Mathematisches Forschungsinstitut Oberwolfach

Report No. 43/2006

Topologie

Organised by
Cameron Gordon (Austin)
Wolfgang Lück (Münster)
Bob Oliver (Paris)

September 17th – September 23rd, 2006

Abstract. The participants in this conference covered all areas of algebraic and geometric topology. The talks covered a wide range of recent developments, such as the Farrell-Jones conjecture, knot theory, geometric group theory, and stable and unstable homotopy theory.

Mathematics Subject Classification (2000): 19xx, 55xx, 57xx.

Introduction by the Organisers

This conference was the fifth in the series of topology conferences organized by Gordon, Lück, and Oliver, and the last in which Lück will be an organizer. This meeting, which currently takes place every second year, is one of the few regularly occurring conferences anywhere which allows researchers from a wide range of areas of topology to meet.

There were about 50 participants in this meeting, including researchers in many different areas of algebraic and geometric topology. This conference was partly funded by the European Commission, which made it possible to invite and support many more young participants — thesis students as well as recent postdocs — than is usually the case.

There were a total of 19 talks at the conference, covering areas such as 3-manifolds and knot theory, geometric group theory, algebraic K- and L-theory, and homotopy theory. Hence it is difficult to separate out themes which covered more than two or three talks. The following is a brief summary of some of the highlights.

Marc Lackenby's talk was about the "folk" conjecture in knot theory that crossing number is additive under connected sum. Clearly $c(K_1 \# K_2) \leq c(K_1) + c(K_2)$; what one wants is an inequality in the other direction. Applying normal surface theory to a suitable handle decomposition of the complement of $K_1 \# K_2$ derived from a minimal crossing diagram, Lackenby shows that for some explicit universal positive constant A, the inequality $c(K_1 \# K_2) \geq A \cdot (c(K_1) + c(K_2))$ holds.

Nathalie Wahl talked about her ongoing joint work with Allen Hatcher on the stability of the homology of the mapping class group of certain 3-manifolds. Namely, she looked at those with n summands of type $S^2 \times S^1$ and s punctures, stabilizing with respect to increasing n. The result is that the i-th homology stabilizes when $n \geq 2i + 2$, which improves considerably the previous stability range.

Also on the subject of 3-manifolds was the talk by Walter Neumann, about his joint work with Jason Behrstock in which they give the quasi-isometry classification of the fundamental groups of graph-manifolds. The result is that for closed non-geometric graph-manifolds there is only one quasi-isometry class, whereas in the bounded case the classification corresponds to the classification of the dual graphs up to so-called bisimilarity (a concept which, interestingly, arises in computer science).

The talks by Thomas Schick and Bernhard Hanke dealt with manifolds with positive scalar curvature. Hanke described conditions under which a closed manifold with fixed point free S^1-action can be shown to have a Riemannian metric with positive scalar curvature. Schick discussed some connections between the nonequivariant problem (existence of positive scalar curvature metrics without a group action) and the Novikov conjecture.

In a different direction, Jesper Grodal and Carles Broto described recent progress on p-completed classifying spaces and related topics. Grodal described work which shows that, for a finite group G and a prime p dividing $|G|$, the fundamental group of the geometric realization of the linking category $\mathcal{L}_p^c(G)$ is in many interesting cases isomorphic to G again. The point here is that the category $\mathcal{L}_p^c(G)$ depends only on the p-completed classifying space $BG_p^\wedge$, and thus that the group G can be "recovered" from this p-completed space in certain favorable cases. Broto described a new class of spaces, classifying spaces of "p-local compact groups," which includes p-completed classifying spaces of compact Lie groups and p-compact groups, for which the spaces have many of the nice homotopy theoretic properties of p-completed classifying spaces of compact Lie groups. Also, in a talk on a related topic, Natalia Castellana described recent work on connected covers of finite H-spaces.

Geometric group theory was represented by the talks of Karen Vogtmann and Mike Davis. Vogtmann talked about joint work with Jim Conant about certain classes defined by Morita in the unstable rational homology of the outer automorphism group of a free group. In particular Conant and Vogtmann reinterpret and generalize these Morita classes, associating a class with every odd-valent graph. Davis talked about his joint work with Dymara, Januszkiewicz and Okun, giving a description of the cohomology module $H^*(W; \mathbb{Z}W)$ of a Coxeter group W with coefficients in the group ring $\mathbb{Z}W$.

Arthur Bartels, in his talk, described the recent proof of the K-theoretic Farrell-Jones conjecture with arbitrary coefficients for subgroups of finite products of word-hyperbolic groups. (Notice that such groups can be very wild.) This has many consequences. It implies the Bass Conjecture, the Kaplansky Conjecture and Moody's induction conjecture for such groups. If G is such a group and torsionfree this says that the Whitehead group of G and the projective class group of $\mathbb{Z}G$ both vanish. Another consequence is that the K-theoretic Farrell-Jones Conjecture with coefficients is true for the examples of groups for which its non-commutative companion, the Baum-Connes Conjecture with coefficients, is known to be false by a result of Higson, Lafforgue, and Skandalis.

Workshop: Topologie

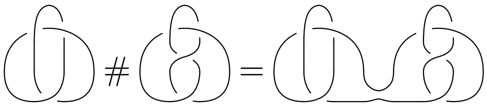
Table of Contents

Abstracts

The crossing number of composite knots

Marc Lackenby

Possibly the simplest invariant of a knot K is its crossing number $c(K)$, which is defined to be the minimal number of crossings in any diagram for K. One of the basic constructions in knot theory is the connected sum of two (oriented) knots K_1 and K_2, denoted $K_1\#K_2$, which is shown below:



The following is a major unsolved problem that is apparently over 100 years old.

Conjecture. *For all (oriented) knots K_1 and K_2*

$$c(K_1\#K_2) = c(K_1) + c(K_2).$$

In my talk, I outlined a proof of a "quasi"-form of this conjecture:

Theorem (L)**.** *For all (oriented) knots K_1 and K_2*

$$\frac{c(K_1)+c(K_2)}{213} \leq c(K_1\#K_2) \leq c(K_1)+c(K_2).$$

This is proved using normal surface theory, which is a classical tool in 3-manifold theory.

Mutliplicative structure in equivariant cohomology

Kathryn Hess

1. Introduction

Let C_*X denote the cubical, integral chain complex of a space X, which admits a natural coassociative and counital comultiplication δ_X, given by the composite chain map

$$C_*X \xrightarrow{C_*\Delta} C_*(X\times X) \xrightarrow{\mathrm{AW}} C_*X\otimes C_*X,$$

where Δ is the usual diagonal map and AW is the natural Alexander-Whitney equivalence. By the Künneth Theorem, if H_*X is torsion free, then δ_X induces a comultiplication $H_*X \to H_*X\otimes H_*X$. In general, δ_X induces a graded commutative multiplication $H^*X\otimes H^*X\to H^*X$, the cup product.

Let E be the total space of a principal G-bundle, where G is a connected topological group. Let Y be any G-space. The multiplication map $\mu : G\times G \to G$

induces the structure of a chain algebra on C_*G, with multiplication map given by the composite

$$C_*G \otimes C_*G \xrightarrow{\mathrm{EZ}} C_*(G \times G) \xrightarrow{C_*\mu} C_*G,$$

where EZ is the natural Eilenberg-Zilber equivalence. The action maps $E \times G \to E$ and $G \times Y \to Y$ similarly induce C_*G-module structures on C_*E and on C_*Y.

Theorem 1.1 (Moore [3]). *There is an isomorphism of graded $\mathbb{Z}$-modules*

$$H_*(E \underset{G}{\times} Y) \cong \mathrm{Tor}_*^{C_*G}(C_*E, C_*Y).$$

The goal of this talk was to explain how to enrich Moore's theorem, obtaining a comultiplicative isomorphism, by taking into account in a coherent manner the comultiplicative structure on C_*G, C_*E and C_*Y, then to analyze in more detail the special case $G = S^1$ and $E = ES^1$. This talk was based on the article [1].

2. Enriching Moore's Theorem

Remark 2.1. *All definitions in this section can be formulated more compactly and more neatly, if less transparently, in terms of* co-rings over operads [2].

We begin by describing the algebraic framework for the enriched version of Moore's theorem.

Definition 2.2. Let (C, d) be a coassociative, counital chain coalgebra, with comultiplication $\delta : C \to C \otimes C$ and counit $\varepsilon : C \to R$. Let $\overline{C} = \operatorname{coker} \varepsilon$. For any $c \in C$, let $\sum_i c_i \otimes c^i$ denote the image of c under the composite $C \xrightarrow{\delta} C \otimes C \twoheadrightarrow \overline{C} \otimes \overline{C}$.

The *cobar construction* on C is the chain algebra $\big(T(s^{-1}\overline{C}), d_\Omega\big)$, where T denotes the free (tensor) algebra functor, s^{-1} denotes desuspension and the differential d_Ω is the derivation specified by

$$d_\Omega s^{-1}c = -s^{-1}(dc) + \sum_i (-1)^{\deg c_i} s^{-1}c_i \otimes s^{-1}c^i.$$

Note that a chain algebra map $\varphi : \Omega C \to \Omega C'$ gives rise to linear maps

$$\{\varphi_k : C \to (C')^{\otimes k} \mid \deg \varphi_k = k-1, k \geq 1\},$$

where, in particular, $\varphi_1 : C \to C'$ is a chain map, and $\varphi_2 : C \to C' \otimes C'$ is a chain homotopy from $(\varphi_1 \otimes \varphi_1)\delta_C$ to $\delta_{C'}\varphi_1$.

Definition 2.3. Let C and C' be coassociative, counital chain coalgebras. A chain map $f : C \to C'$ is a *DCSH-map* if there is a map of chain algebras $\varphi : \Omega C \to \Omega C'$ such that $\varphi_1 = f$, i.e., f is a map of chain coalgebras *up to strong homotopy.*

The next two definitions describe the compatibility we require between multiplicative and DCSH structure. Let $I_{k,n} = \{\vec{\imath} = (i_1, ..., i_k) \in \mathbb{N}^k \mid \sum_j i_j = n\}$ for any $k, n \in \mathbb{N}$.

Definition 2.4. Let H and H' be chain Hopf algebras. A DCSH-map $f : H \to H'$ with corresponding chain algebra map $\varphi : \Omega H \to \Omega H'$ is *multiplicative* if

$$\varphi_{n+1}(xy) = \sum_{\substack{1\le k\le n+1\\ \vec{\imath}\in I_{k,n+1}}} \big(\delta_{H'}^{(i_1)} \otimes \cdots \otimes \delta_{H'}^{(i_k)}\big)\varphi_k(x) * (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})\delta_H^{(k)}(y)$$

for all $n \ge 0$ and all $x, y \in H$, where $*$ denotes multiplication in $(H')^{\otimes n+1}$.

Let H be a chain Hopf algebra. Recall that a *right H-module coalgebra* is a chain complex M that is both an H-module and a coalgebra, where the H-action map $M \otimes H \to M$ is a map of coalgebras.

Definition 2.5. Let $f : H \to H'$ be a multiplicative DCSH-map, with corresponding chain algebra map $\varphi : \Omega H \to \Omega H'$. Let M be a right H-module coalgebra, and let M' be a right H'-comodule algebra. A DCSH-map $g : M \to M'$ with corresponding chain algebra map $\psi : \Omega M \to \Omega M'$ is a *DCSH-module map* if

$$\psi_{n+1}(xy) = \sum_{\substack{1\le k\le n+1\\ \vec{\imath}\in I_{k,n+1}}} \big(\delta_{M'}^{(i_1)} \otimes \cdots \otimes \delta_{M'}^{(i_k)}\big)\psi_k(x) \bullet (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})\delta_H^{(k)}(y)$$

for all $n \ge 0$ and all $x, y \in H$, where $\bullet$ denotes the action of $(H')^{\otimes n+1}$ on $(M')^{\otimes n+1}$.

The enriched version of Moore's theorem can now be stated as follows.

Theorem 2.6. *Given a multiplicative DCSH-quasi-isomorphism $f : H \xrightarrow{\simeq} C_*G$ and a DCSH-module quasi-isomorphism $g : M \xrightarrow{\simeq} C_*E$ with respect to f, there is a DCSH quasi-isomorphism*

$$M \underset{H}{\otimes} C_*Y \xrightarrow{\simeq} C_*(E \underset{G}{\times} Y).$$

In particular, there is an isomorphism of graded algebras

$$H^*\big((M \underset{H}{\otimes} C_*Y)^\sharp\big) \cong H^*(E \underset{G}{\times} Y),$$

where the superscript $\sharp$ denotes the R-linear dual.

3. Homotopy orbits of circle actions

Identification of a special family of primitives in C_*S^1 is the key to applying Theorem 2.6 to computing S^1-equivariant (co)homology.

Proposition 3.1. *There is a set $\{T_k \in C_{2k+1}S^1 \mid k \ge 0\}$ of primitives such that T_0 represents the generator of H_1S^1 and $dT_k = \sum_{i=0}^{k-1} T_i \cdot T_{k-i-1}$ for all k.*

Recall that $H_*BS^1 \cong R[u_2]$ as graded R-modules.

Theorem 3.2. *There is a quasi-isomorphism of chain Hopf algebras*

$$f : \Omega H_*BS^1 \xrightarrow{\simeq} C_*S^1 : s^{-1}(u^k) \mapsto T_{k-1},$$

extending to a DCSH-module quasi-isomorphism with respect to f

$$g : H_*BS^1 \otimes_{t_\Omega} \Omega H_*BS^1 \xrightarrow{\simeq} C_*ES^1,$$

where $H_*BS^1 \otimes_{t_\Omega} \Omega H_*BS^1$ denotes the acyclic cobar construction on H_*BS^1.

Corollary 3.3. *Let Y be a left S^1-space. There is a DCSH-quasi-isomorphism*

$$(H_*BS^1 \otimes_{t_\Omega} \Omega H_*BS^1) \underset{\Omega H_*BS^1}{\otimes} C_*Y \xrightarrow{\simeq} C_*(ES^1 \underset{S^1}{\times} Y) = C_*Y_{hS^1},$$

which gives rise upon dualization to a commutative diagram of cochain algebras

$$\begin{array}{ccccc} (R[u],0) & \xrightarrow{incl.} & (R[u]\otimes C^*Y, D) & \twoheadrightarrow & C^*Y \\ \simeq\uparrow & & \simeq\uparrow & & \| \\ C^*BS^1 & \longrightarrow & C^*Y_{hS^1} & \longrightarrow & C^*Y. \end{array}$$

Here, for all $y \in C^*Y$, $D(u^n \otimes y) = u^n \otimes y + \sum_{k\geq q0} u^{n+k+1} \otimes \omega_k(y)$, where $\omega_k : C^*Y \to C^{*-2k-1}$ is a derivation such that $[d, \omega_k] = -\sum_{i=0}^{k-1} \omega_i \circ \omega_{k-i-1}$. In particular, $\omega_0 : C^*Y \to C^{*-1}Y$ is a chain map inducing the Δ-operation of the Batalin-Vilkoviskiy structure on H^*Y.

4. Open questions

(1) Do results analogous to Proposition 3.1 and to Theorem 3.2 hold for any compact Lie group?
(2) What are the meaning and content of the higher operations ω_k?

References

[1] K. Hess, *A cubical approach to homotopy orbits of circle actions*, available on the arXiv (math.AT/0604592), 2006
[2] K. Hess, P.-E. Parent and J. Scott, *Co-rings over operads characterize morphisms*, available on the arXiv (math.AT/0505559), 2006
[3] J. C. Moore, *Algèbre homologique et homologie des espaces classifiants*, Séminaire Cartan (1959/60), exposé 7.

Cohomology of Coxeter groups with group ring coefficients

Michael Davis
(joint work with Jan Dymara, Tadeusz Januszkiewicz, Boris Okun)

Suppose (W, S) is a Coxeter system. For $T \subset S$, W_T denotes the subgroup generated by T. T is *spherical* if W_T is finite. $\mathcal{S}$ denotes the set of spherical subsets of S.

Let X be a CW complex and $(X_s)_{s\in S}$ a family of subcomplexes. For each $x \in X$, put $S(x) := \{s \in S \mid x \in X_s\}$. Define $\mathcal{U}(W, X)$ $(= \mathcal{U})$ to be the quotient space $(W \times X)/\sim$, where $\sim$ is the equivalence relation defined by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $wW_{S(x)} = w'W_{S(x)}$. W acts on $\mathcal{U}$ and X is a strict fundamental domain (i.e., $\mathcal{U}/W = X$). A W-action on a space is a *reflection group* if it is equivariantly homeomorphic to $\mathcal{U}(W, X)$ for some X. The action is proper and cocompact if and only if X is a finite complex and $S(x)$ is spherical for each $x \in X$. Henceforth, assume this.

