

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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**Mini-Workshop:
The Hauptvermutung for High-Dimensional Manifolds**

Organised by
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August 13th – August 19th, 2006

ABSTRACT. The meeting was devoted to the Kirby-Siebenmann structure theory for high-dimensional topological manifolds and the related disproof of the Hauptvermutung. We found nothing fundamentally wrong with the original work of Kirby and Siebenmann, which is solidly grounded in the literature. Their determination of TOP/PL depends on Kirby's paper on the Annulus Conjecture and his 'torus trick', and the well-known surgery theoretic classification of homotopy tori.

Mathematics Subject Classification (2000): 57A15, 57Q25.

Introduction by the Organisers

The Mini-Workshop *The Hauptvermutung for High-Dimensional Manifolds*, organised by Erik Pedersen (Binghamton) and Andrew Ranicki (Edinburgh) was held August 13th–18th, 2006. The meeting was attended by 17 participants, ranging from graduate students to seasoned veterans.

The manifold Hauptvermutung is the conjecture that topological manifolds have a unique combinatorial structure. This conjecture was disproved in 1969 by Kirby and Siebenmann, who used a mixture of geometric and algebraic methods to classify the combinatorial structures on manifolds of dimension > 4 . However, there is some dissatisfaction in the community with the state of the literature on this topic. This has been voiced most forcefully by Novikov, who has written "In particular, the final Kirby-Siebenmann classification of topological multidimensional manifolds therefore is not proved yet in the literature." (<http://front.math.ucdavis.edu/math-ph/0004012>)

At this conference we discussed a number of questions concerning the Hauptvermutung and the structure theory of high-dimensional topological manifolds. These are our conclusions:

We found nothing fundamentally wrong with the original work of Kirby and Siebenmann [4], which is solidly grounded in the literature. Their determination of TOP/PL depends on Kirby's paper on the Annulus Conjecture and his 'torus trick'. It was noted that Kirby's paper is based on the well-documented work on PL classification of homotopy tori (Hsiang and Shaneson, Wall) and Sullivan's identification of the PL normal invariants with $[-, G/PL]$, but does not depend on any other work of Sullivan, documented or undocumented. This classification can be reduced to the Farrell Fibered Theorem [1], the calculation of $\pi_i(G/PL)$ (Kervaire and Milnor [3]), and Wall's non-simply connected surgery theory [7].

There are modern proofs determining the homotopy type of TOP/PL using either the bounded surgery of Ferry and Pedersen [2] or a modification of the definition of the structure set.

Sullivan's determination of the homotopy type of G/PL , which is well-documented (for instance, in Madsen and Milgram [5]) is used to determine the homotopy type of G/TOP and is fundamental to understanding the classification of general topological manifolds.

The 4-fold periodicity of the topological surgery sequence established by Siebenmann [4, p.283] contains a minor error having to do with base points. This is an easily corrected error, and the 4-fold periodicity is true whenever the manifold has a boundary.

The equivalence of the algebraic and topological surgery exact sequence as established by Ranicki [6] was confirmed.

Sullivan's characteristic variety theorem, however it is understood, is not essential for the Kirby-Siebenmann triangulation of manifolds.

The following papers have been commissioned:

- W. Browder, " PL classification of homotopy tori"
- J. Davis, "On the product structure theorem"
- I. Hambleton, " PL classification of homotopy tori"
- M. Kreck, "A proof of Rohlin's theorem"
- E.K. Pedersen, "Determining the homotopy type of TOP/PL using bounded surgery"
- A. Ranicki, "Siebenmann's periodicity theorem"
- M. Weiss, "Identifying the algebraic and geometric surgery sequences"

The Hauptvermutung website <http://www.maths.ed.ac.uk/~aar/haupt> will record further developments.

REFERENCES

- [1] F. T. Farrell, *The obstruction to fibering a manifold over a circle*, Yale Ph.D. thesis (1967), Indiana Univ. Math. J. **21**, 315-346 (1971)
- [2] S. Ferry and E. K. Pedersen, *Epsilon surgery*, in *Novikov conjectures, Index Theorems and Rigidity*, Vol. 2, LMS Lecture Notes **227**, Cambridge, 167-226 (1995)
- [3] M. Kervaire and J. Milnor, *Groups of homotopy spheres*, Ann. of Maths. **77**, 504-537 (1963)
- [4] R. Kirby and L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Maths. Studies **88**, Princeton University Press (1977)

- [5] I. Madsen and J. Milgram, *The classifying spaces for surgery and cobordism of manifolds*. Ann. of Maths. Studies **92**, Princeton University Press (1979)
- [6] A. Ranicki, *Algebraic L-theory and topological manifolds*, Tracts in Mathematics **102**, Cambridge University Press (1992)
- [7] C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press (1970)

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Abstracts

Topological transversality

ARTHUR BARTELS

In this talk I gave Kirby and Siebenmann’s proof of

Theorem 1 (Topological Transversality Theorem). [1, p.85]

Let C and D be closed subsets of a (metrizable) TOP m -manifold M^m , and let U and V be open neighborhoods of C and D respectively. Let ξ^n be a normal n -microbundle to a closed subset X of a space Y .

Suppose $f : M^m \rightarrow Y$ is a continuous map TOP transverse to ξ on U at ν_0 . Suppose $m \neq 4 \neq m - n$, and either $\partial M \subset C$ or $m - 1 \neq 4 \neq m - 1 - n$.

Then there exists a homotopy $f_t : M \rightarrow Y$, $0 \leq t \leq 1$, of $f_0 = f$ fixing a neighborhood of $C \cup (M - V)$ so that f_1 is transverse to ξ on an open neighborhood of $C \cup D$ at a microbundle ν equal ν_0 near C . Furthermore, if Y is a metric space with metric d , and $\epsilon : M \rightarrow (0, \infty)$ is continuous, then we can require that $d(f_t(x), f(x)) < \epsilon(x)$ for all $x \in M$ and all $t \in (0, 1]$.

using

Theorem 2 (Local Product Structure Theorem). [1, p.36]

Consider the following data : M^m a TOP manifold ; W an open neighborhood of $M \times 0$ in $M \times \mathbb{R}^s$, $s \geq 1$; Σ a CAT structure on W ; $C \subset M \times 0$ a closed subset such that Σ is a product along \mathbb{R}^s near C ; D another closed subset of $M \times 0$; $V \subset W$ an open neighborhood of $D - C$.

Suppose that $m \geq 6$ or $m \geq 5$ and $\partial M \subset C$. Then there exists a concordance $rel (W - V) \cup C$ from Σ to a CAT structure Σ' on W so that Σ' is a product along \mathbb{R}^s near D .

REFERENCES

- [1] Kirby, Robion C. and Siebenmann, Laurence C., *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press (1977).

Microbundles

ALLEGRA E. BERLINER AND STACY L. HOEHN

Given a topological manifold M , we can define what we mean by the “tangent bundle” t_M of M , even if M does not have a smooth structure. To do this, we use microbundles, which were developed by Milnor in [3] and [4].

A topological n -microbundle \mathfrak{X} is a diagram of topological spaces and continuous maps $B \xrightarrow{i} E \xrightarrow{j} B$ with $ji = id_B$. This diagram must satisfy a local triviality condition; namely, for each $b \in B$, there must exist open neighborhoods U of b in

B and V of $i(b)$ in E , with $i(U) \subset V$ and $j(V) \subset U$, as well as a homeomorphism $h : V \rightarrow U \times \mathbb{R}^n$, that make the following diagram commute:

$$\begin{array}{ccc}
 V & \xrightarrow{j|_V} & U \\
 \uparrow i|_U & \searrow h & \uparrow p_1 \\
 U & \xrightarrow{\times 0} & U \times \mathbb{R}^n
 \end{array}$$

PL microbundles are defined analogously to the topological case, using the category of polyhedra and PL maps instead of the category of topological spaces and continuous maps. Since only the behavior of a microbundle near its zero section $i(B)$ matters, microbundles that agree in a neighborhood of $i(B)$ are identified. Thus, the fibres of microbundles are only “germs” of topological spaces.

To every vector bundle ζ with zero section i and projection j , there is a naturally associated microbundle $|\zeta|$. Less trivially, to every topological manifold M there is an associated microbundle \mathfrak{t}_M , called the tangent microbundle of M . This microbundle is given by the diagram $M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$, where Δ is the diagonal map and p_1 is projection onto the first factor. It is shown in [3] that if M is a smooth paracompact manifold with tangent vector bundle τ , $|\tau| \cong \mathfrak{t}_M$.

