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Discrete Differential Geometry

Organised by
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March 5th – March 11th, 2006

Abstract. Discrete Differential Geometry is a broad new area where differential geometry (studying smooth curves, surfaces and other manifolds) interacts with discrete geometry (studying polyhedral manifolds), using tools and ideas from all parts of mathematics. This report documents the 29 lectures at the first Oberwolfach workshop in this subject, with topics ranging from discrete integrable systems, polyhedra, circle packings and tilings to applications in computer graphics and geometry processing. It also includes a list of open problems posed at the problem session.

Mathematics Subject Classification (2000): 52-xx, 53-xx, 57-xx.

Introduction by the Organisers

The workshop *Discrete Differential Geometry*, organized by Alexander I. Bobenko (Berlin), Richard W. Kenyon (Vancouver), John M. Sullivan (Berlin) and Günter M. Ziegler (Berlin), was held March 5th to March 11th, 2006. The meeting was very well attended, with almost 50 participants, from as far away as Australia and China.

Discrete differential geometry is a new and active mathematical terrain where differential geometry (providing the classical theory for smooth manifolds) and discrete geometry (concerned with polytopes, simplicial complexes, etc.) meet and interact. Problems of discrete differential geometry also naturally appear in (and are relevant for) other areas of mathematics. Moreover, the process of discretizing notions, problems and methods from the smooth theory often brings out new connections and interrelations between different areas.

The workshop at Oberwolfach brought together researchers with a wide variety of backgrounds, including of course discrete geometry and differential geometry, but also integrable systems, combinatorics, mathematical physics and geometry processing. The exchange of ideas among different subfields helped to build new bridges between these mathematical communities.

Discrete differential geometry can be said to have arisen from the observation that when a notion from smooth geometry (such as the notion of a minimal surface) is discretized “properly”, the discrete objects are not merely approximations of the smooth ones, but have special properties of their own which make them form in some sense a coherent entity by themselves. The discrete theory would seem to be the more fundamental one: The smooth theory can always be recovered as a limit, while there seems to be no natural way to predict from the smooth theory which discretizations will have the nicest properties.

One case where these ideas seem particularly well-developed is for geometries described by integrable systems. The notion of a discrete integrable system as given by consistency on a cubic lattice has already shed new light on classical, smooth integrable systems.

Another theme which arose repeatedly during the workshop was that of circle patterns and sphere packings. These can be used to discretize conformal maps, isothermic surfaces, and elastic bending energy.

Since a computer works with discrete representations of data, it is no surprise that many of the applications of discrete differential geometry are found within computer science, particularly in the areas of computational geometry, graphics and geometry processing. The workshop brought theoreticians together with people interested in these and other applications.

Workshop: Discrete Differential Geometry

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Abstracts

Discrete Differential Geometry: Consistency as Integrability

Yuri B. Suris
(joint work with Alexander I. Bobenko)

This talk was based on the ongoing textbook with A. Bobenko [1], and aimed at giving an overview of an (integrable part of) discrete differential geometry. We started with recalling the differences between several disciplines with similar names. While *differential geometry* investigates smooth geometric shapes, such as curves and surfaces, with the help of mathematical analysis, and *discrete geometry* studies geometric shapes with finite number of elements, such as polyhedra, with an emphasis on their combinatorial properties, *discrete differential geometry* develops *discrete analogues and equivalents of notions and methods* of the smooth theory. The aims are, on the one hand, a better understanding of the nature and properties of the smooth objects, and, on the other hand, satisfying the needs and requirements of applications in modelling, computer graphics etc.

Then we recalled some stages of the historical development of discrete differential geometry: early work of R. Sauer in 1920-30s on bending of discrete surfaces vs. isometric deformations of smooth surfaces; his and W. Wunderlich's work of 1950 [2, 3] on discrete pseudospheric surfaces; important monograph "Differenzengeometrie" by R. Sauer in 1970 [4]; and the more recent achievements which arose from the interaction of the discrete differential geometry with the theory of integrable systems, and started with the work of A. Bobenko, U. Pinkall of 1995 [5] on discrete pseudospheric surfaces, discrete isothermic surfaces and discrete minimal surfaces, and with the work of A. Doliwa, P. Santini of 1997 [6] on multi-dimensional discrete conjugate and orthogonal nets.

After that, several classes of discrete nets were considered, with an emphasis on the following two basic ideas: first, complicated geometry of the corresponding

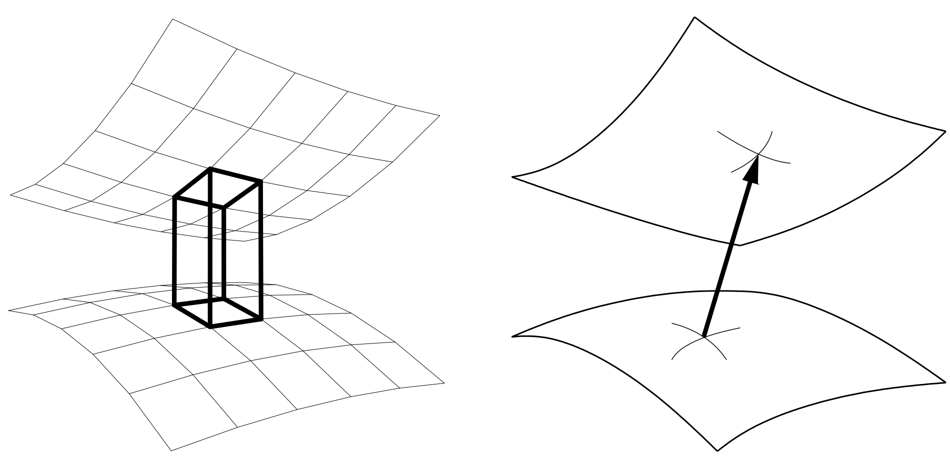

Figure 1. From discrete nets to transformations of smooth nets

smooth nets and their transformations can be most easily comprehended by tracing it back to certain incidence theorems of elementary geometry, and second, these elementary geometric properties ensure the multidimensional consistency of the corresponding discrete nets. It is this multidimensional consistency property which is responsible for the existence of remarkable transformations with permutability properties. Actually, on the discrete level the nets do not differ essentially from their transformations, and the theory becomes complicated only upon the smooth limit accompanied by the break of symmetry between coordinate directions (cf. Fig. 1). Further, the multidimensional consistency is responsible for the possibility to apply the powerful analytic machinery to the geometric problems at hand. In other words, it is the multidimensional consistency property that can and should be interpreted as *integrability* of the corresponding geometries.

The most fundamental class of discrete nets consists of the *discrete conjugate nets* $f : \mathbb{Z}^m \to \mathbb{R}^n$, characterized by the property that four vertices $f(u)$, $f(u+e_i)$, $f(u + e_j)$ and $f(u + e_i + e_j)$ of each elementary quadrilateral lie in a plane. A construction of an elementary hexahedron of a discrete conjugate net with $m = 3$: given 7 points f, f_i, f_{ij} with planar quadrilaterals (f, f_i, f_{ij}, f_j), find the 8th point f_{123} so that the quadrilaterals $(f_i, f_{ij}, f_{123}, f_{ik})$ are planar as well. This problem has a unique solution, since the planes in a three-dimensional space intersect (generically) in exactly one point. In this sense, discrete conjugate nets consist of a 3D system, schematically presented on Fig. 2. This system is 4D con-

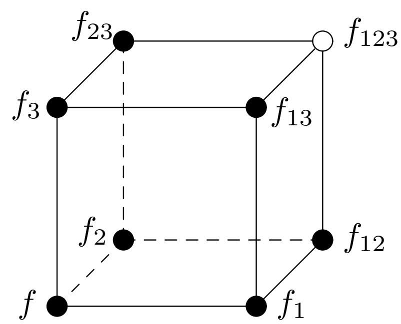

FIGURE 2. 3D system

sistent (and, as a consequence, m-dimensionally consistent for all $m \geq 4$), which is schematically presented on Fig. 3.

Another remarkable class of discrete nets described by a 3D system featuring 4D consistency constitute *discrete asymptotic nets* $f : \mathbb{Z}^m \to \mathbb{R}^3$, characterized by the property that all neighbor points $f(u \pm e_i)$ of $f(u)$ lie in a plane $\mathcal{P}(u)$ through $f(u)$. In this case the underlying incidence theorem is that of MÖBIUS on the pairs of mutually inscribed tetrahedra.

Important reductions of discrete conjugate nets appear if one requires that all points lie on a quadric, or that all elementary quadrilaterals have parallel diagonals (discrete MOUTARD nets). Admissibility of these reductions is again based on some remarkable incidence theorems (the theorem on the 8th associated point in

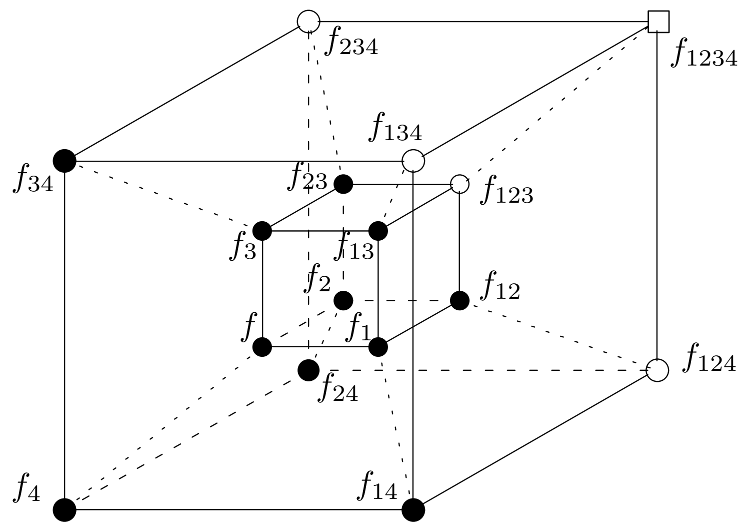

FIGURE 3. 4D consistency

the case of a reduction to a quadric, and the PAPPOS' theorem on quadrangular sets in the discrete MOUTARD case; in the particular case when the quadric under consideration is a sphere, the corresponding incidence theorem is that of MIQUEL). Upon imposing one such reduction, the system remains 3D and inherits the 4D consistency.

If two such reductions are imposed simultaneously, one arrives at discrete nets described by 2D systems featuring the crucial property of 3D consistency. The most prominent example constitute discrete MOUTARD nets in quadrics. Discrete pseudospherical surfaces and discrete isothermic surfaces are particular instances of this construction (in the first case, the quadric is the two-sphere, while in the second – the light cone of the MINKOWSKI space which serves as an ambient space of the projective model of MÖBIUS geometry).

In conclusion, it has been stressed that multidimensional consistency serves as the organizing principle of integrable discrete differential geometry.

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Circle Patterns, Theory and Practice

Boris Springborn

We give a very brief and incomplete overview of the theory of circle patterns and some applications. The roots of this subject area lie in the investigation of polyhedra in hyperbolic 3-space. The vertices need not be contained in hyperbolic space; they may lie on the sphere at infinity (ideal vertices) and even outside (hyperideal vertices). But we will assume that all edges intersect hyperbolic space. At each edge e the adjacent faces enclose a dihedral angle $\phi \in (0, \pi)$. The *exterior* dihedral angle is $\theta = \pi - \phi$. The sum of exterior dihedral angles at the edges incident with a vertex v is $< 2\pi$ ($= 2\pi$, $> 2\pi$) if the vertex is finite (ideal, hyperideal). Andreev classified non-obtuse angled hyperbolic polyhedra with finite and ideal vertices.

Theorem (Andreev [1] [2])**.** *Let P be an abstract polyhedron, $P \neq$ tetrahedron, and let $\theta(e) \in [\frac{\pi}{2}, \pi)$ for each edge e. A hyperbolic polyhedron with finite and ideal vertices of type P with exterior dihedral angles $\theta(e)$ exists, and is then unique, if and only if the following condition is satisfied: For every edge cycle γ in the dual polyhedron P^*,*

$$\sum_{e^* \in \gamma} \theta(e) \leq 2\pi \quad \Longleftrightarrow \quad \gamma \text{ is the boundary of a face of } P^*.$$

The condition implies that all vertices of such a polyhedron have degree 3 or 4, and if the degree is 4 then $\theta(e) = \frac{\pi}{2}$ for all incident edges. Andreev's proof proceeds by the "deformation method"; see also Roeder *et al.* [17].

Hyperbolic planes intersect the infinite boundary in circles, and the angle between the planes equals the intersection angles of the circles. Thus, statements about hyperbolic polyhedra may be formulated in terms of patterns of circles. This idea is due to Thurston, who used this correspondence to prove an existence and uniqueness theorem for circle patterns in surfaces with positive genus.

Theorem (Thurston [22])**.** *Let P be a polyhedral decomposition of a closed surface with positive genus. Suppose each vertex is of degree 3 or 4, and $\theta(e) \in [\frac{\pi}{2}, \pi]$. A corresponding circle pattern with circle intersection angles $\theta(e)$ exists, and is then unique, if and only if the following condition is satisfied: For every null-homotopic edge cycle γ in the dual polyhedron P^*,*

$$\sum_{e^* \in \gamma} \theta(e) \leq 2\pi \quad \Longrightarrow \quad \gamma \text{ is the boundary of a face of } P^*.$$

This theorem allows also exterior intersection angles π, *i.e.* touching circles, and the circle pattern analog of hyperideal vertices. Thurston's theorem implies an old theorem of Koebe [12], which states that for each abstract triangulation of the sphere there exists, uniquely up to Möbius transformations, a packing of circles with the given triangulation as contact graph. Thurston's proof is constructive in that it is based on a numerical algorithm to compute the radius of each circle. This algorithm has been refined and implemented in Stephenson's program `circlepack` [9] [21]. Chow & Luo give an alternative proof of Thurston's theorem which is inspired by the Ricci flow on surfaces [7] .

The circle patterns corresponding (via stereographic projection from a vertex) to convex polyhedra with ideal vertices are planar Delaunay tessellations. These were classified in terms of combinatorial type and dihedral angles by Rivin.

Theorem (Rivin [15]). *Let P be an abstract polyhedron, and let $\theta(e) \in (0, \pi)$ for each edge e. A hyperbolic polyhedron with ideal vertices of type P with exterior dihedral angles $\theta(e)$ exists, and is then unique, if and only if the following condition is satisfied: For every edge cycle γ in the dual polyhedron P^*, $\sum_{e^* \in \gamma} \theta(e) \geq 2\pi$, and equality holds if and only if γ is the boundary of a face of P^*.*

The proof proceeds by a deformation method. Bowditch had also treated Delaunay tessellations of piecewise flat surfaces with isolated cone-singularities [5]. His conditions for existence are very different from Rivin's and their equivalence for flat surfaces is far from obvious. Bao & Bonahon generalized Rivin's theorem to polyhedra with ideal and hyperideal vertices [3]. Such polyhedra correspond to weighted Delaunay tessellations. Recently, Schlenker extended this result by allowing also cone singularities [18].

The existence and uniqueness of circle patterns can also be proved using variational principles. This approach is more constructive and yields efficient numerical algorithms to construct them. Such a variational principle consists in some convex (or concave) function of either the circle radii or of certain angles with the property that its minimum (or maximum) corresponds to a circle pattern. The following table lists known variational principles for different types of circle patterns. They are all related since they can all be derived using Schläfli's differential volume formula for hyperbolic polyhedra.

	packings	ideal/Delaunay	hyperideal/ weighted Delaunay
Variables are radii	Colin de Verdiere [8]	Bobenko & S [4]	
Variables are angles	Bräger [6]	Rivin [14] (euclidean) Leibon [13] (hyperbolic)	S. [20]

Circle packings (touching circles) can be used to approximate conformal maps. This was conjectured by Thurston and first proved by Rodin & Sullivan [16]. Schramm proved an approximation theorem for circle patterns with orthogonally intersecting circles [19]. Circle packings and circle patterns can be seen as discrete analogs of conformal maps [21]. This motivated using circle packings and circle patterns in algorithms that map 3D surface meshes to the plane. The group around Stephenson uses packings to construct planar maps of the surface of the human cerebellum [10]. Kharevych *et al.* propose a method for surface flattening which uses Delaunay type packings [11]. To advance this method was a prime motivation for investigating the variational principle for weighted Delaunay triangulations [20].

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Circle Patterns on Singular Surfaces

Jean-Marc Schlenker

A *circle packing* on the sphere S^2 is a set of oriented circles bounding disjoint open disks. The *incidence graph* of a circle packing is the graph, embedded in S^2, which has a vertex for each circle, and an edge between two vertices if and only if the corresponding circles intersect. Koebe discovered a striking property of circles packings on the sphere: given a triangulation of the sphere, its 1-skeleton is the incidence graph of a circle packing, which is unique up to Möbius transformations.

Results on circle packings were then extended to consider patterns of circle, which we call "ideal" here, which appear in the definition of the Delaunay decomposition of a set of points, and which are related to ideal hyperbolic polyhedra. The circles then bound closed disks which cover the whole surface. The general idea there is that those circle packings are uniquely determined by their combinatorics and by the angles between the circles, results in this direction where obtained in particular by Rivin [Riv94], and Leibon [Lei02a, Lei02b]. This was extended to results concerning "ideal" circle patterns on surfaces with conical singularities, in particular by Bowditch, Rivin, Leibon and by Bobenko & Springborn [BS04]. We consider here a more general notion of circle pattern. Such circle patterns appear in the definition of a weighted Delaunay decomposition, they are related to hyperideal hyperbolic polyhedra just as "ideal" circle patterns are related to ideal polyhedra.

Definition 1. *A* **circle pattern** *on S^2 is a finite family of oriented circles $C_1, \cdots, C_N$. Given a circle pattern, an* **interstice** *is a connected component of the complement of the union of the open disks bounded by the circles. A circle pattern is* **hyperideal** *if:*

- *Each interstice is topologically a disk.*
- *For each $j \in \{1, \cdots, M\}$, corresponding to an interstice which is not a point, there is an oriented circle C'_j, containing I_j, which is orthogonal to all the circles C_i adjacent to I_j.*
- *For all $i \in \{1, \cdots, N\}$ and all $j \in \{1, \cdots, M\}$, if C_i is not adjacent to I_j, then either the interior of C_i is disjoint from the interior of C'_j, or C_i intersects C'_j and their intersection angle is strictly larger than $\pi/2$.*
- *If D is an open disk in S^2 such that:*
  (1) For each $j \in \{1, \cdots, M\}$, either D is disjoint from the interior of C'_j, or ∂D has an intersection angle at least $\pi/2$ with C'_j.
  (2) ∂D is orthogonal to at least 3 of the C'_j.
  then ∂D is one of the C_i.

Such a packing is strictly hyperideal if no interstice is reduced to a point. The circles C_i are called principal circles, while the circles C'_j are the dual circles.

We need one more definition before stating our main result. Let Γ be a graph embedded in a closed surface Σ.

Definition 2. *An admissible domain in* (Σ, Γ) *is a connected open domain* Ω*, which is not a face of* Γ*, such that* $\partial\Omega$ *is a finite union of segments which:*
- *have as endpoints vertices of* Γ*,*
- *either are edges of* Γ *or are contained (except for their endpoints) in an open face of* Γ*.*

To each such admissible domain, we can associate two numbers: its Euler characteristic, $\chi(\Omega)$*, and the number of boundary segments contained in open faces of* Γ*,* $m(\Omega)$*.*

Theorem 1. *Let* Γ *be the 1-skeleton of a cellular decomposition of a closed orientable surface* Σ*. Let* $\kappa : \Gamma_2 \to (-\infty, 2\pi)$ *and let* $\theta : \Gamma_1 \to (0, \pi)$ *be two functions. There exists a flat metric h with conical singularities on* Σ*, with a hyperideal circle pattern* σ *with incidence graph* Γ*, intersection angles given by* θ*, and singular curvatures given by* κ*, if and only if:*

(1) $\sum_{f\in\Gamma_2} \kappa(f) = 2\pi\chi(\Sigma)$,
(2) *for any admissible domain* $\Omega \subset \Sigma$ *:*

$$\sum_{e\in\Gamma_1, e\subset\partial\Omega} \theta(e) \geq (2\chi(\Omega) - m(\Omega))\pi - \sum_{f\in\Gamma_2, f\subset\Omega} \kappa(f) \ ,$$

with strict inequality except perhaps when Ω *is a face of* Γ*.*

The metric h is then unique up to homotheties, and σ *is unique given h.*

Theorem 2. *Let* Σ *be a closed orientable surface, and let* Γ *be the 1-skeleton of a cellular decomposition of* Σ*. Let* $\kappa : \Gamma_2 \to (-\infty, 2\pi)$ *and let* $\theta : \Gamma_1 \to (0, \pi)$ *be two functions. There exists a hyperbolic metric h with conical singularities on* Σ*, with a hyperideal circle pattern* σ *with incidence graph* Γ*, intersection angles given by* θ*, and singular curvatures given by* κ*, if and only if:*

(1) $\sum_{f\in\Gamma_2} \kappa(f) > 2\pi\chi(\Sigma)$,
(2) *for any admissible domain* $\Omega \subset \Sigma$ *:*

$$\sum_{e\in\Gamma_1, e\subset\partial\Omega} \theta(e) \geq (2\chi(\Omega) - m(\Omega))\pi - \sum_{f\in\Gamma_2, f\subset\Omega} \kappa(f) \ ,$$

with strict inequality except perhaps when Ω *is a face of* Γ*.*

h and σ *are then unique.*

There are related results for Euclidean and hyperbolic surfaces with polygonal boundary. The proofs use 3-dimensional hyperbolic geometry and are based on a deformation argument.