For each $w \in W$, put

$$\text{In}(w) := \{s \in S \mid l(ws) < l(w)\}$$
$$\text{In}'(w) := \{s \in S \mid l(sw) < l(w)\},$$

where $l(\ )$ is word length. It is a basic fact that for any w, both $\text{In}(w)$ and $\text{In}'(w)$ ($= \text{In}(w^{-1})$) are spherical subsets of S. Let $A := \mathbf{Z}W$ be the group ring and $\{e_w\}_{w\in W}$ its standard basis. For each $T \in \mathcal{S}$, define elements a_T and h_T in A by

$$a_T := \sum_{w\in W_T} e_w \quad \text{and} \quad h_T := \sum_{w\in W_T} (-1)^{l(w)} e_w.$$

Let A^T denote the right ideal $a_T A$ and H^T the left ideal Ah_T. (If $T \notin \mathcal{S}$, set $A^T = H^T = 0$.) A_T is the set of finitely supported functions on W which are constant on each right coset in $W_T \backslash W$. Put

$$b'_w := a_{\text{In}'(w)} e_w, \qquad b_w := e_w h_{\text{In}(w)}.$$

Then $\{b'_w \mid \text{In}'(w) \supset T\}$ is a basis for A^T and $\{b_w \mid \text{In}(w) \supset T\}$ is a basis for H^T. So, if we define $\widehat{A}^T := \text{Span}\{b'_w \mid \text{In}(w) = T\}$ and $\widehat{H}^T := \text{Span}\{b_w \mid \text{In}(w) = T\}$, we have direct sum decompositions of abelian groups:

$$A^T = \bigoplus_{U\supset T} \widehat{A}^U \quad \text{and} \quad H^T = \bigoplus_{U\supset T} \widehat{H}^U.$$

Theorem.

$$H_*(\mathcal{U}) \cong \bigoplus_{T\in\mathcal{S}} H_*(X, X^T) \otimes \widehat{H}^T$$
$$H_c^*(\mathcal{U}) \cong \bigoplus_{T\in\mathcal{S}} H_*(X, X^{S-T}) \otimes \widehat{A}^T.$$

The first formula was originally proved in [1], the second in [2]. We give a different proof in [6] by using the identifications of these (co)homology groups with certain equivariant (co)homology groups: $H_*^W(\mathcal{U}; \mathbf{Z}W) = H_*(\mathcal{U})$ and $H_W^*(\mathcal{U}; \mathbf{Z}W) = H_c^*(\mathcal{U})$ and then using a direct sum decomposition of the coefficient system on X induced by $\mathbf{Z}W$. This point of view leads to a computation of the W-module structures on $H_*(\mathcal{U})$ and $H_c^*(\mathcal{U})$ in the following sense. We have a decreasing filtration of right W-modules $A = F_0 \supset \cdots \supset F_p$, where $F_p := \sum_{|T|\geq p} A^T$. This leads to a filtration of cohomology. (Similarly, there is a decreasing filtration of left W-modules for homology.) Put

$$A^{>T} := \sum_{U\supsetneq T} A^U \quad \text{and} \quad H^{>T} := \sum_{U\supsetneq T} H^U.$$

With this terminology, we can state the following result of [6].

Theorem. *In filtration degree p, the associated graded term in homology is the left W-module,*

$$\bigoplus_{|T|=p} H_*(X, X^T) \otimes H^T/H^{>T},$$

while in compactly supported cohomology it is the right W-module,

$$\bigoplus_{|T|=p} H_*(X, X^{S-T}) \otimes A^T/A^{>T}.$$

These formulas were suggested by our work in [5] on weighted L^2-cohomology of Coxeter groups (see [3] for an abstract).

If $\mathcal{U}$ is acyclic, then $H_c^*(\mathcal{U}) = H^*(W; \mathbf{Z}W)$. Moreover, there is a particularly nice choice of a contractible $\mathcal{U}$. We usually denote it Σ and its fundamental chamber K [2, 4, 6]. This leads to a formula for $H^*(W; \mathbf{Z}W)$ with each associated graded term a sum of terms of the form $H^*(K, K^{S-T}) \otimes A^T/A^{>T}$. A consequence is the following.

Corollary. $H^*(W; \mathbf{Z}W)$ *is always finitely generated as a W-module.*

Question. Suppose a group Γ is virtually type *FP*. Is $H^*(\Gamma; \mathbf{Z}\Gamma)$ always a finitely generated Γ-module?

References

[1] M.W. Davis *The homology of a space on which a reflection group acts*, Duke Math. J. **55** (1987), 97–104.
[2] ———, *The cohomology of a Coxeter group with group ring coefficents*, Duke Math. J. **91** (1998), 297–314.
[3] ———, *Weighted L^2-cohomology of Coxeter groups*, Oberwolfach Reports **3**, No. 2 (2006).
[4] ———, *The Geometry and Topology of Coxeter Groups*, London Math. Soc. Monograph Series, Vol. 32, Princeton Univ. Press, Princeton, to appear.
[5] M.W. Davis, J. Dymara, T. Januszkiewicz and B. Okun, *Weighted L^2-cohomology of Coxeter groups*, preprint 2004.
[6] ———, *Cohomology of Coxeter groups with group ring coefficents: II*, Algebraic & Geometric Topology **6** (2006), 1289–1318.

Homological stability and a coloring lemma

Nathalie Wahl
(joint work with Allen Hatcher)

Fix a compact, connected, orientable 3-manifold N and let

$$M_{n,s} := N \# (\#_n S^1 \times S^2) \# (\#_s D^3)$$

be the manifold obtained from N by adding n handles and removing s balls. Let $A_{n,s}$ denote the group of components of the diffeomorphisms of $M_{n,s}$ fixing its boundary pointwise, modulo twists along 2-spheres. We consider two stabilization maps $\alpha,\ \beta : A_{n,s} \to A_{n+1,s}$ induced by gluing a twice punctured torus along one of its boundaries, and by gluing a 4-punctured sphere along two of its boundaries respectively. Our main result is the following:

Theorem. *The induced maps $\alpha_*,\ \beta_* : H_i(A_{n,s}; \mathbb{Z}) \to H_i(A_{n+1,s}; \mathbb{Z})$ are isomorphisms when $n \geq 2i + 2$ and surjective when $n \geq 2i + 1$.*

If $\mu : A_{n,s} \to A_{n,s+1}$ and $\eta : A_{n,s} \to A_{n+1,s-1}$ denote the maps induced by gluing a 3-punctured sphere respectively along one and two boundary components, we have $\alpha = \eta\mu$ and $\beta = \mu\eta$. In particular, the above theorem implies that both μ and η also induce isomorphisms in homology in a range and that the groups $H_i(A_{n,s};\mathbb{Z})$ are independent of n and s when $n \geq 2i+2$. This improves the main theorem of [4] and fills in a gap in that paper. It turns out that the maps α and β are more natural than μ and η for proving stability.

When the manifold N we start with is a sphere S^3, we have $A_{n,1} \cong \mathrm{Aut}(F_n)$ and $A_{n,0} \cong \mathrm{Out}(F_n)$, the automorphism and outer automorphism group of the free group F_n. In this case, we recover a result of [1, 2, 3] with a new simpler proof.

We prove the theorem using the action of $A_{n,s}$ on simplicial complexes whose vertices are pairs (S, a), where S is a 2-sphere embedded in $M_{n,s}$ and a is an arc intersecting S transversally exactly once and with boundary points in one (for α) or two (for β) boundary spheres of $M_{n,s}$. Our main tool is a coloring lemma, which allows to "spread the spheres" over simplices, in the spirit of an h-principle.

Let X be a simplicial complex of dimension d and let E be a set of colors. We say that a coloring of a simplex is *good* if all its vertices have different colors. We need the cardinality of E to be at least $d+1$ for the existence of good colorings of X. The following lemma says that $d+1$ is in some sense enough.

Lemma. *Any coloring of X can be modified to a good coloring after subdividing the interior of the badly colored simplices.*

The above mentioned results can be found in [5].

References

[1] A. Hatcher, K. Vogtmann, *Cerf theory for graphs*, J. London Math. Soc. 58 (1998), 633–655.

[2] A. Hatcher, K. Vogtmann, *Homology stability for outer automorphism groups of free groups*, Algebr. Geom. Topol. 4 (2004), 1253–1272.

[3] A. Hatcher, K. Vogtmann, N. Wahl, *Erratum to: Homology stability for outer automorphism groups of free groups [Algebr. Geom. Topol. 4 (2004), 1253–1272] by A. Hatcher and K. Vogtmann*, Algebr. Geom. Topol. 6 (2006), 573-579.

[4] A. Hatcher, N. Wahl, *Stabilization for the Automorphisms of free groups with boundaries*, Geom. Topol. 9 (2005), 1295–1336.

[5] A. Hatcher, N. Wahl, *Stabilization for the Automorphisms of free groups with boundaries, II*, preprint (2006), math.GT/06080333.

Almost complex 4-manifolds with vanishing first Chern class

Stefan Bauer

Any closed and oriented differentiable 4-manifold X can be equipped with a spin^c-structure or equivalently a stably almost complex structure. The former allows for a complex Dirac operator and thus a K-orientation class inducing Poincaré duality in complex topological K-theory. The latter links 4-manifold theory to complex and symplectic geometry. These structures being equivalent is particular to dimension 4 as the natural map between the respective classifying spaces is a 5-equivalence.

There are characteristic classes naturally associated with these structures: A spin^c-structure comes with a first Chern class associated to its complex determinant line bundle and with a first Pontrjagin class reflecting the signature of the manifold via the formula $p_1(X) = 3 \cdot \mathrm{sign}(X)$. A stably almost complex structure comes with a first and a second Chern class. These characteristic classes are related by

$$c_1(X)^2 - 2 \cdot c_2(X) = p_1(X).$$

The second Chern class describes the Euler characteristic of X exactly in the case of an (unstably) almost complex structure. So in the case of an almost complex 4-manifold, vanishing of the first Chern class implies a relation

$$3 \cdot \mathrm{sign}(X) = -2 \cdot \chi(X)$$

between its signature and its Euler characteristic. As the manifold has to be spin, Rochlin's theorem tells its signature to be divisible by 16.

These two conditions are to some extent also sufficient for the existence of an almost complex 4-manifold with vanishing first Chern class. First note that the Betti numbers b_1, b_2^+ and b_2^- of such a manifold are determined by the signature and, say, the first Betti number. Given a number σ divisible by 16 and an integer $b_1 \geq 0$, then one could pose the question whether there exists a 4-manifold realizing these characteristic numbers and which supports an almost complex structure with vanishing first Chern class.

Using Freedman's classification of simply connected topological 4-manifolds, it is easy to construct such manifolds topologically: Start with a simply connected manifold realizing an even intersection form with the correct Betti numbers $b_2^\pm$ and then take connected sum with a suitable number of $S^3 \times S^1$.

When it comes to differentiable 4-manifolds, the picture is more involved: For $\sigma \leq 0$, one may start with a suitable simply connected elliptic surface with signature σ and then take connected sum with suitable numbers of copies of $S^2 \times S^2$ and $S^3 \times S^1$. However, in case $\sigma > 0$, it is only possible to find such manifolds if b_1 is sufficiently large (using connected sums of $K3$-surfaces with reversed orientation and products of spheres).

For an almost complex 4-manifold with vanishing first Chern class to support a complex structure is a severe restriction: The complex surface has to be minimal and of Kodaira dimension at most zero. The list of examples is rather short and known to be complete. It comprises in particular $K3$-surfaces and tori, but also other examples found by and named after Bombieri, Inoue, Hopf and Kodaira. Amongst these surfaces, only the $K3$-surfaces exhibit nonvanishing signature.

Including closed symplectic 4-manifolds into the consideration, a few more examples of such with vanishing first Chern class become available. However, $K3$-surfaces remain the only known examples with nonvanishing signature. The main result presented in the talk relates this more or less empirical fact to Seiberg-Witten theory.

Theorem. [1] *Let X be a closed, almost complex 4-manifold with vanishing first Chern class. If the dimension $b_2^+(X)$ of a maximal positive definite linear subspace in the second cohomology of X satisfies $b_2^+(X) \geq 4$, then the Seiberg-Witten invariant of X is an even number.*

According to a theorem of Taubes [7], the absolute value of the Seiberg-Witten invariant of a symplectic 4-manifold is 1, as soon as $b_2^+(X) \geq 2$. So this theorem applies, in particular, to compact symplectic 4-manifolds.

Corollary. *A closed, symplectic 4-manifold X with torsion first Chern class satisfies the inequality*

$$b_2^+(X) \leq 3.$$

Indeed, if the first Chern class of X is torsion, then there is a finite covering $\tilde{X}$ with vanishing first Chern class, which of course is symplectic. The induced map $H^2(X;\mathbb{R}) \to H^2(\tilde{X};\mathbb{R})$ is injective.

Corollary. *Let X be a closed symplectic 4-manifold with vanishing first Chern class and nonvanishing signature. Then X is an integral homology $K3$-surface. Moreover, the fundamental group of X has no proper subgroup of finite index.*

Indeed, the conditions $3 \cdot \mathrm{sign}(X) = -2 \cdot \chi(X)$ and $b_2^+ \leq 3$ can only be met by manifolds with signature either 0 or -16. Any covering manifold $\tilde{X}$ associated to a subgroup of finite index n of the fundamental group would be compact symplectic with signature $\mathrm{sign}(\tilde{X}) = n \cdot \mathrm{sign}(X)$ and with $c_1(\tilde{X}) = 0$. The corollary follows immediately.

Note that every finitely presented group can be realized as the fundamental group of a symplectic 4-manifold [4]. This leads to the question, whether there exists a symplectic homology-$K3$-surface with vanishing first Chern class and nontrivial fundamental group. Of course one hardly expects a positive answer.

The fundamental group of a symplectic manifold with vanishing first Chern class and vanishing signature has a corresponding property: Any subgroup of finite index has rank at most 4. Of course, this narrows the range of possible fundamental groups of such manifolds. But still there is a considerable gap if one compares with the groups known to be realizable by symplectic manifolds of Kodaira dimension zero.

Partial results with regard to the main theorem were obtained by Morgan-Szabo [6] under the assumption $b_1(X) = 0$ and by Tian-Jun Li [5] under the assumption $b_1(X) \leq 4$.

The proof of the main theorem is modelled on the stable cohomotopy proof [2] of Morgan-Szabo's result. The concept can be explained in a few words: In its stable homotopy interpretation [3], the Seiberg-Witten invariant is the degree of a monopole map. Source and target depend on index data of the given 4-manifold in a controllable way. So it suffices to show that under the assumptions of the theorem there are only maps of even degree between the relevant spaces. This follows from equivariant obstruction theory using the fact that the vanishing of the first Chern class leads to additional symmetry of the monopole map.

References

[1] S. Bauer, *Almost complex 4-manifolds with vanishing first Chern class*, math.GT/0607714, 2006.

[2] S. Bauer, *Refined Seiberg-Witten invariants*, in: S.K. Donaldson, Y. Eliashberg, M. Gromov (Edits.) *Different Faces of Geometry*, Internat. Math. Ser., Kluwer Academic/Plenum Publishers, New York, 2004.

[3] S. Bauer and M. Furuta, *A stable cohomotopy refinement of Seiberg-Witten invariants: I*, Invent. Math. **155**, 1–19, 2004.

[4] R.E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) **142**, 527–595, 1995.

[5] T.J. Li, *Symplectic 4-manifolds with Kodaira dimension zero*, Preprint.

[6] J. Morgan and Z. Szabo, *Homotopy K3-surfaces and mod 2 Seiberg-Witten invariants*, Math. Res. Lett. **4**, 17–21, 1997.

[7] C.H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters **1**, 809–822, 1994.

Uncompleting classifying spaces

Jesper Grodal

(joint work with Bob Oliver)

The information encoded in a finite group G is equivalent to the information encoded in its classifying space BG. In homotopy theory, one can complete a space at a prime p, which produces the p-completed classifying space $BG^\wedge_p$. When G is abelian this procedure simply gives the classifying space BS of the Sylow p-subgroup of G, and all information prime to p is lost. When G is non-abelian the resulting space $BG^\wedge_p$ is a much more complicated object, but still an invariant of the "p-local structure" of the group, suitably defined.

In this talk we showed that often, when G is "non-abelian enough", this p-completion process can in fact be reversed! We explained theorems saying that for many "sufficiently complicated" groups G, the space BG can be recovered from the p-completed space $BG^\wedge_p$ for just a single prime p. In other words the p-local structure in G in fact completely determines its global structure. The approach goes via a certain category $\mathcal{L}_p(G)$ called the p-local finite group of the group (see e.g., [1] or [2] for definitions). We propose the fundamental group $\pi_1(\mathcal{L}_p(G))$ as an interesting invariant of the p-local structure of the group.

For several groups we get "local-to-global" theorems:

Theorem (Grodal-Oliver [3]). *Suppose that G is either*

(1) *A p-solvable group with $O_{p'}(G) = 1$.*
(2) *A finite group of Lie type of rank ≥ 3 with $O_{p'}(G) = 1$.*
(3) *Σ_{p^n} with $n \geq 3$ and $p = 2$ (probably also OK for p odd).*
(4) *Several of the larger sporadic groups for $p = 2$, e.g., the Monster, M_{24}, Co_3, ...*

Then $\pi_1(\mathcal{L}_p(G)) = G$, and in particular G can be recovered from its p-local structure $\mathcal{L}_p(G)$.

For other groups G, such as linear groups over $\mathbb{F}_q$ for q a prime power different from p, easy examples show that the p-local structure cannot determine the group G uniquely. However, our work indicate that even in those cases $\pi_1(\mathcal{L}_p(G))$ can be a sort of "best global approximation" to the p-local finite group $\mathcal{L}_p(G)$, which is an interesting group, though not necessarily finite:

Theorem (Grodal-Oliver [3]). *Suppose that* $G = \mathrm{SO}_n(\mathbb{F}_q)$ *for* $q \equiv 3, 5(8)$, $n \leq 8$, *then* $\pi_1(\mathrm{SO}_n(\mathbb{F}_q)) \cong \mathrm{SO}_n(\mathbb{Z}[\frac{1}{2}])$

References

[1] C. Broto, R. Levi, and B. Oliver, *Homotopy equivalences of p-completed classifying spaces of finite groups*, Invent. math. **151** (2003), 611–664

[2] C. Broto, R. Levi, and B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856

[3] J. Grodal and B. Oliver, *Fundamental groups of p-local finite groups*, In preparation.