There is a space $BTOP$ that classifies topological microbundles over an ENR , i.e. there is a one-to-one correspondence between stable isomorphism classes of topological microbundles over an ENR X and homotopy classes of maps from X to $BTOP$. The analogous classifying space for PL microbundles is BPL ; note that there is a forgetful map from BPL to $BTOP$. Given a topological manifold M , let $\widehat{\mathfrak{t}}_M : M \rightarrow BTOP$ denote the map that classifies \mathfrak{t}_M . The aim of this talk was to show that a lift of $\widehat{\mathfrak{t}}_M$ to a map from M to BPL determines a PL structure on $M \times \mathbb{R}^q$ for some $q \geq 0$ and that, conversely, a PL structure on $M \times \mathbb{R}^q$ determines a lift of $\widehat{\mathfrak{t}}_M$ to a map from M to BPL . The proof of this correspondence is given in Essay IV of [1].

We should note that Kister [2] showed that every microbundle over an ENR admits a fibre bundle which is unique up to isomorphism. Therefore, we could have worked with fibre bundles instead of microbundles. However, it is convenient to work with microbundles directly instead of their associated fibre bundles since, for example, the tangent microbundle is canonically defined while its associated fibre bundle is only defined up to isomorphism.

REFERENCES

- [1] R. Kirby and L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Annals of Math. Studies 88, Princeton University Press (1977).
- [2] J. Kister, *Microbundles are fibre bundles*, Ann. of Math. (2) **80** (1964), 190–199.
- [3] J. Milnor, *Microbundles: Part I*, Topology **3**, Suppl. 1 (1964), 53–80.
- [4] J. Milnor, *Topological manifolds and smooth manifolds*, Proceedings of the International Congress of Mathematicians, Stockholm (1962), 132–138.
- [5] C. Rourke and B. Sanderson, *An embedding without a normal microbundle*, Invent. Math. **3** (1967), 293–299.

Topology in the 1960's: Reminiscences and commentary

WILLIAM BROWDER

By 1954 Pontryagin and Thom had established the inter relation of the theory of smooth manifolds and homotopy theory, via the notion of transversality. Pontryagin saw this relation as a tool for calculation in homotopy theory but it proved of limited use, Thom's calculation of the bordism ring on the other hand opened a fruitful new direction for applying algebraic topology in the study of smooth manifolds.

The discovery of the different smooth structures on S^7 by Milnor in 1957, followed by his introduction of the technique he called 'surgery', revealed a new world of study in the possible classification of smooth manifolds. For the spheres, this program was essentially finished by the paper of Kervaire and Milnor 'Groups of homotopy spheres', which by its rearrangement of the standard phrase 'homotopy groups of spheres' signaled that information was now flowing in a different direction, from algebraic topology to smooth manifold theory.

Smale's h -cobordism theorem and higher dimensional Poincaré Conjecture solidified the connection to manifold topology, and also highlighted the fact that higher dimensional topology is easier than low dimensional. The earlier sphere immersion theorem of Smale followed by the globalization to arbitrary smooth manifolds by Hirsch, opened another route from homotopy to smooth topology, which culminated in the famous ' h -principle' of Gromov.

What is today called PL topology (which had dominated research in geometric topology in the 30's), enjoyed a renaissance in the 50's and 60's, with the embedding paper of Penrose-Whitehead-Zeeman, followed by the unknotting papers of Zeeman, and the PL Poincaré conjecture and engulfing papers of Stallings.

Haefliger had continued the line of Whitney on smooth embedding theory, which was used by Smale.

I began my work in algebraic topology, studying H -spaces and their homology. In one of my early papers 'Torsion in H -spaces' I had proved a Poincaré duality theorem for finite dimensional H -spaces.

At Berkeley in the summer of 1961, I attended a lecture series by Kervaire on his work with Milnor on surgery.

H -space theory was pointed toward the question of comparing finite dimensional H -spaces with Lie groups. On my return to Cornell that Fall I thought that an interesting compromise might be to prove them homotopy equivalent to manifolds.

I set to work to try to adapt the methods of surgery to this problem. This turned out to be rather successful, and the theorem I proved described the homotopy type of smooth 1-connected manifolds of dimension greater than 4, in many cases (except for the notorious Kervaire invariant problem in dimensions $4k + 2$, then called the Arf invariant).

At the same time, S.P. Novikov in Moscow, alerted to the work of Kervaire and Milnor, saw the possibilities of this new technique to classify smooth 1-connected manifolds, and produced his work on surgery on 1-connected manifolds.

In the summer of 1962 I lectured on my theorem in Bonn and Aarhus ('Homotopy type of smooth manifolds', Aarhus Topology Symposium 1962) and then I went to the Stockholm ICM.

Though Novikov was not allowed to attend the Stockholm Congress, he sent a short communication and one of the attending Russians asked me to read it for him, which I did.

During an excursion to Russia to Moscow after the congress, I met Novikov for the first time and we became good friends. In the following years we exchanged reprints and preprints and in the spring of 1967, after much bureaucratic maneuvering, we succeeded in inviting to Novikov to visit Princeton for a few months. More on that later.

My first year at Princeton in 1964, I gave a course on surgery theory and was pleased with a good turnout of students, among them a number who became my thesis students (George Cooke, Norman Levitt, Santiago Lopez de Medrano, Dennis Sullivan and Jack Wagoner).

After my stay at the Institute for Advanced Study in 1963-4, I spent the spring and summer at the Topology Symposium organized by Chris Zeeman in Cambridge. There Moe Hirsch and I discovered we could use smoothing theory of PL manifolds to extend surgery theory to PL manifolds, with the help of Milnor's PL microbundle theory.

Milnor had recently disproved the Hauptvermutung for complexes, leaving open the possibility for manifolds.

When word of Novikov's proof of the topological invariance of the rational Pontryagin classes reached Princeton, and Milnor gave lectures giving the proof using Siebenmann's end theorem, instead of the more ad hoc and much more involved argument of Novikov, the possibility of proving a Hauptvermutung using PL surgery became visible.

The theses of two of my students converged to similar theorems on the Hauptvermutung for 1-connected PL manifolds with some strong restrictions on homology. They came from substantially different points of view, Sullivan from the point of view of analyzing maps into G/PL (or F/PL as it was called then) and Wagoner using surgery in a cell by cell approach in a handlebody decomposition.

Sullivan as a junior faculty member at Princeton in 1966-7, gave a seminar on geometric topology, and issued a set of notes based on it. By the spring of 1967, he had analyzed completely the homotopy type of G/PL , (introducing the method of localization of spaces), having first done the 2-primary case the previous year, and in 1966 the odd primary case (where BO is the answer).

The concept of localization for abelian groups was classical and its application to homotopy groups was introduced by Serre in his thesis and deepened in his later papers and the thesis of John Moore. Sullivan suddenly shifted the focus and localized the whole space!

This new viewpoint was an enormous step forward, and has become (together with his notion of completion) a standard feature of modern homotopy theory.

The construction of localization at a homology theory has been carried out and extended by many authors (see Bousfield for a highly advanced version). The p local homology version is easy to carry out for simply connected spaces using the Postnikov system.

Sullivan's analysis of 2-local G/PL was simple and direct, using the representability of 2-local homology by manifolds, and showed (except for the one nonzero k -invariant in dimension 4), it was a product of Eilenberg-MacLane spaces. The odd primary case is more complicated, using the relation of bordism and K -theory a la Conner-Floyd. This approach is exposed in satisfying detail in the treatise of Madsen-Milgram.

At the heart of Sullivan's attack on the Hauptvermutung, beyond the special situations dealt with in his and Wagoner's theses, is the so called 'Characteristic variety theorem' which asserts that for any PL manifold M there exists a "characteristic variety" $V \rightarrow M$ such that the map of M into G/PL , (corresponding to a homotopy equivalence to another M'), is null homotopic if and only if the composition $V \rightarrow G/PL$ splits, i.e. defines a surgery problem over V which is solvable. Examples of such are the collection of lower dimensional projective spaces for the complex or quaternionic projective spaces, and the splitting would be assured by Novikov's theorem for homeomorphisms or his general technique of proof (Novikov's 'torus trick').

I do not know of an adequate account of the Characteristic Variety Theorem in the published literature.

In Princeton in the spring of 1967, Novikov, set about learning this recent work from Sullivan. He discovered a correction which needed to be made in the statement of the Hauptvermutung theorem dealing with the conditions on the 4th cohomology.

Novikov's 'torus trick': Suppose we have a PL (or smooth) manifold W homeomorphic to $M \times \mathbb{R}^n$, with M a PL (or smooth) closed manifold. We wish to show that W is PL homeomorphic (or diffeomorphic) to some $N \times \mathbb{R}^n$, where N is a PL (or smooth) closed manifold. (This implies the topological invariance of the rational Pontryagin classes but is considerably stronger).

Consider $M \times T^{n-1}$ contained in $M \times \mathbb{R}^n$, where T^{n-1} is the product of $n - 1$ circles, and try to find a PL (or smooth) codimension 1 closed U in W which is homotopy equivalent to $M \times T^{n-1}$, (which is a codimension 1 surgery problem). Taking a cyclic covering of U we get U homotopy equivalent to $M \times T^{n-2}$, etc. thereby unravelling U one dimension at a time to get our desired N .

Milnor in his lectures used Siebenmann's thesis, which generalized to the non simply connected case the theorem of Browder-Levine-Livesay. This made Novikov's proof much more accessible and transparent.

