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Model Theory and Groups

Organised by
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David Marker (Chicago)
Katrin Tent (Bielefeld)
Frank Wagner (Lyon)

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Abstract. The aim of the workshop was to discuss the connections between model theory and group theory. Main topics have been the interaction between geometric group theory and model theory, the study of the asymptotic behaviour of geometric properties on groups, and the model theoretic investigations of groups of finite Morley rank around the Cherlin-Zilber Conjecture.

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Introduction by the Organisers

The workshop *Model Theory and Groups*, organised by Andreas Baudisch (Berlin), David Marker (Chicago), Katrin Tent (Bielefeld) and Frank Wagner (Lyon) was held January 14th–20th, 2007. This meeting focused on interactions between classical model theoretic investigations of groups and their applications to geometric group theory and vice versa. It was well attended with 55 scientists, both model theorists as well as geometric group theorists, including 11 women and a relatively large number of young researchers and students. Needless to say that participants came from a broad geographical background.

For many years groups have played a central role in model theory, both in applied model theory where one is focused on understanding algebraic structures and, more surprisingly, in pure model theory where one is studying structures from an abstract viewpoint.

At first, only the most basic tools from the general theory were needed in applications, but, over the last ten years, some of the most sophisticated ideas from pure model theory have played an important role in applications, most notably

Hrushovski's proof of the Mordell-Lang Conjecture for function fields. The investigation of variations of Mordell-Lang like theorems in different situations played an important role in a number of talks.

Geometric group theory and model theory have started interacting in the context of free groups and surface groups as well as in the study of the asymptotic behaviour of geometric properties on groups. This was a second main topic of the conference which particularly profited from the fact that researchers from different areas attended the meeting and presented their results.

At the core of model theoretic investigations of groups were the reports on groups of finite Morley rank around the Cherlin-Zilber Conjecture which states that every simple group of finite Morley rank is an algebraic group over an algebraically closed field. While originally attempts at proving this conjecture have followed the lines for the classification of algebraic groups, more recent advances have been made by adapting and generalising ideas from the classification of finite simple groups, in particular the study of the 2-Sylow subgroup, which has allowed a distinction into three cases: even characteristic, odd characteristic (including 0) and degenerate (no involutions). The even case is solved, and important progress has been made in the other cases. The recent construction of so-called bad fields, i.e. fields of finite Morley rank with a distinguished multiplicative subgroup also added new impetus to the search for new proofs not involving assumptions on the non-existence of such fields.

The organisers asked Dugald Macpherson and Charles Steinhorn before the conference to give a three-lecture tutorial on asymptotic classes and measurable structures. This is a new development in model theory generalising results on finite and pseudofinite fields. In addition, 27 participants were invited to report on their research (18 long and 9 short talks).

Altogether it was a very successful workshop which inspired a number of new cooperations and further projects. The reader may find here extended abstracts of all talks (in the order in which the talks were given).

Workshop: Model Theory and Groups

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Abstracts

Permutation groups of finite Morley rank

Alexandre V. Borovik
(joint work with Tuna Altınel, Jeff Burdges, Gregory Cherlin)

...a time to plant, and a time
to pluck up that which is planted.
Ecclesiastes 3:2

The aim of the talk is to use questions about permutation groups of finite Morley rank as a testing ground of the power of recent classification results on simple groups of finite Morley rank. Although the complete resolution of the **Algebraicity Conjecture**:

> every infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field,

is still elusive, we are now in position to prove results about groups of finite Morley rank which for decades were deemed completely unaccessible.

The crucial result is the classification of groups of finite Morley rank and even type.

Theorem 1: Even Type Theorem (Altınel, Borovik, Cherlin [1]**)** If a simple group G of finite Morley rank contains an infinite elementary abelian 2-group, then G is a simple algebraic group over an algebraically closed field of characteristic 2.

The second result deals with the so-called degenerate case.

Theorem 2: Degenerate Type Theorem (Borovik, Burdges, Cherlin [2]**)** Let G be a group of finite Morley rank with a finite 2-Sylow subgroup S. Then

$$S \cap G^\circ = 1.$$

We can now move, as promised, to permutation groups of finite Morley rank.

Let G be a group of finite Morley rank acting faithfully, transitively and definably on a set Ω. We say that G is *definably imprimitive* if it preserves a non-trivial definable equivalence relation, and *definably primitive* otherwise. As usual, the latter is equivalent to saying that the action is transitive and that, in addition, the stabiliser of a point G_α, $\alpha \in \Omega$, is a maximal (among definable) subgroups in G.

Theorem 3: Bounds for Primitive Groups (Borovik, Cherlin [3]**)** There exists a function $f(n)$ such that for a definably primitive permutation group (G, Ω) of finite Morley rank,

$$\operatorname{rk} G \leqslant f(\operatorname{rk} \Omega).$$

Surprisingly, the result was unknown even in the case of rational actions of algebraic groups!

The bound $|G| \leqslant n!$ for a finite permutation group on a set of n elements raises a natural question: why, in the context of groups of finite Morley rank, bounds for permutation groups present a problem? The following example shows why: they do not exist without extra assumption about the nature of the action.

Indeed, let K be an algebraically closed field and

$$G = K \oplus \cdots \oplus K$$

be a direct sum of n copies of the additive group of K. Then the following is a faithful action of G (of Zariski dimension n for arbitrary n) on the affine space $\mathbb{A}^2$ (of Zariski dimension 2):

$$(a_1, \ldots, a_n) : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + a_1 x + \cdots + a_n x^n \end{pmatrix}$$

In this example, the action was intransitive. The next example is concerned with transitive, but imprimitive action.

Again, let

$$A = K \oplus \cdots \oplus K$$

be a K-vector space of dimension n. Let $T \simeq K^*$ be a torus acting on A by matrices

$$\mathrm{diag}(t, t^2, \ldots, t^n).$$

Let B be a hyperplane in A in "general position". Then

$$\bigcap_{t \in T} B^t = 0$$

and the right coset action of the natural semidirect product $G = AT$ on $\Omega = G/B$ is faithful and transitive; the dimension of G can be made arbitrary large while the dimension of Ω is 2.

A few words about the proof of the **Bounds for Primitive Groups**. The key idea is that a "very big" primitive permutation group G of a "small" set Ω has to be *generically n-transitive* for some large n, that is, to have a generic orbit on Ω^n. Therefore the proof reduces to proving

Theorem 4: Bounds for degree of generic transitivity. There exists a function $g(n)$ such that for a definably primitive and generically k-transitive permutation group (G, Ω) of finite Morley rank,

$$k \leqslant g(\mathrm{rk}\, \Omega).$$

With some effort, the proof can be reduced to the case when G is simple. Since G is generically k-transitive, it induces the action of the full symmetric group Sym_k on a generic k-tuple $\alpha_1, \ldots, \alpha_k$ of elements from Ω. Involutions from Sym_k can be lifted to involutions in G, thus creating a non-trivial Sylow 2-subgroup in G (quite a lot of now routine, but technical facts about groups of finite Morley rank are used in the process). And here comes the crucial strike: using Even

Type Theorem and Degenerate Type Theorem, we can conclude that G contains a non-trivial 2-*torus*, a divisible abelian 2-group (which means that G behaves, in the sense of Sylow 2-theory, as a simple algebraic group over an algebraically closed field of odd or zero characteristic). At this point, a crucial technical lemma invokes the fact that 2-tori are "rigid":

Lemma. If, in the context of Theorem 4, T is a 2-torus in G and $d(T)$ is its definable closure (that is, the minimal definable subgroup containing T), then

$$\operatorname{rk} d(T)/O(d(T) \leqslant \operatorname{rk} \Omega$$

where $O(d(T))$ is the maximal definable connected subgroup of $d(T)$ without involutions.

And here we start playing a highly technical game typical for proofs in the theory of groups of finite Morley rank and odd type: we show that "small" 2-tori are incompatible with the presence in G of definable sections isomorphic to "big" symmetric groups Sym_k, which gives us a bound on the degree of generic k-transitivity.

References

[1] T. Altınel, A. V. Borovik and G. Cherlin, *Simple groups of finite Morley rank*, Amer. Math. Soc., 2007.

[2] A. V. Borovik, J. Burdges and G. Cherlin, *Involutions in in groups of finite Morley rank of degenerate type*, Selecta Math., to appear.

[3] A. V. Borovik and G. Cherlin, *Permutation groups of finite Morley rank*, to appear.

The Collapse to Infinity

Amador Martin-Pizarro
(joint work with Thomas Blossier)

In [2] Hrushovski disproved the trichotomy conjecture for strongly minimal sets by developing an amalgamation method which turned out to be extremely useful in order to obtain new structures with predescribed geometry, generally wilder than the classical examples.

The amalgamation procedure can be described in the following way: the goal is to construct a countable universal model starting from a given collection of finitely generated structures. In this model there is a unique type (i.e. an orbit under the group of automorphisms) of rank ω. The decisive part (called *collapse*) is to modify this construction in order to algebraize types of finite rank. In order to do so, a collection of representatives (or *codes*) of these types needs to be chosen and one assigns a maximal length of an independence sequence of realizations to each code. The structure obtained after amalgamating again has now finite rank. Note that the prescribed maximal length must reflect any interaction between different codes, since some realizations of one code may yield realizations for another. Using this construction a field of finite rank in positive characteristic equipped with an additive definable proper infinite subgroup was obtained in [1], answering hence negatively the long-standing question, whether or not fields of finite Morley rank in positive characteristic were additively minimal. Note that they are additively minimal in characteristic 0.

During my talk at Oberwolfach, I presented a joint work with T. Blossier, in which a differentially closed field of Morley rank $\omega \cdot 2$ equipped with a distinguished additive subroup of rank ω was obtained by collapsing Poizat's differential red fields in characteristic 0. This is a generalization of the aforementioned collapse construction, by replacing algebraic for finitely dimensional over the field of constants. The proof strongly used results of A. Pillay and W.Y. Pong [3] on ranks of differential groups.

Moreover, using the logarithmic derivative, one obtains fields equipped with a proper divisible (which contains torsion however) multiplicative subgroup. This is neverthless not a bad field (and it cannot be forced to be one) since the field of constants remains definable even in the reduct with a predicate for the multiplicative subgroup.

References

[1] A. Baudisch, A. Martin-Pizarro, M. Ziegler, *Red fields*, to appear at J. Symbolic Logic.

[2] E. Hrushovski, *A new strongly minimal set*, Annals of Pure and Applied Logic, **62**, 147–166, 1993.

[3] A. Pillay, W.Y. Pong, *On Lascar rank and Morley rank of definable groups in differentially closed fields*, J. Symbolic Logic, **67**, vol 3, 1189–1196, 2002.

Model Theory of the Free Group

Anand Pillay

We investigate some stability-theoretic properties of the (complete) theory T_{fg} of nonabelian free groups, on the basis of Sela's recent result that T_{fg} is stable, and assuming some unpublished results of Bestvina and Feighn on "negligible" subsets of finitely generated free groups. My results appear in [1].

Very roughly Bestvina and Feign define a subset X of a finitely generated nonabelian free group $F =< a_1, .., a_n >$ to be negligible, if there is some $N < \omega$ such that for all $\epsilon > 0$ there is a cofinite subset X' of X such that for every word $w \in X'$ all but ϵ of w can be covered by N distinct pairs of proper subwords $w_1, w_1', .., w_N, w_N'$ where $w_i' = w_i$ or its inverse. They prove that for any definable subset X of F either X or its complement are negligible.

We deduce:

Proposition 1. *A definable subset of a nonabelian finitely generated free group F is nonnegligible iff it is generic in the sense of stable group theory (that is finitely many left or right translates of X cover F).*

As Sela pointed out to us, one also easily deduces from Bestvina-Feighn that the only proper definable subgroups of free groups are cyclic, hence the only proper definable subgroups of models G of T_{fg} are abelian. So the maximal abelian subgroups of G are precisely the centralizers of nontrivial elements. Two such centralizers are equal or disjoint. Also G is definably simple.

So $G \models T_{fg}$ resembles a simple bad group of finite Morley rank, with the centralizers playing the role of the Borels.

The main result is:

Proposition 2. *T_{fg} is non CM-trivial.*

The proof is like that for simple bad groups, but we use Proposition 1 in place of Morley rank arguments as a computational tool.

References

[1] Anand Pillay, *Forking in the free group*, arxiv:math.LO/0702134.

Type-definable Groups in $\aleph_0$-stable Continuous Theories are Definable

Itaï Ben Yaacov

Definable sets, as well as more complex definable objects (e.g., groups) play a central and essential role in classical model theory. A fundamental result (or simply the definition) is that a definable set is indeed defined by a first order formula, possibly with parameters.

Continuous first order logic (see [4, 1]) is an extension of classical first order logic, obtained by replacing the two-element set of truth values $\{T, F\}$ with the compact interval $[0, 1]$. It allows to consider various classes of complete metric structures as elementary classes and to study definability therein. However, some things do become more complicated in continuous logic, and in particular the classical notion of a definable set splits in two. First, it can be viewed as no more than a definable function into the set $\{T, F\}$ (or $\{0, 1\}$). As such, the correct analogue is a definable function to $[0, 1]$, which we call a *definable predicate* — it is definable in the sense that its values are either given by a formula, or, at the very worst, by the limit of a uniformly converging sequence of formulae. But when thinking of definable objects, such as groups, there is an essential asymmetry between what is inside (which interests us) and what is outside (about which we could hardly care less, especially if the set is stably embedded). The same asymmetry arises when we wish to quantify over a definable set. In that case the notion of a definable predicate is inadequate and we are led to the following notion of a *definable set*:

Definition. A closed subset X of a metric structure M is *definable* if any of the following equivalent conditions holds:

(1) One can quantify over X. In other words, if $\varphi(x, \bar{y})$ is a definable predicate then so is $\psi(\bar{y}) = \inf_{x \in X} \psi(x, \bar{y})$.
(2) The distance to X is a definable predicate.

The class of definable sets in a continuous structure is far less well-behaved than in classical logic. For example, the family of all definable subsets of M^n does not form a Boolean algebra, as it is not always closed under complement or intersection. Worse than that, it is not at all obvious whether non-trivial definable sets even exist. Indeed, examples exist of theories which do not admit enough definable sets, i.e., where there are distinct types which nonetheless agree on all definable sets (such an example is the theory of real closed fields augmented by a predicate $P(x) = st(|x-a|) \wedge 1$ where $st(\cdot)$ denotes the real standard part and a is some infinite element). As all known examples of this pathology are unstable it make sense to ask whether all stable continuous theories admit enough definable sets.

One of the beautiful aspects of stable group theory in classical logic is the proof that there are also "enough definable groups", namely, that every type-definable group is the intersection of definable subgroups of a definable group. In the case of an $\aleph_0$-stable theory, chain conditions along with the previous general fact yield that every type-definable group is definable. In continuous logic we can prove adequate analogues of the chain conditions for sequences of definable (or type-definable) groups for $\aleph_0$-stable theories, but we do not know to prove enough definable groups exist in stable theories.

The present result is rather a direct proof of the fact that in an $\aleph_0$-stable theory every type-definable group is definable, leaving open the question of the existence of definable groups in general stable theories. We do it in several steps:

We first prove the result under the technical assumption of the invariance of the metric under the group operation, using some technical results concerning metric Morley ranks (i.e., Cantor-Bendixson ranks) from [2].

We next study type-definable groups in continuous theories, and in particular in stable continuous theories. We prove that:

- If the connected component of a type-definable group in a stable theory is definable then so is the entire group.
- A connected type-definable group in a stable theory admits an invariant metric. Such a metric is *partial*, as it is only defined on the group.

Finally, we seek tools allowing us to extend a partial metric on a type-definable set (or group) to a global one. If the set were definable this would not be a problem, but this is precisely the assumption we are not allowed to make. Instead, we generalise results from [3] allowing us to obtain metrics without recourse to quantification. We prove:

Lemma. *Let X be a type-definable set, d_1 a (partial) metric on X. Then there is a continuous function $h\colon [0,1] \to [0,1]$ and a global metric d_2 which extends $h \circ d_1$.*

In particular, if $X = G$ is a group and d_1 is invariant under the operation of G then so is d_2.

