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Stochastic Analysis in Finance and Insurance

Organised by
Dmitry Kramkov, Pittsburgh
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January 27th – February 2nd, 2008

Abstract. This workshop brought together, from all over the world, leading experts and a large number of younger researchers in stochastic analysis and mathematical finance. During a very intense week, participants exchanged many ideas and laid the foundations for new collaborations and further developments in the field.

Mathematics Subject Classification (2000): 60Gxx, 60Hxx, 91Bxx.

Introduction by the Organisers

The workshop *Stochastic Analysis in Finance and Insurance*, organised by Dmitry Kramkov (Pittsburgh), Martin Schweizer (Zürich) and Nizar Touzi (Paris) was held January 27th – February 2nd, 2008. The meeting had a total of 44 participants from all over the world with a good blend of more experienced researchers and many younger participants.

During the five days, there were a total of 29 talks with many lively interactions and discussions. The organisers had to exercise some constraint on the participants in order not to overload the programme, and this prompted many discussions and collaborations during the long lunch breaks and in the evenings.

The topics presented in the talks covered a very wide spectrum. There were some major areas with several talks as well as other more individual contributions pointing towards new developments in mathematical finance. The overall tendency went towards more sophisticated and more realistic models of financial markets; the first generation models with frictionless classical semimartingale prices seem largely understood, and one of the major trends is now towards what might be termed second generation modelling. This includes as major topics *transaction costs* and *large investors and liquidity issues* as well as other nonstandard models or ideas related to such developments. A second major topic revolved around *risk measures or monetary utility functions*, and there were several talks on *option pricing*, on *optimisation problems from finance* and on *credit risk*.

We now give a short overview of the topics covered in the talks, roughly ordered into the themes listed above.

Transaction costs: The classical Merton problem of optimal investment under transaction costs was reconsidered by *Jan Kallsen* who presented a new approach via shadow prices leading to a simpler way of finding the optimal strategy. *Yuri Kabanov* extended the classical hedging theorem under transaction costs from European to the case of American contingent claims.

Large investors and liquidity issues: A model for the optimal liquidation of a large portfolio position was presented by *Alexander Schied*; his results showed that the market impact of such a trader can lead to some unexpected effects. *Mete Soner* studied the problem of superreplicating options in an illiquid market by means of PDE and stochastic control techniques. *Thorsten Rheinländer* developped a new model for utility maximisation by a large trader and showed how this can be modelled via nonlinear stochastic integration theory. A partial equilibrium model for a large investor interacting with other market participants was presented by *Peter Bank*, who emphasised the importance of appropriate financial modelling of gains from trade in continuous time.

Risk measures or monetary utility functions: This topic had the largest number of talks. *Michael Kupper* introduced divergence utilities and showed that this fairly large class can be very well manipulated and leads to explicit solutions for optimisation and risk sharing problems. *Damir Filipović* studied extensions of convex risk measures from the space of all bounded random variables and showed that under law-invariance, the canonical model space consists even of all integrable random variables. *Walter Schachermayer* proved a very new result: He showed that the only time-consistent law-invariant dynamic convex risk measure is the entropic risk measure. *Freddy Delbaen* gave a detailed study of the representation for the penalty function of time-consistent dynamic monetary utility functions with the help of backward stochastic differential equations. Another extension of risk measures to Orlicz hearts was studied by *Patrick Cheridito*, and *Gordan Žitković* introduced maturity-independent risk measures and pointed out some nontrivial existence problems related to this concept.

Option pricing: For a class of stochastic volatility models, *David Hobson* showed how to obtain option price comparisons by means of time changes and other purely probabilistic arguments. *Ludger Rüschendorf* gave a broad overview of methods to obtain comparison results for option prices in large classes of processes and showed several techniques to achieve this goal. *Semyon Malamud* presented a new approach for deriving indifference prices for contingent claims under power utility in discrete-time models having a certain new structure, and pointed out several appealing properties of this class of models. *Vicky Henderson* considered perpetual American options in an incomplete market and determined the optimal exercise strategy when the number of options one owns is infinitely divisible.

Optimisation problems from finance: An overview and some new developments for risk-sensitive portfolio optimisation were presented by *Jun Sekine*. *Paolo Guasoni* studied the problem of finding optimal portfolios and risk premia explicitly in the limit of a long time horizon. *Bruno Bouchard* started with quantile hedging and related problems and embedded these into a general stochastic target problem with controlled probability or controlled losses.

Credit risk: Valuation of credit-sensitive instruments often involves first passage times, and *Tom Hurd* presented new ideas on how these can be handled more explicitly for a fairly large class of jump-diffusion processes. *Ronnie Sircar* gave an asymptotic analysis of multiscale models for multiname credit risk derivatives and illustrated that his approach leads to computationally tractable and yet fairly accurate results when calibrated to market data.

In addition to the above roughly thematically grouped talks, there were presentations that did not fall readily into a particular area; this illustrates the diversity and multiple facets of the field of mathematical finance. *Yannis Karatzas* started the workshop with a very stimulating talk on so-called diverse financial markets and the idea of finding there optimal arbitrage strategies. *Thaleia Zariphopoulou* presented a new way to look at performance measurement in financial markets and formulated this as a novel and intriguing mathematical problem involving a stochastic partial differential equation. *Jakša Cvitanić* gave an overview of recent developments and results on contract theory in continuous time. Motivated by the question of how to model the influence of information on financial markets, *Kostas Kardaras* introduced a topology on σ-fields and on filtrations and presented some first continuity results. *Josef Teichmann* explained how one can compute moments of affine processes in a very easy way; this was motivated by many examples arising in the valuation of financial derivatives. With the goal of modelling both stock prices and the infinite family of all call options in a joint model, *René Carmona* studied dynamic local volatility models and derived the corresponding drift restrictions arising from absence of arbitrage. Another very thought-provoking talk was given by *Denis Talay* who presented some mathematical models and problems connected to so-called technical analysis in financial markets. *Mihai Sîrbu* considered a general semimartingale model and gave some necessary and some sufficient conditions for the validity of a mutual fund theorem.

In contrast to the picture shown on the institute homepage, there was no snow around the institute, as remarked (and regretted) by several participants. On the other hand, this allowed on Wednesday a very pleasant excursion to St. Roman with a very good participation rate.

For us as the organisers, it was (as always) a great pleasure to be at Oberwolfach and to benefit from the excellent infrastructure, support and scientific environment. We thank the Mathematisches Forschungsinstitut Oberwolfach for making this possible, and we are very happy to report that this sentiment was also expressed by all the participants both during and after the conference in many ways. The idea of having a similar workshop in about three years met with very enthusiastic reactions.

Dmitry Kramkov
Martin Schweizer
Nizar Touzi

Workshop: Stochastic Analysis in Finance and Insurance

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Abstracts

Optimal arbitrage

Ioannis Karatzas
(joint work with Daniel Fernholz)

In a Markovian model for an equity market with mean rates of return $\mathrm{b}_i(\mathfrak{X}(t))$ and covariance rates $\mathrm{a}_{ij}(\mathfrak{X}(t))$, $1 \leq i,j \leq n$ for its asset prices $\mathfrak{X}(t) = (X_1(t),\dots,X_n(t))' \in (0,\infty)^n$ at time t, what is the smallest relative amount of initial capital starting with which, and using non-anticipative investment strategies, one can match or exceed the performance of the market portfolio by the end of a given time-horizon $[0,T]$? What are the weights in the different assets of an investment strategy that accomplishes this?

Answers: under appropriate conditions, $U(T,\mathfrak{X}(0))$ and

$$X_i(t)\, D_i \log U\big(T-t,\mathfrak{X}(t)\big) \,+\, \frac{X_i(t)}{X_1(t)+\cdots+X_n(t)}, \qquad i=1,\dots,n,\; t\in[0,T]$$

respectively, where $U : [0,\infty)\times(0,\infty)^n \to (0,1]$ is the smallest non-negative solution of the parabolic partial differential inequality

$$\frac{\partial U}{\partial \tau}(\tau,\mathbf{x}) \,\geq\, \widehat{\mathcal{L}}\, U(\tau,\mathbf{x}), \quad (\tau,\mathbf{x})\in(0,\infty)\times(0,\infty)^n$$

subject to the initial condition $U(0+,\,\cdot\,)\equiv 1$, for the operator

$$\widehat{\mathcal{L}} f \,:=\, \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n x_i x_j\, \mathrm{a}_{ij}(\mathbf{x})\, D^2_{ij} f \,+\, \sum_{i=1}^n \left(\sum_{j=1}^n \frac{x_j\, \mathrm{a}_{ij}(\mathbf{x})}{x_1+\cdots+x_n}\right) x_i D_i f\,.$$

Furthermore, $U(T,\mathbf{x})$ is the probability that the $[0,\infty)^n$-valued diffusion process $\mathfrak{Y}(\cdot) = (Y_1(\cdot),\dots,Y_n(\cdot))'$ with infinitesimal generator $\widehat{\mathcal{L}}$ as above, and starting with the initial configuration $\mathfrak{Y}(0) = \mathfrak{X}(0) = \mathbf{x} \in (0,\infty)^n$, does not hit the boundary of the non-negative orthant $[0,\infty)^n$ by time $t=T$.

It is perhaps worth noting that the answers involve only the covariance structure of the market, not the actual mean rates of return; the rôle of these latter is limited to ensuring that the diffusion $\mathfrak{X}(\cdot)$ lives in $(0,\infty)^n$.

Strong arbitrage relative to the market portfolio exists on the horizon $[0,T]$, if and only if $U(T,\mathfrak{X}(0))<1$; this amounts to failure of uniqueness for the Cauchy problem

$$\frac{\partial U}{\partial \tau}(\tau,\mathbf{x}) \,=\, \widehat{\mathcal{L}}\, U(\tau,\mathbf{x}), \quad (\tau,\mathbf{x})\in(0,\infty)\times(0,\infty)^n \qquad \text{and} \qquad U(0+,\,\cdot\,)\equiv 1\,.$$

As suggested by results in Fernholz & Karatzas (2005, 2008), a sufficient condition for such failure of uniqueness is that there exists a real constant $h>0$, such that

$$(x_1+\cdots+x_n)\sum_{i=1}^n x_i \mathrm{a}_{ii}(\mathbf{x}) - \sum_{i=1}^n\sum_{j=1}^n x_i x_j \mathrm{a}_{ij}(\mathbf{x}) \,\geq h\,(x_1+\cdots+x_n)^2\,, \quad \forall\ \mathbf{x}\in(0,\infty)^n\,;$$

another sufficient condition is that there exists a real constant $h > 0$ with

$$\big(x_1 \cdots x_n\big)^{1/n} \left[\sum_{i=1}^n \mathrm{a}_{ii}(\mathbf{x}) - \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \mathrm{a}_{ij}(\mathbf{x}) \right] \geq h\,(x_1 + \cdots + x_n),\ \forall\, \mathbf{x} \in (0,\infty)^n .$$

As far as we know, this is the first instance that sufficient conditions for non-uniqueness (equivalently, necessary conditions for uniqueness) are obtained for such Cauchy problems.

Consider an "auxiliary market", whose asset prices are given by $\mathfrak{Y}(\cdot) = (Y_1(\cdot), \ldots, Y_n(\cdot))'$. The probabilistic significance of the change of drift inherent in the definition of the operator $\widehat{\mathcal{L}}$, from $\mathrm{b}_i(\mathbf{x})$ for the process $\mathfrak{X}(\cdot)$ to $\sum_{j=1}^n (x_j\, \mathrm{a}_{ij}(\mathbf{x}))/(x_1 + \cdots + x_n)$ for $\mathfrak{Y}(\cdot)$, is that it corresponds to a change of measure which makes the weights $\nu_i(\cdot) := Y_i(\cdot)/(Y_1(\cdot) + \cdots + Y_n(\cdot))$, $i = 1, \ldots, n$ of the auxiliary market portfolio martingales. The financial significance of this change of measure is that it bestows to the auxiliary market portfolio $\underline{\nu}(\cdot) = \big(\nu_1(\cdot), \ldots, \nu_1(\cdot)\big)'$ the so-called *numéraire property:* the ratio of any strategy's performance, relative to the new market with prices $\mathfrak{Y}(\cdot)$, is a supermartingale. This change of measure does not come necessarily from a Girsanov-type (absolutely continuous) transformation; rather, it corresponds to, and represents, the *exit measure* of Föllmer (1972) for an appropriate supermartingale.

The questions raised in this work can be traced back to Fernholz (2002). They are related to the results of Delbaen & Schachermayer (1995), and bear an even closer connection with issues raised in the finance literature under the general rubric of "bubbles". The literature on this latter topic is large, so let us mention the papers by Loewenstein & Willard (2000), Pal & Protter (2007) and, most significantly, Heston et al. (2007), as the closest in spirit to our approach here. Let us also call attention to the recent preprint by Hugonnier (2007), which demonstrates that arbitrage opportunities can arise also in equilibrium models; we also refer to this preprint and to Heston et al. (2007) for an up-to-date survey of the literature on this and related topics.

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On using shadow prices in portfolio optimization with transaction costs

JAN KALLSEN
(joint work with Johannes Muhle-Karbe)

One of the basic questions in mathematical finance is how to choose an optimal investment strategy in a securities market or, more specifically, how to maximize utility from consumption (cf. e.g. [17, 18] for an introduction). This is often called the *Merton problem* because it was solved by Merton [25, 26] for power and logarithmic utility functions in a Markovian Itô process model. In a market with a riskless bank account and one risky asset following a geometric Brownian motion, the optimal strategy turns out to invest a *constant fraction* π^* of wealth in the risky asset and to consume at a rate proportional to current wealth. This means that it is optimal for the investor to keep his portfolio holdings in bank and stock on the so-called *Merton line* with slope $\pi^*/(1-\pi^*)$.

Since then, this problem has been generalized in several ways. One direction has been to consider different market models. Solutions to utility maximization problems are generally obtained by two different methods. One approach is to use stochastic control theory, which leads to Hamilton-Jacobi-Bellman equations (cf. e.g. [3, 10] for Lévy processes and [2, 19] for stochastic volatility models). Alternatively, one can turn to martingale methods which appear in different forms, both in actual computations (cf. e.g. [15] for Lévy processes and [16] for stochastic volatility models) and in general structural results (cf. e.g. [20] and the references therein).

A different generalization of the Merton problem is the introduction of proportional transaction costs. In a continuous time setting this was first done by Magill and Constantinides [24]. Their paper contains the fundamental insight that it is optimal to refrain from transacting while the portfolio holdings remain in a wedge-shaped *no-transaction region*, i.e. while the fraction of wealth held in stock lies inside some interval $[\pi_1^*, \pi_2^*]$. However, their solution was derived in a somewhat heuristic way and also did not show how to compute the location of the boundaries π_1^*, π_2^*.

Mathematically rigorous results were first obtained in the seminal paper of Davis and Norman [9]. They show that it is indeed optimal to keep the proportion of total wealth held in stock between fractions π_1^*, π_2^* and they also prove that these two numbers can be determined as the solution to a free boundary value problem. The theory of viscosity solutions to Hamilton-Jacobi-Bellman equations was introduced to this problem by Shreve and Soner [27] who succeeded in removing several assumptions needed in [9]. Since then, this approach has also been used to obtain optimal portfolios in several variants of the Merton problem with proportional transaction costs, e.g. in the finite horizon case [1, 8, 22], the case of multiple stocks [1], and stocks modeled as jump diffusions [11].

All these articles aiming for the computation of the optimal portfolio employ tools from stochastic control. It seems that martingale methods have so far only been used to obtain structural existence results in the presence of transaction costs. In this context the martingale and duality theory for frictionless markets is often applied to a *shadow price process* $\widetilde{S}$ lying within the bid-ask bounds of the real price process S. Economically speaking, the frictionless price process $\widetilde{S}$ and the original price process S with transaction costs lead to identical decisions and gains for the investor under consideration. This concept has been used in the context of the fundamental theorem of asset pricing (cf. [14] and recently [12, 13]), local risk minimization [21], super-replication [6, 4], and utility maximization [5, 7, 23].

In the present study we reconsider Merton's problem for logarithmic utility and under proportional transaction costs as in [9]. Our goal is threefold. Firstly, we show that the shadow price process can be computed explicitly. More importantly, the shadow price approach can be used to come up with a candidate solution to the utility maximization problem under transaction costs. Finally, we indicate that verification appears — at least for the problem at hand — to be surprisingly simple compared to the very impressive and non-trivial reasoning in [9, 27].