In 1969 Rob Kirby gave a lecture at the Institute for Advanced Study on his theorem that if all homotopy tori were PL homeomorphic, then the Annulus Conjecture would be true. The Annulus Conjecture states that if we have an embedded $S^{n-1} \times [0, 1]$ in \mathbb{R}^n , then it is isotopic to the standard embedding, all in the topological category, (equivalent to the statement that any homeomorphism of \mathbb{R}^n is

stable in the sense of Brown and Gluck). While his exact hypothesis is untrue, his method, based on his torus trick proved inordinately powerful and was in the end successful.

Kirby's torus trick: Let $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a homeomorphism. Immerse the punctured torus $T_0 = T^n - \{\text{point}\}$ into \mathbb{R}^n . Then h pulls back the standard PL (or smooth) structure to a new one on T_0 . The end of T_0 with this structure is still PL equivalent to $S^{n-1} \times \mathbb{R}$, (compare my paper 'Structures on $M \times \mathbb{R}$ '), and thus T_0 compactifies to a PL manifold, homotopy equivalent to the torus T^n . If this new manifold is PL equivalent to T^n , and since PL equivalences are stable, there is a little commutative diagram that shows h is stable.

The problem of the PL classification of homotopy tori was solved by Hsiang-Shaneson and Wall, and as Siebenmann had pointed out, one could change the original immersion in Kirby's argument at will, and in particular compose with finite covers. While there are many different PL homotopy tori, after taking a 2^n fold cover they are all the same. This established the annulus conjecture for $n > 4$ (Kirby).

The application of these ideas to the general problem of existence and uniqueness of PL structures was carried out by Kirby and Siebenmann in a number of papers, as well as their 1977 Annals Study, (see also Lashof-Rothenberg for a different approach).

Novikov has stated in a recent article that Kirby and Siebenmann's work is not fully proved in the literature, because it relies on earlier work of Sullivan which is not complete in the literature.

There is no such dependence.

The proof of the Annulus Conjecture depends on a very well understood argument in surgery theory, and the fact that the torus T^n splits after suspension into the wedge product of spheres. It does not depend on any results of Sullivan, not even his calculation of G/PL , but on earlier work of Kervaire-Milnor calculating the homotopy groups, nor does it use Novikov's topological invariance of Pontryagin classes. It depends strongly on Wall's non simply connected surgery theory, Farrell's fibering theorem (generalizing Browder-Levine), and surgery on PL manifolds. The calculation of the relevant surgery obstruction group was carried out by Shaneson and by Wall.

Though some earlier versions of the triangulation theory used more surgery arguments, Kirby-Siebenmann on page 139 of their book say that the only use they make of surgery theory is in the proof of the Annulus Conjecture. One can confidently say that there are no surgery gaps in this work.

Novikov is therefore also mistaken in his assertion that the Lipschitz structure argument to give an alternate proof of the topological invariance of Pontryagin classes is circular because it uses the Annulus Conjecture. The proof of the latter is independent of Novikov's theorem, as we discussed above.

Siebenmann’s periodicity mistake

DIARMUID J. CROWLEY

In the essay “Periodicity and Topological Surgery” [KS1][pp.277- 283], Siebenmann investigates the 4-fold periodicity in the topological surgery exact sequence and concludes with:

Theorem 1 (Theorem C5). *For any compact TOP manifold X^m , $m \geq 5$, the Sullivan-Wall long exact structure sequence is a long exact sequence of abelian groups, and it is canonically isomorphic to the one for $I^4 \times X^m$. In particular $\mathcal{S}^{TOP}(X) \cong \mathcal{S}^{TOP}(I^4 \times X)$.*

Unfortunately, Siebenmann made a minor mistake involving base points so that the above statement is incorrect when the boundary of X is empty. A correct formulation of topological periodicity is

Theorem 2 (Theorem C5'). *For any compact TOP manifold X^m , $m \geq 5$, the Sullivan-Wall long exact structure sequence is a long exact sequence of abelian groups. There is a canonical sequence of split injective homomorphisms to the sequence for $I^4 \times X$ which are isomorphisms if X has non-empty boundary. If X has no boundary, the homomorphisms fail to be isomorphisms at the normal invariant set where the cokernel is \mathbb{Z} , and possibly at the structure set where the cokernel is \mathbb{Z} or 0 . In particular there is an exact sequence of abelian groups $0 \rightarrow \mathcal{S}^{TOP}(X) \rightarrow \mathcal{S}^{TOP}(I^4 \times X) \rightarrow \mathbb{Z}$.*

Siebenmann’s base point error is in no way deep. Once one is aware of it, one sees that Siebenmann’s proof may be easily corrected to give a correct proof of Theorem C5'. The base point mistake appears in the diagrams in Theorem C4 and in sections §6, §7, §8, §9, and §11. For example, in §6, Siebenmann claims that there is an equivalence $\theta_{0,p} : G/TOP \cong L_{4p}(0)$. However, G/TOP is a connected space whereas $\pi_0(L_{4p}(0)) = \mathbb{Z}$. To see the problem in the statement of Theorem C4, suppose that the boundary of X is empty and follow the (correct) convention that in this case $X_{rel} \partial$ is X with a disjoint base point added, X_+ . Now

$$[I^4 \times X_{rel} \partial, G/TOP] = [X_+, \Omega_4(G/TOP)] = \mathbb{Z} \times [X_+, G/TOP] \neq [X_+, G/TOP]$$

but Theorem C4 claims equality here. In all cases the diagrams given are indeed commutative, but not all the maps are equivalences as claimed. However, when a map fails to be an equivalence, it is always a question of having an equivalence from a space to the base point component of a multi-component space. Thus Siebenmann’s proof can be repaired simply by adding notation for the base point component of a space in the appropriate places.

We conclude with some remarks about the literature which followed and related topics.

1). Siebenmann said of his own proof that it was “... a typical application of F. Quinn’s semi-simplicial formulation of Wall’s surgery” [KS1][p. 277]. A correct statement of topological periodicity appeared in [N] where a detailed account of

Quinn's surgery spaces was also given. There is now also an elegant proof of topological periodicity available using algebraic surgery (see [R] in this report).

2). Using surgery spaces to prove periodicity, means that the periodicity of the structure set itself is rather mysterious: if the boundary of X is non-empty, it is shown that the surgery spaces $\mathcal{S}^{TOP}(X)$ and $\mathcal{S}^{TOP}(I^4 \times X)$ are the homotopy fibers of homotopy equivalent homotopy fibre sequences and so have the same homotopy groups. In particular

$$\mathcal{S}^{TOP}(X) = \pi_0(\mathcal{S}^{TOP}(X)) \cong \pi_0(\mathcal{S}^{TOP}(I^4 \times X)) = \mathcal{S}^{TOP}(I^4 \times X).$$

However, one does not obtain an explicit construction of this map. This issue was addressed by Cappell and Weinberger [CW] where a geometric construction of a map $\mathcal{S}^{TOP}(X) \rightarrow \mathcal{S}^{TOP}(I^4 \times X)$ is given using PL tools, stating that "experts should be able to ... make these ideas (although not details) work topologically". Later Hutt [H] extended the Cappell-Weinberger construction to the topological category and identified the "Cappell-Weinberger map" with Siebenmann's map. However, [CW] is very brief and, prima facie, [H] relies on other unpublished work of the author. It is to be hoped that the work of [CW] and [H] can be expanded upon in the future.

3). One of the features of topological surgery is that the surgery exact sequence is a sequence of groups and homomorphisms. This remains true in the PL category (since G/PL can be given and infinite loop space structure compatible with the infinite loop space structure on G/TOP identified by Siebenmann). One may ask whether this could also be true for the smooth surgery exact sequence (see [N][p.83] for precisely this question). However, it has long been known that there are smooth manifolds which are homotopy equivalent but for which the actions of the L -group are different. *It follows that the smooth surgery exact sequence is not in general a sequence of groups and homomorphisms.* For example, the action of $L_8(0)$ on $S^3 \times S^4$ has an orbit with 28 distinct smooth structures. However, if one takes M to be the total space of an S^3 -bundle over S^4 which is fibre homotopy equivalent to $S^3 \times S^4$ and with first Pontryagin class $\pm 24 \in H^4(M) \cong \mathbb{Z}$, then the action of $L_8(0)$ on M is trivial.

4). There are explicit maps

$$\times \mathbb{C}P^2 : \mathcal{S}^{TOP}(X) \rightarrow \mathcal{S}^{TOP}(X \times \mathbb{C}P^2), \quad E : \mathcal{S}^{TOP}(I^4 \times X \text{ rel } \partial) \rightarrow \mathcal{S}^{TOP}(X \times \mathbb{C}P^2)$$

where the latter map is given by extending by a homeomorphism. One can see that the images of $\times \mathbb{C}P^2$ and E are different but perhaps a procedure involving structures on $X \times \mathbb{C}P^2$ could be used to compare the images and give a simpler more explicit proof of periodicity.