We may now conclude:

Theorem. *A type-definable group in a continuous $\aleph_0$-stable continuous theory is definable.*

Proof. As it is enough to show the connected component of G is definable, we may assume G is connected. It therefore admits an invariant definable metric. Up to a modification, this partial definable metric extends to a global one, call it d_1 (which is invariant on G).

Any two (global) definable metrics are uniformly equivalent (by compactness) and therefore interchangeable. Moreover, as the characterisation of definable sets via quantification does not make direct use of the metric, it does not change if we replace the "standard" metric d with the metric d_1 (demoting d to the status of a mere predicate symbol). We may now apply the first result and conclude. □

As a corollary we can now prove that in an $\aleph_0$-stable continuous theory every type-definable group (not only connected ones) admit an invariant metric, and that any partial metric on such a group extends (without modification) to a global one: indeed, these facts are known for definable groups in arbitrary continuous theories.

References

[1] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Expanded lecture notes for a workshop given in March/April 2005, Isaac Newton Institute, University of Cambridge.

[2] Itaï Ben Yaacov, *Topometric spaces and perturbations of metric structures*, submitted.

[3] ———, *Uncountable dense categoricity in cats*, J. Symbolic Logic **70** (2005), no. 3, 829–860.

[4] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, submitted.

A Trichotomy Theorem for 1-dimensional Types in Reducts of o-minimal Fields

Assaf Hasson

(joint work with A. Onshuus, Y. Peterzil)

Let $\mathcal{N}$ be a structure definable in an o-minimal structure $\mathcal{M}$ (with a fixed interpretation of $\mathcal{N}$ in $\mathcal{M}$). A type p in the structure $\mathcal{N}$ is called *one dimensional* if it contains an $\mathcal{N}$-formula whose $\mathcal{M}$-dimension is one. Notice that the notion of dimension depends on the particular interpretation of $\mathcal{N}$ in $\mathcal{M}$.

We prove the following:

Theorem 1. *Assume that $\mathcal{N}$ is a definable structure in $\mathcal{M}$, an o-minimal expansion of a field, and that p is a complete one-dimensional $\mathcal{N}$-type over a model $\mathcal{N}_0 \prec \mathcal{N}$. Then:*

(1) *p is trivial (with respect to $acl_{\mathcal{N}}$).*

Or, there exists an $\mathcal{N}$-definable equivalence relation E with finite classes such that one of the following holds:

(2) *p is linear, in which case either*
   (a) *p/E is a generic type of a strongly minimal $\mathcal{N}$-definable 1-dimensional G, and the structure which $\mathcal{N}$ induces on G is locally modular. In particular, p is strongly minimal. Or,*
   (b) *there exists an extension $p' \supseteq p$ such that p'/E is a generic type in an $\mathcal{N}$-definable ordered group-interval I, and the structure which $\mathcal{N}$ induces on I is a reduct of an ordered vector space over an ordered division ring.*

(3) *p is rich: There exists an extension $p' \supseteq p$ such that p'/E is a generic type in an $\mathcal{N}$-definable real closed field R and the structure which $\mathcal{N}$ induces on R is o-minimal.*

It follows that a one-dimensional p can be stable only in Case (1), or in Case (2)(a). Moreover, if p is nontrivial and stable then it is necessarily strongly minimal. Indeed, the hardest part of the proof consists in showing:

Theorem 2. *Let $\mathcal{N}$ be interpretable in an o-minimal structure $\mathcal{M}$ and one-dimensional as such. If $\mathcal{N}$ is stable it is 1-based.*

Note, of course, that the assumption that $\mathcal{N}$ is 1-dimensional is crucial, as is witnessed by the structure $(\mathbb{C}, +, \cdot)$, which is naturally interpretable in the field of real numbers.

The proof of the main result is obtained by combining this last theorem with:

Theorem 3 (Hasson-Onshuus). *Let $\mathcal{N}$ be a 1-dimensional structure definable in an o-minimal structure $\mathcal{M}$. For any unstable $X \subseteq N$ there exists $\mathcal{N}$-definable $X_0 \subseteq N$ and equivalence relation E with finite classes such that X_0/E with all its induced $\mathcal{N}$-structure is o-minimal. In particular X_0/E is linearly ordered.*

We strongly believe that Theorem 1 should hold for 1-dimensional types in structures *definable* (i.e. interpretable in the main sort) of an arbitrary o-minimal theory (that is, one not necessarily expanding a field), but the proof is not yet finished. Somewhat more challenging (though possibly not of great importance) would be the generalisation to the case of 1-types of theories *interpretable* in arbitrary o-minimal structures.

Theorem 2 is a "baby version" of a Zilber style trichotomy for minimal stable types for theories interpretable in o-minimal fields. Combined with Theorem **??** it seems plausible that such a trichotomy, if true, could have interesting applications to generalisations of the work of Peterzil-Starchenko (see this volume) on recovering local exponential functions.

Witt Modules

Françoise Point
(joint work with Luc Bélair)

Model theory of Witt vectors in the language of difference valued fields has been investigated by Luc Bélair, Angus Macintyre and Thomas Scanlon [1, 2]. Here, we will consider this theory in the (less expressive) language of valued modules [3].

Let R be a field of characteristic p, then we denote by $W[R]$ the Witt ring over R and its field of fractions by $W(R)$. The Frobenius endomorphism of R ($x \to x^p$) induces a ring endomorphism of $W[R]$; it is called the Witt Frobenius. If R is a perfect field of characteristic p, the corresponding Witt Frobenius is an automorphism of $W[R]$. One can easily extend it to the field of fractions of $W[R]$. Note that the valuation of the image of an element by the Witt Frobenius is equal to the valuation of this element.

More generally, we will consider any valued difference field (K, v, σ) where σ is an isometry of K, namely that $v(k) = v(k^\sigma)$, $k \in K$. Let $\mathcal{O}_K$ be the valuation ring of K, $\bar{K}$ its residue field, and $(\Gamma, +, \leq, 0, 1)$ its value group.

Let A be the skew polynomial ring $K[t;\sigma]$, where the commutation law is given by $k.t = t.k^\sigma$ with $k \in K$. If σ is not the identity on K, then this ring is non commutative, it is an integral domain which is right Euclidean and left Euclidean [4]. We may extend the valuation v on A as follows: $v(\sum_{i=0}^n t^i.a_i) = \min_{i=0}^n\{v(a_i)\}$ ([4], chapter 9, p. 425). Let $A_0 := \mathcal{O}_K[t;\sigma]$ of A. It is an Ore domain ([4], chapter 2). Let $\mathcal{I}$ be the subset of elements of A_0 consisting of the elements of valuation 0.

Definition 1. Let L_A be the language of A-modules and let T_A be the L_A-theory of right A-modules. Let T be the theory T_A together with:

(1) $\forall m\ \exists n\ (m = n.t)$, & $\forall m\ (m.t = 0 \to m = 0)$,
(2) $\forall m\ \exists n\ (n.q(t) = m)$, where $q(t)$ varies over the irreducible polynomials of A with $q(0) \neq 0$.

Lemma 1. *Let F be a field of characteristic p which is p-closed. Let $K := W(F)$, σ be the Witt Frobenius and $A := K[t;\sigma]$. Then, the ring $W[F]$ viewed as an A_0-module where the action of t on $W[F]$ is the action of the Witt Frobenius σ, is divisible with respect to $\mathcal{I}$.*

Proposition 2. *Let F be a field of characteristic p which is p-closed; let $A := W(F)[t;\sigma]$. Then the theory T above is consistent, with model $W(F)$. It admits positive quantifier elimination, namely any positive primitive formula is equivalent to a finite conjunction of atomic formulas. Its completions are obtained by specifying for which irreducible polynomials $q(t)$ with $q(0) \neq 0$ whether $ann(q(t)) \neq \{0\}$, and each completion of T admits quantifier elimination.*

For instance, if $A := W(\tilde{\mathbb{F}}_p)$, then its theory of modules admits quantifier elimination.

Let M be an A-module and we will now suppose that we have a valuation map w on our A-module taking its value in $\Delta \cup \{+\infty\}$, where $(\Delta, \leq)$ is a totally ordered set and $+\infty$ an extra element strictly bigger than Δ. We also assume that we have an action of the group Γ on the ordered set Δ, that we will denote by $+$, so that $\Gamma \subseteq Aut(\Delta, \leq)$. Namely, for any $\delta \in \Delta$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$, then $\delta + \gamma \in \Delta$ and $(\delta + \gamma_1) + \gamma_2 = \delta + (\gamma_1 + \gamma_2)$. Moreover, this action respects the order: for any $\delta_1, \delta_2 \in \Delta$ and $\gamma_1, \gamma_2 \in \Gamma$, if $\Delta \models \delta_1 \leq \delta_2$ and $\Gamma \models \gamma_1 \leq \gamma_2$, then $\delta_1 + \gamma_1 \leq \delta_2 + \gamma_2$.

From now on, we will assume that the two-sorted structure $((\Delta, \leq), (\Gamma, +, -, 0, \leq), +)$ satisfies the axioms above that we will denote by T_Δ.

Definition 2. Let T_w be the following theory of valued A-module $(M, \Delta \cup \{\infty\}, w)$ with M an A-module, $w(0) = +\infty$ and $w : M - \{0\} \to \Delta$, satisfying in addition:

(1) $M \models T_A$,
(2) $((\Delta, \leq), (\Gamma, +, -, 0, \leq), +) \models T_\Delta$, where $\Gamma = v(A)$,
(3) $\forall m_1 \in M \forall m_2 \in M \; w(m_1 + m_2) \geq \min\{w(m_1), w(m_2)\}$, $w(0) = +\infty$,
(4) $\forall m \in M \; w(m.t) = w(m)$,
(5) $\forall m \in M \; w(m.\lambda) = w(m) + v(\lambda)$, for all $\lambda \in K - \{0\}$.

First, we will work in the setting of abelian structures, so we will introduce another (less expressive) language.

Let M be a valued A-module. We will define in M a set of subgroups V_γ, $\gamma \in v(A)$, with $V_\gamma := \{m \in M : w(m) \geq \gamma\}$. Such a language has been considered by T. Rohwer in his thesis [6] when he investigated the additive theory of Laurent series field $\mathbb{F}_p((t))$.

On each of the predicates V_γ, with $\gamma \in \Gamma$, we say the following:

(1) $\forall m \;\; (V_\gamma(m) \leftrightarrow V_\gamma(m.t))$,
(2) $\forall m_1 \, \forall m_2 \;\; (V_\gamma(m_1) \;\&\; V_\gamma(m_2) \to V_\gamma(m_1 + m_2))$,
(3) $\forall m \;\; (V_\gamma(m) \to V_{\gamma+v(\lambda)}(m.\lambda))$, for any $\lambda \in K$,
(4) $\forall m \;\; (V_\gamma(m) \to V_{\gamma+v(q(t))}(m.q(t)))$, where $q(t) \in K[t, \sigma]$,
(5) $\forall m \in V_\gamma \; \exists n \in V_\gamma \; n.q(t) = m$ for all $q(t) \in \mathcal{I}$.

Let T_V be the theory T_A together with the above scheme of axioms (1) up to (4). Let T_V^* be the theory T_V together with axiom scheme (5). Let $K = W(F)$, where F is p-closed, then K is a model of T_V^*.

Proposition 3. *The theory T_V^* admits quantifier elimination, up to index sentences.*

Now, we will consider the two-sorted theory of valued modules. We consider two cases, either $(\Delta, \leq)$ is a densely totally ordered set, or Γ has a smallest strictly positive element and we have the following condition on the action: Δ satisfies the following: $\forall \delta_1 \exists \delta_2 \forall \delta_3 \; (\delta_2 > \delta_1 \;\&\; (\; \delta_3 > \delta_1 \to (\delta_2 \leq \delta_3 \;\&\; \delta_2 = \delta_1 + 1)) \; (\star)$.

Let T_{dense} be the theory T_Δ together with the axioms stating that $(\Delta, \leq)$ is dense and let $T_{discrete}$ be the theory T_Δ together with Γ is discretely ordered, Δ has a smallest positive element and satisfies the axiom $(*)$ above.

Proposition 4 ([5]). *Suppose that* $(\Delta, \leq)$ *satisfies either* T_{dense} *or* $T_{discrete}$. *Then the structure* $< (\Delta, \leq), (\Gamma, +, -, \leq, 0, 1), + >$ *admits quantifier elimination in the sort* Δ.

Definition 3. Let $T_w^*(discrete)$ (respectively $T_w^*(dense)$ be the following theory of valued A-modules:

(1) T_w,
(2) $(\Delta, \leq) \models T_{discrete}$ (respectively $(\Delta, \leq) \models T_{dense}$),
(3) Divisibility axioms: $\forall u_1 \in M - \{0\}\ \exists u \in M\ (u_1 = u.q(t)\ \&\ w(u) = w(u_1))$, with $q(t) \in \mathcal{I}$,
(4) Annihilator axioms: given $\gamma \in \Delta$ and a finite number of elements $p_0(t), p_1(t), \cdots, p_n(t)$ of $\mathcal{I}$, with $deg(p_0(t)) > deg(p_1(t)) \geq deg(p_2(t)) \cdots$:

$$\begin{aligned} \forall u_1, \cdots, \forall u_m \exists u \in M\ :\ & (\bigwedge_{i=1}^{n} w(u_i) = \gamma) \\ & \rightarrow\ (u.p_0(t) = 0\ \&\ \bigwedge_{i=1}^{m} w(u.p_i(t)) = \gamma\ \&\ w(u.p_i(t) + u_i) = \gamma). \end{aligned}$$

Let $K := \prod_U^{\omega} W(\tilde{\mathbb{F}}_p)$, where U is a non-principal ultrafilter on the set of prime numbers, then K is a model of $T_w^*(discrete)$.

Proposition 5. *The two-sorted theory* $T_w^*(discrete)$ *(respectively* $T_w^*(dense)$*), admits quantifier elimination.*

References

[1] Bélair, L., Macintyre, A., *L'automorphisme de Frobenius des vecteurs de Witt*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 1, 1-4.
[2] Bélair, L., Macintyre, A., Scanlon T., *Model Theory of Frobenius on Witt Vectors*, to appear in Amer. J. Math.
[3] Bélair L., Point F., *Elimination des quantificateurs dans les équations aux différences sur les vecteurs de Witt*, in preparation.
[4] Cohn P. M., *Skew fields*, Encyclopedia of mathematics and its applications, edited by G.-C. Rota, volume 57, Cambridge University Press, 1995.
[5] van den Dries, Lou, *Quantifier elimination for linear formulas over ordered and valued fields*, Proceedings of the Model Theory Meeting (Univ. Brussels, Brussels/Univ. Mons, Mons, 1980), Bull. Soc. Math. Belg. Sér. B 33 (1981), no. 1, 19-31.
[6] Rohwer T., *Valued difference fields as modules over twisted polynomial rings*, PhD thesis, University of Illinois at Urbana-Champaign, 2003.

Asymptotic Classes and Measurable Structures I–III

Dugald Macpherson and Charles Steinhorn

The inspiration for the work discussed in this three-lecture tutorial is the theorem of Chatzidakis, van den Dries, and Macintyre [1] which establishes the uniform asymptotic behavior of definable sets in finite fields in terms of notions of dimension and measure. This theorem suggests the definition of a *1-dimensional asymptotic class* of finite structures in a first order language L (see [6]). Essentially one requires the same kind of asymptotic uniformities for definable sets in the structures in the class. The second author's student, R. Elwes [2], has extended this definition to that of an *N-dimensional asymptotic class* (of finite structures).

An infinite ultraproduct of an N-dimensional asymptotic class has a supersimple theory of SU-rank $\leq N$. The dimension and measure on the class additionally yield a corresponding (definable) measure on the definable sets in the ultraproduct, as well as any elementarily equivalent structure. This suggests the notion of a *measurable* supersimple theory.