The slightly more involved case of power utility is the subject of current reserach and is not treated here. However, the use of shadow prices for portfolio selection is not limited to the particular model of [9]. We confine ourselves to the present setup because it allows best to present the key idea underlying our approach.

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Comparison results for stochastic volatility models via coupling

DAVID HOBSON

The aim of this paper is to investigate the properties of stochastic volatility models, and to discuss to what extent, and with regard to which models, properties of the classical exponential Brownian motion model carry over to a stochastic volatility setting. The properties of the classical model of interest include the fact that the discounted stock price is positive for all t but converges to zero almost surely, the fact that it is a martingale but not a uniformly integrable martingale, and the fact that European option prices (with convex payoff functions) are convex in the initial stock price and increasing in volatility. We give examples of stochastic volatility models where these properties continue to hold, and other examples where they fail. The main tool is a construction of a time-homogeneous autonomous volatility model via a time change.

More specifically, our aim is to construct a pair (S_t, V_t) on a suitable probability space such that

$$\begin{aligned} &S_0 = s > 0 && dS_t = \sigma(S_t)V_t dB_t^S \\ &V_0 = v \geq 0 && dV_t = \alpha(V_t)dB_t^V + \beta(V_t)dt \\ &dB_t^S dB_t^V = \rho(V_t)dt, \end{aligned} \tag{1}$$

in such a way that we can provide useful couplings, from which it will be possible to derive comparison results. Here S is the discounted stock price and V is the volatility, and we work under a martingale measure $\mathbb{Q}$.

Theorem 1. *Suppose that $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t>0}, \mathbb{Q})$ is a Brownian filtration, satisfying the usual conditions. Suppose that the SDE*

$$\begin{aligned} &X_0 = s && dX_t = \sigma(X_t)dB_t^X \\ &Y_0 = v && dY_t = \frac{\alpha(Y_t)}{Y_t}dB_t^Y + \frac{\beta(Y_t)}{Y_t^2}dt \\ &dB_t^X dB_t^Y = \rho(Y_t)dt \end{aligned} \tag{2}$$

has a unique strong solution, up to the first explosion time ε. Define $\Gamma_t = \int_0^t Y_s^{-2}ds$, and set $A \equiv \Gamma^{-1}$. Then $S_t \equiv X_{A_t}$ and $V_t \equiv Y_{A_t}$ solve (1).

More precisely, let $\zeta = \lim_{t\uparrow\varepsilon}\Gamma_t \leq \infty$, so that $A_\zeta = \varepsilon$, and for $t \leq \zeta$ set $\mathcal{F}_t = \mathcal{G}_{A_t}$ and define

$$B_t^S = \int_0^{A_t} \frac{dB_u^X}{Y_u} \qquad\qquad B_t^V = \int_0^{A_t} \frac{dB_u^Y}{Y_u}.$$

Then, for $t \leq \zeta$, (B_t^S) and (B_t^V) are $(\mathcal{F}_t)$-Brownian motions and the pair $(S_t \equiv X_{A_t}, V_t \equiv Y_{A_t})$ is a weak solution to (1).

The idea is that by analysing the autonomous diffusion (Y_t) and the time-change (A_t) we can derive results for the discounted price process (S_t) which may be hard to obtain directly. This proves useful for the study of many classical stochastic volatility models from the literature, and in generating new models which can be used to provide counterexamples. The models of Hull-White, Heston and Lewis are all amenable to study with this approach, and determining their properties reduces to studying the properties of Bessel processes of different dimensions. By analysing whether these processes (and additive functionals of them) explode to zero or infinity we can deduce properties of the price process under a martingale measure.

The time-change construction can also be used to study the properties of option prices under $\mathbb{Q}$. Since equivalent martingale measures are characterised by different drifts β (but σ and α must necessarily be identical under all EMMs) we can deduce comparison theorems for stochastic volatility models. For example:

Theorem 2. *Consider a pair of stochastic volatility models indexed by $i = 0, 1$ which differ only in the form of the drift on volatility:*

$$\begin{aligned} &i = 0,1 && S_0 = s, V_0 = v \\ &dS_t = \sigma(S_t)V_t dB_t^S, && dV_t = \alpha(V_t)dB_t^V + \beta^{(i)}(V_t)dt, \\ & && dB_t^S dB^V = \rho(V_t)dt. \end{aligned}$$

Denote by $(S^{(0)}, V^{(0)})$ and $(S^{(1)}, V^{(1)})$ the solutions under the two different models. Suppose that $S^{(i)}$ is a true martingale in each case.

Suppose that for each model the corresponding time-changed stochastic differential equation

$$\begin{aligned} &i = 0,1 && X_0 = s, Y_0 = v \\ &dX_t = \sigma(S_t)dB_t^X, && dY_t = (\alpha(Y_t)/Y_t)dB_t^Y + (\beta^{(i)}(Y_t)/Y_t^2)dt, \\ & && dB_t^X dB_t^Y = \rho(Y_t)dt, \end{aligned}$$

has a strong solution.

Suppose that $\beta^{(0)}(y) \leq \beta^{(1)}(y)$ for all y. Then for any convex Φ,

$$\mathbb{E}[\Phi(S_T^{(0)})] \leq \mathbb{E}[\Phi(S_T^{(1)})].$$

Proofs of the analogous result for one-dimensional diffusions rely on the notion that for convex payoffs the option price is a convex function of the underlying. This is sometimes called the propagation of convexity effect. The proof of Theorem 2 is based on a comparison of time-changes and does not rely on this effect. Indeed, there is no propagation of convexity in a general stochastic volatility models unless $\sigma(s) = s$ or $\rho = 0$. We give an example where convexity fails in the paper.

Some variations of risk-sensitive portfolio optimization

Jun Sekine

Risk-sensitive portfolio optimization (abbreviated to RSPO, hereafter) treats

$$\sup_{\pi \in \mathcal{A}_T} \frac{1}{\gamma} \log E e^{\gamma \log X_T^{x,\pi}} = \sup_{\pi \in \mathcal{A}_T} \frac{1}{\gamma} \log E (X_T^{x,\pi})^\gamma \tag{1}$$

for a given $T \in \mathbb{R}_{>0}$, and

$$\sup_{\pi \in \mathcal{A}} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log E e^{\gamma \log X_T^{x,\pi}} = \sup_{\pi \in \mathcal{A}} \overline{\lim_{T\to\infty}} \frac{1}{\gamma T} \log E (X_T^{x,\pi})^\gamma. \tag{2}$$

Here, $X^{x,\pi} := (X_t^{x,\pi})_{t \geq 0}$ is the wealth process of a self-financing investor with the initial capital $x > 0$ and the dynamic trading strategy $\pi := (\pi_t)_{t \in [0,T]}$, and $\mathcal{A}_T$ and $\mathcal{A}$ are certain spaces of trading strategies satisfying the admissibility condition

$$X^{x,\pi} \geq 0 \tag{3}$$

on the time interval $[0, T]$ and $[0, \infty)$, respectively. Several tractable examples and interesting applications of RSPO have been studied:

(i) Explicit representations of optimal trading strategies are obtained for market models described by linear Gaussian factor processes with the help of solutions to differential/algebraic Riccati equations (Bielecki-Pliska, 1999; Kuroda-Nagai, 2002; Fleming-Sheu, 2002; S, 2006; Davis-Lleo 2006, for example).

(ii) Large deviations control problems, i.e., for a given $k \in \mathbb{R}$,

$$\sup_{\pi\in\mathcal{A}} \overline{\lim_{T\to\infty}} \frac{1}{T} \log P\left(X_T^{x,\pi} \geq \mathrm{e}^{kT}\right) \quad \text{and} \quad \inf_{\pi\in\mathcal{A}} \underline{\lim_{T\to\infty}} \frac{1}{T} \log P\left(X_T^{x,\pi} \leq \mathrm{e}^{-kT}\right)$$

are studied via the "dual" optimization problem (2) (Pham, 2003; Hata-Nagai-Sheu, 2007, for example).

(iii) Drawdown constraint problem, i.e., (2) with the constraint

$$X_t^{x,\pi} \geq \alpha S_t^0 \max_{s\in[0,t]} \left(\frac{X_s^{x,\pi}}{S_s^0}\right) \quad \text{for all } t \geq 0,$$

($\alpha \in [0,1)$, S^0 is the bank-account process), is studied via a certain transformation, and the explicit representations of the optimal solutions are obtained in some concrete examples (Grossman-Zhou, 1993; Cvitanić-Karatzas, 1995; S, 2006, for example).

In this talk, as a different variation of RSPO, we treat RSPO with a floor on a long time-horizon, i.e., for a given $x > 0$ and $\gamma \in (-\infty, 1)$, we are interested in (2) with the constraint

$$X^{x,\pi} \geq K, \tag{4}$$

where $K := (K_t)_{t\geq 0}$ is a given nonnegative adapted floor process. We denote by $\mathcal{A}_T^K(x)$ and $\mathcal{A}^K(x)$ the totalities of admissible trading strategies satisfying the constraint (4) on the time interval $[0,T]$ and $[0,\infty)$, respectively. This problem is interpreted as an infinite time horizon version of CRRA-utility maximization with the constraint (4), which is studied in El Karoui-Jeanblanc-Lacoste (2005).

To state our result, we define the processes $\hat{X}^{(T)} := (\hat{X}_t^{(T)})_{t\in[0,T]}$ and $\hat{X}^{(\infty)} := (\hat{X}_t^{(\infty)})_{t\in[0,\infty)}$ by

$$\hat{X}_t^{(T)} := (x - \bar{K}_0)\bar{X}_t^{(T)} + \bar{K}_t \quad \text{and} \quad \hat{X}_t^{(\infty)} := (x - \bar{K}_0)\bar{X}_t^{(\infty)} + \bar{K}_t.$$

Here, $\bar{K} := (\bar{K}_t)_{t\geq 0}$ is the minimal super-replication of the floor K, written as

$$\bar{K}_t := X_t^{\bar{K}_0,\eta}$$

with some $\eta \in \mathcal{A}_T$ and $x \geq \bar{K}_0 > 0$, and $\bar{X}^{(T)} := (\bar{X}_t^{(T)})_{t\in[0,T]}$ (resp. $\bar{X}^{(\infty)} := (\bar{X}_t^{(\infty)})_{t\in[0,\infty)}$) is the optimal wealth process for the finite (resp. infinite) time-horizon problem (1) (resp. (2)) with the initial capital 1 and the constraint (3) (i.e., *without floor*) so that

$$\sup_{\pi\in\mathcal{A}_T}\frac{1}{\gamma}\log\mathbb{E}\left(X_T^{1,\pi}\right)^\gamma = \frac{1}{\gamma}\log\mathbb{E}\left(\bar{X}_T^{(T)}\right)^\gamma,$$
$$\sup_{\pi\in\mathcal{A}}\overline{\lim_{T\to\infty}}\frac{1}{\gamma T}\log\mathbb{E}\left(X_T^{1,\pi}\right)^\gamma = \overline{\lim_{T\to\infty}}\frac{1}{\gamma T}\log\mathbb{E}\left(\bar{X}_T^{(\infty)}\right)^\gamma.$$

We assume the existence of these optimal processes $\bar{X}^{(T)}$ for all $T\in\mathbb{R}_{>0}$ and $\bar{X}^{(\infty)}$. Recall $\hat{X}^{(T)}$ and $\hat{X}^{(\infty)}$ satisfy the floor constraint (4). We then obtain the following.

Theorem 1. *For any $\pi\in\mathcal{A}_T^K(x)$,*

$$\frac{1}{\gamma}\log E\left(X_T^{x,\pi}\right)^\gamma \le \frac{1}{\gamma}\log E\left(\hat{X}_T^{(T)}\right)^\gamma + \frac{x}{x-K_0},$$

and so

$$\overline{\lim_{T\to\infty}}\sup_{\pi\in\mathcal{A}_T^K}\frac{1}{\gamma T}\log E\left(X_T^{x,\pi}\right)^\gamma = \overline{\lim_{T\to\infty}}\frac{1}{\gamma T}\log E\left(\hat{X}_T^{(T)}\right)^\gamma.$$

Further, it holds that

$$\sup_{\pi\in\mathcal{A}^K}\overline{\lim_{T\to\infty}}\frac{1}{\gamma T}\log E\left(X_T^{x,\pi}\right)^\gamma = \overline{\lim_{T\to\infty}}\frac{1}{\gamma T}\log E\left(\hat{X}_T^{(\infty)}\right)^\gamma.$$

Here, we note that the optimal solution for (2) with the constraint (4) is not uniquely determined. For example, considering a financial market model consisting of the bank-account process S^0 and the risky assets prices process $S := (S^1,\dots,S^n)^\top$, we see the following.

Theorem 2. *Assume K/S^0 is nonincreasing. Let $\bar{\pi}^{(\infty)} := (\bar{\pi}_t^{(\infty)})_{t\ge 0}\in\mathcal{A}$ be the optimal trading strategy for the problem (2) with the constraint (3) so that $\bar{X}^{(\infty)} = X^{1,\bar{\pi}^{(\infty)}}$. Define the process $\check{X} := (\check{X}_t)_{t\ge0}$ by the stochastic differential equation*

$$\begin{aligned} d\check{X}_t &= \left(\check{X}_t - K_t\right)\sum_{i=1}^n\left(\bar{\pi}_t^{(\infty)}\right)^i\frac{dS_t^i}{S_{t-}^i} + \left\{\check{X}_t - \left(\check{X}_t-K_t\right)\sum_{i=1}^n\left(\bar{\pi}_t^{(\infty)}\right)^i\right\}\frac{dS_t^0}{S_t^0}, \\ \check{X}_0 &= x. \end{aligned}$$

Then $\check{X}$ is optimal for the problem (2) with the constraint (4).

Remark 1. Optimal solutions $\bar{X}^{(\infty)}$ and $\check{X}$ for the RSPO with floor may be interpreted as generalizations of CPPI (Constant Proportion Portfolio Insurance), studied by Black, Jones, Perold, Sharpe (1987, 1988, 1992), etc.: e.g., $\bar{X}^{(\infty)}$ is the combination of the (minimal replication of the) floor, $\bar{K}$, and the "optimized cushion", $(x-\bar{K}_0)\bar{X}$.

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Portfolios and risk premia for the long run

Paolo Guasoni
(joint work with Scott Robertson)

This paper develops a method to derive optimal portfolios and risk premia explicitly in a general diffusion model, for an investor with power utility and in the limit of a long horizon. The market has several risky assets and is potentially incomplete. Investment opportunities are driven by, and partially correlated with, state variables which follow an autonomous diffusion. The framework nests models of stochastic interest rates, return predictability, stochastic volatility, and correlation risk.

In models with several assets and a single state variable, long-run portfolios and risk premia admit explicit formulas up to the solution of a linear ordinary differential equation, which characterizes the principal eigenfunction and the corresponding eigenvalue of an elliptic operator. Multiple state variables lead to a partial differential equation, which is solvable for many models of interest.

For different values of the relative risk aversion parameter, the paper derives the long-run portfolio, its implied risk premia and pricing measure, and their performance on a finite horizon.

1. Model

Consider a financial market with a risk-free asset S^0 and n risky assets $S = (S^1, \dots, S^n)$. Investment opportunities (i.e. interest rates, expected returns and covariances) depend on k state variables $Y = (Y^1, \dots, Y^k)$, which capture the effect of economic fundamentals:

$$\begin{aligned} \frac{dS_t^0}{S_t^0} =& r(Y_t)dt, \\ \frac{dS_t^i}{S_t^i} =& r(Y_t)dt + dR_t^i, && 1 \leq i \leq n. \end{aligned}$$

Cumulative excess returns $R = (R^1, \dots, R^n)$ and state variables follow the diffusion

$$\begin{aligned} dR_t^i =& \mu_i(Y_t)dt + \sum_{j=1}^{n} \sigma_{ij}(Y_t)dZ_t^j, && 1 \leq i \leq n, \\ dY_t^i =& b_i(Y_t)dt + \sum_{j=1}^{k} a_{ij}(Y_t)dW_t^j, && 1 \leq i \leq k, \\ d\langle Z^i, W^j\rangle_t =& \rho_{ij}(Y_t)dt, && 1 \leq i \leq n, 1 \leq j \leq k, \end{aligned}$$

where $Z = (Z^1, \dots, Z^n)$ and $W = (W^1, \dots, W^k)$ are multivariate Brownian motions. This model is well-defined if the corresponding martingale problem identifies a unique probability measure P on the canonical space $\Omega = C([0, \infty), \mathbb{R}^{n+k})$ of continuous paths, endowed with its Borel σ-algebra. The market is in general incomplete, and the symmetric, positive definite matrix $\Upsilon'\Sigma^{-1}\Upsilon$ represents the covariance of hedgeable shocks. It also gauges the degree of incompleteness of the market, highlighting two extremes: complete markets ($\Upsilon'\Sigma^{-1}\Upsilon = A$) and fully incomplete markets ($\Upsilon'\Sigma^{-1}\Upsilon = 0$).

