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Arbeitsgemeinschaft: Percolation

Organised by
Vincent Beffara (Lyon),
Jacob van den Berg (Amsterdam),
Federico Camia (Amsterdam).

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Abstract.

Percolation as a mathematical theory is more than fifty years old. During its life, it has attracted the attention of both physicists and mathematicians. This is due in large part to the fact that it represents one of the simplest examples of a statistical mechanical model undergoing a phase transition, and that several interesting results can be obtained rigorously.

In recent years the interest in percolation has spread even further, following the introduction by Oded Schramm of the Schramm-Loewner Evolution (SLE) and a theorem by Stanislav Smirnov showing the conformal invariance of the continuum scaling limit of two-dimensional critical percolation. These results establish a new, powerful and mathematically rigorous, link between lattice-based statistical mechanical models and conformally invariant models in the plane, studied by physicists under the name of Conformal Field Theory (CFT).

The Arbeitsgemeinschaft on percolation has attracted more than thirty participants, most of them young researchers, from several countries in Europe, North America, and Brazil. The main focus has been on recent developments, but several classical results have also been presented.

Mathematics Subject Classification (2000): 60K35, 82B43, 82B27, 60G18.

Introduction by the Organisers

Percolation as a mathematical theory was introduced by Broadbent and Hammersley [4, 5] about fifty years ago to model the spread of a gas or a fluid through a porous medium. To model the randomness of the medium, they declared the edges of the d-dimensional cubic lattice independently *open* (to the passage of the gas or fluid) with probability p or *closed* with probability $1-p$. Since then, many variants of this simple model have been studied, attracting the interest of both mathematicians and physicists.

Mathematicians are interested in percolation because of its deceiving simplicity which hides difficult and elegant results. For the physicists, percolation is maybe the simplest statistical mechanical model undergoing, as the value of the parameter p is varied, a phase transition with all the standard features typical of critical phenomena (scaling laws, a conformally invariant scaling limit, universality). On the applied side, percolation has been used to model the spread of a disease, a fire or a rumor, the displacement of oil by water, the behavior of random electrical circuits, and more recently the connectivity properties of communication networks.

The work of mathematicians has concentrated on the behavior of the model both at the critical point p_c and away from it. However, despite the fact that we have had for some time a good understanding of the *subcritical* ($p < p_c$) and *supercritical* ($p > p_c$) phases (see [11, 9, 2]), a complete and rigorous understanding of the "critical behavior" has proved more difficult and until recently seemed to be out of reach (despite various important achievements–see, e.g., [12] and again [11, 9, 2] as general references).

Meanwhile, the problems encountered by mathematicians did not prevent the physicists from studying the critical point and its vicinity using theoretical physics methods. This enterprise was particularly successful in two dimensions where the (powerful but not mathematically rigorous) tools of Conformal Field Theory (CFT) produced many predictions describing the behavior of the model at p_c or as $p \to p_c$, including various critical exponents.

Recently, the introduction by Oded Schramm [17] of the Schramm-Loewner Evolution (SLE) has provided a new powerful and mathematically rigorous tool to study continuum scaling limits of critical lattice models. While CFT focuses on correlation functions, SLE describes the behavior of macroscopic random curves present in those models, such as percolation cluster boundaries. There is a one-parameter family of SLEs, indexed by a positive real number κ, which appears to contain essentially all possible candidates for the conformally invariant scaling limits of interfaces from two-dimensional critical systems. One exciting aspect of SLE techniques applied to the study of critical lattice models is the fact that it combines methods from at least three different areas of mathematics, i.e., discrete probability, stochastic processes, and complex analysis.

Thanks to the work of Lawler, Schramm, Werner on SLE (see [13, 20] and references therein) and of Smirnov [18], in recent years tremendous progress has been made in the study of two-dimensional critical percolation. The main power of SLE stems from the fact that it allows to compute different quantities (for example, percolation crossing probabilities and various percolation critical exponents [14, 19]), thus giving a mathematically rigorous confirmation of the predictions made by physicists using CFT methods. Moreover, SLE has provided a totally new perspective, which has resulted in a much deeper understanding of the *random geometry* of the scaling limit of two-dimensional critical percolation (see [6, 7, 8]).

Two-dimensional critical percolation and its continuum scaling limit have been the main theme of the Arbeitsgemeinschaft, which has also included a series of three lectures on SLE by Vincent Beffara. However, although progress in this area represents maybe the single most exciting recent development within the field, since its introduction percolation has continued to produce a wealth of beautiful results and has been an important paradigm for the behavior of other random systems and an important tool for the study of various other models. Today, it is still a very active area of research, strategically placed at the interface between probability and statistical physics. For this reason, the Arbeitsgemeinschaft has touched upon other important recent developments in areas such as percolation in high dimensions [3, 10] and on trees and nonamenable graph [15, 16], and Voronoi percolation [1].

In order to make the lectures accessible to all the participants, the topics of the first two talks were carefully selected to provide a concise introduction to some of the main techniques and results on which modern percolation theory is built. A successful open problem session on Wednesday afternoon was attended by most of the participants. On Friday afternoon, the program was complemented by a series of six short talks by some of the participants on their own research topics.

The Arbeitsgemeinschaft has attracted more than thirty researchers from various European countries, North America, and Brazil. The majority were advanced Ph.D. students and young postdocs, which has resulted in an informal and lively atmosphere that has proved very effective for learning and exchanging ideas. Most of the talks have been followed by many questions and interesting discussions. The meeting has also profited from the presence of some more senior researchers who have been instrumental in leading the discussions following some of the more technical talks.

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Workshop: Arbeitsgemeinschaft: Percolation

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Abstracts

Introductory talk I

Jacob van den Berg

Consider the graph of which the vertices are the integer points in the plane, and where each vertex has an edge with each of the four vertices to which it has distance 1. This graph is called the square lattice. Suppose each edge, independently of the others, is declared open (1) with probability p and closed (0) with probability $1-p$. Percolation theory, introduced by Broadbent and Hammersley in the mid-fifities as a model of a porous medium, studies the connectivity properties of this random network. An *open path* is a path in the graph of which each edge is open. One of the first questions concerned the probability that from a given vertex, say the vertex $0 := (0,0)$, infinitely many vertices can be reached by open paths:

$$\theta(p) := P_p(\exists \text{ an infinite open path starting in } 0).$$

It is easy to see that $\theta(p)$ is non-decreasing in p and that $\theta(0) = 0$ and $\theta(1) = 1$. This leaves open the possibility that $\theta(p) > 0$ for all $p > 0$ or that $\theta(p) = 0$ for all $p < 1$, which would not be very interesting. (In fact, the latter is what happens if we do percolation on the integer line instead of on the square lattice). However, as Broadbent and Hammersley showed, this model has an interesting critical behaviour: There is a value p_c, *strictly* between 0 and 1, such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. One part of this result follows from the following claim

$$\theta(p) = 0 \text{ for all } p < \frac{1}{4}. \qquad (1)$$

This claim (1) can be proved quite easily, as follows: Let A_n denote the event that there is an open self-avoiding (s.a.) path of length n starting in 0. (The length of a path is the number of edges in the path). Obviously $\theta(p) \le P_p(A_n)$, for each n. Further, it is also clear that the number of s.a. paths of length n starting in 0 is at most 4^n. Hence, for each n,

$$\theta(p) \le P_p(A_n) \le (4p)^n.$$

If $p < 1/4$, the r.h.s. goes to 0 as $n \to \infty$. This proves (1)

The other part of the above mentioned Broadbent-Hammersley result follows immediately from

$$\theta(p) > 0 \text{ for all } p > \frac{3}{4}. \qquad (2)$$

Part of the proof of (2) is very similar to that of (1), but an additional feature is needed, involving the notion of *duality.* In general, the dual lattice of a planar lattice is the lattice whose vertices correspond with the midpoints of the faces of the original lattice, and where two vertices have an edge if their corresponding faces in the original lattice are adjacent. Thus, in the case of the square lattice, the dual lattice is again a square lattice: it is the original lattice shifted over the vector $(1/2, 1/2)$. (This self-duality is special; for instance, the dual lattice of the regular triangular lattice is not triangular but hexagonal). Since each edge in the dual lattice corresponds with exactly one edge in the original lattice, a configuration in the original lattice induces a configuration on the dual lattice: simply declare a dual edge open if and only if the corresponding original edge is open. The importance of duality for percolation comes from the following fact: Each finite open cluster (maximal connected open subgraph) in S is surrounded (in fact, 'bordered') by a circuit of closed edges in the dual lattice. So to prove (2) it suffices to show that for $p > 3/4$ the probability that there is a closed circuit around 0 in the dual lattice is smaller than 1. Finally, that the latter is indeed smaller than 1 can be proved in a way very similar to the proof of (2), namely by counting the number of circuits (around 0, or more generally around a given box) of a certain length.

The above model is called bond percolation (on the square lattice). Analogously, one can assign random states to the vertices instead of the edges. Then we speak of site percolation. Of course percolation can be (and has been) studied on many other graphs, for instance the three-dimensional cubic lattice.

In the sixties and seventies much effort was made in the determination (or approximation) of the value of p_c. The famous proof by Kesten (in 1980) that for bond percolation on the square lattice $p_c = 1/2$ is not only interesting in itself but involved powerful techniques which turned out to be of much more general use in percolation.

Another important object of study is the asymptotic behaviour (for large n) of the probability $\pi(n)$ that there is an open path from 0 to some point at distance larger than n from O. If $p < p_c$, $\pi(n)$ goes to 0 exponentially. (On the square lattice this result was strongly connected with the result, and its proof, that $p_c = 1/2$; More generally, for the cubic lattice in any dimension, this exponential decay was proved by Menshikov and, independently by Aizenman and Barsky around 1986). Very different is the case $p = p_c$. It was predicted by physicists around 1980 that in that case $\pi(n)$ behaves asymptotically as a power of n; in particular, for the square lattice and other 'nice' planar lattices it was predicted that $\pi(n)$ behaves asymptotically as $n^{-5/48}$. Arount 1997 a rigorous mathematical proof of such result still looked hopeless. However, only about five years later, the breakthroughs by Lawler, Schramm and Werner and by Smirnov (which are extensively discussed in other talks in this Arbeitsgemeinschaft) led to a rigorous proof (for site percolation on the triangular lattice) of this and many other so-called power laws.

The rest of this introductory talk is used to state three of the main basic techniques, the FKG and BK inequalities and Russo's formula. These results, each of which involves the notion of increasing events, hold in a much more general context than percolation, but here I will state them in the language and context of percolation. First, an event A is said to be *increasing* if it has the property that if a configuration ω belongs to A, then any new configuration which can be obtained from ω by making one or more closed edges open, also belongs to A.