The L-theory of $\mathrm{PSL}_2(\mathbb{Z})$ and connected sums of manifolds

James F. Davis

(joint work with Frank Connolly)

This is an application of a program to prove the Farrell-Jones isomorphism conjecture in L-theory for groups which act properly, co-compactly by isometries on non-positively curved manifolds and is work in progress.

Fix a closed manifold $X^n = X_1 \# X_2$ and $n > 2$. Let $\pi_1 X = G_1 * G_2$.

The *connected sum problem* is: Is every homotopy equivalence $h : M \to X$ splittable? I.e., is $M = M_1 \# M_2$ and $h \simeq h_1 \# h_2$ for suitable homotopy equivalences h_i?

Metatheorem (Stallings, Browder, Cappell). *For* $n > 4$, *the answer depends only on* G_1, G_2, *the orientation characters, and* n (mod 4).

Answers:

- Yes, if $\pi_1 = 1$ (Browder)
- No, for $X = \mathbb{R}P^{4k+1} \# \mathbb{R}P^{4k+1}$ (Cappell)
- Yes, if $n = 4k + 3$, X orientable (Cappell)
- Yes, if M and X are h-cobordant (Stallings). In fact he showed $\mathrm{Wh}(G_1 * G_2) = \mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2)$

Cappell showed:

$$\widetilde{L}_n(G_1 * G_2) = \widetilde{L}_n(G_1) \oplus \widetilde{L}_n(G_2) \oplus \mathrm{UNil}_n(\mathbb{Z}; \widehat{\mathbb{Z}G_1}, \widehat{\mathbb{Z}G_2}),$$

where $\widehat{\mathbb{Z}G}$ is the $\mathbb{Z}[\mathbb{Z}_2]$-module $\mathbb{Z}[G - e]$ with involution $g \mapsto g^{-1}$.

He also showed that $\mathrm{UNil}_n(\mathbb{Z}; \widehat{\mathbb{Z}G_1}, \widehat{\mathbb{Z}G_2})$ vanishes if G_1 and G_2 have no elements of order 2, and that $\mathrm{UNil}_{n+1}(\mathbb{Z}; \widehat{\mathbb{Z}G_1}, \widehat{\mathbb{Z}G_2}) = 0$ iff every homotopy equivalence h is splittable.

We reduce the connected sum problem to very special fundamental groups: There is a splitting of abelian groups with involution

$$\widehat{\mathbb{Z}G} = \bigoplus_{g^2=1} \mathbb{Z}g \oplus \bigoplus_{g^2\neq 1} (\mathbb{Z}g + \mathbb{Z}g^{-1}).$$

In other words,

$$\widehat{\mathbb{Z}G} = \bigoplus \mathbb{Z}\text{'s} \oplus \bigoplus H\text{'s},$$

where $H = \mathbb{Z} \oplus \mathbb{Z}$ with the involution switching the two summands.

Lemma (CD)**.** $\mathrm{UNil}_n(\mathbb{Z}; P, Q)$ *is symmetric, bilinear in P and Q:*

$$\mathrm{UNil}_n(\mathbb{Z}; P, Q) = \mathrm{UNil}_n(\mathbb{Z}; Q, P),$$
$$\mathrm{UNil}_n(\mathbb{Z}; P \oplus P', Q) = \mathrm{UNil}_n(\mathbb{Z}; P, Q) \oplus \mathrm{UNil}_n(\mathbb{Z}; P', Q).$$

Thus the connected sum problem reduces to three special cases:

- The group $\mathrm{UNil}(\mathbb{Z}; H, H) \subset \widetilde{L}(\mathbb{Z}_3 * \mathbb{Z}_3)$ vanishes by Cappell.
- $\mathrm{UNil}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \subset \widetilde{L}(\mathbb{Z}_2 * \mathbb{Z}_2)$ is computed recently using algebra ($\mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group).
- The computation of $\mathrm{UNil}(\mathbb{Z}; \mathbb{Z}, H) \subset \widetilde{L}(\mathbb{Z}_2 * \mathbb{Z}_3)$ requires geometry ($\mathbb{Z}_2 * \mathbb{Z}_3$ is the modular group $\mathrm{PSL}_2(\mathbb{Z})$).

The L-theory of the infinite dihedral group was computed by Cappell, Connolly-Ranicki, Connolly-Davis, Banagl-Ranicki:

$$\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \begin{cases} 0 & n \equiv 0, 1 \pmod 4 \\ (\mathbb{Z}_2)^\infty & n \equiv 2 \pmod 4 \\ (\mathbb{Z}_2)^\infty \oplus (\mathbb{Z}_4)^\infty & n \equiv 3 \pmod 4. \end{cases}$$

One consequence of this computation is: There exists $M^4 \simeq \mathbb{R}P^4 \# \mathbb{R}P^4$ which is no (non-trivial) connected sum. This uses $0 \neq \mathrm{UNil}_3(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \cong \mathrm{UNil}_5(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}^-)$ and Wall realization.

Theorem (CD)**.** *The Farrell-Jones isomorphism conjecture in L-theory for $\mathbb{Z}_2 * \mathbb{Z}_3$ implies*

$$\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, H) = \bigoplus_{\Gamma} \mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}),$$

where Γ is the set of conjugacy classes of maximal infinite dihedral subgroups of the group $\mathbb{Z}_2 * \mathbb{Z}_3$.

This solves the connected sum problem:

- Every homotopy equivalence $h : M^n \to X_1 \# X_2$ is splittable when $n \equiv 0, 3 \pmod 4$.
- When $n \equiv 1, 2 \pmod 4$ and π_1 has some 2-torsion there are obstructions.

To prove the isomorphism conjecture in L-theory one uses the isomorphism $\mathbb{Z}_2 * \mathbb{Z}_3 \cong \mathrm{PSL}_2(\mathbb{Z})$. Then Γ corresponds to reciprocal geodesics in the orbifold $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$. For more on these see *Letter to J. Davis about reciprocal geodesics* [5].

Currently we have proven the isomorphism conjecture in L-theory for crystallographic groups and the isomorphism conjecture for negatively curved groups is work in progress.

References

[1] W. Browder, *Embedding* 1*-connected manifolds*, Bull. Amer. Math. Soc. **72** (1966), 225–231; Erratum: ibid. **72** (1966), 736.

[2] S. E. Cappell, *Manifolds with fundamental group a generalized free product. I.*, Bull. Amer. Math. Soc. **80** (1974), 1193–1198.

[3] S. E. Cappell, *Unitary nilpotent groups and Hermitian K-theory. I.*, Bull. Amer. Math. Soc. **80** (1974), 1117–1122.

[4] F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (1993), 249–297.

[5] P. Sarnak. *Letter to J. Davis about reciprocal geodasics*, (2005), available online at http://www.math.princeton.edu/sarnak/SarnakDavisLtr05.pdf

[6] J. Stallings, *Whitehead torsion of free products*, Ann. of Math. (2) **82** (1965) 1965 354–363.

Quasi-isometric classification of graph manifolds groups

Walter D. Neumann

(joint work with Jason Behrstock)

When we speak of quasi-isometry of groups the metric in question is the word metric with respect to some finite generating set. Two groups are "weakly commensurable" if they can be made isomorphic by taking finite index subgroups of quotients by finite normal subgroups. Weakly commensurable finitely generated groups are quasi-isometric.

A compact 3-manifold M (possibly with boundary) is "geometric" if $M - \partial M$ admits a geometric structure in Thurston's sense. By Perelman it is now known that any irreducible 3-manifold has a "geometric decomposition" – a decomposition along tori and Klein bottles into geometric pieces.

There is a considerable literature on quasi-isometric rigidity and classification of 3-manifold groups. The rigidity results can be briefly summarised:

Theorem. *If G is a group which is quasi-isometric to the fundamental group of a 3-manifold M then G is weakly commensurable with $\pi_1(M')$ for some 3-manifold M'. Moreover, M' is irreducible respectively geometric if and only if M is.*

This quasi-isometric rigidity for geometric 3-manifold groups is the culmination of the work of many authors, key steps being provided by Gromov-Sullivan, Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb, Rieffel, Schwartz [2, 3, 5, 7, 12, 13]. The reducible case reduces to the irreducible case using Papasoglu and Whyte [11] and the irreducible non-geometric case is considered by Kapovich and Leeb [7].

The classification results in the geometric case can be summarized:

Theorem. *There are exactly seven quasi-isometry classes of fundamental groups of closed geometric 3-manifolds, namely any such group is quasi-isometric to one of the eight Thurston geometries (*$\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{E}^3$, Nil, $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{\mathrm{PSL}}$, Sol, $\mathbb{H}^3$*) but the two geometries* $\mathbb{H}^2 \times \mathbb{E}^1$ *and* $\widetilde{\mathrm{PSL}}$ *are quasi-isometric.*

If a geometric manifold M has boundary, then it is either Seifert fibered and its fundamental group is quasi-isometric (indeed commensurable) with $F_2 \times \mathbb{Z}$, or it is hyperbolic, in which case quasi-isometry also implies commensurability [13].

The talk reported progress on classification in the non-geometric case. The geometric version of JSJ decomposition (e.g., [10]) gives uniqueness up to isotopy for the minimal decomposition along tori and Klein bottles of an irreducible non-geometric 3-manifold M into geometric pieces. These geometric pieces are either hyperbolic or Seifert fibered. As a first step to classification, it is not hard to see that the existence of pieces of either type in the decomposition is a quasi-isometry invariant [6, 8]. We describe the case that there are no hyperbolic pieces, so M is a "graph manifold" in the sense of Waldhausen.

We associate to the geometric JSJ decomposition of M its *two-colored decomposition graph* $\Gamma(M)$ which has a vertex for each Seifert piece and an edge for each decomposing torus or Klein bottle; vertices of $\Gamma(M)$ are colored **b**lack or **w**hite according to whether the Seifert piece includes a boundary component of M or not (**b**ounded or **w**ithout boundary). A two-colored tree is similarly associated to the decomposition of the universal cover $\tilde{M}$ into its fibered pieces. This infinite valence two-colored tree is denoted $\mathrm{BS}(M)$, since it is the Bass-Serre tree corresponding to the graph of groups JSJ-decomposition of $\pi_1(M)$.

The Bass-Serre tree $\mathrm{BS}(M)$ can be constructed directly from the decomposition graph $\Gamma = \Gamma(M)$ by first replacing each edge of Γ by a countable infinity of edges with the same endpoints and then taking the universal cover of the result, so we also call it $\mathrm{BS}(\Gamma)$. If $\mathrm{BS}(\Gamma_1) \cong \mathrm{BS}(\Gamma_2)$ we say Γ_1 and Γ_2 are *bisimilar*.

A *weak covering map* $\phi\colon \Gamma \to \Gamma'$ between two-colored graphs is a color-preserving graph homomorphism satisfying: for any vertex v of Γ and every edge e' at $\phi(v)$, there is at least one edge e at v mapping to e'. An example of such a map is the map that collapses any multiple edge of Γ to a single edge. Any covering map of irreducible non-geometric graph manifolds induces a weak covering map of their two-colored decomposition graphs.

If a weak covering map $f\colon \Gamma \to \Gamma'$ exists then clearly $\mathrm{BS}(\Gamma) = \mathrm{BS}(\Gamma')$, so Γ and Γ' are bisimilar. In fact, the equivalence relation generated by the relation of existence of a weak covering map is bisimilarity. In this form the equivalence relation is known to computer scientists, from whom the "bisimilarity" terminology is borrowed (they call weak covering "bisimulation").

Proposition. *Consider countable connected graphs with at least one edge. Then each equivalence class of two-colored graphs includes two characteristic elements: a unique tree that weakly covers every element in the class (the Bass-Serre tree); and a unique minimal element, which is weakly covered by all elements in the class.*

For example, if all the vertices of a graph have the same color, then the minimal graph for its bisimilarity class is a single vertex with a loop attached and the Bass-Serre tree is the single-colored regular tree of countably infinite degree.

Our main theorem is:

Theorem. *The following are equivalent for irreducible non-geometric graph manifolds M and M':*

(1) *$\tilde{M}$ and $\tilde{M}'$ are bilipschitz homeomorphic.*
(2) *$\pi_1(M)$ and $\pi_1(M')$ are quasi-isometric.*
(3) *$\mathrm{BS}(M)$ and $\mathrm{BS}(M')$ are isomorphic as two-colored trees.*
(4) *The minimal two-colored graphs in the bisimilarity classes of the decomposition graphs $\Gamma(M)$ and $\Gamma(M')$ are isomorphic.*

One can list minimal two-colored graphs of small size, yielding, for instance, that there are exactly $2, 6, 26, 199, 2811, 69711, 2921251, 204535126, \dots$ quasi-isometry classes of fundamental groups of non-geometric graph manifolds that are composed of at most $1, 2, 3, 4, 5, 6, 7, 8, \dots$ Seifert pieces.

The theorem gives just one quasi-isometry class of closed non-geometric graph manifolds: the minimal two-colored graph is a single white vertex with a loop. This answers a question of Kapovich and Leeb. Similarly, there is just one quasi-isometry class of non-geometric graph manifolds that have boundary components in every Seifert piece: the minimal graph is a single black vertex with a loop.

The latter answers a question of Bestvina on RAAGs (right-angle Artin groups). In unpublished work with Kleiner and Sageev he has proved quasi-isometric rigidity results for RAAGs whose presentation graphs are far from being trees. He asked what happens for RAAGs whose presentation graphs are trees. RAAGs with small presentation trees (diameter ≤ 2) give groups of the form (free)$\times\mathbb{Z}$, while larger trees give fundamental groups of non-geometric graph manifolds with boundary components in every Seifert component. Thus, excluding the "small" cases, there is just one quasi-isometry class of RAAGs with presentation trees, answering Bestvina's question and showing that quasi-isometry behavior of his RAAGs is dramatically different from those with presentation trees.

An Artin group is quasi-isometric to a 3-manifold group if and only if the components of the presentation graph are trees or 2-weighted triangles, in which case the manifold is a graph-manifold so our results can be applied [1] (strengthening a result of Gordon [4]).

Commensurability classification for non-geometric graph manifold groups is far from understood but is known to be much richer than quasi-isometry [9], and both classifications for general Artin groups are wide open.

References

[1] J. Behrstock and Walter D. Neumann. Quasi-isometric classification of graph manifolds groups. Preprint, arXiv:math.GT/0604042, 2006.

[2] J. Cannon and D. Cooper. A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three. Trans. AMS, **330** (1992), 419–431.

[3] Alex Eskin, David Fisher, and Kevin Whyte. Quasi-isometries and rigidity of solvable groups. Preprint, arXiv:math.GR/0511647, 2005.
[4] C. McA. Gordon. Artin groups, 3-manifolds and coherence. (Bol. Soc. Mat. Mexicana (3) **10** (2004), Special Issue in Honor of Francisco "Fico" Gonzalez-Acuna, 193–198.
[5] M. Gromov. Groups of polynomial growth and expanding maps. IHES Sci. Publ. Math., **53** (1981) 53–73.
[6] M. Kapovich and B. Leeb. On asymptotic cones and quasi-isometry classes of fundamental groups of nonpositively curved manifolds. Geom. Funct. Anal. **5** (1995), 582–603.
[7] M. Kapovich and B. Leeb. Quasi-isometries preserve the geometric decomposition of Haken manifolds. Invent. Math. **128** (1997), 393–416.
[8] M. Kapovich and B. Leeb. 3-manifold groups and nonpositive curvature. Geom. Funct. Anal. **8** (1998), 841–852.
[9] Walter D. Neumann. Commensurability and virtual fibration for graph manifolds. Topology **36** (1997), 355–378.
[10] Walter D. Neumann and G.A. Swarup. Canonical decompositions of 3-manifolds. Geometry and Topology **1** (1997), 21–40.
[11] P. Papasoglu and K. Whyte. Quasi-isometries between groups with infinitely many ends. *Comment. Math. Helv.*, **77 (1)** (2002) 133–144.
[12] E. Rieffel. Groups quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$. *J. London Math. Soc. (2)*, **64 (1)** (2001) 44–60.
[13] R. Schwartz. The quasi-isometry classification of rank one lattices. *IHES Sci. Publ. Math.*, **82** (1996) 133–168.

p-local compact groups

Carles Broto
(joint work with Ran Levi, Bob Oliver)

In 1994 Dwyer and Wilkerson [3] defined p-compact groups as p-local homotopy theoretic analogue of compact and connected Lie groups, where p is a fixed prime number. We later defined p-local finite groups [1] as the p-local objects that model the p-local structure of finite groups. The aim of this talk is to define the concept of p-local compact group, as a unifying theory that gathers together p-local finite groups and p-compact groups.

In [1], a p-local finite group is defined as a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p-group, $\mathcal{F}$ a saturated fusion system over S, and $\mathcal{L}$ a centric linking system associated to $\mathcal{F}$. In case of p-local compact groups the finite p-group S is replaced by a discrete p-toral group, and the fusion and linking systems is adapted to the new situation with small changes. Precise definitions follow below.

Write $\mathbb{Z}/p^\infty = \bigcup_{n\geq 1} \mathbb{Z}/p^n \cong \mathbb{Z}[\frac{1}{n}]/\mathbb{Z}$.