An investor maximizes expected power utility $U(x) = \frac{x^p}{p}$ from terminal wealth, by trading in the market according to a portfolio strategy $(\pi_t^i)_{t\geq 0}^{1\leq i\leq n}$, which represents the proportions of wealth in each risky asset. The investor observes economic variables Y and asset prices S, therefore π is adapted to the filtration generated by (R, Y).

2. Main Result

In the limit of a long horizon, optimal portfolios and risk premia are governed by the quasilinear partial differential equation (where $q = p/(p-1)$)

$$pr - \frac{1}{2}q\mu'\Sigma^{-1}\mu + \frac{1}{2}\nabla v' \left(A - q\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla v + \nabla v'\left(b - q\Upsilon'\Sigma^{-1}\mu\right) + \frac{1}{2}\operatorname{tr}\left(AD^2v\right) = \lambda.$$

Assume that this equation admits a solution $v \in C^2(E, \mathbb{R})$ for some $\lambda \in \mathbb{R}$, and that the auxiliary model

$$\begin{cases} dR_t = \frac{1}{1-p}\left(\mu + \Upsilon \nabla v\right) + \sigma d\hat{Z}_t \\ dY_t = \left(b - q\Upsilon'\Sigma^{-1}\mu + (A - q\Upsilon'\Sigma^{-1}\Upsilon)\nabla v\right) dt + a d\hat{W}_t \end{cases}$$

is well-defined for some probability $\hat{P}$ equivalent to P. The paper provides sufficient conditions which guarantee that the optimal long-run portfolio is

$$\hat{\pi} = \frac{1}{1-p}\Sigma^{-1}\left(\mu + \Upsilon \nabla v\right),$$

while the q-optimal long-run martingale measure follows the dynamics

$$\begin{cases} dR_t = \sigma d\tilde{Z}_t \\ dY_t = \left(b - \Upsilon'\Sigma^{-1}\mu + \left(A - \Upsilon'\Sigma^{-1}\Upsilon\right)\nabla v\right) dt + a d\tilde{W}_t. \end{cases}$$

In the case of a single state variable, the main differential equation becomes a linear ODE through the change of variable $\phi = \exp(\frac{v}{\delta})$. Then the problem reduces to the search of a principal eigenfunction.

The paper also studies the finite-horizon performance of long-run optima, by providing upper bounds for the welfare loss in terms of a certainty equivalent rate.

Stochastic partial differential equation and portfolio choice

Thaleia Zariphopoulou
(joint work with Marek Musiela)

We propose new ways of measuring the performance of investment strategies under uncertainty. Traditionally, how well the investor does is assessed through expected utility criteria, typically formulated via a deterministic, concave and increasing function of terminal wealth. A key element of this approach is the a priori choice of both the horizon and the associated risk preferences. The optimal solution (value function) has been widely studied under rather general modeling assumptions. Its fundamental properties, consequences of the dynamic programming principle, are the supermartingality for arbitrary investment strategies and martingality at an optimum. The value function, then, serves as the intermediate (indirect) utility in the relevant market environment.

An alternative approach is proposed which offers flexibility with regards to the aforementioned a priori choices while preserving the natural optimality properties of the value function process (martingality at an optimum and supermartingality away from it). In contrast to the existing framework, the utility is specified for today and not for a (possibly remote) future time. The performance measurement criterion is defined in terms of a stochastic process, called *forward performance process,* defined on $[0, \infty)$ and indexed by a wealth argument.

Several difficulties are encountered due to the fact that the associated stochastic optimization problems are posed "inversely in time" and, thus, existing techniques in portfolio choice might not be directly applicable. In a model of k stocks driven by a d-dimensional Brownian motion, it turns out that the forward performance process solves the stochastic partial differential equation

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda + a\right)\right|^{2}}{\mathcal{A}^{2}U}dt + a \cdot dW,$$

with $U\left(x,0\right) = u_0\left(x\right)$ increasing and concave. The coefficients σ and λ represent, respectively, the volatility matrix and a market price of risk. The volatility process a is $(\mathcal{F}_t)$-adapted and d-dimensional, and the operator $\mathcal{A}$ stands for the spatial partial derivative, $\mathcal{A} = \frac{\partial}{\partial x}$.

The novel element in the above formulation is the volatility process. We note that in the traditional maximal expected utility problems, the process a is uniquely determined. However, in the forward context, the choice of the volatility needs to be judicious.

A special class of forward processes are the ones corresponding to the zero volatility case, $a\left(x,t\right) = 0$, $t > 0$. Then, the problem simplifies considerably. The forward solution is given by

$$U\left(x,t\right) = u\left(x, \int_0^t \left|\sigma\sigma^{+}\lambda\right|^2 ds\right), \quad t > 0,$$

with $u : \mathbb{R} \times [0, +\infty)$ solving the fully non-linear equation $u_t u_{xx} = \frac{1}{2}u_{xx}^2$ and $u(x,0) = u_0(0)$.

When $a = U\phi$, $\phi \in \mathcal{F}_t$ $\left(\phi \in \mathcal{R}^d\right)$, the solution is given by

$$U\left(x,t\right) = u\left(x, \int_0^t \left|\sigma\sigma^{+}\left(\lambda + \phi\right)\right|^2 ds\right) Z,$$

with Z solving $dZ = Z\phi \cdot dW$, $Z_0 = 1$. If the volatility is chosen by $a = -xU_x\delta$, $\delta \in \mathcal{F}_t$ $\left(\delta \in \mathcal{R}^d\right)$, the solution is

$$U = u\left(\frac{x}{Y}, \int_0^t \left|\sigma\sigma^{+}\lambda - \delta\right|^2 ds\right),$$

where Y solves $dY = Y\delta\left(\lambda dt + dW\right)$ with $Y_0 = 1$.

The forward process corresponding to the more general volatility choice $a = -xU_x\delta + \phi U$ is given by

$$U\left(x,t\right) = u\left(\frac{x}{Y}, \int_0^t \left|\sigma\sigma^{+}\left(\lambda + \phi\right)\right|^2 ds\right) Z,$$

with the processes Y and Z as above. One might interpret Y as a benchmark (or numeraire) while Z as a process that gives flexibility in terms of the investor's views of upcoming market movements.

The optimal policies for the above family of solutions can be explicitly calculated, in a stochastic feedback form, via the risk tolerance process which is functionally related to a fast diffusion equation. The optimal wealth process is also explicitly given in terms of a space-time harmonic function evaluated at initial wealth and market-specific processes. This explicit representation enables us to obtain various distributional characteristics of the optimal wealth and portfolios.

Optimal portfolio liquidation: market impact models and optimal control

Alexander Schied
(joint work with Torsten Schöneborn)

Problem description. A common problem for stock traders is to unwind large block orders of shares. These can comprise a major part of the daily traded volume of shares and create substantial impact on the asset price. The overall costs of such a liquidation can be significantly reduced by splitting the order into smaller pieces that are placed during a certain time period. In our work, we approach the corresponding optimization problem by maximizing the expected utility of the revenues from selling a position of $x > 0$ shares until time $T \in [0, \infty]$. The investor thus chooses a liquidation strategy that we describe by the number X_t of shares held at time t and that satisfies the boundary condition $X_0 = x$ and $X_T = 0$. We assume that $t \mapsto X_t$ is bounded, predictable, and absolutely continuous with derivative $\dot{X}_t$. By $\mathcal{X}(x, T)$ we denote the class of all these strategies X. We consider one of the standard models for dealing with the price impact of such a liquidation strategy, namely the continuous-time model introduced by Almgren [3]. It is also the basis for optimal execution algorithms that are widely used in practice. In this model, the price process is of the form

$$P_t = P_0 + \sigma B_t + bt + \gamma(X_t - X_0) + h(\dot{X}_t)$$

when the strategy X is used. Here, $P_0, \gamma > 0$, and $b, \sigma \in \mathbb{R}$ are constants, B is a standard Brownian motion starting at $B_0 = 0$, and h is a function such that $f(x) := xh(x)$ has superlinear growth and is positive, strictly convex, and continuously differentiable. The revenues from using the strategy $X \in \mathcal{X}(x, T)$ are given by

$$(1)\quad R_T(X) = \int_0^T (-\dot{X}_t) P_t \, dt = R_0 + \sigma \int_0^T X_t \, dB_t + b \int_0^T X_t \, dt + \int_0^T f(\dot{X}_t) \, dt,$$

where $R_0 := P_0 x - \frac{\gamma}{2} x^2$. In the sequel, $u : \mathbb{R} \to \mathbb{R}$ will be a strictly concave and increasing utility function. The problem of the investor can now be formulated as

$$(2)\qquad \text{maximize the expected utility } \mathbb{E}[\, u(R_T(X)) \,] \text{ over } X \in \mathcal{X}(x, T).$$

Finite time horizon and CARA utility. Liquidation problems in practice often need to be completed within several days or even hours. Assuming $T < \infty$ is therefore a natural constraint. When setting up (2) as a stochastic control problem with control $X \in \mathcal{X}(x, T)$ and controlled diffusion process

$$R_t(X) = R_0 + \sigma \int_0^t X_s \, dB_s + b \int_0^t X_s \, ds + \int_0^t f(\dot{X}_s) \, ds, \qquad 0 \le t \le T,$$

we face the difficulty that the class $\mathcal{X}(x, T)$ of admissible controls depends on both x and T. Control problems of this type are known as *finite-fuel control problems*, as Ioannis Karatzas kindly pointed out during the workshop in Oberwolfach.

Heuristic arguments suggest that the value function

$$v(T, X_0, R_0) := \sup_{X \in \mathcal{X}(x,T)} \mathbb{E}[\, u(R_T(X))\,]$$

should satisfy the Hamilton-Jacobi-Bellman (HJB) equation

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_{\xi} \big(\xi v_X + v_R f(\xi)\big) \tag{3}$$

with singular initial condition

$$\lim_{T \downarrow 0} v(T, X, R) = \begin{cases} u(R) & \text{if } X = 0, \\ -\infty & \text{otherwise.} \end{cases} \tag{4}$$

The following result is proved in [6]:

Theorem 1. For $u(x) = -e^{-\alpha x}$ there exists a unique optimal strategy X^*, which is a deterministic function of time. Moreover, the value function is a classical solution of the Cauchy problem (3), (4).

If X is deterministic, then $R_T(X)$ is normally distributed, and we obtain

$$\mathbb{E}[\, u(R_T(X))\,] = -\mathbb{E}[\, e^{-\alpha R_T(X)}\,] = -\exp\Big(-\alpha \mathbb{E}[\, R_T(X)\,] + \frac{\alpha^2}{2} \mathrm{var}\,(R_T(X))\Big).$$

Finding the optimal liquidation strategy is thus reduced to the problem of finding the deterministic strategy that maximizes a mean-variance functional. The maximization of the mean-variance functional over deterministic strategies is in turn equivalent to the minimization of the action functional $\int_0^T L(X_t, \dot{X}_t)\, dt$ where the Lagrangian is given by $L(q,p) = \frac{1}{2}\alpha\sigma^2 q^2 + bq + f(p)$. This is a classical problem and can be solved by standard calculus of variations (at least if f satisfies some additional regularity conditions).

Infinite time horizon and general utility functions. The results stated in this section are taken from and proved in [7]. We need to make some simplifying assumptions to solve the problem for general utility functions. The main simplifying assumption is $T = \infty$. It implies that v is only a function of X and R. Hence the time derivative in (3) vanishes, so that v_X can take over this role. Moreover, the singularity in (4) is no longer present. Assuming $b = 0$ will then automatically guarantee liquidation, because a risk-averse investor will try to be neither long nor short in the risky asset, due to the martingale dynamics of its price. We also assume linear temporary impact, $h(x) = \lambda x$ for some $\lambda > 0$, but this assumption is not essential and can be relaxed. The HJB equation then becomes

$$\frac{\sigma^2}{2} X^2 v_{RR} = \inf_c \big(c v_X + \lambda v_R c^2\big), \qquad v(0, R) = u(R), \tag{5}$$

or, in reduced form, $v_X^2 = -2\lambda\sigma^2 X^2 v_R v_{RR}$.

Theorem 2. Assume that $u \in C^6(\mathbb{R})$ with absolute risk aversion

$$A(R) := -\frac{u''(R)}{u'(R)} \in [A_{\min}, A_{\max}], \qquad R \in \mathbb{R},$$

where $0 < A_{\min} \leq A_{\max} < \infty$. Then the value function is a classical solution of (5). Moreover, the a.s. unique optimal control $(\hat{\xi}_t)$ is Markovian and given in feedback form by $\hat{\xi}_t = c(X_t^{\hat{\xi}}, R_t^{\hat{\xi}})$ where c is the minimizer in (5).

The optimal control c can also be described in terms of another PDE and without any reference to (5): The transformed optimal control $\tilde{c}(Y,R) := c(\sqrt{Y}, R)/\sqrt{Y}$ is the unique classical solution of the Cauchy problem

$$\tilde{c}_Y = -\frac{3}{2}\lambda \tilde{c}\tilde{c}_R + \frac{\sigma^2}{4\tilde{c}}\tilde{c}_{RR}, \qquad \tilde{c}(0,R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \tag{6}$$

in the class of functions with values in $[A_{\min}, A_{\max}]$. This observation allows to conduct a sensitivity analysis of the dependence of the optimal strategy on the various model parameters. For instance, one can show that $c(X,R)$ is increasing in R if $A(R)$ is increasing (i.e., u is of IARA type). In this case, the corresponding strategy is *aggressive in the money.* Conversely, if $A(R)$ is decreasing (i.e., u is of DARA type), then $c(X,r)$ is decreasing, hence *passive in the money.*

We also study the monotonicity with respect to the other model parameters. Surprisingly, it turns out that the optimal strategy can sometimes be *decreasing* as a function of the amount X of shares to be liquidated. For DARA utility functions it can also be both decreasing and increasing in the temporary impact parameter λ. If, however, the utility function u is IARA, then the optimal control c is decreasing in λ.

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On divergence utilities

Michael Kupper
(joint work with Alexander Cherny)

We investigate a class of concave monetary utility functions, which we call *divergence utilities*, and solve several optimization problems. These monetary utilities are truly concave, i.e., not coherent. On the economical side, these are nothing but the *translation invariant hulls* of the expected utilities. In a sense, they are between coherent utilities and expected ones. Thus, the representation

$$u_G(X) = \inf_{Z \in \mathcal{P}} \mathbb{E}\left[ZX + G(Z)\right], \quad X \in L^0, \tag{1}$$

shows their relation to coherent utilities, while the representation

$$u_G(X) = \max_{c \in \mathbb{R}} (c - \mathbb{E}\left[F(c - X)\right])$$

shows their relation to expected utilities. Here $\mathcal{P}$ denotes the set of all probability densities and G is a mapping from $\mathbb{R}_+$ to $\mathbb{R}$ which satisfies some extra conditions such as differentiability. F is the convex conjugate of G. A typical example is the *entropic utility function* which corresponds to the case $G(X) = x \ln(x)$. The $AV@R$ can be viewed as a limiting case. Such representations are also studied by P. Cheridito and T. Li, A Ben-Tal and M. Tebouille as well as A. Schied. On the mathematical side, we observe that the class of divergence utilities possesses some nice properties like strict concavity (up to the cash direction, which guarantees the uniqueness of a solution for optimization problems), strict monotonicity, and second-order monotonicity (which is useful in finding the solution in some optimization problems).

We basically need these representations to give an explicit solution of the portfolio optimization problem as well as an explicit solution of the risk sharing problem and provide a utility contribution formula.