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Discrete Models of Isoparametric Hypersurfaces in Spheres

Wolfgang Kühnel
(joint work with Thomas F. Banchoff)

We present polyhedral models for isoparametric families in the sphere with at most three principal curvatures. Each member of the family (including the analogues of the focal sets) is tight in the boundary complex of an ambient convex polytope. In particular, the tube around the real (complex) Veronese surface is represented as a tight polyhedron in 5-space (8-space). The examples are based on the Bier sphere triangulation of S^4 or S^7, respectively. In the 4-dimensional case there are simplicial branched coverings of these triangulations in the complex projective plane and in $S^2 \times S^2$ which are branched precisely along the polyhedral analogues of the Veronese surface.

By a theorem of E. Cartan [4] all *isoparametric families* of hypersurfaces in the sphere with at most three principal curvatures are given by the following list:
(1) tubes around a point in S^{n-2}
(2) tubes around a great sphere $S^k \subset S^{n-2}$ where $1 \leq k \leq n-4$
(3) tubes around any of the Veronese-type standard embeddings of the projective planes $\mathbb{R}P^2 \to S^4$, $\mathbb{C}P^2 \to S^7$, $\mathbb{H}P^2 \to S^{13}$, or $\mathbb{O}P^2 \to S^{25}$.
In these three cases we have 1, 2 or 3 constant principal curvatures, respectively. In addition isoparametric hypersurfaces have the geometric property of tightness.

Main Theorem 1 *In each of the cases above (except possibly for the case of $\mathbb{O}P^2$) there is a* simplicial $(n-2)$-sphere *in Euclidean n-space satisfying the following:*

(1) *It contains two* disjoint simplicial subcomplexes *triangulating the two* focal sets *of the isoparametric family as a kind of "top" and "bottom" of the simplicial n-sphere (in the case of $\mathbb{H}P^2$ a complete proof is not available),*
(2) *each member of the isoparametric family corresponds to a* slice *through this $(n-2)$-sphere between top and bottom,*
(3) *each member of the family (including the focal sets) is a* tight *polyhedral submanifold in the boundary complex of a certain convex $(n-1)$-polytope. So in particular the real Cartan hypersurface is tight in the boundary complex of a 5-polytope, the complex Cartan hypersurface is tight in the boundary complex of an 8-polytope.*

The construction will make use of the following three ingredients:

1. Higher-dimensional octahedra (cross polytopes)
2. Tight triangulations of the projective planes over $\mathbb{R}$ and $\mathbb{C}$,
3. Sarkaria's deleted join of a simplicial complex with itself, and the Bier sphere.

Definition (1) The *deleted join* $K *_\Delta K$ of a simplicial complex K with itself is a part of the ordinary join of two disjoint copies K_1 and K_2 of K where we take the join of only those two simplices in K_1 and K_2, respectively, which are disjoint in K. So in particular, each vertex of K leads to a missing edge (a *diagonal*) in $K *_\Delta K$.

(2) Similarly we have the *deleted join* $K *_\Delta K^*$ of an n-vertex simplicial complex K with its combinatorial Alexander dual K^*. The vertex set of the deleted join will be denoted by $\{1, 2, \dots, n, \overline{1}, \overline{2}, \dots, \overline{n}\}$ with diagonals $1\overline{1}, 2\overline{2}, \dots, n\overline{n}$.

Theorem 2 *For any given simplicial complex K with n vertices the deleted join of K with its combinatorial Alexander dual K^* is a triangulated $(n-2)$-sphere with at most $2n$ vertices. It is called the* Bier sphere *$Bier_n(K)$ after Thomas Bier. After subdivision, the Bier sphere coincides with the first barycentric subdivision of an $(n-1)$-simplex, see* [5].

Theorem 3

(1) [1] *Any combinatorial $2k$-manifold with $n = 3k + 3$ vertices (which is not a sphere) satisfies $k = 0, 1, 2, 4, 8$ and, in addition, the following* combinatorial complementarity condition*:*

- Any subset of vertices spans a simplex in the triangulation if and only if the complementary subset does not.

In particular, if K denotes the simplicial complex triangulating the manifold, then we have the (Alexander) self-duality $K^* = K$. *Moreover K is $(k+1)$-neighborly meaning that any $(k+1)$-tuple of vertices spans a simplex in K.*

(2) [3] *In the cases $k = 0, 1, 2, 4$ there exists such a combinatorial manifold with $n = 3, 6, 9, 15$ vertices, respectively. It is unique for $k = 0, 1, 2$ and not unique for $k = 4$. For $k = 8$ the existence is still open.*

Corollary 4 *If K denotes any simplicial complex triangulating a combinatorial $2k$-manifold with $n = 3k + 3$ vertices which is not a sphere, then the deleted join $Bier_n(K) = K *_\Delta K$ is a combinatorial sphere of dimension $n - 2$ with $2n$ vertices. It can be regarded as a subcomplex of the cross polytope β_n. The family of all slices between top and bottom constitutes a polyhedral analogue of the isoparametric family in these cases, i.e., for $k = 0, 1, 2, 4$. (For $k = 8$ this depends on the existence or non-existence of a 27-vertex triangulation of a 16-manifold which is not a sphere.)*

It was pointed out by Massey [6] that a number of interesting 4-manifolds (among them the complex projective plane) are quotients of $S^2 \times S^2$. In particular,

$\mathbb{C}P^2$ is the quotient of $S^2 \times S^2$ by the involution $\tau(x,y) = (y,x)$, and the 4-sphere is the quotient of $\mathbb{C}P^2$ modulo complex conjugation σ where $\sigma[z_0, z_1, z_2] = [\overline{z_0}, \overline{z_1}, \overline{z_2}]$.

By a *branched simplicial k-sheeted covering* between two d-manifolds we mean a simplicial mapping which is simultaneously a branched k-sheeted covering. In particular, it is required that the preimage of any (open) d-simplex consists of k disjoint (open) d-simplices and that there is no collapsing of lower-dimensional simplices. Then the branch locus is a simplicial subcomplex of each of the two triangulated d-manifolds.

Proposition 5 *There is a branched simplicial 2-sheeted covering from a triangulated $\mathbb{C}P^2$ onto a triangulated 4-sphere which is branched along a subcomplex isomorphic to $\mathbb{R}P^2_6$. We can denote it – by slight abuse of notation – as follows:*

$$\mathbb{C}P^2_{18} := S^2_{12} *_\Delta \mathbb{R}P^2_6 \longrightarrow \mathbb{R}P^2_6 *_\Delta \mathbb{R}P^2_6.$$

Here S^2_{12} denotes the icosahedral triangulation of the 2-sphere with its 2-fold simplicial covering $S^2_{12} \longrightarrow \mathbb{R}P^2_6$. The complex $S^2_{12} *_\Delta \mathbb{R}P^2_6$ does not literally denote the deleted join but the join where each simplex is deleted which involves one vertex of $\mathbb{R}P^2_6$ and any of the two corresponding antipodal vertices of the icosahedron S^2_{12}.

Proposition 6 *There is a branched simplicial 2-sheeted covering from a triangulated $S^2 \times S^2$ onto a triangulated $\mathbb{C}P^2$ which is branched along a subcomplex isomorphic to the icosahedral triangulation of S^2. We can denote it – with the same remark as in Proposition 5 above – as follows:*

$$(S^2 \times S^2)_{24} := S^2_{12} *_\Delta S^2_{12} \longrightarrow S^2_{12} *_\Delta \mathbb{R}P^2_6$$

where $S^2_{12} \longrightarrow \mathbb{R}P^2_6$ denotes the same 2-fold simplicial covering as above.

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Closed Polyhedral 3-Manifolds with $K \geq 0$

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(joint work with Vsevolod V. Shevchishin)

Let a closed 3-manifold be glued (isometrically, face-to-face) from polyhedra such that for any edge the sum of dihedral angles around the edge is $\leq 2\pi$. We show that in this case the manifold can carry a Riemannian metric of nonnegative sectional curvature such that it is close to the initial metric in the Gromov-Hausdorff distance. In view of results of Hamilton [Ha1, Ha2] this implies that the manifold is homeomorphic to the quotient of one of the spaces S^3 or $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by a group of fixed-point-free isometries in the standard metric.

A (convex) *polyhedron* (of dimension 3) is the convex hull of finitely many points in $\mathbb{R}^3$ such that not all of them lie on the same plane. Every polyhedron carries the induced metric (= distance-function). *Polyhedral* 3-manifolds are those glued from polyhedra using face-to-face isometries.

From the definition it follows that the faces, the edges, and the vertices of one polyhedron are glued with those of other polyhedra, which make the notions *edge* and *vertex* of the polyhedral manifold well defined.

We say that a polyhedral manifold has *nonnegative curvature*, if it has nonnegative curvature in the sense of Alexandrov. By Globalisation Theorem [B-G-P], the latter is equivalent to the following condition: for every edge the sum of the dihedral angles around this edge $\leq 2\pi$, see also [Mi].

An edge will be called *essential*, if the sum of the dihedral angles around this edge is strictly less than 2π. A vertex will be called *essential*, if it is the endpoint of at least three essential edges.

Main Theorem. *Let M be a closed polyhedral 3-dimensional manifold of nonnegative curvature. Then, the manifold is homeomorphic to the quotient of one of the spaces S^3 or $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by a group of fixed point free isometries in the standard metric. In particular, M can be finitely covered by S^3, $S^2 \times S^1$, or $S^1 \times S^1 \times S^1$. Moreover, the existence of an essential edge implies that the manifold can be finitely covered by S^3 or $S^2 \times S^1$, and the existence of an essential vertex implies that the manifold can be finitely covered by S^3.*

Corollary. (Independently obtained by Lutz & Sullivan [Lu-Su]) *Let M be a triangulated closed 3-manifold. Assume for every edge of the triangulation the number of the simplices containing this edge is less than six. Then, M can be finitely covered by S^3.*

The proof of Corollary is deduced from main Theorem as follows. One realizes each tetrahedron P_i of the triangulation as a *regular tetrahedron*, i.e., a tetrahedron in $\mathbb{R}^3$ such that the length of every edge is equal to 1. Then, all gluing functions will automatically be isometric, and our manifold with the induced metric becomes a polyhedral manifold. Using the hypotheses one can show that for every edge the sum of dihedral angles containing this edge is at most $5 \times \arccos(1/3) \approx$

$352.66\frac{2\pi}{360} < 2\pi$. So every edge is essential, and hence every vertex is essential as well, and Main Theorem applies.

Scheme of the proof of Main Theorem. For a given closed polyhedral manifold M of nonnegative curvature we construct a Riemannian metric of nonnegative sectional curvature on M. Moreover, we show that in the presence of at least one essential edge the constructed metric is not everywhere flat, and in the presence of at least one essential vertex there exists a point on M at which the sectional curvature is positive in every two-dimensional direction. Then, our theorem follows from the famous results of Hamilton [Ha1, Ha2].

The idea is rather natural. According to Cheeger [Che] (see Remark 6 there), it was suggested by Gromov directly after Hamilton proved his results. However, implementation of this idea turned out to be difficult, see for example [Pe] where a few approaches were discussed, and [B-G-P], where it was explicitly written that it is hard to smooth a polyhedral metric keeping control over the lower curvature bound.

If there are no essential vertices, then the construction of a smooth Riemannian metric of nonnegative sectional curvature is easy. If such a vertex exists, our construction of such a metric consists of three steps.

Step 1: We show that (in the presence of at least one essential vertex) it is possible to construct a polyhedral metric on the manifold such that

- all polyhedra are tetrahedra,
- all edges are essential.

The main tool of the construction is Alexandrov's embedding theorem from [Al].

Step 2: We change the metric on the manifold replacing every tetrahedron T by a spherical tetrahedron T_R lying in the sphere S^3 of sufficiently large radius R such that the lengths of edges of T coincide with those of T_R. If R is big enough, such a spherical tetrahedron T_R exists. Moreover, the gluing mappings remain to be isometric, so that the metric of the spherical tetrahedra induces a new metric on M. If R is huge enough, all edges remain essential, so that the new metric has Alexandrov curvature $\geq 1/R^2$. Then, we smooth this metric near the edges. In order to do this, we use an *Ansatz* involving a function of two variables such that the positivity of the sectional curvature of the metric is equivalent to the concavity of the function. Then, we construct an appropriate concave function by gluing from three pieces. As result we obtain a metric of nonnegative curvature which is a Riemannian metric outside the vertices and has conical singularities near the vertices.

Step 3: We smooth the metric near the vertices using the Alexandrov embedding theorem [Al] and regularity results of Pogorelov [Po].

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Algebraic Topology on Polyhedral Surfaces from Finite Elements

Max Wardetzky

(joint work with Klaus Hildebrandt and Konrad Polthier)

It has been known to the numerics community for some time that discretizations of smooth differential complexes such as the de Rham complex yield very stable methods for approximating solutions to partial differential equations (cf. Arnold [1]). Among the most notable such discretizations are *Whitney elements*. Given a locally finite C^∞ triangulation of a smooth manifold M, Whitney [11] defined a certain linear map W from the simplicial cochains C^q induced by this triangulation to $L^2\Lambda^q$, a chain map in the sense that $\mathrm{d}\,W = W\,\delta$, where d is the Cartan outer differential. Dodziuk and Patodi [4, 5] observed that this map together with a Riemannian metric g on a compact smooth manifold M gives rise to a positive definite inner product on simplicial cochains, and hence a discrete Hodge decomposition (using the inner product on simplicial cochains to define adjoint operators to the simplicial coboundary operators). Then under (suitable) refinement of the triangulation of M, the discrete Hodge decomposition converges to the smooth one on (M, g). Recently Wilson [12] has extended these results by a (converging) discrete wedge product on simplicial cochains and a (converging) combinatorial Hodge star operator.

Whitney elements are piecewise linear by construction. Here we report on a different development using *piecewise constant* vector fields (or one-forms) on compact *polyhedral surfaces*. The function spaces corresponding to a discrete Hodge decomposition then turn out to be a mixture of *conforming* and *nonconforming* linear elements. For sequences of polyhedral surfaces whose *positions and normals* converge to the positions and normals of a compact smooth surface embedded in $\mathbb{E}^3$, we report on a convergence result for the corresponding discrete Hodge decompositions and Hodge star operators. The proof is mainly based on showing that the convergence results of Dodziuk/Patodi and Wilson remain valid if one works with variable (and converging) metrics (M, g_n), instead of a fixed one. The motivation to investigate into piecewise constant structures here is that *piecewise constant harmonic fields* come in pairs of a conforming and a nonconforming version, much like linear models of discrete minimal surfaces [9] which also turn out to come in pairs of a conforming and a conjugate nonconforming minimal surface. Finally we remark that one finds strong similarities between the current analytic approach of discretizing function spaces (using the duality between conforming and nonconforming elements) and an algebraic approach (using the duality between primal and dual graphs), such as pursued by Desbrun et al. [3], Mercat [8], Dynnikov/Novikov [6], and others.

By a *polyhedral surface* M_h we mean the result of isometrically gluing flat Euclidean triangles along their boundaries such that the result is homeomorphic to a topological 2-manifold. As usual, h denotes the *mesh size* (a notation which goes back at least to [2]). We only consider orientable surfaces. The Euclidean structure on triangles induces a Euclidean cone structure on M_h. The triangulation gives rise to the following function spaces:

$$S_h = \{u \in C^0(M_h) \,|\, u \text{ is linear on triangles}\},$$

$$S_h^* = \{u \in L^2(M_h) \,|\, u \text{ is linear on triangles and continuous at edge midpoints}\},$$

$$\mathfrak{X}_h = \{X \text{ is tangential and constant on all individual triangles}\}.$$

Clearly, $S_h \subset S_h^*$. The space S_h is called *conforming*, and S_h^* is called *nonconforming*. Finally, $\mathfrak{X}_h$ denotes the space of *piecewise constant vector fields*. The cone metric on M_h induces a L^2-inner product on each of these spaces.

The *gradient* of a function in S_h or S_h^* is well-defined on triangles and takes values in $\mathfrak{X}_h$. Let div denote the adjoint operator to grad : $S_h \to \mathfrak{X}_h$ with respect to the L^2-inner products. Similarly, let div^* denote the adjoint operator to grad : $S_h^* \to \mathfrak{X}_h$. *Complex multiplication* J acts on $\mathfrak{X}_h$ by rotation by $\pi/2$ on each individual triangle. Set curl $= -\,\text{div} \circ \text{J}$, and $\text{curl}^* = -\,\text{div}^* \circ \text{J}$. It is not difficult to see that for $X \in \mathfrak{X}_h$, the terms $\text{curl}^* X$ and $\text{div}^* X$ are measures for the tangential and normal jumps of X across edges of M_h, respectively. If M_h is *closed* (has empty boundary), one obtains the following (mutually L^2-adjoint) chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_h & \xrightarrow{\text{grad}} & \mathfrak{X}_h & \xrightarrow{\text{curl}^*} & S_h^* & \longrightarrow & 0 \\ 0 & \longleftarrow & S_h & \xleftarrow{\text{div}} & \mathfrak{X}_h & \xleftarrow{\text{J grad}} & S_h^* & \longleftarrow & 0. \end{array}$$

Lemma. *The homology groups for (each of) the above chain complexes are isomorphic to the respective simplicial homology groups. This gives the following two discrete Hodge decompositions of* $\mathfrak{X}_h$*:*

$$\begin{aligned} \mathfrak{X}_h &= \operatorname{im}\operatorname{grad}_{|S_h} \oplus \operatorname{im}\mathrm{J}\operatorname{grad}_{|S_h^*} \oplus \ker\operatorname{curl}^* \cap \ker\operatorname{div} \\ &= \operatorname{im}\mathrm{J}\operatorname{grad}_{|S_h} \oplus \operatorname{im}\operatorname{grad}_{|S_h^*} \oplus \ker\operatorname{div}^* \cap \ker\operatorname{curl}, \end{aligned}$$

where the second row is the J*-transformed version of the first.*

By construction, the sum is orthogonal with respect to the L^2-inner product on $\mathfrak{X}_h$. The space $\mathcal{H}(M_h;\mathbb{R}) = \ker\operatorname{curl}^* \cap \ker\operatorname{div}$ is termed *conforming harmonic*, and the space $\mathcal{H}^*(M_h;\mathbb{R}) = \ker\operatorname{div}^* \cap \ker\operatorname{curl}$ is termed *nonconforming harmonic*. The dimension of each of these spaces equals twice the genus of M_h. Note that complex multiplication J acts as a linear *isomorphism* between these two spaces. In a similar fashion to [12], one defines a *discrete Hodge star operator* on $\mathcal{H}(M_h;\mathbb{R})$ by first applying J and then L^2-projecting back to $\mathcal{H}(M_h;\mathbb{R})$,

$$\star : \mathcal{H}(M_h;\mathbb{R}) \longrightarrow \mathcal{H}(M_h;\mathbb{R}).$$

In other words, if $X \in \mathcal{H}(M_h;\mathbb{R})$, then $\star X$ is the conforming harmonic part of $\mathrm{J}(X)$. Note that $\star\star \neq -\mathrm{Id}$. However, $\star$ is still an isomorphism. There exists a similar nonconforming version.