REFERENCES

[CW] S. Cappell and S. Weinberger, *A geometric interpretation of Siebenmann's periodicity phenomenon*, Geometry and topology (Athens, Ga., 1985), 47-52, Lecture Notes in Pure and Appl. Math., 105, Dekker, New York, (1987).
 [H] S. Hutt, *On Siebenmann periodicity*, Pacific J. Math. **183**, no. 2, (1998)291-303.
 [KS1] R. C. Kirby. and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Maths. Studies, **88**, Princeton University Press (1977).
 [N] A. J. Nicas, *Induction theorems for groups of homotopy manifold structures*, Mem. Amer. Math. Soc. **39** no. 267 (1982).
 [R] A. A. Ranicki, *The manifold Hauptvermutung and Siebenmann periodicity from the algebraic surgery point of view*, in this report.

The Product Structure Theorem

JAMES F. DAVIS

1. PRODUCT STRUCTURE THEOREM

Notation: Greek letters: Γ, Σ for PL manifolds.

Roman letters: e.g. M, N for topological manifolds.

Definition 3. Two PL -structures Γ_0, Γ_1 on M are *concordant* if there is a PL -structure Σ on $M \times I$ which restricts to Γ_i on $M \times i$. Write $\Gamma_0 \sim \Gamma_1$.

Definition 4. $\tau_{PL}(M) =$ concordance classes of PL -structures on M .

Assumption: M is a topological manifold of dimension greater than 4.

Product Structure Theorem. • *Existence:* Let Σ be a PL -structure on $M \times \mathbb{R}$. Then there is a PL -structure Γ on M so that $\Sigma \sim \Gamma \times \mathbb{R}$.

• *Uniqueness:* If Γ_0 and Γ_1 are PL -structures on M , and if $\Gamma_0 \times \mathbb{R} \sim \Gamma_1 \times \mathbb{R}$, then $\Gamma_0 \sim \Gamma_1$.

Corollary 5. $\tau_{PL}(M) \rightarrow \tau_{PL}(M \times \mathbb{R})$ is a bijection.

2. INGREDIENTS

- (i) Stable homeomorphism theorem ([1],[3])
 [Requires PL -surgery and the computation of $L_*(\mathbb{Z}^n)$]
- (ii) Concordance implies isotopy ([2])
 [Requires the s -cobordism theorem and $Wh(\mathbb{Z}^n) = 0$]

Stable Homeomorphism Theorem. $\pi_0(TOP(n)) = \mathbb{Z}_2$, i.e. every orientation-preserving homeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is isotopic to identity.

Concordance Implies Isotopy. Let Σ be a PL -structure on $M \times I$ and Γ_0 its restriction to $M = M \times 0$. Then there is an isotopy $h_t : M \times I \rightarrow M \times I$ with $h_0 = Id$ and $h_1 : \Gamma_0 \times I \rightarrow \Sigma$ a PL -isomorphism. (Furthermore h_t is close to the identity.)

Corollary 6. $\Gamma_0 \sim \Gamma_1$ implies $\Gamma_0 \cong_{PL} \Gamma_1$.

Isotopy Implies Concordance. : Given two PL -structures Γ_0 and Γ_1 on M , and an isotopy $h_t : M \rightarrow M$ with $h_0 = Id$ and $h_1 : \Gamma_0 \cong \Gamma_1$, then $h^*(\Gamma_0 \times I)$ is a concordance $\Gamma_0 \sim \Gamma_1$ where $h(m, t) = (h_t(m), t)$.

3. PST \Rightarrow CLASSIFICATION THEOREM

Classification Theorem.

- *Existence:* M admits a PL -structure if and only if $\tau_M : M \rightarrow BTOP$ lifts to BPL .
- *Uniqueness:* If M admits a PL -structure, $\tau_{PL}(M) = [M, TOP/PL]$, i.e. homotopy classes of lifts of τ_M to BPL .

Proof. (Existence) \Rightarrow : Microbundles, etc.

\Leftarrow : Embed $M \hookrightarrow \mathbb{R}^N$, N large. Then

$$M \times \mathbb{R}^N = \tau_M \oplus \nu_M$$

and

$$\begin{array}{ccc} \tau_M \oplus \nu_M & \longrightarrow & \tau_M \\ \downarrow & & \downarrow \\ \nu_M & \xrightarrow{\pi} & M \end{array}$$

So $M \times \mathbb{R}^N = \pi^*\tau_M$ is a PL -bundle (since τ_M lifts) over a PL -manifold ν_M (an open set in \mathbb{R}^N), so $M \times \mathbb{R}^N$ is PL . By PST M is PL .

(Uniqueness) For M a PL -manifold, I indicate the inverse to the obvious map $\tau_{PL}(M) \rightarrow [M, G/TOP]$. A map $M \rightarrow TOP/PL$ gives a f.p. homeomorphism

$$\begin{array}{ccc} & & M \times \mathbb{R}^N \\ & & \downarrow \\ E & \xrightarrow{h} & M \end{array}$$

where $E \rightarrow M$ is a PL -bundle over a PL -space. By PST, we have a PL -structure on M . □

The Product Structure Theorem is a special case of the Classification Theorem.

4. TWO CONSEQUENCES OF THE CLASSIFICATION THEOREM

Proposition 7. [2, p.301] *Any compact topological manifold has the homotopy type of a finite simplicial complex.*

Proof. Embed the manifold M^n in \mathbb{R}^{n+k} and let E be the normal D^k -bundle (Hirsch, Annals 1966). Then E is a parallelizable topological manifold, hence by the classification theorem admits a PL -structure, hence is a finite simplicial complex. □

Proposition 8. [2, p.301] *Every compact topological manifold has a preferred simple homotopy type, i.e. given a homeomorphism $h : M \rightarrow M'$ of compact manifolds and two normal disk bundles E_M and $E_{M'}$ triangulated as above, the map*

$$E_M \xrightarrow{\pi} M \xrightarrow{h} M' \xrightarrow{i} E_{M'}$$

is a simple homotopy equivalence.

This is a consequence of uniqueness theorems for normal disk bundles.

5. PROOF OF PST ASSUMING CII

Step 1: State relative versions of PST and CII

Step 2: Uniqueness follows from a relative form of existence.

Step 3: By chart-by-chart induction, reduce to $M = \mathbb{R}^n$.

We next deal with the PST for $M = \mathbb{R}^n$, $n > 4$. In fact we prove

Theorem 9. $\tau_{PL}(\mathbb{R}^n) = *$ for $n > 4$.

Lemma 10. *Any two PL-structures on \mathbb{R}^n are PL-isomorphic.*

Proof. Let Γ be a PL-structure on \mathbb{R}^n . Use Browder's End Theorem to complete it to a PL-structure Σ on the disk. The complement of the interior of a simplex in \mathbb{R}^n is an PL h-cobordism, coning off on the disk given a PL-isomorphism to the standard D^n . □

Proof of the PST for $M = \mathbb{R}^n$. Let Γ be a PL-structure on \mathbb{R}^n . Then, by the lemma, there is a homeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ so that $\Gamma = h^* \text{std}$. By the stable homeomorphism theorem, $\pi_0(TOP(n)) = \mathbb{Z}/2$, so h is topologically isotopy to i , the identity, or r , reflection through a hyperplane. Thus $h^* \text{std}$ is isotopic to $i^* \text{std} = \text{std}$ or $h^* \text{std}$. Apply isotopy implies concordance. □

REFERENCES

- [1] Kirby, Robion C., *Stable homeomorphisms and the annulus conjecture*, Ann. of Maths. **89**, 575–582 (1969)
- [2] Kirby, Robion C. and Siebenmann, Laurence C., *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Maths. Studies **88**, Princeton University Press (1977).
- [3] Quinn, Frank, *Ends of maps. III. Dimensions 4 and 5*, J. Differential Geom. **17**, 503–521 (1982)

The classification of homotopy tori

IAN HAMBLETON

A key ingredient in the work of Kirby and Siebenmann was the classification of PL -structures on the n -torus T^n , or more generally on $T^n \times D^k$, relative to the standard structure on $\partial(T^n \times D^k) = T^n \times S^{k-1}$. The results needed were proved using the surgery exact sequence of Browder, Novikov, Sullivan and Wall [2]. Let $\mathcal{S}^{PL}(T^n \times D^k, T^n \times S^{k-1})$ denote the equivalence classes of PL -structures on $T^n \times S^{k-1}$, relative to the boundary. The following result was proved by C. T. C. Wall [2, 15A], and at the same time by W.-C. Hsiang and J. Shaneson [1].

Theorem 1. *There is a bijection*

$$\mathcal{S}^{PL}(T^n \times D^k, T^n \times S^{k-1}) \cong H^{3-k}(T^n; \mathbf{Z}/2),$$

for $n + k \geq 5$, which is natural under finite coverings.