We study the *portfolio optimization problem*

$$u_G(W + X) \longrightarrow \max, \quad X \in \mathcal{A}.$$

W denotes the initial endowment (which may be random). The set $\mathcal{A}$ (representing a financial market) consists of all (discounted) terminal cash flows with zero-cost trading opportunities. In fact, this problem possesses two economic interpretations: first, as the optimization problem for an agent possessing the wealth W and employing the monetary utility u_G; second, as the problem of superhedging a contingent claim $-W$ through the no-good-deals pricing technique associated with the convex risk measure $\rho_G(X) = -u_G(X)$. For example, W could be an insurance portfolio. The insurance company tries to minimize their risk (modeled through $\rho_G = -u_G$) by investing in a financial market. The optimal hedging portfolio is given as the solution of the above optimization problem.

The solution of the portfolio optimization problem is based on a duality approach.

We further study the *risk sharing problem* based on divergence utilities: Suppose that we have N agents in the economy, and the n-th agent assesses the risk by the risk measure $\rho_{G_n}(X) = -u_{G_n}(X)$. Let X be the total endowment of the agents. The problem of risk sharing is

$$\sum_{n=1}^{N} u_{G_n}(X_n) \longrightarrow \max, \quad \sum_{n=1}^{N} X_n = X. \tag{2}$$

The sum $\sum_n u_{G_n}(X_n)$ is understood as $-\infty$ if any of the summands equals $-\infty$. Under some extra integrability condition on the total endowment we present an explicit formula for the optimal risk sharing.

Contract theory in continuous time

Jakša Cvitanić
(joint work with Xuhu Wan, Jianfeng Zhang)

Company executives are often given options which they are free to exercise at any time during a given time period. The possibility of exercising early (being paid early) is definitely beneficial for executives, but is it beneficial for the company? We develop a general contract theory with random time of payment, which enables us to address questions like this in standard, stylized continuous-time principal-agent models, in which the agent can influence the drift of the process by her unobservable effort, while suffering a certain cost. The agent is paid only once, at a random time, which is not quite the case in the executive compensation example, but it is in the same spirit.

In our general model, the timing of the payment depends crucially on the "outside options" of the agent and of the principal. By outside options we mean the benefits and the costs the agent and the principal will be exposed to after the payment has occurred. In our general framework, we model these as stochastic processes which are flexible enough to include a possibility of the agent leaving the company, maybe being replaced by another agent and maybe not, or the agent staying with the company and applying substandard effort, or the agent being retired with a severance package or regular annuity payments, or any other modeling of the events taking place after the payment time. In addition, when we add adverse selection (unknown agent's type) to the model, we also allow for the possibility that the agent increases the earnings either by manipulation or by skill, or both.

The paper that started the continuous-time principal-agent literature is Holmström and Milgrom (1987). That paper considers a model with moral hazard, lump-sum payment at the end of the time horizon, and exponential utilities. Because of the latter, the optimal contract is linear. Their framework was extended by Schättler and Sung (1993, 1997), Sung (1995, 1997), Detemple, Govindaraj and Loewenstein (2001). See also Dybvig, Farnsworth and Carpenter (2001), Hugonnier and Kaniel (2001), Müller (1998, 2000), and Hellwig and Schmidt (2003). The papers by Williams (2004) and Cvitanić, Wan and Zhang (2005) (henceforth CWZ 2005) use the stochastic maximum principle and forward-backward stochastic differential equations to characterize the optimal compensation for more general utility functions under moral hazard. Cvitanić and Zhang (2007) (henceforth CZ 2007) consider adverse selection in the special case of separable and quadratic cost function on the agent's action. Another paper with adverse selection in continuous time is Sung (2005), in the special case of exponential utility functions and with only the initial and the final value of the output being observable. A continuous-time paper which considers an optimal random time of retiring the agent is Sannikov (2007). Moreover, He (2007) has extended Sannikov's work to the case of the agent controlling the size of the company.

The present paper extends CWZ (2005) and CZ (2007) to the possibility of the contract payoff being paid at a random time, which we call payment time, exercise time, or stopping time. If we do not restrict the set of allowable contract payoffs, the principal can "force" the agent to exercise at a time of the principal's choosing by an appropriate payoff design. We show that this design can be accomplished in a natural way and often leads to simple looking contracts in which the agent is paid a low contract value unless she waits until the output hits a certain level.

We discuss now the main contributions and results of our paper. First, we find general necessary conditions for the hidden action case, with arbitrary utility functions for the principal and the agent and a separable cost function for the agent. As usual in dynamic stochastic control problems of this type, the solution to the agent's problem depends on her "value function", that is, on her remaining expected utility process (what Sannikov 2007 calls "promised value"). However, this process is no longer a solution to a standard backward stochastic differential equation (BSDE), but a reflected BSDE, because of the optimal stopping component. The solution to the principal's problem depends, in general, not only on his and the agent's remaining expected utilities, but also on the remaining expected ratio of marginal utilities (which is constant in the first-best case, with no moral hazard).

We describe more precisely how to find the optimal solution, including the optimal stopping time, in a variation on the classical Holmström-Milgrom (1987) set-up, with exponential utilities and quadratic cost. It turns out that under a wide range of "stationarity conditions", it is either optimal to have the agent be paid right away (to be interpreted as the end of the vesting period), or not be paid early, but wait until the end. In other words, it is often not optimal for the principal that the agent be given an option to exercise the payment at a random time. For example, if the risk aversions are small and the "total output process", which is the sum of the output plus the certainty equivalents of the outside options, is a submartingale (has positive drift), then it is optimal not to have early payment. If the agent is risk-neutral, in analogy with the classical models, the principal "sells the whole firm" to the agent, in exchange for a possibly random payment at the optimal stopping time in the future. Moreover, the agent would choose the same optimal payment time as the principal, even if she was not forced to do so.

We are able to provide semi-explicit results also for non-exponential utilities, assuming that the cost function of the agent is quadratic and separable. The generality of utilities in which we work is possible because with the quadratic cost function the agent's optimal utility and the principal's problem can both be represented in a simple form which involves explicitly the contracted payoff only, and not the agent's effort process. The ratio of the marginal utilities of the principal and the agent depends now also on the principal's utility. The optimal payoff depends in a nonlinear way on the value of output at the time of payment, and the optimal payment time is determined as a solution to an optimal stopping problem of a standard type. In an example with a risk-neutral principal and a log agent, the optimal exercise time is much more complex than in the exponential utilities case. It is the time when the maximum is reached by a certain nonlinear function of the value of output plus the value of the principal's outside option. The function itself depends on the parameters driving not only the output and the principal's outside option processes, but also the agent's outside option process.

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Convex risk measures beyond L^∞ (the canonical model space for law-invariant convex risk measures is L^1)

Damir Filipović
(joint work with Gregor Svindland)

Convex risk measures are best known on L^∞. Indeed, Artzner et al. [1] introduced the seminal axioms of coherence, which then were further generalized to the convex case by Föllmer and Schied [5] and Frittelli and Rosazza-Gianin [6], on L^∞. However, there is a growing mathematical finance literature dealing with the extension of convex risk measures beyond L^∞, see e.g. [2, 3, 7, 8]. This extension is vital since important risk models, such as normally distributed random variables, are not contained in L^∞.

We first discuss some topological properties of convex risk measures on L^p and derive a dichotomy: A closed convex risk measure ρ on L^p is either finite and continuous on L^p or its domain has empty interior. We then discuss several attempts to extend a given convex risk measure on L^∞ to L^p. It turns out that such extensions – if they exist – are not unique in general. We define the L^p-closure of a given function f on L^∞ as the greatest closed convex function on L^p majorized by f on L^∞. If a closed convex extension of f to L^p exists, then the L^p-closure of f coincides with f on L^∞ and is the greatest closed convex extension of f to L^p. As an example, we show that the L^∞-closure of value at risk (VaR) equals minus infinity. Hence there exists no finite closed convex function majorized by VaR.

In the last part of the talk, we show that there is a one-to-one relation between law-invariant closed convex functions on L^∞ and L^1. In particular, every law-invariant closed convex function f on L^∞ is the restriction of a unique law-invariant closed convex function on L^1 (which coincides with the L^1-closure of f). We conclude that the canonical model space for law-invariant convex risk measures is L^1.

This talk is based on a joint paper with Gregor Svindland [4].

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On the ordering of option prices

Ludger Rüschendorf

The aim of this paper is to unify and extend some of the various results on the ordering of option prices in exponential semimartingale models. This is a problem of interest not only for financial applications, but similar questions also arise in the ordering properties of Markovian networks or in the problem to develop consistency results for risk measures for portfolio vectors with respect to dependence orders. We consider these questions in the framework of multivariate semimartingale models.

The following comparison result for two homogeneous Markov processes was recently established. Let X, Y be Feller processes with values in a LCCB space E, typically $E = \mathbb{R}^k$ with transition operators $S = (S_t)_{t\geq 0}$, $T = (T_t)_{t\geq 0}$ on $C_0(E)$ and infinitesimal generators A, B with domains D_A, D_B. Assume that $\mathcal{F} \subset C_0(E)$, $\mathcal{F} \subset D_A \cap D_B$. $\mathcal{F}$ generates a partial order $\leq_{\mathcal{F}}$ on $M^1(E, \mathfrak{B})$ defining $P \leq_{\mathcal{F}} Q$ if $\int f\, dP \leq \int f\, dQ$ for all $f \in \mathcal{F}$.

Theorem 1 (Comparison result for Markov processes, Rüschendorf (2007))**.** Assume that the processes X, Y satisfy the following two conditions:

(1) Y is 'stochastically monotone', i.e., $f \in \mathcal{F}$ implies that $S_t f \in \mathcal{F}$.
(2) For $f \in \mathcal{F}$ holds $Af \geq Bf$ $[P^{X_0}]$.

Then $T_t f \geq S_t f$ $[P^{X_0}]$, $\quad \forall t \geq 0$.

The proof of Theorem 1 uses a similar idea as in the classical result of Liggett (1985) on the association of stochastic processes. By a coupling argument it can be seen that Theorem 1 implies as particular case the Liggett result. It can be used to establish dependence ordering results but also to establish convex ordering results of interest for risk measures.

For the case of Lévy processes, the stochastic monotonicity condition is obviously satisfied for several natural orders like convex, directionally convex, supermodular or increasing stochastic order. The second condition on the comparison of the infinitesimal generators is also necessary (and thus is necessary and sufficient) in this case.

Via a stochastic analysis approach based on Itô's formula and a general version of Kolmogorov's backward PDE, Theorem 1 has further been generalized to the comparison of general multivariate semimartingale models with some Markovian semimartingale models in some recent joint work with J. Bergenthum (2006–2007). This extends previous results in El Karoui, Jeanblanc-Picqué and Shreve (1998) and Bellamy and Jeanblanc (2000) for diffusion and stochastic volatility models and of Gushchin and Mordecki (2002) for one-dimensional semimartingales.

Suppose that S, S^* are two continuous-time semimartingales with differential local characteristics $(b, c, K), (b^*, c^*, K^*)$, where S^* is Markovian. We denote by $\mathcal{G}(t, s) = E^*(g(S^*_T) \mid S^*_t = s)$ the propagation operator (value process) in the Markovian model. The basic role in the stochastic analysis approach to comparison theorems is played by the linking process $\mathcal{G}(t, S_t)$. This process forms a link between the value processes in the S and in the S^* models since

$$\mathcal{G}(0, s) = E^*(\mathcal{G}(S^*_T) \mid S^*_0 = s) = E^* g(S^*_T) \quad \text{if } S^*_0 = s$$
$$\text{and } \mathcal{G}(T, S_T) = g(S_T).$$

As a consequence, the essential step in order to obtain a comparison result is to establish that the linking process is a sub- (resp. a super-)martingale. This property implies that

$$Eg(S_T) = E\mathcal{G}(T, S_T) \geq E\mathcal{G}(0, S_0) = E^* g(S^*_T)$$

assuming that $S_0 = s$. This sub- (super-)martingale property is derived under conditions similar to (1), (2) in Theorem 1 in the papers mentioned above, together with several applications to Lévy processes, stochastic volatility models and others.

In particular we develop in these papers some new methods which allow to establish in some examples the basic 'propagation of convexity property' respectively in the general case the 'propagation of ordering' property. The proof of these results makes essential use of a reduction to discrete time by the Euler approximation scheme and then using results on optimal couplings for the discrete-time Markov operators. This stochastic analysis approach has also been extended to derive comparison results for path-dependent options like for instance lookback, Asian, American, and barrier options (see Bergenthum and Rüschendorf (2007)).

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First passage problems for jump diffusions

Tom Hurd
(joint work with Alexey Kuznetsov and Zhuowei Zhou)

Classic problems such as barrier options, perpetual American options and structural credit models like the Black-Cox model [2] boil down to computation of the first passage time t^* for an underlying process. When this process is a jump diffusion or Lévy process, computing first passage is in general difficult and little can be done analytically. In this talk I focussed on processes which are expressible as Brownian motion W subjected to time change by an independent increasing process G [3]. Important examples abound, such as the variance gamma [5] and normal inverse Gaussian models [1].

For such processes one can define an alternative notion of first passage that I call "first passage of the second kind" $\tilde{t}$. In words, $\tilde{t}$ is the first time that the time change G exceeds the first passage time t^*_W of the Brownian motion W. It agrees with t^* if G is continuous, but in general $\tilde{t} \leq t^*$. In some contexts, for example, structural credit models, $\tilde{t}$ is arguably as natural as t^*. The main advantage compared to t^* is that the probabilities of $\tilde{t}$ can computed by iterated expectations in terms of the density of G via

$$P\left[\tilde{t} > t\right] = E\big[P[G_t < t^*_W | G_t = g]\big].$$

Since the distributions of t^*_W are well known, this is generally easier than computing t^*. This relative ease of computation was illustrated by numerical results on barrier option pricing in a variance gamma stock model and bond pricing in a variance gamma structural credit model.

Then I discussed the theoretical question of how $\tilde{t}$ relates to t^*. After some thought one realizes that there is an explicit relation between them. If one denotes by $t^*(x,\omega)$ the first passage time for $X := W \circ G$ to cross 0 for a sample path ω starting from a level $x > 0$, then one has $s^1(x,\omega) := \tilde{t}(x,\omega)$ as the first term of a sequence of stopping times s^n that satisfy an iteration

$$s^n(x,\omega) = s^1(x,\omega) + s^{n-1}(X_{s^1+},\omega')\mathbf{1}_{\{X_{s^1+}>0\}}, \quad n \geq 2.$$

Here ω' is the sample path ω shifted backwards in time by the amount $s^1 := s^1(x,\omega)$. I then proved that $\lim_{n\to\infty} s^n = s^*$ almost surely (in fact the sequence becomes constant in a finite number of steps almost surely). Similar logic implies that the joint probability distribution functions $p^*(x,y,t)$ and $\tilde{p}(x,y,t)$ of (X_{t^*+}, t^*) and $\left(X_{\tilde{t}+}, \tilde{t}\right)$ defined for each initial condition $X_0 = x$ satisfy an equation

$$p^*(x,y,t) = \tilde{p}(x,y,t)\mathbf{1}\{y \leq 0\} + \int_0^\infty ds \int_0^\infty dz\ \tilde{p}(x,z,s)\ p^*(z,y,t-s)$$

that can be solved by a convergent iteration. In other words, $\tilde{t}$ is the first term in a "geometric" sequence converging to t^*. It is also computable. For example, if G is a Lévy subordinator with log-characteristic function $\Psi(u) := \frac{1}{t}\log E[e^{iuG_t}]$ and W is Brownian motion with drift β, the joint PDF $\tilde{p}(x,y,t)$ of $\left(X_{\tilde{t}+}, \tilde{t}\right)$ is given by

$$\frac{e^{\beta(y-x)}}{(2\pi)^2 i}\iint_{\mathbb{R}^2} dz_1 dz_2 \left(\frac{\Psi(z_2)-\Psi(z_1)}{z_1-z_2}\right)\left(\frac{e^{-|y|\sqrt{\beta^2+2iz_2}-x\sqrt{\beta^2+2iz_1}+s\Psi(z_1)}}{\sqrt{\beta^2+2iz_2}}\right).$$

Some aspects of first passage problems for time-changed Brownian motion can be found in [4]. Other results are work in progress.

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Indifference pricing for power utilities

Semyon Malamud

(joint work with Eugene Trubowitz, Mario V. Wüthrich)

Almost all existing work on utility indifference pricing is done in the framework of an exponential utility of terminal wealth. See, e.g., [2], [4], [5]. The main problem with the exponential utility, already mentioned by Rouge and El Karoui [4], is that the indifference price is independent of the initial wealth of the seller/buyer.