The FKG inequality says that if A and B are increasing events, then

$$P_p(A \cap B) \geq P_p(A)P_p(B).$$

The BK inequality also involves two increasing events but goes in the other direction: It says that if A and B are increasing events, then the probability that A and B 'occur for disjoint reasons', is at most $P_p(A)P_p(B)$. Instead of giving a general and formal definition of 'to occur for disjoint reasons' I give an example which is typical in percolation: Let v, w, x, y be vertices and let A be the event that there is an open path from v to w and B the event that there is an open path from x to y. Then the event

$$\{A \text{ and } B \text{ occur for disjoint reasons}\}$$

is the event that there are disjoint open paths from v to w and from x to y.

Russo's formula involves another notion that needs explanation. Let A be an event involving only finitely many edges, say $e_1, \cdots e_n$. Let ω be a configuration. We say that edge e_i is pivotal (w.r.t. A and ω), if either ω or $\omega^{(i)}$ (but not both) is in A; here $\omega^{(i)}$ is the configuration obtained from ω by 'flipping' the state of e_i. Let the random variable $N(A)$ denote the number of pivotal edges for A. Russo's formula says that if A is increasing, then

$$\frac{d}{dp}P_p(A) = E_p(N(A)),$$

where E_p stands for the expectation w.r.t. the distribution P_p.

Most of this introductory talk is based on Chapters 1 and 2 in [1]

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Introductory talk II: Uniqueness of the infinite cluster and RSW theory

Michael Damron

The first part of the talk follows the Burton and Keane proof [1]. Consider the probability space $(\Omega = \{0,1\}^{\mathbb{E}^d}, F, \mathbb{P})$ where $\mathbb{E}^d$ is nearest neighbor edges of $\mathbb{Z}^d$, F = cylinder sets, and $\mathbb{P}$ is translation invariant. We also suppose that $\mathbb{P}$ satisfies the following property:

Let $e \in \mathbb{E}^d$ and let $T_e = \sigma(T_f : f \neq e)$. For any element $\omega \in \Omega$ declare the edge e occupied if $\omega(e) = 1$ and vacant if $\omega(e) = 0$. We say that the measure $\mathbb{P}$ satisfies the **finite energy property** if

$$0 < \mathbb{P}(\omega(e) = 1 | T_e) < 1$$

Note that this is equivalent to the corresponding event with the edge e vacant.

Fact: Let Λ be a finite box and $\phi : \Omega \to \Omega$ be such that for all $f \notin \Lambda$, $\omega(f) = (\phi(\omega))(f)$. If $\mathbb{P}$ has the finite energy property and $\mathbb{P}(A) > 0$ then $\mathbb{P}(\phi(A)) > 0$.

We will use this in the proof of the following theorem. Let $N(\omega)$ be the number of infinite (1) clusters in the configuration ω.

Theorem: $\mathbb{P}(\{\omega : N(\omega) > 1\}) = 0$

By ergodic decomposition we assume that the measure $\mathbb{P}$ is ergodic so that N is a.s. constant. If we connect distinct infinite clusters with (1) paths in a large box we see that N cannot be any finite number larger than 1. To show that N cannot be infinite, we make a definition. For any vertex $v \in \mathbb{Z}^d$ we cal v a **triple point** if the following three conditions hold: (a) v is in an infinite (1) cluster, (b) exactly 3 edges adjacent to v are occupied, and (c) the removal of v and all its adjacent edges splits the infinite (1) cluster, of which v is a member, into 3 infinite clusters. We count the number of triple points in a finite box Λ with $|\Lambda| = n$ and let $\Lambda \to \mathbb{Z}^d$. Translation invariance gives the expected number of such points grows like n^d but, on the other hand, the tree-like structure of triple points gives that the number cannot grow faster than n^{d-1}. Details can be found in [1].

The second part of the talk concerns the Russo-Seymour-Welsh estimates. We consider Bernoulli site perclation on $\mathbb{Z}^2$ with probability p of each site having a marking of (+) and with probability $1-p$ a marking of (-). Let $\mathbb{P}_p$ be the product measure associated with the model. We say that two sites are (+) connected if there exists a nearest-neighbor path of vertices, each with a (+) marking, connecting the two sites. We say that two sites are (-) *connected if they are connected by a nearest-neighbor *path of vertices (diagonal connection is allowed), each with a (-) marking. Define the rectangle $R(a,b) = [-a,a] \times [-b,b]$ and the event $C(a,b) = \{\text{there is a (+) path connecting the left and right sides of } R(a,b)\}$. Let $\theta(p)$ be the probability that the origin is in an infinite connected (+) cluster.

Theorem: If $\theta(p) > 0$ then

$$\lim_{n\to\infty} \mathbb{P}_p(C(n,n)) = 1$$

$$\lim_{n\to\infty} \mathbb{P}_p(C(2n,n)) = 1$$

$$\lim_{n\to\infty} \mathbb{P}_p(C(3n,n)) = 1$$

We finish with two definitions. Let C be the finite connected (+) cluster of the origin and let $\bar{C}$ be the finite *connected (-) cluster of the origin. Let $S(p) = \mathbb{E}_p(|C|)$ and let $\bar{S}(p) = \mathbb{E}_p(|\bar{C}|)$.

Theorem: If $\theta(p) > 0$ then

$$\max(S(p), \bar{S}(p)) < \infty$$

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Mini-course on SLE processes

Vincent Beffara

SLE_κ is a two-dimensional stochastic process defined as the solution to Loewner's equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \beta_t}$$

in the complex upper-hal plane, where $\beta_t = \sqrt{\kappa} B_t$ is a real-valued Brownian motion with variance parameter $\kappa > 0$. It was introduced by Oded Schramm as a candidate for the scaling limit of various critical, discrete two-dimensional models of statistical physics, such as self-avoiding walks, spanning trees, percolation and the Ising model. It has since then been proved to indeed be the scaling limit of such models in several cases, notably (in the framework of this Arbeitsgemeinschaft) critical site percolation on the triangular lattice, as shown by Stanislav Smirnov.

The aim of this mini-course was to provide a quick introduction to SLE for non-specialists, focusing on the case of critical percolation; as such, it articulates with a few other talks of the week to produce a (mostly) self-contained proof of convergence of the critical percolation exploration process to an SLE with $\kappa = 6$.

As can easily be seen from the definition, the domain of g_t is the complement in the upper-half plane $\mathbb{H}$ of a random compact set K_t, and g_t maps $H_t := \mathbb{H} \setminus K_t$ conformally to $\mathbb{H}$. A very technical result by Rohde and Schramm is the existence of a *trace* of the process; more precisely, wih probability 1, there exists a continuous curve γ in $\bar{\mathbb{H}}$ such that the following happens: For every positive t, H_t is the (unique) unbounded connected component of $\mathbb{H} \setminus \gamma([0, t])$.

The structure of the process strongly depends on the value of the parameter κ: almost surely,

- If $0 < \kappa \leqslant 4$, then γ is a simple curve (*i.e.*, it has no double point); it has Hausdorff dimension $1 + \kappa/8$;
- If $4 < \kappa < 8$, γ has double points but no self-crossing. It still has Hausdorff dimension, and thus Lebesgue measure 0, but K_t itself has positive measure; $\bigcup K_t$ is the whole half-plane;
- If $\kappa \geqslant 8$, then γ is space-filling, in the sense that it almost surely contains every point of $\bar{\mathbb{H}}$.

One of the main advantages of SLE is that it relates two-dimensional geometrical properties of random curves to features of a one-dimensional process (the driving process β_t), which makes it amenable to computations using standard tools of stochastic analysis. Two representative examples of such computations are the following.

- Take $\kappa > 4$, so that γ is sure to hit the real axis. The probability that it touches the half-line $(x, +\infty)$ before the half-line $(-\infty, y)$ (for $y < 0 < x$) can be expressed as the solution of a differential equation in $x/(x-y)$, obtained via Itô's formula. In the case $\kappa = 6$, one obtains Cardy's formula, describing crossing probabilities for critical percolation in large rectangles, and this, together with Smirnov's result, is enough to obtain a statement of convergence for cluster boundaries in the scaling limit. (It is not enough to obtain convergence of the exploration process to the SLE trace though, as explained in a separate talk this week.)
- One can describe the way γ approaches a small, fixed disk contained in the upper-half plane, and estimate hitting probabilities for SLE through a leading eigenvalue computation applied to a one-dimensional diffusion. As a result, one can compute the so-called *bichromatic two-arm exponent* of percolation, which happens to be 1/4, *i.e.* $2 - d$ where $d = 7/4$ is the Hausdorff dimension of the curve.

Another, more natural way of computing critical exponents is to use another version of Loewner's equation known as *radial SLE*. It is defined by solving Loewner's equation in a punctured disk instead of a half-plane; the differential equation is then

$$\partial_t g_t(z) = g_t(z) \frac{e^{i\beta_t} + g_t(z)}{e^{i\beta_t} - g_t(z)},$$

where again β_t is real-valued Brownian motion with appropriate variance.

Some of the critical exponents can then be seen as leading eigenvalues of the generators of various one-dimensional diffusions on the interval $[0, 2\pi]$ — this was explained in more detail in another talk of the meeting.

Good references on SLE processes are multiplying at an exponential rate; a good idea is to refer to the book of Lawler [1] and perform a breadth-first search from there. As another possible starting point, one can choose Werner's Saint Flour lecture notes [2] instead.

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$p_c + p_c^* = 1$

András Bálint

We consider Bernoulli site percolation on the square lattice $\mathbb{Z}^2$: we declare each vertex open with probability p, closed with probability $1-p$, independently for different vertices. Writing p_c for the critical value, and p_c^* for the critical value of Bernoulli site percolation on the matching lattice of $\mathbb{Z}^2$ (which can be obtained from $\mathbb{Z}^2$ by adding edges along the diagonals), our goal is to prove $p_c+p_c^*=1$. We follow the proof described in [3]. A simple but crucial observation for the proof is that whatever the configuration of states of the vertices is, fixing a rectangle R, there is either an open horiztonal crossing or a closed vertical $*$-crossing in R, but never both. The other tools we use in the proof are Russo's formula, a Russo-Seymour-Welsh theorem, and a finite size criterion (fsc) stating that having high enough probability for a crossing in a rectangle in the long direction implies positive probability for percolation of the origin. We prove fsc as in [3], by a coupling of the site percolation with a 1-dependent bond percolation, however, the formulation of the argument is taken from [1].