Remarks:

1. All fake tori are parallelisable and smoothable.
2. For each fake structure there is a finite covering (of degree $\leq 2^n$) which is standard.
3. After the work of Kirby and Siebenmann it was shown that

$$\mathcal{S}^{TOP}(T^n \times D^k, T^n \times S^{k-1}) = 0$$

for $n + k \geq 5$, so all the fake PL -structures are homeomorphic to the standard structure.

In the talk I described a method for factoring the PL -surgery obstruction map through an “assembly” map (based on the work of Quinn and Ranicki). The computation of the surgery exact sequence and the determination of the structure set follows from a purely algebraic calculation of the assembly map, taking advantage of the Shaneson-Wall codimension 1 splitting theorem [2, 12B].

REFERENCES

- [1] W.-C. Hsiang and J. L. Shaneson, *Fake tori*, Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), Markham, Chicago, Ill., 1970, pp. 18–51.
- [2] C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, 1970.

The homotopy type of G/PL and the characteristic variety theorem

QAYUM KHAN AND TIBOR MACKO

To determine the structure set of a manifold via the surgery exact sequence one needs to understand the normal invariants. The theme of the talk was to describe two approaches to this, both due to Sullivan, a homotopy theoretic and a geometric approach.

1. Homotopy theory (TM). For a *CAT* manifold X , where $CAT = DIFF$, PL , or TOP , transversality yields isomorphisms

$$\mathcal{N}_\partial^{CAT}(X \times D^k) \cong [X \times D^k, \partial(X \times D^k); G/CAT, *],$$

where G/CAT is the homotopy fiber of the map $BCAT \rightarrow BG$. To determine this group one needs to understand the homotopy type of G/CAT . The following result for $CAT = PL$ was first obtained by Sullivan in the unpublished notes [Su]. A published proof appears in the book of Madsen and Milgram [MM, chapter 4]. Localization of spaces is used in the statement:

Theorem A ([Su],[MM]). *There are compatible homotopy equivalences*

$$\begin{aligned} G/PL_{(2)} &\simeq F \times \prod_{i \geq 2} K(\mathbb{Z}_{(2)}, 4i) \times K(\mathbb{Z}_2, 4i - 2), \\ G/PL_{(odd)} &\simeq BO_{(odd)}, \\ G/PL_{(0)} &\simeq BO_{(0)} \simeq \prod_{i \geq 1} K(\mathbb{Q}, 4i), \end{aligned}$$

where F is a 2-stage Postnikov system $K(\mathbb{Z}_{(2)}, 4) \times_{\beta Sq^2} K(\mathbb{Z}_2, 2)$.

We gave a sketch of the proof. The ingredients are:

- The surgery exact sequence for $X = D^k$ shows that the homotopy groups of the spaces on the left hand side and the right hand side in Theorem A are isomorphic. It remains to find the maps realizing the isomorphisms.
- These maps are constructed as representatives of the cohomology classes which arise from functionals on the homology or the real K -theory of G/PL using suitable universal coefficient theorems.
- Thom and Conner-Floyd provide a relation of the oriented bordism spectra to the Eilenberg-MacLane spectra when localized at 2 and to the real K -theory when localized at odd primes ([Th], [CF]).
- Finally, one uses the surgery obstruction map

$$\sigma: \mathbf{MSO}_*(G/PL) \rightarrow L_*(\mathbb{Z}[\pi_1(X)])$$

and the results of Thom and Conner-Floyd to obtain the functionals. At odd primes a surgery product formula is also used.

2. Geometry (QK). The geometric approach to computing normal invariants goes under the name of the *characteristic variety theorem*. It states that elements of the group of normal invariants $\mathcal{N}^{PL}(X)$ are detected by a collection of simply-connected surgery obstructions, over \mathbb{F}_2 and over \mathbb{Z} modulo various r , of restrictions to various singular \mathbb{Z}_r -submanifolds. We do not address the question of realizability of obstructions.

Definition ([Su]). Let $r \geq 0$. A \mathbb{Z}_r -manifold is a pair $(Y, \delta Y)$ consisting of a compact, connected, oriented manifold Y and an orientation-preserving identification $\partial Y \cong \bigsqcup_r \delta Y$. We denote \overline{Y} as the quotient of Y by the r -to-1 map from ∂Y to δY . The pair $(Y, \delta Y)$ possesses an *orientation class* $[Y] \in H_{\dim(Y)}(\overline{Y}; \mathbb{Z}_r)$. A *singular \mathbb{Z}_r -submanifold in a topological space X* is a pair $((Y, \delta Y), \beta)$ consisting of a \mathbb{Z}_r -manifold $(Y, \delta Y)$ and a continuous map $\beta: \overline{Y} \rightarrow X$. There is also a notion of

cobordism of n -dimensional singular \mathbb{Z}_r -manifolds in X , with the cobordism group denoted by $\Omega_n(X, \mathbb{Z}_r)$.

Dennis Sullivan asserted a less precise version of the following theorem in his unpublished notes [Su, Thm. II.4']. Also, the proof there is not rigorous. We clarify the statement:

Theorem B ([Su]). *Let X be a closed PL-manifold of dimension $n > 0$. Then there exist $N \geq 0$, odd $M > 0$, and an injective function*

$$\begin{aligned} \Sigma : \mathcal{N}^{PL}(X) &\longrightarrow \prod_{0 < 4i-2 \leq n} \text{Hom}(\Omega_{4i-2}(X, \mathbb{Z}_2), \mathbb{Z}_2) \\ &\times \prod_{0 < 4i \leq n} \text{Hom}(\Omega_{4i}(X, \mathbb{Z}), \mathbb{Z}_{(2)}) \times \prod_{0 < 4i \leq n} \text{Hom}(\Omega_{4i}(X, \mathbb{Z}_{2^N}), \mathbb{Z}_{2^N}) \\ &\times \prod_{0 \leq j} \text{Hom}(\Omega_{4j}(X, \mathbb{Z}), \mathbb{Z}_{(odd)}) \times \prod_{0 \leq j} \text{Hom}(\Omega_{4j}(X, \mathbb{Z}_M), \mathbb{Z}_M) \end{aligned}$$

given by the surgery obstructions σ_ of the restrictions to singular \mathbb{Z}_r -submanifolds in X :*

$$(f : M \rightarrow X, \xi) \longmapsto (((Y, \delta Y), \beta) \mapsto \sigma_*(\beta^* f \rightarrow \bar{Y}, \nu_\beta \oplus \beta^* \xi) \pmod r).$$

The second part of the talk outlined Sullivan's sketch of proof. Using the homotopy type of G/PL stated in Theorem A, Sullivan constructs another injective function Σ' with the same domain and codomain. The rigorous proof of the statement that $\Sigma = \Sigma'$ (hence that Σ is injective) would require the characteristic classes formulae for surgery obstructions [TW], as well as the addressing of certain issues arising from the different shape of the universal coefficient theorem for real K -theory. The latter seems to be the most controversial part of the argument.

REFERENCES

- [CF] P.E.Conner and E.E.Floyd, *The Relation of Cobordism to K-theories*, LNM **28**, Springer-Verlag, New York, (1966).
- [MM] I.Madsen and R.J.Milgram, *The Classifying Spaces for Surgery and Cobordism of Manifolds*, Annals of Mathematics Studies **92**, Princeton University Press, Princeton, NJ, (1979).
- [Su] D.P.Sullivan, *Triangulating and smoothing homotopy equivalences and homeomorphisms*, Geometric Topology Seminar Notes (1967), reprinted in The Hauptvermutung Book, *K*-Monographs in Mathematics **1**, Kluwer, (1996), 69–103.
- [TW] L.Taylor and B.Williams, *Surgery spaces: formulae and structure*, LNM **741**, Springer, Berlin, (1979), 170–195.
- [Th] R.Thom, *Quelques propriétés globales des variétés différentiables*, Comm. Math. Helv. **28**, (1954), 17–86.

Milnor's counter-example to the Hauptvermutung

ANDREW KORZENIEWSKI

The (non-manifold) Hauptvermutung states that if two simplicial complexes X_1 and X_2 are homeomorphic as topological spaces, then after suitable subdivision they are isomorphic as simplicial complexes. In 1961 Milnor [1] shows that this is not the case by constructing a counter-example. We outline his construction here.

Let L_1 and L_2 be lens spaces which are homotopy equivalent but not homeomorphic. We take the product $L_1 \times \sigma^n$ and cone off the boundary $L_1 \times \partial\sigma^n$ to form a new complex X_1 ; we form the complex X_2 from L_2 by the same process. By a Theorem of Mazur the complexes X_1 and X_2 are homeomorphic. The spaces L_1 and L_2 are distinguished by means of their Reidemeister torsion which depends only on the simplicial structures of these lens spaces. Milnor constructs a similar torsion invariant for the spaces X_1 and X_2 which distinguishes their simplicial structures, hence proving that the spaces X_1 and X_2 are a counter-example to the Hauptvermutung.

REFERENCES

- [1] J. Milnor, *Two complexes which are homeomorphic but combinatorially distinct*, Ann. of Math (2) **74** (1961), 575–590.