For non-exponential utilities, there are only approximate results, based on Taylor polynomial approximations. The most general result in this direction is due to Kramkov and Sîrbu [1]. They obtained a second order Taylor expansion of the indifference prices for an arbitrary utility function when the *claim size is small.*

We introduce a new technique that allows us to analyze indifference prices with intertemporal consumption (not only final wealth), arbitrary utilities, and in the case when the *claim size is large.* We work exclusively in discrete time. Our method is based on an explicit, recursive procedure for constructing optimal consumption streams for a large class of incomplete markets. This class is very easy to describe. In addition to the underlying filtration $\mathcal{G} = (\mathcal{G}_t)$, there is another, additional filtration $\mathcal{H} = (\mathcal{H}_t)$, referred to as the hedgeable filtration. It has the following properties:

(1) $\mathcal{G}_{t-1} \subset \mathcal{H}_t \subset \mathcal{G}_t$;

(2) all asset prices are adapted to $\mathcal{H}$;

(3) each $\mathcal{H}_{t+1}$-measurable payoff X can be replicated by a $\mathcal{G}_t$-measurable investment at time t.

It is not difficult to see that (1) any diffusion driven, incomplete market model can be discretized to fit into this class; (2) there exists a unique state price density process $\mathbf{M} = (M_t)$ adapted to $\mathcal{H}$. This is the state price density process corresponding to the minimal martingale measure. Then, the first order conditions for an agent maximizing

$$E\left[\sum_{t=0}^{T} e^{-\rho t} u(c_t)\right]$$

take the form

$$E[e^{-\rho t} u'(c_t) M_t^{-1} | \mathcal{H}_t] = e^{-\rho(t-1)} u'(c_{t-1}) M_{t-1}^{-1},$$

that is, $\left(e^{-\rho t} u'(c_t) M_t^{-1}\right)$ is a martingale with respect to $(\mathcal{H}_{t+1})$. This special structure of first order conditions allows us to construct the optimal consumption stream by an explicit, recursive procedure. A special case of this recursive structure has been discovered by Musiela and Zariphopoulou [3].

Let now the agent be endowed with a random stream $(w_0, \mathbf{w})$ of income (or liabilities). Let $\mathbf{c} = (c_t)_{t \geq 1}$ be the optimal consumption stream. The recursive structure allows us to write it as an explicit function of c_0 and the endowment $\mathbf{w}$. Thus, $\mathbf{c} = \mathbf{c}(c_0, \mathbf{w})$ and $c_0 = c_0(w_0, \mathbf{w})$. This map has many remarkable properties. Let

$$\mathbf{P_c} = \frac{\partial \mathbf{c}}{\partial \mathbf{w}}$$

be the derivative (Jacobian) of the map with respect to the endowment. Then,

$$\mathbf{P}_\mathbf{c}^2 = \mathbf{P_c}$$

for any $\mathbf{w}$. That is, this **derivative is always a projection!** Define a scaled inner product

$$\langle \mathbf{Z}, \mathbf{Y} \rangle_c = \sum_{t=1}^{T} e^{-\rho t} E\big[(-u''(c_t)) Z_t Y_t\big] \qquad \text{and} \qquad \|Z\|_c^2 = \langle Z, Z \rangle_c.$$

Then, $\mathbf{P_c}$ is orthogonal in this inner product. This orthogonal projection property is crucial for the indifference pricing.

From now on we will discuss exclusively the CRRA utilities given by $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$. Most results can be extended to general utilities. Suppose that an agent with initial wealth W sells insurance against a stream of claims $\mathbf{Y} = (Y_t)_{t \geq 1}$ for the indifference price

$$\pi = \pi(W, \mathbf{Y})$$

so that

$$(1)\quad c_0(W,0)^{1-\gamma} + E\left[\sum_{t=1}^{T} e^{-\rho t} c_t^{1-\gamma}(c_0,0)\right] = c_0(W+\pi,-\mathbf{Y})^{1-\gamma} + E\left[\sum_{t=1}^{T} e^{-\rho t} c_t^{1-\gamma}(c_0,-\mathbf{Y})\right].$$

Using the projection property of $\mathbf{P_c}$, we obtain remarkable algebraic identities and prove

Theorem Let $\mathfrak{d} = (e^{-\rho t})_{t\geq 1}$. We have

$$(2)\qquad \frac{\partial \pi_0}{\partial \mathbf{Y}} = \frac{\partial w_0(\mathbf{w})}{\partial \mathbf{w}} = \mathfrak{d}(\mathbf{c}/c_0)^{-\gamma},$$

and

$$(3)\qquad \frac{\partial^2 \pi_0}{\partial \mathbf{Y}^2}(\mathbf{y},\mathbf{y}) = \gamma c_0^{\gamma} \left\langle \mathbf{P_c}(\mathbf{y}), \mathbf{y}\right\rangle_c + \gamma c_0^{\gamma} \frac{\left\langle \mathbf{P_c}(\mathbf{c}), \mathbf{y}\right\rangle_c^2}{c_0^{1-\gamma} + \|(I-\mathbf{P_c})\mathbf{c}\|_c^2}.$$

It is easy to see that $\pi(\lambda W, \lambda \mathbf{Y}) = \lambda \pi(W, \mathbf{Y})$. Thus, the case of large $\mathbf{Y}$ is equivalent to the case of small W. Let $\mathbf{Y}^u$ be the *upper hedging price* for the claims stream $\mathbf{Y}$. Since the utility is defined on the half line (no bankruptcy), it is easy to see that

$$\mathbf{Y}^u - W \leq \pi(W,\mathbf{Y}) \leq \mathbf{Y}^u.$$

Consequently,

$$\lim_{W\to 0} \pi(W,\mathbf{Y}) = \mathbf{Y}^u.$$

Interestingly enough, for small W, π is *only defined for* $\gamma \geq 1$. We want to understand the asymptotic behavior of π as $W \to 0$.

Theorem

$$\pi - \mathbf{Y}^u = \begin{cases} -W + B_1(\mathbf{Y})W^{\alpha(\mathbf{Y})} + o(W^{\alpha(\mathbf{Y})}), & \gamma = 1, \\ (-1 + A(\mathbf{Y}))W + B_2(\mathbf{Y})W^{\gamma} + O(W^2), & 1 < \gamma < 2, \\ (-1 + A(\mathbf{Y}))W + B_3(\mathbf{Y})W^2 + O(W^{\min\{3,\gamma\}}), & 2 \leq \gamma. \end{cases}$$

The coefficients A, B_1, B_2, B_3 and the power $\alpha > 1$ are computed explicitly. Furthermore, if $1 < \gamma \leq 2$, then

$$\begin{aligned} &\mathbf{Y}^u - W(1 - A(\mathbf{Y})) \leq \pi(W,\mathbf{Y}) \\ &\leq \mathbf{Y}^u - W\left(1 - A(\mathbf{Y})\left(1 - \left(c_0^{\infty}\right)^{\gamma-1} B(\mathbf{Y}) W^{\gamma-1}\right)^{\frac{1}{1-\gamma}}\right) \end{aligned}$$

always. This is a **sharp, non-perturbative** bound.

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Asymptotic analysis of multiscale intensity models for multiname credit derivatives

Ronnie Sircar

(joint work with Evan Papageorgiou)

1. Introduction

The pricing of collateralized debt obligations (CDOs) and other basket credit derivatives is contingent upon (i) a realistic modeling of the firms' default times and the correlation between them, and (ii) efficient computational methods for computing the portfolio loss distribution from the individual firms' default time distributions. *Factor models*, a widely-used class of pricing models, are computationally tractable despite the large dimension of the pricing problem, thus satisfying issue (ii), but to have any hope of calibrating CDO data, numerically intense versions of these models are required. We revisit the intensity-based modeling setup for basket credit derivatives and, with the aforementioned issues in mind, we propose improvements (a) via incorporating fast mean-reverting stochastic volatility in the default intensity processes, and (b) by considering homogeneous groups within the original set of firms. This can be thought of as a hybrid of top-down and bottom-up approaches. We refer to [1] for the details.

2. Homogeneous groups and multiscale stochastic volatility default intensities

Suppose we have a portfolio of n firms and we are interested in pricing the tranches of a CDO written on it. In practice, n is of the order of $50 - 600$. Let us define the *loss process* $L = (L_t)$, where L_t counts the number of defaulted firms up to time t. The pricing of the CDO tranches requires the law of the process L, or under weaker assumptions, the probability loss distribution $m \mapsto P\{L_t = m\}$ at a finite set of times.

2.1. **Factor models and name grouping.** In the usual context of intensity-based models for multi-name credit derivatives pricing introduced by [2] and later extended by [3], we assume that a firm's default is the first arrival from a *Cox process*. We propose dividing the firms into groups of similar credit quality. This reduces the dimensional burden, while preserving much of the heterogeneity of the portfolio constituents which is important to retain for CDO valuation. Specifically, we make the following assumptions:

(1) There are k homogeneous groups of firms ($k < n$), and each of the n firms belongs in one group only. We denote by n_i the number of firms in the i-th group, $i = 1, \dots, k$.
(2) Within each group, the firms share a common intensity of default; therefore given a default from the group, each (remaining) firm within that group is equally likely to be the defaulted one.
(3) Let $L^i = (L^i_t)_{t\geq 0}$ be the loss process corresponding to the firms of group i, $i = 1, \dots, k$, and obviously the total loss process $L = L^1 + \cdots + L^k$. For $i = 1, \dots, k$, we define $\lambda^i = (\lambda^i_t)_{t\geq 0}$ to denote the default intensity process shared by the firms of group i.
(4) The default intensity process shared by all firms of the homogeneous group i, $\lambda^i = (\lambda^i_t)_{t\geq 0}$ is given by
$$\lambda^i_t = X^i_t + c_i\, Z_t, \quad i = 1, \dots, k.$$
The *idiosyncratic* factors $X^1, \dots, X^k$ are independent from each other and independent of the *systematic* factor Z. The processes $X^1, \dots, X^k$ and Z are non-negative almost surely, and the parameters $c_1, \dots, c_k$ are positive constants.

2.2. **Stochastic volatility on the default intensity.** We model the processes X^i and Z via *square-root diffusions* and we introduce stochastic volatility in the idiosyncratic processes $X^1, \dots, X^k$ in the following manner (for fixed $i = 1, \dots, k$):
$$\begin{aligned} dX^i_t &= \alpha_i(\bar{x}_i - X^i_t)\, dt + f_i(Y^i_t)\sqrt{X^i_t}\, dW^i_t, \\ dY^i_t &= \frac{1}{\varepsilon} X^i_t(\bar{y}_i - Y^i_t)\, dt + \frac{\nu_i\sqrt{2}}{\sqrt{\varepsilon}}\sqrt{X^i_t}\, dW^{Y^i}_t, \\ dZ_t &= \alpha_z(\bar{z} - Z_t)\, dt + \sigma_z\sqrt{Z_t}\, dW^Z_t. \end{aligned}$$
Here, W^i and W^{Y^i} are Wiener processes such that
$$d\langle W^i, W^{Y^i}\rangle_t = \rho_i\, dt, \quad d\langle W^i, W^{Y^j}\rangle_t = 0, \quad d\langle W^i, W^j\rangle_t = 0, \quad d\langle W^{Y^i}, W^{Y^j}\rangle_t = 0,$$
for $i = 1, \dots, k$, $j = 1, \dots, k$, $i \neq j$, and ρ_i in $[-1, 1]$. Due to the factor model assumption (4) the Wiener process W^Z is independent of W^i and W^{Y^i}. When ε is small the volatility driving factors Y^i are fast mean-reverting, and we work in this regime. The effect is to thicken the upper tail of the loss distribution, so that the (small) probability of losses in the range 10–30% is enhanced. We employ a singular perturbation asymptotic analysis of the type used in option pricing in

[4] to obtain near closed-form expressions for loss distributions and hence CDO tranche premia. This is important for efficient calibration to market data.

3. Data Calibration and Conclusions

Table 1 shows the performance of the model we propose (HGSV) against some recent alternatives. The proposed setup with seven homogeneous groups fits the equity and the three mezzanine tranches very closely, but the senior tranche (15–30%) is underpriced.

Table 1. Market spreads for the Dow Jones CDX Series 4 index on August 23, 2004, and fit comparisons between the homogeneous groups with stochastic volatility model (HGSV) and five other popular models. The equity tranche is quoted in percentage points and all the other tranches in basis points (1 basis point is equal to 1/10000).

	Tranche					
	0–3%	3–7%	7–10%	10–15%	15–30%	RMSE
Mid-market spread	40%	312.5	122.5	42.5	12.5	
Bid/ask spread	2%	15	7	7	3	
HGSV	**40.8%**	**311.5**	**123.4**	**40.4**	**1.6**	**1.64**
Jump-diffusion intensities	46.9%	340.2	119.7	61.9	14.3	2.17
Pure-diffusion intensities	49.3%	442.9	94.9	16.8	0.4	5.34
Gaussian copula	46.8%	474.4	131.8	36.9	2.9	5.3
RFL Gaussian copula	48.6%	334.9	125.5	66.5	9.2	2.59
Double-t copula	45.1%	367.0	114.9	54.9	20	2.44

Source: Last five rows [3].

The flexibility of the group structure allows the HGSV model to fit the equity and mezzanine tranches considerably better than all the other models. However, the senior tranche remains underpriced, even after an exhaustive survey of the model's free parameters. Considering the rich structure of the proposed framework we claim that there is still a considerable *liquidity premium* for the virtually impossible losses. This premium is exogenous to the credit premium dictated by *arbitrage-free* pricing models. One way to model this through *utility indifference valuation* is studied in [5].

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A topology on information

Constantinos Kardaras

A major contribution of the axiomatization of probability theory, as set forth in [5], is modeling concretely the concept of *information* available to observers of random experiments. This is done by introducing the concept of a *σ-algebra* over the set of possible outcomes. In the presence of a dynamic time-flow component, one is led to considering *filtrations*, i.e., increasing sequences of σ-algebras.

Lately there has been considerable interest on the effect that different levels of information have on the optimal decisions of financial agents. The optimality criteria are usually given via the *expected utility maximization* paradigm, probably the most important agent-specific problem of economic theory. The bulk of the work up to the present has been devoted to quantifying the value of additional information that a more informed agent (insider) has with respect to an uninformed one — see, for example, [1, 2] and the references therein. The question that is tackled here is: *How can one define a reasonable topological structure on information, as represented by filtrations, in order to make the outputs of the utility maximization problem (optimal wealth processes, optimal consumption streams, etc.) continuous with respect to the input information?*

First, the static one-time-period problem is considered. On a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, consider the set $\Sigma = \Sigma(\mathcal{G})$ of all σ-algebras dominated by a σ-algebra $\mathcal{G}$, representing complete information. Each $\mathcal{F} \in \Sigma$ represents the information of some agent at time zero. The full information $\mathcal{G}$, that is assumed to contain the prices of some liquid assets traded in the market, will be revealed at some fixed future date. Simple reasoning shows that a *minimal* topological structure on Σ in order to make the utility maximization problems continuous with respect to $\mathcal{F} \in \Sigma$ is the weakest (coarsest) topology that makes all mappings $\Sigma \ni \mathcal{F} \mapsto \mathbb{P}[A \mid \mathcal{F}] \in \mathbf{L}^0$ continuous for all $A \in \mathcal{G}$. Here, $\mathbf{L}^0$ is the class of all (equivalent modulo $\mathbb{P}$) finite-valued, $\mathcal{G}$-measurable random variables equipped with convergence in $\mathbb{P}$-probability. Hints on the usefulness of this topology can be traced as back as in Exercise IV.3.2, page 124 in [7]; for a more detailed treatment, see [6].

The main result that is presented concerns a very convenient characterization of convergence of σ-algebras $(\mathcal{F}^n)_{n\in\mathbb{N}}$ to some limiting $\mathcal{F} \in \Sigma$ in terms of convergence of the corresponding random variable spaces $(\mathbf{L}^0(\mathcal{F}^n))_{n\in\mathbb{N}}$ to $\mathbf{L}^0(\mathcal{F})$. This is the vessel that allows to show continuity of the utility maximization problem with respect to information for the one-time-period problem, as long as utilities are defined on the whole real line and agents do not face any (hard) credit-limit constraints. For utilities defined only on the positive real line, such as the logarithmic utility, further assumptions on the supports of the conditional distributions of the asset prices given the agent's information have to be made in order to make sure that the limiting information set does not allow for extra investment abilities. For this last point, an easy counterexample is presented, where the value function (the *indirect utility* of the economic agent) is not continuous, but only *upper* semi-continuous with respect to information in the aforementioned topology. This upper semi-continuity property of the indirect utility is shown to always be valid.