As $p_c+p_c^* \geq 1$ follows easily from the fact that in case of $p > p_c$, there is $\mathbb{P}_p$-a.s. at least one open circuit surrounding the origin, we focus on proving $p_c + p_c^* \leq 1$. By assuming the opposite, we may choose values $p_1 < p_2$ between $1 - p_c^*$ and p_c. Following Kesten's idea [2], we consider an increasing event A, namely that there is an open horizontal crossing in a large square S, and show that the probability of A increases too rapidly in the interval $p \in [p_1, p_2]$. We do that by giving the uniform lower bound $1/(p_2 - p_1)$ on the expected number of pivotal vertices for A (which holds for all $p \in [p_1, p_2]$) by showing that the probability of finding the lowest horizontal open crossing π and several closed $*$-paths (i.e. paths in the matching lattice) from the top of S to π is high enough. Then Russo's formula ensures that

$$\min_{r\in[p_1,p_2]} \frac{d}{dp}\mathbb{P}_r(A) > \frac{1}{p_2-p_1}.$$

This leads to the desired contradiction as it implies

$$\mathbb{P}_{p_2}(A) - \mathbb{P}_{p_1}(A) \geq \ +(p_2-p_1) \min_{p\in[p_1,p_2]} \frac{d}{dp}\mathbb{P}_r(A) > 1.$$

Finally, we note that essentially the same proof works in case of Bernoulli bond percolation on $\mathbb{Z}^2$ (in fact, the proof in the bond percolation case [2] chronologically preceded the proof in the site percolation case [3]). This yields that the sum of the critical value and the critical value for the dual lattice is 1. As $\mathbb{Z}^2$ and its bond dual are isomorphic, the two critical values are equal. We obtain this way Kesten's celebrated result that the critical value for Bernoulli bond percolation on the square lattice equals 1/2.

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Cardy's formula on the triangular lattice

Vincent Vargas

The purpose of this talk is to present Smirnov's proof of Cardy's formula on the triangular lattice. More specifically, we consider site percolation on the triangular lattice at critical probability $p_c = 1/2$ with mesh size $\delta > 0$. We will call black sites open and white sites closed. We will denote by $\mathbb{P}^\delta$ the underlying probability measure. Let (D, P_1, P_2, P_3, P_4) be a counterclockwise marked Jordan domain. For all i, let A_i denote the arc of ∂D oriented counterclockwise that starts at point P_i and ends at point P_{i+1}. For all i, we consider the probability $\mathbb{P}^\delta(A_i \to A_{i+2})$ that there exists an open (black) path from A_i to A_{i+2}. As δ goes to 0, the probability $\mathbb{P}^\delta(A_1 \to A_3)$ converges to a limit $f(D, P_1, P_2, P_3, P_4)$. This limit is conformally invariant; if ϕ is a conformal map that sends the marked domain (D, P_1, P_2, P_3, P_4) onto the marked domain $(\phi(D), \phi(P_1), \phi(P_2), \phi(P_3), \phi(P_4))$, one has the identity:

$$f(D, P_1, P_2, P_3, P_4) = f(\phi(D), \phi(P_1), \phi(P_2), \phi(P_3), \phi(P_4)).$$

In the equilateral triangular case, one gets the following simple and explicit expression:

$$f(D, P_1, P_2, P_3, P_4) = d(P_3, P_4)/d(P_3, P_1),$$

where d is just the euclidean distance.

The idea of the proof of this result, as given by Smirnov, is to consider point P_4 as a complex variable and to work in the 3-marked Jordan domain (D, P_1, P_2, P_3). Then one can define similarly as above three arcs $(\tilde{A}_i)_{1 \leq i \leq 3}$ between the points (P_1, P_2, P_3). The key idea is to consider three applications $(g_i^\delta)_{1 \leq i \leq 3}$ defined on D and given by:

$$g_i^\delta(z) = \mathbb{P}^\delta(\exists \text{ black crossing } \tilde{A}_{i-1} \to \tilde{A}_{i+1} \text{ that separates } z \text{ from } \tilde{A}_i).$$

Then, it is obvious that $f(D, P_1, P_2, P_3, P_4)$ is the limit as δ goes to 0 of $g_2^\delta(P_4)$. The problem is therefore to study the family $(g_i^\delta)_{1 \leq i \leq 3}$. This study is divided into two parts:

- Part 1: the family $(g_i^\delta)_{1 \leq i \leq 3}$ is equicontinuous (key theorems used: Ascoli, Russo-Seymour-Welsh estimates).
- Part 2: Characterize the limit family $(g_i)_{1 \leq i \leq 3}$(key theorems used: Morera's theorem, color swapping lemma, Russo-Seymour-Welsh estimates).

Part 2 is the most original part of Smirnov's proof. By using a the color swapping lemma along with discrete complex integration in a very elegant way, it is possible to show that on a triangle T:

$$\forall 1 \leq i \leq 3, \quad \int_T g_{i+1}(z)dz = j \int_T g_i(z)dz,$$

where j is the cube root of unity. The relations above along with the boundary conditions uniquely determine the family $(g_i)_{1\leq i\leq 3}$.

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Scaling relations for $2D$-percolation

Artem Sapozhnikov

In my talk I will summarize the results obtained in the paper [1].

We consider an independent site or bond percolation on a periodic graph in two dimensions. Let C_x be the open cluster at x. The size of a cluster at x is denoted by $|C_x|$. We say that two sites x and y are connected, $x \leftrightarrow y$ if they belong to the same open cluster. One of the main characteristics of the percolation process is the percolation probability

$$\theta(p) = \mathbf{P}_p(|C_0| = \infty).$$

It is well known that there is a critical probability $p_c = \sup\{p\ :\ \theta(p) = 0\} \in (0,1)$ such that there is almost surely no infinite open cluster for $p \leq p_c$ and there is the unique infinite open cluster for $p > p_c$. The average size of a finite cluster at the origin is

$$\chi^f(p) = \mathbf{E}_p\{|C_0|; |C_0| < \infty\} = \sum_{n=1}^{\infty} n\mathbf{P}_p(|C_0| = n).$$

For $p < p_c$, $\chi^f(p)$ coincides with the expected cluster size $\mathbf{E}_p|C_0|$. The correlation length is defined as

$$\xi(p) = \left(\frac{1}{\chi^f(p)} \sum_y |y|^2 \mathbf{P}_p(0 \leftrightarrow y, |C_0| < \infty)\right)^{\frac{1}{2}},$$

where $|y| = \max\{|y_i|, i = 1,2\}$ for $y = (y_1, y_2)$.

It is conjectured in the physics literature that the above characteristics obey the power law behaviour as $p \to p_c$:

$$\theta(p) \approx (p-p_c)^{\beta},\ \text{for}\ p > p_c;$$
$$\chi^f(p) \approx |p-p_c|^{-\gamma};$$
$$\xi(p) \approx |p-p_c|^{-\nu},$$

where $A(p) \approx |p-p_c|^{\zeta}$ stays for $\log A(p)/\log|p-p_c| \to \zeta$ as $p \to p_c$. Moreover, it is believed that the corresponding exponents are universal for a general class of periodic graphs. In [1] it is shown that given $\xi(p) \approx |p-p_c|^{-\nu}$ as $p \to p_c$ and $\mathbf{P}_{p_c}(0 \leftrightarrow \partial B(n)) \approx n^{-\delta}$, $\theta(p)$ and $\chi^f(p)$ have the above mentioned asymptotic with $\beta = \nu\delta$

and $\gamma = 2(\nu - \beta)$. It is also shown in [1] that $\xi(p) \approx |p - p_c|^{-\nu}$ as $p \to p_c$ is implied by $\mathbf{P}_{p_c}$(there are 4 alternating paths (2 open and 2 occupied) from 0 to $\partial B(n)) \approx n^{-\delta_4}$. These results allowed to relate the behaviour of the process near the critical point with its behaviour at the criticality.

In the talk I will give the proofs of the above mentioned relations between the critical exponents.

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Convergence of 2D critical percolation to SLE_6

Michael J. Kozdron

The goal of this lecture is to explain the convergence of the critical site percolation exploration path on the triangular lattice to the trace of chordal SLE_6 for Jordan domains; detailed lecture notes [4] are available from the speaker's website. Our primary reference is the recent paper by Camia and Newman [2] and we cite many results verbatim from that work. The speaker makes no claims of originality, but it is hoped that by highlighting some key elements of the proof in a slightly different way than is done in [2], the interested party can use this present work as a companion to help increase his or her understanding of [2]. At various times in the lecture we will be a little casual with certain hypotheses. We hope that this lack of precision will allow us to better capture the key ideas of [2].

It should be noted that a recent preprint by W. Werner [9] contains lecture notes from a short course given at the 2007 IAS/Park City Mathematics Institute on Statistical Mechanics. Lecture 3 in those notes is concerned with a proof of this convergence result, but Werner follows a different approach than Camia and Newman. In fact, Werner's notes [9] contain six lectures and a set of exercises on critical site percolation on the triangular lattice that coincide with the topic of this Arbeitsgemeinschaft; we highly recommend reading them.

The primary theorem that we will be concerned with is the following precise formulation of the convergence of the percolation exploration path to SLE_6 as given by the theorem below.

Let $\mathcal{T}$ denote the standard two-dimensional triangular lattice with lattice spacing 1, and let $\mathcal{H}$ denote the hexagonal lattice which is dual to $\mathcal{T}$. For $\delta > 0$, we write $\delta\mathcal{H}$ to denote the hexagonal lattice with lattice spacing δ. Let $D \subset \mathbb{C}$ be a bounded, simply connected Jordan domain. That is, D is a simply connected domain whose boundary ∂D is a Jordan curve (i.e., ∂D is a simple closed curve which is homeomorphic to the unit circle). Suppose that $D^\delta \subset \delta\mathcal{H}$ is a Jordan set which approximates D. That is, D^δ is a simply connected subset of the hexagonal lattice with lattice spacing δ whose external site boundary is a simple closed loop of hexagons such that D^δ is a discrete approximation to D. Suppose further that a, $b \in \partial D$ are distinct boundary points, and let a^δ, $b^\delta \in \partial D^\delta$ be the corresponding external boundary vertices (or e-vertices). Without being more precise about this exact approximation, we denote by (D, a, b) the simply connected Jordan domain with two distinguished boundary points, and let its δ-scale approximation be denoted by $(D^\delta, a^\delta, b^\delta)$. Essentially, we think of choosing $D^\delta \equiv D \cap \delta\mathcal{H}$. (But this may not produce a simply connected D^δ so we do need to be careful.) A bit more technically, we assume that D^δ, a^δ, and b^δ are chosen so that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ in the Carathéodory sense as $\delta \downarrow 0$. If we now consider D^δ with distinguished e-vertices a^δ and b^δ, then we can see that these two distinguished boundary points partition the (topological) boundary of D^δ into two disjoint arcs. Associate to all external boundary hexagons on one of the arcs the colour "red" and associate to all boundary hexagons on the other arc the colour "white." (The colour red shows up clearly when an electronic version of this note is displayed on screen. However, in this printed version, red appears as grey instead.) Perform critical site percolation on D^δ; that is, for each remaining interior hexagon colour it either red with probability 1/2 or white with probability 1/2. There will be a resulting *interface* joining a^δ with b^δ; that is, a simple path connecting a^δ to b^δ with the property that all hexagons on one side of the path will be white while all hexagons on the other side of the path will be red. We call this path/interface the (critical site) percolation exploration path and denote it by $\gamma^\delta_{D,a,b}$. As $\delta \downarrow 0$, it is this path that converges to chordal SLE_6 in D from a to b.

Figure 1 shows schematically one way of producing the percolation exploration path. Given the realization of the percolation configuration with the boundary conditions (as shown on the left of Figure 1) we then "swallow any islands" by swapping the colour of an "island" with the colour of the "ocean" surrounding it. This produces two disjoint sets—one coloured red and the other coloured white. The percolation exploration path is exactly the interface between these two sets. If we now delete all of the hexagons, then what remains is the percolation exploration path as shown on the right of Figure 1.