Natural examples of asymptotic classes include: finite fields, carefully chosen classes of envelopes of a smoothly approximable structure [2], finite cyclic groups, and Paley graphs. In addition to measurable structures obtained from asymptotic classes, applying a result of Hrushovski provides other examples that arise by taking the fixed point set of a generic automorphism of a strongly minimal set satisfying suitable hypotheses (see [4] and [2]).

Another student of the second author, M. Ryten, has shown [7] that any family of finite simple groups of fixed Lie type forms an asymptotic class, from which it follows (by results of J. Wilson [9]) that any pseudofinite simple group is measurable. It is reasonable to conjecture a converse to the latter, at least under the hypothesis of pseudofiniteness. There are also the beginnings of a theory of measurable groups of low dimension. For example, every 1-dimensional measurable group is finite-by-abelian-by-finite, and there is a corresponding statement for 1-dimensional asymptotic classes of groups [6] (see also [3]).

Ryten proves his theorem by showing that any class of finite simple groups of fixed Lie type is bi-interpretable, uniformly over parameters, either with the class of finite fields or with a class of finite difference fields of the form

$$\mathcal{C}_{m,n,p} := \{(\mathbb{F}_{p^{kn+m}}, \mathrm{Frob}^k) : k > 0\},$$

where m, n, p are fixed. The asymptotic theory of $\mathcal{C}_{m,n,p}$ is shown to be interpretable in $ACFA$, the theory of algebraically closed fields with a "generic" automorphism. Combining this with results of Hrushovski [5] on the nonstandard Frobenius and some joint work with Tomašić [8], Ryten proves that $\mathcal{C}_{m,n,p}$ is a 1-dimensional asymptotic class.

The first two lectures in this tutorial introduce and survey results about asymptotic classes and measurable structures; the third has Ryten's thesis as its focus.

References

[1] Z. Chatzidakis, L. van den Dries, A.J. Macintyre, *Definable sets over finite fields*, J. Reine Angew. Math. **427** (1992), 107–135.
[2] R. Elwes, *Dimension and measure in finite first-order structures*, Ph.D. thesis, University of Leeds, 2005.
[3] R. Elwes and M. Ryten, *Measurable groups of low dimension*, in preparation.
[4] E. Hrushovski, *Pseudofinite fields and related structures*, in *Model theory and applications* (Edited by L. Bélair, Z. Chatzidakis, P. D'Aquino, D. Marker, M. Otero, F. Point, A. Wilkie), Quaderni di Matematica, vol. 11, Caserta, 2005, 151–212.
[5] E. Hrushovski, *The Elementary Theory of the Frobenius Automorphisms*, arxiv:math.LO/0406514.
[6] H.D. Macpherson, C. Steinhorn, *One-dimensional asymptotic classes of structures*, Trans. Am. Math. Soc. , to appear.
[7] M. Ryten, *Results around asymptotic and measurable groups*, Ph.D. thesis, University of Leeds, in preparation.
[8] M. Ryten, I. Tomašić , *ACFA and measurability*, Selecta Mathematica, to appear.
[9] J.S. Wilson, *On pseudofinite simple groups*, J. London Math. Soc. (2) **51** (1995), 471–490.

Recovering a Local Exponential Map

Ya'acov Peterzil
(joint work with Sergei Starchenko)

Let $\langle G, \oplus\rangle$ be an n-dimensional abelian Lie group definable in some o-minimal expansion M of the field of real numbers. By basic Lie theory, there is an analytic local isomorphism π between $\langle \mathbb{R}^n, +\rangle$ and G, unique up to conjugation by an element of $Gl(n, \mathbb{R})$. We formulate sufficient conditions under which π can be definably recovered (in some neighborhood of 0) in the structure M.

Let $X \subseteq \mathbb{R}^n$ be a semi-algebraic curve in the domain of π, and let $\Gamma = \pi(X)$.

Theorem. *Assume that Γ is definable in the structure M. Then either Γ is contained in a finite union of cosets of M-definable 1-dimensioinal local subgroups of G, or there is a linear subspace H of $\mathbb{R}^n$ of dimension at least 2 such that $\pi|H$ is locally definable in the structure M.*

The motivation for this theorem, and the crucial elements of the proof come from Hrushovski's treatment of the Mordell-Lang conjecture [1]. Indeed, combining the above result with a theorem of Pila and Wilkie [2] we obtain the following corollary, with "Diaphontine flavor": We denote by $Tor_n(G)$ the subgroup of all elements of G whose order divides n.

Theorem. *Let G be a compact abelian Lie group, definable in an o-minimal expansion M of $\mathbb{R}_{an}$. Let $\pi : \mathbb{R}^n \to G$ be a partial local Lie isomorphism, as before. Assume that $\Gamma \subseteq G$ is a definable curve containing many torsion points in the following sense: There exists an $\epsilon > 0$ such that*

$$\overline{\lim_{n\to\infty}} \frac{|\Gamma \cap Tor_n(G)|}{n^\epsilon} = \infty.$$

Then:

(i) Γ *contains a coset of an* M*-definable 1-dimensional local subgroup of* G*, OR*
(ii) there exists a linear subspace $H \subseteq \mathbb{R}^n$ *of dimension at least two, such that* $\pi|H$ *is locally definable in the structure* $\langle \mathbb{R}; <, +, \cdot, \langle G, \oplus \rangle, \Gamma \rangle$.

Notice that if G above is a semi-abelian variety over the complex numbers, and Γ is itself a semialgebraic curve containing many torsion points, (in the sense of the theorem) then only option (i) is possible, because π is a transcendental map.

The idea of the proof of the last theorem is to pull back Γ, using π, into $\mathbb{R}^n$ and then use the Pila-Wilkie result on the intersection of $\pi^{-1}(\Gamma)$ with $\mathbb{Q}^n$. The fact that Γ has many n-torsion points implies that $\pi^{-1}(\Gamma)$ contains many rational points of height n. This in turn implies, using Pila-Wilkie, that Γ contains a semialgebraic curve. We can now apply our first theorem.

References

[1] E. Hrushovski, *Mordell-Lang conjecture for function fields*, Journal AMS **9** (1996), 667–690.
[2] J. Pila and A. J. Wilkie *The rational points of a definable set*, Duke Math. J. **133**, no. 3 (2006), 591–616.

Finitely Generated Simple Groups of Infinite Commutator Width

Alexey Muranov

The commutator width of a group G is the maximum of the commutator lengths of elements of its derived subgroup $[G, G]$, and the commutator length of an element $g \in [G, G]$ is the minimal number of commutators sufficient to express g as their product.

In [7], Oystein Ore conjectured that the commutator width of every non-cyclic finite simple group is 1. This question still remains open, though it was shown by John Wilson in [8], using the classification of finite simple groups, that there exists an upper bound (not found explicitly) on commutator widths of all finite simple groups. It was pointed out by Martin Isaacs in [4] that no simple groups, finite or infinite, were known at that time to have commutator width greater than 1.

Jean Barge and Étienne Ghys showed in [1] that there exist (infinitely generated) simple groups of surface diffeomorphisms for which the commutator width is infinite. Other similar groups were studied in [3].

Finitely generated infinite simple groups of infinite commutator width, as well as boundedly simple groups of large finite commutator width, are constructed in [6] using methods of the small-cancellation theory.

References

[1] Jean Barge and Étienne Ghys, *Cocycles d'Euler et de Maslov [Euler and Maslov cocycles]*, Math. Ann. **294** (1992), no. 2, 235–265, in French.
[2] Andrew J. Duncan and James Howie, *The genus problem for one-relator products of locally indicable groups*, Math. Z. **208** (1991), no. 2, 225–237.

[3] Jean-Marc Gambaudo and Étienne Ghys, *Commutators and diffeomorphisms of surfaces*, Ergod. Th. & Dynam. Sys. **24** (2004), 1591–1617.
[4] I. Martin Isaacs, *Commutators and the commutator subgroup*, Amer. Math. Monthly **84** (1977), no. 9, 720–722.
[5] V. D. Mazurov and E. I. Khukhro (eds.), *The Kourovka Notebook: unsolved problems in group theory*, 14th augmented ed., Russian Acad. of Sci. Siber. Div., Inst. Math., Novosibirsk, 1999, translated from Russian.
[6] Alexey Muranov, *Finitely generated infinite simple groups of infinite commutator width*, to appear in Internat. J. Algebra Comput., arXiv.org preprint: `arxiv:math.GR/0608688`.
[7] Oystein Ore, *Some remarks on commutators*, Proc. Amer. Math. Soc. **2** (1951), no. 2, 307–314.
[8] John S. Wilson, *First-order group theory*, Infinite groups '94 (Berlin—New York) (Francesco de Giovanni and Martin L. Newell, eds.), Walter de Gruyter & Co., 1996, Proceedings of the international conference held in Ravello, Italy, May 23–27, 1994, pp. 301–314.

Groups of Finite Morley Rank with Solvable Local Subgroups

Eric Jaligot
(joint work with Adrien Deloro)

In the Classification of the Finite Simple Groups, the study of finite simple groups whose proper subgroups are all solvable, the *minimal simple* groups, has been a cornerstone in the whole process. The *local analysis* of these groups, done by J. Thompson originally for the Odd Order Theorem, has been used to get a classification in presence of involutions. This has then been slightly generalized, with only very few additional groups, to a classification of nonsolvable finite groups in which normalizers of nontrivial abelian subgroups are all solvable. This full classification appeared in a series of papers starting with [Tho68]

The work presented here is an analog of this final transfer in the context of groups of finite Morley rank. Indeed, a large body of results have been obtained in the last few years concerning *solvable* and *minimal connected simple* groups of finite Morley rank, i.e. connected simple groups of finite Morley rank in which proper definable connected subgroups are all solvable. The present work can be considered as a "collapse" on these two classes of the more general class of groups of finite Morley rank naturally defined by mimicing the finite case. More precisely, and as we prefer to work with connected groups throughout, we say that a group of finite Morley rank is

- *locally solvable* if $N(A)$ is solvable for any nontrivial definable abelian subgroup A.
- *locally*$^\circ$ *solvable* if $N(A)$ is solvable for any nontrivial definable connected abelian subgroup A.
- *locally solvable*$^\circ$ if $N^\circ(A)$ is solvable for any nontrivial definable abelian subgroup A.
- *locally*$^\circ$ *solvable*$^\circ$ if $N^\circ(A)$ is solvable for any nontrivial definable connected abelian subgroup A.

These definitions should not astound with the more standard notion of groups in which all *finitely generated* subgroups are solvable, even though this standard notion corresponds in the finite Morley rank context to that of solvable groups, and hence is just a special case of the classes of groups introduced here.

In this context, the local analysis takes a peculiar form relying on connectedness and on the existence of a well behaved graduated unipotence theory, as developed in Burdges' thesis. Technically, the main property given by local solvability is a uniqueness lemma.

Uniqueness Lemma. *Let G be a locally$^\circ$ solvable$^\circ$ group of finite Morley rank, $\tilde{q} = (q, r)$ a unipotence parameter with $r > 0$, and U a Sylow $\tilde{q}$-subgroup of G. Assume that U_1 is a nontrivial definable $\tilde{q}$-subgroup of U containing a nonempty (possibly trivial) subset X of G such that $d_q(C^\circ(X)) \leq r$. Then U is the unique Sylow $\tilde{q}$-subgroup of G containing U_1, and in particular $N(U_1) \leq N(U)$.*

This lemma has important consequences on intersections of Borel subgroups, giving notably either their "almost disjointness" or their "fusion" in most interesting situations. When such a dichotomy fails, the analysis of [Bur07] delineates with precision intersections of such Borel subgroups. This analysis, originally done in the minimal connected simple context, generalizes very naturally in the context of locally$^\circ$ solvable$^\circ$ groups, with very similar conclusions except a very few additional phenomena.

With this local analysis we continue an intensive study of the class of groups considered, and we prove the following theorems in presence of involutions.

Mixed type theorem. *Let G be a locally solvable$^\circ$ group of finite Morley rank of mixed type. Then G° is solvable.*

Even type theorem. *Let G be a locally solvable$^\circ$ group of finite Morley rank of even type. Then either G° is solvable or $G \simeq PSL_2(K)$ for some algebraically closed field K of characteristic* 2.

Both the mixed and the even type theorem are proved by the techniques used in the context of simple groups, though it is much simpler with local solvability$^\circ$.

For groups of odd type the situation is much more complicated. In the minimal connected simple situation, the algebraic case has first been partially studied in the unpublished [Jal00]. Then the model-theoretic simplifying assumption of nonappearance of so-called bad fields has been adopted to reduce the size of an overambitious project at that time to manageable size. This gave a classification in [CJ04], both in the algebraic case and in the nonalgebraic case, and with a strong reduction of possibilities in the second case. This has later been reworked in the thesis of the second author, under the supervision of the first, without the simplifying assumption on fields occuring. This gave in [Del07a] and [Del07b], using bounds obtained in [BCJ07], very similar conclusions in the general case.

Odd type theorem (Algebraic case). *Let G be a locally solvable$^\circ$ group of finite Morley rank of odd type and Prüfer rank one. If i is a toral involution such that $C^\circ(i) < B$ for some Borel subgroup B, then $G \simeq PSL_2(K)$ for some algebraically closed field K of characteristic different from 2.*

Odd type theorem (Nonalgebraic case). *Let G be a locally solvable$^\circ$ group of finite Morley rank of odd type. If G° is nonalgebraic, then it is solvable or centralizers$^\circ$ of toral involutions, which are conjugate, are Borel subgroups. Furthermore in the second case, either the Prüfer rank is one and the Weyl group has order one or two, or the Prüfer rank is two and the Weyl group has order three.*

The full proof of these results is quite lengthy and involves large repetitions of [BCJ07], [Del07a], and [Del07b] from the minimal connected simple case. Beyond the need of a global and coherent rewriting of the proofs existing in this case, to get the fullest expectable classification, the present work has the following other important aspects.

- It provides an entirely uniform version of the uniqueness lemma, giving dichotomies on intersections of Borel subgroups independant of the characteristics involved, contrarily to the previous versions which have unfortunately always appeared in specific contexts only: [Jal00, Lemme 2.14] for the original form, in any characteristic but unpublished, [CJ04, Proposition 3.11 and Lemma 3.12] in absence of bad fields, [Bur07, Lemma 2.1] in positive characteristic, and [Del07a, §3.2] in characteristic 0.
- It provides the new reduction to [BCJ07] needed for the bound on the Prüfer rank, and which did not follow from the existing theory.
- More interestingly, it gives a simplification, or rather a standardization of the proofs with a global form of the concentration argument discovered in [Del07a].
- It leaves aside any simplicity assumption, the concentration argument of [Del07a] being adapted to the local$^\circ$ solvable$^\circ$ context.
- Last but not least, it provides an intensive study of the new configurations appearing in the locally$^\circ$ and nonlocally solvable$^\circ$ case, pushing them eventually to a point where they won't create serious damages to the above classification.

References

[Bur07] J. Burdges. The Bender method in groups of finite Morley rank. J. Algebra, to appear, 2007.

[BCJ07] J Burdges, G. Cherlin, and E. Jaligot. Minimal connected simple groups of finite Morley rank with strongly embedded subgroups. J. Algebra, to appear, 2007.

[CJ04] G. Cherlin and E. Jaligot. Tame minimal simple groups of finite Morley rank. *J. Algebra*, 276(1):13–79, 2004.

[Del07a] A. Deloro. Groupes simples connexes minimaux algébriques de type impair. J. Algebra, to appear, 2007.

[Del07b] A. Deloro. Groupes simples connexes minimaux non-algébriques de type impair. Submitted, 2007.

[Jal00] E. Jaligot. *FT*-groupes. Prépublications de l'institut Girard Desargues, UPRESA 5028, Janvier 2000, n° 33, 2000. `http://math.univ-lyon1.fr/~jaligot/FT-groupes.pdf`
[Tho68] J. G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. *Bull. Amer. Math. Soc.*, 74:383–437, 1968.