A topology for general discrete-time filtrations is straightforward to define; one simply considers the product topology on the cross-product time-information components. Continuity of the expected utility maximization problem in general discrete-time models follows from the analysis of the one-time-period model without any extra technicalities.

For continuous-time filtrations the situation turns out to be more complicated because the time-index set is uncountable. Convergence in the product topology fails to translate to convergence of predictable processes with respect to the corresponding filtrations; this is of utmost importance from a financial point of view, since predictable processes represent the possible investment strategies of economic agents. One therefore *starts* by defining the topology *directly* on the σ-algebra of predictable processes on $\Omega \times [0, T]$ under a suitable measure, where T is some finite financial planning time-horizon. Under the financially sound condition of *No Free Lunch with Vanishing Risk* (see [3] for the exact definition of this concept), one establishes continuity with respect to information of the Doob-Meyer decomposition for continuous-path asset price processes. This, in turn, allows to prove a joint continuity result with respect to information *and* probability changes of *the numéraire portfolio*, since the solution of this problem is given in closed form in terms of the predictable characteristics of the asset-price processes. (For definitions and results concerning the numéraire portfolio, which is a generalization of the log-optimal portfolio, see [4] and references therein.)

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Perpetual American options in incomplete markets: The infinitely divisible case

Vicky Henderson
(joint work with David Hobson)

We consider the problem of determining the optimal exercise strategy for a number of American options in an incomplete market. The problem is incomplete since the agent is restricted from trading in the underlying asset itself. In a complete market, the optimal exercise time of American options does not depend on the quantity of options held. However, in an incomplete market this is no longer the case and we show the holder of a number of American options in an incomplete market would prefer to exercise options inter-temporally, rather than exercising all options at one time.

We provide an explicit analysis of the situation where options are infinitely divisible, so that the agent can exercise fractions of options continuously over time. Our assumptions that the holder of the options has exponential utility and the options are infinite maturity have the advantage of enabling us to solve the problem in closed form. We derive the optimal exercise boundary and show that it is a convex function of the asset price. The optimal policy is to exercise just enough options to stay below the boundary.

Our model lends itself naturally to the study of exercising executive stock options, since executives typically receive American style call options on the stock of their company. See Murphy [6] for an overview and statistics of executive compensation. Executives receiving options cannot sell them and are prohibited from short selling stock. Our model is also relevant for modelling real investment decisions (see Dixit and Pindyck [1], McDonald and Siegel [7] for the classic models). Real assets are often not traded and the manager making an investment decision faces unhedgeable risks and an incomplete market, see Henderson [3].

In the special case of complete markets, the perpetual American option problem was solved by McKean [8] (see also Merton [9]). Furthermore, under the assumption of infinite maturity, exponential utility, and incomplete markets, Henderson [3] solves in closed form the American option exercise problem with perfect indivisibility (or equivalently, the situation with only one option). Grasselli and Henderson [2] studied the problem where the options are finitely divisible (the options can only be exercised in whole units). This can be solved in closed form and can be thought of as an approximation of the problem in this paper.

We also consider the situation in which the agent has access to a financial market on which there trades an instrument which is correlated with the underlying asset. (For stock options it is appropriate to assume that the executive is forbidden contractually from trading on his own stock, but it is also realistic to assume that he might both wish and have the opportunity to hedge his option risk using a market index.) We show that by a suitable transformation the problem can be reduced to the case without a hedging asset. Issues arise in this set-up regarding consistency of utility functions, see Henderson and Hobson [4] for details.

We consider an agent who holds θ units of an American style claim with payoff upon exercise of $C(Y)$ per-unit claim, where Y denotes the price of an underlying. The claim is infinitely divisible and the act of exercising is irreversible. By assumption the agent is not able to trade in the asset Y itself, either for contractual reasons (in the case of executive stock options) or because it is not a financial asset. Since standard hedging arguments cannot be used, the agent faces an incomplete market. We specialise to the perpetual option situation by assuming the claim has infinite maturity. The agent's objective is to maximise the expected utility of wealth, where wealth accrues from the gains received upon option exercise. Denote by Θ_t the number of options remaining at time t, and suppose $\Theta_0 = \theta$. The utility maximising agent with initial wealth x solves

$$\max_{(\Theta_t)\in S,\Theta_0=\theta} \mathbb{E}U\left(x + \int_{t=0}^{\infty} C(Y_t)|d\Theta_t|\right) \tag{1}$$

where S is the set of positive decreasing processes $(\Theta_t)_{t\geq 0}$. The problem in (1) can equivalently be written as

$$\max_{(\tau_\phi)_{0\leq\phi\leq\theta},\tau_\phi\in\mathcal{T}} \mathbb{E}U\left(x + \int_{\phi=0}^{\theta} C(Y_{\tau_\phi})d\phi\right)$$

where $\mathcal{T}$ is the family of decreasing stopping times parametrised by the quantity ϕ which represents the number of unexercised options, so that $\tau_\phi = \inf\{t : \Theta_t \leq \phi\}$.

We will assume exponential utility of the form $U(x) = -e^{-\gamma x}/\gamma$. Throughout the paper we work with zero interest rates for simplicity, which is equivalent to taking the risk-less bond as numeraire. We are interested in the American call option, so we take $C(Y) = (Y - K)^+$, and we assume that the underlying asset Y follows exponential Brownian motion

$$\frac{dY}{Y} = \nu dt + \eta dW$$

with constant drift ν and volatility η. We suppose that $\nu \leq \eta^2/2$, else (Y_t) grows to infinity almost surely, and the problem is degenerate.

Let

$$V = \max_{\tau_\phi\in\mathcal{T}} \mathbb{E}U\left(x + \int_0^{\theta} C(Y_{\tau_\phi})d\phi \,\middle|\, Y_0 = y, \Theta_0 = \theta\right). \tag{2}$$

Note that by the Markov property, $V = V(x, y, \theta)$. By the properties of exponential utility, we expect that the value function factorises, so that $V = -\frac{1}{\gamma}e^{-\gamma x}\Lambda(y, \theta)$ for some function Λ. We present the solution to the agent's problem in the following theorem, which relies on the definition of a key function.

Definition 1. Let $\beta = 1 - 2\nu/\eta^2$ and suppose $\beta > 0$. For $\beta > 1$ define $E(\beta) = \beta/(\beta - 1)$, and set $E(\beta) = \infty$ otherwise. For $1 < y < E(\beta)$ define

$$I(y) = \frac{2}{(y-1)} - (1+\beta)\ln\left(\frac{y}{y-1}\right) + i_{(\beta>1)}\left[(1+\beta)\ln\beta - 2(\beta-1)\right],$$

where i is the indicator function, and for $\beta > 1$ and $y \geq E(\beta)$ set $I(y) = 0$. Finally, let J be the inverse to I with $J(0) = E(\beta)$ for $\beta > 1$ and $J(0) = \infty$ otherwise.

Theorem 2. Suppose $\beta > 0$. For $0 < y < \infty$ and $0 \leq \theta < \infty$ define
(3)

$$\Lambda(y,\theta) = \begin{cases} 1 - y^{\beta} J(\gamma\theta K)^{-(\beta+1)} K^{-\beta}(\beta - (\beta-1)J(\gamma\theta K)) & y \leq KJ(\gamma\theta K) \\ \beta e^{-(y/K-1)(\gamma\theta K - I(y/K))}(1 - K/y) & KJ(\gamma\theta K) < y < KE(\beta) \\ e^{-\gamma(y-K)\theta} & KE(\beta) \leq y \text{ (if } \beta > 1). \end{cases}$$

Then

$$V = V(x,y,\theta) = -\frac{1}{\gamma} e^{-\gamma x} \Lambda(y,\theta) \tag{4}$$

and the optimal strategy is to take

$$\Theta_t = \frac{1}{\gamma K} I\left(\frac{1}{K} \max_{0 \leq s \leq t} Y_s\right). \tag{5}$$

Remark 3.
(i) If the initial option holdings θ satisfy $\theta > \frac{1}{\gamma K} I(Y_0/K)$ then the optimal strategy involves exercising a tranche of the initial holdings to reduce the holdings to $\Theta_{0+} = \frac{1}{\gamma K} I(Y_0/K)$. Thereafter the optimal strategy is to exercise just enough options to remain in the region $\Theta_t \leq \frac{1}{\gamma K} I(Y_t/K)$. Since Θ has to be decreasing, this means that (Θ_t) is given by (5). As a result the optimal exercise strategy is a singular control.
(ii) For $\beta > 1$, $KE(\beta)$ is the threshold for the perpetual American call option problem with strike K for a risk neutral agent. The threshold used by a risk neutral agent is independent of quantity. For $\beta < 1$ it is never optimal for the risk neutral agent to exercise options and the problem is degenerate. Thus, when we introduce incompleteness and risk aversion into the model, the set of parameter values for which we get a non-degenerate problem (with finite exercise thresholds) expands. A similar phenomenon was found in Henderson [3] for the perfectly indivisible case.

Rather than solve the HJB equation – which appears to be a non-trivial exercise since it involves determining the form of both the optimal boundary, and the value function on the boundary – we solve for the value function for an arbitrary exercise boundary and then use calculus of variations to determine the optimal boundary.

The details of the solution are contained in the paper by Henderson and Hobson [5].

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How to calculate moments of affine processes in a very easy way

Josef Teichmann
(joint work with Christa Cuchiero, Damir Filipović, Martin Keller-Ressel)

We introduce a class of Markovian stochastic processes, called m-polynomial, where the calculation of (mixed) moments up to order m only requires the computation of matrix exponentials. This class contains all affine processes where moments up to order $m+\epsilon$ exist, as well as processes with quadratic squared diffusion coefficient. Furthermore, Lévy-driven SDEs with affine vector fields are also included. This setting extends therefore the class of analytically tractable processes beyond the affine one. The applications range from pricing and hedging related issues to statistical GMM estimation.

We consider stochastically continuous, time-homogeneous Markov processes $X := (X_t^x)_{t\geq 0, x\in S}$ with values in a state space $S \subset \mathbb{R}^N$, a closed subset of $\mathbb{R}^N$. When we speak of polynomials $f : S \to \mathbb{R}$ we mean the restriction of a polynomial on $\mathbb{R}^N$ to S. We denote the finite dimensional vector space of all polynomials of degree less than or equal to m by $\mathrm{Pol}_{\leq m}(S)$ and the finite dimensional vector space of polynomials whose degree is precisely m by $\mathrm{Pol}_m(S)$.

Definition 1. Let $m \geq 0$ be fixed. We call a stochastically continuous Markov process m-**polynomial** if, for $f \in \mathrm{Pol}_{\leq m}(S)$ and for all $t \geq 0$,

$$P_t f(x) := E(f(X_t^x)), \quad x \in S$$

is a well-defined polynomial on S of degree less than or equal to the degree of f. A process is called **polynomial** if it is m-polynomial for all $m \geq 0$.

Under additional moment conditions, the following two theorems characterize the Markov semi-group (P_t) and its infinitesimal generator $\mathcal{A}$ on $\mathrm{Pol}_{\leq k}(S)$ for $0 \leq k \leq m$.

Theorem 1. Let $m \geq 0$ be fixed and assume that X is m-polynomial on S. Assume furthermore that X admits moments of order $m+\epsilon$ for some $\epsilon > 0$. Then $P_t : \mathrm{Pol}_{\leq k}(S) \to \mathrm{Pol}_{\leq k}(S)$ is a well-defined analytic semi-group on $\mathrm{Pol}_{\leq k}(S)$ for $0 \leq k \leq m$.

The above property is due to the finite dimensionality of $\mathrm{Pol}_{\leq k}(S)$ and the continuity of $t \mapsto P_t f$ for polynomials f of degree less than or equal to m. If X additionally admits the Feller property, the form of its infinitesimal generator is precisely known.

Theorem 2. If (P_t) is a Feller semi-group on S and the assumptions of Theorem 1 hold, then for $f \in \mathrm{Pol}_{\leq m}(S)$ the generator $\mathcal{A}$ of P_t is of the form

$$\begin{aligned} \mathcal{A}f(x) &= \sum_{j,\ell=1}^{N} a_{j\ell}(x)\frac{\partial^2 f(x)}{\partial x_j \partial x_\ell} + \sum_{j=1}^{N} b_j(x)\frac{\partial f(x)}{\partial x_j} - cf(x) \\ &+ \int_{S\setminus\{0\}} \left(f(x+\xi) - f(x) - \sum_{j=1}^{N} \chi_j(\xi)\frac{\partial f(x)}{\partial x_j} \right) N(x, d\xi), \end{aligned} \tag{1}$$

where $N(x,\cdot)$ is a Radon measure on $S\setminus\{0\}$ such that $\int (f(x+\xi)-f(x))N(x,d\xi)$ is a polynomial in x of degree less than or equal to the degree of f; $a(x)$ is a polynomial of degree 2 at most, $b(x)$ is an affine function, c a non-negative constant and χ a truncation function.

Since the generator of affine processes (see [1]) and Lévy-driven stochastic differential equations is of the form (1), the following two corollaries can be considered as consequences of the previous theorem.

Corollary 1. Every regular affine process X with constant killing rate admitting moments up to order $m+\epsilon$ is an m-polynomial process on $S = \mathbb{R}_+^n \times \mathbb{R}^{N-n}$. In particular, we have that $P_t \, \mathrm{Pol}_{\leq k}(S) \subseteq \mathrm{Pol}_{\leq k}(S)$ for all $t \geq 0$ and $0 \leq k \leq m$.

Corollary 2. Let $L^1, \ldots, L^d$ denote a d-dimensional Lévy process with generating triplet (A, ν, b) and let $V_1, \ldots, V_d$ be N-dimensional affine functions on some closed subset S. Every process X which solves a stochastic differential equation of the type

$$dX_t^x = \sum_{i=1}^{d} V_i(X_t^x) dL_t^i, \quad X_0^x = x \in S,$$

in $S \subset \mathbb{R}^N$ and which leaves S invariant is m-polynomial if it admits moments up to order $m+\epsilon$. If $V_1, \ldots, V_d$ are linear in the state variable X, then we even have $P_t \, \mathrm{Pol}_k(S) \subseteq \mathrm{Pol}_k(S)$ for all $0 \leq k \leq m$ and all $t \geq 0$.

The practical consequences of the previous observations are the following: Under the assumptions of Theorem 1 we can find for any $f \in \mathrm{Pol}_{\leq k}(S)$ a polynomial $f_t \in \mathrm{Pol}_{\leq k}(S)$ such that $f_0 = f$ and the process $(f(t, X_t))_{t\geq 0}$ is a martingale. The coefficients of the time-dependent polynomial f_t are in fact quasi-exponentials in time and can be calculated by exponentiation of a non-autonomous matrix.

Hence for processes satisfying the assumptions of Theorem 1 there are claims with explicit pricing and hedging formulae; therefore either approximation techniques for arbitrary claims, or variance reduction techniques for arbitrary claims apply.

Furthermore, this method also gives rise to new techniques for pricing and estimation issues, which are then applicable to a larger class than the affine one.

Related work has been done for the Jacobi process in [3] and for one-dimensional processes in [4]. In both cases estimation of parameters for statistical purposes was the aim. The construction of time-dependent space-time harmonic polynomials has been worked out for Lévy processes by W. Schoutens and J. Teugels. Our work generalizes this to processes satisfying the assumptions of Theorem 1.

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Hedging of American options under transaction costs

Yuri Kabanov

(joint work with Dimitri De Vallière, Emmanuel Denis)

Let $G = (G_t)_{t\in[0,T]}$ be an adapted set-valued càdlàg process whose values are proper polyhedral cones in $\mathbf{R}^d$ containing $\mathbf{R}^d_+$. Let U be an $\mathbf{R}^d$-valued càdlàg process bounded from below in the sense of partial ordering induced by G, i.e., $Y_t + c\mathbf{1} \in G_t$ where c is a constant and $\mathbf{1} = (1, ..., 1)$. As in [2], we define the set $\mathcal{Y}^b$ of predictable processes Y of bounded variation, bounded from below and such that $\dot{Y}^c_\tau$ and the jump $\Delta^+ Y_\tau = Y_{\tau+} - Y_\tau$ take values in the cone $-G_\tau$ whatever is the stopping time τ. The jump $\Delta Y_\sigma = Y_\sigma - Y_{\sigma-}$ is supposed to take values in $-G_{\sigma-}$ for any predictable stopping time σ. We denote above by $\dot{Y}^c$ the Radon–Nikodým derivative of the continuous component Y^c of Y with respect to the total variation process of Y^c.