Theorem (Camia and Newman [2]**).** *Let (D, a, b) be a Jordan domain with two distinct selected points on its boundary ∂D, and suppose that $D^\delta \subset \delta\mathcal{H}$ are Jordan sets with two distinct selected e-vertices a^δ, $b^\delta \in \partial D^\delta$ such that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ as $\delta \downarrow 0$. If $\gamma^\delta_{D,a,b}$ denotes the percolation exploration path inside D^δ from a^δ to b^δ, then $\gamma^\delta_{D,a,b}$ converges in distribution as $\delta \downarrow 0$ to $\gamma_{D,a,b}$, the trace of chordal SLE_6 inside D from a to b.*

There are essentially two main parts to the proof. The first is a characterization of SLE_6, and the second is the fact that any subsequential limit of the exploration path satisfies this characterization. The actual proof of the theorem is relatively short *once all of the preliminary lemmas and preparatory theorems have been established.*

Proof. Consider $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ and $\gamma^\delta_{D,a,b}$, the percolation exploration path. The law of $\gamma^\delta_{D,a,b}$ is a distribution on curves. An earlier result of Aizenman

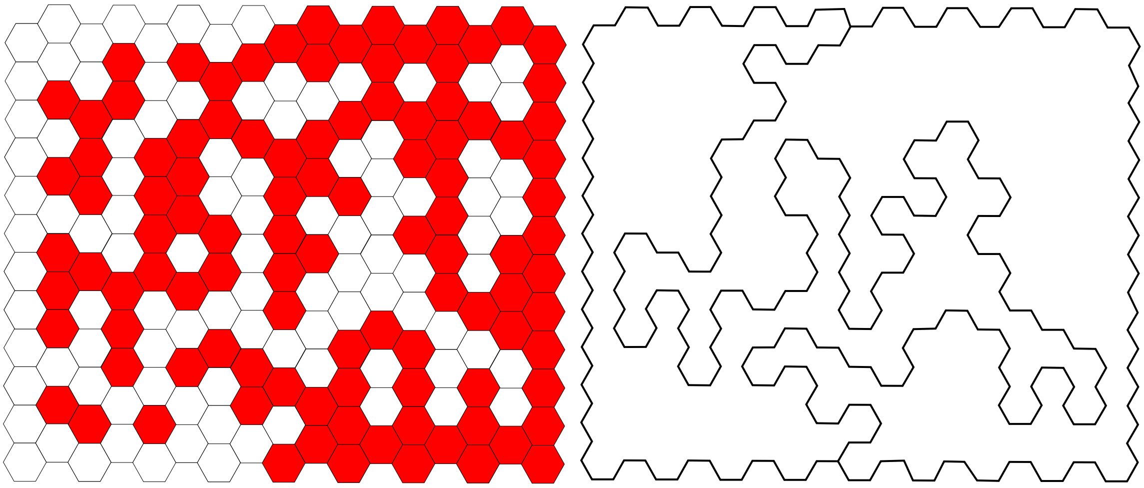

Figure 1. On the left is the realization of the percolation configuration with the imposed boundary conditions, and on the right is the resulting exploration path.

and Burchard [1] (in particular, Theorem A.1) is that this family $\gamma^{\delta}_{D,a,b}$ converges in distribution along subsequential limits $\delta_k \downarrow 0$ to the law of some curve γ. Since the filling of any subsequential limit $\tilde{\gamma}_{D,a,b} \equiv \lim_{\delta_k \downarrow 0} \gamma^{\delta_k}_{D,a,b}$ satisfies the spatial Markov property and the hitting distribution of $\tilde{\gamma}$ is determined by Cardy's formula, it follows that the limit is unique and that the law of $\gamma^{\delta}_{D,a,b}$ converges as $\delta \downarrow 0$ to the law of $\gamma_{D,a,b}$, the trace of chordal SLE$_6$ in D from a to b. □

Remark. As a historical note, we mention that a beautiful argument due to Schramm [6] showed that if the scaling limit of the exploration path exists and is conformally invariant, then it must be SLE$_\kappa$ for some κ. The value $\kappa = 6$ is then obtained by noting that Cardy's formula is satisfied only by SLE$_6$. The proof of this result was announced by Smirnov in 2001 [7], although a detailed proof did not appear until 2005 in an appendix of a preprint by Camia and Newman. Their paper [2], based on that appendix, presents that proof in an essentially self-contained form. We also mention that convergence of the exploration path to SLE$_6$ was used by Smirnov and Werner [8]; and Lawler, Schramm, and Werner [5] to rigorously derive the values of various percolation critical exponents. Camia and Newman also used the convergence to obtain the full scaling limit of critical percolation in two dimensions. Lectures by P. Nolin and C. Hongler during the current Arbeitsgemeinschaft will discuss these critical exponents and the full scaling limit, respectively.

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Convergence of percolation to CLE_6

Clément Hongler

Consider site percolation on a planar graph. Color the open sites in white and the closed ones in black. Such a percolation configuration can described by the set of its black and white clusters. The boundaries of these clusters form a collection of nested loops on the dual graph. For instance, if we perform site percolation on the triangular lattice, we get a collection of loops on the hexagonal lattice. Here we are interested in describing this set of loops in the case of critical site percolation on the triangular lattice at the *scaling limit* (when the mesh size of the lattice tends to 0).

The proof of this result relies mainly on the fact that in this model the discrete path called *exploration path* that separates black and white sites (in the case of a domain with one arc on the boundary colored white and the other black) converges in distribution to the continuum process SLE_6 as the mesh tends to 0. We give an algorithm using iteratively the exploration path that exhausts the set of loops and construct in a similar way the limit measure using SLE_6 instead of the discrete path.

Basically take a domain with monochromatic boundary (say white), create artificial boundary conditions pretending part of the boundary is of the other color (say black), launch an exploration process in this artificial setup, get some subdomains (the connected components of the complementary of the path in the domain) which have either monochromatic boundary conditions or mixed ones (one arc white and one arc black). In the domains with mixed boundary, launch another exploration process and eventually get smaller subdomains all with monochromatic boundary. In all the monochromatic domains obtained at the first or second step, iterate inductively the algorithm, until having exhausted all loops.

From now, it should be intuitive that the same algorithm using SLE_6 and iterated infinitely many times is a good candidate for being the limit of the discrete algorithm. Mainly two things are to be checked in order to get convergence. The first is to verify that the subdomains given by the exploration path and SLE_6 are the same in the limit. The second is to ensure that the number of steps needed to explore all the loops of a given size is stochastically bounded, uniformly with respect to the mesh size; that is, that the number of steps needed to do so does not blow up.

Finally we give some basic properties of the limit, which is a measure on the set of infinite collection of nested loops in our domain. The first is its conformal invariance: if the domain is mapped conformally onto another domain, the measure in the image domain will be the same (in law) as the image by the conformal map of the measure in the starting domain. The second is the so-called restriction-renewal property: if one removes a collection of loops, what remains is the same as independent copies of the measure in the subdomains which are the complementary of the removed collection.

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One–arm exponent for 2D percolation at the critical point

Pierre Nolin

The setting is two-dimensional site percolation, on the triangular lattice. We discuss the so-called "arm events": these are exceptional events referring to the existence of some number j of disjoint monochromatic crossings – "arms" – of annuli of the form $S_{n,N} := \{z : n < |z| < N\}$ $(n < N)$. These events are useful because they can be combined together, and they turn out to be instrumental to study critical and near-critical percolation. Their asymptotic behavior can be described precisely using SLE(6): they follow power laws, governed by the "arm exponents".

We focus here on the case $j = 1$: this is the one–arm event, simply denoting the existence of a black crossing $S_n \rightsquigarrow S_N$. We follow the proofs of [1, 3]. A similar derivation (with some non-trivial differences however), still using SLE(6), can be made for a larger number of arms, but only if we assume the existence of at least one black arm and one white arm: these are the "multichromatic arm exponents", computed in [2].

We first need to introduce the radial SLE, with parameter $\kappa = 6$. There exists a simple relation between radial and chordal SLE(6): up to some disconnection time, their laws (as curves) are the same.

For percolation, we define a "radial" exploration process, the scaling limit of which is radial SLE(6). In an annulus, the existence of a black crossing is related to the fact that this exploration process, starting from the outer boundary, does not close any counter-clockwise loop before reaching the inner boundary. The probability of this last event is roughly the same as the probability that a radial SLE(6) does not close any counter-clockwise loop before some given time.

We then perform computations for this process (for which we can use stochastic calculus), and we get that for percolation, the probability to cross an annulus with radii r and R tends to $f_1(r/R)$ when the mesh size decays to 0, for some function f_1 satisfying

$$f_1(\eta) \sim \eta^{5/48+o(1)}$$

as $\eta \to 0^+$.

The property of quasi-multiplicativity for percolation then allows to return to the discrete setting and we obtain the desired estimate for one arm:

$$\mathbb{P}_{1/2}(0 \rightsquigarrow \partial S_N) = N^{-5/48+o(1)}$$

as $N \to \infty$.

We finally review some consequences of these derivations on the characteristic functions used to describe percolation (like the density of the infinite cluster or the characteristic length), more precisely how the corresponding critical exponents can be derived.

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Lace expansion and percolation in high dimensions

Markus Heydenreich

Consider bond percolation on the hypercubic lattice $\mathbb{Z}^d$, $d \geq 2$, where we fix a parameter $p \in [0,1]$, and make every bond in the graph (independently of other bonds) *open* with probability p, and *closed* otherwise. The connected component in the subgraph of open bonds are called *clusters*. For two sites $x, y \in \mathbb{Z}^d$, we write $\{x \leftrightarrow y\}$ for the event that the two are in the same cluster, i.e., they are connected via a path of open bonds. The probability of this event is denoted by $\tau_p(x,y)$. Let further denote

$$\chi(p) := \sum_{x \in \mathbb{Z}^d} \tau_p(0,x)$$

the *expected cluster size* or *susceptibility*, and $\theta(p)$ the probability that the origin belongs to a cluster of infinitely many sites. A fundamental question is the behaviour of the model around the critical point

$$p_c := \inf\{p \mid \theta(p) > 0\} = \sup\{p \mid \chi(p) < \infty\},$$

where equality between the two characterizations of p_c is due to Menshikov [10] and Aizenman–Barsky [1].