Definition. A discrete p-toral group P is a group that fits in an extension

$$P_0 \longrightarrow P \longrightarrow P/P_0,$$

where $P_0 \cong (\mathbb{Z}/p^\infty)^r$ and P/P_0 is a finite p-group. We call P_0 the identity component, r the rank of P, and $\pi_0(P) = P/P_0$ the group of components.

Discrete p-toral groups are characterized by being artinian and locally finite p-groups. The identity component of a discrete p-toral group is a characteristic subgroup that can also be described as the subset of all infinitely p-divisible elements of P and also as the minimal subgroup of finite index in P. We measure the size of P as $|P| = (r, |\pi_0(P)|)$ with lexicographical order. The class of discrete p-toral groups is closed under taking subgroups, quotients, and extensions.

The p-completed classifying space of a discrete p-toral group P_0 of rank r with trivial group of components is $BP_0{}^\wedge_p \simeq (BT^r)^\wedge_p$, where $T^r = (S^1)^r$ is a torus of rank r. So, in general, if P has rank r, then $BP^\wedge_p$ fits in a fibration sequence

$$(BT^r)^\wedge_p \longrightarrow BP^\wedge_p \longrightarrow B\pi_0(P),$$

where $T^r = (S^1)^r$ is a torus of rank r. Thus $BP^\wedge_p$ is a p-compact toral group [3]. In fact, all p-compact toral groups arise in this manner.

Saturated fusion systems are defined in a way similar to the finite case. A fusion system $\mathcal{F}$ over a discrete p-toral group S consists of a set $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ for every pair P, Q of subgroups of S such that

$$\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$$

and form a category where every morphism decomposes as an isomorphism followed by an inclusion. A fusion system is called saturated if it satisfies some extra axioms. Now, to the axioms of saturation due to Puig [4, 1] for fusion systems over finite p-groups we add a new axiom that concerns infinite subgroups of S. We include here the complete set of axioms.

Definition. Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S.

(1) A subgroup $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ that are $\mathcal{F}$-conjugate to P.

(2) A subgroup $P \leq S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ that are $\mathcal{F}$-conjugate to P.

(3) We will say that $\mathcal{F}$ is *saturated* if the following axioms are satisfied:

(I) Sylow Axiom: If $P \leq S$ is fully normalized in $\mathcal{F}$, then P is fully centralized in $\mathcal{F}$, $\mathrm{Out}_{\mathcal{F}}(P)$ is finite, and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p \mathrm{Out}_{\mathcal{F}}(P)$.

(II) Extension Axiom: If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set

$$N_\varphi = \big\{ g \in N_S(P) \bigm| \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P) \big\},$$

then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

(III) Continuity Axiom: If $P_1 \leq P_2 \leq P_3 \leq \cdots$ is an increasing sequence of subgroups of S, with $P_\infty = \bigcup_{n=1}^\infty P_n$, and $\varphi \in \mathrm{Hom}(P_\infty, S)$ is any homomorphism such that $\varphi|_{P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all n, then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$.

The definitions of $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups in a fusion system $\mathcal{F}$ over a finite p-group carry over to the case of fusion systems over discrete p-toral groups. Indeed, it is proved that there is a finite number of $\mathcal{F}$-conjugacy classes of subgroups which are $\mathcal{F}$-centric and $\mathcal{F}$-radical and then that Alperin's fusion theorem holds.

Like in the finite case, a *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c,$$

and "distinguished" monomorphisms $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the same axioms (A), (B), and (C), as in the p-local finite group case [1, Definition 1.7].

Definition. A p-local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The classifying space is the p-completed nerve of the centric linking system $|\mathcal{L}|^{\wedge}_p$.

The algebraic structure of a p-local compact group is completely determined by its classifying space. However some important differences with the finite case are worth mentioning. The following is a key theorem.

Theorem. *If $(S, \mathcal{F}, \mathcal{L})$ is a p-local compact group, then:*

(a) *For any discrete p-toral group Q, $\mathrm{Rep}(Q, \mathcal{L}) \xrightarrow{\cong} [BQ, |\mathcal{L}|^{\wedge}_p]$, where we define $\mathrm{Rep}(Q, \mathcal{L}) = \mathrm{Hom}_{\mathcal{F}}(Q, S)/\mathcal{F}$-conjugacy.*
(b) *If $\rho \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ and $\rho(Q)$ is $\mathcal{F}$-centric, then*

$$BZ(\rho(Q))^{\wedge}_p \simeq \mathrm{map}(BQ, |\mathcal{L}|^{\wedge}_p)_{\theta\circ B\rho}\,,$$

where $\theta\colon BS \to |\mathcal{L}|^{\wedge}_p$ is the canonical map induced by the distinguished homomorphism δ_S.

Part (a) of the theorem shows that the fusion system can be obtained from the classifying space. However, in part (b) we observe that $\rho(Q)$ being discrete p-toral might generate non-trivial homotopy groups in dimension 2 in the mapping spaces $\mathrm{map}(BQ, |\mathcal{L}|^{\wedge}_p)_{\theta\circ B\rho}$. For this reason we have to develop a new strategy in order to recover the centric linking system from the classifying space.

The orbit category $\mathcal{O}^c(\mathcal{F})$ of a fusion system $\mathcal{F}$ over a discrete p-toral group S is the category with objects all subgroups of S that are $\mathcal{F}$-centric, and where $\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) = \mathrm{Rep}_{\mathcal{F}}(P, Q)$; that is, $\mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}}(P, Q)$. The classifying space functor induces a homotopy functor $\mathbf{B}\colon \mathcal{O}^c(\mathcal{F}_0) \longrightarrow \mathrm{hoTop}$ defined by setting $\mathbf{B}(P) = BP$.

Given a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$, the left homotopy Kan extension of the constant point functor along the projection $\mathcal{L} \to \mathcal{O}^c(\mathcal{F})$, defines a rigidification $\widetilde{B}$ of $\mathbf{B}$, that in turn, produces a homotopy colimit decomposition $\mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})} \widetilde{B} \simeq |\mathcal{L}|$. A *rigidification* of the homotopy functor $\mathbf{B}$ is a lifting $\widetilde{B}\colon \mathcal{O}^c(\mathcal{F}) \longrightarrow \mathrm{Top}$ of $\mathbf{B}$ to the category of topological spaces, together with a natural homotopy equivalence of functors (in hoTop) from $\mathbf{B}$ to $\mathrm{ho} \circ \widetilde{B}$.

It turns out that this construction can be reversed and we obtain a one-to-one correspondence

$$\left\{\begin{array}{c}\text{linking systems}\\ \text{associated to } \mathcal{F}_0\\ \text{up to isomorphism}\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{rigidifications } \mathcal{O}(\mathcal{F}_0) \to \text{Top}\\ \text{of the homotopy functor } \mathbf{B}\\ \text{up to natural homotopy equivalence}\end{array}\right\}.$$

We can also look at the p-completion $\mathbf{B}_p^\wedge$ of $\mathbf{B}$ that assigns $BP_p^\wedge$ to a $\mathcal{F}$-centric subgroup P of S. Obstruction theory shows that the above set is also in one-to-one correspondence with rigidifications of $\mathbf{B}_p^\wedge$, up to natural homotopy equivalence. From any of these three sets, one obtains the classifying space, and conversely, a classifying space associated to a saturated fusion system over a discrete p-toral group S determines a rigidification of $\mathbf{B}_p^\wedge$, and therefore an associated centric linking system. It follows that the centric linking system is determined, up to isomorphism, by the homotopy type of the classifying space. In particular:

Theorem. *If $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are two p-local compact groups such that $|\mathcal{L}|_p^\wedge \simeq |\mathcal{L}'|_p^\wedge$, then $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are isomorphic as p-local compact groups.*

References

[1] C. Broto, R. Levi, B. Oliver, The homotopy theory of fusion systems, Journal Amer. Math. Soc. 16 (2003), 779–856.

[2] C. Broto, R. Levi, B. Oliver, Discrete models for the p-local homotopy theory of compact Lie groups and p-compact groups. Preprint, 2006.

[3] W. Dwyer, C. Wilkerson, Homotopy fixed-point methods for Lie groups and finite loop spaces, Annals of math. 139 (1994), 395–442.

[4] L. Puig, Unpublished notes.

Trace invariants for fixed and periodic points

Bjørn Jahren

Let $f : X \to X$ be a self–map of a finite complex. The fundamental *Nielsen equivalence relation* between fixed points of f is defined by: $x \sim x'$ if and only of there is a path ω from x to x' such $f\omega \simeq \omega$ (homotopy relative endpoints). This relation is classically motivated and studied via covering spaces, but my interest in the subject comes from the observation that it also has the following interpretation:

Think of the homotopy $f\omega \simeq \omega$ as a homotopy from the constant path in x to the constant path in x' through paths from points y to $f(y)$, and let $\Lambda(f)$ be the space of such paths. Then the fixed point set $\mathrm{Fix}(f)$ of f can be identified with the subset of constant paths in $\Lambda(f)$, and Nielsen equivalence of two fixed points just says that the points are in the same component of $\Lambda(f)$. Note that $\Lambda(f) \approx \mathrm{Map}_{\mathbb{N}}(\mathbb{R}, X)$, i.e., the homotopy fixed point set of the additive monoid of natural numbers, acting by f. Hence Nielsen theory can be considered a very early instance of a "lim = holim" problem. I will discuss how this point of view leads to new interpretations of classical invariants, with indications of how this can be generalized to give new invariants for fixed and periodic points of parametrized maps. Most of this is work in progress, partly joint with Bruce Williams.

The most important invariant in the classical theory is the *Nielsen–Reidemeister trace* $\rho(f)$ — an obstruction to finding a map homotopic to f without fixed points and a refinement of the Lefschetz number of f. The trace $\rho(f)$ lies in a free $\mathbb{Z}$–module $R(f) = \mathbb{Z}G_\phi$, where G_ϕ is the set of twisted equivalence classes of $G = \pi_1(X, x_0)$, depending on a choice of basepoint x_0 and a path from x_0 to $f(x_0)$. The dependence of these choices is a nuisance, but our first observation is:

Lemma. $G_\phi \approx \pi_0(\Lambda(f))$.

It follows that $R(f)$ is isomorphic to $H_0(\Lambda(f))$, which does not depend on any choices, and it is natural to ask if there is a definition of $\rho(f)$ directly as an element of $H_0(\Lambda(f))$ which avoids choices. I will describe two such definitions. Assume in the following, for simplicity, that X is an oriented, closed, connected manifold of dimension n.

Note that $\Lambda(F)$ is defined so that the left square in the following diagram is a pullback:

$$\begin{array}{ccccc}
\Lambda(f) & \longrightarrow & X^I & \xleftarrow{\widetilde{\Delta}} & X \\
{\scriptstyle \mathrm{ev}_0}\big\downarrow & & {\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)}\big\downarrow & & {\scriptstyle =}\big\downarrow \\
X & \xrightarrow{\Gamma(f)} & X \times X & \xleftarrow{\Delta} & X
\end{array}$$

Here ev_t is evaluation at t and $\Gamma(f)(x) = (x, f(x))$. Moreover, Δ is the diagonal map, and $\widetilde{\Delta}$ maps a point x to the constant path at x.

Recall that the Lefschetz number can be thought of as the homology intersection of the diagonal and the graph of f in $X \times X$. The idea is to lift this to the upper horizontal line in the diagram, and "intersect" X and $\Lambda(f)$ in X^I.

In the first approach, joint work with my student Ingrid Seem, we approximate the diagram by a diagram of covering spaces. The vertical maps are fibrations, and we may construct associated covering spaces by "collapsing components of the fibers" — producing a diagram

$$\begin{array}{ccccc}
\overline{\Lambda(f)} & \xrightarrow{\gamma} & \overline{X^I} & \xleftarrow{\overline{\Delta}} & X \\
{\scriptstyle p}\big\downarrow & & {\scriptstyle p'}\big\downarrow & & {\scriptstyle =}\big\downarrow \\
X & \xrightarrow{\Gamma(f)} & X \times X & \xleftarrow{\Delta} & X
\end{array}$$

Now consider the following sequence of maps:

$$H_n(X) \overset{\overline{\Delta}_*}{\to} H_n(\overline{X^I}) \overset{\mathrm{PD}}{\leftarrow} H^n_c(\overline{X^I}) \overset{\gamma^*}{\to} H^n_c(\overline{\Lambda(f)}) \overset{\mathrm{PD}}{\to} H_0(\overline{\Lambda(f)}) \overset{\approx}{\leftarrow} H_0(\Lambda(f)).$$

PD is Poincaré duality, so this defines a homomorphism $H_n(X) \to R(f)$.

Theorem (J.–Seem [4]). *This composition maps the fundamental class $[X] \in H_n(X)$ to $\rho(f) \in Rf$.*

Remark. *If we choose a basepoint, there is a natural identification of* $\overline{X^I}$ *with the more familiar* $\tilde{X} \times_{\pi_1(X)} \tilde{X}$*, where* $\tilde{X}$ *is the universal covering space of* X*.*

The second approach deals directly with X and $\Lambda(f)$ as subspaces of X^I and is based on the fact that $\Lambda(f)$ essentially has an n–dimensional normal bundle neighborhood $\mathcal{U}$ in X^I, gotten by pulling the normal bundle of $\Gamma(f)(X)$ in $X \times X$ back over the map p. Then we have a map $X \to X^I \to X^I/(X^I - \mathcal{U})$. But $X^I/(X^I - \mathcal{U})$ is equivalent (deforms to) to the Thom space of the normal bundle, so composing with the Thom isomorphism, we get a homomorphism

$$H_n(X) \to H_n(X^I/(X^I - \mathcal{U}) \approx H_0(\Lambda(f)) \approx R(f)$$

Theorem. *This homomorphism maps the fundamental class* $[X]$ *to* $\rho(f)$*.*

These approaches to $\rho(f)$ have the advantage that they do not require any assumption about the fixed points of f, e.g., that they should be isolated. Homotopy invariance is also obvious. Moreover, it is easy to extend to families of maps, and both approaches can be adapted to give invariants for *coincidences* of maps. But for these generalizations the second approach is the most interesting, since the invariants will lie in higher homology groups, and $\Lambda(f)$ has more interesting homology than $\overline{\Lambda(f)}$.

There is yet another interpretation of $R(f)$, namely as the framed bordism group $\Omega_0^{\mathrm{fr}}(\Lambda(f)) \approx \pi_0(Q(\Lambda(f)_+)$. As an element here, $\rho(f)$ can be defined using transversality and the Thom–Pontrjagin construction. It also appears as an application of the "homotopy intersection theory" in [2]. This is the interpretation we will use in our study of *periodic points.*

Periodic orbits

I conclude with the following brief sketch of basic ideas behind ongoing joint work with Bruce Williams.

Periodic points are fixed points of powers of f, and $\rho(f^n)$ is an obstruction to finding a map homotopic to f^n without fixed points, hence also an obstruction to finding a map homotopic to f without points of period n. But since points of period n also have period mn, there should be some kind of structure on the set of invariants reflecting this. More specifically, the sets $\mathrm{Fix}(f^n)$ have natural actions of the cyclic group C_n such that $\mathrm{Fix}(f^{mn})^{C_m} = \mathrm{Fix}(f^n)$, and we would like some refinement of homotopy fixed point sets $\Lambda(f^n)$ with a similar structure. The idea is to study n–periodic orbits instead, considered as elements of X^n.

Let $f^{(n)} : X^n \to X^n$ be defined by

$$f^{(n)}(x_1, \dots, x_n) = \big(f(x_n), f(x_1), \dots, f(x_{n-1})\big).$$

Then the fixed points of $f^{(n)}$ are precisely the n–periodic orbits of f. Moreover, the C_n–action on X^n lifts to an action on $\Lambda(f^{(n)})$ such that $\Lambda(f^{(mn)})^{C_m} = \Lambda(f^{(n)})$, restricting to $\mathrm{Fix}(f^{(mn)})^{C_m} = \mathrm{Fix}(f^{(n)})$.

Because of the C_n–action, it is now more natural to consider $\rho(f^{(n)})$ in C_n–equivariant stable homotopy $Q^{C_n}(\Lambda(f))$.

Proposition. (1) *$\rho(f^{(n)})$ can naturally be chosen to lie in $Q^{C_n}(\Lambda(f))^{C_n}$.*

(2) *Restriction to C_m–fixed points defines a map*

$$Q^{C_{mn}}(\Lambda(f))^{C_{mn}} \to Q^{C_n}(\Lambda(f))^{C_n}$$

taking $\rho(f^{(mn)})$ to $\rho(f^{(n)})$.

Note that this is very reminiscent of the system of restriction maps of the cyclotomic structure used in the construction of topological cyclic homology [3]. Unfortunately, there does not seem to be an analogue of the Frobenius maps in the present situation. But, using the Segal–tom Dieck splitting, we get more information on the system of invariants $\rho(f^{(n)})$.

Let p be a prime an consider the numbers p^k, $k = 1, 2, \ldots$. Then

$$Q^{C_{p^n}}(\Lambda(f^{(p^n)}))^{C_{p^n}} \simeq \prod_{l \leq n} Q\big(\Lambda(f^{(p^l)})_{hC_{p^l}}\big).$$

Corollary. *There is an invariant $\rho^\infty(f) = (\rho_1, \rho_2, \ldots) \in \prod_{l \geq 0} Q(\Lambda(f^{(p^l)})_{hC_{p^l}})$ such that ρ_k represents (in π_0) an obstruction to deforming f to a map without points of period exactly p^k*

Remark. *These results are closely related to results obtained by other methods by R. McCarthy and Y. Iwashita* [1].