A Proof of Rohlin's Theorem and the Computation of the Low Dimensional Spin Bordism Groups

MATTHIAS KRECK

Theorem 1. (*Rohlin 1952*) *The signature of a closed 4-dimensional smooth Spin manifold M is divisible by 16.*

This result is proven by Rohlin in the last paper of a series of four papers. On the way he gives elementary proofs of the result that oriented bordism groups are trivial in dimension 1, 2, and 3 and \mathbb{Z} in dimension 4. These proofs are rather short. Kirby [1] gave detailed elementary proofs of these results, and Teichner [2] applied this to give a new beautiful proof of Rohlin's theorem. In this note I sketch Teichner's proof.

The main input is a generalized Atiyah-Hirzebruch spectral sequence called the James spectral sequence. We only need the following special case. Let X be a CW-complex and $w : X \rightarrow K(\mathbb{Z}/2, 2)$ a fibration (a class in $H^2(X; \mathbb{Z}/2)$). Let $w_2 : BSO \rightarrow K(\mathbb{Z}/2, 2)$ the classifying map of the second Stiefel-Whitney class. We pull the fibration $w : X \rightarrow K(\mathbb{Z}/2, 2)$ and w_2 back to obtain a fibration denoted $X(w)$ over BSO .

We denote the normal bordism group of this fibration by $\Omega_n(X(w))$. The James spectral sequence computes this group. It's E_2 term is $H_i(X; \Omega_j^{Spin})$.

Now we consider the special case, where $X = K(\mathbb{Z}/2, 2)$ and w is the identity. Then $X(w)$ is nothing but the trivial fibration BSO over BSO and so

$$\Omega_n(K(\mathbb{Z}/2, 2), id) = \Omega_n,$$

the oriented bordism group. The James spectral sequence computes this well known group in terms of the more complicated $Spin$ bordism groups. We want to argue backwards and compute Ω_n^{Spin} through Ω_n in small dimension. We list the homology groups of $K(\mathbb{Z}/2, 2)$:

$$\begin{aligned} H_2(K(\mathbb{Z}/2, 2); \mathbb{Z}) &= \mathbb{Z}/2, & H_3(K(\mathbb{Z}/2, 2); \mathbb{Z}) &= \{0\}, \\ H_4(K(\mathbb{Z}/2, 2); \mathbb{Z}) &= \mathbb{Z}/4, & H_5(K(\mathbb{Z}/2, 2); \mathbb{Z}) &= \mathbb{Z}/2. \end{aligned}$$

Using this and the vanishing of Ω_n for $1 \leq n \leq 3$ one immediately sees from the James spectral sequence that

$$\Omega_1^{Spin} \cong \mathbb{Z}/2, \quad \Omega_2^{Spin} \cong \mathbb{Z}/2.$$

The case of Ω_3^{Spin} is not obvious since there is an unknown differential. But we use the fact that $\Omega_3^{Spin} = \{0\}$. For example this follows from Rohlin's elementary computation of framed bordism in dimension 3, which implies that all 3-manifolds are framed bordant to a framing on S^3 . This implies that all 3-dimensional $Spin$ -manifolds are bordant to S^3 with some $Spin$ -structure. But there is a unique $Spin$ -structure on S^3 .

Now we look at the line in the James spectral sequence computing Ω_4 . All d_2 -differentials on this line are given by reduction mod 2 (if necessary) composed by the dual of $x \mapsto Sq^2x + wx$. One easily checks that they are all zero. The case of $H_5(X; \mathbb{Z}) \rightarrow H_3(X; \mathbb{Z}/2)$ is a bit more complicated. But the 5-manifold $SU(3)/SO(3)$ realizes the non-trivial element in $H_5(X; \mathbb{Z}) = \mathbb{Z}/2$ implying that all differentials starting from $H_5(X; \mathbb{Z}/2)$ are trivial.

Thus, the entries in the E_∞ -term on this line are $\Omega_4^{Spin}, \mathbb{Z}/2, \mathbb{Z}/2$ and $\mathbb{Z}/4$. The spectral sequence gives an exact sequence

$$0 \rightarrow \Omega_4^{Spin} \rightarrow \Omega_4 \rightarrow C \rightarrow 0,$$

where C has order 16. Now we use that $\Omega_4 \cong \mathbb{Z}$ and obtain an exact sequence

$$C \rightarrow \Omega_4^{Spin} \rightarrow \Omega_4 \rightarrow \mathbb{Z}/16 \rightarrow 0.$$

This implies Rohlin's theorem. Our arguments furthermore imply:

Theorem 2. $\Omega_1^{Spin} \cong \mathbb{Z}/2, \Omega_2^{Spin} \cong \mathbb{Z}/2$ and $\Omega_4^{Spin} \cong \mathbb{Z}$.

REFERENCES

- [1] Kirby, Robion C. *The topology of 4-manifolds*. Lecture Notes in Mathematics, 1374. Springer-Verlag, Berlin, 1989. vi+108 pp.
- [2] Teichner, Peter. *On the signature of four-manifolds with universal covering spin*. Math. Ann. 295 (1993), no. 4, 745–759

TOP/PL using bounded surgery

ERIK KJÆR PEDERSEN

The talk describes how TOP/PL is determined by Ferry and Pedersen, using bounded surgery theory. Consider the Browder, Novikov, Sullivan, Wall surgery exact sequence for the PL case:

$$\begin{aligned} \rightarrow [\Sigma(M/\partial M), G/PL]_* \rightarrow L_{n+1}(\mathbb{Z}\pi) \rightarrow \\ \mathcal{S}_{PL}^h(M \text{ rel } \partial) \rightarrow [M/\partial M, G/PL]_* \rightarrow L_n(\mathbb{Z}\pi) \end{aligned}$$

Here M is a given n -dimensional PL -manifold. This is an exact sequence when $n \geq 5$. We note however that all the terms are defined also for $n \leq 4$, and the only map that is not defined is the action of the L -group on the structure set, which is only partially defined. In the case of M an n -dimensional disk we get a map $\pi_n(G/PL) \rightarrow L_n(\mathbb{Z})$ which is an isomorphism for $n \geq 5$ by the Poincaré conjecture. For $n = 2$ it is the Arf invariant map, and since the Arf invariant can be realized by a normal map $T^2 \rightarrow S^2$ it is an isomorphism. For $n = 4$ the map is given by the index divided by 8, and Rohlin’s theorem implies the map is multiplication by 2.

Crossing with \mathbb{R}^k maps this sequence into the bounded surgery exact sequence parameterized by \mathbb{R}^k , and if we make sure that $n + k \geq 5$ this sequence is exact. The map crossing with \mathbb{R}^k is shown to be an isomorphism of L -groups by Ranicki, and on the normal invariant term it is obviously an isomorphism since \mathbb{R}^k is contractible. We can thus compute the maps in the bounded surgery exact sequence, and it follows that the bounded structure set for $D^n \times \mathbb{R}^k$ has one element for $n \neq 3$ and $n + k \geq 5$, and two elements when $n = 3$ and $n + k \geq 5$. It is easy to construct a map from $\pi_n(\widetilde{TOP}(k), \widetilde{PL}(k))$ to this \mathbb{R}^k -bounded structure set, and using a trick blowing up the metric of D^n near infinity of \mathbb{R}^k this map is seen to be a monomorphism. In doing this, the usual application of local contractibility of the space of homeomorphism is replaced by an Alexander isotopy, and the usual classification of tori is replaced by the classification of \mathbb{R}^k up to \mathbb{R}^k -bounded homotopy equivalence. To see that $\pi_3(\widetilde{TOP}(k), \widetilde{PL}(k))$ is nontrivial for $n + k \geq 6$ we need to use Quinn’s end theorem in the simply connected case. Notice we lose one dimension here since we can not use Quinn’s end theorem to produce a 4-dimensional PL end.

Handlebody decompositions of high-dimensional *TOP* manifolds

ULRICH PENNIG

A handlebody decomposition of a *CAT* manifold W on a *CAT* submanifold $M \subset W$ ($CAT \in \{DIFF, PL, TOP\}$) is a filtration of W the form

$$M = M_0 \subset M_1 \subset M_2 \subset \cdots \subset \bigcup_i M_i = W ,$$

such that each submanifold M_i is obtained from M_{i-1} by attaching a handle, i.e. for $H_i = M_i \setminus M_{i-1}$ the following *CAT* isomorphism holds:

$$(H_i, H_i \cap M_{i-1}) \cong (D^k, S^{k-1}) \times D^{m-k} \quad \text{for some } 0 \leq k \leq m.$$

In the category of differentiable manifolds the existence of such decompositions is shown using the arguments of Morse theory. The case of *PL* handlebodies is based on the second derived of a triangulation of (W, M) , in which the handles are easily identified (see [2]). However, in the case of high-dimensional (i.e. $\dim(W) \geq 6$) *TOP* manifolds, the existence proof for handlebody decompositions requires more elaborate theorems, like a "local version" of the *product structure theorem* (LPST) and the *concordance implies isotopy theorem* (CII). It was published in the book of Kirby and Siebenmann in 1977 (see [1]) and was presented in this talk. Its basic idea is to cover W by open sets that carry *PL* structures and contain a closed covering of W . To locally reduce the *TOP* to the *PL* case of the handlebody theorem the *PL* structures on the overlaps of the open regions have to be taken care of. This is where an LPST argument enters, which requires a careful treatment in the case of 6-dimensional manifolds with boundary. An application emphasizing the importance of handlebody decompositions is given by a result of M.Cohen and Sanderson: A compact *TOP* manifold that is a handlebody is homeomorphic to the mapping cylinder $\text{Map}(f)$ of a map $f : \partial W \rightarrow X$ into a finite *CW* complex X .