Convergence. Let (M, g) be compact smooth surface embedded into $\mathbb{E}^3$ which inherits its metric structure from ambient space. A polyhedral surface M_h in a (small enough) tubular of M is a *normal graph* if M_h can be viewed as a section in the normal bundle of M. A sequence of normal graphs $\{M_n\}$ converges *totally normally* ([7][10]) to M if the positions of M_n converge in Hausdorff distance and the normals of M_n converge in L^∞ to those of M. Using the pullback from M_n to M, the surface M inherits a sequence of cone metrics $\{g_n\}$ coming from $\{M_n\}$.

Lemma. *If* $M_n \to M$ *totally normally, and* X, Y *are vector fields on* M *then*

$$\sup_{X,Y} \left\| \frac{|g_n(X,Y) - g(X,Y)|}{\|X\|_g \cdot \|Y\|_g} \right\|_\infty \longrightarrow 0.$$

Under the pullback from M_n to M, our objects are defined a.e. on M. In particular, let Π_n be the L^2-projections of smooth vector fields on M to piecewise constant fields on M associated with a totally normally converging sequence $\{M_n\}$. Then:

Theorem. *In* $L^2(M)$*, the components of the discrete Hodge splittings of* $\Pi_n(X)$ *converge to the components of the smooth Hodge splitting of* X*. Moreover, if* $\mathfrak{h}$ *is harmonic on* (M, g) *and* $\mathfrak{h}_n$ *is the conforming harmonic part of* $\Pi_n(\mathfrak{h})$ *then* $\star_n \mathfrak{h}_n$ *converges to* $\star\mathfrak{h}$*. Finally,* $\mathcal{H}^*(M_n;\mathbb{R})$ *tends to* $\mathcal{H}(M_n;\mathbb{R})$*, insofar as* $\mathrm{J}_n\, \mathfrak{h}_n \to \star_n \mathfrak{h}_n$*.*

On the one hand the proof is based showing that the convergence results proved in [4][5][12] remain true for variable and converging metrics g_n, and on the other hand on relating the Hodge splitting of Whitney elements to the Hodge splitting of piecewise constant elements. In a similar fashion one obtains convergence for the spectral decomposition of Laplacians. For details we refer to [10].

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Alexandrov Theorem, Weighted Delaunay Triangulations, and Mixed Volumes

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(joint work with Alexander I. Bobenko)

Theorem (A. D. Alexandrov, 1942)**.** *Let S be a 2-sphere equipped with a convex polyhedral metric. Then there is a convex polytope P in $\mathbb{R}^3$ with boundary isometric to S. Besides, P is unique up to a rigid motion.*

A polyhedral metric is one modelled locally on the Euclidean plane and Euclidean cones. It is called convex if all cone angles are less than 2π. The surface of a convex polytope provides an example of a convex polyhedral metric; the only cone singularities are vertices. The fact that the edges cannot be distinguished intrinsically is the main difficulty in reconstructing the polytope from the metric of its boundary.

The uniqueness part of Alexandrov theorem was essentially obtained by Cauchy. The existence is the more difficult part. Alexandrov's original proof [1, 2] is elegant but non-constructive. In his book [2] Alexandrov poses the problem of finding a constructive, possibly variational, proof. Alexandrov's student Volkov gave a proof partly of this sort which can be found in [3]. The idea is to adjust some set of parameters one by one (like in Thurston's proof of circle packing theorem).

In the talk we present another proof of the existence part of Alexandrov's theorem that leads to an algorithm for constructing a polytope for a given metric. In our proof we come up with weighted Delaunay triangulations. Our other tool are mixed volumes, which play also a crucial role in the proof of Minkowski theorem. We relate the total curvature of a polytope to the volume of the dual polyhedron, see Lemma 5 below. This reveals a connection between the Alexandrov and Minkowski theorems.

An outline of the proof follows.

Fix a convex polyhedral metric S. Singularities of S will be indexed by $i, j, \ldots$. By δ_i we denote the angular defect of i-th singularity. A geodesic triangulation T of S is a subdivision of S in triangles by Euclidean geodesics with endpoints in singularities. Edges are denoted by e. Multiple edges and loops are allowed.

Definition. *A generalized polytope is a polyhedral complex homeomorphic to a ball, which is glued from pyramids with a common apex and has boundary isometric to S. Formally it is a couple (T, r) where T is a geodesic triangulation of S, and $r = (r_i)$ is an assignment of positive numbers, called radii, to singularities. The complex is glued from pyramids with side lengths r_i based on triangles of T.*

Let θ_e be the dihedral angle of a generalized polytope $P = (T, r)$ at an edge e of T. If all $\theta_e \leq 2\pi$ then P is called convex. Let κ_i be 2π minus the total dihedral angle around the edge that joins the apex to i-th singularity. We call κ_i curvature. If we achieve $\kappa_i = 0$ for all i, then P becomes a convex polytope with a distinguished interior point.

The algorithm works as follows. Begin with a convex generalized polytope $P(0) = (T_D, R)$, where T_D is a Delaunay triangulation of S, and all r_i equal to a sufficiently large number R. Then every κ_i is slightly less than δ_i. Start to deform the radii so that all κ_i's decrease proportionally:

$$\kappa_i(t) = (1 - t) \cdot \kappa_i(0), \text{ for } t \in [0, 1)$$

As $t \to 1$, the corresponding convex generalized polytope $P(t)$ converges to a convex polytope that we were looking for. In order that $P(t)$ remains convex, at certain moments changes of triangulation T (so called flips) has to be performed.

The following lemmas allow us to control the triangulation.

Lemma 1. *If (T, r) is a convex generalized polytope, then T is the weighted Delaunay triangulation of S with weights r_i^2 at singularities. The converse is true provided that pyramids over triangles of T with side lengths r_i exist.*

Lemma 2. *The weighted Delaunay triangulation with given weights is unique (up to flat edges), if it exists. It can be obtained from any triangulation by consecutive flipping of non-Delaunay edges. The space of admissible weights as well as of those set of radii for which pyramids exist can be described explicitly.*

Thus radii can be viewed as coordinates on the space $\mathcal{P}(S)$ of convex generalized polytopes with boundary S.

The following lemma allows us to deform the curvatures κ_i as we like, as long as they satisfy certain inequalities and as long as we stay inside $\mathcal{P}(S)$.

Lemma 3. *The Jacobian* $\left(\frac{\partial \kappa_i}{\partial r_i}\right)$ *is non-degenerate if* $0 < \kappa_i < \delta_i$.

This can be interpreted as infinitesimal rigidity of convex generalized polytopes under given assumptions on curvatures. The proof of Lemma 3 is based on the following three facts.

Lemma 4. *The Jacobian* $\left(\frac{\partial \kappa_i}{\partial r_i}\right)$ *coincides with the Hessian* $\left(\frac{\partial^2 \mu}{\partial r_i \partial r_j}\right)$, *where*

$$\mu(P) = \sum_i r_i \kappa_i + \sum_e \ell_e(\pi - \theta_e)$$

is the total curvature of a convex generalized polytope P, ℓ_e *is the length of the edge* e.

Lemma 5. *Let* P^* *be the convex generalized polyhedron dual to convex generalized polytope* P. *Then volume of* P^* *and total curvature of* P *have same Hessians:*

$$\frac{\partial^2 \mu}{\partial r_i \partial r_j}(P) = \frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j}(P^*)$$

The construction of P^* generalizes the classical construction of polar dual. As opposed to P whose curvature is concentrated along segments joining the apex to vertices, the curvature of P^* sits on perpendiculars to the faces. Coordinates for P^* are heights $h_i = r_i^{-1}$.

Lemma 6. *For a convex generalized polyhedron with positive curvatures and positive edge lengths and face areas the Hessian* $\left(\frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j}\right)$ *has signature* $(1, n-1)$.

Positivity of curvatures and lengths and areas for P^* can be followed from inequalities $0 < \kappa_i < \delta_i$ for P.

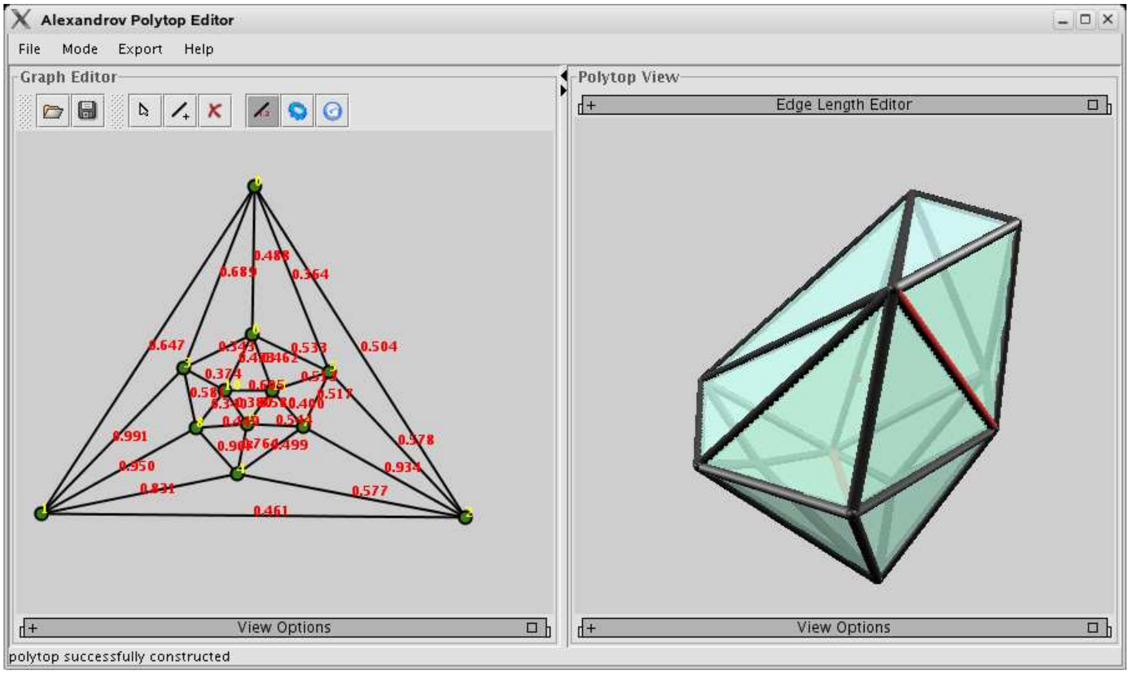
Note an interesting fact

$$\left(\frac{\partial^2 \mu}{\partial r_i \partial r_j}\right) \sim R^{-1} \cdot \Delta_S$$

for $P = (T_D, R)$ as $R \to \infty$. Here Δ_S is the discrete Laplace operator associated with the Delaunay triangulation of S.

The last ingredient of the proof is to show that the pyramids of $P(t)$ don't degenerate when we follow the path $\kappa_i(t) = (1-t)\kappa_i(0)$.

The proof provides an algorithm for constructing the polytope. A computer program implementing this algorithm was written by Stefan Sechelmann.



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A Spectral Invariant of Graphs

Yves Colin de Verdière

In 1976, S. Cheng [1] proved the following very nice result:

The multiplicity of the first nonzero eigenvalue of the Laplacian for any Riemannian metric on S^2 is ≤ 3. The proof easily extends to Schrödinger operators $\Delta_g + V$.

Upper bounds were later found for the other surfaces by G. Besson, N. Nadirashvili and B. Sévennec [10]. As a result, the multiplicity of the second eigenvalue of any Schrödinger operator on a closed surface X of Euler characteristic $\chi(X) \geq -3$ is bounded by $C(X) - 1$, where $C(X)$ is its chromatic number. All proved upper bounds are compatible with the following conjecture I have made: *for any manifold, the maximal multiplicity of the second eigenvalue of any Schrödinger operator is $C(X) - 1$.*

The fact that $C(X) - 1$ is always a sharp lower bound for the maximal multiplicity and, in particular, the fact that there is no such upper bound for manifold of dimension ≥ 3 is a result of the study of a similar problem for graphs.

Several years ago [2], I introduced invariants of graphs which are coming from spectral theory. If $G = (V, E)$ is a finite graph with $V = \{\text{vertices}\}$ and $E = \{\text{edges}\}$, one of these invariants, which I called $\mu(G)$ ($\in \mathbf{N}$), is a "topological" invariant:

- $\mu(G) \leq 3$ *iff* G *is planar*
- $\mu(G) \leq 4$ *iff* G *admits a linkless embedding in* $\mathbf{R}^3$ *(Lovasz-Schrijver* [9]*)*

Roughly speaking, $\mu(G)$ is the largest multiplicity of the second eigenvalue of a Schrödinger operator on G satisfying a *structural stability* assumption which is formulated in terms of transversality.

The main property of μ is the following:

If G *is a minor of* G'*, then* $\mu(G) \leq \mu(G')$*;* similarly, *if* G *embeds into a manifold* X*, then there exists a Schrödinger operator on* X *whose second eigenvalue has multiplicity* $\mu(G)$*.*

The main ingredient is a relationship between minors of graphs and singular limits of operators on it. This is a very simple case of what people in functional analysis call Γ-convergence of a sequence of unbounded operators. It involves looking at "graphs" of symmetric operators as Lagrangian spaces and taking limits of them in the Grassmanian of Lagrangian subspaces of $T^\star\mathbf{R}^V$.

I introduced in [6] another invariant ν which turns out to be related with the tree-width.

Another related topic studied in [3, 4] is a discrete Dirichlet-to-Neumann map, which, in the case of graph, is just the map which to a given electrical potential on the boundary vertices associates the outgoing currents.

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Stress Matrices and M Matrices

Robert Connelly
(joint work with Károly Bezdek)

1. Introduction

In [2] a connection is made between what are called "M" matrices, as used in Colin de Verdière's theory of graph invariants, and stress matrices as used in rigidity theory in [1]. Following a description of stress matrices and their properties relevant to rigidity theory, it is shown how a theorem of László Lovász [2], using results about M matrices, implies a conjecture about the global rigidity of certain tensegrity frameworks by Károly Bezdek.

2. Stress Matrices

Given a finite graph $G = (V, E)$ without loops or multiple edges, where V is the set of n vertices labeled $1, \dots, n$ and E the edges, a stress matrix Ω is a symmetric n-by-n matrix, where the off-diagonal entries are denoted as $-\omega_{ij}$ and the following conditions hold:

(1) When $i \neq j$ and $\{ij\}$ is not in E, then $\omega_{ij} = 0$.
(2) $[1, 1, \dots, 1]\Omega = 0$.

Condition (2) defines the diagonal entries of Ω in terms of the off-diagonal entries. The i-th row and column of Ω correspond to the i-th vertex.

Consider a configuration of points $\mathbf{p} = (p_1, \dots, p_n)$, where each p_i is in Euclidian d-dimensional space $\mathbb{E}^d$. Form the d-by-n *configuration matrix* $P = [p_1, p_2, \dots, p_n]$, where each p_i is regarded a column of P. The configuration $\mathbf{p}$ is said to be in *equilibrium* with respect to the stress $\omega = (\dots, \omega_{ij}, \dots)$ if $P\Omega = 0$. This is equivalent to the vector equation for each vertex i, $\sum_j \omega_{ij}(p_j - p_i) = 0$. Some basic properties of stress matrices, which can be found in [1], are in the following proposition.

Proposition. *If the configuration* $\mathbf{p}$ *is in equilibrium with respect to the stress* ω*, then the following hold:*

(1) The dimension of the affine span of the configuration $\mathbf{p}$ *is at most* $n - 1 -$ *rank* Ω.
(2) If the dimension of the affine span of the configuration $\mathbf{p}$ *is exactly* $n - 1 -$ *rank* Ω*, and* $\mathbf{q} = (q_1, \dots, q_n)$ *is another configuration in equilibrium with respect to* ω*, then* $\mathbf{q}$ *is an affine image of* $\mathbf{p}$.

If Condition (2) in the Proposition holds for a configuration $\mathbf{p}$, then we say $\mathbf{p}$ is *universal* with respect to ω. It is easy to see that if a configuration $\mathbf{p}$, with a d-dimensional affine span, is not universal for a given equilibrium stress ω, then there is a configuration $\mathbf{q}$, whose affine span is at least $(d + 1)$-dimensional, that projects orthogonally onto $\mathbf{p}$, and which is in equilibrium with respect to ω as well.

3. Global Rigidity

Suppose that the edges of a graph G are labeled either a cable or a strut. We say a configuration $\mathbf{q}$, corresponding to the vertices V, is *dominated by* the configuration $\mathbf{p}$ if the cables of $\mathbf{q}$ are not increased, and struts are not decreased in length. We call $G(\mathbf{p})$ a *tensegrity*, and if every configuration in $\mathbb{E}^d$ that is dominated by $\mathbf{p}$ is congruent $\mathbf{p}$, we say $G(\mathbf{p})$ is *globally rigid in* $\mathbb{E}^d$.

If $v_1, v_2, \dots$ are vectors in $\mathbb{E}^d$, we say that they lie on a *conic at infinity* if for all i, there is a non-zero d-by-d symmetric matrix C such that $v_i^T C v_i = 0$, where $()^T$ is the transpose. The following fundamental result can be found in [1].

Theorem 1. *If a configuration* $\mathbf{p}$ *in* $\mathbb{E}^d$ *has an equilibrium stress* ω*, with* $\omega_{ij} > 0$ *for cables,* $\omega_{ij} < 0$ *for struts, (called a* proper stress *for* $G = (V, E)$*) such that*
(1) the member directions $p_i - p_j$*, for* $\{ij\}$ *in* E*, do not lie on a conic at infinity,*
(2) the matrix Ω *is positive semi-definite, and*
(3) the configuration $\mathbf{p}$ *is universal with respect to* ω*,*
then $G(\mathbf{p})$ *is globally rigid in* $\mathbb{E}^N$*, for all* $N \geq d$*.*

Any configuration that satisfies the hypothesis above is called *super stable*.

4. Lovász's Result

The following result of Lovász in [2] has a situation that satisfies all the conditions of Theorem 1 except condition (3). Condition (1) is easy to verify.

Theorem 2. *If a tensegrity framework* $G(\mathbf{p})$ *is defined by putting cables for the edges of a convex* 3*-dimensional polytope* P *and struts from any interior vertex to each of the vertices of* P*, then the configuration* $\mathbf{p}$ *has a proper equilibrium stress* ω*, and any such non-zero stress has a stress matrix* Ω *with exactly one negative eigenvalue and* 4 *zero eigenvalues.*

If one takes the stress matrix Ω from Theorem 2 and removes the row and column corresponding to the central vertex, then one gets an M matrix as used in the definition of Colin de Verdière's number defined as a graph invariant. The problem is to get rid of the offending negative eigenvalue.

5. The Conjecture

Suppose P is a convex polytope in $\mathbb{E}^3$ and one creates a tensegrity $G(\mathbf{p})$ by assigning the vertices of P as the vertices of a configuration for G, the edges of P as the cables for $G(\mathbf{p})$, and assigning struts as some of the internal diagonals such that $G(\mathbf{p})$ has a proper equilibrium stress. Then it appears that the resulting stress matrix Ω satisfies the conditions for being super stable, but no proof is known in general. Károly Bezdek specialized that conjecture to the case when the polytope P is centrally symmetric. With the help of Theorem 2 by Lovász, we can prove that conjecture, which is the following.

Theorem 3. *For any 3-dimensional centrally symmetric convex polytope P, the associated tensegrity $G_P(\mathbf{p})$, with struts between all antipodal vertices, has a stress ω such that $G_P(\mathbf{p})$ is super stable. Furthermore any such proper equilibrium stress for $G_P(\mathbf{p})$ is such that it serves to make $G_P(\mathbf{p})$ super stable.*

Proof. Let ω' denote any non-zero proper stress determined by the conclusion of Theorem 2 for the centrally symmetric polytope P with the central vertex as the interior point. Let $\hat{\omega}'$ denote the stress on $G_P(\mathbf{p})$ obtained by replacing each cable and strut stress with the stress on its antipode. Then $\hat{\omega}'$ is an equilibrium stress for $G_P(\mathbf{p})$ as well. Hence $\omega' + \hat{\omega}'$ is a proper equilibrium stress for $G_P(\mathbf{p})$, where stresses on antipodal cables and struts are equal. So we assume without loss of generality that the stresses in ω' are symmetric, and Theorem 2 assures us that the associated stress matrix has only one negative eigenvalue.