Our interest is the critical and near-critical behaviour of percolation, and we use the notion of *critical exponents* to describe it. To this end, we consider the exponents γ and β given by

$$\begin{aligned} \chi(p) &\asymp \frac{1}{(p_c - p)^\gamma} \qquad \text{as } p \nearrow p_c, \\ \theta(p) &\asymp (p - p_c)^\beta \qquad \text{as } p \searrow p_c. \end{aligned}$$

We emphasize that if β exists and is strictly positive, then $\theta(p_c) = 0$. It is believed that critical exponents are *universal*, i.e., minor modifications of the model, like changes in the underlying lattice, leave the critical exponents unchanged (though p_c differs). Their values depend on the dimension d. However, it is predicted that there is an *upper critical dimension* $d_c = 6$, such that the critical exponents take the same value for all $d > d_c$. These values are the *mean-field* values of the critical exponents, and for percolation these mean-field values are $\gamma = 1$ and $\beta = 1$, which coincide with the corresponding critical exponents obtained for percolation on an infinite regular tree, see [6, Section 10.1]. Nevertheless, a rigorous proof for these mean-field values is established only for $d \geq 19$, and that is what we are focussing now.

We introduce the triangle diagram

$$\nabla_p := \sum_{x,y \in \mathbb{Z}^d} \tau_p(0,x)\, \tau_p(x,y)\, \tau_p(y,0),$$

and say that the *triangle condition* is satisfied whenever the critical triangle diagram ∇_{p_c} is finite. This triangle condition plays a key role in the understanding of high-dimensional percolation. In particular, the triangle condition implies $\gamma = 1$ (as proven by Aizenman–Newman [2]) and $\beta = 1$ (Barsky–Aizenman [3]). For a textbook-style proof of these facts we refer to [11, Section 9].

Theorem 1 (Hara–Slade 1990). *For bond-percolation on $\mathbb{Z}^d$ in dimension $d \geq 19$, the triangle condition $\nabla_{p_c} < \infty$ is satisfied.*

The proof uses the *lace expansion*, a technique invented by Brydges and Spencer [5] to study weakly self-avoiding walk. They use an algebraic expansion involving a certain class of graphs called *laces*, which gave the lace expansion its name. In 1990, Hara and Slade [7] presented the lace expansion for percolation, where the algebraic expansion is replaced by an inclusion-exclusion expansion.

Here is a brief sketch of some ideas from the proof of Theorem 1. We first use an inclusion-exclusion expansion of the two-point function $\tau_p(x,y)$ to obtain

(1)
$$\tau_p(x,y) = \delta_{x,y} + (\tau_p * J_p)(x,y) + \left(\Pi_p^{(M)} * J_p * \tau_p\right)(x,y) + \Pi_p^{(M)}(x,y) + R_p^{(M)}(x,y),$$

where

- $\delta_{x,y}$ refers to the Kronecker delta-function;

- $J_p(x,y) = \begin{cases} p & \text{if } (x,y) \text{ is a bond,} \\ 0 & \text{otherwise;} \end{cases}$
- $\Pi_p^{(M)}(x,y)$ is obtained as an alternating sum of certain configurations arising from the inclusion-exclusion expansion;
- $R_p^{(M)}(x,y)$ is a remainder term, and is proven to vanish as $M \to \infty$.

The superscript M indicates the level of expansion. The J_p-terms arise from *pivotal* bonds in the expansion.

As a second step, we bound $\Pi_p^{(M)}(x,y)$ and $R_p^{(M)}(x,y)$ in terms of the triangle diagram ∇_p. These bounds are called *diagrammatic bounds* since the quantities $\Pi_p^{(M)}(x,y)$ and $R_p^{(M)}(x,y)$ can be represented by certain diagrams, and the underlying structure expressed with the help of these diagrams is heavily used to obtain the bounds.

We finally use a so-called *bootstrap argument*. Suppose we could show the following "improvement of the bounds": If (for any $p < p_c$) the triangle diagram ∇_p is smaller than some uniform constant, say $\nabla_p < 4$, then indeed it must be small than 3. Since ∇_p is monotone in p and $\nabla_0 = 1$, this implies $\nabla_p < 3$ for all $p \in [0, p_c]$, and in particular $\nabla_{p_c} < \infty$. Unfortunately, we cannot quite show the improvement of the bounds directly for the triangle diagram. Instead, we consider the Fourier transform $\hat{\tau}_p$ of the two-point function τ_p. A Fourier version of Eq. (1) and the diagrammatic bounds imply an improvement of the bounds for certain functionals of $\hat{\tau}_p$. In turn, the bounds on these functionals of $\hat{\tau}_p$ give rise to sufficient bounds on the triangle diagram.

Recall that the triangle condition is conjectured to hold for $d > 6$. To support this conjecture, Hara–Slade [7] prove this fact for a spread-out version of the model, where more bonds are added to the lattice. More precisely, consider a lattice with bonds between any two sites that are of distance at most L from each other, where L is a parameter of the model. For this spread-out model it is proven that the triangle condition is satisfied for $d > 6$ provided that L is sufficiently large.

The talk is mainly based on the work by Borgs et al. [4], where the lace expansion is applied to percolation on *finite* graphs, but their clever use of trigonometric bounds applies verbatim to the infinite setting. The interested reader might consult the Saint-Flour lecture notes by Gordon Slade [11, Section 10] for the expansion and the diagrammatic bounds, and [8] for an implementation of the bootstrap argument.

An alternative approach to the lace expansion is the *inductive method*, where the bootstrap argument is replaced by induction. This approach is outlined in the lecture notes by van der Hofstad [9].

There are many extensions of the method leading to a variety of results for high-dimensional percolation, like identification of the scaling limit of the incipient infinite cluster, asymptotics for p_c as $d \to \infty$, and various other critical exponents. We refer to [11] for references and discussion.

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Sharp thresholds

Jeffrey E. Steif

While sharp thresholds are very relevant in percolation, they can be better appreciated in a more general context. The context here will be that we have an increasing event $A \subseteq \{0,1\}^n$ and we look at how $P_p(A)$ behaves as a function of p and in particular whether for large n a 'sharp threshold' arises meaning that this function goes very quickly from being small to being large. (P_p here is product measure with density p.)

One of the key concepts that is relevant to sharp thresholds and is a very natural and interesting concept in itself is the notion of influence. For $i = 1, \dots, n$, the influence of i on A at level p, denoted by $I_i^p(A)$ is the P_p-probability of the event that 'i is pivotal for A' meaning that flipping the value X_i changes whether A occurs or not. We mention right away that Russo's formula (also proved by Margulis) tells us that the derivative of $P_p(A)$ is simply

$$\sum_i I_i^p(A).$$

Fix p at the moment to be $1/2$ and we ask if there is some i with large influence? No if A is the empty set and so to avoid trivial matters, we assume now $P_{1/2}(A)$

is bounded between (say) 1/4 and 3/4. A natural quantity to look at is

$$J_n := \inf_{A:P_{1/2}(A)\in[1/4,3/4]} \sup_i I_i^{1/2}(A)$$

and to see how it behaves as n goes to ∞.

Taking n odd and letting A be the majority event (meaning there are more 1's than 0's), it is easy to see (using 1-d random walk results) that each influence is of order $1/\sqrt{n}$ and so J_n behaves at most like $1/\sqrt{n}$. An edge-isoperimetric inequality for the cube shows that J_n is at least $1/n$ (the sum of the influences is the same as the edge boundary). A more sophisticated example by Ben Or and Linial shows that J_n is at most $\log(n)/n$. For this, break $1,\ldots,n$ into subintervals of length $\log_2(n) - \log_2 \log_2(n)$ and let A be the event that at least one of these subintervals is all 1's.

Kahn, Kalai and Linial (using a beautiful argument using Fourier analysis and Beckner's inequality) show in fact that this is sharp meaning J_n is at least $c\log(n)/n$. They showed in fact that for all events A and for all p, there is an i such that

$$I_i^p(A) \geq c\log(n)/n \min\{P_p(A), 1-P_p(A)\}.$$

Friedgut and Kalai pointed out that this result together with Russo's formula fairly easily leads to a sharp threshold for invariant events A meaning that A is invariant under a transitive action of $1,\ldots,n$. (Note that in this case all the influences are the same.) More precise, $P_p(A)$ goes from ϵ to $1-\epsilon$ within an interval of length $c\log(1/2\epsilon)/\log n$. This holds for graph properties in particular since they are invariant under the transitive subgroup of permutations of the edges coming from permutations of the vertices.

Friedgut and Kalai conjectured that for graph properties, the threshold interval should be much smaller, of order $1/(\log n)^2$. Later Bourgain and Kalai proved this and related the threshold interval with how big of a subgroup leaving A invariant exists.

Of course, not all events A have large threshold such as $\{X_1 = 1\}$ but Friedgut and Bougain have results that sort of say that if there is not a sharp threshold, then there is a good reason for it meaning that the event more or less depends on a fixed number of variables.

Much earlier, Russo obtained an approximate 0-1 law which basically said that if all influences (both as p and i vary) are small, then there is a sharp threshold. The proof was more qualitative and was not based on Fourier analysis as these later proofs are.

Finally, we mention that one of the key steps in Kesten's proof that the critical parameter is 1/2 in 2D was the following. He showed that had the critical value been larger, then in the interval between 1/2 and p_c, a certain crossing event would have a large threshold which he proved by showing with his hands that the sum of the influences is large. Russo had proofs which also explained how this threshold was a key part of the proof and later on Bollobas and Riordan explained how one could get this needed sharp threshold out of the Friedgut-Kalai threshold result.

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An exposition of the Bollobás-Riordan proof that $p_c = 1/2$ for random Voronoi percolation in the plane

Jakob Erik Björnberg

The Voronoi percolation model in $\mathbb{R}^2$ is defined as follows: let $p \in [0,1]$ and let Z^+, Z^- be independent Poisson processes of rates p and $1-p$ respectively. The Voronoi cell of $z \in Z := Z^+ \cup Z^-$ by is the set of points $x \in \mathbb{R}^2$ closer to z than to any other point of Z. Call z and its cell black if $z \in Z^+$ and white otherwise. We are interested in the proability that there is an infinite path of adjacent black cells, one of which contains the origin. Bollobás and Riordan proved in [1] that this probability is positive iff $p > 1/2$.

The fact that no percolation occurs for $p \leq 1/2$ may be considered standard, because the famous argument of Zhang applies to this model. The major achievement of [1] is to show that percolation occurs for $p > 1/2$. The main steps are similar to those for bond percolation on $\mathbb{Z}^2$, but the first two steps are (much) harder:

(1) Find a Russo-Seymour-Welsh (RSW) type result at $p = 1/2$, saying roughly that the probability of crossing a long rectangle does not vanish as the size of the rectangle gets large;
(2) Moving up to an arbitrary $p > 1/2$, use a sharp-threshold result to show that with the new parameter such crossings are extremely likely;
(3) Finally paste together crossings of rectangles, obtaining the result by comparison with a “known” model (in this case 1-independent percolation on $\mathbb{Z}^2$).