In the context of models of financial markets with transaction costs, G_t is the solvency cone when assets are expressed in terms of physical units (in the specific notations $G_t = \widehat{K}_t$). The process U is interpreted as the pay-off of an American option while the elements of $\mathcal{Y}^b$ are admissible self-financing portfolios. The set

$$\Gamma := \{x \in \mathbf{R}^d : \ \exists Y \in \mathcal{Y}^b \text{ such that } x + Y_t + U_t \in G_t\}$$

describes the initial endowments from which one can start a portfolio process hedging the given American option. Our main result is a “dual” description of Γ, usually referred to as a "hedging theorem". Namely, we show that under appropriate assumptions

$$\Gamma = \{x \in \mathbf{R}^d : \ \bar{Z}_0^\nu x \geq E^\nu Z U \quad \forall Z \in \mathcal{Z}(G^*, \nu), \ \forall \nu\},$$

where ν is a probability measure on $[0, T]$ and $\mathcal{Z}(G^*, \nu)$ is the set of adapted càdlàg processes Z such that Z and $\bar{Z}^\nu$ evolve in G^*. The notation E^ν stands for the integral with respect to $dPd\nu$, and $\bar{Z}^\nu$ is the optional projection of the process $\int_{[t,T]} Z_s \nu(ds)$, i.e., an optional process such that for every stopping time τ we have

$$\bar{Z}_\tau^\nu = E\left(\int_{[\tau,T]} Z_s \nu(ds) \middle| \mathcal{F}_\tau \right).$$

The main theorem of our study [3] extends the hedging result for the American option in discrete time established in [1]. Its hypotheses are the same as in the paper [2] where the hedging theorem was proven for European options.

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Local volatility dynamic models

René Carmona

(joint work with Sergey Nadtochyi)

The talk is concerned with the characterization of arbitrage free dynamic stochastic models for the equity markets when Itô stochastic differential equations are used to model the dynamics of a set of basic instruments including, but not limited to, the underliers.

Absence of static arbitrage can be read from properties of the surface of call option prices (see for example [4], [10] or [5]) which is usually coded by the so-called implied volatilities. We argue that the attempts to define and analyze arbitrage free stochastic dynamics for the implied volatility surface (see for example [11], [3], [8], [9], [12] or [13]) all suffer from unsurmountable technical complications, and we explain why the local volatility surface (as originally introduced in [7]) is the natural basic object to code the information available on the market, and define a stochastic dynamic model. In this sense, the talk is in the spirit of [6].

We follow the philosophy of the HJM approach to bond markets, and we present a large class of dynamic models for which we characterize absence of arbitrage by a drift condition and a spot consistency condition for the coefficients of the local volatility dynamics. We also derive simple consequences of our drift condition and we concentrate on the special case of stochastic volatility models.

The *HJM-like point of view* of the talk was advocated in the survey paper [1] and the original results presented should appear in [2].

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The only time-consistent law-invariant dynamic convex risk measure is the entropic one

Walter Schachermayer

(joint work with Michael Kupper)

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space such that $\mathcal{F} = \sigma(\cup_{k\in\mathbb{N}}\mathcal{F}_k)$. Each sigma-field $\mathcal{F}_t$ is either atomless or generated by finitely many atoms which all have the same probabilities. We further assume that for each $t \in \mathbb{N}$ there exists a Bernoulli random variable b_{t+1} taking the values -1 and 1, each with probability 1/2, such that b_{t+1} is independent of $\mathcal{F}_t$. A dynamic risk measure is a family of functions $\rho_t : L^\infty(\mathcal{F}) \to L^\infty(\mathcal{F}_t)$ such that ρ_t is normalized, monotone, convex and translation invariant. We call this family time-consistent if $\rho_0(X) = \rho_0(-\rho_t(X))$ for all $t \in \mathbb{N}$. We prove that for every time-consistent dynamic risk measure ρ_t

for which ρ_0 is law-invariant (i.e. $\rho_0(X) = \rho_0(Y)$ if X and Y have the same law), there exists $\gamma \in [0, \infty]$ such that

$$\rho_t(X) = \frac{1}{\gamma} \ln \mathbb{E}\left[\exp(-\gamma X) \mid \mathcal{F}_t\right] \quad \text{for all } t \in \mathbb{N}$$

which is called the entropic risk measure. The limiting cases $\gamma = 0$ and $\gamma = \infty$ correspond to the negative expected value and the worst case risk measure, respectively. The idea of the proof is to define $\gamma(\varepsilon)$ implicitly through

$$\rho_0(\varepsilon b_1) = \frac{1}{\gamma(\varepsilon)} \ln \mathbb{E}\left[\exp(-\gamma(\varepsilon)\varepsilon b_1)\right], \quad \text{for all } \varepsilon > 0,$$

taking a converging sequence $\gamma(\varepsilon_k) \to \bar{\gamma} \in [0, \infty]$ as ε_k tends to zero and approximating general distributions with the Bernoulli random variables $\varepsilon_k b_1, \varepsilon_k b_2, \ldots$ such that $\rho_0(X) = 1/\bar{\gamma} \ln \mathbb{E}\left[\exp(-\bar{\gamma} X)\right]$ for all $X \in L^\infty(\mathcal{F})$.

The representation of the penalty function for time consistent utility functions

Freddy Delbaen
(joint work with Emanuela Rosazza-Gianin and Shige Peng)

The space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ denotes a probability space on which the filtration of a d-dimensional Brownian motion W is defined. We emphasize that we work with a finite time horizon $T < \infty$. We use utility functions u that without further notice will satisfy the following properties:

1. $u\colon L^\infty \to \mathbb{R}$, i.e., the utility function is defined for bounded random variables. An extension to other spaces is possible but will not be done here; we refer to the talk of Cheridito.
2. The function u is monetary, i.e., $u(\xi + a) = u(\xi) + a$ for $a \in \mathbb{R}$ and $\xi \in L^\infty$.
3. $u(0) = 0$ and for $\xi \geq \eta$ we have $u(\xi) \geq u(\eta)$.
4. The function u is concave.
5. The Fatou property: If (ξ_n) is a uniformly bounded sequence of random variables tending in probability to ξ, then $u(\xi) \geq \limsup u(\xi_n)$.

We remark that such utility functions are not of von Neumann-Morgenstern type. Because of the Fatou property we can recover u by looking at the weak* (i.e., $\sigma(L^\infty, L^1)$-) closed convex set of admissible elements

$$\mathcal{A}_0 = \{\xi \mid u(\xi) \geq 0\}, \qquad L^\infty_+ \subset \mathcal{A}_0, \qquad u(\xi) = \sup\{a \mid u(\xi - a) \in \mathcal{A}_0\}.$$

The dual representation of u is given by the Fenchel-Legendre transform as shown by Föllmer and Schied. The transform c is $+\infty$ except for probabilities $\mathbb{Q}$ that are absolutely continuous with respect to $\mathbb{P}$, and

$$c(\mathbb{Q}) = \sup\{\mathbb{E}_\mathbb{Q}[\xi] \mid \xi \in \mathcal{A}_0\}, \qquad u(\xi) = \inf_{\mathbb{Q}}\{\mathbb{E}_\mathbb{Q}[\xi + c(\mathbb{Q})]\}.$$

We make the standing assumption that $c(\mathbb{P}) = 0$. This is not completely innocent. It is equivalent to saying that for all $\xi \in \mathcal{A}_0$ we must have $\mathbb{E}_\mathbb{P}[\xi] \geq 0$. Eventually we might replace the condition with the existence of an equivalent $\mathbb{Q}$ such that $c(\mathbb{Q}) = 0$. It is possible to give easy examples of Fatou utility functions with $\inf_{\mathbb{Q}} c(\mathbb{Q}) = 0$ but such that for all $\mathbb{Q} \ll \mathbb{P}$ we have $c(\mathbb{Q}) > 0$.

With $\mathcal{A}_0$ we can associate elements that are acceptable at a stopping time $\sigma \leq T$ via

$$\mathcal{A}_\sigma = \{\xi \in L^\infty(\mathcal{F}_\sigma), \xi \in \mathcal{A}_0\}, \qquad \mathcal{A}^\sigma = \{\xi \mid \text{ for all } A \in \mathcal{F}_\sigma : \mathbf{1}_A\xi \in \mathcal{A}_0\}.$$

We say that the utility function u satisfies the decomposition property if $\mathcal{A}_0 = \mathcal{A}_\sigma + \mathcal{A}^\sigma$. For each such stopping time we can also define a utility function

$$u_\sigma(\xi) = \operatorname{ess\,sup}\{\eta \in L^\infty(\mathcal{F}_\sigma) \mid \xi - \eta \in \mathcal{A}_\sigma\}.$$

It turns out that the decomposition property is equivalent to the time consistency, first introduced by Koopmans in 1960. This time consistency means that for $\sigma \leq \tau$ stopping times and for ξ, η random variables, $u_\tau(\xi) \geq u_\tau(\eta)$ implies that $u_\sigma(\xi) \geq u_\sigma(\eta)$. Time consistent utility functions were extensively studied in the literature starting with a series of papers by Epstein and co-authors. We also introduce the intermediate penalty functions

$$c_{\sigma,\tau}(\mathbb{Q}) = \operatorname{ess\,sup}\{\mathbb{E}_{\mathbb{Q}}[\xi \mid \mathcal{F}_\sigma] \mid \xi \in \mathcal{A}^\sigma \cap \mathcal{A}_\tau\}.$$

Of course we can represent the utility functions u_σ by their transforms via

$$c_\sigma(\mathbb{Q}) = \operatorname*{ess\,sup}_{\mathbb{Q}\sim\mathbb{P}}\{\mathbb{E}_{\mathbb{Q}}[\xi \mid \mathcal{F}_\sigma] \mid \xi \in \mathcal{A}^\sigma\}.$$

We find $u_\sigma(\xi) = \operatorname{ess\,inf}_{\mathbb{Q}\sim\mathbb{P}}\{\mathbb{E}_{\mathbb{Q}}[\xi + c_\sigma(\mathbb{Q}) \mid \mathcal{F}_\sigma]\}$. The decomposition property translates into a property called cocycle property: for $\sigma \leq \tau$ stopping times we have for $\mathbb{Q} \sim \mathbb{P}$ that $c_\sigma(\mathbb{Q}) = c_{\sigma,\tau}(\mathbb{Q}) + \mathbb{E}_{\mathbb{Q}}[c_\tau(\mathbb{Q}) \mid \mathcal{F}_\sigma]$. Using the cocycle property, J. Bion-Nadal was able to prove that there is an adapted càdlàg process denoted by $u_.(\xi)$ that describes all the random variables $u_\sigma(\xi)$. The main result is now the structure of the penalty function. Because of the additivity in the cocycle property and because the process $c_t(\mathbb{Q})$ is a $\mathbb{Q}$-supermartingale of class D, one can guess that there is a representation as a potential. More precisely there is a function

$$f\colon \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}_+ \cup \{\infty\}$$

such that for all $x \in \mathbb{R}^d$, the process $f(.,.,x)$ is predictable and such that for all (t,ω) the function $f(t,\omega,.)$ is lower semi-continuous, proper convex in x. Moreover $f(.,.,0) = 0$. If $\mathbb{Q} \sim \mathbb{P}$ is given by the density process $\mathcal{E}(q \cdot W)$, then the penalty is given by

$$c_t(\mathbb{Q}) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T f_u(q_u)\,du \,\middle|\, \mathcal{F}_t\right].$$

From this we can deduce part of the result presented by Schachermayer. If the utility function u is such that $u(\xi)$ only depends on the law of ξ, then the function f will not depend on t. But also it will not depend on ω (by taking random variables that have the same law but with different realisations on Ω). Because of the scaling property of BM, f must have the form $f(q) = \gamma|q|^2$ where γ is a constant. If $\gamma = +\infty$ we find the expected value, if $\gamma = 0$ we find the essential infimum of ξ. Otherwise we find the entropic risk measures since – by Girsanov's theorem –

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T |q_u|^2\,du\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right].$$

The proof relies on an earlier result of Coquet-Hu-Mémin-Peng, where the same theorem is proved under the additional assumption that for all $k \in \mathbb{R}$, the set $\{\mathbb{Q} \mid c_0(\mathbb{Q}) \leq k\}$ is weakly compact in L^1. Using the representation of c as a potential (Rao's theorem) and using some truncation, one can then remove the weak compactness assumption. It is also possible to reprove the result of Coquet-Hu-Mémin-Peng using Rao's theorem.

A stochastic target approach for quantile hedging and related problems

Bruno Bouchard

(joint work with Nizar Touzi and Romuald Elie)

Let $T > 0$ be a finite time horizon and $W = \{W_t,\ 0 \leq t \leq T\}$ a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbb{F} = \{\mathcal{F}_t,\ 0 \leq t \leq T\}$ the $\mathbb{P}$-augmentation of the filtration generated by W.

Given $\nu \in \mathcal{U}_o$, the collection of progressively measurable processes with values in a given subset U of $\mathbb{R}^d$, $t \in [0,T]$ and $z = (x,y) \in \mathbb{R}^d \times \mathbb{R}$, we define $Z^\nu_{t,z} := (X^\nu_{t,x}, Y^\nu_{t,z})$ as the $\mathbb{R}^d \times \mathbb{R}$-valued solution of the stochastic differential equation

$$\begin{aligned} dX(r) &= \mu(X(r),\nu_r)dr + \sigma(X(r),\nu_r)dW_r, \\ dY(r) &= \mu_Y(Z(r),\nu_r)dr + \sigma_Y(Z(r),\nu_r)dW_r, \quad t \leq r \leq T \end{aligned} \tag{1}$$

satisfying the initial condition $Z(t) = (X(t), Y(t)) = (x, y)$. Here, μ_Y, σ_Y, μ and σ are assumed to be locally Lipschitz continuous. We denote by $\mathcal{U}$ the subset of elements of $\mathcal{U}_o$ for which (1) admits a strong solution for all given initial conditions.

Given $u \in U$, we denote by $\mathcal{L}^u$ the Dynkin operator associated to the controlled diffusion Z, i.e.,

$$\mathcal{L}^u\varphi(t,x) := \partial_t\varphi(t,x) + \mu(x,u)\cdot D\varphi(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}(x,u)D^2\varphi(t,x)\right].$$

Let G be a measurable map from $\mathbb{R}^{d+1}$ into $\mathbb{R}$ such that $(x,y) \in \mathbb{R}^d \times \mathbb{R} \mapsto G(x,y)$ is non-decreasing in its y-component, and define the **stochastic target problem**

$$V(t,x) := \inf\left\{y \in \mathbb{R}\ :\ G\left(X^\nu_{t,x}(T), Y^\nu_{t,x,y}(T)\right) \geq 0 \quad \mathbb{P}\text{-a.s. for some } \nu \in \mathcal{U}\right\}.$$

The first aim of this paper is to provide a characterization of the value function V as a discontinuous viscosity solution of

$$\sup\left\{\mu_Y(x,\varphi(t,x),u) - \mathcal{L}^u\varphi(t,x)\ :\ u \in \mathcal{N}_0\left(x,\varphi(t,x),D\varphi(t,x)\right)\right\} = 0,$$

where we define the set $\mathcal{N}_0(x,y,q) := \{u \in U\ :\ N^u(x,y,q) = 0\}$ via the function $N^u(x,y,q) := \sigma_Y(x,y,u) - \sigma(x,u)^{\mathrm{T}}q$. We also provide suitable boundary conditions.

The novelty with respect to the previous works of [1], [3] and [4], see also the references therein, is that the set U in which the controls take their values is not bounded. This makes the derivation of the PDE and the boundary conditions significantly more difficult and requires the use of the notion of discontinuous viscosity solutions for discontinuous operators.