Suppose that i and j correspond to antipodal vertices of P. Let $\omega'_{i0} = \omega'_{j0} < 0$ be the stresses from the central vertex to the i and j vertices coming from the stress ω'. Form a small tensegrity $G_{ij}(p_i, p_j, 0)$ with just three vertices i, j, and the central vertex 0, where $\{i, j\}$ is a strut, while $\{0, i\}$ and $\{0, j\}$ are cables. Let $\omega_{ij} = 2\omega'_{i0} = 2\omega'_{j0} < 0$, and replace ω'_{i0} and ω'_{j0} with $-\omega'_{i0}$. It is easy to check that this is an equilibrium stress for $G_{ij}(p_i, p_j, 0)$ whose associated stress matrix is positive semi-definite. Extend this to all the vertices of G by having all other stresses 0. The associated stress matrix Ω'_{ij} defined on all the vertices of G is still positive semi-definite. But now $\Omega' + \Omega'_{ij}$ has its $(0, i)$ and $(0, j)$ entry 0. Let $\Omega' + \sum_{ij} \Omega'_{ij} = \Omega$, where the sum is over all antipodal vertices $\{i, j\}$. We obtain a stress matrix Ω corresponding to a stress ω, where all the $\omega_{0i} = 0$ and $\omega_{ij} < 0$ for pairs $\{i, j\}$ of antipodal vertices. Otherwise $\omega_{ij} = \omega'_{ij}$. Since Ω is obtained by adding a positive semi-definite matrix $\sum_{ij} \Omega'_{ij}$ to Ω', none of the eigenvalues of Ω' decrease. It is clear that the stress ω is an equilibrium stress for a configuration whose affine span is 4-dimensional, since the central vertex can be displaced into $\mathbb{E}^4$, where it has essentially been disconnected from all the other vertices. So the 0 eigenvalues of Ω must stay at 0, while the negative eigenvalue must increase to provide the extra 0. If we remove the central vertex, then the resulting framework with antipodal vertices connected by struts is super stable, as desired.

It clear that the above process can be reversed, starting with an arbitrary equilibrium stress ω for the centrally symmetric polytope P to obtain a proper stress for a tensegrity as in Theorem 2. So any such proper equilibrium stress ω for struts connecting antipodal vertices of P will be super stable. □

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The 'Consistency Approach': Application to the Discrete Differential Geometry of 'Plated' Membranes in Equilibrium

Wolfgang K. Schief

The aim of the talk is to present a non-technical exposition of work published in [14] which embraces elements of classical shell membrane theory, discrete differential geometry and integrable systems.

The theory of integrable systems (soliton theory) is multifaceted and extends to a variety of areas in mathematics and physics. Amongst others, there have been two important recent developments which are both geometric in nature but which have otherwise unfolded independently. On the one hand, it has become evident that the area of 'discrete differential geometry', which seeks to identify and make use of canonical discrete analogues of differential geometric objects, provides key new insight into the origins of integrable systems and their integrability-preserving discretisations (see the monograph [4] and references therein). On the other hand, until recently, the nonlinear equations descriptive of solitonic behaviour in physical systems had been derived by approximation or expansion methods (except for the Ernst equation of general relativity [7]). However, it turns out that, remarkably, there exists 'exact' hidden integrable structure in diverse areas of nonlinear continuum mechanics such as hydrodynamics, magnetohydrodynamics, the kinematics of fibre-reinforced materials and elastostatics of shell membranes (see the review article [13] and references therein).

In this talk, the above-mentioned two strands are brought together. Thus, we present a discrete model of (shell) membranes in equilibrium together with their resultant internal stress distributions in the absence of external forces. The discrete membranes are composed of planar quadrilateral elements ('plates') which are not entirely arbitrary but may be inscribed in circles. The latter property is motivated by the fact that in discrete differential geometry [4] and computer-aided surface design [6] quadrilaterals inscribed in circles have been identified as canonical discrete analogues of surface 'patches' which are bounded by pairs of lines of curvature. In mathematical terms, the mid-surfaces of the 'plated' membranes therefore constitute standard discrete curvature lattices. We derive a set of equilibrium equations which reduces in the natural continuum limit to that associated with classical membranes [10]. It is noted that finite element modelling of plates and shells based on 'discrete Kirchhoff techniques' [3] has been a subject of extensive research.

We show that, in the case of vanishing 'shear stresses', the equilibrium equations for plated membranes admit a parameter-dependent linear representation (Lax pair) [1] which may be used to construct explicitly large classes of plated membranes in equilibrium via an associated Bäcklund transformation [11]. This is achieved by adopting the 'Consistency Approach' which has recently been investigated in detail in connection with the algebraic and geometric isolation and classification of discrete integrable systems [2, 5, 8, 9] and discrete geometries which exhibit underlying integrable structure (cf. Abstract by Yu.B. Suris). Thus, we first demonstrate that it is *consistent* to demand that the equilibrium equations regarded as relations between objects (such as stresses and discrete tangent vectors) which are 'attached' to the quadrilaterals of a ('horizontal') lattice of $\mathbb{Z}^2$ combinatorics hold on the faces of a three-dimensional extension, that is, a $\mathbb{Z}^3$ lattice. The relations on the 'vertical' faces then turn out to be linearisable and give rise to the above-mentioned Lax pair.

The results presented here constitute the discrete analogue of the recent observation that classical membranes on which the principal lines of stress coincide with the lines of curvature are integrable [12] and highlights the validity of the standard discrete model of lines of curvature.

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S-CMC Surfaces

Tim Hoffmann

A discretization of surfaces of constant mean curvature (cmc surfaces) is proposed. Smooth cmc surfaces are described by the SinhGordon equation and are known to be isothermic, that is they allow conformal parameterization by curvature lines. There is a characterization of cmc surfaces by some isothermic properties, that we will state below.

Isothermic surfaces in $\mathbb{R}^n$ can be characterized by the fact that they have a lift

$$f \mapsto \hat{f} = \Big(\frac{1+|f|^2}{2}, f, \frac{1-|f|^2}{2}\Big)$$

into the light cone of Minkowski $\mathbb{R}^{n+2}$ that solves the Moutard equation [4]

$$f_{xy} = \lambda f$$

Two classical transformations for isothermic surfaces are of special interest here: The dual (or Christoffel) transformation and the Darboux transformation: A dual surface f^* of an isothermic surface f in conformal curvature line parameterization is given by the condition

$$df^* = \frac{f_x}{\|f_x\|^2}dx - \frac{f_y}{\|f_y\|^2}dy.$$

The Darboux transformation is characterized by the fact that the surface and its transform both envelope a special sphere congruence.

Note that in the light cone description the lift of a Darboux transform is a Moutard transform for the lift of the original surface.

Now cmc surfaces can be characterized by the fact that they possess a dual surface that is a Darboux transform as well.

There is an integrable discretization of the Moutard equation [6]:

$$F + F_{12} = \lambda F_1 + F_2.$$

Solutions to this equation in the Minkowski light cone give rise to discrete isothermic surfaces in the sense Bobenko and Pinkall defined in [2]: The surfaces are build from quadrilaterals with real cross-ratios that factor (in particular this implies con-circular vertices).

S-isothermic surfaces can be viewed as nonlinear deformations of the above discrete isothermic surfaces: They are generated by solutions to the discrete Moutard equation in the space like unit sphere [3]. In $\mathbb{R}^n$ they are build form sphere configurations [3, 5]: Each vertex of the lattice corresponds to a sphere, the four spheres around an elementary quadrilateral have a common orthogonal circle, and the inversive distances of neighboring spheres are equal for opposite edges. A discretization of minimal surfaces using this notion was proposed in [1].

s-isothermic surfaces possess dual and Darboux transforms and thus allow the definition of s-cmc surfaces as s-isothermic surfaces that have a dual that is a Darboux transform as well. Interpreting the radii of the spheres and circles as metric of the surface one can derive a discrete SinhGordon equation:

For for the radius R of a circle and the radii r_i of its four neighboring spheres the equation reads:

$$R^2 = \frac{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}{r_1 + r_2 + r_3 + r_4} \tag{1}$$

This is of course the same equation one finds for Schramm type circle patterns [7]. For a central sphere of radius r and the four neighboring circles with radii R_i on the other hand one finds:

$$\prod_{k=1}^{4} \frac{\sqrt{(B^2 - R_k^2)(A^2 + r^2)} - i\sqrt{(A^2 - R_k^2)(B^2 + r^2)}}{\sqrt{(B^2 - R_k^2)(A^2 + r^2)} + i\sqrt{(A^2 - R_k^2)(B^2 + r^2)}} \; = \; 1. \tag{2}$$

Inserting the former into the latter equation gives rise to an equation for nine sphere radii only. It can be easily shown, that this equation has a maximum principle.

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Surface Curvature Lines and Circle Patterns

Sergey P. Tsarev

(joint work with Alexander I. Bobenko)

Discrete conjugate nets, defined as mappings $\mathbf{Z}^2 \longrightarrow \mathbf{R}^3$ with the condition that each elementary quadrangle is flat, and discrete nets of curvature lines, defined as discrete conjugate nets with additional property of circularity of the four vertices of every elementary quadrangle, play an important role in the contemporary discrete differential geometry (see e.g. [1] for a review) and applications to CAGD.

In this talk we formulate results about the order of approximation of discrete nets to a given smooth conjugate net or the net of curvature lines on a smooth surface that can be achieved. Roughly speaking, our results show that the previously known upper bounds ([2]) can be made one order better for conjugate nets; for discrete nets of curvature lines one can impose one additional geometric requirement: all vertices of approximating discrete circular nets should lie on the original smooth surface.

We use the following terminology and notations: $d(A, S)$ is the distance from a point A to another point S (or a line, plane, etc.). A point A is said to be ε^k-close to S if $d(A, S) < C\varepsilon^k$ for $\varepsilon \to 0$ and we write in this case $d(A, S) \preceq \varepsilon^k$. A is said to lie at ε^k-distance from S, if $d(A, S) \sim \varepsilon^k$.

Suppose that a smooth surface α parameterized locally by some curvilinear net of conjugate lines is given: $\alpha : \Omega \longrightarrow \mathbf{R}^3$, $\Omega \subset \mathbf{R}^2 = \{(u, v)\}$. Take some initial point $\alpha_1 = \alpha(u_0, v_0)$ and points $\alpha_2 = \alpha(u_0 + \varepsilon, v_0)$, $\alpha_3 = \alpha(u_0, v_0 + \varepsilon)$ at ε-distance on the two conjugate lines of the net on α and let $\alpha_4 = \alpha(u_0 + \varepsilon, v_0 + \varepsilon)$ be the fourth point of the curvilinear quad on α. Using the conjugacy condition $\alpha_{uv} = a(u, v)\alpha_u + b(u, v)\alpha_v$ and its derivatives, one can easily estimate the distance from this fourth point to the plane π defined by α_1, α_2, α_3:

Theorem 1. *For any arbitrary smooth conjugate net α, $d(\alpha_4, \pi) \preceq \varepsilon^4$.*

One can show that for generic conjugate nets, $d(\alpha_4, \pi) \sim \varepsilon^4$.

Using this result we may inductively construct for any $\varepsilon > 0$ an approximating discrete conjugate net α^ε, starting from the initial point α_1 and two series of points $\alpha_{1,i}$, $\alpha_{2,i}$ at ε-distances on two curvilinear coordinate lines on α passing through α_1: for every three ε-close points we choose the fourth point α^ε_{ij} on α^ε to be the projection of the corresponding point $\alpha_{ij} = \alpha(u_0 + i\varepsilon, v_0 + j\varepsilon)$ in $\alpha(\Omega)$ onto the plane passing through the three available points. The technique of [2] suffices to prove the following:

Theorem 2. *For a disc $\alpha : \Omega \to \mathbb{R}^3$ and sufficiently small $\varepsilon > 0$, there exists a discrete conjugate net α^ε, such that $d(\alpha_{ij}, \alpha^\varepsilon_{ij}) \preceq \varepsilon^3$.*

For a given smooth α parameterized by curvature lines one can prove a similar result, if we take the circle ω passing through the points α_1, α_2, α_3 defined as above.

Theorem 3. *For arbitrary smooth net of curvature lines in a neighborhood of a non-umbilic point α_1 on α, $d(\alpha_4, \omega) \preceq \varepsilon^3$.*

A bit more careful investigation shows that one can give the following remarkable bound for the fourth point M of *intersection* of ω and α (in a neighbourhood of a non-umbilic point for sufficiently small ε there is always exactly one such point M in addition to the chosen α_1, α_2, α_3):

Theorem 4. *Under the assumptions of Theorem 3, $d(\alpha_4, M) \preceq \varepsilon^3$.*

Again, one can prove a global approximation result, inductively constructing a discrete circular net starting as above for the case of conjugate nets and taking the fourth point α^ε_{ij} of intersection of α and the infinitesimal circle passing through the three already constructed ε-close points on α. This fourth point will be close to the initial node $\alpha_{ij} = \alpha(u_0 + i\varepsilon, v_0 + j\varepsilon)$ of the smooth curvature line net on α:

Theorem 5. *For a disc $\alpha : \Omega \to \mathbb{R}^3$ and sufficiently small $\varepsilon > 0$, there exists a discrete circular net α^ε with all its points α^ε_{ij} on α, such that $d(\alpha_{ij}, \alpha^\varepsilon_{ij}) \sim \varepsilon^2$.*

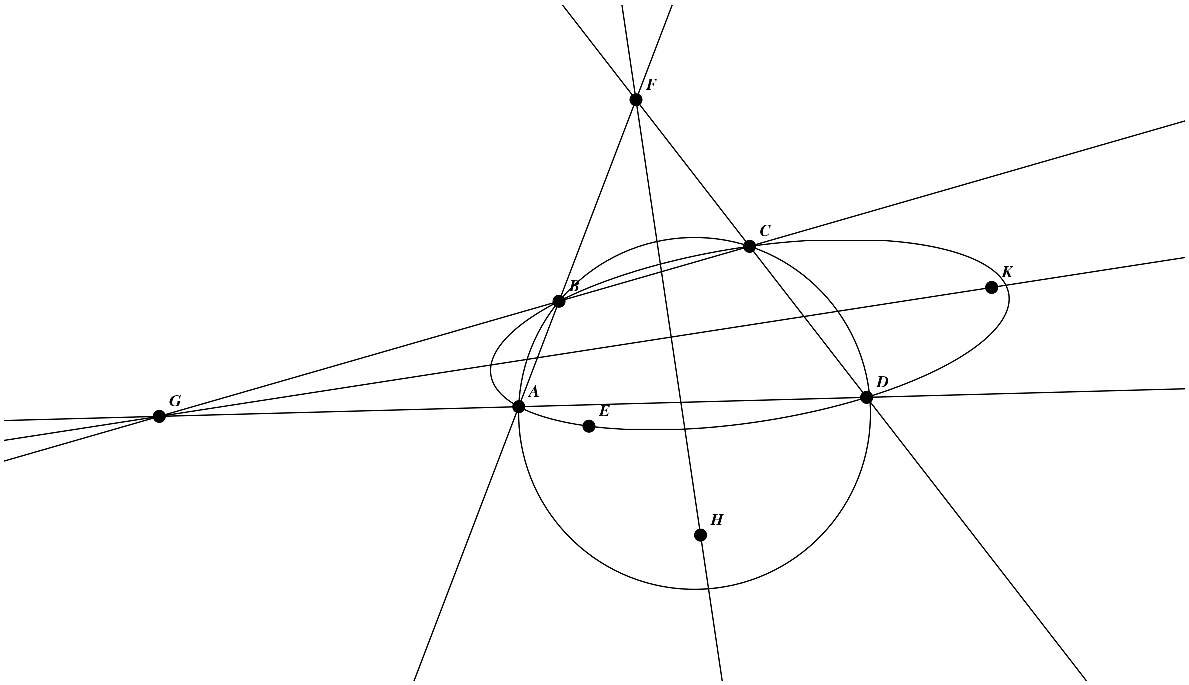

Figure 1.

In [2] one can find a similar result with the same order ε^2 of approximation but without the condition $\alpha_{ij}^{\varepsilon} \in \alpha$.

A. Doliwa proposed to define *directions* of "discrete curvature lines" for an elementary circular quadrangle $ABCD$ (see Figure 1) as the directions of the *bisectors* GK and FH of the angles AGB and BFC. It is easy to prove that GK and FH are orthogonal for arbitrary quadrangle $ABCD$ inscribed in a circle. In fact GK and FH approximate the directions of smooth curvature lines on a surface: if we take any non-umbilic point P on a smooth surface α and a plane π_ε, intersecting α, parallel to the tangent plane π at P, $d(P, \pi_\varepsilon) \sim \varepsilon^2$, then the intersection line $\delta = \alpha \cap \pi_\varepsilon$ is ε^2-close to the Dupin indicatrix of α at P; if we take then an *arbitrary* circle ω in π_ε, intersecting δ in points A, B, C, D (see Figure 1), then the directions of the bisectors GK, FH approximate the principal directions of the Dupin indicatrix:

Theorem 6. *For any non-degenerate plane quadric the bisectors GK, FH (shown on Figure 1 for the elliptic case) are parallel to the axes of the quadric.*

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Some Applications of the Cosine Law

Feng Luo

In the discrete approach to smooth Riemannian metrics on surfaces, the basic building blocks are sometimes taken to be triangles in constant curvature spaces. The smooth surfaces are approximated by the polyhedral surfaces which are the isometric gluing of Euclidean (or spherical or hyperbolic) triangles. The metric on a polyhedral surface is determined by the lengths of its edges and the discrete curvature at a vertex is defined to be 2π less the sum of inner angles at the vertex. In this setting edge lengths and inner angles of triangles correspond to the metrics and its curvatures. For triangles in hyperbolic, spherical and Euclidean geometries, edge lengths and inner angles are related by the cosine law. Thus cosine law should be considered as the metric-curvature relation. From this point of view, the derivative of the cosine law is probably an analogy of the Bianchi identity in Riemannian geometry. The talk is focused on some applications of the derivatives of the cosine law.

The main result of the talk is the construction of two continuous families of functions defined on the space of all geometric triangles. Two of members of one family were discovered by Y. Colin de Verdiere in [CV] and G. Leibon [Le] motivated by the 3-dimensional Schlaefli volume formula. They were used by Colin de Verdiere as the energy function in a variational framework which proves Thurston's circle packing theorem and by G. Leibon for Delaunay triangulations of hyperbolic. Another function in the family was discovered Richard Kenyon and his coworkers in a statistical model. The Legendre transforms of two other members of the family give the energy functions used by I. Rivin [Ri] and W. Braegger [Br] for their study of Euclidean polyhedral surfaces. In the same fashion, each function in the continuous families produces a local rigidity result for polyhedral surfaces and a variational framework on triangulated surfaces. See also [BS].

We also discussed an application of a function in the family to Teichmuller theory [Lu2].

Most of the computations involving derivative cosine law can be found in [CL], [Lu1], [Lu2] and [Lu3].

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Discrete Curvature Flows and Laplacians

David Glickenstein

In two dimensions, a natural analogue of curvature on a piecewise Euclidean surface is to consider at every vertex i the curvature $K_i = 2\pi - \sum \gamma_i$, where γ_i are each of the angles at vertex i. In this setting, B. Chow and F. Luo's combinatorial Ricci flow [1] takes the form

$$\frac{dr_i}{dt} = -K_i r_i \tag{1}$$

where r_i are weights at each point such that the distance between vertex i and vertex j in a triangle containing both is $r_i + r_j$. Each r_i can be considered the radius of a circle centered at the vertex i so that these circles form a packing, except for the fact that there is may be a cone point inside each circle. The curvature is the obstruction from it being a real packing at each point (if $K_i = 0$ then the circle centered at i really is a flat circle).