The main focus of this talk is to give an exposition of the RSW-result (step one) in [1]; in light of the other recent applications of this argument [3, 4] we feel that an awareness of the techniques employed would benefit all students of percolation. Before we give some more details about this part of the argument, let us first indicate quite briefly what the main problems (and their solutions) are for the second step. In order to be able to apply one of the currently known sharp threshold theorems, it is necessary to transfer the problem to a discrete product space. It is thus necessary to find an event A, say, such that (i) A has probability that is not too small, and (ii) even a "cruder" (discretized) version of A implies the existence of rectangle crossings in the continuum. It turns out that the right event A to consider is the existence of a "robust" rectangle crossing, i.e. one that is not sensitive to small perturbations of the points of Z. To achieve properties (i) and (ii) it is necessary to increase p in two steps. Property (i) transpires after the first increase, because it turns out that one can couple the processes with the higher and lower parameters in such a way that a crossing in the lower process implies a *robust* crossing in the higher—showing this is the hardest part of the whole argument. Once this is done, an application of the Friedgut-Kalai theorem during the second increase boosts the probability of a crossing to almost 1.

Now we give the main ideas for the RSW part of the argument. Let $h(\rho, s)$ denote the probability that there is a black path crossing the rectangle $[0, \rho s] \times [0, s]$ horizontally.

Theorem 2. *Let* $p = 1/2$*, and* $\rho > 1$*. Then* $\limsup_{s\to\infty} h(\rho, s) > 0$.

This implicitly uses the fact that $h(1, s) = 1/2 > 0$ for all s, but a result analogous to Theorem 2 holds for arbitrary values of p; see also [3] for some extensions.

Here is an outline of the proof of Theorem 2. We assume the conclusion to be false, i.e. that for some $\varepsilon > 0$ we have $\lim_s h(1+\varepsilon, s) = 0$. The first step is to use the FKG inequality to see that in fact the limit is zero for *all* $\varepsilon > 0$. Using symmetry and the FKG inequality again, this statement may be strengthened to say that with high probability (whp) as $s \to \infty$, any horizontal crossing of some rectangle (of arbitrary dimensions) has very tightly controlled maximal and minimal heights. In fact, given $\delta > 0$, any such crossing is up to δs just as "tall" as it is "wide". Now consider the fact that for a path to traverse a horizontal distance of s, it must first traverse a distance of $0.99s$. The bound just mentioned applied to the thinner rectangle will contradict the bound for the wider one (for δ small enough) *unless* the path goes back almost to where it started before completing the longer crossing. This is because when it first crosses the narrow rectangle the path has to have maximum height almost exactly $0.495s$, but the maximum height of the whole path must be almsot exactly $0.5s$, i.e strictly higher. It follows that whp any crossing of a rectangle that is s wide contains two disjoint subcrossings of rectangles that are $0.99s$ wide. See Figure 1. This argument may of course be repeated for each of these subpaths, giving the absurd (but not quite contradictory!) conclusion that every crossing contains 16 disjoint crossings of rectangles that are $0.96s$ wide. This indicates that the length of a crossing of a recangle grows very fast as a function of the width of the rectangle—indeed, too fast.

To rigorously prove this, the next crucial ingredient is some form of asymptotic independence. We use the fact that an appropriate (but small) modification $\tilde{L}$ of the length L of the shortest horizontal crossing of a rectangle is a random variable that is independent for two rectangles that are distance $\Omega(s)$ apart. If a crossing of a rectangle of width s and height $2s$ is shorter than x say, then one of its 16

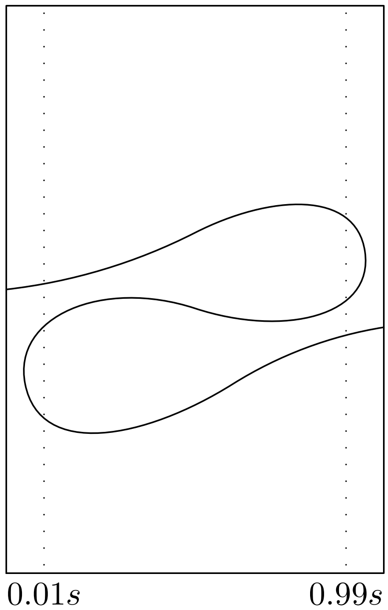

Figure 1. The key step to arrive at the contradiction which proves the result is to show that any crossing contains two disjoint subcrossings of rectangles almost as wide.

subcrossings must be shorter than $x/16$. It is possible to show, using again the strict control on the vertical span of crossings, that this slightly shorter crossing must in fact traverse two of a bounded number of rectangles of with $0.47s$ and height $2 \cdot 0.47s$, that are at least $0.01s$ apart. Since $\tilde{L}$ for these small rectangles are independent, this "squares" the probability of $\tilde{L}$ being small. This type of argument lets us obtain that if the width of our rectangle goes up by a factor $(1/0.47) \approx 2$ then the length of a shortest crossing goes up by a factor 16. But using elementary arguments one may show that $\tilde{L}$ is whp $o(s^3)$, finally giving the desired contradiction.

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Multiscale methods

Bernardo N.B. de Lima

In this talk, we consider an oriented percolation model in a random environment, this model was proposed and studied by Kesten, Sidoravicius and Vares in [4].

Consider the following North-East oriented site percolation model on $\mathbb{Z}_+^2$. Given the parameters δ, p and $\Delta \in [0,1]$, for each line $H_i := \{(x,y) \in \mathbb{Z}_+^2; x+y=i\}$, $\forall i \in \mathbb{Z}_+$ we associate the 0-1 ('good' or 'bad') random variable, ξ_i, independently of each other and

$$P(\xi_i = 1) = \delta = 1 - P(\xi_i = 0), \ \forall i \in \mathbb{Z}_+.$$

Given the configuration of good (0) and bad (1) lines, (ξ_i), we declare each vertex of $\mathbb{Z}_+^2$ as 'open' or 'closed', independently each other, with the following conditional distribution

$$P_{\delta,p,\Delta}(v \text{ is open}|\xi_i, \forall i) = \begin{cases} \Delta, & \text{if } \xi_{\|v\|} = 1 \\ p, & \text{if } \xi_{\|v\|} = 0 \end{cases} \tag{1}$$

If Y_v is the random variable that indicates if v is open or closed, we can observe that Y_v and Y_u are positive correlated whenever the vertices v and u belong to the same line. For this reason the Peierls argument doesn't work, then to prove phase transition in this model a multiscale renormalization scheme is used to prove the following result

Theorem [4] Given any $\Delta > 0$ and $p > p_c$ one can find some $\delta^*(p,\Delta) > 0$ such that

$$P_{\delta,p,\Delta}(0 \leftrightarrow \infty) > 0, \ \forall \delta < \delta^*.$$

Here p_c is the percolation threshold for the ordinary independent site percolation on $\mathbb{Z}_+^2$. This result has connections to several questions in probability theory: Winkler's problem of compatibility of sequences, existence of percolation of arbitrary words (introduced in [1]) when p is close to 1 and the question of percolation on the Random Stretched Lattice in dimension two (this model was introduced and studied in [3]).

The proof of this theorem is rather involved and at this talk we will discuss the main ideas on a simplified version of the problem, the so called 'hierarchical model', and with p being close enough to 1. The main reference for this simplified version is the Part II of [2].

Given some large natural number L, define the sequence $(\tilde{\xi}_j^L)_j$ as

$$\tilde{\xi}_j^L := \max\{k \in \mathbb{Z}_+; \ L^k | j \ \}.$$

The L-Hierarchical binary sequence $(\xi_j^L)_j$ is obtained from $(\tilde{\xi}_j^L)_j$ replacing each k ($k \geq 2$) by a string of k consecutive 1's and shifting the rest of the sequence $k-1$ units to the right.

Now, given the L-Hierarchical sequence, $(\xi_j^L)_j$, and the parameters p and $\Delta \in (0,1)$, we declare each vertex $v \in \mathbb{Z}_+^2$ as open or closed, independently of each other, with probability defined by

$$P_{L,p,\Delta}(v \text{ is open}) = \begin{cases} \Delta, \text{ if } \xi^L_{\|v\|} = 1 \\ p, \text{ if } \xi^L_{\|v\|} = 0 \end{cases} \tag{2}$$

Introducing several new definitions : renormalized sites, seeds, s-passability, s_ρ-dense kernel... we gave the main ideas of the multiscale renormalization scheme and proved the analogous statement in the Hierarchical context.

Theorem There exists some $p^* < 1$, such that, given any $\Delta > 0$ and $p > p^*$ one can find some $L^*(p, \Delta) > 0$ large enough, such that

$$P_{L,p,\Delta}(0 \leftrightarrow \infty) > 0, \ \forall L > L^*.$$

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Overview of percolation on nonamenable graphs

Alexander Fribergh

The study of percolation on general graphs has developed during the last twenty years. The aim of this talk is to give ideas about the methods which are available to study percolation on such graphs. The main reference for this talk is the book [1].

We need to restrict slightly the term general graphs, most study has been done on graphs that look the same from every vertex. These graphs are called transitive.

Definition : *If G has the property that for every pair of vertices x, y, there is an automorphism of G that takes x to y, then G is called transitive.*

The notion of transitivity is enough to get a very nice result on the number of infinite clusters.

Theorem : *If G is a transitive connected graph, the number of infinite clusters is constant a.s. and equal to either* 0, 1 *or* ∞.

One subclass of great importance arises from group theory. Let Γ be a finitely generated group and $S = \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ a symmetric set of generators.

Definition : *The Cayley graph of* Γ *is the graph* $X(\Gamma, S) := (V, E)$ *with* $V := \Gamma$ *and* $[g, h] \in E$ *iff* $g^{-1}h \in S$.

A notion of great importance concerning percolation is amenability and constitutes the heart of this talk. This has links to the growth rate of the graph.

Definition : *We define the edge-isoperimetric constant of* $G = (V, E)$ *to be*

$$\iota_E(G) = \inf\left\{\frac{|\partial_E K|}{|K|},\ \emptyset \neq K \subset V \text{ is finite}\right\},$$

where $\partial_E K = \{[u, v] \in E, u \in K, v \in V - K\}$.

Then we can define amenability

Definition : *A graph* G *is said to be amenable if* $\iota_E(G) = 0$. *Otherwise it is nonamenable.*

Nonamenable graphs grow very quickly, so there is space for multiple infinite clusters to coexist. This is not possible on an amenable graph.

Theorem : *If* G *is a connected transitive amenable graph, then* $\mathbf{P}_p$*-a.s., there is at most one infinite cluster.*

As a consequence the transition phase on an amenable graph is very simple. In fact even on nonamenable graphs the transition phase is not very complicated. Let us set

- $p_c(G) := \sup\{p \geq 0,\ \theta(p) = 0\}$ where $\theta(p) := \mathbf{P}[0 \leftrightarrow \infty]$.
- $p_u(G) := \inf\{p \leq 1,\ \text{ there exists a unique } p\text{-infinite cluster}\}$.

Those two thresholds are enough to describe the transition phase using the following theorem.

Theorem :*Let* G *be a transitive graph. Set* $p_2 > p_1 > p_c(G)$. *In standard coupling, every infinite* p_2*-cluster contains an infinite* p_1*-cluster.*

On nonamenable Cayley graphs most proofs are related to invariant percolation. Let Γ be a group that acts by automorphisms on a graph G.