Ax-Schanuel Conditions in Positive Characteristic and Formal Maps

Piotr Kowalski

The following theorem was proved by Ax in [1]:

Theorem 1. *Let (K, ∂) be a differential field of characteristic 0 and C its field of constants. For any $x_1, \ldots, x_n \in K^*$ and $y_1, \ldots, y_n \in K$, if we have*

$$(*) \qquad \frac{\partial x_1}{x_1} = \partial y_1, \ldots, \frac{\partial x_n}{x_n} = \partial y_n$$

and $\dim_{\mathbb{Q}}(\partial y_1, \ldots, \partial y_n)$*, then* $\operatorname{trdeg}_C(\bar{x}, \bar{y}) \geq n+1$.

If $C = \mathbb{C}$, the differential equation in the statement of the above theorem may be regarded as the differential equation of the analytic homomorphism

$$\exp : \mathbb{G}_a^n(\mathbb{C}) \to \mathbb{G}_m^n(\mathbb{C}).$$

For a commutative algebraic group A we have a differential homomorphism

$$l_\partial A : A(K) \to T_0 A(K)$$

called *logarithmic derivative.* For algebraic groups A, B and any analytic homomorphism $\phi : A(\mathbb{C}) \to B(\mathbb{C})$, we may consider the differential equation of ϕ:

$$(**) \qquad \phi'(l_\partial A(x)) = l_\partial B(y).$$

Then $(*)$ is the special case of $(**)$, if we take $\phi = \exp$. In the case of an arbitrary field if we want to stay with this interpretation we need to replace the notion of an analytic homomorphism with the notion of a formal isomorphism.

Ax in [2] and Kirby in his Ph.D. thesis [4] extended Theorem 1 to the case of the formal isomorphism $\exp : \mathbb{G}_a^n \to A$ where A is an n-dimensional semi-abelian variety. Bertrand [3] extended it further to the case of the formal exponential map into a commutative algebraic group with no $\mathbb{G}_a$ quotient. The same statement was also known for the case of a formal automorphism of the multiplicative group given by rising to an irrational power.

I have proved a theorem (Theorem 2 below) which deals with an arbitrary formal isomorphism, which should be "far" from being algebraic. It includes all the known cases as well as some new cases in characteristic 0, but more importantly it extends the result to the positive characteristic case. It is phrased in the language of Hasse-Schmidt derivations, since the statement would be meaningless in terms of ordinary derivations (a differential field is algebraic over its constants if the characteristic is greater than 0).

Let $D = (D_i)_{i<\omega}$ be a Hasse Schmidt derivation on a field K and let C be its field of absolute constants. For A, a commutative algebraic group over C, we have a pro-unipotent group U_A being an inverse limit of all arc spaces of A at 0. We again have a homomorphism definable in (K, D):

$$l_D A : A(K) \to U_A(K).$$

Any formal isomorphism $\phi : A \to B$ induces an isomorphism $U_\phi : U_A \to U_B$ and we can consider a Hasse-Schmidt differential equation of ϕ:

$$(***) \qquad U_\phi(l_D A(x)) = l_D B(y).$$

Since in the characteristic 0 case, any derivation gives a unique Hasse-Schmidt derivation, the equation $(**)$ is a special case of $(***)$.
We say that a formal isomorphism between algebraic groups is *nowhere algebraic*, if it is not algebraic after the composition with any non-trivial algebraic homomorphism.
The afore mentioned theorem is the following:

Theorem 2. *Let A, B be commutative algebraic groups over C and $\phi : A \to B$ a formal isomorphism which is nowhere algebraic. Assume $(a, b) \in A(K) \times B(K)$ satisfies $(***)$ for ϕ and $\mathrm{trdeg}_C(a, b) \leq n$. Then, there is a proper algebraic subgroup $B_0 < B$ defined over C and $c \in B(C)$ such that $b \in B_0(K) + c$.*

The conclusion in Theorem 2 is not symmetric, since the notion of nowhere algebraic is not symmetric. However, the most common case is when both ϕ and ϕ^{-1} are nowhere algebraic and then we clearly get a symmetric statement.
There are some open problems which should be considered.

(1) Can one get in the conclusion of Theorem 2 a subgroup of A as well? I could not find any counterexamples.
(2) In characteristic 0, Theorem 1 implies “weak CIT” [5]. Can Theorem 2 (for a formal map between the multiplicative group and an ordinary elliptic curve) be used for a proof of “weak CIT_p” [6]?
(3) What is the right equivalent of $\dim_{\mathbb{Q}}$ to rephrase Theorem 2 in a more Schanuel-style form as in Theorem 1? In the case of a formal map from (2) above, $\dim_{\mathbb{F}_p}$ may be used, but one probably needs an appropriate stronger notion.
(4) Generalize Theorem 1F of Ax from [2] to our context.

References

[1] James Ax, *On Schanuel's conjectures*, Annals of Mathematics **93**(2) (1971), 252–267.
[2] James Ax, *Some topics in differential algebraic geometry. I Analytic subgroups of algebraic groups*, Amer. J. Math. **94** (1972), 1195–1204.
[3] D. Bertrand, *Schanuel's conjecture for non-isoconstant elliptic curves over function fields*, submitted to the MAA 2005 Cambridge Proceedings
[4] Jonathan Kirby, *The Theory of Exponential Differential Equations*, Oxford, PhD thesis, (2006).
[5] Bruno Poizat, *L'egalite au cube*, J. Symbolic Logic, **66**(4) (1990), 1647–1676.
[6] Boris Zilber, *Analytic and pseudo-analytic structures*, In Logic Coll. 2000 Paris, Lect.Notes in Logic v. 19, (2005), eds. R.Cori, A.Razborov, S.Todorcevic and C.Wood, AK Peters, Wellesley, Mass. 392–408.

Isomorphism of Complete Local Noetherian Rings and Strong Approximation

Lou van den Dries

About a year ago Angus Macintyre raised the following question. Let A and B be complete local noetherian rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ such that the rings $A/\mathfrak{m}^n$ and $B/\mathfrak{n}^n$ are isomorphic for every natural number n. Does it follow that the rings A and B are isomorphic?

I showed that the answer is *yes* if the residue field is algebraic over its prime field. The proof uses a strong approximation theorem of Pfister and Popescu [4], or rather a variant of it obtainable by a method due to Denef and Lipshitz [2].

Ofer Gabber gave a negative answer to Macintyre's question in the general case, by examples in equicharacteristic 0 with residue field of transcendence degree 1 over $\mathbb{Q}$, and examples in equicharacteristic $p > 0$ with residue field of infinite transcendence degree over $\mathbb{F}_p$.

These examples are not integral domains, and it seems that the problem remains open in that case. Cutkosky mentioned to me that [1] yields a positive answer when A and B are reduced, equicharacteristic, equidimensional formal germs of isolated singularities.

References

[1] S. Cutkosky and H. Srinivasan, An intrinsic criterion for isomorphism of singularities, Am. Journal of Math. **115** (1993), 789–821.

[2] J. Denef and L. Lipshitz, Ultraproducts and Approximation in Local Rings II, Math. Ann. **253** (1980), 1–28.

[3] H. Matsumura, *Commutative Algebra, 2nd edition*, Mathematics Lecture Note Series, Benjamin, 1980.

[4] G. Pfister and D. Popescu, Die strenge Approximationseigenschaft lokaler Ringe, Inv. Math. **30** (1975), 145–174.

[5] J.-P. Serre, *Corps Locaux.* Hermann, Paris, 1968.

Recovering Fields from Galois Groups

Jochen Königsmann

1. Introduction

If two fields K and K' are isomorphic then so are their absolute Galois groups $G_K := Gal(K^{sep}/K)$ and $G_{K'}$. The converse holds, if K and K' are global fields (Neukirch, Ikeda, Uchida 70's) or fields finitely generated over $\mathbf{Q}$ (Pop [P]).

In general, the converse does not hold: to any field K there is a field K' with $G_K \cong G_{K'}$, but $K \not\cong K'$. E.g., if $char\, K = 0$, take $K' := K((\Gamma))$, the field of formal Laurent series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ with coefficients $a_\gamma \in K$ and with well-ordered support $\{\gamma \in \Gamma \mid a_\gamma \neq 0\}$, where Γ is any divisible ordered abelian group with $\sharp\Gamma > \sharp K$.

However, replacing G_K by $G_{K(t)/K}$, the absolute Galois group of the rational function field $K(t)$ over K, does the trick:

Main Theorem. *Assume that K and K' are perfect fields and that K allows finite extensions of degree > 2 and prime to char K. Then for any function field F'/K' one has*

$$K(t)/K \cong F'/K' \iff G_{K(t)/K} \cong G_{F'/K'}.$$

In particular,

$$K \cong K' \iff G_{K(t)/K} \cong G_{K'(t)/K'}.$$

Here a *function field F'/K'* is any finitely generated field extension F' of K' of transcendence degree ≥ 1 in which K' is relatively algebraically closed.

We conjecture that the theorem remains true when $K(t)$ is replaced by any function field F in one variable over K:

Main Conjecture (K and K' as in the Main Theorem)**.** *Let F/K be a function field in one variable and let F'/K' be any function field. Then the canoncial map*

$$\mathbf{Isom}(F/K, F'/K') \to \mathbf{Isom}(G_{F/K}, G_{F'/K'})/\mathbf{Inn}(G_{F'})$$

is bijective.

Here an isomorphism $\phi : F/K \to F'/K'$ is a field isomorphism $F \to F'$ inducing via restriction an isomorphism $K \to K'$, and an isomorphism $\psi : G_{F/K} \to G_{F'/K'}$ is an isomorphism $G_F \to G_{F'}$ of profinite groups inducing via the canonical restriction epimorphisms $pr_{F/K} : G_F \to Gal(FK^{sep}/F) \cong G_K$ and $pr_{F'/K'}$ an isomorphism $G_K \to G_{K'}$.

The conjecture is proved if K and K' are finitely generated over $\mathbf{Q}$ (Pop [P]) or if $K = K'$ is a sub-p-adic field (Mochizuki [Mo]). Except for $F = K(t)$, the conjecture is still open even if K is finitely generated over $\mathbf{Q}$ (e.g. $K = \mathbf{Q}$), but K' is arbitrary. This case could be settled by proving either of the two following conjectures:

Conjecture. *A field K is finitely generated over $\mathbf{Q}$ iff char $K = 0$, $cd(G_{K(\sqrt{-1})}) < \infty$ and for any finite extension L/K and for any smooth projective curve $\mathcal{C}$ over L, the set $\mathcal{C}(L)$ of L-rational points on $\mathcal{C}$ is finite.*

Conjecture. *If K is finitely generated over $\mathbf{Q}$ and if K' is a field with $G_{K'} \cong G_K$, then K' has a henselian valuation with residue field K and divisible value group.*

Note that the second of these conjectures would also imply the birational Section Conjecture in Grothendieck's anabelian geometry (cf. [K2]).

The Main Conjecture becomes false if any of the assumptions on K and K' are violated.

2. Galois code for rational points

Any Galois characterization of function fields begins with a Galois code for rational points, i.e. a group theoretic characterization of decomposition subgroups. We achieve this in the setting of the Main Conjecture using our general Galois code for henselian valued fields. This code is roughly (and slightly incorrectly) summarized in the following three principles (for any field E):

I: *Non-trivial abelian normal subgroups of G_E are inertias.*
II: *If a Sylow-extension of E is henselian, then so is E.*
I + II = III: *E is henselian iff some Sylow subgroup of G_E has a non-trivial normal abelian subgroup.*

For the precise statments see Theorem 1 in [K1] or Theorem 5.4.3 in [EP]. ¿From this one obtains the

Key Lemma (for simplicity we assume K is non-henselian of $char\, K = 0$)**.** *Let F/K be as in the Main conjecture, so $F = K(\mathcal{C})$ is a function field of a smooth projective K-curve $\mathcal{C}$. Then the map*

$$\begin{array}{rccl} \phi: & \mathcal{C}(K) & \to & \left\{ \begin{array}{l} \textit{conjugacy classes of} \\ \textit{maximal subgroups } D \leq G_F \\ \textit{with } D \cong \hat{\mathbf{Z}} \rtimes G_K \textit{ and } pr_{F/K}(D) = G_K \end{array} \right\} \\ & P & \mapsto & [D_P] \end{array}$$

is a bijection. Here D_P is a decomposition subgroup of G_F w.r.t. (the valuation v_P on F corresponding to) P.

Note that the fixed field of D_P is a henselization of (F, v_P).

If K is henselian or $char\, K \neq 0$ we also have a group theoretic description of the elements in $im\, \phi$. This is, however, much more involved.

As a consequence one gets a Galois code for

$$\mathcal{C}(K^{sep}) = \bigcup_{L/K} \mathcal{C}(L)$$

(together with the G_K-action) with L ranging over all finite separable extensions of K, and hence a Galois code for $Div(F/K^{sep})$ and $Div^0(F/K^{sep})$ as G_K-modules.

3. Encoding + and · via elliptic curves

To prove the Main Theorem we let $F = K(t)$ and we use the Galois code from the Key Lemma for the points in $\mathbf{P}^1(K)$. We fix any three of them and call them 0, 1 and ∞. Now we consider certain elliptic curves $\mathcal{E}/K$ for which the function field $F' := K(\mathcal{E})$ is a quadratic extension of F ramified at exaclty 4 points in $\mathbf{P}^1(K)$ (always including ∞). These function fields F' are seen by $G_{F/K}$, because $G_{F/K}$ recognizes ramification. Now one has to establish a Galois code for the principal divisors of F'/K in order to obtain a Galois code for $\oplus_{\mathcal{E}}$ via

$$(\mathcal{E}, \oplus_{\mathcal{E}}) \cong Div^0(F')/\{\text{principal divisors}\}.$$

A suitable choice of such $\mathcal{E}$'s then gives $+_K$ and $\cdot_K$ (from the addition formulas for $\oplus_{\mathcal{E}}$).

4. Applications

As a model theoretic variation of the Main Theorem (same assumptions on K and K') one gets

Theorem. *K and $G_{K(t)/K}$ are biinterpretable.*
In particular,

$$\begin{array}{rcl} K \equiv K' & \Longleftrightarrow & G_{K(t)/K} \equiv G_{K'(t)/K'} \\ K \text{ is decidable} & \Longleftrightarrow & G_{K(t)/K} \text{ is decidable} \end{array}$$

Here one considers K in the 1-st order language of fields, and $G_{K(t)/K}$ in an adaptation of the 1-st order language for profinite groups introduced by Cherlin, van den Dries and Macintyre (cf. [C]).

As a consequence one obtains (via the Haar measure on G_F) a measure for definable sets in K.

The hope is to use this model theoretic variant of the Main Theorem to show decidability of the perfect hull of $\mathbf{F}_p((t))$ and undecidability for $\mathbf{C}(x)$. It may also prove useful for Hilbert's 10th problem over $\mathbf{Q}$.

References

[C] Z. Chatzidakis: *Properties of forking in ω-free pseudo-algebraically closed fields*, J. Symb. Logic **67(3)** (2002), 957–996.

[EP] A. J. Engler and A. Prestel: *Valued Fields*, Springer 2005.

[K1] J. Koenigsmann: *Encoding valuations in absolute Galois groups*, in F.-V.+S. Kuhlmann, M. Marshall (eds): *Valuation theory and its applications*, Fields Institute Comm. **33** (2003), 107–132.

[K2] J. Koenigsmann: *On the section conjecture in anabelian geometry*, J. reine angew. Math. **588** (2005), 221–235.

[Mo] S. Mochizuki: *The local pro-p anabelian geometry of curves*, Invent. math. **138** (1999), 319–423.

[P] F. Pop: *On Grothendieck's conjecture of birational anabelian geometry I*, Ann. of Math. **138** (1994), 145–182, and *II*, Heidelberg-Mannheim Preprint Series Arithmetik II, No **16**, April 1995.