In three dimensions, a natural analogue of the curvature is $K_i = 4\pi - \sum \alpha_i$ where α_i are the solid (trihedral) angles at vertex i. One can use the same evolution equation on tetrahedra whose geometry is determined by weights at the four vertices. Since K_i looks like a kind of scalar curvature, the flow (1) is referred to as the combinatorial Yamabe flow [3] [4]. This curvature had previously been studied by D. Cooper and I. Rivin [2].

In both 2D and 3D it can be seen that the evolution of curvature takes the form

$$\frac{dK_i}{dt} = \sum_{j \sim i} a_{ij} \left(K_j - K_i \right),$$

which appears to have the form of a discrete Laplace operator, which we call $\triangle K_i$. With some work it can be shown that the coefficients in both cases have a natural expression as the "volume" of the geometric dual to the edge divided by the "volume" (length, actually) of the edge. In 2D, the dual of a triangle is its incenter and in 3D the dual of a tetrahedron is the center of the sphere tangent to all of its edges (this exists because of the fact that the tetrahedra are determined by the four weights r_i not by edge lengths). This gives a nice geometric interpretation of the Laplace operator.

I derived the geometric form of the Laplace operator and showed that if the flow does not cause tetrahedra to collapse, the flow will converge to constant curvature. A major tool used in 2D which is lacking in 3D is the maximum principle for the

Laplace operator. That is because the weights a_{ij} are necessarily positive in the 2D case but not in the 3D case.

The geometric interpretation of the Laplacians derived from combinatorial Yamabe flow leads one to perhaps try to define the Laplacian geometrically. Given a Euclidean triangulation of an n-dimensional manifold, if one has a consistent way to define the geometric dual of all of the simplices, then a Laplacian can be defined as

$$\triangle f_i = \sum_{j\sim i} \frac{|\bigstar\{i,j\}|}{|\{i,j\}|}(f_j - f_i)$$

where f is a function on the vertices, $|\{i,j\}|$ is the length of the edge from vertex i to vertex j, and $|\bigstar\{i,j\}|$ is the $(n-1)$-dimensional volume of the geometric dual to the edge $\{i,j\}$ (see, for instance, Hirani [6]). For consistency it is sometimes necessary to allow $|\bigstar\{i,j\}|$ to be negative. It turns out that to define the geometric duality it is sufficient to define centers of edges with a consistency condition on each triangle. Then there are a lot of different possibilities, such as the one described in the previous section for circle packings (the distance from a fixed vertex to the center each edge containing is the same) and one which leads to Delaunay triangulations (the distance from a vertex to the center of an edge is half the length of that edge). In two dimensions, the condition that $|\bigstar\{i,j\}| \geq 0$ is the condition that the triangulation is weighted Delaunay (also called "regular"). Once we have this intuition, it is possible to generalize some results of Delaunay triangulations to weighted Delaunay, for instance the observation of Rippa that the Dirichlet energy

$$-\sum f_i \triangle f_i$$

is optimized on Delaunay triangulations [7]. In higher dimensions, the condition $|\bigstar\{i,j\}| \geq 0$ does not appear to be the same as being weighted Delaunay (which would be that the dual of the $(n-1)$-dimensional simplices has positive length). This work is found in the preprint [5].

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Symmetry in Densest Packings

Charles Radin

Hilbert's eighteenth problem is to classify the symmetry groups of the densest packings, of Euclidean space $\mathbb{E}^n$ and hyperbolic space $\mathbb{H}^n$, by spheres or polyhedra [1]. The motivation was to encourage the study of interesting subgroups of the isometry groups of these spaces [2].

Hilbert assumed that the symmetry groups would be crystallographic. This was consistent with the very few examples of densest packings then known: that for unit circles in $\mathbb{E}^2$, for unit spheres in $\mathbb{E}^3$, for spheres of special radii in $\mathbb{H}^2$, and for certain tilings of $\mathbb{E}^n$ and $\mathbb{H}^n$. (Note that the densest packings are unknown even for such "simple" cases as regular pentagons or heptagons in $\mathbb{E}^2$, or regular tetrahedra in $\mathbb{E}^3$.) Since Hilbert's time a few more examples have been successfully analyzed. One class is "aperiodic tilings", for instance the Schmitt tilings of $\mathbb{E}^3$, based on a unit cube modified by the addition of certain bumps and dents on its sides [3]. These tilings exhibit some symmetry, but first of all the group is not crystallographic, and second the symmetry is not that of the tilings directly but of the invariant probability measures on the set of all the Schmitt tilings [4]. (This was widely exploited for other tilings in mathematical models of quasicrystals; the probability measures were there interpreted as diffraction intensities [4].) In a related investigation, the densest packings of spheres of generic (fixed) radius in $\mathbb{H}^n$ were shown not to have crystallographic symmetry [5]. Again the analysis required the mathematics of invariant probability measures on the space of packings of optimal density.

The focus of this lecture is the old example of the densest packings of $\mathbb{E}^3$ by unit spheres. The densest packings are not unique but can be understood as made by nestling successive planar layers of spheres together. Each succeeding layer can be added in one of two ways, so the resulting structure is highly nonunique.

The nonuniqueness of these densest packings poses a difficulty in the context of Hilbert's problem, as it is unclear how to associate a symmetry in this example. A solution was proposed by Stillinger *et al.* in 1968 [6], based on the natural connection between these sphere packings and a certain mathematical model of material crystals, the so-called "hard sphere" model of classical statistical mechanics. This formalism requires analysis of the uniform probability distributions, μ_d, on the sets X_d of all packings of $\mathbb{E}^3$, of density d, by unit spheres. It was argued that although at optimal density there may not be a unique symmetry, at densities just below optimum one symmetry is probably picked out by the averaging. (There have been several attempts in the past 40 years to determine which symmetry is indeed picked out, though the result does not appear to be conclusive [7].) There is a more compelling feature of this approach than the resolution of the nonuniqueness; numerical simulations show that the distributions μ_d behave smoothly in d, as d is decreased from its optimal value, but only to some special density d_c at which point a "phase transition" is found, indicated by nonanalyticity in d, at which the crystalline symmetry begins to disappear.

The behavior described above suggests that the whole phenomenon of the symmetry of densest packings actually extends to packings of high but not necessarily optimum density. This could lead to a different understanding of the mechanism producing these symmetries.

The phase transition discussed above has an analog in packings of unit circles in $\mathbb{E}^2$, again shown by numerical simulations. There are no proofs of the phase transitions in either the 2- or 3-dimensional hard sphere model. However in recent joint work with L. Bowen, R. Lyons and P. Winkler [8], [9] we have proven the existence of this phase transition in packings of $\mathbb{E}^2$ not by unit circles but unit hexagons which have special bumps and dents on their edges, so-called "zipper molecules". To be specific, we prove the:

Theorem. Let μ_d be the uniform probability distribution on the space of packings of $\mathbb{E}^2$, of density d, by zipper molecules. Let S^d be the subset of those packings in which there is an infinite cluster of molecules linked together - that is, with the bumps of one molecule inside dents of a neighbor. Then there exist densities $0 < d_1 < d_2 < 1$ such that $\mu_d(S^d) = 1$ for $d_2 < d < 1$ while $\mu_d(S^d) = 0$ for $0 < d < d_1$.

(The resulting nonanalyticity is evident.)

The moral we draw from this result is twofold. First, in continuation of the previous results on densest packings indicated above, the introduction of a uniform probability distribution on a space of packings of optimal density (or more generally any fixed density) allows new techniques for analyzing densest packings. A second moral is that symmetry may not be the only, or easiest, way to understand what is special about densest packings; what we are using is not their symmetry but a form of connectedness familiar from percolation theory.

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Open Problems in Discrete Differential Geometry

Collected by Günter Rote

Problem 1 (Walter Whiteley)**.** Rigidity is preserved under polarity: If a (non-convex) triangulated polyhedron is infinitesimally rigid, then its polar polytope is infinitesimally rigid as well. (The polar polytope might be self-crossing.)

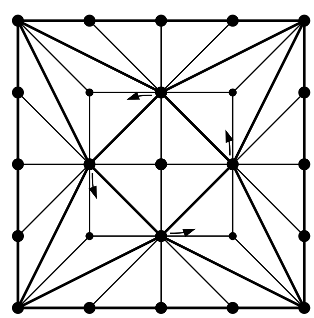
Can this statement be extended to smooth surfaces, with an appropriate definition of polar surfaces? (In the smooth category, one may distinguish between static and first-order rigidity.)

Note (Jean-Marc Schlenker): There is a result which somehow answers this. It can be found on the top of p. 20 of a fairly recent preprint: `arXiv:math.DG/0205305(v5)`, "Hyperbolic manifolds with convex boundary". Somehow that part did not make it into the final, published version of the preprint (Inventiones math. **163** (2006), 109–169).

Problem 2 (Günter M. Ziegler)**.** Does a 4-connected triangulated plane graph with a fixed outer face (with n vertices and $3n-6$ edges) always have a straight-line drawing with all interior triangles of equal area?

Problem 3 (Richard Kenyon)**.** Given a triangulation of a quadrilateral where all triangle areas are equal, is the configuration rigid under the constraint that the triangle areas must remain constant?

The example below is not *infinitesimally rigid*: If the inner square is rotated (and the small vertices follow accordingly), the derivative of the area is zero. The example can be enclosed in a larger square with no vertices on the boundary edges.



Walter Whiteley has shown (and outlined the proof during his talk) that triangulations which triangulate a triangle, with specified but not necessarily equal areas, are first-order rigid (and therefore rigid) for all *generic* realizations. Consequently, the areas are also independent, in the sense that, in a neighborhood of a given generic triangulation, one can change the interior areas by small amounts and realize these areas in a unique way.

He also pointed out that a triangulation $T(p)$ has a flex u if and only if the triangulations $T(p+u)$ and $T(p-u)$ have the same areas.

Problem 4 (Richard Kenyon)**.** Let M be a closed polyhedral surface homeomorphic to S^2 which is entirely composed of equal regular pentagons. If M is immersed in 3-space, is it necessarily the boundary of a union of solid dodecahedra that are glued together at common facets? The pentagonal faces may intersect each other (and the "union of solid dodecahedra" must be defined appropriately) but two different faces are not allowed to coincide.

(The corresponding question for equal *squares* has an affirmative answer.)

Problem 5 (John M. Sullivan)**.** Does every 3-manifold have a triangulation where each edge has degree 5 or 6? Does it have such a triangulation with the additional requirement that no triangle has more than one degree-6 edge (a so-called "TCP triangulation")?

Note: Only spherical manifolds can have triangulations where all edges have degree at most 5. (Compare the reports by Matveev and Lutz.) On the other hand, it is known that all three-manifolds can be triangulated using edge degrees 4, 5 and 6, see Brady, McCammond, Meier, *Bounding edge degrees in triangulated 3-manifolds*, Proc. Amer. Math. Soc. **132** (2004), 291–298.

Problem 6 (Wolfgang Kühnel)**.** Is there a direct geometric proof of the following theorem: Every simply-connected n-dimensional simplicial (or even polyhedral) manifold in which all ridges have positive curvature is homeomorphic to S^n. (The statement follows from a result of Cheeger.)

The following three problems are variations of the long-standing open question whether the boundary of a convex 3-polytope can be unfolded into the plane without self-overlap by cutting is along edges.

Problem 7 (Alexander I. Bobenko)**.** Can the boundary of a convex polytope be unfolded into the plane without self-overlap by cutting the surface along edges of the Delaunay triangulation T of the boundary? (T can have loops and multiple edges, but the faces are triangles.)

In particular, if all faces of the polytope are acute triangles, can it be unfolded by cutting along the edges of the polytope?

Problem 8 (Jeff Erickson)**.** Let T be an *arbitrary* triangulation on the boundary of a convex polytope whose vertices are the vertices of the polytope and whose edges are geodesics. Can the surface be unfolded without self-overlap by cutting it along edges of T? (The answer is yes for a tetrahedron.)

Problem 9 (Bob Connelly)**.** Can one cut the boundary of a convex polytope along edges and *start* an unfolding motion during which all faces remain on the convex hull ("blossoming"), at least for some short (infinitesimal) period?

Problem 10 (Günter M. Ziegler)**.** Let $f(2n+1)$ the smallest difference in area between the largest and the smallest triangle in a triangulation of the unit square with $2n+1$ triangles. Vertices on the boundary edges are allowed. Find (asymptotic) bounds on $f(2n+1)$.

It is known that $f(2n+1) > 0$ (Monsky, using 2-adic valuations), and $f(2n+1) = O(1/n^2)$ is straightforward. It can also be established that $f(2n+1) = \Omega(1/2^{c^n})$, using separation bounds for algebraic numbers.

Problem 11 (Rade T. Živaljević)**.** Is it true that a graph $G = (V, E)$ admits an embedding in 3-space (with curved edges permitted) without a (strong) *quadrisecant line* if and only if it is $\frac{1}{2}$-planar?

A strong quadrisecant line is a line that intersects 4 vertex-disjoint edges of G (in their interior). A graph is $\frac{1}{2}$*-planar* if it does not contain two vertex-disjoint non-planar subgraphs.

Remarks: Both classes of graphs are closed under minors (exercise) and an associated question is to determine/compare collections of the corresponding "forbidden minors". The first property arose in the context of studying "planarity on vector bundles" (R. Živaljević, The Tverberg-Vrećica problem and the combinatorial geometry on vector bundles, *Israel J. Math.* **111** (1999), 53–76). It is known (Corollary 3.5, loc. cit.) that the first property implies the second, so the conjecture is that the opposite implication is true as well. It is also interesting (J. Sullivan) to compare these classes with graphs which admit an embedding in the 3-space without a (weak) quadrisecant, that is a line which has at least 4 intersection points with the graph. "Forbidden graphs" in this case include all graphs which have two linked cycles (alternatively a knotted cycle) in each embedding, for example K_6, cf. E. Denne, Y. Diao, J. Sullivan, Quadrisecants give new lower bounds for the ropelength of a knot, *Geometry & Topology* **10** (2006), 1–26, and references therein.

Problem 12 (Konrad Polthier)**.** Consider an unbounded triangulated surface in 3-space which is intersected by every vertical line exactly once (the graph of a bivariate function). The vertices can be moved freely to minimize the surface area. If the position of the vertices is critical with respect to total area, does it follow that the surface must be a plane?

The same question may be asked for a bounded surface with a fixed plane boundary curve. (The "discrete maximum principle" does not apply here: the vertex height which minimizes the area of the faces for a given position of its neighbors may be higher than all its neighbors.)

Problem 13 (Ken Stephenson)**.** In any triangulation of the torus, a simple Euler characteristic count shows that the average degree (the average number of edges meeting at a vertex) is precisely 6. Suppose for a given triangulation T, all vertices are of degree 6; by flipping one edge in the pair of faces to which it belongs, one can change this so there are two degree 5 vertices and two degree 7 vertices, all the rest remaining degree 6. Does there exist a triangulation T in which there is *one* degree 5 vertex, *one* degree 7 vertex, and all the rest have degree 6? (Note: this is easily done for a Klein bottle.)

Ivan Izmestiev, Günter Rote, and John Sullivan have answered this negatively. If we build the surface from equilateral triangles with the same side length, we get a Riemannian metric on the torus with two cone singularities. In each shortest system of (non-trivial) loops based on the vertex of degree 7, the angle between the two ends of the loop arriving at the vertex must be at least π on each side. On the other hand, the loops are geodesics and respect the Euclidean structure of the triangular grid, and therefore the angle must be a multiple of $\pi/3$. One can then derive a contradiction.

Problem 14 (John M. Sullivan and Jeff Erickson)**.** Is there an acute triangulation of a cube? (This means a subdivision into tetrahedra whose dihedral angles are strictly less than a right angle.) Note that the square can be acutely triangulated, but this requires eight triangles; it is not hard to see that an acute triangulation of the cube would require hundreds of tetrahedra. Acute triangulations of three-space and of a slab are known; none is possible for higher-dimensional space (and hence for a higher-dimensional slab or a hypercube).

Similarly, is there a non-trivial acute triangulation of the regular tetrahedron? Does *any* 3-polytope have more than one acute triangulation (ignoring trivial symmetries)?

Discrete Differential Geometry of Polygons and the Simulation of Fluid Flow

Ulrich Pinkall

The velocity vector field v of an ideal incompressible fluid moving in 3-space is completely determined by its vorticity $\operatorname{curl} v$. An important special situation occurs when the vorticity is concentrated in the tubular neighborhood of a number of closed curves (called vortex filaments), which form a link in the sense of knot theory. In the limit of infinitely thin vortex filaments one obtains from the Euler equations of fluid flow a purely local evolution equation for space curves: $\dot{\gamma} = \gamma' \times \gamma''$ (the so-called smoke ring flow). The smoke ring flow turns out to be a completely integrable Hamiltonian system. Here we present a discretization of the smoke ring flow as an evolution equation for polygons (again an integrable system). Perturbing this system to account for the long-range interactions between vortex filaments given by the Biot-Savart formula we obtain a realistic model for the motion of real vortex filaments. The motion of the whole fluid can easily be reconstructed from the motion of the filaments. We present a computer implementation that allows simulation of complicated fluid flow in real time.

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Piecewise Linear Morse Theory

Günter Rote

Classical Morse Theory [8] considers the topological changes of the level sets $M_h = \{ x \in M \mid f(x) = h \}$ of a smooth function f defined on a manifold M as the height h varies. At *critical points*, where the gradient of f vanishes, the topology changes. These changes can be classified locally, and they can be related to global topological properties of M. Between critical values, the level sets vary smoothly.

This talk concerns Morse Theory of *piecewise linear* functions, and in particular, the "uninteresting" part of Morse theory, the level sets *between* the critical values, where "nothing happens". Spatial data coming from data acquisition processes (like medical imaging) or numerical simulations (like fluid dynamics) need to be represented for the purpose of storage on a computer, visualization, or further processing. Commonly they are represented as piecewise linear functions. My interest in Morse theory arose out of a fast and simple algorithm [4] for constructing the *contour tree* (or *Reeb graph*) of a piecewise linear function, a tree that represents how the connected components of the level sets, the *contours*, split and merge, are created and destroyed. While writing up this algorithm, I felt that I should say something about the obvious absence of topological changes when passing over "non-critical" vertices, but I could not find any results in the literature that I could readily apply. The results below are a contribution towards the foundations of Morse theory for piecewise linear functions of up to three variables.

1. Results and Open Questions

We assume that the domain M is a triangulation of a convex region in $\mathbb{R}^3$. The function f is given at the vertices and extended to M by linear interpolation. For simplicity, we restrict our attention to vertices in the interior of M. For our purposes, the *link* of a vertex v is the graph consisting of the neighbors of v, with an edge between two neighbors u and w if the triangle uvw is in the triangulation. The upper (lower) link is generated by the vertices whose value is bigger (smaller) than $f(v)$. We assume that no two vertices have the same value.

Theorem 1. *Let v be a vertex in the interior of M. The topology of the level sets M_h is the same for all values h in a sufficiently small interval $f(v) - \varepsilon \le h \le f(v) + \varepsilon$ if and only if the upper and the lower link are both non-empty and connected.*

The criterion can be adapted for boundary vertices, and also for two-dimensional domains. If the condition if the theorem is fulfilled, we call v a regular (or ordinary) point, otherwise it is a *critical point*, and $f(v)$ is a *critical value*.

Theorem 2. *If the interval $[a, b]$ contains no critical value, then there is an isotopy between all level sets in this range, i. e., a continuous bijection*

$$g\colon M_b \times [a, b] \to \{ x \in M \mid a \le f(x) \le b \}$$

that is level-preserving: $f(g(x, h)) = h$.

For a fixed height h, the homeomorphism $g(\cdot, h)$ between M_b and M_h is piecewise linear. However, the isotopy is not piecewise linear when regarded on its domain $M_b \times [a, b]$. (It should not be too difficult to strengthen the proof to achieve this.)

The proof is given in the appendix of [4] by an explicit construction: very roughly, the upper link of v is embedded as a planar straight-line graph inside the convex polygon whose sides correspond to the tetrahedra incident to v that are intersected by the level set through v. This graph has the same face structure as the level set above v. As the level set proceeds downwards towards v, the graph of the upper link shrinks towards the center, and at v, the result is a wheel.

The part of the proof that relies on drawing a graph with straight lines does not carry over to higher dimensions. An alternative approach that has not been tried might be to use a sequence of elementary subdivision operations (by inserting a new vertex into a cell) and their inverse "welding" operations [5, Theorem II.11].