References

[1] Y. Iwashita *The Lefschetz trace in algebraic K–theory*, Ph. D. thesis, UIUC (1999)

[2] J.R. Klein, E.B. Williams *Homotopy intersection theory, I*, arXiv:math.AT/0512479 (2005)

[3] I. Madsen *Algebraic K-theory and traces*, Current Development in Mathematics, International Press Inc, Boston (1995), 191–321

[4] I. Seem, *Homotopifikspunkter og Nielsen teori*, Master thesis (Norwegian), University of Oslo (2002)

Resolutions of unbounded chain complexes

Jérôme Scherer

(joint work with Wojciech Chachólski and Wolfgang Pitsch)

Here is the setup of the problem. Let $\mathcal{A}$ denote a bicomplete abelian category and $\mathrm{Ch}(\mathcal{A})$ the category of unbounded chain complexes in $\mathcal{A}$. We fix a class $\mathcal{W}$ of objects, which we wish to consider as injectives in $\mathcal{A}$ (the dual case where one works with projective objects can be handled in the same way). A morphism $f : X_\bullet \to Y_\bullet$ of chain complexes is a $\mathcal{W}$-*equivalence* if, for any object $W \in \mathcal{W}$, the morphism $f^* : \mathcal{A}(Y_\bullet, W) \to A(X_\bullet, W)$ is a quasi-isomorphism in $\mathrm{Ch}(\mathbf{Z})$.

In [2], Christensen and Hovey work in the projective case. In order to do "relative homological algebra", in particular to construct projective resolutions in this setting, they analyze when $\mathrm{Ch}(\mathcal{A})$ can be given a Quillen model structure [3], where of course the weak equivalences should be the $\mathcal{W}$-equivalences. They succeed to do so in many cases, for example when $\mathcal{A}$ is the category of bimodules over a

k-algebra A and $\mathcal{W}$ is the class of free A-bimodules. In this case they show that the class of homotopy classes $[\Sigma^n A, N] \cong HH^n(A; N)$ for any (chain complex of) A-bimodule N.

Our point of view is that for most applications one does not really need to know $\mathrm{Ch}(\mathcal{A})$ is a model category. We will instead provide a "model approximation", roughly speaking a model category which is easier to deal with, and in which $\mathrm{Ch}(\mathcal{A})$ can be embedded. In this way important characteristics of model categories still hold:

(1) One can form the homotopy category $\mathrm{Ho}(\mathrm{Ch}(\mathcal{A}))$ and deduce that homotopy classes $[X_\bullet, Y_\bullet]$ form a set.
(2) One can construct injective (fibrant) resolutions and thus construct right derived functors in the relative setting.

Definition. Let $\mathcal{C}$ be a category with a distinguished class of weak equivalences. A model category $\mathcal{M}$ together with a pair of adjoint functors

$$l : \mathcal{C} \rightleftarrows \mathcal{M} : r$$

forms a *right model approximation* if

(1) the left adjoint l sends weak equivalences to weak equivalences,
(2) the right adjoint r sends weak equivalences between fibrant objects to weak equivalences,
(3) for a fibrant object $X \in \mathcal{M}$, if a morphism $lA \to X$ is a weak equivalence, then so is its adjoint $A \to rX$.

The algorithm to build a (fake) fibrant replacement for an object $A \in \mathcal{C}$ is then as follows. Push A into the model category $\mathcal{M}$ with the functor l. Using the model category structure in $\mathcal{M}$ construct a fibrant replacement $lA \to R(lA)$. Take the adjoint $A \to r(R(lA))$. This is not really a fibrant replacement in $\mathcal{C}$, not only because there is no model structure on $\mathcal{C}$, but more seriously because the lifting property with respect to acyclic cofibrations might fail, for some obvious choice of cofibrations.

Example. When $\mathcal{D}$ is a model category and I is a small category, it is not known whether the category $\mathcal{C} = \mathrm{Fun}(I, \mathcal{D})$ of $\mathcal{D}$-valued diagrams indexed by I form a model category. There is however an obvious choice of weak equivalences, namely the natural transformations of diagrams $F \to G$ such that $F(i) \to G(i)$ is a weak equivalence in $\mathcal{D}$ for all $i \in I$. We proved in [1] that there *always* exists a model approximation. Consider the simplex category of the nerve $N(I)$, where the morphisms are generated by the face and degeneracy maps and define $\mathcal{M}$ to be the full subcategory $\mathrm{Fun}^b(N(I)^{op}, \mathcal{D})$ of functors indexed by $N(I)^{op}$ that are *bounded*, in the sense that degeneracies induce isomorphisms.

The projection functor $\varepsilon : N(I)^{op} \to I$ yields a pair of adjoint functors

$$\varepsilon : \mathrm{Fun}(I, \mathcal{D}) \rightleftarrows \mathrm{Fun}^b(N(I)^{op}, \mathcal{D}) : \varepsilon^k,$$

where ε^k denotes the right Kan extension. This forms a model approximation. In particular the homotopy limit of any functor $F : I \to \mathcal{D}$ can be computed by taking the limit of the (fake) fibrant replacement:

$$\operatorname{holim}_I F = \lim_I \varepsilon^k (R(\epsilon F)).$$

Let us now go back to chain complexes. To do relative homological algebra, we will exhibit a model approximation for $\mathrm{Ch}(\mathcal{A})$. The only standing assumption is that there are "enough injective objects". More precisely we will assume that any object $A \in \mathcal{A}$ admits a $\mathcal{W}$-monomorphism $A \to W$ to some object of the class $\mathcal{W}$. Whereas unbounded chain complexes are difficult to handle, bounded ones are tame. Let us denote by $\mathrm{Ch}(\mathcal{A})_{\leq 0}$ the category of bounded above chain complexes (differentials have degree -1 and the chain complexes are zero in positive degrees).

Proposition. *The category* $\mathrm{Ch}(\mathcal{A})_{\leq 0}$ *admits a model category structure where the weak equivalences are the* $\mathcal{W}$*-equivalences.*

The model approximation for $\mathrm{Ch}(\mathcal{A})$ will be constructed out of bounded chain complexes by taking towers. The objects of the category $\mathrm{Tow}(\mathrm{Ch}(\mathcal{A})_{\leq 0})$ of towers are sequences $X^0_\bullet, X^1_\bullet, X^2_\bullet, \dots$ of chain complexes in $\mathrm{Ch}(\mathcal{A})_{\leq 0}$ together with structure maps $\Sigma X^{i+1}_\bullet \to X^i_\bullet$ for all $i > 0$, which should remind the reader of spectra. Weak equivalences are levelwise $\mathcal{W}$-equivalences.

Proposition. *The category* $\mathrm{Tow}(\mathrm{Ch}(\mathcal{A})_{\leq 0})$ *is a model category.*

The structure maps provide towers in $\mathcal{A}$ of the form

$$X^m_n \leftarrow X^{m+1}_{n-1} \leftarrow X^{m+2}_{n-2} \leftarrow \dots$$

for all $n \leq 0$ and $m \geq 0$. Taking the limit yields thus a functor

$$\lim : \mathrm{Tow}(\mathrm{Ch}(\mathcal{A})_{\leq 0}) \to \mathrm{Ch}(\mathcal{A})$$

to unbounded chain complex, which is right adjoint to the truncation functor. We are now ready to state our main result.

Theorem. *The pair of adjoint functors* $\mathrm{Trunc} : \mathrm{Ch}(\mathcal{A}) \rightleftarrows \mathrm{Tow}(\mathrm{Ch}(\mathcal{A})_{\leq 0}) : \lim$ *forms a model approximation.*

To conclude, let us remark that when $\mathcal{A}$ is the category of R-modules and the class $\mathcal{W}$ consists of all injective R-modules, then the fibrant resolution we construct coincides with the "right injective resolutions" Spaltenstein constructs in [4]. In his work there is no mention of the word model category.

References

[1] W. Chachólski and J. Scherer, *Homotopy Theory of Diagrams*, Mem. Amer. Math. Soc. **155** no. 736 (2002), ix+90 pp.
[2] D. Christensen and M. Hovey, *Quillen model structures for relative homological algebra*, Math. Proc. Cambridge Philos. Soc. **133** (2002), 261–293.
[3] D.G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics **43** (1967), iv+156 pp.
[4] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), 121–154.

On the cohomology of highly connected covers of finite H-spaces

Natàlia Castellana
(joint work with Juan A. Crespo, Jérôme Scherer)

Consider a finite complex X and an integer n. Does its n-connected cover $X\langle n\rangle$ satisfy any cohomological finiteness property? Of course, some restriction on the fundamental group will be needed, as the universal cover of $S^1 \vee S^2$ is an infinite wedge of copies of S^2.

The kind of prototypical theorems we have in mind are the Evens-Venkov result [4, 10] that the cohomology of a finite group is Noetherian, the analog for p-compact groups obtained by Dwyer and Wilkerson [3], and the fact that the mod p cohomology of an Eilenberg-Mac Lane space $K(A, n)$, with A abelian of finite type, is finitely generated as an algebra over the Steenrod algebra, which can easily been inferred from the work of Serre [8] and Cartan [1].

This last observation leads us to ask first whether or not the mod p cohomology of a finite Postnikov piece is also finitely generated as an algebra over the Steenrod algebra and second, since a finite complex X and its n-connected cover $X\langle n\rangle$ only differ in a finite number of homotopy groups, if $H^*(X\langle n\rangle; \mathbb{F}_p)$ satisfies the same property.

In this research project we offer a positive answer when X is an H-space, based on the analysis of the fibration $P \to X\langle n\rangle \to X$, where P is a finite Postnikov piece. In fact, we prove a strong closure property for H-fibrations.

Theorem. *Let $F \to E \to B$ be an H-fibration in which both $H^*(F; \mathbb{F}_p)$ and $H^*(B; \mathbb{F}_p)$ are finitely generated unstable algebras. Then so is $H^*(E; \mathbb{F}_p)$.*

This applies in particular to highly connected covers of finite H-spaces, see next corollary.

Corollary. *Consider an H-space X with finite mod p cohomology. Then the mod p cohomology of its n-connected cover $X\langle n\rangle$ is a finitely generated $\mathcal{A}_p$-algebra.*

In our previous work [2], we proved that the theorem holds whenever the base space is an Eilenberg-Mac Lane space. The proof relied mainly on Smith's work [9] on the Eilenberg-Moore spectral sequence. He shows that given an H-fibration $F \xrightarrow{i} E \xrightarrow{\pi} K(A, n)$, where A is either $\mathbb{Z}/p$ or a Prüfer group $\mathbb{Z}_{p^\infty}$ and $n \geq 2$, there is a coexact sequence of Hopf algebras

$$\mathbb{F}_p \longrightarrow H^*(E)//\pi^* \xrightarrow{i^*} H^*(F) \longrightarrow R \longrightarrow \mathbb{F}_p,$$

and R is described in turn by a coexact sequence of Hopf algebras

$$\mathbb{F}_p \longrightarrow \Lambda \longrightarrow R \longrightarrow S \longrightarrow \mathbb{F}_p,$$

where Λ is an exterior algebra, and $S \subseteq H^*(K(A, n-1))$ is a Hopf subalgebra.

Our strategy is the same and we need to analyze carefully certain Hopf subalgebras of $H^*(F; \mathbb{F}_p)$. Observe that the property for an unstable algebra K to be a finitely generated $\mathcal{A}_p$-algebra is equivalent to say that the module of the undecomposable elements QK is finitely generated as unstable module. It is often better to work with this module because it is smaller than the whole algebra and, above all, the category of unstable modules is locally Noetherian [5].

The main difficulty with the functor $Q(-)$ is the failure of left exactness. To what extent this functor is not left exact is precisely measured by André-Quillen homology $H_*^Q(-)$. We can briefly recall from Schwartz's book [7] how one computes André-Quillen homology in our setting. The symmetric algebra comonad $S(-)$ yields a simplicial resolution $S^\bullet(A)$ for any commutative algebra A. The André-Quillen homology group $H_i^Q(A)$ is the i-th homology group of the complex obtained from $S^\bullet(A)$ by taking the module of indecomposable elements (and the differential is the usual alternating sum). Because the Steenrod algebra acts on the symmetric algebra via the Cartan formula, the $\mathbb{F}_p$-vector space $H_i^Q(A)$ is equipped with an action of $\mathcal{A}_p$ This yields the same unstable module $H_i^Q(A)$ as the derived functor computed with a resolution in the category of unstable algebras [7, Proposition 7.2.2]. In our setting we can compute these unstable modules.

Proposition. *Let A be a Hopf algebra which is a finitely generated unstable $\mathcal{A}_p$-algebra. Then $H_0^Q(A) = QA$ and $H_1^Q(A)$ are both finitely generated unstable modules. Moreover, $H_i^Q(A) = 0$ for $i > 1$.*

This proposition yields then our main algebraic structural result about the category of unstable Hopf algebras.

Theorem. *Let B be a Hopf algebra which is a finitely generated unstable $\mathcal{A}_p$-algebra. Then so is any unstable Hopf subalgebra of B.*

For plain unstable algebras, this is false, as pointed out to us by Hans-Werner Henn. Consider the unstable algebra $H^*(\mathbf{C}P^\infty \times S^2; \mathbf{F}_p) \cong \mathbf{F}_p[x] \otimes E(y)$, where both x and y have degree 2. Turn the ideal generated by y into an unstable subalgebra by adding 1. This is isomorphic, as an unstable algebra, to $\mathbf{F}_p \oplus \Sigma^2\mathbf{F}_p \oplus \Sigma^2\widetilde{H}^*(\mathbf{C}P^\infty; \mathbf{F}_p)$, which is not finitely generated.

All this discussion leads naturally to ask whether the same statement holds for more general spaces.

Question. Let X be a finite space with $\pi_1 X$ finite and $n \geq 1$. Is $H^*(X\langle n\rangle; \mathbf{F}_p)$ finitely generated as an algebra over the Steenrod algebra?

References

[1] H. Cartan, *Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1954/1955. Algèbres d'Eilenberg-MacLane et homotopie, exposés 2 –16*, Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955.

[2] N. Castellana, J. A. Crespo, and J. Scherer, *Deconstructing Hopf spaces*, Invent. Math. (2006), DOI: 10.1007/s00222-006-0518-8.

[3] W. G. Dwyer and C. W. Wilkerson. *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. (2), **139** (1994), 395–442.

[4] L. Evens, *The cohomology ring of a finite group*, Trans. Amer. Math. Soc. **101** (1961), 224–239.

[5] J. Lannes and S. Zarati. *Sur les $\mathcal{U}$-injectifs*, Ann. Sci. École Norm. Sup. (4) **19**(2) (1986), 303–333.
[6] H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) **120**(1) (1984), 39–87.
[7] L. Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL (1984).
[8] J.-P. Serre. *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comment. Math. Helv. **27** (1953), 198–232.
[9] L. Smith. *Lectures on the Eilenberg-Moore spectral sequence* Lecture Notes in Mathematics, Vol. **134** (1970), Springer-Verlag, Berlin.
[10] B. B. Venkov. *Cohomology algebras for some classifying spaces*, Dokl. Akad. Nauk SSSR **127** (1959), 943–944.

Orbivariant K-theory

André Henriques

Orbispaces are spaces with extra structure. The main examples come from topological group actions $X \circlearrowleft G$ and are denoted $[X/G]$, their underlying space, or *coarse moduli space* being X/G. By definition, every orbispace is locally of the form $[X/G]$, but the group G might vary.

We shall work with orbispaces whose coarse moduli spaces are CW-complexes, and whose stabilizer groups are compact Lie groups. We also require the stratification of the coarse moduli space by the type of stabilizer group to be compactible with the CW structure. A convenient model for an orbispace is then given by a *topological groupoid* [2, 3].

An orbispace always comes with a map to its coarse moduli space. By a *suborbispace* $\mathfrak{X}' \subset \mathfrak{X}$, we shall mean an orbispace obtained by pulling back along a subspace of the coarse moduli space.

If $\mathfrak{X}$ is an orbispace modeled by a topological groupoid $\mathcal{G}$, then a vector bundle over $\mathfrak{X}$ is a vector bundle over the space of objects of $\mathcal{G}$ equipped with an action of the arrows of $\mathcal{G}$. It is tempting to define K-theory as the Grothendieck group of vector bundles. But, as shown in [1], this is not always a good idea, even if $\mathfrak{X}$ is compact. For example, one needs the following condition to prove excision:

Definition 1 (Lück, Oliver [1]). An orbispace $\mathfrak{X}$ has enough vector bundles if for every suborbispace $\mathfrak{X}' \subset \mathfrak{X}$ and every finite dimensional vector bundle V on $\mathfrak{X}'$, there exists a finite dimensional vector bundle W on $\mathfrak{X}$ and a linear embedding $V \hookrightarrow W$.

This condition is not always satisfied (see Example 3, and [1, Section 5]).