Definition

- *A probability measure on the subgraphs of* G *is a* Γ*-invariant percolation model if it is invariant under the action of* Γ.
- *An invariant percolation model* $\mathbf{P}$ *is a bond percolation model if* $\mathbf{P}[V(\omega) = V(G)] = 1$.

The invariant percolation is used with the mass-trasport principle. It is a central technique which appeared in 1998 in the context of percolation on nonamenable graphs. It is more or less the only existing technique.

Definition : *A function $m(x, y, \omega) \in [0, \infty]$ is diagonally invariant if it verifies that $m(x, y, \omega) = m(\gamma x, \gamma y, \gamma\omega)$ for all $\gamma \in \Gamma$.*

Theorem :
For $m(., ., .)$ diagonally invariant, we have

$$\forall x \in \Gamma, \sum_{y \in \Gamma} M(x, y) = \sum_{y \in \Gamma} M(y, x),$$

where $M(x, y) = \mathbf{E}[m(x, y, \omega)]$ for any invariant $\mathbf{P}$.

This theorem is the key tool in the proof of the following theorem.

Theorem : *If G is a nonamenable Cayley graph, then $\theta(p_c(G)) = 0$.*

The proof of this result can be found in [2]. Through this presentation the main results available nowadays are presented.

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Short talk I: Mean field forest fire model

Balázs Ráth

(joint work with Bálint Tóth)

We consider continuous-time Markov process with a finite state space: the set of all simple graphs on n vertices. At a certain time an edge may be occupied or vacant. Vacant edges become occupied with rate $\frac{1}{n}$ independently. At the same time lightnings strike each vertex independently with rate $\lambda(n)$. If a lightning strikes a vertex, the fire spreads along the edges of the connected component of that vertex, and burns them. The number of vertices remain unchanged, but the connected component that was hit is turned immediately into a set of isolated vertices.

The special case $\lambda(n) \equiv 0$ is the well-known Erdős-Rényi random graph process. Let

$$v_k^n(t) = \frac{1}{n} \cdot \text{the number of vertices contained in components of size } k \text{ at time } t$$

It is natural to expect that the law of large numbers hold: $\lim_{n\to\infty} v_k^n(t) = v_k(t)$ where $v_1(t), v_2(t), \ldots$ are deterministic functions of t.

In the Erdős-Rényi case, the phase transition of the random graph can be observed by looking at $\big(v_k(t)\big)_{k=1}^{\infty}$: for $t < 1$, we have $\sum_{k=1}^{\infty} v_k(t) = 1$ and $v_k(t)$ decays exponentially. This is the subcritical phase. For $t > 1$, $\sum_{k=1}^{\infty} v_k(t) < 1$, indicating the presence of the giant component in the supercritical phase. $v_k(t)$ decays exponentially in the supercritical phase as well. $t = 1$ is the critical time:

$v_k(t) \asymp k^{-\frac{3}{2}}$ in the critical phase. $\frac{3}{2}$ is the critical exponent of the cluster size distribution in mean field percolation.

The critical forest fire model is characterized by $\frac{1}{n} \ll \lambda(n) \ll 1$. If the rate of forest fires is in this regime, then the fire has no effect on small components in the limit, but destroys giant components immediately. The deterministic functions $v_k(t)$ are identical to that of the Erdős-Rényi limit in the subcritical phase, but evolve differently after the critical time: for every $t > 1$, we have $\sum_{k=1}^{\infty} = 1$ and $v_k(t) \asymp k^{-\frac{3}{2}}$, this phenomenon is called self-organized criticality. The functions $v_k(t)$ can be identified as the solution of the critical controlled Smoluchowski coagulation equation.

Short talk II: A phase transition between strong amenability and anchored expansion

Florian Sobieczky

Certain percolative partial graphs of the horocyclic product of two homogeneous trees are considered. Removal of edges by a Bernoulli bond-percolation process is carried out only on a subset of the set of edges. By its construction, the connected component containing a preassigned root is the horocyclic product of two random trees, sampled from the augmented Galton Watson measure [1, 2]. Given that the percolation results from the horocyclic product of two trees realized as samples of two independent augmented Galton Watson measures with equal growth, we show almost sure strong amenability. For another subclass of these random product graphs, sufficient closeness to an unsymmetric horocyclic product guarantees anchored expansion [3, 4]. This implies the existence of a phase-transition between strong amenability and weak non-amenability (=anchored expansion).

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Short talk III: Uniqueness of the infinite cluster for oriented percolation and the contact process.

Jan Swart

Recall that a graph is called *transitive* if for each two vertices there exists an automorphism of the graph that maps the first vertix onto the second. There exists an extensive literature about (unoriented) percolation on general transitive graphs. In particular, it is known that one has uniqueness of the infinite cluster whenever the graph is amenable, while it is conjectured (and proved in several special cases) that on any nonamenable graph there exists an intermediate parameter regime where there are infinitely many infinite clusters.

One wonders if it is possible to state and prove an analogue result for oriented percolation, or for the closely related graphical representation of the contact process, which we recall now. Let E and V be the edge and vertex sets of a transitive graph G, respectively, and let $E \times \mathbb{R} := \{(i,t) : i \in E,\ t \in \mathbb{R}\}$. We interpret t as the time coordinate, which is usually plotted upwards. For each $i \in E$, at times chosen according to an independent rate one Poisson processes, we draw a recovery symbol $*$ at the point (i,t). For each ordered pair (i,j) such that $[i,j] \in V$, at times chosen according to an independent Poisson processes with rate $\lambda \geq 0$, we draw an arrow from (i,t) to (j,t). We write $(i,t) \rightsquigarrow (j,u)$ if there exists an upward path in $E \times \mathbb{R}$ from (i,t) to (j,u) that may jump from one vertex to another over arrows but must avoid recovery symbols. Now, for each $A \subset E$,

$$\eta_t^A := \{j \in E : \exists i \in A \text{ s.t. } (i,0) \rightsquigarrow (j,t)\} \tag{1}$$

defines a Markov process $(\eta_t^A)_{t \geq 0}$, taking values in the subsets of E, started in the initial state $\eta_0^A = A$. This process is called the (nearest-neighbor) *contact process* on G with infection rate λ.

For given $(i,t) \in E \times \mathbb{R}$, let

$$\vec{C}_{(i,t)} := \{(j,u) : (i,t) \rightsquigarrow (j,u)\} \tag{2}$$

denote the oriented cluster at (i,t). We write $(i,t) \rightsquigarrow \infty$ if $\vec{C}_{(i,t)}$ is infinite. For the nearest-neighbor process on $\mathbb{Z}$, it has been shown in [Swa05, Lemma 4] (see also [WZ06, Thm 1.4] for an analogue statement concerning oriented percolation) that

$$\vec{C}_{(i,t)} \mathbin{\triangle} \vec{C}_{(j,s)} \text{ is compact a.s. on the event that } (i,t) \rightsquigarrow \infty \text{ and } (j,s) \rightsquigarrow \infty. \tag{3}$$

Here $A \mathbin{\triangle} B := (A \cup B) \backslash (A \cap B)$ denotes the symmetric difference between two sets A and B. Equation (3) says that if (i,t) and (j,s) each belong to an infinite cluster, then these infinite clusters are eventually equal.

Grimmett and Hiemer [GH02] have proved the much weaker statement that for the nearest-neighbor process on $\mathbb{Z}^d$

$$\begin{gathered} \exists (k,u) \text{ s.t. } (i,t) \rightsquigarrow (k,u),\ (j,s) \rightsquigarrow (k,u), \text{ and } (k,u) \rightsquigarrow \infty \\ \text{a.s. on the event that } (i,t) \rightsquigarrow \infty \text{ and } (j,s) \rightsquigarrow \infty. \end{gathered} \tag{4}$$

This says that each two infinite clusters contain a third infinite cluster in their intersection.

It is an interesting question if one can prove either weak cluster uniqueness in the sense of (4) or strong cluster uniqueness in the sense of (3) for more general graphs G, and what should be the right condition on G. It seems that the Burton-Keane proof [BK89] (which proves cluster uniqueness for unoriented percolation on any transitive amenable graph) cannot be adopted to the oriented setting. Indeed, one may wonder if amenability is the right property to look at. A simple subadditivity argument shows that each contact process has a well-defined exponential growth rate:

$$r := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[|\eta_t^A|\big] \qquad (A \text{ finite, nonempty}). \tag{5}$$

It is known that there exist amenable graphs with exponential growth (e.g., the lamplighter group). Here are two open problems: does cluster uniqueness (weak or strong) hold if the exponential growth rate r from (5) is zero? If the compact process survives (i.e., percolates) on an exponentially growing graph, is r then necessarily positive? For nonamenable graphs, I hope to answer the latter question positively in work in progress [Swa05].

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Short talk IV: Maximal flow in first passage percolation

Marie Théret

The model of first passage percolation that we study is the following: on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$, we consider a family of independent and identically distributed variables $(t_e, e \in \mathbb{E}^d)$, where t_e is a non negative real number. We interpret t_e as the capacity of the edge e, i.e., the maximal amount of fluid that can cross the tube e per unit of time. This yields a straightforward definition of the maximal flow ϕ_B, the maximal amount of liquid that can pass through B from its bottom to its top per unit of time using only edges inside B, under the condition of capacity on these edges and without loss of liquid inside the box; vertices at the top and the bottom of B can been seen as sources and sinks. The max-flow min-cut theorem provides an equivalent expression for ϕ_B, namely the minimal capacity of a set of edges (i.e. the sum of the capacities of the edges that belong to the set) that disconnects the bottom from the top of B. If we identify the dual of an edge as a plaquette of side length one, orthogonal to it and that cuts it in its middle, we can interpret such separating set of edges as a separating surface of plaquettes.

In 1987 Kesten proved a law of large number for ϕ_B (see [1]): in dimension 3, under some conditions (exponential moment, $\mathbb{P}(t_e = 0)$ small enough, height of the box not too big), ϕ_B grows linearly in $surf(B)$, the surface of the basis of B. Our work addresses the large deviations for $\phi_B/surf(B)$. The upper large deviations are of volume order (see [2]). This can be proved by comparing the separating surface of plaquettes in the box which minimizes the capacity with a separating surface of plaquettes in an infinite cylinder whose intersection with the cylinder's boundary is prescribed. The capacity of this second object is sub-additive, because the fixed boundary conditions allow us to glue together separating surfaces in adjacent cylinders. We then use the classical Cramér Theorem to conclude. In fact, we obtain a large deviation principle.

The lower deviations are of surface order, and they are more complicated to study. A partial result is that the probability that the renormalized flow is very small decays exponentially fast with the surface of the basis of B, under very weak conditions on $\mathbb{P}(t_e = 0)$ and the height of B. This can be shown in the case of Bernoulli percolation by an argument of coarse graining, and can then be carried over to our model with a general law for the capacities (see [3]). This result is about to be extended thanks to certain concentration inequalities (this is a joint work with Raphaël Rossignol), and it tunrs out that a large deviation principle holds here too.