There is a natural conjecture for the extension of the characterization of critical points to 4 dimensions: the link of a vertex is a 3-sphere, and for a regular vertex, it should be necessary and sufficient that the level set through this vertex cuts this 3-sphere into two 3-balls that are glued together along a 2-sphere forming their common boundary. This condition is straightforward to test. In five and higher dimensions, the problem of recognizing a critical point becomes more difficult, and it is probably even undecidable, for some high enough dimension.

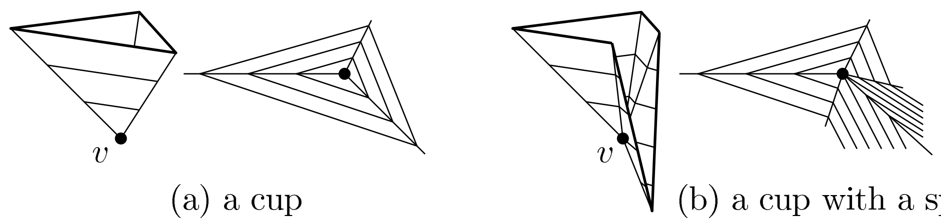
2. Related Literature

Interestingly, in Morse Theory for *continuous* functions [9], the criterion for the *definition* of a regular point v is just a local version of the *conclusion* of our Theorem 2: the existence of an isotopy between level sets in the neighborhood of v, i. e., some height-preserving homeomorphism between some neighborhood of v and the Cartesian product of a manifold with an interval of height values.

Tom Banchoff introduced Morse theory for piecewise linear functions in a widely known and often cited paper [2] about critical points, which even contains a *Critical Point Theorem*, without ever defining critical points, however. The results concern the Euler characteristic of the manifold and its relation to an appropriately defined index of a critical point. They remain at the level of counting, and no connection to the topology of level sets is made.

Morse Theory for piecewise linear functions has also been treated by Brehm and Kühnel [3, Section 2]; see also Kühnel [7, Chapter 7] for a more detailed account. Critical points are defined and the topology changes at these points are analyzed at the level of homology. For our case of two and three dimensions, this implies that regular points, where the homology is trivial, do not incur a topology change when the level set passes them, and there is a piecewise linear homeomorphism between different level sets [5]. However, the existence of this homeomorphism does not lead to the isotopy of Theorem 2.

In a related paper, Agrachev, Pallaschke and Scholtes [1] get a conclusion like in Theorem 2 under a stronger condition. To classify a vertex v in a piecewise linear function as a regular vertex, they require the existence of a direction that has a positive scalar product with the gradients of f on all cells incident to v. The example of a bivariate function in the figure shows that this is stronger than necessary. The graph of the function has three faces forming a "cup". The figure shows a side view and a top view with level lines. The vertex v at the bottom is a minimum and therefore certainly not regular. If we add a "spout" that makes it possible for the water to flow out, we only add more faces incident to v, and this cannot make the vertex regular, according to this definition. However, one sees that the level lines are isotopic as they pass through v.



(a) a cup (b) a cup with a spout

The *Stratified Morse Theory* of Goresky and MacPherson [6] seems like a natural candidate to apply to our setting: It treats smooth functions on non-smooth manifolds. If we look at the height function on the graph of the function f, which is a polyhedral hyper-surface in four dimensions, we are precisely in the situation that we need. One problem is that all vertices v of the domain are regarded as critical points by definition. The theory makes statements about the nature of the topology changes in this case, but only about the homeomorphism type of the level sets above ($M_{f(v)+\varepsilon}$) and below ($M_{f(v)-\varepsilon}$) the value $f(v)$, and thus they cannot be used straight from the book in order to get our results about the isotopy *across* the value $f(v)$. Moreover, this theory depends on heavy tools like René Thom's isotopy lemma, which is a powerful and deep statement with a very long proof (by John Mather) that has never been formally published. It might be rewarding to try to follow the proofs of [6] for the special case considered here, possibly replacing the application of Thom's Lemma by something that can be proved directly.

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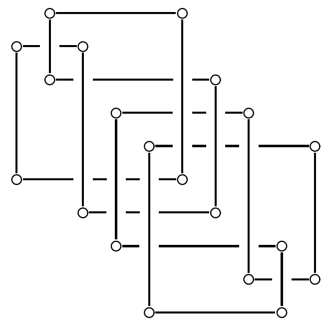
Towards a Fast Algorithm for Recognizing the Unknot

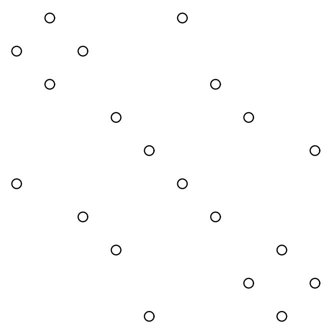
Ivan Dynnikov

A rectangular diagram of a knot or link is a picture like this:



In all crossings, the vertical line must be overcrossing. Collinear edges are forbidden.

To specify a rectangular diagram, it suffices to provide the positions of its vertices. For example, the diagram above can be given as



or, equivalently, as the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Rectangular diagrams give rise to a nice combinatorial formalism for describing isotopy classes of links. Any link type can be presented by a rectangular diagram, and any two different representations are equivalent modulo elementary moves that include:

- cyclic permutations of vertical and horizontal edges (columns and rows, respectively, of the corresponding matrix);
- exchanges of non-interleaved neighboring edges (neighboring columns or rows of the matrix);
- stabilizations and destabilizations.

Rows (columns) with non-zero entries at positions i, j and k, l are called interleaved if $i < k < j < l$ or $k < i < l < j$. A stabilization can be defined geometrically as replacing a vertex with coordinates (x, y) by three vertices $(x + \varepsilon, y)$, $(x, y + \delta)$, $(x+\varepsilon, y+\delta)$, where ε and δ are small non-zero numbers (not necessarily positive). The inverse operation is called destabilization.

The complexity of a rectangular diagram is defined as (one half of) the number of edges. Cyclic permutations and exchanges do not change it, whereas destabilizations reduce it. It was shown in [1] that any rectangular diagram of the unknot can be turned into a trivial diagram (a rectangular) by cyclic permutation, exchanges, and destabilizations (i.e., no stabilization needed). The number of diagrams that can be obtained from any given one by elementary moves excluding stabilizations is obviously finite, so we get a simple algorithm for recognizing the unknot: first, present the given knot in the rectangular form, then search all possible sequences of cyclic permutations, exchanges, and destabilizations.

Though this algorithm looks very simple and close in nature to the way a human recognizes unknots (untangling a rope), there is no optimistic upper bound for its running time. The only upper bound known so far is the total number of diagrams of complexity not exceeding the original one, which is of order n^n. The idea now is to find a larger family of moves such that: 1) any rectangular diagram of the unknot can simplified by a move from that family in just one step; 2) the moves that can be applied to a given diagram can be searched quickly, i.e., in polynomial time.

For a while, a hope was that flypes, which are introduced in [2], could be such a family. The number of flypes that can be applied to a given diagram of complexity n is of order n^6, but, jointly with Jean-Marie Droz, we have proved that a simplifying flype (if existing) can be found in n^4 operations. This made it realistic to implement a flype-based simplifying procedure on a computer and try it for rather large ($n = 100$) examples. In this way a rectangular diagram of the unknot has been found that does not admit any simplifying flype, showing that flypes are not yet sufficient for our purpose. However, the example, which has 45 vertical edges, still can be easily simplified by hand, which suggests that some simple idea that might finally make the algorithm polynomial can be out there.

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Linear Versus Piecewise Linear Embeddability of Simplicial Complexes

Ulrich Brehm
(joint work with Karanbir S. Sarkaria)

Determining the minimal dimension m such that a given simplicial complex K embeds piecewise linearly in $\mathbb{R}^m$ is in many cases an important but very difficult problem of algebraic topology. Piecewise linear embeddability of K means linear embeddability of the r-th derived complex $K^{(r)}$ for some r.

The first example of a triangulated 2-dimensional manifold with boundary which does not embed linearly in $\mathbb{R}^3$ has been constructed in [1]:

Theorem 1. *There exists a triangulated Möbius strip with* 9 *vertices which does not immerse linearly in* $\mathbb{R}^3$.

An immersion is a locally injective continuous mapping, thus intersections of triangles not sharing a vertex are allowed.

A detailed proof of Theorem 1 (see [1]) was given in the talk because it already contains the basic ideas for the higher dimensional analogues.

The following higher-dimensional analogue of Theorem 1 is a counterexample to a conjecture of Grünbaum in [2] (choosing $m = 2n$):

Theorem 2. *For each* $n \geq 2, m \geq 3, r \geq 0$ *with* $n \leq m \leq 2n$ *there exists an* n*-dimensional simplicial complex* K *which embeds piecewise linearly in* $\mathbb{R}^m$ *but its* r*-th derived complex* $K^{(r)}$ *does not embed linearly in* $\mathbb{R}^m$.

Note that the dimension bounds are best possible since each n-dimensional simplicial complex always embeds linearly in $\mathbb{R}^{2n+1}$ (merely choose the vertices in general position) and each planar graph ($n = 1, m = 2$) embeds linearly in $\mathbb{R}^2$.

For the proof of Theorem 2 one first observes that it is sufficient to consider the cases $m = 2n$ and $m = 2n - 1$.

In the case $m = 2n$ (where $n \geq 2$) we consider $Sk_n(\Delta^{2n+2})$, the n-skeleton of a $(2n+2)$-simplex. This complex does not embed piecewise linearly in $\mathbb{R}^{2n}$ but after removing an n-simplex s the complex $L = Sk_n(\Delta^{2n+2}) \setminus \{s\}$ embeds piecewise linearly in $\mathbb{R}^{2n}$, however, for such an embedding the boundary complexes ∂s and ∂t are linked with linking number $\neq 0$, where $t = \text{vert}\, \Delta^{2n+2} \setminus s$ is the complementary simplex of s in Δ^{2n+2}. Note that ∂s is an $(n-1)$-sphere and ∂t is an n-sphere.

Now the idea is to construct $K = L \cup M$ in such a way that K still embeds piecewise linearly in $\mathbb{R}^{2n}, L_0 \cap M = \partial s$ and M contains a simplicial n-sphere $\partial\tilde{s}$ with $n+2$ vertices which is homologous to u times ∂s (choosing $u > 1$). Now ∂t and $\partial\tilde{s}$ have linking number at least u (in absolute value) under any piecewise linearly embedding $e : K \to \mathbb{R}^{2n}$.

The number of simplices contained in the disjoint simplicial complexes $(\partial\tilde{s})^{(r)}$ and $(\partial t)^{(r)}$ is bounded in terms of r.

From this it follows that under any linear embedding of the union of these spheres in $\mathbb{R}^{2n}$ the absolute value of the linking number is also bounded by a constant depending only on r.

Choosing u bigger than this constant in the construction of K we get that $K^{(r)}$ does not embed linearly in $\mathbb{R}^{2n}$.

For the case $m = 2n-1$ one proceeds almost exactly in the same way using the results in [3]. We merely replace $Sk_n(\Delta^{2n+2})$ in the construction by any complementary n-dimensional complex $\tilde{L}$ with $2n+2$ vertices, where complementary means that for each subset $\sigma \subseteq \text{vert}\,\tilde{L}$ exactly one of the sets σ or $\text{vert}\,\tilde{L} \setminus \sigma$ is a simplex of $\tilde{L}$.

A generalization of Theorem 1 to higher dimensional manifolds with boundary is

Theorem 3. *For each $n = 2^k, k \geq 1$ there exists an n-dimensional simplicial complex K which embeds piecewise linearly but not linearly in $\mathbb{R}^{2n-1}$, where K is homeomorphic to the manifold with boundary obtained from the real projective space $\mathbb{R}\mathrm{P}^n$ by deleting an open n-dimensional ball.*

Open problem: *Can the minimal dimensions for piecewise linear embeddability and linear embeddability of a simplicial complex differ by more than one?*

Of particular interest are the low dimensional cases of n-dimensional simplicial complexes with $n = 2$ and $n = 3$.

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Computing Discrete Shape Operators on General Meshes

Denis Zorin

(joint work with E. Grinspun, Y. Gingold, J. Reisman)

Discrete curvature and shape operators, which capture complete information about directional curvatures at a point, are essential in a variety of applications: simulation of deformable two-dimensional objects, variational modeling and geometric data processing. In many of these applications, objects are represented by meshes. Currently, a spectrum of approaches for formulating curvature operators for meshes exists, ranging from highly accurate but computationally expensive methods used in engineering applications to efficient but less accurate techniques popular in simulation for computer graphics.

We propose a simple and efficient formulation for the shape operator for variational problems on general meshes, using degrees of freedom associated with normals. On the one hand, it is similar in its simplicity to some of the discrete curvature operators commonly used in graphics; on the other hand, it passes a number of important convergence tests and produces consistent results for different types of meshes and mesh refinement.

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Discrete CMC Surfaces in Space Forms

Udo Hertrich-Jeromin

(joint work with Wayne Rossman, relying on recent work by Fran Burstall, David Calderbank and Susana Santos)

We are concerned with the following question:

How to define discrete cmc surfaces in spaces of constant curvature?

More precisely: we are seeking an "integrable discretization" of the notion of a surface of constant mean curvature ("cmc") H in any space form Q^3_κ (here κ denotes the curvature of the space form), where "integrable discretization" in this context means that the defined class of discrete surfaces/nets should have a transformation theory that is similar to the smooth case (see Y. Suris' talk for more details). In particular, we seek a class of discrete nets that allows for "Bäcklund transformations" and "Lawson correspondence", as in the smooth case, and we hope to see a remnant of the harmonic map associated family that drives the usual integrable systems approach to constant mean curvature surfaces as well.

A second requirement for the sought definition is that it shall be a "unified definition", that is, one that works in all space forms and for all values of the mean curvature alike. We will briefly discuss this issue below.

Observations and known results.

A key observation is that smooth cmc surfaces in space forms are isothermic, that is, they can be conformally parametrized by their curvature lines, they are "divisible into infinitesimal squares by their curvature lines".

Smooth isothermic surfaces have a particularly rich transformation theory and the Bäcklund transformation and Lawson correspondence for cmc surfaces are special cases of the Darboux and Calapso (or "T"-) transformations for isothermic surfaces, respectively (see [7]).

This transformation theory carries over completely to the discrete case (see [5] and [7]) when a (discrete) isothermic net is defined as a net for which the cross ratios of elementary quadrilaterals factorise into two functions of one variable or, equivalently, the product of four adjacent cross ratios (taken in an appropriate way) is 1 (cf. [2]).

These facts led to definitions of discrete minimal and cmc $H \neq 0$ nets in Euclidean space (see [1] and [4]) as well as to the definition of horospherical nets as the discrete analogue of cmc $H = 1$ surfaces in hyperbolic space: these can equivalently be defined as Darboux transforms of spherical isothermic nets or as Calapso transforms of discrete minimal nets in Euclidean space (see [5] and [6]).

In fact, the Calapso transformation can be used to carry the definitions in Euclidean space over to other space forms in all cases where $H^2 + \kappa \geq 0$ (which is a conserved quantity for the Lawson correspondence). A discrete analogue of Bianchi's permutability theorem for the Darboux and Calapso transformations will then provide "intrinsic" definitions via "double Darboux transforms".

The indicated definition will provide "integrable discretizations" in the covered cases but they fail to provide a "unified" approach that would work in all situations (the problem is the lack of the existence of a parallel cmc surface in certain situations, i.e., the fact that Bonnet's theorem fails).

Polynomial conserved quantities.

Unlike other transformations of surfaces, the Calapso transformations can be realized by a loop of maps $\lambda \mapsto T_\lambda$ from the isothermic surface/net into the Möbius group of the conformal 3-sphere (or n-sphere) in the discrete and smooth cases alike. The Darboux transformations now appear as those surfaces/nets that are mapped to a single point by T_λ.

This loop of Möbius transformations can be lifted into the Lorentz group $O_1(5)$ of 5-dimensional (or, $(n+2)$-dimensional) Minkowski space. Thus, in the smooth case, the above characterization of the Darboux transforms of an isothermic surface is nothing but the integrated form of Darboux's linear system (cf. [7]).

Burstall/Calderbank/Santos (see [3] and [8]) now define a polynomial conserved quantity ("pcq") as a polynomial $P(\lambda) = \sum_{k=0}^{n} P_k \lambda^k$ (where the coefficients P_k are maps from the surface into Minkowski space) that satisfies Darboux's linear system for all λ, i.e., that satisfies $T_\lambda \cdot P(\lambda) \equiv \mathit{const}$ when the constants of integration for T_λ are suitably chosen. They observe that a (smooth) isothermic surface has constant mean curvature in a space form exactly when it has a linear conserved quantity $P(\lambda) = Q + \lambda Z$. Furthermore, the class of isothermic surfaces with pcq's is invariant under the Calapso transformation and has Bäcklund transformations that appear as special cases of the Darboux transformation.

Discrete cmc nets in space forms.

We use this characterization of smooth cmc surfaces as isothermic surfaces that have a linear conserved quantity as a definition in the discrete case: "an isothermic net is cmc if it has a linear conserved quantity $Q + \lambda Z$".

This definition is obviously a unified definition. Also, it turns out to provide an integrable discretization in the aforementioned sense: we can show, for isothermic nets with pcq's of any degree, that the Calapso transformation preserves the class (this is straightforward from the properties of the Calapso transformation), thus providing an analogue of the Lawson correspondence in the linear case, and that suitably chosen Darboux transformations also preserve the class, hence providing a notion of Bäcklund transformation. Further, the usual Bianchi permutability theorem holds for the Bäcklund transformation defined in this way, which is a first step towards a better understanding of the geometry of transformation and of the defined class of discrete nets.

Finally, using the fact that Q (being constant) models the ambient space form of the discrete cmc net and Z models its central sphere congruence (indeed, at least generically, Z gives rise to a discrete normal field in the sense of Schief [9]), we can prove that our definition generalizes the known definitions of discrete minimal and cmc nets in Euclidean space (and of all definitions obtained from these via Lawson correspondence).

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Triangulated Surfaces and Higher-Dimensional Manifolds

Frank H. Lutz
(joint work with Jürgen Bokowski, Stefan Hougardy, Thom Sulanke, John M. Sullivan, and Mariano Zelke)

We discuss and survey three different schemes for the enumeration of triangulated manifolds (cf. [8], [9], [11], [12], and [14]):

- generation from irreducible triangulations,
- (isomorphism free) lexicographic enumeration,
- strongly connected enumeration.

With implementations of the second scheme, all triangulated surfaces with up to 12 vertices have been enumerated [8], [13], [14]. There are 865, 20, and 821 vertex-minimal triangulations of the orientable surfaces of genus 2, 3, and 4 with 10, 10, and 11 vertices, respectively. All these examples have geometric realizations as polyhedra in $\mathbb{R}^3$. The respective realizations were obtained by

- randomly choosing coordinates [8],
- geometric construction [1],
- enumeration of small coordinates [5], [6],
- simulated annealing based on the intersection edge functional [7].

For the orientable surface of genus 5 there are 751593 different 12-vertex triangulations [14] of which at least 15 are realizable [7] and at least 3 are non-realizable [10].

By Steinitz' theorem (cf. [15]), every triangulation of the 2-sphere can be realized as the boundary of a simplicial 3-polytope. The realizability of triangulations of the torus was conjectured by Duke [3] and Grünbaum [4]. Our computational results in combination with the non-realizability results of Bokowski and Guedes de Oliveira [2] and Schewe [10] give rise to:

Conjecture. *Every triangulation of an orientable surface of genus* $1 \leq g \leq 4$ *is geometrically realizable.*

By strongly connected enumeration we further obtained all triangulations of 3-manifolds with edge degree at most five [9]: Altogether, there are 4787 such examples, $4761 \times S^3$, $22 \times \mathbb{R}\mathbf{P}^3$, $2 \times L(3,1)$, $1 \times L(4,1)$, and $1 \times S^3/Q$, with the 600-cell as the largest example. (It was shown independently by Matveev and Shevchishin that all triangulated 3-manifolds with edge degree at most five are spherical and have at most 600 tetrahedra.)