Pseudo-finite Dimensional Representations of $sl_2(k)$

Angus Macintyre
(joint work with Sonia L'Innocente)

In a very original paper Herzog [1] undertook the analysis of the theory of all finite-dimensional modules over the Lie algebra $sl_2(k)$ over an algebraically closed field k of characteristic 0. Model-theoretically, there is one such theory for each k, but there are no essential differences between them, beyond the minor expressive power of extra constants as k increases. For algorithmic purposes one will naturally restrict to the case of countable k where the Lie algebra can always be given computably.

One passes to the formalism of modules over the universal enveloping algebra U which is an Ore domain. The main novelty of Herzog's work is to produce (not very explicitly) a von Neumann regular ring $U^{'}$ which is an epimorphic extension of U, and which encodes the behaviour of p.p definable maps on finite dimensional U-modules.

The pseudo-finite dimensional modules are the models of the theory of finite-dimensional modules. It turns out that they form an axiomatizable class of $U^{'}$-modules. But since $U^{'}$ is not given computably by Herzog, it remains a problem to decide if the theory of finite-dimensional modules is decidable.

Our work, still in progress, seeks to present $U^{'}$ computably. This involves an "unwinding" of Herzog's proof (which is already very subtle and ingenious). He makes essential (and surprising) use of the fraction field of U, which is not pseudo-finite dimensional. We effectivize this part, and then use it systematically in the unwinding. Our main contribution, going beyond Herzog, is to see that the phenomenon of uniform boundedness which he establishes is, in almost all cases, an instance of a much deeper phenomenon connected to Siegel's Theorem in diophantine geometry. For most idempotents in $U^{'}$ one can decide if they are trivial assuming the decision problem for curves. For the remainder (which constitute a computable set) the corresponding decision problem comes down to problems about genus zero plane curves. There have been a number of serious errors in the literature about these. It seems to us that we can use recent corrections to these to show decidability of the theory of finite-dimensional modules, assuming the decision problem for curves.

References

[1] I. Herzog, *Pseudo-finite dimensional representations of $sl(2,k)$*, Sel. Math. **7** (2001), 241-290.

Model Theory of Universal Covering Spaces

Misha Gavrilovich

We tried to argue that the notion of a path (up to homotopy) on a complex algebraic variety is essentially algebraic, and can be characterised by its basic algebraic properties; we did so via an attempt to define a natural, discrete language to consider the universal covering space of a complex algebraic variety, and prove that the corresponding discrete, algebraic structure on the space could be characterised by certain simple properties, reflecting certain analytic properties of the covering space, but also this language is essentially equivalent to the natural language describing the notion of a path (up to homotopy) on an algebraic variety. Both the proof of the characterisation and the equivalence of the languages describing the universal covering space and paths, require some understanding of geometry and arithmetics of the underlying algebraic variety X. Indeed, full results are only known for X being the multiplicative group of a field, and were obtained by Zilber [1]; for X an elliptic curve only partial results are known [3]. In these cases, the universal covering space is essentially either the complex exponential map or the Weierstrass map. Indeed, the study of the complex exponential map was the motivation of Zilber [1, 2]; he formulated his result as that simple algebraic properties charactercise uniquely the exponential map as a map from an abelian group on the multiplicative group of a field. For elliptic curves, the characterisation is not unique but rather it admits finitely many variants corresponding to different embeddings of the curve into the field of complex numbers. This charactercisation is in fact a categoricity result in infinitary $L_{\omega_1\omega}$-logic.

However, in the talk I tried to bring attention to the following remark. Many basic facts used in the proof are either of the form that certain *topological* property of *algebraic* objects is actually algebraic (a complex algebraic morphism between varieties being a topological unramified covering; a Zariski connected component is a connected component in complex topology), or that the only obstructions to the existence of an algebraic object are of topological, homotopy nature: for a complex algebraic variety, its finite topological unramified covering carries a structure of an algebraic variety, unique up to a translation. There is another example which I did not have time to talk about and which relates to the arithmetic issues appearing in the proof. It is Kummer theory which is used to prove the existence of a prime model; roughly, it gives some conditions for the existence of Galois automorphisms of certain field extensions; I wanted to say that these conditions can also be interpreted using the language of homotopy theory.

References

[1] B. Zilber, *Covers of the multiplicative group of an algebraically closed field of characteristic zero*, to appear in JLMS.

[2] B. Zilber, *Pseudo-exponentiation on algebraically closed fields of characteristic zero*, Annals of Pure and Applied Logic **132** (2004), 67–95

[3] M. Gavrilovich, *Model Theory of the Universal Covering Spaces of Complex Algebraic Varieties*, DPhil thesis, Oxford, 2006

Asymptotic Cones of Nilpotent Groups

Guennady Noskov

According to Gromov's metaphor, the asymptotic cone of a metric space X captures the intuitive notion of looking at X from infinitely far away. Informally we take a "limit" of sequence of spaces $\frac{1}{n}X$ with metric scaled by $n \in \mathbb{N}$. The idea of asymptotic cone were first used by Gromov in [5], where he constructed limit spaces of Cayley graphs of finitely generated groups in order to prove that groups with polynomial growth are virtually nilpotent. Van den Dries and Wilkie in [4] gave a precise definition of the notion inspired by ideas of mathematical logic and nonstandard analysis. An extensive treatment of asymptotic cones is given by Gromov in [6]. There is an astounding interplay between the properties of the space and that of its asymptotic cone. For example, a group is virtually nilpotent if and only if all its asymptotic cones are locally compact [Drutu,Gromov].

In the talk we address the following questions: How far away is a metric space X from its asymptotic cone $\mathcal{C}X$? When X is within a finite Hausdorff distance from $\mathcal{C}X$?

The questions turn out to be nontrivial already in the simplest case of abelian groups with an invariant metric. The following result can be interpreted as a finiteness of a Hausdorff distance between $\mathbb{Z}^n$ and $\mathcal{C}\mathbb{Z}^n$.

Theorem 1. *(D. Burago* [3]*, Abels-Margulis* [1]*). For any proper coarsely geodesic length function* $|\cdot|$ *on* $\mathbb{Z}^n, n \geq 1,$ *there exists a constant* C *such that*

$$|x| - C \leq \lim_{n\to\infty} \frac{|nx|}{n} \leq |x|, \ \forall x \in \mathbb{Z}^n.$$

Indeed, the function $\lim_{n\to\infty} \frac{|nx|}{n}$ on $\mathbb{Z}^n$ extends homogeneously to $\mathbb{Q}^n$ and then continuously to $\mathbb{R}^n$.The space $\mathbb{R}^n$ with a resulting stable norm $\|\cdot\|$ is the asymptotic cone $\mathcal{C}\mathbb{Z}^n$ and the result shows the finiteness of Hausdorff $d_H(\mathbb{Z}^n, \mathcal{C}\mathbb{Z}^n)$, where $\mathbb{Z}^n$ is considered with the length function $|\cdot|$ and $\mathbb{R}^n$ is considered with the stable norm.

The next interesting case to look is that of finitely generated nilpotent groups. P.Pansu in [9] gave a detailed description of the geometry of the asymptotic cone of a finitely generated nilpotent group Γ: it is a graded nilpotent Lie group G_∞ with a certain leftinvariant metric d_∞. While G_∞ depends only on Γ, the metric d_∞ depends on the choice of a norm on Γ which may be for example a word norm associated to a generating system. In any case d_∞ is a Carnot-Caratheodory-Finsler (=subfinslerian) metric associated to a leftinvariant subbundle of TG_∞.

There exist two finitely generated nilpotent groups Γ_1, Γ_2 (in every dimension ≥ 7) which have isomorphic graded Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, but different Betti numbers b_2 (Y.Benoist, exposition in [10]). It follows that their asymptotic cones $Lie\,(\mathfrak{g}_1)\,, Lie\,(\mathfrak{g}_2)$ are isomorphic as Lie groups. Nevertheless Γ_1, Γ_2 are even not quasi-isometric because they have distinct Betti numbers [10]. If both Γ_1 and Γ_2 were finite distance from $Lie\,(\mathfrak{g}_1)\,, Lie\,(\mathfrak{g}_2)$ respectively, then since $Lie\,(\mathfrak{g}_1) \simeq Lie\,(\mathfrak{g}_2)$ they would be quasi-isometric, contradicting the previous claim. It follows that at least one of the groups Γ_1, Γ_2 is infinite distance away from its asymptotic cone. The evident reason for this phenomenon is that Γ_1, Γ_2 are not graded.

The main result of our work is

Theorem 2. *Let* Γ *be a finitely generated torsionfree 2-step nilpotent group with a horizontal word metric* d *and let* (G_∞, d_∞) *be the corresponding asymptotic cone. Then* $|d - d_\infty|$ *is bounded. In particular, the asymptotic cone of* Γ *is within a finite Hausdorff distance from* Γ.

The theorem is also proven in the case of 3-dimensional Heisenberg group by S.Krat [7, 8]. The debt we owe to the papers [11, 2] can not be overestimated. In particular the consideration of "$\mathbb{R}$-word metrics" is essential in the proof. These are Lie theoretic analogs of word metrics in abstract groups. Namely we consider a precompact generating set A in the Lie algebra $\mathfrak{g}$ of a Lie group G. We prove that then A is an $\mathbb{R}$-generating set for G in the following sense: for each $g \in G$ there is a curve from 1 to g, which is a finite concatenation of finite pieces of translated 1-parameter subgroups $e^{ta}, a \in A$. Infimizing the naturally defined length of such a curve, we obtain the $\mathbb{R}$-word length of g.

In the course of the proof we obtain also the following result, which is of independent interest:

Theorem 3. *Any $\mathbb{R}$-word metric on any real Lie group is of Carnot-Caratheodory-Finsler type.*

References

[1] Abels, Herbert and Margulis, Gregory, *Coarsely geodesic metrics on reductive groups,* in "Modern dynamical systems and applications", Cambridge Univ. Press, 2004, pp.163–183,

[2] Berestovskiĭ, V. N., *Homogeneous manifolds with an intrinsic metric.* {*I*}, Sibirsk. Mat. Zh., v.29, 1988, pp.17–29,

[3] Burago, D. Yu.,*Periodic metrics*, in "Representation theory and dynamical systems", Adv. Soviet Math., v.9, 1992, pp.205–210,

[4] van den Dries, L. and Wilkie, A. J.,*Gromov's theorem on groups of polynomial growth and elementary logic*,J. Algebra,v.89,1984, pp.349–374,

[5] M.Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. v.53, 1981, pp.53–73

[6] Gromov, M., *Asymptotic invariants of infinite groups*, in "Geometric group theory, Vol. 2 (Sussex, 1991)", Cambridge Univ. Press,v.182, 1993, pp.1–295,

[7] Krat, S. A., *Asymptotic properties of the Heisenberg group*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), v.261,1999, pp.125–154,

[8] Krat, S. A., *On pairs of metrics invariant under a cocompact action of a group*, Electron. Res. Announc. Amer. Math. Soc., v.7, 2001, pp.79–86,

[9] Pansu, Pierre, *Croissance des boules et des géodésiques fermées dans les nilvariétés*, Ergodic Theory Dynam. Systems, v.3, 1983, pp. 415–445,

[10] Shalom, Yehuda, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. , v.192, 2004, pp.119-185,

[11] Stoll, Michael, *On the asymptotics of the growth of 2-step nilpotent groups*, J. London Math. Soc. (2), v.58,1998, pp.38–48.

On the Dynamics of Automorphisms of Free Groups

Gilbert Levitt

(joint work with Martin Lustig)

Let G be a finitely generated group. Let α be an automorphism of G, and $g \in G$. What can be said about the sequence $\alpha^n(g)$ as $n \to +\infty$?

• The element g may be periodic (there is $q \geq 1$ such that $\alpha^q(g) = g$). In this case, one may ask what can be said about the period (the smallest q) in a given G, or for a given α.

• If g is not periodic, then $\alpha^n(g)$ becomes long. One may ask how it grows.

• Does the sequence $\alpha^n(g)$ have a limit (in a suitable sense)? Or rather: does α have a power β such that all sequences $\beta^n(g)$ have a limit?

It turns out that the last question has a positive answer in a non-abelian free group F_p, and a negative answer in an abelian group Z^p, but the talk focused on growth.

In the case of Z^p, it is easy to see that $\alpha^n(g)$ always grows like $n^d\lambda^n$ for some $d \in N$ and $\lambda \geq 1$. The same result is true in the case of F_p, but one needs the train tracks of Bestvina-Feighn-Handel in order to control cancellation.

When an automorphism α of F_p is induced by a homeomorphism φ of a compact surface Σ, it follows from Nielsen-Thurston theory that the growth of $\alpha^n(g)$ (or rather of its conjugacy class) is always linear or exponential (linear growth comes from Dehn twists, exponential growth from pseudo-Anosov components).

An automorphism such that $a \mapsto a$, $b \mapsto ba$, $c \mapsto cb$, $d \mapsto dc$ is not induced by such a φ, because $\alpha^n(c)$ grows quadratically and $\alpha^n(d)$ grows cubically. Adding generators, one sees that there is an automorphism α of F_p and elements g_i ($1 \leq i \leq p-1$) such that $\alpha^n(g_i)$ grows like n^i. In general, we prove that, given an automorphism of F_p, the number of different growth types (d, λ) is at most $p-1$.

One may ask whether there may exist $p-1$ different exponential growth types λ_i^n. If one first considers automorphisms induced by homeomorphisms of surfaces, one is led to ask how many pseudo-Anosov components there may be in the Nielsen-Thurston reduction of a given homeomorphism of a compact surface Σ with fundamental group F_p. It turns out that the answer is not $p-1$ (because there is no pseudo-Anosov homeomorphism on a thrice-punctured sphere), but the integral part of $\frac{3p-2}{4}$. This maximal value is achieved by a homeomorphism whose pseudo-Anosov components are carried by once-punctured tori and four-punctured spheres.

We prove that, given any automorphism of F_p, the number of exponential growth types is at most $\frac{3p-2}{4}$.

Jets and Prolongations

Thomas Scanlon
(joint work with Rahim Moosa)

In joint work with Rahim Moosa, we develop a general theory of jet and prolongation spaces for $\mathcal{D}$-schemes, generalizing the construction of jet spaces for differential and difference varieties of finite dimension of Pillay-Ziegler.

Univers $\aleph_1$-catégoriques. Un prérequis à la communication de Monsieur Martin Hils

Bruno Poizat

L'article [1] introduit les univers, et pose dix problèmes à leur propos.

1. Quelques définitions

- *Univers* d'une structure M (de langage $\mathcal{L}$) = l'ensemble des parties de M^n, pour chaque $n \in \omega$, qui sont définissables *avec paramètres* dans M. Les relations de $\mathcal{L}$ sont considérées comme un système générateur de l'univers.
- *Transformation* de la structure M, de langage $\mathcal{L}$, en la structure M', de langage $\mathcal{L}'$ = une bijection entre M et M' qui, pour chaque n, échange les parties définissables de M^n et celles de M'^n.

  Exemples. (1) *Un isomorphisme entre deux structures de même langage.*
  (2) *Une permutation de M définissable.*

- *Structures semblables* (dans des langages peut-être différents) = ont des extensions élémentaires transformables l'une en l'autre.

Remarque. *La notion d'extension élémentaire ne dépend pas du langage choisi pour engendrer le petit univers. La similitude, comme la transformabilité, sont en fait des propriétés des univers.*

Exemple. *Deux structures élémentairement équivalentes. Mais, attention ! : deux univers semblables ne sont pas nécessairement engendrés par des structures élémentairement équivalentes.*

Théorème. *La similitude est une relation d'équivalence.*

Démonstration. Il faut voir que c'est transitif ; M, de langage $\mathcal{L}$, est semblable à M' de langage $\mathcal{L}'$, et à M" de langage $\mathcal{L}$" ; on peut trouver une extension élémentaire N' de M dont l'univers est celui d'une extension élémentaire de M', et une extension élémentaire N" de M dont l'univers est celui d'une extension élémentaire de M" ; N' et N" ont une extension élémentaire commune N dans le langage $\mathcal{L}$; dans le langage $\mathcal{L}'$, N est extension élémentaire de M', et dans le langage $\mathcal{L}$", N est extension élémentaire de M". □

- *Univers mince* = engendré par un ensemble fini de relations = engendré par une seule.