Theorem 2. *Let $\mathfrak{X}$ be a compact orbispace (i.e., its coarse moduli space is compact). Then the following are equivalent:*

(1) *$\mathfrak{X}$ is a global quotient by a compact Lie group, i.e., $\mathfrak{X} = [X/G]$ for some compact Lie group G acting on a compact space X.*
(2) *$\mathfrak{X}$ has enough vector bundles.*

(3) *There exists a vector bundle W on $\mathfrak{X}$ such that for every point x the action of* $\mathrm{Aut}(x)$ *on W_x is faithful.*

Proof. 1. $\Rightarrow$ 2. Let $X \circlearrowleft G$ be such that $\mathfrak{X} \simeq [X/G]$, and let $X' \subset X$ be the G-invariant subspace corresponding to $\mathfrak{X}' \subset \mathfrak{X}$. Let V be a vector bundle on $\mathfrak{X}'$ and let $\tilde{V}$ be the corresponding G-equivariant vector bundle on X'. It is well known that any equivariant vector bundle $\tilde{V}$ on a compact space X' embeds in one of the form $X' \times M$, where M is a representation of G. Let W be the vector bundle on $\mathfrak{X}$ corresponding to $X \times M \to X$. Since $\tilde{V}$ embeds in $X \times M$, the bundle V embeds in W.

2. $\Rightarrow$ 3. Suppose that $\mathfrak{X}$ has enough vector bundles, and let $\{U_i\}$ be a finite cover of $\mathfrak{X}$ such that $U_i \simeq [X_i/G_i]$. Let M_i be faithful representations of G_i, and let V_i be the vector bundles on U_i corresponding to $X_i \times M_i \to X_i$. Since M_i is faithful, the stabilizer groups act faithfully on the fibers of V_i. Let W_i be vector bundles on $\mathfrak{X}$ such that $V_i \hookrightarrow W_i$, and let $W := \bigoplus W_i$. Clearly, the stabilizer groups act faithfully on the fibers of W.

3. $\Rightarrow$ 1. Let P be the total space of the frame bundle of W. The stabilizer groups act faithfully on the fibers of W, hence they act freely on the fibers of P. Having no stabilizer groups, P is a space. We have $\mathfrak{X} = [P/O(n)]$ and so $\mathfrak{X}$ is a global quotient. $\square$

Example 3. Let $P \to S^3$ be the principal BS^1-bundle classified by

$$1 \in [S^3, B(BS^1)] = \mathbb{Z}.$$

Then $\mathfrak{X} := [P/ES^1]$ does not have enough vector bundles.

Proof. We show that $\mathfrak{X}$ is not a global quotient by a compact Lie group. Indeed, suppose that $\mathfrak{X} = [X/G]$. Since $\mathfrak{X} \to S^3$ is homotopically non-trivial, the map $X \to S^3$ needs to be be a non-trivial G-fiber bundle with fiber $S^1 \backslash G \circlearrowleft G$. Let $H = \mathrm{Aut}_G(S^1 \backslash G)$ be the structure group of that bundle. All compact Lie groups have trivial π_2, therefore $[S^3, BH] = \pi_3 BH = \pi_2 H = 0$. The bundle $X \to S^3$ is trivial, a contradiction. $\square$

More generally, any S^1-gerbe whose class in H^3 is non-torsion is an orbispace without enough vector bundles.

Since there exist orbispaces without enough vector bundles, vector-bundle-K-theory is not a cohomology theory. So we need another definition for K-theory of orbispaces. Our preferred one, inspired by [4], is the following:

Definition 4. Let $\underline{\mathbb{C}} := \mathbb{C} \times \mathfrak{X}$ be the trivial bundle. A cocycle for $K^0(\mathfrak{X})$ is a chain complex of $\underline{\mathbb{C}}$-modules (not necessarily locally constant) which is locally quasi-isomorphic to a bounded complex of finite dimensional vector bundles. Two K^0-cocycles on $\mathfrak{X}$ represent the same element in $K^0(\mathfrak{X})$ if they extend to a K^0-cocycle on $\mathfrak{X} \times [0, 1]$.

If $\mathfrak{X}$ is a space (by which we mean that $\mathfrak{X}$ is a CW-complex, not necessarily compact) this definition recovers the usual topological K-theory of $\mathfrak{X}$.

References

[1] Wolfgang Lučk and Bob Oliver. The completion theorem in K-theory for proper actions of a discrete group. *Topology*, 40(3):585–616, 2001.
[2] David Metzler. Topological and smooth stacks. *math.DG/0306176*, 2003.
[3] Ieke Moerdijk. Orbifolds as groupoids: an introduction. Orbifolds in mathematics and physics (Madison, WI, 2001). *Contemp. Math.*, 310:205–222, 2002.
[4] Robert W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. *The Grothendieck Festschrift, Vol. III*, Progr. Math., 88, Birkhäuser, Boston, MA, pages 247–435, 1990.

The Farrell-Jones conjecture in algebraic K-theory for word-hyperbolic groups

Arthur Bartels
(joint work with Wolfgang Lück and Holger Reich)

Let G be group and R be a ring. Roughly speaking, the Farrell-Jones conjecture in algebraic K-theory [2] says that the algebraic K-theory of the group ring RG can be computed from knowlege of the algebraic K-theory of RV, where V runs over the family $\mathcal{VC}\mathrm{yc}$ of virtually cyclic subgroups of G. More precisely, it asserts that the assembly map

$$\alpha_{\mathcal{VC}\mathrm{yc}} \colon H_*^G(E_{\mathcal{VC}\mathrm{yc}}G; \mathbf{K}_R) \to K_*RG$$

is an isomorphism.

In this talk I discussed applications and the proof of the following result.

Theorem. *The Farrell-Jones conjecture in algebraic K-theory holds for word-hyperbolic groups.*

This result implies that for a torsion-free word-hyperbolic group the Whitehead group $\mathrm{Wh}(G)$ and the reduced class group $\tilde{K}_0(\mathbb{Z}G)$ are trivial. There are also applications to the Bass conjecture for word-hyperbolic groups, the Kaplansky conjecture and Waldhausen Nil-groups. For example, if G is a torsion-free word-hyperbolic group that is in addition sofic and F is a skew-field, then there are no non-trivial idempotents in FG, i.e., the Kaplansky conjecture holds in this situation.

The proof uses controlled algebra to express the assembly map $\alpha_{\mathcal{VC}\mathrm{yc}}$ as a forget-control map, a transfer argument and a mixture of large scale and small scale geometry to "gain control". An important ingredient is the following equivariant version of the fact that word-hyperbolic groups have finite asymptotic dimension.

Theorem. *Let G be a word-hyperbolic group and $\overline{X}$ be the compactification of the Rips complex X of G. Then there is a number $N \in \mathbb{N}$ depending only on G, such that for every $R > 0$ there is a cover $\mathcal{U}_R$ of $G \times \overline{X}$ by open sets with the following properties:*

(1) $\dim \mathcal{U}_R \leq N$;

(2) *for every* $x \in \overline{X}$ *the Lebesgue number of the induced cover of* $G \times \{x\}$ *is at least* R*;*
(3) *for* $U \in \mathcal{U}_R$ *and* $g \in G$ *we have* $g(U) \in \mathcal{U}_R$*, where we consider the diagonal action of* G *on* $G \times \overline{X}$*;*
(4) *for* $U \in \mathcal{U}_R$ *the set* $\{g \mid gU \cap U \neq \emptyset\}$ *is a virtually cyclic subgroup of* G*.*

The proof of this theorem uses Mineyev's contruction of a flow space for hyperbolic groups [3] and a generalization of the long and thin cells of Farrell and Jones [1].

References

[1] F. T. Farrell and L. E. Jones, *K-theory and dynamics I*, Ann. of Math. (2), 124(3):531–569, 1986.
[2] F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc., 6(2):249–297, 1993.
[3] I. Mineyev, *Flows and joins of metric spaces*, Geom. Topol., 9:403–482 (electronic), 2005.

Morita cycles in the homology of $\mathrm{Out}(F_n)$

Karen Vogtmann
(joint work with James Conant)

The abelianization map from a finitely-generated free group F_n to the free abelian group $\mathbb{Z}^n$ induces a map on outer automorphism groups which is an isomorphism onto $GL(n,\mathbb{Z})$ for $n = 2$, but not for any $n > 2$. There are many other representations from finite-index subgroups of $\mathrm{Out}(F_n)$ onto to arithmetic groups, but we know that $\mathrm{Out}(F_n)$ itself is *not* arithmetic for any $n \geq 3$. Although not arithmetic, it shares a large number of properties with arithmetic groups. Many of these features are cohomological, including strong cohomological finiteness properties and (virtual) duality.

In this talk I described three chain complexes which can be used to study the rational homology of $\mathrm{Out}(F_n)$ and gave some indication of how they are related. I then used them to study certain cycles defined by S. Morita in [8] (see also [9]), proving in particular that they are unstable in the strongest possible sense.

The first of the three chain complexes is obtained by considering the action of $\mathrm{Out}(F_n)$ on the spine of Outer space. This spine is a finite-dimensional, contractible cubical complex on which $\mathrm{Out}(F_n)$ acts with finite point stabilizers and compact quotient, so that the quotient is a rational $K(\pi,1)$ for $\mathrm{Out}(F_n)$. The chain complex is the CW-chains on the quotient; the k-chains are indexed by pairs consisting of a graph G and a forest $F \subset G$ with k edges. If we consider basepointed graphs, we get an analogous cube complex whose quotient computes the homology of $\mathrm{Aut}(F_n)$. The Degree theorem of [4] proves that this cube complex contains small invariant subcomplexes A_k which are k-dimensional and $(k-1)$-connected, so act as skeleta for the entire complex. The quotient of A_k by $\mathrm{Aut}(F_n)$ is not only small, making low-dimensional homology calculations possible, but it is independent of n for $n > 5k/4$, proving that the $(k-1)$-st rational homology of $\mathrm{Aut}(F_n)$ is independent of n in this range. Using homotopy-theoretic methods, S. Galatius has recently completely determined the stable rational homology, which turns out to be trivial in all dimensions [3].

The second chain complex is the Chevalley-Eilenberg complex for the Lie algebra of an infinite-dimensional symplectic Lie algebra ℓ_∞ defined by M. Kontsevich in [6, 7]. Kontsevich proved that the homology of ℓ_∞ is basically equal to the direct sum of the homologies of the groups $\mathrm{Out}(F_n)$ for all n. S. Morita recognized the positive part of the finite approximation ℓ_g of ℓ_∞ to be the same as the Lie algebra $\mathfrak{h}_{g,1}$ associated to a surface of genus g with one boundary component; he then used a certain "trace" function which he had defined on $\mathfrak{h}_{g,1}$ to define cocycles on the chain complex [8]. He proved that the first of these cocycles gives a non-trivial cohomology class, which by Kontsevich's theorem corresponds to an element of $H_4(\mathrm{Out}(F_4))$. It was known that $H_4(\mathrm{Out}(F_4)) \cong \mathbb{Q}$ [5], so that this element gives all of the homology in this rank and dimension.

The third chain complex is a graphical interpretation of the subcomplex of symplectic invariants in the Chevalley-Eilenberg complex. Since the symplectic Lie algebra is simple, this subcomplex is quasi-isomorphic to the entire complex. Following Kontsevich, we use Weyl's invariant theory to identify symplectic invariants with pairs consisting of a trivalent graph G and a forest $F \subset G$, where the forest must contain all of the vertices of G. When Morita's cocycles are reinterpreted on the subcomplex of symplectic invariants, they have a very simple description in terms of graphs, and it was shown in [1] that the second cocycle in Morita's series also gives a non-trivial cohomology class, this time corresponding to an element of $H_8(\mathrm{Out}(F_6))$. Recently Ohashi computed that $H_8(\mathrm{Out}(F_6)) \simeq \mathbb{Q}$ [10], so again this element gives all of the homology in this rank and dimension.

The graphical version of Morita's cocycles can be realized back in the original chain complex for the spine of Outer space, where it is obvious that they lift to cycles for the homology of $\mathrm{Aut}(F_n)$. By Galatius' theorem, these cycles must vanish eventually under the stabilization maps $H_k(\mathrm{Aut}(F_n)) \to H_k(\mathrm{Aut}(F_{n+1}))$, but in general they are in dimensions far below the known stable range. We prove that in fact they vanish after a single stabilization, so that they are very unstable indeed. The talk ended with the conjecture that all of the Morita classes are non-trivial; also with speculation that the Morita classes may generate all of the unstable rational homology of $\mathrm{Out}(F_n)$ and that the map $\mathrm{Out}(F_n) \to GL(n, \mathbb{Z})$ described in the first paragraph of this abstract may map Morita classes non-trivially in some dimensions.

References

[1] J. Conant and K. Vogtmann, *Morita classes in the homology of* $\mathrm{Aut}(F_n)$ *vanish after one stabilization.* math.AT/0606510.

[2] J. Conant and K. Vogtmann, *Morita classes in the homology of automorphism groups of free groups*, Geom. Topol., 8 (2004), pp. 1471–1499 (electronic).

[3] S. Galatius. in preparation.

[4] A. Hatcher and K. Vogtmann, *Cerf theory for graphs*, J. London Math. Soc. (2), 58 (1998), pp. 633–655.
[5] ———, *Rational homology of* Aut(F_n), Math. Res. Lett., 5 (1998), pp. 759–780.
[6] M. Kontsevich, *Formal (non)commutative symplectic geometry.* The Gelfand Mathematical Seminars, 1990–1992, 173–187, Birkhäuser Boston, Boston, MA, 1993.
[7] M. Kontsevich, *Feynman diagrams and low-dimensional topology.* First European Congress of Mathematics, Vol. II (Paris, 1992), 97–121, Progr. Math., 120, Birkhï¿user, Basel, 1994.
[8] ———, *Structure of the mapping class groups of surfaces: a survey and a prospect*, in Proceedings of the Kirbyfest (Berkeley, CA, 1998), vol. 2 of Geom. Topol. Monogr., Geom. Topol. Publ., Coventry, 1999, pp. 349–406 (electronic).
[9] S. Morita, *Cohomological structure of the mapping class group and beyond*, 2005, arXiv:math.GT/0507308.
[10] R. Ohashi, in preparation.

Positive scalar curvature with symmetry

Bernhard Hanke

We develop equivariant analogues of the construction techniques introduced by Gromov-Lawson and Schoen-Yau for positive scalar curvature metrics. Part of the nonequivariant discussion can be translated more or less directly to the equivariant context. As shown in [1], this applies in particular to the surgery principle [2], which states that the class of smooth manifolds admitting metrics of positive scalar curvature is closed under surgery of codimension at least 3. However, the following equivariant bordism principle requires a refined argument because it is based on handle cancelation techniques that in general do not carry over to the equivariant world (which is illustrated by the failure of equivariant analogous of the h- and s-cobordism theorems):

Theorem A. *Let G be a compact Lie group and let Z be a compact connected oriented G-bordism (with an orientation preserving G-action) between the closed G-manifolds X and Y. Assume the following:*

i. *The cohomogeneity of Z is at least* 6*,*
ii. *the inclusion of maximal orbits $Y_{\max} \hookrightarrow Z_{\max}$ is a (nonequivariant)* 2*-equivalence (i.e., a bijection on π_0, an isomorphism on π_1 and a surjection on π_2),*
iii. *each singular stratum of codimension* 2 *in Z meets Y.*

Then, if X admits a G-invariant metric of positive scalar curvature, the same is true for Y.

We remark that by a classical result of Lawson-Yau [3], closed connected effective G-manifolds admit G-invariant metrics of positive scalar curvature if the identity component of G is non-abelian. Hence, Theorem A is useful mainly for finite or for toral G.

Our Theorem A is almost a direct analogue of the corresponding nonequivariant result [6, Theorem 3.3]. In particular, the dimension restriction i. and the connectivity restriction for the inclusion $Y_{\max} \hookrightarrow Z_{\max}$ stated in point ii. translate to analogous requirements in the nonequivariant setting if $G = \{1\}$. However, if G is not trivial, we need an additional assumption on codimension-2 singular strata.

Theorem A is useful for constructing equivariant metrics of positive scalar curvature only if it can be combined with powerful structure results for geometric equivariant bordism groups implying that the manifold X in Theorem A can be assumed to admit an equivariant positive scalar curvature metric under general assumptions on the manifold Y. Two main difficulties occur at this point. Firstly, explicit geometric generators of equivariant bordism groups are known only in a very limited number of cases. Secondly, whereas conditions i. and ii. in Theorem A can be achieved under fairly general assumptions on the manifold Y (by performing appropriate surgeries on $Z_{\max}$), it is a priori not clear under what circumstances condition iii. holds.

We present a solution to the last mentioned problem if $G = S^1$ and the G-action on Z is fixed point free. The idea we use is to alter a given bordism Z by cutting out equivariant tubes connecting Y with each of the codimension-2 singular strata in Z that are disjoint from Y. This replaces the bordism Z and the manifold Y by other manifolds Z' and Y' so that each codimension-2 singular stratum in Z' meets Y'. In particular, Theorem A can be applied to Z' (after some more manipulations of Z', but we omit these details here). We must now understand how Y can be recovered from Y'. A closer inspection of the situation shows that Y' is obtained from Y by adding certain codimension-2 singular strata with finite isotropies. Conversely, Y can be reconstructed from Y' by a kind of codimension-2 surgery process that removes these additional singular strata and puts back free ones instead. We show by a somewhat involved geometric argument that this surgery step preserves the existence of S^1-invariant positive scalar curvature metrics under fairly general assumptions. Roughly speaking, we replace the "bending outwards" process in the surgery step due to Gromov-Lawson and Schoen-Yau by a "bending inwards" process. We remark that this kind of positive scalar curvature preserving codimension-2 surgery only works under the additional S^1-symmetry on Z.