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Short talk V: Fluctuation estimates for last-passage percolation

Márton Balázs

(joint work with Eric Cator, Timo Seppäläinen)

We study the last-passage growth model on the planar integer lattice with exponential weights. We change the rates of the exponential weights on the axes in order to form boundary conditions that represent the equilibrium exclusion process as seen from a particle right after its jump. The variance of the last-passage time in a characteristic direction is of order $t^{2/3}$. With more general boundary conditions that include the rarefaction fan case it is also possible to show that the last-passage time fluctuations are still of order $t^{1/3}$, and also that the transversal fluctuations of the maximal path have order $t^{2/3}$. We adapted and then built on a recent study of Hammersley's process by Cator and Groeneboom, and also utilized the competition interface introduced by Ferrari, Martin and Pimentel. The arguments are entirely probabilistic, and no use is made of the combinatorics of Young tableaux or methods of asymptotic analysis.

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Short talk VI: One-dimensional transient random walks in random environment in the sub-ballistic regime

Olivier Zindy
(joint work with Nathanaël Enriquez, Christophe Sabot)

The main result about transient RWRE with zero asymptotic speed was obtained by Kesten, Kozlov and Spitzer in [3] who proved that, when normalized by a suitable power of n, the hitting time of the level n converges towards a positive stable law whose index corresponds to the power of n lying in the normalization. In [1], our purpose is to characterize this positive stable law.

The proof chooses a radically different approach than previous ones dealing with the transient case. While the proof in [3] is mainly based on the representation of the trajectory of the walk in terms of branching processes in random environment (with immigration), our approach relies heavily on Sinai's interpretation of a particle living in a random potential. However, in the recurrent case, the potential one has to deal with is a recurrent random walk and Sinai introduces a notion of valleys which does not make sense anymore in our setting where the potential is a (let's say negatively) drifted random walk. Therefore, we introduce a different notion of valley which is closely related to the excursions of this random walk above its past minimum. It turns out that a result of Iglehart gives an equivalent of the tail of the height of these excursions. Now, as soon as one can prove that the hitting time of the level n can be reduced to the time spent by the random walk to cross the high excursions of the potential above its past minimum, between 0 and n, which are well separated in space, an i.i.d. property comes out, and the problem is reduced to the study of the tail of the time spent by the walker to cross a single excursion.

It turns out that this tail involves the expectation of the functional of some meander associated with the random walk defining the potential. Now, this functional is itself related to the constant that appears in Kesten's renewal theorem. These last two facts are contained in [2]. Now, in the case when the transition probabilities follow some Beta distribution a result of Chamayou and Letac gives an explicit formula for this constant which yields finally an explicit formula for the parameter of the positive stable law which is obtained at the limit.

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Open problem: Is there a RSW theorem for anisotropic percolation?

Jakob Erik Björnberg

We consider bond percolation on the square lattice with horizontal and vertical edge parameters p_h and p_v, respectively, satisfying $p_h + p_v = 1$. This model is critical and self-dual, see [1, Section 11.9]. Let $H(a, b)$ be the event that there is an open horizontal crossing of the rectangle $[0, a] \times [0, b]$. It is a well-known and extremely important fact that in the case $p_h = p_v = 1/2$, we have

- *An a-priori bound* saying that $P(H(n, n)) \geq 1/2 > 0$ for all n, and
- *A conditional extension result* saying that, *if* an a-priori bound of the form $P(H(n, n)) \geq c_1 > 0$ does hold, *then* for $k = 1, 2, \ldots$ there are constants $c_k > 0$ such that also $P(H(kn, n)) \geq c_k$.

Our question is simply: do corresponding results hold for general p_h, p_v as above?

Some clarifications are necessary. On the one hand, it is not hard to show that a conditional extension result for crossings of rectangles does hold (only horizontal reflection symmetry is needed, so the method due to Smirnov explained in [2, Section 1.3] works). But for squares there is *no known* a-priori bound. On the other hand, for shapes other than squares an a-priori bound *is* known: letting

$$D_n = \{(x, y) \in \mathbb{R}^2 : |x| + |y - 1/2| \leq n + 1/2\}$$

be an "off-set diamond", we have that there is probability exactly 1/2 of having an open crossing from the upper left to the lower right side, for all n. But for this shape there is no known extension result to, say, "elongated diamonds". So the question may be rephrased: is there a class of "shapes" such that an a-priori bound and some form of conditional extension result hold simultaneously?

This question is vague about what classes of shapes we wish to consider. Let us illustrate what we want by considering rectangles again. In Kesten's book [3, Lemma 7.1], it is shown that there are constants $A, B, c_1 > 0$ and a function $f(n)$ satisfying $A \log n \leq f(n) \leq e^{Bn}$, such that for large enough n,

$$P(H(n, f(n))) \geq c_1 \qquad \text{and} \qquad P(H(f(n), n)) \geq c_1.$$

Rectangles of this form have aspect ratio $n/f(n)$ depending on n. This a-priori bound was sufficiently strong for Kesten to deduce that the critical surface is indeed $p_h + p_v = 1$, but is less useful for studying scaling limits of the model. Yet another version of our question may therefore be: can the bounds on $f(n)$ be improved? Can we take f such that $n/f(n)$ has a nontrivial limit?

Finally, note that the a-priori bound for diamonds coupled with horizontal reflection symmetry do not necessarily imply results of the form we want. To show this, Vincent Beffara gave the example of a type of percolation on the triangular lattice, where each horizontal line is open or closed with probability 1/2.

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An open problem in anisotropic percolation

Bernardo N.B. de Lima

This open problem proposed by E. Andjel some years ago is described as follows. We consider an anisotropic independent bond percolation model on $\mathbb{Z}_+^2$, i.e. we suppose that the vertical edges of $\mathbb{Z}_+^2$ are open with probability p and closed with probability $1-p$, while the horizontal edges of $\mathbb{Z}_+^2$ are open with probability αp and closed with probability $1-\alpha p$, with $0 < p, \alpha < 1$. Let $x = (x_1, x_2) \in \mathbb{Z}_+^2$, with $x_1 < x_2$, and $x' = (x_2, x_1) \in \mathbb{Z}_+^2$. It is natural to ask how the two point connectivity function $P_{p,\alpha}(0 \leftrightarrow x)$ behaves, and whether anisotropy in percolation probabilities implies the strict inequality $P_{p,\alpha}(0 \leftrightarrow x) > P_{p,\alpha}(0 \leftrightarrow x')$. In general, this question is open and the note [1] gives an affirmative answer in some regions of the parameters involved.

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Reporter: Bálint Vágvölgyi

Participants

Dr. Marton Balazs
Institute of Mathematics
Budapest University of Technology and Economics
1.Egry Jozsef U., Bldg. H
H-1111 Budapest

Andras Balint
Faculteit Wiskunde en Informatica
Vrije Universiteit Amsterdam
De Boelelaan 1081 a
NL-1081 HV Amsterdam

Dr. Vincent Beffara
Mathematiques
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Prof. Dr. Jacob van den Berg
Centrum voor Wiskunde en Informatica
Kruislaan 413
NL-1098 SJ Amsterdam

Jakob Björnberg
Statistical Laboratory
Centre for Mathematical Sciences
Wilberforce Road
GB-Cambridge CB3 OWB

Dr. Federico Camia
Faculteit Wiskunde en Informatica
Vrije Universiteit Amsterdam
De Boelelaan 1081 a
NL-1081 HV Amsterdam

Dr. Massimo Campanino
Dipartimento di Matematica
Universita degli Studi di Bologna
Piazza di Porta S. Donato, 5
I-40126 Bologna

Michael Damron
Courant Institute of Mathematical Sciences
New York University
251, Mercer Street
New York , NY 10012-1110
USA

Maxime Delebassee
Mathematiques
Universite Paul Sabatier
118, route de Narbonne
F-31062 Toulouse Cedex 9

Prof. Dr. Gerd Faltings
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Alexander Fribergh
107, rue Vendome
F-69006 Lyon

Dr. Urs Hackstein
Institut f. Reine Mathematik
Universität Ulm
Helmholtzstr. 18
89081 Ulm

Hadrian Heil
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Markus O. Heydenreich
Department of Mathematics and Computer Science
Eindhoven University of Technology
Postbus 513
NL-5600 MB Eindhoven

Clement Hongler
Departement de Mathematiques
Universite de Geneve
Case Postale 64
2-4 rue du Lievre
CH-1211 Geneve 4

Dr. Michael Kozdron
Dept. of Mathematics & Statistics
University of Regina
Regina , SK S4S 0A2
CANADA

Sebastian Lück
Abteilung Stochastik
Universität Ulm
89069 Ulm

Dr. Michael Müger
Dept. of Mathematics
Radboud Universiteit Nijmegen
Postbus 9010
NL-6525 ED Nijmegen

Gregor Noetzel
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Dr. Pierre Nolin
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Dr. Bernardo Nunes Borges de Lima
Departamento de Matematica-ICEX
Universidade Federal de Minas Gerais
CP 702
Av. Antonio Carlos 662
Belo Horizonte -MG 31270-901
BRAZIL

Balazs Rath
Institute of Mathematics
Budapest University of Technology and Economics
1.Egry Jozsef U., Bldg. H
H-1111 Budapest

Damien Regnault
Mathematiques
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Dr. Silke Rolles
Zentrum Mathematik
TU München
85747 Garching

Remy Sanchis
Mathematics Department
UFMG
Av. Antonio Carlos, 6627
Caixa Postal 702
Belo Horizonte , MG 30123-970
BRAZIL

Artem Sapozhnikov
Centrum voor Wiskunde en Informatica
Kruislaan 413
NL-1098 SJ Amsterdam

Prof. Dr. Dierk Schleicher
School of Engineering and Science
Jacobs University Bremen
Postfach 750561
28725 Bremen

Dr. Florian Sobieczky
Institut für Mathematik
Technische Universität Graz
Steyrergasse 30
A-8010 Graz

Prof. Dr. Jeffrey Steif
Dept. of Mathematics
Chalmers University of Technology
S-41296 Gothenburg

Dr. Ronfeng Sun
Institut für Mathematik
Fakultät II; Sekr. MA 7-5
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin

Dr. Jan M. Swart
UTIA AV CR
Pod vodarenskou vezi 4
18208 Praha 8
Czech Republic

Dr. Marie Theret
Ecole Normale Superieure
DMA
45 rue d'Ulm
F-75230 Paris Cedex 05

Ralf Thiedmann
Abteilung Stochastik
Universität Ulm
89069 Ulm

Balint Vagvolgyi
Faculteit Wiskunde en Informatica
Vrije Universiteit Amsterdam
De Boelelaan 1081 a
NL-1081 HV Amsterdam

Dr. Vincent Vargas
Laboratoire de Probabilites et Modeles
Aleatoires - UMR 7599
Universite Paris VII - Denis Diderot
2, place Jussieu - Boite Courrier 7012
F-75251 Paris Cedex 05

Dr. Olivier Zindy
Weierstrass Institute of Applied
Analysis and Stochastics(WIAS)
Mohrenstr. 39
10117 Berlin