Théorème. *Si U est engendré par une relation r satisfaisant un énoncé $\varphi(r)$, il en est de même pour chacun de ses semblables.*

Démonstration. Soient U' semblable à U, et U" extension élémentaire commune. On trouve dans U" une relation r', provenant de U', qui permet de définir r, et toutes les relations de U" (avec des paramètres dans U"), si bien que r' engendre U' (avec des paramètres dans U' !). Dans U", on trouve des paramètres permettant d'interdéfinir avec r' une relation satisfaisant φ ; on en trouve aussi dans U'. □

Remarque. *Si U' est oméga-saturé (une fois un langage générateur fini fixé), on peut lui trouver une génératrice élémentairement équivalente à r.*

2. Le problème de la classification des univers

- *Classification des structures* : décrire à isomorphisme près, quand c'est possible, toutes les structures élémentairement équivalentes à une structure donnée.
- *Classification des univers* : décrire à transformation près, quand c'est possible, toutes les structures semblables à une structure donnée.

Ces problèmes sont liés, mais non équivalents ; tout d'abord parce que des univers semblables ne sont pas nécessairement engendrés par des structures élémentairement équivalentes (question d'omniprésence de la génératrice) ; et ensuite parce que deux relations élémentairement équivalentes non isomorphes peuvent avoir des univers semblables (question du caractère classifiant de la génératrice).

Théorème. *Un univers mince est oméga-catégorique si et seulement si c'est l'univers d'une structure de langage fini oméga-catégorique.*

Démonstration. Quand on fixe un langage générateur fini, il n'y a qu'une famille dénombrable de types ; en conséquence la structure est oméga-saturée, et isomorphe à chacune de ses extensions élémentaires dénombrables. □

Remarque. *Il existe des univers oméga-catégoriques dénombrablement engendrés qui interprètent des relations non oméga-catégoriques.*

Théorème. *Si un univers mince est catégorique en un cardinal non dénombrable, il est catégorique en tout cardinal non dénombrable (et chacune de ses relations génératrices est indénombrablement catégorique).*

Démonstration. Grâce à la construction d'Ehrenfeucht, l'univers est oméga-stable ; on voit ensuite que tous ses semblables en un certain cardinal non dénombrable sont saturés, une fois un langage générateur fini fixé ; d'après le Théorème de Morley, cela se produit en tout cardinal non dénombrable. Si r engendre U et U' est semblable à U et oméga-saturé, U' est engendré par une relation r' élémentairement équivalente à r. □

Rappelons le Théorème de Baldwin et Lachlan, connu au Qazaqstan sous le nom de Théorème de Mustafin et Taimanov : Si une structure (de langage fini ou dénombrable) est indénombrablement catégorique, mais pas oméga-catégorique, ses équivalentes élémentaires dénombrables sont : $M_0, M_1, \dots M_n, \dots M_\omega$, où M_0 est le modèle premier, M_n est le modèle de dimension n sur le modèle premier, et M_ω est le modèle saturé.

Est-il valable pour les univers minces ?

Dans ce cas, il y a au moins deux (il n'est pas sûr qu'il y en ait trois !) semblables dénombrables, dont le saturé ; les non-saturés sont les univers du modèle premier d'une relation génératrice du saturé ; ils forment une famille finie ou dénombrable.

Il n'est pas clair qu'il y en ait une infinité, car pourquoi M_n et M_{n+m} ne sont-elles pas transformables l'une en l'autre ? Il n'est pas clair non plus qu'ils soient totalement ordonnés par plongement élémentaire.

Martin Hils va présenter (voir l'exposé suivant) un contre-exemple où ils forment une chaîne de type $\mathbb{Z}$.

References

[1] B. Poizat, *À la recherche de la structure intrinsèque de l'Univers*, Théorie des modèles au Qazaqstan (en russe), à la mémoire de A. D. Tajmanov, Ecostudy, Almaty, 2006.
[2] B. Poizat, *Univers positifs*, J. Symbolic Logic **71**(3) (2006), 969–976.

Semifree Actions of Free Groups

Martin Hils

We give an example of a thin universe which is $\aleph_1$-categorical and where the countable universes similar to it are ordered (with respect to elementary embeddability) like $(\mathbb{Z}\cup\{\infty\}, <)$. In fact, we show in [1] that this the case for the universe of a free F_k-action, F_k being the free group on k generators ($1 < k < \omega$). Thus, the Baldwin-Lachlan theorem does not hold for universes. This answers negatively a question raised in [2] (see also Poizat's contribution in this volume).

It is shown in [2] that there is an example of a two-dimensional thin universe having exactly 2 similar non-transformable countable universes. In the corresponding structure, both dimensions are given by minimal types with a trivial pregeometry. The study of the universe of a free G-action grew out of the attempt to find a similar example which is uncountably categorical or even strongly minimal. Since a free group action is the prototype of a trivial strongly minimal theory, it seemed to be very natural to look at these universes.

In what follows, to simplify the exposition, we assume that G is a finitely presented infinite group.

Definition.

- Let (G, X_i, φ_i) be infinite G-actions ($i = 1, 2$), where $\varphi_i : G \to Sym(X_i)$. A bijection $F : X_1 \to X_2$ is called a *transformation* if $F(\varphi_1(g)) \sim \varphi_2(g)\ \forall g \in G$ (i.e. coincide almost everywhere; note that it is sufficient to check this on a set of generators of G).
- An infinite G-action (G, X, φ) is *semifree* if after adding a finite number of regular G-orbits it can be transformed into a free G-action.
- G is called *classifying* if the following holds: whenever the free G-action with n regular orbits is transformable into the free G-action with m regular orbits (where $m, n \in \mathbb{N}$), then $m = n$.
- G is called *ubiquitous* if every semifree G-action can be transformed into a free G-action.

The following basic result shows that the above notions are a perfect translation of their analogues for universes, and so the classification problem for the universe of a free G-action can be phrased in group-theoretical terms:

Fact. *The category of semifree G-actions (up to transformations as defined above) is in 1:1 correspondence with the category of universes similar to a free G-action (up to transformations of universes).*

There are some rather general results concerning the classification problem. Recall that for a (finitely generated) goup G, the *number of ends* is defined to be the number of ends of its Cayley graph with respect to some finite set of generators (this is welldefined).

Proposition.
(1) *If the number of ends of G is 1, then G is classifying and ubiquitous.*
(2) *Every amenable group is classifying.*
(3) *Every abelian group is classifying and ubiquitous.*

Theorem (Classification of semifree actions of free groups)**.**
Let $1 < k < \omega$. Then, the countable semifree F_k-actions form a chain of the form $(\mathbb{Z} \cup \{\infty\}, <)$. In particular, F_k is classifying but not ubiquitous.

Sketch of proof. One first shows that for every $m, n \in \mathbb{N}$, with $m > 0$, the F_k-action with m regular and n trivial orbits is semifree.

Then, looking at the growth of the borders of finite (connected) sets in the Cayley graphs of the respective actions, it is not hard to see that the integer $m - n(k-1)$ is an invariant up to transformation. Moreover, an explicit construction of a transformation between too F_k-actions of this kind having the same invariant can be given.

Finally, one shows that every countable semifree F_k-action whose universe is not ω-saturated can be transformed to one of the above form. □

Note that F_ω is not classifying. Indeed, all countable semifree F_ω-actions are transformable. So the corresponding universe is totally categorical (and has non-ω-categorical reducts).

Open questions.

(1) *Is there a finitely presented (or at least a finitely generated) group which is not classifying?*
(2) *Is every finitely presented amenable group ubiquitous?*

References

[1] M. Hils, *Semifree Actions of Free Groups*, to appear in Archive Math. Logic.
[2] B. Poizat: *À la recherche de la structure intrinsèque de l'univers*, Teoriia modelei v Kazaxstane, Sbornik naucnyx rabotposviaçennyx pamiati A.D. Taimanova, Eco Study, Almaty, **2006**, 339–388.

Mild K-manifolds

Sergei Starchenko
(joint work with Ya'acov Peterzil)

We fix an o-minimal expansion $\mathcal{R}$ of a real closed field R and denote by K its algebraic closure $R(\sqrt{-1})$. By *definable* we will always mean definable in $\mathcal{R}$.

We refer to [1, 2, 3] for basic definitions and properties of K-holomorphic functions.

A *K-manifold* is a definable set M, equipped with a finite cover of definable sets $M = \bigcup_i U_i$, each of which is in definable bijection with an open subset of K^n such that the transition maps are K-holomorphic maps between open subsets of K^n. K-holomorphic maps between K-manifolds are defined using the charts of the manifold.

A K-analytic subset of a K-manifold M is a definable set $A \subseteq M$, such that at every point $z \in M$, the set A is given, locally near z, as the zero set of some K-holomorphic function.

Let M be a K-manifold. We will denote by $\mathcal{A}(M)$ the first order structure whose universe is M and basic relations are K-analytic subsets of Cartesian powers of M.

Definition. A K-manifold M is called *mild* if the structure $\mathcal{A}(M)$ admits a quantifier elimination.

Example. If M is a compact complex manifold then M, considered as a $\mathbb{C}$-manifold definable in the structure $\mathbb{R}_{an}$, is mild.

The above example generalizes to the non-standard setting.

Theorem 1. *If M is a definably compact K-manifold then M is mild.*

The following theorem is a generalization of Zilber's observation to mild manifolds.

Theorem 2. *If M is a mild K-manifold then the structure $\mathcal{A}(M)$ has finite Morley rank, and it is a Zariski structure.*

Our next theorem is one of the main sources of mild manifolds.

Theorem 3. *Let M be a mild K-manifold and $A \subseteq M$ a K-analytic subset of M. Then the set $reg(A)$ of smooth points of A is a mild K-manifold.*

As in the case of compact complex manifolds we also have a strong version of Zilber's trichotomy.

Theorem 4. *Let M be a mild K-manifold such that the structure $\mathcal{A}(M)$ is strongly minimal. If $\mathcal{A}(M)$ is not locally modular then M is K-biholomorphic with an algebraic curve.*

References

[1] Y. Peterzil and S. Starchenko, *Expansions of algebraically closed fields in o-minimal structures*, Selecta Math. (N.S.) **7** (2001), no. 3, 409–445.

[2] Y. Peterzil and S. Starchenko, *Expansions of algebraically closed fields II: Functions of several variables*, J. Math. Log. **3** (2003), no. 1, 1–35.

[3] Y. Peterzil and S. Starchenko, *Complex analytic geometry in a nonstandard setting*, Preprint.

On Analogies between Algebraic Groups and Groups of Finite Morley Rank

Tuna Altinel

(joint work with Jeffrey Burdges)

Linear algebraic groups and infinite groups of finite Morley rank have many analogous properties. This is what is emphasized in the strongest possible way by the central problem in the analysis of groups of finite Morley rank, namely the *Cherlin-Zilber algebraicity conjecture* which states that an infinite simple group of finite Morley rank is a linear algebraic group over an algebraically closed field. In the last fifteen years an increasing number of affirmative answers have been given to special cases of this conjecture. Nevertheless major portions of the problem remain open and counterexamples are not unexpected.

In recent years, results which elucidate strong analogies between algebraic groups and groups of finite Morley rank without proving specific isomorphism theorems have reappeared in the area. Such theorems are reminiscent of the early work of Daniel Lascar and Bruno Poizat later developed by Frank Wagner, and the most important examples are in [4], [6], [1] and [2].

An interesting common point of the main theorems in these four papers is that they permit the introduction of an abstract notion of *Weyl group* which corresponds in the algebraic category to the usual Weyl group. More precisely, there are several possible definitions, and especially [6] and [2] provide keys to the relationships among these possibilites. In this context the centralizers of *tori* are of importance. The following is known for all connected linear algebraic groups:

Fact 1 ([5, Theorem 22.3])**.** *Let G be a connected linear algebraic group and S be a torus in G. Then $C_G(S)$ is connected.*

In my talk, I first gave a detailed history of related results on groups of finite Morley rank. Then I stated and sketched the proof of the following weak analogue of Fact 1.

Theorem 2. *Let G be a connected group of finite Morley such that if A is a non-trivial definable solvable subgroup of G then $N_G^\circ(A)$ is solvable. If T is a decent torus in G then $C_G(T)$ is connected.*

A *decent torus* in a group of finite Morley rank is a definable, divisible, abelian subgroup which is the *definable hull* of its torsion [3]. Any group of finite Morley rank which contains a nontrivial copy of the Prüfer p-group for some prime p contains a nontrivial decent torus, namely the definable hull of this divisible abelian p-subgroup.

In the ongoing work with Jeffrey Burdges, we have obtained various applications of this theorem. One of them is motivated by the following well-known theorem about connected linear algebraic groups:

Fact 3 ([5, Theorem 22.2]). *Let G be a connected linear algebraic group and B a Borel subgroup of G. Then the union of all conjugates of B is G.*

I exposed in Oberwolfach the following weak analogue again restricted to severe minimal conditions:

Theorem 4. *Let G be a connected group of finite Morley rank such that if A is a non-trivial definable solvable subgroup of G then $N^{\circ}_{G}(A)$ is solvable. Then every element of G belongs to a Borel subgroup.*

In the minimal context described by the hypotheses of the above theorems, one is also able to prove results about the Weyl groups in groups of finite Morley rank. Moreover, this same context allows sufficient generality for analyzing minimal simple groups of finite Morley rank together with the action of a definable group of automorphisms. I stated in my talk some of the results which we have obtained in these rich directions. Since they are more technical and in the process of being improved, I do not give precise statements.

The joint work with Burdges which I exposed in Oberwolfach on 19 january 2007 is part of an improving process of which one main final objective would be to prove the above results at the right level of generality. We are currently working towards achieving this objective.

References

[1] A. Borovik, J. Burdges, and G. Cherlin. Involutions in groups of morley rank of degenerate type. To appear in *Selecta Mathematica*.

[2] J. Burdges and G. Cherlin. Semisimple torsion in groups of finite morley rank. in preparation.

[3] G. Cherlin. Good tori in groups of finite morley rank. *J. Group Theory*, 8:613–621, 2005.

[4] O. Frécon and E. Jaligot. The existence of Carter subgroups in groups of finite Morley rank. *J. Group Theory*, 8:623–633, 2006.

[5] J. E. Humphreys. *Linear Algebraic Groups*. Springer Verlag, Berlin-New York, second edition, 1981.

[6] E. Jaligot. Generix never gives up. *J. Symbolic Logic*, 71:599–610, 2006.

Groups of Finite Quantifier-free Rank

Abderezak Ould Houcine

In a preceding work [1], we have studied existentially closed (abbreviated e.c.) CSA-groups. This was motivated by a question of Poizat whenever an e.c. CSA-group is a bad group and by the fact that a bad group of Morley rank 3 is a CSA-group. We showed in [3] that these groups are not superstable. This is due to the existence of a lot of quantifier-free types.

One defines a quantifier-free rank (abbreviated QF-rank), by analogy with the Morley rank restricted to quantifier-free formulas (abbreviated QF-formulas).

Definition. Let $\mathcal{M}$ be a model and let $X \subseteq \mathcal{M}$ be a QF-definable set. We define $QF(X) = RC(X)_{\mathcal{M}^*}$, where $\mathcal{M}^*$ is an $\aleph_0$-saturated elementary extension of $\mathcal{M}$ and $RC(X)_{\mathcal{M}^*}$ is the Cantor rank of X in $\mathcal{M}^*$ relatively to the boolean algebra of QF-definable subsets. We define $QF(\mathcal{M}) = QF(x = x)$.

It follows that a subgroup of a group of finite Morley rank has a finite QF-rank. So in any tentative of construction of groups of finite Morley rank from classes of finitely generated groups, it is necessary to take finitely generated groups of finite QF-rank.