With the help of this surgery method, we conclude that the original manifold Y admits an invariant metric of positive scalar curvature if the manipulated one Y' admits such a metric. Arguing in this rather roundabout manner, assumption iii. of Theorem A is no longer a true obstacle against the construction of equivariant positive scalar curvature metrics on fixed point free S^1-manifolds. This insight is now combined with a classical theorem of Ossa [4], which states that fixed point free S^1-manifolds (satisfying the additional technical hypothesis formulated in the next theorem) are S^1-boundaries, to complete the proof of the following result, an equivariant version of the well known existence result of positive scalar curvature metrics on closed simply connected non-spin manifolds of dimension at least 5 due to Gromow and Lawson [2]:

Theorem B. *Let M be a connected closed oriented fixed point free S^1-manifold so that all normal bundles around H-fixed components ($H \subset S^1$ being a closed subgroup) in M are complex S^1-bundles. If the dimension of M is at least 6,* *the union of maximal orbits of M is simply connected and does not admit a spin structure, then M admits an S^1-invariant metric of positive scalar curvature.*

We remark that no additional assumption on codimension-2 singular strata in M is needed. It is not clear at present to what extent Ossa's theorem can be generalized to the spin case so that we leave the discussion of a corresponding S^1-equivariant analogue of Stolz' theorem [5] for later consideration.

References

[1] L. Bérard Bergery, *Scalar curvature and isometry group*, in: *Spectra of Riemannian manifolds*, Kaigai Publications, Tokyo (1983), 9–28.
[2] M. Gromov, H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. Math. **111** (1980), 209–230.
[3] H. B. Lawson, S. T. Yau, *Scalar curvature, non-abelian group actions and the degree of symmetry of exotic spheres*, Comment. Math. Helv. **49** (1974), 232-244.
[4] E. Ossa, *Fixpunktfreie S^1-Aktionen*, Math. Ann. **186** (1970), 45–52.
[5] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. Math. **136** (1992), 511–540.
[6] S. Stolz, *Positive scalar curvature metrics – existence and classification questions*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2, Zürich (1994), 625–636.

The functor category $\mathcal{F}_{\text{quad}}$

Christine Vespa

1. Introduction

In recent years, one of the categories of functors which has been studied is the category $\mathcal{F}(p)$ of functors from the category $\mathcal{E}^f$ of finite dimensional $\mathbb{F}_p$-vector spaces to the category $\mathcal{E}$ of all $\mathbb{F}_p$-vector spaces, where $\mathbb{F}_p$ is the prime field with p elements. One algebraic motivation for the special interest in the category $\mathcal{F}(p)$ follows from the link with the stable cohomology of general linear groups. In fact, Dwyer proved in 1980 [3] that, for any finite objects P and Q of $\mathcal{F}(p)$ (i.e., P and Q admit finite composition series), the following inverse system

$$\cdots \to H^k(GL_{n+1}(\mathbb{F}_p), M_{n+1}) \to H^k(GL_n(\mathbb{F}_p), M_n) \to \cdots,$$

where $M_n = \operatorname{Hom}_{\mathcal{E}}(P(\mathbb{F}_p^n), Q(\mathbb{F}_p^n))$, stabilizes. In 1999, Suslin proved in the appendix of [4] and, independently, Betley showed [2] that there exists an isomorphism between the stable value of the previous system and the extension groups $\operatorname{Ext}^k_{\mathcal{F}(p)}(P, Q)$. Whereas the stable cohomology groups are not readily accessible by direct computation, this theorem reduces their determination to homological algebra calculations in the category $\mathcal{F}(p)$ where we have powerful computational tools.

Then, it is natural to seek to construct categories of functors for other families of algebraic groups and, in particular, the orthogonal groups.

This report presents the construction of a category of functors $\mathcal{F}_{\text{quad}}$ which has some good properties as a candidate for the orthogonal group over the field with two elements and to give several results about the structure of this category. Henceforth, we will denote the category $\mathcal{F}(2)$ by $\mathcal{F}$.

We refer the interested reader to [5, 6, 7, 8] for details.

2. Definition of the category $\mathcal{F}_{\text{quad}}$

Let $\mathcal{E}_q$ be the category having as objects finite dimensional $\mathbb{F}_2$-vector spaces equipped with a non-degenerate quadratic form and with morphisms linear maps which preserve the quadratic forms. Observe that all morphisms of $\mathcal{E}_q$ are injective linear maps; however, the constructions which relate the category $\mathcal{F}$ and the stable cohomology of general linear groups use, in an essential way, the existence of retractions in the category $\mathcal{E}^f$. As a consequence, to consider similar constructions in the category $\mathcal{F}_{\text{quad}}$, we have to add orthogonal projections formally to $\mathcal{E}_q$. For this, we imitate the construction of the coSpan-category $\text{coSp}(\mathcal{C})$ of a category $\mathcal{C}$ equipped with pushouts introduced by Bénabou [1]. In this construction, the pushout is used to define the composition. However, the category $\mathcal{E}_q$ does not admit pushouts. To resolve this difficulty, we define the notion of a pseudo-pushout in $\mathcal{E}_q$.

Definition. Let $f : V \to W \simeq V \perp V'$ and $g : V \to X \simeq V \perp V''$ be morphisms of $\mathcal{E}_q$. The pseudo-pushout of f and g is the object, unique up to isometry, $X \underset{V}{\perp} W \simeq V \perp V' \perp V''$ in $\mathcal{E}_q$.

This definition uses the non-degeneracy of the spaces in an essential way and allows the construction of a category $\mathcal{T}_q$ which generalises the category of Bénabou.

Definition. The category $\mathcal{T}_q$ is the category having as objects those of $\mathcal{E}_q$ and, for V and W objects of $\mathcal{T}_q$, $\text{Hom}_{\mathcal{T}_q}(V, W)$ is the set of equivalence classes of diagrams in $\mathcal{E}_q$ of the form $V \xrightarrow{f} X \xleftarrow{g} W$ for the equivalence relation generated by the following relation $\mathcal{R}$: $V \xrightarrow{f} X_1 \xleftarrow{g} W \quad \mathcal{R} \quad V \xrightarrow{u} X_2 \xleftarrow{v} W$ if there exists a morphism α in $\mathcal{E}_q$ such that $\alpha \circ f = u$ and $\alpha \circ g = v$. The composition is given by the pseudo-pushout.

For a morphism $f : V \to W$ of $\mathcal{E}_q$, the morphism of $\mathcal{T}_q$ represented by the diagram: $W \xrightarrow{\text{Id}} W \xleftarrow{f} V$ is a retraction of that represented by the diagram: $V \xrightarrow{f} W \xleftarrow{\text{Id}} W$.

Definition. The category $\mathcal{F}_{\text{quad}}$ is the category of functors from $\mathcal{T}_q$ to $\mathcal{E}$.

The category $\mathcal{F}_{\text{quad}}$ is abelian and has enough injective and projective objects. The Yoneda lemma gives a set of projective generators of $\mathcal{F}_{\text{quad}}$ indexed by a set $\mathcal{R}$ of representatives of isometry classes of non degenerate quadratic spaces. These projective objects will be denoted P_V, for an element V of $\mathcal{R}$.

3. Some results about the simple objects of $\mathcal{F}_{\text{quad}}$

A first family of simple objects of $\mathcal{F}_{\text{quad}}$ is obtained thanks to the following theorem, which relates the category $\mathcal{F}_{\text{quad}}$ to the category $\mathcal{F}$.

Theorem. *There exists a fully-faithful, exact functor* $\iota : \mathcal{F} \hookrightarrow \mathcal{F}_{\text{quad}}$ *which preserves the simple objects.*

To define the subcategory $\mathcal{F}_{\text{iso}}$ of $\mathcal{F}_{\text{quad}}$, we need to introduce the category $\mathcal{E}_q d$ having as objects finite dimensional $\mathbb{F}_2$-vector spaces equipped with a (possibly degenerate) quadratic form and with morphisms, injective linear maps which preserve the quadratic forms. Since $\mathcal{E}_q^{\text{deg}}$ admits pullbacks, the category of spans $\text{Sp}(\mathcal{E}_q^{\text{deg}})$ is defined, where the Span-category $\text{Sp}(\mathcal{C})$ of a category $\mathcal{C}$ equipped with pullbacks is the dual construction of the coSpan-category. The category $\mathcal{F}_{\text{iso}}$ is defined to be the category of functors from $\text{Sp}(\mathcal{E}_q^{\text{deg}})$ to $\mathcal{E}$. The category $\mathcal{F}_{\text{iso}}$ is related to the category $\mathcal{F}_{\text{quad}}$ by the following theorem.

Theorem. *There exists a fully-faithful, exact functor* $\kappa : \mathcal{F}_{\text{iso}} \to \mathcal{F}_{\text{quad}}$ *which preserves the simple objects.*

We obtain the following classification of the simple objects of the category $\mathcal{F}_{\text{iso}}$.

Theorem. *There is a natural equivalence* $\mathcal{F}_{\text{iso}} \simeq \prod_{V\in\mathcal{S}} \mathbb{F}_2[O(V)]-\text{mod}$ *of categories; where* $\mathcal{S}$ *is a set of representatives of isometry classes of quadratic spaces (possibly degenerate) and* $O(V)$ *is the orthogonal group associated to the quadratic space* V.

The object of $\mathcal{F}_{\text{iso}}$ which corresponds, via this equivalence, to the module $\mathbb{F}_2[O(V)]$ is denoted by Iso_V and will be called the *isotropic functor* associated to the quadratic space V.

To exhibit the existence of other simple objects in the category $\mathcal{F}_{\text{quad}}$ we decompose certain projective generators. We denote by H_0 (resp. H_1) the non-degenerate quadratic space of dimension two such that the Arf invariant is equal to 0 (resp. to 1). We obtain the following decompositions of the projective objects P_{H_0} and P_{H_1}, in which new functors $\text{Mix}_{0,1}$ and $\text{Mix}_{1,1}$ of $\mathcal{F}_{\text{quad}}$, called *mixed functors*, appear and where Q is the projective object of $\mathcal{F}$ given by the Yoneda lemma and associated to the vector space $\mathbb{F}_2^{\oplus 2}$.

Theorem. *We have the following direct sum decompositions:*

$$P_{H_0} = \iota(Q) \oplus (\text{Mix}_{0,1}^{\oplus 2} \oplus \text{Mix}_{1,1}) \oplus \kappa(\text{Iso}_{H_0}); P_{H_1} = \iota(Q) \oplus \text{Mix}_{1,1}^{\oplus 3} \oplus \kappa(\text{Iso}_{H_1}).$$

For $\varepsilon \in \{0,1\}$, the functor $\text{Mix}_{\varepsilon,1}$ is neither an object of $\mathcal{F}$, nor an object of $\mathcal{F}_{\text{iso}}$. The functor $\text{Mix}_{\varepsilon,1}$ is isomorphic to a subfunctor of $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\text{Iso}_{(x,\varepsilon)})$ where $\text{Iso}_{(x,\varepsilon)}$ is the isotropic functor associated to the degenerate quadratic space of dimension one, generated by x and such that $q(x) = \varepsilon$.

The decompositions of the projective objects P_{H_0} and P_{H_1} into indecomposable summands give rise to a complete classification of the simple functors of $\mathcal{F}_{\text{quad}}$ which are non-zero on at least one of the spaces H_0 or H_1. The definition of polynomial functor extends to $\mathcal{F}_{\text{quad}}$. An important application of the previous classification is the description of the polynomial functors of the category $\mathcal{F}_{\text{quad}}$.

Theorem. *The polynomial functors of* $\mathcal{F}_{\text{quad}}$ *are in the image of the functor* ι.

4. Open questions

(1) Does the analogue, for the orthogonal groups over $\mathbb{F}_2$, of the system described in the introduction stabilize?
(2) What is the analogue of the Betley-Suslin theorem for the category $\mathcal{F}_{\text{quad}}$?

References

[1] J. Bénabou, *Introduction to bicategories*, Reports of the Midwest Category Seminar (1967), 1-77.
[2] S. Betley, *Stable K-theory of finite fields*, K-Theory 17 (1999), 103-111.
[3] W.G. Dwyer, *Twisted homological stability for general linear groups*, Ann. of Math. 111 (1980), 239-251.
[4] V. Franjou, E.M. Friedlander, A. Scorichenko, A. Suslin, *General linear and functor cohomology over finite fields*, Ann. of Math. 150 (1999), 663-728.
[5] C. Vespa, *La catégorie de foncteurs $\mathcal{F}_{\text{quad}}$ sur les espaces quadratiques sur* $\mathbb{F}_2$, C. R. Acad. Sci. Paris, Ser. I 343 (2006).
[6] C. Vespa, *The functor category $\mathcal{F}_{quad}$*, arXiv:math.AT/0606484 (2006).
[7] C. Vespa, *The mixed functors of the category $\mathcal{F}_{quad}$: a first study*, in preparation.
[8] C. Vespa, *On projective functors in the category $\mathcal{F}_{quad}$*, in preparation.

Positive scalar curvature and the Novikov conjecture

Thomas Schick
(joint work with Bernhard Hanke, Dieter Kotschick, John Roe)

Let M be a closed smooth spin manifold of dimension m. There are important obstructions to the existence of a Riemannian metric on M with positive scalar curvature, based on the Dirac operator. Particularly powerful among them is the K-theoretic index in the (real) K-theory of the (real) reduced C^*-algebra of the fundamental group, $\alpha_{\text{red}}(M) \in \text{KO}_m(C^*_{\text{red}}\pi_1 M)$, a certain completion of the group ring $\mathbb{R}\pi_1 M$. Closely related is $\alpha_{\max}(M) \in \text{KO}_m(C^*_{\max}\pi_1 M)$, mapped to $\alpha_{\text{red}}(M)$ under the canonical homomorphism, both constructed in [10, 8, 9].

However, work of Gromov-Lawson [4, 3] provides other invariants which also use the index of Dirac operators, e.g., based on enlargeability, to be explained below.

Question. How much information is contained in $\alpha_{\text{red}}(M)$ (or $\alpha_{\max}(M)$)?

Answer. Stephan Stolz [12] proves that the *strong Novikov conjecture* (i.e., the injectivity of the Baum-Connes map $\text{KO}_*(B\pi_1 M) \to \text{KO}_*(C^*_{\text{red}}\pi_1 M)$ for torsion-free $\pi_1 M$) implies the stable Gromov-Lawson-Rosenberg conjecture: $0 = \alpha_{\text{red}}(M)$ if and only if the product of M with a sufficiently high power of the Bott-manifold admits a metric of positive scalar curvature. This implies that, as long as $\pi_1(M)$ satisfies the strong Novikov conjecture, $\alpha_{\text{red}}(M)$ contains *all* index theoretic obstructions to positive scalar curvature on M.

(Note however, that the counterexamples to the unstable Gromov-Lawson-Rosenberg conjecture of [11, 1] show that additional, non index-theoretic, obstructions, exist as well.)

The goal of the talk is to relate $\alpha_{\mathrm{red}}(M)$ to enlargeability without assuming the Novikov conjecture. We need the following definition:

Definition. A manifold M as above is called *enlargeable* if for every $\varepsilon > 0$ there is a connected covering $M_\varepsilon \to M$ and a map $f_\varepsilon \colon M_\varepsilon \to S^m$ with the following properties:

(1) f_ε is constant outside a compact subset of M_ε.
(2) The degree of f_ε is not zero.
(3) f_ε is ε-Lipschitz (where we use the lift of a fixed Riemannian metric on M, and the standard metric on S^m).

Examples of enlargeable manifolds are manifolds with non-positive sectional curvature.

Gromov and Lawson [4] introduced enlargeability and proved that enlargeable spin manifolds do not admit a metric of positive scalar curvature, by producing bundles of very small curvature such that the twisted Dirac operator has non-trivial index.

Theorem. (1) *(Hanke-S.* [5, 6]*): If M is an enlargeable spin manifold, then $0 \neq \alpha_{\max}(M) \in K_m(C^*_{\max}\pi_1 M)$.*
(2) *(Hanke-Kotschick-Roe-S.): If M is an enlargeable spin manifold, then $0 \neq \alpha_{\mathrm{red}}(M) \in K_m(C^*_{\mathrm{red}}\pi_1 M)$.*

Of course, the second result implies the first one. However, the first easily generalizes to other situations, where bundles with small curvature are used to produce obstructions to positive scalar curvature, e.g., if M is only area-enlargeable. The proof of the second result, on the other hand, makes essential use of the special geometric situation given by enlargeability. It uses methods from coarse geometry [7].

We actually derive (along the way) that enlargeability obstructs positive scalar curvature on the universal covering of M for any metric coarsely equivalent to a $\pi_1(M)$-invariant metric.

Note that the result implies the Novikov conjecture, provided we can represent every class of $\mathrm{KO}_*(B\Gamma)$ by fundamental classes of enlargeable spin manifolds with fundamental group Γ.

Question. (1) Gromov's notion of infinite (stable) K-area [2] is a particularly efficient way to obstruct positive scalar curvature, based on the index of the Dirac operator twisted with almost flat bundles. Does this imply $\alpha_{\max}(M) \neq 0$? It is quite likely that one can prove this rather easily using the methods of [6]. A problem is that Gromov treats stable K-area using the family index theorem; for this, a suitable connection to $\alpha_{\max}$ has not yet been worked out.
(2) Does this (at least in special cases like for area-enlargeability) carry over to $\alpha_{\mathrm{red}}(M)$? This seems to be completely open at the moment.
(3) How much of the Novikov conjecture follows from the Theorem, i.e., how much of $\mathrm{KO}_*(B\Gamma)$ can be represented by fundamental classes of enlargeable spin manifolds with fundamental group Γ?

