notion of dimension-curvature bounds obtained in this framework are a priori global in nature and do not allow for simple localization unless additionl geodesic convexity of the respective domain is assumed.

In particular it is not at all obvious how to obtain local Poincare type inequalities which are known as a crucial ingredient for regularity and heat kernel estimates on metric measure spaces. For instance, the ad hoc technique of expanding the global logarithmic Sobolev inequality gives only a global Poincare estimate in the positive curvature case [3].

We show how to derive local Poincare inequalities from the related concept of the qualitative Measure Contraction Property (MCP) introduced in [5] and which is implied by the curvature-dimension bounds under weak additional assumptions. The MCP condition is a substitute for the Jacobian estimate for the exponential map in the smooth Riemannian case and it suffices to carry out the argument in [1] in order to obtain the 'segment inequality'. The latter implies a weak (1,1)-Poincare inequality in the sense of upper gradients.

The result is very similar to the work [6] which assumes a 'strong local doubling condition'. A different approach to local Poincare inequalities based on the notion of 'democratic coupling' in the non negative curvature case is taken in [4].

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The spectrum of the averaging operator on a network

Wolfgang Woess
(joint work with Donald I. Cartwright)

Let X be a countable, connected graph with symmetric neighbourhood relation $\sim$ and without loops and multiple edges. We view it as a one-complex, where each edge is a (homeomorphic) copy of the unit interval and edges are glued together at common endpoints (vertices). We write X^0 for the vertex set and X^1 for the one-skeleton of X. Every point in X^1 is of the form (xy, t), the point at distance t from x on the non-oriented edge $[x, y] = [y, x]$, where $0 \le t \le 1$, and $x, y \in X^0$, $x \sim y$. Thus, $(xy, 0) = x$ and $(xy, t) = (yx, 1-t)$. The discrete graph metric $d(\cdot,\cdot)$ on the vertex set has a natural extension to X^1. We equip each edge $[x, y]$ with a positive *conductance*

$\mathsf{c}(xy) = \mathsf{c}(yx)$. On X^0, we consider the discrete measure m^0, where $\mathsf{m}^0(x) = \sum_{y:y\sim x} \mathsf{c}(xy)$, and assume that

$\mathsf{m}^0(x) < \infty$ for all $x \in X^0$. On X^1, we introduce the

continuous weighted "Lebesgue" measure m^1 which at the point

(xy, t) is given by $\mathsf{c}(xy)\cdot dt$, if $0 < t < 1$ (the vertex set has m^1-measure 0). The pair (X, c), together with these measures, is called a *network*. On a network, there are three natural operators.

The first is the *transition operator* P acting on functions $g : X^0 \to \mathbb{C}$ by

$$Pg(x) = \frac{1}{\mathsf{m}^0(x)} \sum_{y:y\sim x} \mathsf{c}(xy)\, g(y)\,.$$

The second is the *Laplace operator* Δ. It can be defined via Dirichlet form theory, or by considering the space of all continuous functions $F : X^1 \to \mathbb{C}$ which are twice differentiable in the interior of each edge and satisfy the Kirchhoff equations

$$\sum_{y:y\sim x} \mathsf{c}(xy)\, F'(xy, 0+) = 0 \quad \text{for all } x \in X^0\,.$$

We then have

$$\Delta F(xy, t) = F''(xy, t)\,,$$

the 2nd derivative with respect to $t \in (0, 1)$, and Δ has to be closed suitably. See e.g. Cattaneo [1], Solomyak [3] or Eells and Fuglede [2] for precise details.

The third operator, and main object of the talk, is the

averaging operator A over balls of radius 1. It acts on locally integrable functions $F : X^1 \to \mathbb{C}$ by

$$AF(xy, t) = \frac{1}{\mathsf{m}^0(x)} \sum_{u\sim x} \mathsf{c}(xu) \int_0^{1-t} F(xu, s)\, ds + \frac{1}{\mathsf{m}^0(y)} \sum_{v\sim y} \mathsf{c}(yv) \int_0^{t} F(yv, s)\, ds\,.$$

In the *regular* case, i.e., when $\mathsf{m}^0(\cdot)$ is constant, this is just the m^1-average of F over the ball with radius 1 centered at (xy, t).

We are interested in the relation between the spectra of the operators A and P. Cattaneo [1] has given a complete description of the H^2-spectrum of Δ in terms of the ℓ^2-spectrum of P. Our plan is to describe the L^2-spectrum of A in terms of the ℓ^2-spectrum of P.

This refers to the (complex) Hilbert spaces $L^2(X^1, \mathsf{m}^1)$ and $\ell^2(X^0, \mathsf{m}^0)$. The inner product of the latter is given by

$$\langle g_1, g_2 \rangle = \sum_{x \in X^0} g_1(x)\overline{g_2(x)}\, \mathsf{m}^0(x)\,,$$

and P is self-adjoint with $\|P\| \le 1$ on this space.

Analogously, the inner product on $L^2(X^1, \mathsf{m}^1)$ is

$$\langle F_1, F_2 \rangle = \frac{1}{2} \sum_{x \in X^0} \sum_{y \in X^0 : y \sim x} \mathsf{c}(xy) \int_0^1 F_1(xy, t)\overline{F_2(xy, t)}\, dt\,.$$

Again, A is self-adjoint with norm bounded by 1 on $L^2(X^1, \mathsf{m}^1)$.

There is a large body of literature on the spectrum of transition (resp. adjacency and discrete Laplace) operators on finite graphs.

Since not much work has been done regarding the spectra of averaging operators on networks, it appears to be useful to have a method for translating the spectrum of P into the spectrum of A. Our main results are the following.

Theorem 1. *The spectrum of A is*

$$\mathsf{spec}(A) = \{0\} \cup \left\{ \frac{\sin\omega}{\omega} : \omega \in \mathbb{R} \setminus \{0\}\,,\ \cos\omega \in \mathsf{spec}(P) \right\} \cup \{1 : 1 \in \mathsf{spec}(P)\}\,.$$

Corollary. *Let $\rho = \rho(P)$ denote the spectral radius of P. Then the spectral radius of A is*

$$\rho(A) = \begin{cases} 1\,, & \text{if} \quad \rho = 1\,, \\ \sqrt{1-\rho^2}\big/\arccos(\rho)\,, & \text{if} \quad \rho < 1\,. \end{cases}$$

Let $\mathsf{spec}_p(P)$ denote the point spectrum of P, i.e., the set of $\ell^2(X^0, \mathsf{m}^0)$-eigenvalues of P.

Theorem 2. *We have*

$$\mathsf{spec}_p(A) \setminus \{0\} = \big\{1 : 1 \in \mathsf{spec}_p(P)\big\} \cup \left\{ \frac{\sin\omega}{\omega} : \omega \in \mathbb{R} \setminus \{0\}, \cos\omega \in \mathsf{spec}_p(P) \right\}.$$

Moreover, $0 \in \mathsf{spec}_p(A)$ *unless* $\mathsf{m}^0(X^0) = \infty$ *and* X *is a tree with the property that after removal of any edge, at least one of the two connected components is recurrent.*

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