REFERENCES

- [1] Kirby, Robion C. and Siebenmann, Laurence C., *Foundational essays on topological manifolds, smoothings and triangulations*, Princeton University Press (1977).
- [2] Rourke, C. P. and Sanderson, B. J., *Introduction to piecewise-linear topology*, Springer-Verlag (1972).

Periodicity and the Hauptvermutung

FRANK QUINN

A "relaxed" version of the structure set $\widehat{\mathcal{S}}(M)$ was described for which the surgery exact sequence is exact in all dimensions. Homology equivalence over $\mathbb{Z}[\pi_1 M]$ is allowed to get exactness in dimension 3, and the Poincaré homology sphere P represents a nontrivial element in $\mathcal{S}^{PL}(S^3)$. Carefully controlled connected sums with $S^2 \times S^2$ are allowed in dimension 4. The Poincaré conjecture

in dimensions $\neq 3$ (stable in dimension 4) then gives a quick demonstration that $G/PL \rightarrow \mathbb{L}$ has fiber $K(\mathbb{Z}/2, 3)$ over the basepoint component.

The result that $G/TOP \rightarrow \mathbb{L}$ is an equivalence to the basepoint component comes from topological transversality, the high-dimensional topological Poincaré conjecture (the 3-dimensional Poincaré conjecture is actually irrelevant here), and Freedman’s result that the Poincaré sphere bounds a contractible topological manifold. Freedman’s work can be avoided using the “double suspension” result that $(\text{cone}(P)) \times \mathbb{R}$ is a topological manifold (there is a direct proof of this case).

The Siebenmann periodicity $\mathcal{S}^{TOP}(M) \rightarrow \mathcal{S}^{TOP}(M \times D^4)$ is a formal consequence of periodicity in \mathbb{L} , the equivalence of G/TOP with the basepoint component of \mathbb{L} , and a space formulation of surgery. The periodicity map is a bijection if M has boundary, and an injection with cokernel at most \mathbb{Z} if M is closed. The cokernel comes from the fact that G/TOP is the *basepoint* component of \mathbb{L} , and $\pi_0\mathbb{L} \simeq \mathbb{Z}$.

Use of the relaxed structure set also provides an explicit description of PL homotopy tori. A decomposition $T^n = T^3 \times T^{n-3}$ gives a homology equivalence

$$(T^3 \# P) \times T^{n-3} \rightarrow T^n$$

where as above P denotes the Poincaré homology sphere. These clearly realize the appropriate PL normal invariants. These are canonically h -cobordant (by a “plus” construction) to the required homotopy equivalences.

A sketch was also given of a much more direct approach to the Hauptvermutung using controlled topology. This uses “Ends of maps I”, which appeared in the *Annals* just two years after the publication of the Kirby-Siebenmann book.

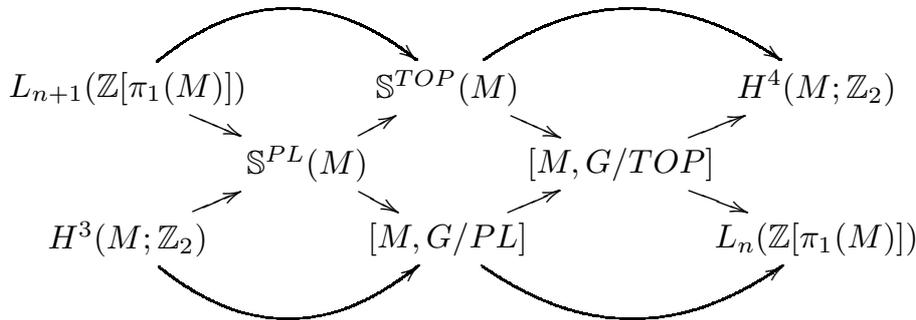
The manifold Hauptvermutung and the Siebenmann periodicity from the algebraic surgery point of view.

ANDREW RANICKI

The manifold Hauptvermutung is the conjecture that every homeomorphism $h : L \rightarrow M$ of compact n -dimensional PL manifolds is homotopic to a PL homeomorphism – see [3] for the background. This conjecture has been known to be false ever since the 1969 work of Kirby and Siebenmann [1], with counterexamples in every dimension $n \geq 5$. The original counterexamples were constructed geometrically. Counterexamples may also be constructed using algebra and the realization theorem of the Wall non-simply-connected surgery obstruction theory, as follows.

For $n \geq 5$ the structure sets $\mathcal{S}^{PL}(M)$, $\mathcal{S}^{TOP}(M)$ of a closed n -dimensional PL manifold M are abelian groups which fit into a commutative braid of exact

sequences



with

$$H^3(M; \mathbb{Z}_2) \rightarrow \mathbb{S}^{PL}(M) ; x \mapsto s(g : L \rightarrow M)$$

sending $x \in H^3(M; \mathbb{Z}_2) = [M, TOP/PL]$ to the structure invariant $s(g) \in \mathbb{S}^{PL}(M)$ of a homotopy equivalence $g : L \rightarrow M$ of PL manifolds such that

$$\begin{aligned} \nu_M - (g^{-1})^* \nu_L &= [x] \\ &\in \text{im}([M, TOP/PL] \rightarrow [M, BPL]) = \ker([M, BPL] \rightarrow [M, BTOP]) \end{aligned}$$

and g is homotopic to a homeomorphism h .

Let $M = T^n$, so that $\pi_1(M) = \mathbb{Z}^n$, $\mathbb{S}^{TOP}(M) = 0$. The morphism

$$\begin{aligned} L_{n+1}(\mathbb{Z}[\mathbb{Z}^n]) &= \bigoplus_{k=0}^n \binom{n}{k} L_{k+1}(\mathbb{Z}) \rightarrow \mathbb{S}^{PL}(M) \\ &= H^3(M; \mathbb{Z}_2) = \binom{n}{3} L_4(\mathbb{Z}) / 2L_4(\mathbb{Z}) \end{aligned}$$

is just the projection, with $E_8 = 1 \in L_4(\mathbb{Z}) = \mathbb{Z}$ mapping to the generator of $L_4(\mathbb{Z}) / 2L_4(\mathbb{Z}) = \mathbb{Z}_2$. For any $x \neq 0 \in H^3(M; \mathbb{Z}_2)$ there exists

$$y \neq 0 \in \binom{n}{3} L_4(\mathbb{Z}) \subset L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$$

with image $x \in \mathbb{S}^{PL}(M) = H^3(M; \mathbb{Z}_2)$. Realize $y \in L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$ as the rel ∂ surgery obstruction of a normal map in the PL category

$$(f; g, 1) : (K^{n+1}; L^n, M^n) \rightarrow M^n \times (I; \{0\}, \{1\})$$

such that $g : L \rightarrow M$ is a homotopy equivalence with structure invariant

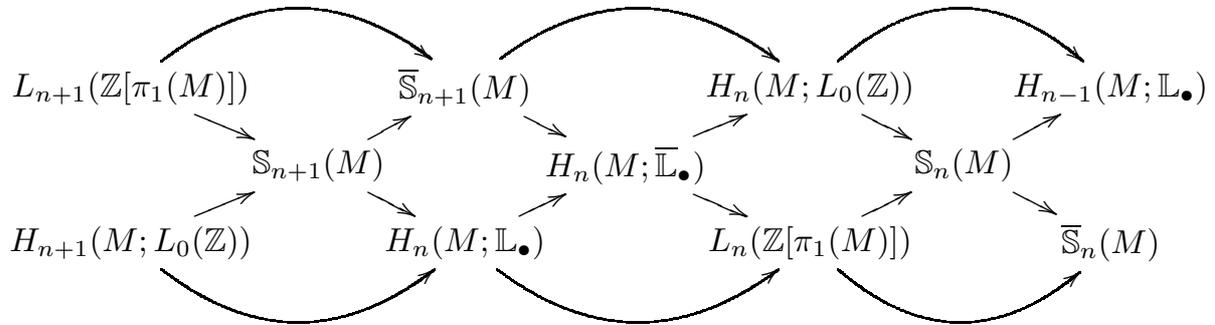
$$s(g) = x \neq 0 \in \mathbb{S}^{PL}(M) = H^3(M; \mathbb{Z}_2),$$

and g is homotopic to a homeomorphism h . For any $T^3 \subset M$ such that $\langle x, [T^3] \rangle = 1 \in \mathbb{Z}_2$ it is possible to choose f PL transverse at $T^3 \times I \subset M \times I$. The 4-dimensional normal map in the PL category

$$(f; g, 1)| : (W^4; \tau^3, T^3) = (f; g, 1)^{-1}(T^3 \times (I; \{0\}, \{1\})) \rightarrow T^3 \times (I; \{0\}, \{1\})$$

has the quadratic form E_8 as simply-connected kernel, and $g| : \tau^3 = T^3 \# \Sigma^3 \rightarrow T^3$ is a homology equivalence, with Σ^3 the Poincaré homology 3-sphere. Now g is not homotopic to a PL homeomorphism by Rohlin's theorem, so that the homeomorphism h is a counterexample to the manifold Hauptvermutung.