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Nonrealizability of Triangulated Surfaces

Lars Schewe

In this talk we report on progress in finding nonrealizable triangulations of surfaces; that is, triangulations of orientable, closed surfaces that do not admit a polyhedral embedding without self-intersections in $\mathbb{R}^3$.

The first such example was found by Bokowski and Guedes de Oliveira [2]. Using a similar method we have the following new results:

(1) No triangulation of a surface of genus 6 using only 12 vertices admits a polyhedral embedding in $\mathbb{R}^3$.
(2) There exist at least three triangulations of a surface of genus 5 using only 12 vertices that do not admit a polyhedral embedding.
(3) For every $g \geq 5$ we can construct an infinite family of triangulations of a surface of genus g none of which admit a polyhedral embedding in $\mathbb{R}^3$.

The last result is especially interesting in comparison to the results of Hougardy, Lutz, and Zelke (cf. the talk of Frank Lutz). They found polyhedral embeddings for some triangulations of a surface of genus 5 using only 12 vertices and found that all minimal vertex triangulations of surfaces of genus $g \leq 4$ admit polyhedral embeddings.

Bokowski and Guedes de Oliveira showed that one special triangulation with 12 vertices of a surface of genus 6 does not admit a polyhedral embedding in $\mathbb{R}^3$. To achieve this result they used an algorithm to generate oriented matroids and showed that no oriented matroid was admissible for the triangulation in question.

Oriented matroids (as a general reference we recommend [1]) serve as a combinatorial model of the point configuration of the vertices of the surface. From this model we can read off which triangles and edges intersect each other. We call an oriented matroid admissible for a given triangulation if the triangles do not intersect each other (on the level of oriented matroids). It is known that the existence of such an admissible oriented matroid is a necessary condition for the realizability of the given triangulation. This allows us to show that a triangulation is nonrealizable by showing that no admissible oriented matroid exists. Bokowski and Guedes de Oliveira gave an algorithm that given a triangulation generates all admissible oriented matroids. With this algorithm they could show that the triangulation they studied did not admit an oriented matroid.

One of the drawbacks of this method of generating admissible oriented matroids was the enormous amount of CPU-time needed for the algorithm. They used about four months (on different machines) to check all possibilities. This made it infeasible to check all 59 combinatorially distinct minimal vertex triangulations of a surface of genus 6.

The results mentioned above were achieved using a new algorithm to generate oriented matroids that are admissible for a given triangulation. The idea of this algorithm is to transform this problem into an instance of the well-known satisfiability problem (SAT) and then solve the resulting SAT instance using freely available SAT solvers.

With the new algorithm it is possible to check the triangulations in question in reasonable time (the example of Bokowski and Guedes de Oliveira takes about two hours on a single machine). We obtained result (3) by showing that after removing one triangle from one of the triangulations mentioned in (2) the resulting complex is still nonrealizable. Thus, we can construct new nonrealizable triangulations by taking connected sums with this triangulation.

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Discrete Conformal Structures

Kenneth Stephenson

Circle packings are configurations of circles with specified patterns of tangency. Such configurations were introduced into complex function theory through a 1985 conjecture of Thurston and its subsequent proof by Burt Rodin and Dennis Sullivan. Since then a rather comprehensive *discrete function theory* has emerged by defining discrete analytic functions as maps between circle packings [4, 5]. These maps manifest geometric behavior which not only parallels that of classical analytic functions, but also converges to it under appropriate refinement. A key feature of the discrete theory is its computability and the consequent experimental possibilities. The research described in this talk goes more directly to the geometry of the surfaces which underly analytic functions — namely, conformal geometry — and specifically its discretization.

A short list of what one expects when "discretizing" a classical topic would include: (1) geometric intuition — the essential global feel of the topic should be evident; (2) the discretization should provide models for a significant range of classical objects; (3) a refinement procedure should be available to allow the discrete model to incorporate more features of the classical; and (4) the discrete objects should converge under refinement to their classical counterparts.

The classical topic here is Riemann surfaces. These are largely smoke and mirrors: the conformal structure of a Riemann surface is whatever remains invariant under the conformal transition maps of an atlas. In short, the web of consistencies *is* the conformal structure. In the discrete case, geometric consistency is to be associated with "discrete" conformal transition maps — that is, maps between circle packings. In particular, this talk is intended to provide evidence in support of this definition:

Definition 1. *A* discrete conformal structure *for an oriented topological surface $\mathcal{S}$ is an abstract simplicial 2-complex K which is (isomorphic to) a triangulation of $\mathcal{S}$.*

A discrete conformal structure K on a surface allows for direct hands-on control of that surface (as well as a useful marking). In particular, each circle packing for K may be viewed as imposing a geometry on $\mathcal{S}$. The existence and properties of circle packings are well established and they are (largely) computable in practice, so one can manipulate $\mathcal{S}$. The issue is the extent to which these geometries on $\mathcal{S}$ are *discrete conformal*; that is, the extent to which their behaviors mimic what one expects in the classical setting and whether they converge appropriately under refinement.

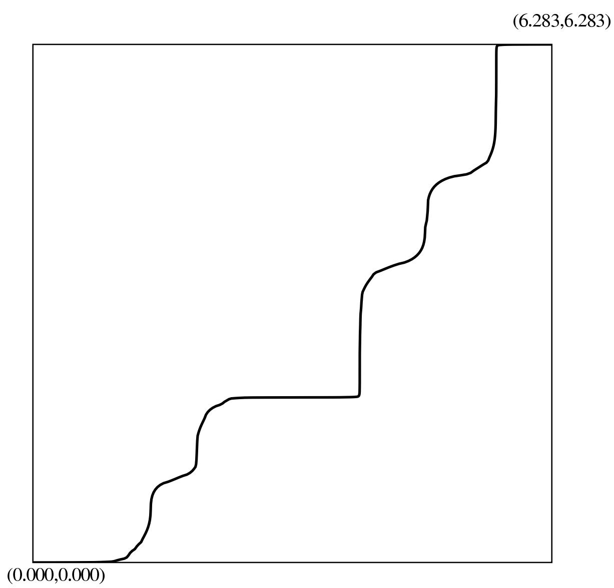
The talk surveys various situations where one can both apply and visualize concrete constructions with discrete conformal structures. Some are classical, some are recent. The talk begins, however, by surveying some of the basics of circle packing to illustrate the available discrete machinery. Thus one has the Discrete Uniformization Theorem which states that associated with every triangulation K

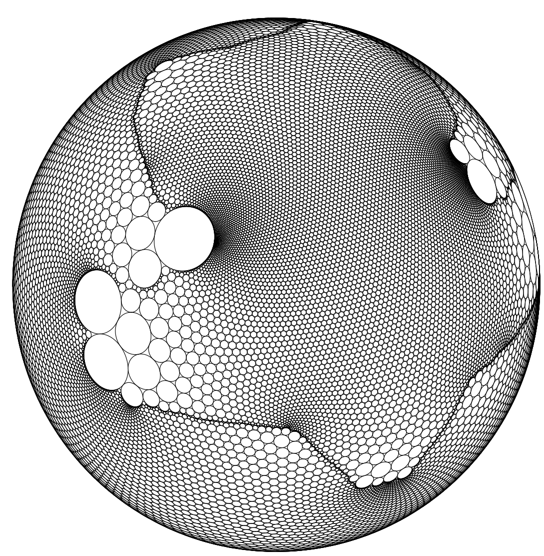
of an oriented topological surface $\mathcal{S}$ is an essentially unique classical conformal structure on $\mathcal{S}$ which supports a circle packing (in its intrinsic metric of constant curvature) with the combinatorics of K. In other words, K does induce a geometry. Moreover, if K has a boundary, then there is considerable plasticity available: one can realize circle packings for K satisfying specified boundary conditions in terms of radii or angle sums (and branching, though not relevant to this talk). These provide the discrete analogue of Neumann and Dirichlet boundary conditions.

The examples start with dessins d'enfants, a theory initiated by Grothendieck in which closed Riemann surfaces are created from simple drawings and their triangulations [1]. The classical conformal structure is defined by making each face from a unit-sided equilateral triangle, so it is fundamentally combinatoric. If such triangulations are refined and circle packed, the resulting discrete conformal structures provide the only general method to approximate the classical conformal structure. Examples of genus 0 and 2 are illustrated.

The next example illustrates conformal tiling, a topic due to Jim Cannon, Bill Floyd, and Walter Parry [2]. Here "subdivision" rules define surfaces as collections of abstract polygonal faces. Given an abstract such tiling, one can add a barycenter to each face and circle pack the resulting triangulation to obtain a geometric tiling. As the faces undergo successive subdivisions, one can watch the resulting geometries. The central issue is whether internal consistencies emerge in the geometry which reflect the combinatorics of the subdivision rules, such things as conformal type and internal self-similarities.

Next the classical topic of conformal welding. An illustration is given in the figure below. The classical theory of conformal welding, which associates *welding functions*, orientation preserving homeomorphisms of the unit circle, with *welding curves*, simple closed curves in the plane, has recently been proposed by Mumford [3] as an approach to analyzing "shape" for plane curves.





In this illustration, the welding map represented by the self-map of $[0, 2\pi]$ given on the left is converted to the owl-shaped welding curve on the sphere on the right.

One simply attaches two triangulated, circle packed copies of the unit disc using the welding map to determine how boundary vertices of the two triangulations are associated. The result is a triangulated sphere, and when circle packed on the Riemann sphere, the common boundary vertices take the owl shape (see Williams [6]. Circle packing can equally well be used to move from a shape to the associated welding map on the unit circle.

Additional examples are discussed involving computations of discrete extremal length, harmonic measures, ad hoc constructions of analytic function image surfaces, pairs-of-pants constructions of closed surfaces, and conformal mapping of piecewise affine surfaces embedded in 3-space. In all these cases one again sees the close parallels between the discrete and classical objects, the advantage on the discrete side being that the objects are constructible. An application of the last topic, the conformal flattening of embedded surfaces, is illustrated with "brain mapping" work (joint with Monica Hurdal). Besides providing a example of uses outside mathematics, this work illustrates the extreme complications that can be handled in circle packing constructions.

In closing, it is observed that for essentially all the situations that are illustrated it has been proven that the discrete conformal objects converge under refinement to their classical counterparts. This faithfulness and the intuitive and hands-on nature of the discrete model are sufficient that one might (discreetly) describe classical conformal structures as those imposed by "infinitesimal" triangulations of their surfaces.

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Discrete Riemann Surfaces, Linear and Non-Linear

Christian Mercat
(joint work with Alexander Bobenko and Yuri Suris)

To the first two orders, discrete holomorphic functions are either complex linear or Möbius transformations. We discretize this notion to complex functions of vertices $\diamondsuit_0$ of an oriented quad-mesh $\diamondsuit$. First, fix a direct embedding of this quad-mesh into the complex plane by a complex function of the vertices $Z:\diamondsuit_0\to C$, corresponding to the identity map. We will say that another function is *linear holomorphic* (with respect to Z) if and only if the ratio along the diagonals are the same as Z on each quadrilateral:

$$\forall(x,y,x',y')\in\diamondsuit_2,\ \frac{f(y')-f(y)}{f(x')-f(x)}=\frac{Z(y')-Z(y)}{Z(x')-Z(x)}=i\,\rho_{(x,x')}.$$

Likewise we will say that a function is *cross-ratio preserving* if the cross-ratio on each quadrilateral is the same as the one given by Z:

$$\frac{f(x)-f(y)}{f(y)-f(x')}\frac{f(x')-f(y')}{f(y')-f(x)}=\frac{Z(x)-Z(y)}{Z(y)-Z(x')}\frac{Z(x')-Z(y')}{Z(y')-Z(x)}=q_{(x,y,x',y')}.$$

An important class of cross-ratio preserving maps are given by circle patterns with prescribed intersection angles. In the form of a *Hirota system*, a connection between the two notions can be described: A discrete function F is cross-ratio preserving if its exterior differential can be written, on each edge $(x,y)\in\diamondsuit_1$ as $F(y)-F(x)=f(x)f(y)\big(Z(y)-Z(x)\big)=:\int_{(x,y)}f\,dZ$ for a function f. The constraint on f is the Morera equation: $\oint f\,dZ=0$ around every quadrilateral. Looking at logarithmic derivatives $f_\varepsilon=f\times(1+\varepsilon g)$ of f that still form a Hirota system, one finds that g should be linear holomorphic with respect to F: $\frac{g(y')-g(y)}{g(x')-g(x)}=\frac{F(y')-F(y)}{F(x')-F(x)}$. Linear constraints can as well be reformulated as a Morera equation $\oint f\,dZ=0$ not for a multiplicative but for an additive coupling between functions and 1-forms: $\int_{(x,y)}f\,dZ:=\frac{f(x)+f(y)}{2}\big(Z(y)-Z(x)\big)$.

We will be interested in so called *critical* reference maps Z composed of rhombi, the length $\delta=|Z(y)-Z(x)|$ is constant for all edges $(x,y)\in\diamondsuit_1$. Then these two constraints are *integrable*, meaning that they give rise to a well-defined Bäcklund (or Darboux) transform: given a linear holomorphic (resp. cross-ratio preserving) map f and a starting point $O\in\diamondsuit_0$, one can define a 2-parameters family $f_{\lambda,\mu}$ of deformations of f that still fulfill the same linear (resp. cross-ratio) condition and starting value μ at the point O. This is done by viewing the solution f lying on a ground level and building "vertically", above each edge (x,y), a quadrilateral on which the same kind of equation will be imposed: $\frac{f_{\lambda,\mu}(x)-f(y)}{f_{\lambda,\mu}(y)-f(x)}=\frac{\lambda+Z(x)-Z(y)}{\lambda+Z(y)-Z(x)}$ for the linear constraint, resp. $\frac{f_{\lambda,\mu}(x)-f(x)}{f(x)-f(y)}\frac{f(y)-f_{\lambda,\mu}(y)}{f_{\lambda,\mu}(y)-f_{\lambda,\mu}(x)}=\frac{\lambda^2}{\big(Z(y)-Z(x)\big)^2}$ for the cross-ratio constraint.

In [5], based on the point of view of integrable systems, we define discrete holomorphicity in $\mathbb{Z}^d$, for $d>1$ finite, equipped with rapidities $(\alpha_i)_{1\le i\le d}$ and the tools of integrable theory yields interesting results like a *Lax pair* governing a *moving frame* $\Psi(\cdot,\lambda) : \mathbb{Z}^d \to GL_2(\mathbb{C})[\lambda]$ and *isomonodromic* solutions like the *Green function* found by Kenyon [4].

An important tool of the linear theory is the existence of an explicit basis of discrete holomorphic functions. In the rhombic case, for a discrete holomorphic function f, the 1-form $f\,dZ$ is holomorphic (it is closed and its ratios on two dual diagonals are equal to the reference ratios) and can be integrated, yielding back a holomorphic function. Differential equations can be setup, producing explicit formulae for *exponentials and polynomials* which are shown to form a basis of discrete holomorphic functions.

Lots of results of the continuous theory can be extended to this discrete setting: There exists a *Hodge star*: $* : C^k \to C^{2-k}$, defined by $\int_{(y,y')} *\,\alpha := \rho_{(x,x')} \int_{(x,x')} \alpha$; the discrete *Laplacian* is written as usual $\Delta := d\,d^* + d^* d$ with $d^* := -*\,d\,*$, which reads as weighted differences around neighbours $\Delta\, f(x) = \sum \rho_{(x,x_k)}(f(x) - f(x_k))$. The weights are given by the usual cotan formula.

A wedge product defines an L^2 norm for functions and forms by $(\alpha,\beta) := \iint_{\diamond_2} \alpha \wedge *\,\bar{\beta}$. The norm of df is called the *Dirichlet* energy of the function f, $E_D(f) := \|df\|^2 = (df,\, df) = \frac{1}{2}\sum_{(x,x')\in\Lambda_1} \rho(x,x')\,|f(x') - f(x)|^2$. The *conformal energy* of a map measures its conformality defect $E_C(f) := \frac{1}{2}\|df - i * df\|^2$. They are related through $E_C(f) = E_D(f) - 2\mathcal{A}(f)$ just as in the continuous case.

The Hodge star decomposes forms into exact, coexact and harmonic ones, the harmonic being the orthogonal sum of holomorphic and anti-holomorphic ones. A Weyl's lemma and a Green's identity are found.

Nonclosed 1-forms with prescribed diagonal ratios define *meromorphic* forms and the holonomy around a quadrilateral is called its *residue*. The compact case is covered with flat atlases of critical maps for a given euclidean metric with conic singularities. *Meromorphic* forms of prescribed holonomies and poles are defined and are used to form a basis of the space of holomorphic forms. It is $2g$-dimensional on a genus g surface, that is *twice* as large as the continuous case, defining two period matrices instead of one. This difference is explained by the doubling of degrees of freedom, and partially solved through continuous limit theorems: the two period matrices converge to the same limit when refinements of quad-meshes for a given Euclidean metric with conic singularities are taken. Every holomorphic function can be approximated by a converging sequence of discrete holomorphic functions on refinements of critical quad-meshes.

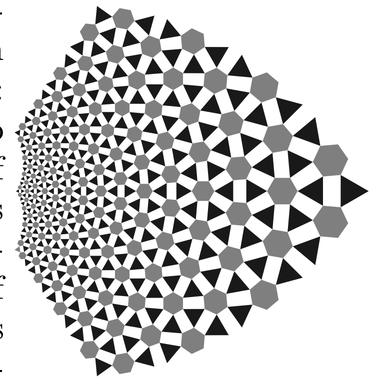
The Green function and potential allow one to setup a Cauchy integral formula giving the value at a point (in fact its average at two neighbours x, y) as a contour integral: $\oint_{\partial D} f\, dG_{x,y} = 2i\pi\, \frac{f(x)+f(y)}{2}$.

We define a derivation with respect to Z by $\partial : C^0(\diamond) \to C^2(\diamond)$ with

$$\partial f = \Big[(x,y,x',y') \longmapsto -\frac{i}{2\mathcal{A}(x,y,x',y')} \oint_{(x,y,x',y')} f\,d\bar{Z} = \frac{(f(x')-f(x))(\bar{y}'-\bar{y}) - (\bar{x}'-\bar{x})(f(y')-f(y))}{(x'-x)(\bar{y}'-\bar{y}) - (\bar{x}'-\bar{x})(y'-y)}\Big],$$

where $Z(x)$ is simply written x; and likewise $\bar\partial f$ with $f dZ$. A holomorphic function f satisfies $\bar\partial f \equiv 0$ and $\partial f(x,y,x',y') = \frac{f(y')-f(y)}{y'-y} = \frac{f(x')-f(x)}{x'-x}$. The Jacobian $J = |\partial f|^2 - |\bar\partial f|^2$ relates the areas $\iint_{(x,y,x',y')} df \wedge \overline{df} = J \iint_{(x,y,x',y')} dZ \wedge \overline{dZ}$.

Following Colin de Verdière and Kenyon, a geometrical interpretation of linear discrete holomorphicity is enlightening. As circle patterns with prescribed angles can be checked by eye, so can be a linear holomorphic map: The quad-mesh $\diamondsuit$, when bipartite, decomposes into two dual graphs Γ and Γ^* whose edges are dual diagonals of each quadrilateral. Around each vertex $x \in \Gamma_0$, there is a polygon, image of the dual face $x^* \in \Gamma_2^*$ by the reference map Z. Consider the identity map as a picture of all these polygons shrunk by a factor half. It represents both dual graphs at the same time as matching polygons. A map $f : \diamondsuit_0 \to \mathbb{C}$ is *discrete holomorphic* if and only if every polygon x^*, centered at $f(x)$, scaled and turned according to $\partial f(x)$, form into a polygonal pattern of the same combinatorics as the reference polygonal pattern, made of *similar* polygons.


The discrete exponential as a polygonal pattern on the triangular/hexagonal lattice

The *dilatation* coefficient of a discrete map f is defined as $D_f := \frac{|\partial f|+|\bar\partial f|}{|\partial f|-|\bar\partial f|}$. We will call f *quasi-conformal* when $D_f \geq 1$, that is $|\bar\partial f| \leq |\partial f|$. It can be written in term of the *complex dilatation*: $\mu_f = \frac{\bar\partial f}{\partial f} = \frac{(f(x')-f(x))(y'-y)-(x'-x)(f(y')-f(y))}{(f(x')-f(x))(\bar y'-\bar y)-(\bar x'-\bar x)(f(y')-f(y))}$.