As linear groups are subgroups of groups of finite Morley rank, they have a finite QF-rank and this gives a model-theoretic context in which we can study them.

We discuss here some results about groups of finite QF-rank [2]. First we have the following important tool.

Theorem 1. *Let $\mathcal{M}$ be a model of ordinal QF-rank. Let $\phi(\bar{x})$ and $\psi(\bar{x}, \bar{y})$ be a QF-formulas. Then there exists a finite sequence $\bar{a}_{ij} \in \phi(\mathcal{M})$, such that the set*

$$\{\bar{b} \in \mathcal{M} \mid QF(\phi(\bar{x}) \wedge \psi(\bar{x}, \bar{b})) = QF(\phi)\},$$

is QF-definable in $\mathcal{M}$ by $\bigvee_{i=1}^{i=n} (\bigwedge_{j=1}^{j=m} \psi(\bar{a}_{ij}, \bar{y}))$. □

As in the context of groups of finite Morley rank one introduces: stabilizer relatively to the action of G on the set of QF-1-types, QF-generic sets, QF-connectedness and so on ...

Some results are generalizable but by using methods which are different from those in the finite Morley rank case. For instance, for any QF-1-type p, $stab(p)$ is QF-definable and if X is a QF-generic set then a finite number of translates of X cover G. As an application we have:

Theorem 2. *Let $\mathcal{M}$ be a nonabelian model of the universal theory of free groups. Then $\mathcal{M}$ is QF-connected, $\mathcal{M}$ is e.c. in $\mathcal{M} * \mathbb{Z}$, and every QF-definable proper subgroup of $\mathcal{M}$ is abelian.* □

Here $\mathcal{M} * \mathbb{Z}$ denotes the free product of $\mathcal{M}$ and $\mathbb{Z}$. The fact that $\mathcal{M}$ is QF-connected and $\mathcal{M}$ is e.c. in $\mathcal{M} * \mathbb{Z}$ can be deduced from [4, 5].

In that context the following questions are open.

Question 3. *If $\mathcal{M}$ is a nonabelian model of the universal theory of free groups, is $\mathcal{M} \preceq \mathcal{M} * \mathbb{Z}$?*

Question 4. *If $\mathcal{M}$ is a nonabelian model of the universal theory of free groups such that $\mathcal{M} \preceq \mathcal{M} * \mathbb{Z}$, is any proper definable subgroup of $\mathcal{M}$ abelian ?*

The answer to the question 3 is positive when $\mathcal{M}$ is finitely generated and this was pointed out to me by Sela. The answer to the question 4 is positive if $\mathcal{M}$ is a nonabelian free group and this was anounced by Bestvina and Feighn.

In dealing with finitely generated groups of finite QF-rank the following question arises naturally.

Question 5. *Is there any alternative for groups of finite QF-rank which looks like the Tits alternative?*

First one looks at locally (solvable-by-finite) groups.

Theorem 6.
- *A locally (solvable-by-finite) group of finite QF-rank is either locally finite or has a nontrivial normal abelian subgroup.*
- *A locally solvable group of finite QF-rank is solvable.* □

It follows that the solvable radical $R(G)$ is solvable (which is in fact definable by equations), and if G is locally (solvable-by-finite) then $G/R(G)$ is locally finite. We have also that if $H \leq G$ is a QF-definable subgroup then $N_G(H)$ is QF-definable. This generalizes known properties of linear groups.

Under additional hypotheses one can get an alternative. Recall that a group is called locally graded if every nontrivial finitely generated subgroup has a proper subgroup of finite index. For instance locally residually finite groups are locally graded as well as linear groups.

Theorem 7. *Let G be a locally graded group of finite QF-rank. Then either G is solvable-by-finite or there is an elementary extension $G \preceq K$ such that K contains a free semigroup.* □

In the context of p-groups we have.

Theorem 8. *A p-group of finite QF-rank is either nilpotent-by-finite or has a series $G_0 \trianglelefteq G_1 \leq G$, where G_1 is QF-definable, G_0 is equationally-definable in G_1, and G_1/G_0 is a nonabelian CSA-group. In particular a bad p-group interprets a CSA bad p-group.* □

Question 9. *Is there a CSA p-group of finite QF-rank ?*

As noticed above the study of groups of finite QF-rank was motivated by the understanding of subgroups of groups of finite Morley rank and to get several information in order to use an amalgamation method.

One looks initially at Fraïssé limits of classes of finitely generated groups having the amalgmation property. Let $\mathcal{K}$ be a countable class of finitely generated groups having the hereditary property (abbriviated HP) and the amalgamation property (abbriviated AP). Since one can amalgamate over the trivial group, $\mathcal{K}$ has the joint embedding property and by Fraïssé's theorem, $\mathcal{K}$ has an ultrahomogenous Fraïssé limit which will be denoted by $\widehat{\mathcal{K}}$.

Definition. Le $\mathcal{K}$ be a class of groups.

(1) $\mathcal{K}$ is said to be *closed under free products* if whenever $A, B \in \mathcal{K}$, $A * B \in \mathcal{K}$.

(2) $\mathcal{K}$ is said to be *closed under HNN-extensions* if whenever $A \in \mathcal{K}$ such that A contains two isomorphic finitely generated subgroups A_1, A_2 then the HNN-extension $A^* = \langle A, t | A_1^t = A_2 \rangle$ is also in $\mathcal{K}$.

Theorem 10. *Let $\mathcal{K}$ be a countable class of finitely generated groups having HP and AP.*

(1) *Suppose that $\mathcal{K}$ satisfies: if $A, B \in \mathcal{K}$ then $A * B \models Th_\forall(\widehat{\mathcal{K}})$. Then $\widehat{\mathcal{K}}$ is not superstable.*

(2) *If $\mathcal{K}$ is closed under free products, then $\widehat{\mathcal{K}}$ is simple and not superstable.*

(3) *If $\mathcal{K}$ is closed under HNN-extensions, then $\widehat{\mathcal{K}}$ is simple and unstable.* □

We notice that every e.c. group is the Fraïssé limit of its age.

References

[1] E. Jaligot, A. Ould Houcine, *Existentially closed CSA-groups*, Journal of Algebra, **280** (2004), 772–796.

[2] A. Ould Houcine, *Algebraic geometry over groups and quantifier-free rank*, work in progress.

[3] A. Ould Houcine, *On superstable CSA-groups*, to appear in Annals of Pure and Applied Logic.

[4] O. Kharlampovich, A. Myasnikov, *Implicit function theorem over free groups*, Journal of Algebra, **290** (2005), 1–203.

[5] G. Baumslag, A. Myasnikov, V. N. Remeslennikov, *Algebraic geometry over groups I: Algebraic sets and ideal theory*, Journal of Algebra, **219** (1999), 16–79.

Topological Dynamics and Model Theory

Ludomir Newelski

Assume G is a group definable in a sufficiently saturated first-order structure M. $\mathfrak{C}$ denotes a monster model.

The investigation of combinatorial properties of coverings of groups by countably many type-definable sets hinted that there should be a notion of a "large" subset of a group, generalizing that of a generic subset, present in the stable context [1]. Consequently, in [2] I proposed a notion of a weak generic set and type (in a group).

Namely, we say that a set $U \subseteq G$ is *(left) generic* if some finitely many left translates of U cover G. We say that U is *weak generic* if for some non-generic $V \subseteq G$ we have that $U \cup V$ is generic. The notion of a weak generic type is defined accordingly. Unlike generic types, complete weak generic types always exist.

This notion was subsequently used to show some combinatorial properties of countable coverings of groups by type-definable sets. Eventually it turned out however that the proper set-up for these considerations is that provided by topological dynamics. In fact, topological dynamics puts in proper perspective also some classical results on generic types in stability theory.

Topological dynamics deals with actions of a group G on a compact space X, by homeomorphisms. In this case, X is called a *G-flow*. A closed invariant subset Y of X is called a subflow of X. Of particular interest are minimal subflows (with respect to inclusion), whose elements are called *almost periodic*.

In the model theoretic context the natural G-flow to consider is the space of types $S_G(M)$ of elements of $G(\mathfrak{C})$ over M, on which G acts by left translation. It turns out that the set of weak generic types in $S_G(M)$ is the topological closure of the set of almost periodic types.

A new ingredient added in the model-theoretic context to topological dynamical investigations is a comparison of G-flows $S_G(N)$ for various elementary extensions N of M. This involves some extension properties of weak generic and almost periodic types.

I have constructed an example, where there is a weak generic type that is not almost periodic, and also an example, where a weak generic type that is not almost periodic has an extension (over a larger model) to an almost periodic type. This example is obtained by expanding the additive group of rationals by some "semi-generic" predicates, and the resulting structure has simple theory of SU-rank 1.

Still no example is known, where the theory is simple and weak generic types differ from forking-generic types. (In a simple theory, every weak generic type is a forking-generic type.)

Maybe the most intriguing notion in topological dynamic is the Ellis semigroup. This is just the closure (in the topology of pointwise convergence) of the set of homeomorphisms of a G-flow X, induced by elements of G, and the semi-group operation is the composition of functions. It turns out that this semi-group contains a "characteristic" subgroup related (conjecture) to the group G/G^{00}. I managed to interpret the Ellis semigroup model-theoretically. These results appear in [3].

Also, the whole new set-up may be extended to generalize forking.

References

[1] L. Newelski, M. Petrykowski, *Coverings of groups and types*, Proc. London Math.Soc. (2) **71** (2005) no.1, 1-21.

[2] L. Newelski, M. Petrykowski *Weak generic types and coverings of groups I*, Fund. Math. **191** (2006), 201-225.

[3] L. Newelski, *Topological dynamics of definable groups actions*, J. Symb. Logic, submitted.

Division Points on Subvarieties of Isotrivial Semiabelian Varieties

Rahim Moosa
(joint work with Dragos Ghioca)

In its most general form, the Mordell-Lang theorem in characteristic 0 was proved by McQuillan [5] and states that if G is a semiabelian variety over $\mathbb{C}$, $\Lambda \leq G(\mathbb{C})$ is a finite rank subgroup, and $X \subset G$ is an irreducible subvariety whose intersection with Λ is Zariski dense, then X is a translate of an algebraic subgroup of G. (Recall that a subgroup is said to be of *finite rank* if it is contained in the divisible hull of a finitely generated subgroup.) This fails when $\mathbb{C}$ is replaced by a field of positive characteristic. In [1], Abramovich and Voloch formulate and conjecture a function-field version of the Mordell-Lang statement in positive characteristic. The conjecture is proved, using model-theoretic techniques, by Hrushovski:

Theorem 1 (Mordell-Lang [3]). *Suppose L is an algebraically closed field of characteristic $p > 0$, G is a semiabelian variety defined over L, $X \subset G$ is an irreducible subvariety defined over L, and $\Lambda \leq G(L)$ is a subgroup of the prime-to-p divisible hull of a finitely generated subgroup of $G(L)$. If $X(L) \cap \Lambda$ is Zariski dense in X, then X is special. That is, $X = g + h^{-1}(X_0)$ where $g \in G(L)$, $h : G' \to G_0$ is a surjective homomorphism from an algebraic subgroup G' of G to a semiabelian variety G_0 defined over $\mathbb{F}_p^{\mathrm{alg}}$, and X_0 is a subvariety of G_0 also defined over $\mathbb{F}_p^{\mathrm{alg}}$.*

Besides the necessary modification of the conclusion from "translate of algebraic subgroup" to "special" (note that translates of algebraic subgroups are special), this theorem differs from the characteristic 0 version in that it only applies to subgroups of the *prime-to-p* divisible hull of a finitely generated group. Indeed, the more general statement is not accessible by the methods of [3] and remains open:

Conjecture 2 (Full Mordell-Lang). *Suppose $\Lambda \leq G(L)$ is a finite rank subgroup and $X \subset G$ is an irreducible subvariety. If $X(L) \cap \Lambda$ is Zariski dense in X, then X is special.*

Here is a summary of what we accomplish in [2]:

I. We reduce Conjecture 2 to the case of $\Lambda \leq G(K^{\mathrm{per}})$ where K is a finitely generated field and $K^{\mathrm{per}} := \{a \in K^{\mathrm{alg}} \mid a^{p^n} \in K \text{ for some } n \in \mathbb{N}\}$ is the *perfect closure* of K. This is done by combining model theoretic methods of Scanlon [8] with an idea of Rössler's [7]. As a consequence, using a result of Kim's [4], we resolve the curve case of Conjecture 2.

II. We resolve Conjecture 2 for semiabelian varieties defined over finite fields. Besides (I), we make use of a uniform description, obtained by Scanlon and the second author [6], of sets of the form $X(L) \cap \Gamma$ where Γ is a finitely generated subgroup of $G(L)$ that is invariant under the Frobenius endomorphism of G.

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III. Still in the isotrivial case, we extend the results of [6] to give an explicit description of sets of the form $X(L)\cap\Gamma^{\mathrm{div}}$ where Γ is a finitely generated subgroup of $G(L)$ that is invariant under the Frobenius endomorphism of G and $\Gamma^{\mathrm{div}} := \{g \in G(L) \mid ng \in \Gamma \text{ for some } n \in \mathbb{Z}\}$ is the *divisible hull* of Γ. See Theorem 3.20 of the paper for a precise statement.

References

[1] D. Abramovich and J.F. Voloch, *Toward a proof of the Mordell-Lang conjecture in characteristic p.*, International Mathematics Research Notices (1992), no. 5, 103–115.

[2] D. Ghioca and R. Moosa, *Division points on subvarieties of semi-abelian varieties*, International Mathematics Research Notices (2006), 1–23, Article ID 65437.

[3] E. Hrushovski, *The Mordell-Lang conjecture for function fields*, Journal of the American Mathematical Society **9** (1996), no. 3, 667–690.

[4] M. Kim, *Purely inseparable points on curves of higher genus*, Mathematical Research Letters **4** (1997), no. 5, 663–666.

[5] M. McQuillan, *Division points on semi-abelian varieties*, Inventiones Mathematicae **120** (1995), no. 1, 143–159.

[6] R. Moosa and T. Scanlon, *F-structures and integral points on semiabelian varieties over finite fields*, American Journal of Mathematics **126** (2004), 473–522.

[7] D. Rössler, *An afterthought on the generalized Mordell-Lang conjecture*, To appear in the proceedings of the workshop on model theory, algebraic and analytic geometry (Newton Institute, Cambridge, July 11–15, 2005).

[8] T. Scanlon, *Positive characteristic manin-mumford theorem*, Compositio mathematica **141** (2005), no. 6, 1351–1364.

Cotorsion Groups and Modules

Philipp Rothmaler

Consider the following problem (for the terminology refer to [3]). Given any ring R and a class of left R-modules, $\mathcal{X}$, what does it mean for R-Mod, the category of left R-modules, that every pp sort φ/ψ that does not open up in (a member of) $\mathcal{X}$ satisfies the descending chain condition for pp formulas in R-Mod (*dcc*, henceforth)?

For short, this property of the ring R will be denoted by *co-$\mathcal{X}$-dcc.* Typically, $\mathcal{X}$ will be axiomatized by pp implications, hence by the shutting of pp pairs (i.e. $\mathcal{X}$ will be an axiomatizable class closed under product). Then one may reformulate the problem within the left Ziegler spectrum, ${}_R\mathrm{Zg}$, of R, for then the intersection of $\mathcal{X}$ with ${}_R\mathrm{Zg}$ is a typical closed set of ${}_R\mathrm{Zg}$ and co-$\mathcal{X}$-dcc is the same as co-$({}_R\mathrm{Zg} \cap \mathcal{X})$-dcc.

Note, co-$\mathcal{X}$-dcc does not mean that every point outside $\mathcal{X}$ in the Ziegler spectrum is totally transcendental, but just that the particular sort in M^{eq} that defines the neighborhood is totally transcendental. However, the property in question indicates that, somehow, R-Mod is controlled by $\mathcal{X}$ and totally transcendental pieces.