(4) What is the relation of $\alpha_{\mathrm{red}}(M)$ or of $\alpha_{\max}(M)$ to the codimension 2 obstructions invented by Gromov-Lawson in [4] (particularly powerful for 3-dimensional aspherical manifolds)?

References

[1] William Dwyer, Thomas Schick, and Stephan Stolz. Remarks on a conjecture of Gromov and Lawson. In *High-dimensional manifold topology*, pages 159–176. World Sci. Publishing, River Edge, NJ, 2003.

[2] M. Gromov. Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In *Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993)*, volume 132 of *Progr. Math.*, pages 1–213. Birkhäuser Boston, Boston, MA, 1996.

[3] Mikhael Gromov and H. Blaine Lawson, Jr. Spin and scalar curvature in the presence of a fundamental group. I. *Ann. of Math. (2)*, 111(2):209–230, 1980.

[4] Mikhael Gromov and H. Blaine Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.

[5] Bernhard Hanke and Thomas Schick. Enlargeability and index theory. *J. Differential Geometry*, 74(2):293–320, 2006. arXiv:math.GT/0403256.

[6] Bernhard Hanke and Thomas Schick. Enlargeability and index theory: infinite covers. arXiv:math.GT/0604540, to appear in K-theory, 2006.

[7] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[8] Jonathan Rosenberg. C^*-algebras, positive scalar curvature and the Novikov conjecture. II. In *Geometric methods in operator algebras (Kyoto, 1983)*, volume 123 of *Pitman Res. Notes Math. Ser.*, pages 341–374. Longman Sci. Tech., Harlow, 1986.

[9] Jonathan Rosenberg. C^*-algebras, positive scalar curvature, and the Novikov conjecture. *Inst. Hautes Études Sci. Publ. Math.*, (58):197–212 (1984), 1983.

[10] Jonathan Rosenberg. C^*-algebras, positive scalar curvature, and the Novikov conjecture. III. *Topology*, 25(3):319–336, 1986.

[11] Thomas Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, 37(6):1165–1168, 1998.

[12] Stephan Stolz. Manifolds of positive scalar curvature. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, pages 661–709. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

Symmetric powers of spheres

Neil P. Strickland

(joint work with Johann Sigurdsson)

This is a preliminary report on a project to understand, extend and consolidate a dense network of connections between various phenomena in stable homotopy theory. In this abstract we map out certain parts of this network.

One way into the maze is to consider symmetric powers of spheres. Given a finite dimensional vector space V, we have a sphere $S^V = V \cup \{\infty\}$. We can then form the n'th unstable symmetric power $(S^V)^n/\Sigma_n$. This is the V'th space in an orthogonal prespectrum, which we call $\mathrm{SP}^n(S^0)$. By the Dold-Thom theorem [4], the colimit $\mathrm{SP}^\infty(S^0) = \lim_{\to n} \mathrm{SP}^n(S^0)$ is just the integer Eilenberg-MacLane spectrum H. This gives an interesting filtration of H, whose quotient spectra $\overline{\mathrm{SP}}^n(S^0) = \mathrm{SP}^n(S^0)/\mathrm{SP}^{n-1}(S^0)$ have a variety of different descriptions.

One theme of our work is to try to provide uniform combinatorial models for as many spaces as possible. For the $\mathrm{SP}^n(S^0)$ themselves, the best we can do is as follows. Let $\mathcal{F}$ be the category of finite sets and bijections. This is symmetric monoidal under the disjoint union, so it has a K-theory spectrum $K(\mathcal{F})$, which is well-known to be equivalent to S^0. Next, a *multiset* means a finite set A in which each element a is assigned a multiplicty $\nu(a) > 0$. A morphism from (A, ν) to (B, μ) is a function $f\colon A \to B$ that is bijective up to multiplicity, in the sense that $\sum_{f(a)=b} \nu(a) = \mu(b)$. We write $\mathcal{M}$ for the category of multisets, and $\mathcal{M}_k$ for the subcategory where each point has multiplicity at most k. A theorem of Kathryn Lesh [10] then shows that $K(\mathcal{M}_k) = \mathrm{SP}^k(S^0)$, whereas $K(\mathcal{M}) = \mathrm{SP}^\infty(S^0) = H$.

Using Lesh's theorem and its proof, we find that $\overline{\mathrm{SP}}^n(S^0)$ is the unreduced suspension of $B(\mathcal{M}_n^{n-1})$, where $\mathcal{M}_n^{n-1}$ is the category of multisets of total multiplicity n and maximum multiplicity less than n. We also have $B(\mathcal{M}_n^{n-1}) \simeq S(\infty W_n)/\Sigma_n$, where $W_n = \{x \in \mathbf{R}^n \mid \sum_i x_i = 0\}$ and $S(\cdot)$ denotes the unit sphere. For another description, let $\mathcal{P}(A)$ denote the partially ordered set of partitions of a finite set A. This has a smallest element $\bot$ (the partition as a single block) and a largest element $\top$ (the partition into blocks of size 1). This implies that the geometric realisation $P(A) = |\mathcal{P}(A)|$ is contractible. However, we can define a subcomplex $\dot{P}(A)$ as follows. The simplices in $P(A)$ are indexed by chains $\sigma = \{\pi_0 < \pi_1 < \cdots < \pi_d\}$ in $\mathcal{P}(A)$, and we let $\dot{P}(A)$ denoe the union of those simplices σ for which $\sigma \not\supseteq \{\bot, \top\}$. We then put $\widehat{P}(A) = P(A)/\dot{P}(A)$. We also write $WA = \{x\colon A \to \mathbf{R} \mid \sum_a x(a) = 0\}$, and note that Σ_A acts on the (based) space $S^{WA} \wedge \widehat{P}(A)$, so we can form the homotopy orbit space $(S^{WA} \wedge \widehat{P}(A))_{h\Sigma_A}$. A theorem of Arone and Dwyer [1] tells us that this is equivalent to $\overline{\mathrm{SP}}^n(S^0)$, where $n = |A|$.

This allows us to make contact with a number of other interesting ideas. Firstly, it is known that the space $\widehat{P}(A)$ is a wedge of spheres, with only one nontrivial homology group, which is closely related to the operad for Lie algebras. There are some issues about grading, signs and duality here. The nicest way to package them is to consider instead the spectra $Q(A) = F(\widehat{P}(A), S^{WA})$. The theorem is then that $H_*Q(A) = \mathrm{Lie}(A)$, concentrated in degree zero. Here $\mathrm{Lie}(A)$ is the space of Lie words in variables $\{x_a \mid a \in A\}$ that involve each variable exactly once, or in other words, the A'th space in the operad for Lie algebras. This suggests that the spectra $Q(A)$ should also form an operad. This is in fact the case, as follows from the work of Michael Ching [3]. One way to see this is to show that the contractible spaces $P(A)$ form an operad, and that the operad structure maps are open embeddings away from $\dot{P}(A)$. We can thus do a Pontrjagin-Thom collapse to make the based spaces $\widehat{P}(A)$ into a cooperad. The spaces S^{WA} also form an operad in a natural way, and by combining these structures we make Q

into an operad. To get the operad structure on $P(A)$, one can show that $P(A)$ is homeomorphic to a certain space of rooted trees, where the leaves are labelled by the elements of A, each internal edge is assigned a length, and the distance from the root to any leaf is equal to one. The operad composition is then given by a kind of grafting. Alternatively, one can show that $P(A)$ is homeomorphic to the space of maps $h\colon C(A) \to [0,1]$ (where $C(A)$ is the set of nonempty subsets of A) such that $h(\{a\}) = 0$ and $h(U \cup V) = \max(h(U), h(V))$ whenever $U \cap V \neq \emptyset$. (This avoids the slightly fiddly equivalence relations implicit in the tree description.)

The spectra $Q(A)$ also appear in Goodwillie calculus. (This is the reformulation by Arone and Mahowald [2] of a result of Brenda Johnson [6].) The upshot is that for any space X, there is a natural tower of spaces

$$0 = X_0 \leftarrow \Omega^\infty \Sigma^\infty X = X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

with inverse limit X, such that the fibre of $X_n \to X_{n-1}$ is $\Omega^\infty(F(\widehat{P}(n), X^{(n)})_{h\Sigma_n})$. In particular, when $X = S^1$ we have $X_n = \Omega^\infty \Sigma Q(n)$. We have still not fully understood the relationship between this appearance of $Q(A)$ in Goodwillie calculus and its previous appearance (in a slightly different form) in the symmetric power filtration. One of our main tasks is to elucidate this. In both appearances, the rôle of the operad structure is mysterious.

So far we have worked integrally, but many more interesting phenomena appear if we localise at a prime p (which we do implicitly from now on). Firstly, it is known that $\overline{\mathrm{SP}}^n(S^0) = 0$ unless n is a power of p, so we need only consider the spectra $H(k) = \mathrm{SP}^{p^k}(S^0)$. These filter the integer Eilenberg-MacLane spectrum H, and there is an analogous filtration of the mod p Eilenberg-MacLane spectrum $\overline{H}$ by subspectra $\overline{H}(k)$. It is known that the resulting filtration of the Steenrod algebra $\overline{H}^*\overline{H}$ is by the length of admissible monomials. There is some interesting algebra related to the dual filtration of $\overline{H}_*\overline{H}$, which we have not yet fully understood. It was shown by Arone and Dwyer that the spectra $L(k) = \Sigma^{-k} H(k)/H(k-1) = \Sigma^{-k}\overline{\mathrm{SP}}^{p^k}(S^0)$ can be described in terms of the poset of subgroups of $(\mathbf{Z}/p)^k$ and thus in terms of the Steinberg module and Hecke algebra for $\mathrm{GL}_k(\mathbf{Z}/p)$. This makes contact with various other applications of the Steinberg module in topology, notably in various papers of Mitchell, Priddy and/or Kuhn (see [11, 12, 7], for example). As the spectra $H(k)$ give a multiplicative filtration of the ring spectrum H, we find that the quotients $L(*)$ form a differential graded ring object in the homotopy category of spectra, so the spaces $\Omega^\infty L(*)$ give a differential graded ring in the homotopy category of spaces. It is known that this object is chain homotopy equivalent to $\mathbf{Z}$ (considered as a discrete ring in grading zero); this was conjectured by Whitehead, and later proved by Kuhn and Priddy [9].

Now let $K(n)$ denote Morava K-theory. One can show that the chain complex $K(n)_*L(*)$ is contractible, either by a direct argument, or by applying the Bousfield-Kuhn functor [8] to the Whitehead conjecture. Moreover, one can show that the groups $K(n)_*L(k)$ are built from the Steinberg modules for elementary abelian subgroups of the formal group for $K(n)$. Similar descriptions can be given for many other parts of the theory, either in Morava K-theory itself, or in the approximation given by the Hopkins-Kuhn-Ravenel generalised character theory [5]. This in turn leads to many connections with the study of power operations in Morava E-theory.

References

[1] G.Z. Arone, W.G. Dwyer, *Partition complexes, Tits buildings and symmetric products*, Proc. London Math. Soc. (3) **82** (2001), no. 1, 229–256.

[2] G. Arone, M. Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743–788.

[3] M. Ching, *Bar constructions for topological operads and the Goodwillie derivatives of the identity*, Geom. Topl. **9** (2005), 833–933 (electronic).

[4] A. Dold, R. Thom, *Quasifaserungen und unendliche symmetrische produkte*, Ann. of Math. **67** (1958), 239–281.

[5] M.J. Hopkins, N.J. Kuhn, D.C. Ravenel, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Soc. **13** (2000), no. 3, 553–594 (electronic).

[6] B. Johnson, *The derivatives of homotopy theory*, Trans. Amer. Math. Soc. **347** (1995), no. 4, 1295–1321.

[7] N.J. Kuhn, *The modular Hecke algebra and Steinberg representation of finite Chevalley groups*, J. Algebra **91** (1984), no. 1, 125–141. With an appendix by Peter Landrock.

[8] N.J. Kuhn, *Morava K-theories and infinite loop spaces*, Algebraic topology (Arcata, CA, 1986), Lecture Notes in Mathematics, vol. 1370, Springer, Berlin (1989), 243–257.

[9] N.J. Kuhn, S.B. Priddy, *The transfer and Whitehead's conjecture*, Math. Proc. Cambridge Philos. Soc. **98** (1985), no. 3, 459–480.

[10] K. Lesh, *A filtration of spectra arising from families of subgroups of symmetric groups*, Trans. Amer. Math. Soc. **352** (2000), no. 7, 3211–3237.

[11] S.A. Mitchell, *Finite complexes with $A(n)$-free cohomology*, Topology **24** (1985), no. 2, 227–246.

[12] S.A. Mitchell, S.B. Priddy, *Stable splittings derived from the Steinberg module*, Topology **22** (1983), no. 3, 285–298.

Reporter: Clara Löh

Participants

Noe Barcenas Torres
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Arthur Bartels
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Stefan Alois Bauer
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld

**Prof. Dr.
Carl-Friedrich Bödigheimer**
Mathematisches Institut
Universität Bonn
Beringstr. 6
53115 Bonn

Prof. Dr. Michel Boileau
Mathematiques
Laboratoire Topologie et Geometrie
Universite Paul Sabatier
118 route de Narbonne
F-31062 Toulouse Cedex

Prof. Dr. Martin R. Bridson
Department of Mathematics
Imperial College London
Huxley Building
GB-London SW7 2AZ

Prof. Dr. Carles Broto
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus UAB
E-08193 Bellaterra

Prof. Dr. Natalia Castellana
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus UAB
E-08193 Bellaterra

Christophe Cazanave
Mathematiques
Universite Paris Nord
Centre Scientifiques et Polytechn.
Av. J. B. Clement
F-93430 Villetaneuse

Prof. Dr. James F. Davis
Dept. of Mathematics
Indiana University
Bloomington, IN 47405
USA

Prof. Dr. Michael W. Davis
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus, OH 43210-1174
USA

Prof. Dr. Anand N. Dessai
Fachbereich 10 - Mathematik und
Informatik
Universität Münster
Einsteinstr. 62
48149 Münster

Aurelien Djament
Mathematiques
LAGA - CNRS:UMR 7539
Universite Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
F-93430 Villetaneuse

Dr. Vivien Easson
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Prof. Dr. Cameron M. Gordon
Department of Mathematics
University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
USA

Prof. Dr. Jesper Grodal
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
USA

Dr. Joachim Grunewald
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Bernhard Hanke
Mathematisches Institut
Universität München
Theresienstr. 39
80333 München

Prof. Dr. Hans-Werner Henn
U.F.R. de Mathematique et
d'Informatique
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg -Cedex

Dr. Andre Henriques
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Kathryn Hess
Ecole Polytechnique Federale de
Lausanne
SB IGAT GR - HE
BCH 5103
CH-1015 Lausanne

Prof. Dr. Björn Jahren
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
N-0316 Oslo

Ramon Jimenez-Esparza
I.R.M.A.
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg -Cedex

Dr. Michael Joachim
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Klaus Johannson
Fachbereich Mathematik
Universität Frankfurt
Postfach 111932
60054 Frankfurt am Main

Prof. Dr. Marc Lackenby
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Gerd Laures
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum

Prof. Dr. Ran Levi
Department of Mathematics
University of Aberdeen
GB-Aberdeen AB24 3UE

Clara Löh
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Wolfgang Lück
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Walter David Neumann
Dept. of Mathematics
Barnard College
Columbia University
New York, NY 10027
USA

Prof. Dr. Dietrich Notbohm
Department of Mathematics
University of Leicester
University Road
GB-Leicester, LE1 7RH

Prof. Dr. Robert Oliver
Departement de Mathematiques
Institut Galilee
Universite Paris XIII
99 Av. J.-B. Clement
F-93430 Villetaneuse

Prof. Dr. Geoffrey M.L. Powell
Mathematiques
LAGA - CNRS:UMR 7539
Universite Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
F-93430 Villetaneuse

Prof. Dr. Andrew A. Ranicki
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings,Mayfield Road
GB-Edinburgh, EH9 3JZ

Holger Reeker
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum

Prof. Dr. Holger Reich
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Birgit Richter
Department Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg

Prof. Dr. Albert Ruiz
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus UAB
E-08193 Bellaterra

Prof. Dr. Jerome Scherer
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus UAB
E-08193 Bellaterra

Prof. Dr. Thomas Schick
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Dr. Dirk Schütz
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham, DH1 3LE

Dr. Björn Schuster
FB C: Mathematik u. Naturwissensch.
Bergische Universität Wuppertal
42097 Wuppertal

Prof. Dr. Wilhelm Singhof
Mathematisches Institut
Heinrich-Heine-Universität
Gebäude 25.22
Universitätsstraße 1
40225 Düsseldorf

Prof. Dr. Neil P. Strickland
Dept. of Pure Mathematics
Hicks Building
University of Sheffield
GB-Sheffield S3 7RH

Dr. Christine Vespa
Mathematiques
Universite Paris Nord
Centre Scientifiques et Polytechn.
Av. J. B. Clement
F-93430 Villetaneuse

Prof. Dr. Elmar Vogt
Institut für Mathematik II (WE2)
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Prof. Dr. Karen L. Vogtmann
Department of Mathematics
Cornell University
Malott Hall
Ithaca NY 14853-4201
USA

Prof. Dr. Nathalie Wahl
Department of Mathematics
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen

Prof. Dr. Sarah Whitehouse
Dept. of Pure Mathematics
Hicks Building
University of Sheffield
GB-Sheffield S3 7RH