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Some Observations from the (Projective) Theory of Rigidity

Walter Whiteley

Following along with the material presented in talks at this workshop, it became clear that there are a surprising number of important connections between key questions in field of Discrete Differential Geometry and the field of Rigidity Theory. In this talk, we outlined a selection of these connections. The material on moving between metrics is joint work with Franco Saliola [3, 4], and the final section on area constraints is new joint work with Bernd Schulze [6]. Other results come from the several decades of collaboration in the Structural Topology Research Group [1, 2]. For those with an interest in this larger selection of background material, we note that the full set of articles from the journal Structural Topology / Topologie Structurale is now available on the web from: *http://www-iri.upc.es/people/ros/StructuralTopology/*.

One of the key themes is that, in spite of appearances, the key first-order, or infinitesimal concepts are projectively invariant, and there can be substantial additional insight from working with this projective geometry in the analysis [1, 8, 9]. Because the talks at the workshop on discrete integrable systems emphasized the role of key projective configurations in the constructions, it is likely that these concepts are also projectively invariant and further connections to rigidity and associated geometric constraints can be anticipated and should be explored.

The first section of the talk outlined the rigidity matrices, and associated connections for infinitesimal rigidity of frameworks. One key emphasis was that both statics (the row dependences and row rank of the rigidity matrix) and first-order kinematics (column dependencies and column rank) can be expressed in projective form. If one examines the linear transformations which take the rigidity matrices over Euclidean spaces to Rigidity Matrices over Spherical or Hyperbolic metrics, one finds the matrix is decomposes into a a series of location specific blocks, one for each vertex, that has the impact of switching the definition of 'perpendicular' to the underlying 'projective motion' as a weighted hyperplane [4]. This, in turn, gives the simple translation of infinitesimal rigidity of a configuration in one metric to infinitesimal rigidity of a configuration in another metric which shares the same underlying projective geometry.

We also used this projective connection to connect first-order results for circle configurations in the plane, with intersection angles as constraints, with first-order results for corresponding points in Euclidean 3-space, which represent the circle $x^2 + y^2 - 2bx - 2cx + d$ with the point (b, c, d) [3]. The underlying connection actually passes through stereographic lifting to the sphere, and then duality to the 'de Sitter distance' of points in the exterior of the sphere: a connection studied as Laguerre Geometry, though it was also directly explored by Pedoe.

We also explored a second geometric construction which also appeared both implicitly and explicitly in other talks at the workshop: parallel drawing of a configuration. In the plane, with specified edges (essentially a framework), one can seek new configurations where the designated edges are parallel to the original edges. This might be a trivial change (translation or dilation) or a non-trivial change. The surprise is that in the plane, the search for parallel drawings is an isomorphic problem to the search for first-order motions of the framework [5, 8]. This is an old engineer trick arising in folklore techniques developed at the drafting tables in the 19th century. In 3-space the connection to first-order motions is more tenuous, but it was noted that it is also connected to problems of integrable systems for configurations of plane quadrilaterals forming a surface or disc in 3-space. In general, parallel drawing in any dimension is the projective dual of the concepts of lifting and projection for polyhedral scenes into the next lower dimension, studied in scene analysis. The known fast combinatorial algorithms for determining the space of parallel drawings at generic configurations come from studies in that field [7]. We also note that parallel redrawing is a natural concept in studies of Minkowski decomposability of polyhedra and polytopes, and in the study of reciprocal diagrams for configurations in all dimensions [2]. Again, the parallel drawing properties of a configuration are projectively invariant (in Euclidean spaces) and this underlying projective geometry offers additional insight.

Finally, we presented the solution to the infinitesimal version of one of the problems from the problem session on Wednesday evening. At that session, it was asked whether a plane configuration of a triangulated triangle, constrained to hold the areas of the interior triangles fixed (area one or even more general areas) would be unique up to area preserving maps (translation, affine transformations with determinant 1). We described an approach using infinitesimal motions and 'rigidity like' matrices which represent the constraints of equal area [6]. This model of the problem presents a lot of analogs to results in plane and spatial rigidity.

One analog is a necessary counting condition for local uniqueness (rigidity), and for independence:
With $|T|$ triangles and $|V$ vertices, we need $|T| \geq 2|V| - 5$ for local uniqueness. Moreover, for independence of the constraints, all non-empty subsets T' of triangles must satisfy $|T'| \leq 2|V'| - 5$.
On counting, a triangulated triangle has exactly the right number of triangles to be independent and locally unique. We *conjecture* that these counts are also sufficient for local uniqueness when the vertices are in generic position. We note that, like the counts for rigidity in the plane (and unlike the counts for rigidity in 3-space) these counts do generate a matroidal independence structure on 3-uniform hypergraphs (collections of abstract triangles) in the plane ([9] Appendix A).

We outlined an inductive proof that a triangulated triangle gives a locally unique pattern, with fixed areas, provided the vertices are generic. The proof begins with the simplest case: one triangle on three vertices, and then works up by a process called vertex splitting which adds one new vertex and two new triangles at each stage, while preserving the null-space of the corresponding matrix, at generic configurations.

We also noted that these area constraints have an averaging property (similar to averaging for first-order rigidity). If we have two configurations p and q which give the same areas a pattern (T, V), then the same pattern at the configuration

$\frac{p+q}{2}$ has an infinitesimal motion $p-q$ which preserves the area. Conversely, a first-order deformation u of a pattern (T,V) at p will generate two patterns at $p+u$ and $p-q$ which have the same areas (though these are different than the areas of the original). In this manner, it is possible to create a number of first-order flexible configurations on a triangulated triangle, but it we do not know of any finitely flexible patterns for a triangulated triangle.

In closing, we returned to the central theme: these concepts have a core projective content which gives simplicity to the analysis and gives insight for the solution of associated geometric problems.

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Computing Quadrilateral and Conical Meshes

Johannes Wallner

(joint work with Helmut Pottmann and Wenping Wang)

It is well known that the network of principal curvature lines can be discretized by circular meshes, i.e., quadrilateral meshes with planar faces, where the vertices of each face are co-circular [1]. A new discretization is *conical meshes*, where we require faces adjacent to a vertex to be co-conical; to be precise: the oriented planes which carry those faces are tangent to an oriented cone of revolution [2]. In the smooth case, infinitesimally neighbouring surface normals along a principal curvature line are co-planar (this is a characterization) – in the discrete case, neighbouring axes of circles/cones of a circular/conical mesh are co-planar.

In the S^3 model of Möbius geometry, co-circular vertices lie in $S^3 \cap U$, with $\dim U = 2$. Analogously, in the Blaschke cylinder model $S^2 \times \mathbb{R}$ of Laguerre geometry, co-conical faces appear as points which lie in $(S^3 \times \mathbb{R}) \cap U$, with $\dim U = 2$. In this way, both the circular and conical meshes appear as quadrilateral meshes in the appropriate geometric model. Möbius/Laguerre transformations transform circular/conical meshes into meshes of the same property, an important example of a Laguerre transformation being the offsetting operation. The latter leads to applications of conical meshes in architectural design.

For a conical mesh, the unit normal vectors of the faces constitute a circular mesh in S^2, which implies that 3D consistency of the conical condition follows directly from Miquel's theorem. It is interesting to note that the rhombic networks of [3] which are models of surfaces of constant curvature have diagonals which constitute a mesh which is both circular and conical.

We express planarity/circularity/conicality of a mesh in terms of the angles $\phi_{e,f}$ enclosed by edges e, f of the mesh ($0 \leq \phi_{e,f} \leq \pi$): A face with boundary edges $e_1, \dots, e_n$ is planar and convex $\iff \sum \phi_{e_i, e_{i+1}} = (n-2)\pi$ (Fenchel's theorem). In the case $n = 4$, it is in addition circular $\iff$ the sums of opposite angles equal π. If $e_1, \dots, e_4$ are the edges emanating successively from a vertex, then this vertex is conical $\iff \phi_{e_1,e_2} + \phi_{e_3,e_4} = \phi_{e_2,e_3} + \phi_{e_4,e_1}$ (Lexell's theorem).

By summing up the squares of these conditions we arrive at a nonnegative geometry functional $F_G(v_1, \dots)$ on the vertices where $F_G = 0$ characterizes meshes of the required properties. In order to perturb a given mesh such that it becomes planar/circular/conical, we numerically optimize in the space of vertices such that $F_G \to 0$ and in addition $F_P, F_F \to \min$, where F_P and F_F are nonnegative functionals expressing distance from a target surface and mesh fairness, resp.

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An Approach to Combinatorial Holonomy

Michael Joswig

Introduction. The classical holonomy group of a Riemannian manifold is a linear group generated by translating a frame of reference at a base point along closed smooth curves. Holonomy is related to curvature, and the holonomy group captures the obstruction to certain embeddability problems. The purpose of this talk is to survey recent results in geometric and topological combinatorics which have been obtained by transferring these concepts to a combinatorial level.

Combinatorial Manifolds. Let Δ be a simplicial complex. A *perspectivity* is a local reflection of one facet of Δ to an adjacent one. This defines a bijection between the vertex sets of any two adjacent facets. Concatenating perspectivities leads to *projectivities*, and the set of all projectivities with respect to closed paths in the dual graph of Δ which start and end at a fixed facet σ_0 form a group, the *group of projectivities* $\Pi(\Delta, \sigma_0)$. If the dual graph of Δ is connected then the isomorphism type of the group of projectivities does not depend on the base facet σ_0; in this case we write $\Pi(\Delta)$.

A first application of projectivities is to a very special class of coloring problems. A d-dimensional simplicial complex Δ is *balanced* if its vertices can be colored with $d+1$ colors such that vertices sharing an edge receive different colors, that is, the chromatic number of the 1-skeleton equals $d+1$. Clearly, since the vertices in a d-face form a clique of size $d+1$, this is a lower bound for the chromatic number of the graph of any d-dimensional simplicial complex. Under mild connectivity assumptions, which are satisfied, for instance, if Δ is a combinatorial manifold, it is easy to see that Δ is balanced if and only if $\Pi(\Delta)$ is the trivial group; see [4]. The following result has been proved a number of times, and ideas can be traced back to Heawood's 1897 paper on the 4-color problem.

Theorem. *A simply connected combinatorial manifold (with or without boundary) is balanced if and only if each interior face of codimension* 2 *is contained in an even number of facets.*

This result is instrumental, for instance, in obtaining bounds on the dimension of a real torus acting freely on moment angle manifolds defined by simple polytopes. For this and a suitable generalization to the not simply connected case, see [4].

Balanced triangulations of lattice polytopes can lead to non-trivial lower bounds on the number of real roots of certain sparse polynomial systems. Let P be a d-dimensional lattice polytope with a regular and balanced lattice triangulation Δ such that Δ uses all lattice vertices inside P. Since Δ is balanced its dual graph is bipartite. Call the facets in the bipartition 'black' and 'white', respectively. Then the *signature* $\sigma(\Delta)$ is the absolute value of the difference between the number of black facets (with odd normalized volume) and the number of white facets (with odd normalized volume). A *Wronski* system for (P, Δ) is a system of d real polynomials in d indeterminates such that all polynomials have P as its Newton polytope. There are certain additional constraints, which we omit here, on the coefficients of the polynomials which are related to the triangulation Δ. The following result, due to Soprunova and Sottile [7], generalizes the basic fact that a real univariate polynomial of odd degree has at least one real root.

Theorem. *A generic Wronski polynomial system for* (P, Δ) *(which satisfies certain additional geometric properties) has at least* $\sigma(\Delta)$ *real roots.*

We give an example. Consider the lattice triangle $T = \operatorname{conv}\{(0,0),(2,0),(0,2)\}$ with the balanced and regular lattice triangulation shown in the figure (left). All triangles in the triangulation have normalized volume 1, and hence the signature equals $4-2=2$. A Wronski system for T consists of two polynomials of type $a(1+xy)+b(x+y^2)+c(x^2+y)$, where a, b, c are real parameters (corresponding to the color classes of vertices in the triangulation) still to be chosen. For instance,

$(a_1, b_1, c_1) = (1, -1, 2)$ and $(a_2, b_2, c_2) = (-1, 3, 5)$ yields a generic Wronski system for T with two real roots, depicted in the figure (right).

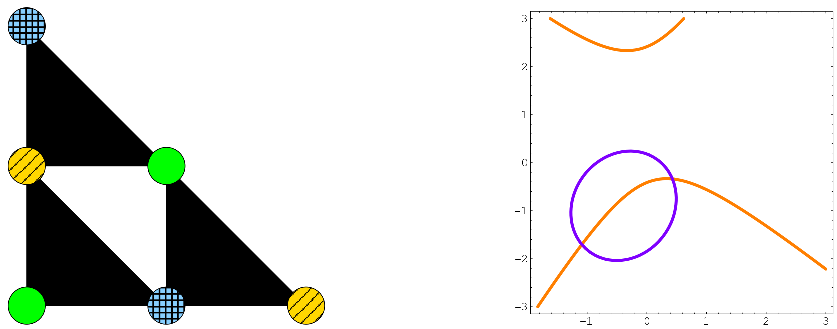

FIGURE 1. Lattice triangle with a balanced regular lattice triangulation and two conics corresponding to a Wronski system.

The special case of Wronski systems corresponding to products of lattice polytopes is discussed in [5].

It is an interesting question to ask, what happens if the group of projectivities of a combinatorial manifold Δ is not trivial? An answer is given in [3]: The group of projectivities can be read as a monodromy group, and this way it defines a branched covering over Δ where the branch locus is formed of those interior codimension-2-faces which are contained in an odd number of facets. This branched covering is called *partial unfolding*. If the branch locus itself is a manifold then the covering space is also a manifold. An interesting fact is that, from the topological point of view, these very special combinatorially defined branched coverings are quite common in the following sense.

Theorem. *For each closed oriented manifold M with $d := \dim M \leq 3$ there is a triangulation Δ of the d-sphere such that M is homeomorphic with the partial unfolding of Δ.*

Holonomy of Groupoids. Recently, Živaljević suggested to reformulate the above approach to combinatorial holonomy in the language of category theory, see [8, 9]. A *groupoid* $\mathcal{C} = (\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$ is a small category with invertible morphisms. For an object $x \in \mathrm{Obj}(\mathcal{C})$ we call $\mathrm{Hol}(\mathcal{C}, x) = \mathcal{C}(x, x)$ the *holonomy group* of $\mathcal{C}$ at x. Clearly, $\mathrm{Hol}(\mathcal{C}, x)$ is independent of x if the category $\mathcal{C}$ is connected.

The first example of this kind justifies the name: Let M be a *Riemannian manifold*. We obtain a groupoid $\mathcal{R}$ with object set M and morphisms between $x, y \in M$ given by the linear isomorphisms of the tangent spaces $T_xM \to T_yM$ induced by parallel transport along piecewise smooth paths from x to y. Then $\mathrm{Hol}(\mathcal{R}, x)$ is the usual holonomy group of M at x.

For the second example let Δ be a simplicial complex. Then $\mathcal{S}_k$ is a groupoid, where the objects are the k-faces of Δ, and where the morphisms are the projectivities between faces in the k-skeleton $\Delta_{\leq k}$. In this case $\mathrm{Hol}(\mathcal{S}_k, \sigma) = \Pi(\Delta_{\leq k}, \sigma)$ for each k-face $\sigma \in \Delta_{\leq k}$.

A third example was studied independently by Bolker, Guillemin, and Holm [2]. Let $\Gamma = (V, E)$ be a (connected and undirected) d-regular graph. We let $\mathrm{Obj}(\mathcal{G}) = \{E(v) : v \in V\}$, where $E(v)$ is the set of edges containing v. Moreover, we let $\mathrm{Mor}(\mathcal{G}) = \{\nabla_{v,w} : \{v,w\} \in E\}$ with bijections $\nabla_{v,w} : E(v) \to E(w)$ satisfying $\nabla_{v,w}\{v,w\} = \{v,w\}$ and $\nabla_{v,w} = \nabla_{w,v}^{-1}$. The holonomy group $\mathrm{Hol}(\mathcal{G})$ plays a role in questions concerning the parallel redrawings of Γ.

One of the most striking application of this approach to combinatorial holonomy up to now is a new proof of the Lovász Conjecture on lower bounds for the chromatic number of graphs. The conjecture was settled in the affirmative by Babson and Kozlov [1], and the new proof is due to Živaljević [8, 9] and Schultz [6]. Let Γ be a finite graph and C_{2r+1} a cycle of odd length $2r+1$, then the *Hom-complex* $\mathrm{Hom}(C_{2r+1}, \Gamma)$ is a cell complex capturing the different ways of how to map the odd cycle C_{2r+1} to the graph Γ.

Theorem. *If* $\mathrm{Hom}(C_{2r+1}, \Gamma)$ *is* k*-connected then* $\chi(\Gamma) \geq k + 4$.

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Edge Subdivision Schemes and the Construction of Smooth Vector Fields

Peter Schröder

(joint work with Ke Wang, Weiwei, Yiying Tong, Mathieu Desbrun)

Subdivision schemes are a broadly deployed tool in all areas of geometric modeling and computer graphics [20, 16]. Their foremost benefit is the ease with which they accommodate the construction of smooth surfaces in the arbitrary topology setting. They also offer many favorable computational properties for applications ranging from surface compression [5] to physical modeling [4]. Their mathematical properties are by now well understood [10, 11, 18, 19] and a large variety of subdivision schemes and extensions have been developed. Broadly, subdivision schemes are classified as either primal (*e.g.*, Catmull-Clark [1], Loop [7], and $\sqrt{3}$ [6]) with vertices carrying the data and faces being split, or dual (*e.g.*, Doo-Sabin [2] and dual-$\sqrt{3}$ [8]) in which data lives at faces and vertices are split.

In this talk we present a novel class of subdivision schemes which carry scalar coefficients on *edges* from the coarser mesh into scalar coefficients on edges in the refined (face split) mesh. The method can be viewed as constructing higher regularity bases for *discrete differential* 1*-forms* in the arbitrary topology 2-manifold (with boundaries) setting. Given the metric induced by an underlying surface, 1-forms then yield smooth tangent vector fields, which are useful in many computer graphics applications including texture synthesis [15], fluid simulation [14], crowd animation [12], and shading [13]. In particular the *design* of vector fields is greatly facilitated by the intuitive relationship between coefficients and the resulting vector field.

Approach and Contributions. Our construction is based on treating vertex-, edge-, and face-based subdivision schemes as a *triple* of schemes linked through Stokes' theorem, ensuring that the spaces spanned by the underlying bases form a chain complex. Given the (purely topological) exterior derivative operator d this amounts to requiring that the subdivision operators S_0, S_1, and S_2 (for the vertex-, edge-, and face-based schemes respectively) satisfy commutative relations with respect to d

$$d\,S_0 = S_1 d \quad \text{and} \quad d\,S_1 = S_2 d. \tag{1}$$

In words: taking the exterior derivative of a 0-form (vertex-based) subdivision scheme is equivalent to first taking differences—assigning values to edges from their endpoints—and then applying the edge-based subdivision scheme. Similarly, taking the exterior derivative of a 1-form (edge-based) subdivision scheme is the same as first computing signed sums around the boundary of each face followed by application of the face-based subdivision scheme. This generalizes the well known *formule de commutation* [3] to the bivariate setting.

Applying this line of reasoning to piecewise linear subdivision recovers the well known Whitney forms [17]. Asking for smoother bases over arbitrary triangulations leads to considering Loop subdivision for S_0. For S_2 one may then choose (a generalization of) half-box splines [9]. With S_0 and S_2 fixed in this manner, S_1 follows *uniquely* using Eq. 1. It too is (a generalization of) a piecewise polynomial spline scheme, which we describe for the first time.

More generally, one may begin with a desired support (stencil size) and symmetries for the 0-form (vertex-based) subdivision scheme and then derive fully parameterized families of subdivision scheme triples from Eq. 1. While we demonstrate this only in the case of Loop (and triangles), the approach applies equally well to other settings, *e.g.*, quadrilaterals with Catmull-Clark for S_0 and Doo-Sabin for S_2.

The preprint (`http://multires.caltech.edu/pubs/FormSubdivision.pdf`) of the same title contains all the details of the construction.

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