Clearly, one can always pass to such an axiomatizable $\mathcal{X}$ by passing to the model class of the theory of the original class — without affecting the co-$\mathcal{X}$-dcc. E.g., starting from the class $\flat$ of all flat left R-modules, which may or may not be axiomatizable, the models of the theory of $\flat$, which we call *pseudoflat*, form such an axiomatizable class $\mathcal{X}$, while co-$\flat$-dcc and co-$\mathcal{X}$-dcc are the same in that case. It is this case that will eventually lead to cotorsion modules. But first note that the co-$\flat$-dcc takes place in two extreme types of ring: when $\mathcal{X} = \flat$ is everything (that is, R is von Neumann regular) and when the totally transcendental part is everything (that is, when R is left pure-semisimple). The co-$\flat$-dcc may be thought of as to combine these two types of building block.

The ring $\mathbb{Z}$ of integers is of neither type, and in fact, it does not have co-$\flat$-dcc. Indeed, flat=torsionfree here, and the torsionfree points form a closed set in $_{\mathbb{Z}}\mathrm{Zg}$ (consisting of $\mathbb{Q}$ and the p-adics). Its complement consists of the finite indecomposables and the Prüfer groups. Considering $\bigoplus_i \mathbb{Z}/2^i\mathbb{Z}$, it is easy to see that, although it shuts in the torsionfree groups, the sort $2x = 0/x = 0$ does not have the dcc.

An observation going back to the original work of Ziegler [5] is that pp pairs with the dcc do not open up in cokernels of pure-injective envelopes. We prove the converse, which yields the first answer to the problem.

Theorem 1. *Given a class of left R-modules, $\mathcal{X}$, axiomatized by pp implications, $R-\mathrm{Mod}$ has the co-$\mathcal{X}$-dcc if and only if all cokernels of pure-injective envelopes are in $\mathcal{X}$.*

In the particular case mentioned, we obtain

Corollary 2. *$R-\mathrm{Mod}$ has the co-$\flat$-dcc if and only if all cokernels of pure-injective envelopes are pseudoflat.*

A very useful concept is that of pure-injective module [3]. But it has one defect, it is not closed under extension. Cotorsion modules constitute a generalization of pure-injective modules that make up for this defect. A *cotorsion module* is a module C for which every short-exact sequence $0 \to C \to M \to F \to 0$ with F flat splits [4]. Since every short exact sequence ending in a flat is pure, every pure-injective is indeed cotorsion.

Due to the proof of Enochs' Flat Cover Conjecture [1], which is equivalent to the existence of cotorsion envelopes [4], there has recently been a growing interest in cotorsion modules. One of our main concerns is to understand the difference between cotorsion and pure-injective. So it is natural to ask how strong a condition on the ring it is that there be *none*. Xu proved that every cotorsion left R-module is pure-injective if and only if cokernels of pure-injective envelopes are flat [4]. This brings us back to the co-$\flat$-dcc.

Corollary 3. *If every cotorsion left R-module is pure-injective then every left R-module has the co-flat-dcc. If R is right coherent, the converse is also true.*

(The second half follows from the well-known fact that pseudoflat=flat if and only if R is right coherent.)

As a consequence, there are cotorsion groups that are not pure-injective. Two questions spring to mind.

Questions. *Are there non-coherent rings over which every cotorsion module is pure-injective? What interesting properties of the ring are equivalent to the condition that all cokernels of pure-injective envelopes be pseudoflat?*

This is joint work with Ivo Herzog, OSU Lima, [2].

References

[1] E. E. Enochs, O. M. G. Jenda, *Relative Homological Algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter 2000.
[2] I. Herzog, Ph. Rothmaler, *When cotorsion modules are pure-injective*, in preparation.
[3] M. Prest, *Model Theory and Modules*, London Math. Soc. Lecture Note Series, vol. 130, Cambridge University Press 1987.
[4] J. Xu, *Flat Covers of Modules*, Lecture Notes in Mathematics, vol. 1634, Springer-Verlag 1996.
[5] M. Ziegler, *Model theory of modules*, Ann. Pure Appl. Logic **26** (1984), 149–213.

Definable Relations in the Real Field with a Subgroup of the Unit Circle

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(joint work with Boris Zilber)

In 2003 Boris Zilber [1] showed that the structure $(\mathbb{R}, \mathbb{U})$, the ordered field of reals $\mathbb{R}$ augmented by the binary predicate for the group of complex roots of unity $\mathbb{U}$, is 'tame': *every definable relation in the stucture is a Boolean combination of existentially definable relations.* (Here complex numbers are identified with pairs of real numbers in a usual way.) The proof used the so called *Lang property* of the group $\mathbb{U}$, a known deep number-theoretic result, which, roughly speaking, says that any polynomial relation between elements of $\mathbb{U}$ is equivalent to a positive Boolean combination of monomial relations with coefficients in $\mathbb{U}$.

He suggested to obtain a similar result for $\mathbb{U}$ replaced with an infinite cyclic group Γ of the unit circle $\mathbb{S}$. The motivation for the question came from the realization that the quotient-space $\mathbb{S}/\Gamma$ is interesting for mathematical physics: it is related to the quantum torus, an important in noncommutative geometry quantum space based on a certain noncommutative C^*-algebra. There is a hope that understanding of model-theoretic properties of the structure induced on $\mathbb{S}/\Gamma$ from $(\mathbb{R}, \Gamma)$ can help in the study of the quantum torus.

The conjectured result about infinite cyclic subgroups of $\mathbb{S}$ seemed plausible because they satisfy the Lang property like $\mathbb{U}$ does; in fact, any finite rank subgroup of $\mathbb{C}^*$ is known to have the Lang property. (An abelian group Γ is of *finite rank* iff Γ has a finitely generated subgroup G such that Γ/G is a torsion group.) However, the proof in [1] uses that $\mathbb{U}$ is divisible and so is easy to treat. Still, in this talk we show that the result of [1] can be generalized to arbitrary finite rank subgroups Γ of $\mathbb{S}$; for the proof see [2]. We don't know whether the assumption $\Gamma \leq \mathbb{S}$ cannot be dropped, even though we heavily use it in the proof.

In [3] A. Günaydın and L. van den Dries, among other things, obtained a similar result for the field of reals with a multiplicative group of finite rank. Note, however, that they considered the real field augmented by a subgroup of $\mathbb{R}^*$ but not of $\mathbb{C}^*$ as we do, which makes essential difference: it is more difficult to treat the binary predicate than the unary one.

Here is our main result.

Theorem 1. *Let Γ be a finite rank subgroup of $\mathbb{S}$. The definable relations in the structure $(\mathbb{R}, <, +, \cdot, 0, 1, \Gamma)$ are exactly the Boolean combinations of relations of the form*

$$(\star) \qquad \exists x_1 y_1 \dots x_n y_n \left(P(x_1, y_1, \dots, x_n, y_n, \boldsymbol{v}) \wedge \bigwedge_{i=1}^{n} (x_i, y_i) \in \Gamma\right),$$

where P is a semi-algebraic relation on $\mathbb{R}$.

In fact, besides the assumption $\Gamma \leq \mathbb{S}$, the proof uses only the properties (i)–(iii) of Γ from the proposition below.

Proposition 2. *Let Γ be a finite rank subgroup of $\mathbb{C}^*$. Then*

(i) *Γ is at most countable;*
(ii) *Γ modulo the nth powers is finite, for each $n > 0$,*
(iii) *for every algebraic set $V \subseteq \mathbb{C}^n$ the trace $V \cap \Gamma^n$ is definable in the group Γ by a positive quantifier-free formula with parameters.*

Here (i) is almost obvious, (ii) is an exercise in abelian group theory, and (iii) is a consequence of the following result [4]:

Fact 3. *Let Δ be a finite rank subgroup of $(\mathbb{C}^*)^n$. Then for every algebraic set $V \subseteq \mathbb{C}^n$ the trace $V \cap \Delta$ is a finite union of cosets of subgroups of the form $H \cap \Delta$, where H is a subgroup of $(\mathbb{C}^*)^n$ defined by finitely many equations of the form $X_1^{m_1} \dots X_n^{m_n} = 1$ with $m_i \in \mathbb{Z}$.*

Fact 3 is a special case of a powerful result of diophantine geometry about traces of subvarieties of a semi-abelian variety on finite rank subgroups (proven in the 90s by efforts of many number-theorists, and now known as the generalized Mordell-Lang conjecture). Following A. Pillay, we call (iii) the Lang property.

The group Γ is not assumed to be divisible, but we are able to overcome this by using a much weaker property (ii).

Here is a more precise version of the main result.

Theorem 4. *Let Γ be an infinite subgroup of $\mathbb{S}$ with the properties* (i)–(iii)*, and $\Gamma^{\mathrm{re,\,im}} = \{a, b : (a, b) \in \Gamma\}$. Let M be the expansion of the structure*

$$M_0 = (\mathbb{R}, <, +, \cdot, 0, 1, \Gamma, c)_{c \in \Gamma^{\mathrm{re,\,im}}}$$

by all relations of the form $(\star)$ with P defined over $\Gamma^{\mathrm{re,\,im}}$. Then M admits quantifier elimination.

We find a complete axiom system for the theory of M and show that it admits quantifier elimination.

For every polynomial $f(\boldsymbol{X})$ over $\mathbb{Z}$ fix a positive quantifier-free formula $\theta_f(\boldsymbol{X})$ in the language $\{\cdot, ^{-1}, 1, g\}_{g\in\Gamma}$ such that for any tuple $\boldsymbol{z}$ in Γ

$$\mathbb{C} \models f(\boldsymbol{z}) = 0 \leftrightarrow \theta_f(\boldsymbol{z});$$

such θ_f can be chosen by the Lang property (iii).

For a real closed field R, let C be the algebraic closure of R identified with R^2, and S the unit circle in C. For $Z \subseteq C$, put $Z_{\rm re}\{x \in R : \exists y\,(x, y) \in Z\}$. Consider the class of all structures N in the language of M such that:

(1) N satisfies all the quantifier-free sentences that hold in M.

(2) The reduct of N to the language of ordered rings is a real closed field.

(3) The set $\Gamma(N)$ is a subgroup of S elementarily equivalent to Γ.

(4) $\Gamma(N)$ is dense in S.

(5) For any polynomial $f(X, Y_1, \ldots, Y_n, Z_1, \ldots, Z_m)$ over $\mathbb{Z}$ with $\deg_X f > 0$, and any $\boldsymbol{c} \in R^m$ the set

$$\{a \in R : f(a, \boldsymbol{b}, \boldsymbol{c}) \neq 0 \text{ for all } \boldsymbol{b} \in \Gamma(N)^n_{\rm re}\}$$

is dense in R.

(6) for every polynomial $f(\boldsymbol{X})$ over $\mathbb{Z}$, and every tuple $\boldsymbol{z}$ in $\Gamma(N)$

$$C \models f(\boldsymbol{z}) = 0 \leftrightarrow \theta_f(\boldsymbol{z}).$$

The structure M is a member of the class. Indeed, clearly, M satisfies (1–3) and (6). It is well-known that any infinite subgroup is dense in $\mathbb{S}$; so (4). Any interval in $\mathbb{R}$ is uncountable, but Γ is countable and so for any finite subset A of $\mathbb{R}$ there are only countably many elements algebraic over $\Gamma_{\rm re} \cup A$; so (5).

Let N be a member of the class. For $g = (a, b) \in \Gamma$, let $g_N = (a_N, b_N)$. It is easily seen that $g \mapsto g_N$ is a pure embedding of the group Γ into the group $\Gamma(N)$; we denote $\Gamma_N = \{g_N : g \in \Gamma\}$.

It is easily seen that the class is first order axiomatizable; let T be its theory.

Theorem 5. *The theory T is complete and admits quantifier elimination.*

Quantifier elimination for T implies Theorem 4, and, due to (1), completeness of T. We prove quantifier elimination by proving submodel completeness of T, which follows from the proposition below. For a model N of T denote by N_0 its reduct to the language of the structure M_0; clearly, any elementary map in N_0 is an elementary map in N.

Proposition 6. *Let N and N' be $(2^{\aleph_0})^+$-saturated models of T. Then there exists a back-and-forth system $\mathcal{S}$ from N_0 to N'_0 such that any finite partial isomorphism from N to N' extends to a member of $\mathcal{S}$.*

The construction of $\mathcal{S}$ (and the proof that it does work) is rather complicated. Let $\mathcal{S}_0$ be the set of all partial maps β from R to R' which are elementary over $\Gamma^{\rm re,\,im}$ and such that there exist

- a finite subset A of R algebraically independent over $\Gamma(N)$ in C, and a group H of cardinality at most $2^{\aleph_0}$ with $\Gamma_N \leq H \leq \Gamma(N)$ such that H has a divisible torsion-free complement in $\Gamma(N)$, and
- A' and H' for N' with similar properties,

for which $\mathrm{dom}(\beta) = A \cup H_{\mathrm{re}}$, $\beta(A) = A'$, and $\beta(H_{\mathrm{re}}) = H'_{\mathrm{re}}$. Then we define $\mathcal{S}$ as $\{\bar{\beta} : \beta \in \mathcal{S}_0\}$, where $\bar{\beta}$ is the unique elementary map extending β to $\mathrm{acl}_R(\mathrm{dom}(\beta))$.

References

[1] Boris Zilber, *Complex roots of unity on the real plane*, 2003 (unpublished notes available at www.maths.ox.ac.uk/∼zilber).

[2] Oleg Belegradek and Boris Zilber, *Definable relations in the real field with a distinguished subgroup of the unit circle*, preprint NI06008-MAA, Isaac Newton Institute for Mathematicas Sciences, 2006, www.newton.cam.ac.uk/preprints/NI06008.pdf.

[3] Ayhan Günaydın and Lou van den Dries, *The field of real and complex numbers with a small multiplicative group*, Proc. London Math. Soc. (3) **93** (2006), 43–81.

[4] Michel Laurent, *Equations diophantiennes exponentielles*, Invent. Math. **78** (1984), 299–327.

Superrosy Groups and Fields with NIP and FSG

Krzysztof Krupiński
(joint work with Clifton Ealy and Anand Pillay)

The general goal is to apply some methods from stable groups (e.g. forking calculus) to show structural results about groups which are not necessarily stable but in which we still have a notion of independence relation satisfying some minimal list of nice properties (like transitivity or the existence of independent extensions). Such groups are called rosy groups. Having such an independence relation, wlog we can replace it by the coarsest relation, which is called thorn-independence.

We concentrate on structural results about superrosy groups with small thorn U-rank (the rank defined by means of thorn independence relation in the same way as Lascar U-rank is defined by means of forking independence), additionally assuming NIP (non independence property) and FSG (finitely satisfiable generics). In particular, our results and conjectures generalize some results about superstable groups and definably connected, definably compact groups definable in o-minimal structures.

Theorem 1. *Each goup with NIP, FSG and of thorn U-rank 1 is abelian-by-finite.*

Conjecture 2. *Each group with NIP, having hereditarily FSG and of thorn U-rank 2 is solvable-by-finite.*

We also consider a weaker conjecture.

Conjecture 3. *Each group with NIP, having hereditarily FSG, of thorn U-rank 2 and such that $G = G^{00}$ is solvable.*

We have proved the following partial result concerning Conjecture 3.

Proposition 4. *Suppose G has NIP, hereditarily FSG, is of thorn U-rank 2 and $G = G^{00}$. Then either G is solvable or there are nontrivial elements $a, b \in G$ such that $C(a)$ is finite and $C(b)$ is infinite. More precisely, assume G is not solvable. Then $Z(G)$ is finite. So replacing G by $G/Z(G)$, we can assume that $Z(G)$ is trivial. Then G has infinitely many involutions and all involutions have thorn U-rank at most 1; so every involution has infinite centralizer. Moreover, there are involutions i, j such that $C(ij)$ is finite and thorn U-rank of ij is 2.*

Finally we have the following theorem.

Theorem 5. *Each superrosy field K with NIP and such that $(K, +)$ has FSG is algebraically closed.*

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