The (corrected) Siebenmann periodicity theorem $\mathbb{S}_\partial^{TOP}(M) \cong \mathbb{S}_\partial^{TOP}(M \times D^4)$ ([1, C.5,p.283]) for an n -dimensional topological manifold M with nonempty boundary ∂M (and as ever $n \geq 5$) has the following algebraic surgery interpretation. For any space M there is defined a commutative braid of exact sequences



with L_\bullet (resp. \bar{L}_\bullet) the 1- (resp. 0-) connective L -spectrum of \mathbb{Z} (Ranicki [2, 25.1]). In particular, if M is an n -dimensional CW complex then

$$\mathbb{S}_{n+k}(M) = \bar{\mathbb{S}}_{n+k}(M) \quad (k \geq 2)$$

and there is defined an exact sequence

$$0 \rightarrow \mathbb{S}_{n+1}(M) \rightarrow \bar{\mathbb{S}}_{n+1}(M) \rightarrow H_n(M; L_0(\mathbb{Z})) \rightarrow \mathbb{S}_n(M) \rightarrow \bar{\mathbb{S}}_n(M) \rightarrow \dots$$

Let now M be a connected n -dimensional manifold with (possibly empty) boundary and $n \geq 5$. By [2, 18.5,25.4] there is defined an isomorphism between the 1-connective algebraic surgery exact sequence and the topological surgery exact sequence for the rel ∂ structure set

$$\begin{array}{ccccccc} \dots & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \longrightarrow & \mathbb{S}_\partial^{TOP}(M) & \longrightarrow & [M, G/TOP] \longrightarrow L_n(\mathbb{Z}[\pi_1(M)]) \\ & & \parallel & & \downarrow \cong & & \downarrow \cong \\ \dots & \longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \longrightarrow & \mathbb{S}_{n+1}(M) & \longrightarrow & H_n(M; L_\bullet) \longrightarrow L_n(\mathbb{Z}[\pi_1(M)]) \end{array}$$

with

$$\mathbb{S}_\partial^{TOP}(M \times D^k) = \begin{cases} \mathbb{S}_{n+1}(M) & \text{if } k = 0 \\ \mathbb{S}_{n+k+1}(M) = \bar{\mathbb{S}}_{n+k+1}(M) & \text{if } k \geq 1, \end{cases}$$

with an exact sequence

$$0 \rightarrow \mathbb{S}_\partial^{TOP}(M) \rightarrow \mathbb{S}_\partial^{TOP}(M \times D^4) \rightarrow H_n(M; L_0(\mathbb{Z})) \rightarrow \mathbb{S}_n(M) \rightarrow \bar{\mathbb{S}}_n(M) .$$

If M is closed then $H_n(M; L_0(\mathbb{Z})) = L_0(\mathbb{Z}) = \mathbb{Z}$. For example, the case $M = S^n$

$$0 \rightarrow \mathbb{S}^{TOP}(S^n) = 0 \rightarrow \mathbb{S}_\partial^{TOP}(S^n \times D^4) \rightarrow L_0(\mathbb{Z}) \rightarrow \mathbb{S}_n(S^n) \rightarrow \bar{\mathbb{S}}_n(S^n) = 0$$

gives the canonical counterexample to the Siebenmann periodicity theorem

$$\mathbb{S}^{TOP}(S^n) = 0 \neq \mathbb{S}_\partial^{TOP}(S^n \times D^4) = \bar{\mathbb{S}}_{n+1}(S^n) = L_0(\mathbb{Z}) .$$

On the other hand, the periodicity theorem holds in the case $M = T^n$

$$0 \rightarrow \mathbb{S}^{TOP}(T^n) = 0 \rightarrow \mathbb{S}_\partial^{TOP}(T^n \times D^4) = 0 \rightarrow L_0(\mathbb{Z}) \rightarrow \mathbb{S}_n(T^n) \rightarrow \bar{\mathbb{S}}_n(T^n) = 0 ,$$

with

$$\mathbb{S}^{TOP}(T^n) = \mathbb{S}_\partial^{TOP}(T^n \times D^k) = 0 \quad (k \geq 0).$$

If M has non-empty boundary then $H_n(M; L_0(\mathbb{Z})) = 0$ and the periodicity theorem holds, with

$$\mathbb{S}_\partial^{TOP}(M) = \mathbb{S}_\partial^{TOP}(M \times D^4).$$

REFERENCES

- [1] R. Kirby and L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. Math. Stud. **88**, Princeton University Press (1977)
- [2] A. Ranicki, *Algebraic L-theory and topological manifolds*, Cambridge University Press (1992)
- [3] A. Ranicki (ed), *The Hauptvermutung Book*, K-Monograph in Mathematics 1, Kluwer (1996)

Identifying the algebraic and topological surgery exact sequences

MICHAEL WEISS

For a closed m -dimensional topological manifold M^m , the *geometric surgery fibration sequence* has the form

$$\mathcal{S}(M) \rightarrow \mathcal{N}(M) \rightarrow \mathbb{L}_m(M)$$

where $\mathcal{S}(M)$ is the block structure space, $\mathcal{N}(M)$ is the space of degree 1 normal maps to M and $\mathbb{L}(M)$ is the appropriate L -theory space, so $\pi_k \mathbb{L}_m(M) = L_{k+m}(\pi_1 M)$ for $k \geq 0$. These spaces are geometric realizations of simplicial sets (often without degeneracy operators). In particular $\mathbb{L}_m(M)$ is defined following Ranicki, so that the 0-simplices are certain chain complexes with m -dimensional nondegenerate quadratic structure.

One of the main points was to prove that this sequence is indeed a homotopy fibration sequence if $m \geq 5$ (using ideas related to the concept of a *Kan fibration*). The *algebraic surgery fibration sequence* has the form

$$\text{hofiber}(\alpha) \longrightarrow \mathbb{H}_m(X; \mathbb{L}_\bullet(\star)) \xrightarrow{\alpha} \mathbb{L}_m(X)$$

where X is a simplicial complex or Δ -set. The spaces in it are geometric realizations of simplicial sets without degeneracy operators. The details are as in Ranicki's blue book, *Algebraic L-theory and topological manifolds*.

Using a homotopy equivalence $M \rightarrow X$ (where M^m is a closed manifold and X is a simplicial complex) and working with Poincaré's *dual cell decomposition* of X , and making full use of topological transversality, one can produce a map from the geometric surgery sequence of M to the algebraic surgery sequence of X . It was the second main point of the talk to show that this is "almost" an equivalence for $m \geq 5$, except for a well-known small deviation (caused by the fact that the comparison map from G/TOP to $\mathbb{L}_0(\star)$ is not a homotopy equivalence but a 0-connected Postnikov cover). The proof can be given by induction on the number of handles in a topological handlebody decomposition of M . Of course, this strategy makes it necessary to generalize the statement from closed manifolds to compact manifolds with boundary.

A user's guide to the algebraic theory of surgery

MASAYUKI YAMASAKI

I gave a brief introduction to the algebraic theory of surgery. The talk was based on Andrew Ranicki's blue book [2].

First I described the 4-periodic \mathbb{L} -spectrum and discussed that the 4-periodicity comes from the double skew-suspension. Next I mentioned the assembly map and the 4-periodic algebraic surgery exact sequence. Then I discussed about the q -connected versions of these. Finally I quoted a theorem due to Aravinda, Farrell and Roushon [1] which says that the assembly map $H_i(E(K); \mathbb{L}_\bullet(\mathbb{Z})) \rightarrow L_i(\pi_1(E(K)))$ is an isomorphism for every $i \in \mathbb{Z}$, where $E(K)$ denotes the exterior of a non-split link K , and showed how the authors used algebraic theory of surgery to prove this.

REFERENCES

- [1] C. S. Aravinda, F. T. Farrell and S. K. Roushon, *Surgery groups of knot and link complements*, Bull. London Math. Soc. **29** (1997), 400 – 406.
- [2] A. A. Ranicki, *Algebraic L-theory and topological manifolds*, Tracts in Math. **102** (Cambridge Univ. Press, Cambridge, 1992).

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