This study is motivated by a more general formulation of the above problem as a **stochastic target problem with controlled probability**, namely

$$\bar{V}(t,x,p) := \inf\left\{y \in \mathbb{R} \;:\; \mathbb{P}\left[G\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0\right] \geq p \quad \text{for some } \nu \in \mathcal{U}\right\},$$

which can be converted into the class of (standard) stochastic target problems presented above. Indeed, let us introduce an additional controlled state variable valued in $[0,1]$ defined by

$$P^{\alpha}_{t,p} = p\ ,\ \ dP^{\alpha}_{t,p}(u) = P^{\alpha}_{t,p}(u)\left(1 - P^{\alpha}_{t,p}(u)\right)\alpha_u \cdot dW_u\ ,\quad u \in [t,T]\ , \tag{2}$$

where the additional control α is an $\mathbb{F}$-progressively measurable real valued process. Denoting by $\bar{X} := (X, P)$, $\bar{U} = U \times \mathbb{R}^d$, by $\bar{\mathcal{U}}$ the corresponding set of admissible controls, and setting

$$\bar{G}(\bar{x}, y) := \mathbb{I}_{\{G(x,y)\geq 0\}} - p\ ,\ y \in \mathbb{R}\ ,\ \bar{x} := (x,p) \in \mathbb{R}^d \times [0,1]\ ,$$

one can indeed show that

$$\bar{V}(t,x,p) \;=\; \inf\left\{y \in \mathbb{R} \;:\; \bar{G}\left(\bar{X}^{\bar{\nu}}_{t,x,p}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0 \text{ for some } \bar{\nu} = (\nu,\alpha) \in \bar{\mathcal{U}}\right\}\ .$$

This allows us to introduce a new direct dynamic programming approach for such control problems, known in finance as quantile hedging problems. To our knowledge, this is the first time that such a direct approach is used in the literature.

This idea can be further extended. Let $\ell : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing function, and denote by

$$L \;:=\; \overline{\mathrm{conv}}\left(\ell \circ G(\mathbb{R}^d \times \mathbb{R})\right)$$

the closed convex hull of the image of $\ell \circ G$. For $p \in L$, we can then define **the target reachability problem with controlled loss** by

$$\bar{V}^{\ell}(t,x,p) \;:=\; \inf\left\{y \in \mathbb{R} \;:\; \mathbb{E}\left[\ell \circ G\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p \ \text{ for some } \nu \in \mathcal{U}\right\}\ .$$

As previously, we introduce an additional controlled state variable valued in L defined by

$$P^{\alpha}_{t,p} = p\ ,\ \ dP^{\alpha}_{t,p}(u) = \alpha_u \cdot dW_u\ ,\quad u \in [t,T]\ , \tag{3}$$

where the additional control α is an $\mathbb{F}$-progressively measurable real-valued process such that P^{α} takes values in L $\mathbb{P}$-a.s. We next denote by $\bar{Z} := (Z, P)$, $\bar{U} = U \times \mathbb{R}^d$, $\bar{\mathcal{U}}$ the corresponding set of admissible controls, and set

$$\bar{G}^{\ell}(\bar{x}, y) := \ell \circ G(x,y) - p\ ,\ y \in \mathbb{R}\ ,\ \bar{x} = (x,p) \in \mathbb{R}^d \times L\ .$$

We then have

$$\bar{V}^{\ell}(t,x,p) \;=\; \inf\left\{y \in \mathbb{R} \;:\; \bar{G}^{\ell}\left(\bar{X}^{\bar{\nu}}_{t,x,p}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0 \text{ for some } \bar{\nu} = (\nu,\alpha) \in \bar{\mathcal{U}}\right\}.$$

Concrete examples where the associated PDE can be solved explicitly are presented.

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(Technical) Analysis of technical analysis in finance

Denis Talay

The aim of the lecture is to present some effects of misspecifications of financial models and to propose a mathematical framework to analyze financial strategies which, issued from technical analysis, do not use any mathematical model.

In finance modelling issues are much more complex than in physics for – at least – the following reasons. First, no physical law helps the modeller to choose a particular dynamics to describe the time evolution of market prices or indices. The real market is incomplete and arbitrages occur. Therefore, the modeller has a high degree of freedom to mathematically describe the market in order to compute optimal portfolio allocations or risk measures. For example, to compute price options and deltas, practitioners and quants find it convenient to suppose that the no arbitrage and completeness hypotheses prevail: In diffusion models, this assumption constrains the dimension and the algebraic structure of the volatility matrix, so that the model used to hedge may not exactly fit the market data.

Second, statistical procedures issued from the theory of statistics of random processes and based upon historical data may be extremely inaccurate because of the lack of data. For example, an accurate parametric estimation of a volatility matrix requires that the asset price is observed at very high frequencies. Of course, it would be unclever to use historical data only to calibrate financial models; in order to calibrate a stock price model, the practitioners do not actually consider the past prices of the stock only, but also use other available informations such as past prices of derivatives on this stock (see, e.g., papers and references in [3]). However, the stationarity of the market during the observation period remains questionable, and error estimates for complex calibration methods are not available in the literature.

Third, the modeller needs to design and calibrate models by using one single history of the market.

In addition, model uncertainties also occur in the numerical resolution of partial differential equations (PDEs) related to option pricing or optimal portfolio allocation. Commonly used stochastic models in finance actually lead to consider processes whose time marginal laws have unbounded supports. Consequently the PDEs are posed in unbounded domains and artificial boundary conditions are necessary.

Consequently model misspecifications cannot be avoided, which leads to model risk. The specificity and definitions of model risk are not universally admitted; see the extended introduction and list of references in Cont [5]. During this lecture we limit ourselves to a particular restricted family of questions: How to evaluate – and possibly control – the impact of certain model uncertainties on profit and losses of hedging portfolios, or on portfolio management strategies?

We start with illustrating the difficulty to construct a reliable market model by presenting recent results on one of the very first steps of the modelling process, namely, the design of the driving noise of the dynamics of the assets under consideration.

We then propose a tentative methodology to compare the performances of financial strategies derived from (misspecified) mathematical models and strategies which, derived from technical analysis, avoid modelling and calibration issues.

Limitations of statistical procedures based on historical data

In the literature numerous papers analyze parametric and non-parametric estimators for the coefficients of stochastic differential equations and, more specifically, for the parameters of stochastic models in finance.

However, only few papers deal with the following problem: How to calibrate the noise driving the processes modelling stock prices, interest rates, etc.? In an impressive paper Aït-Sahalia and Jacod [2] recently constructed and analyzed a rule to decide whether a price process X observed at discrete times is continuous or jumps at least once during the observation time interval. The authors prove several limit theorems which allow them to construct levels of tests based on their test statistics. The asymptotic variances can be estimated by means of the discrete time observations of X. Thus it is possible to construct real tests for the null hypothesis that X is discontinuous as well as for the null hypothesis that X is continuous. For precise critical regions, asymptotic levels and power functions, we refer to [2].

We then summarize a paper by Jacod, Lejay, and Talay [8] on statistical procedures to determine an explanatory Brownian dimension of a stochastic model from a discrete-time observation. By ‘explanatory Brownian dimension r_B’ of an Itô process we (informally) mean that a model driven by an r_B-dimensional Brownian motion satisfyingly fits the information conveyed by the observed path, whereas increasing the Brownian dimension does not bring a better fit.

More precisely, suppose that we observe a path of the process

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s,$$

where B is a standard q-dimensional Brownian motion, (b_s) is a predictable $\mathbb{R}^d$-valued locally bounded process, σ is a $d \times q$ matrix-valued adapted and càdlàg process. Set $c_s := \sigma_s \sigma_s^\star$.

We propose procedures to estimate the maximal explanatory rank of c_s on the basis of the observation of $X_{iT/n}$ for $i = 0, 1, \dots, n$ and discuss the theoretical justifications which we have obtained so far. We also discuss a few open questions.

Empirical studies show that if the process (X_t) is observed at low frequencies, the tests that we developed may lead to erroneous conclusions. In any case, the transformation of the real Brownian dimension into an explanatory one induces a specific model risk.

A stochastic game to face model risk

Consider the market model

$$\begin{cases} dS_t^i &= S_t^i [b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dB_t^j] \quad \text{for } 0 \le i \le n, \\ \\ dP_t &= P_t \sum_{i=1}^n \pi_t^i \left[b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dB_t^j \right] + r P_t \left(1 - \sum_{i=1}^n \pi_t^i\right) dt. \end{cases}$$

Here $\{\pi^i\}$ is the set of prescribed strategies. Consider $u(\cdot) := (b(\cdot), \sigma(\cdot))$ as the market's control process.

In [6] Cvitanić and Karatzas have studied the dynamic measure of risk

$$\inf_{\pi(\cdot) \in \mathcal{A}(x)} \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu (F(X^{x,\pi}(T))),$$

where $\mathcal{A}(x)$ denotes the class of admissible portfolio strategies starting from the initial wealth x, and $\mathbb{E}_\nu$ denotes the expectation under the probability $\mathbb{P}_\nu$ for all ν in a suitable set. All the measures $\mathbb{P}_\nu$ have the same risk-neutral equivalent martingale measure, which implies that the trader (or the regulator) is concerned by model risk on stock appreciation rates. For numerical methods related to this approach, see, e.g., Gao et al. [7].

We briefly present a somewhat different approach aimed to compute the minimal amount of money and dynamic strategies which allow the financial institution to (approximately) contain the worst possible damage due to model misspecifications for volatilities, stock appreciation rates, and yield curves. In this approach the trader acts as a minimizer of the risk whereas the market systematically acts as a maximizer of the risk. Thus the model risk control problem can be set up as a two-player zero-sum stochastic differential game problem. We recall a result obtained by [12]: The model risk value function is the unique viscosity solution (in a suitable space) to an Hamilton-Jacobi-Bellman-Isaacs equation.

Model risk and technical analysis

In view of the difficulty to calibrate models (and even their Brownian dimension as shown previously), and to compute reserve prices to balance model misspecifications, practitioners use various rules to rebalance their portfolios. These rules usually come either from fundamental economic principles, or from mathematical approaches derived from mathematical models, or technical analysis approaches. Technical analysis, which provides decision rules based on past price behavior, avoids model specification and thus model risk. Blanchet et al. [4] propose a framework allowing one to compare the performances obtained by strategies derived from erroneously calibrated mathematical models and the performances obtained by technical analysis techniques.

In the lecture we focus on the following situation. Consider an asset whose instantaneous expected rate of return changes at an unknown random time, and a trader who aims to maximize his/her utility of wealth by selling and buying the asset. The benchmark performance results from a strategy which is optimal when the model is perfectly specified and calibrated. We can compare to this benchmark the performances resulting from optimal rules but erroneous parameters, and the performances resulting from technical analysis indicators.

The real market is described by

$$\begin{cases} dS_t^0 &= S_t^0 r dt, \\ dS_t &= S_t \left(\mu_2 + (\mu_1 - \mu_2)\mathbb{I}_{(t \leq \tau)}\right) dt + \sigma S_t dB_t. \end{cases}$$

Here, the Brownian motion (B_t) and the change time τ are independent, and τ follows an exponential law with parameter λ.

We start with describing one of the technical analysis rules which are applied in the context of instantaneous rates of return changes. We explicit the logarithmic utility of the wealth W_T at time T owing to a result due to Yor [13].

Then the performance of the technical analysis strategy is compared to the benchmark performance, i.e., the optimal wealth of a trader who perfectly knows the parameters μ_1, μ_2, λ and σ. This trader's strategy is constrained to be adapted with respect to the filtration

$$\mathcal{F}_t^S := \sigma\left(S_u, 0 \leq u \leq t\right)$$

generated by (S_t). To explicit the logarithmic utility of the wealth W_T at time T of this 'perfect' trader, we follow and adapt the methodology developed in Karatzas and Shreve [10], and introduce an auxiliary unconstrained market.

For general utilities the optimal strategy cannot be explicited. It thus is worth considering the case of a trader who chooses to reinvest the portfolio only once, namely at the time where the change time τ is optimally detected owing to the price history. We suppose that the reinvestment rule is the same as the technical analyst's one: At the detected change time from μ_1 to μ_2, all the portfolio is reinvested in the risky asset. The stopping rule Θ^K which minimizes the expected miss $\mathbb{E}|\Theta - \tau|$ over all stopping rules Θ with $\mathbb{E}(\Theta) < \infty$ has been studied by Shiryaev [11] and Karatzas [9].

In practice, one cannot estimate λ, μ_1 and σ with good accuracy, and the value of μ_2 cannot be determined a priori. Therefore traders believe that the stock price is

$$dS_t = S_t\left(\overline{\mu}_2 + (\overline{\mu}_1 - \overline{\mu}_2)\mathbb{I}_{t \leq \overline{\tau}}\right) dt + \overline{\sigma} S_t dB_t,$$

where the law of $\overline{\tau}$ is exponential with parameter $\overline{\lambda}$. We describe the value of a misspecified optimal allocation strategy, the erroneous stopping rule and the corresponding wealth process.

Finally, we examine the following question: Is it better to invest according to a mathematical strategy based on a misspecified model, or according to a strategy based upon technical analysis rules?

It appears that, even in the logarithmic utility case, the explicit formulae for the wealth are too complex to allow analytical comparisons. However, Monte Carlo simulations on case studies show that the technical analyst overperforms misspecified optimal allocation strategies when the parameter λ is underestimated. We have looked for other cases where the technical analyst is able to overperform the misspecified optimal allocation strategies. Other numerical studies show that a single misspecified parameter is not sufficient to allow the technical analyst to overperform the 'model and detect' traders. Astonishingly, other simulations show that the technical analyst may overperform the misspecified optimal allocation strategy but not the misspecified 'model and detect' strategy. One can also observe that when μ_2/μ_1 decreases, the performances of well specified and misspecified 'model and detect' strategies decrease.

Our conclusion is that more mathematics is necessary to understand the performances of portfolios based on technical analysis rules, and to construct strategies which are robust to model misspecifications.

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Pricing in the illiquid market of Çetin-Jarrow-Protter

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(joint work with Umut Çetin, Nizar Touzi)

For a given contingent claim $g(S_T)$ and a stock price process S, we consider the minimal super-replication price

$$v(t,s) := \inf\{z \ : \ \exists Y \in A_{t,s} \text{ s.t. } Z_T^{t,s,z,Y} \geq g(S_T^{t,s}) \text{ a.s. }\},$$

where as usual

$$dS_u = S_u \sigma dW_u,$$

the admissible set $A_{t,s}$ is given in [1], and the "wealth" process Z is given by

$$Z_T^{t,s,z,Y} = z + \int_t^T Y_u dS_u - \int_t^T \frac{1}{4\ell(S_u)} \, d[Y,Y]_u^c,$$

and the function $\ell(s) > 0$ is a measure of liquidity of the market. This function is defined in the model of Çetin-Jarrow-Protter [1, 2]. Under our assumptions, admissible portfolio processes satisfy

$$dY_u^c = \alpha_u \, du + \Gamma_u \, dS_u.$$

Hence, $[Y,Y]_u^c = \sigma^2 S_u^2 \Gamma_u^2 \ du$. We prove that the minimal price is the unique solution of

$$-v_t + \sup_{B \geq 0} \left\{ -\frac{1}{2}\sigma^2 s^2 (v_{ss} + B) - \frac{1}{\ell}\sigma^2 s^2 (v_{ss} + B)^2 \right\} = 0$$

on $t < T$ with terminal data $v(T,s) = g(s)$. In particular, there is a liquidity premium.

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Utility maximization for a large trader

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(joint work with Jan Kallsen)

One of the classic but unrealistic assumptions of many studies in mathematical finance is that all traders are assumed to be ‘small’, i.e., are just acting as price takers. This has been relaxed in several studies allowing for a situation where there is in addition to a pool of small traders also one ‘large’ investor around who faces different prices for her transactions depending on the trade size. We focus here on the two recent papers Çetin, Jarrow and Protter [2], and Bank and Baum [1], and refer to these also for a detailed overview of the literature dealing with the large trader problematic.

The former study postulates the existence of a stochastic supply curve governing the transactions of the large trader. Once the transaction has been settled, however, the actual price remains unchanged, hence the large trader does not move prices in this framework. Arbitrage opportunities can then be excluded by assuming the existence of a martingale measure for the point of the supply curve corresponding to the zero net trade. In contrast, in Bank and Baum [1] the prices itself are modelled as a random field indexed by the large trader's position in the asset: If the large trader changes her position at time t from ϑ to ϑ' units, the price reacts instantaneously by moving from P_t^{ϑ} to $P_t^{\vartheta'}$. Mathematically, the resulting value process of the large investor then has to be modelled by a nonlinear stochastic integral (sometimes also called line integral); the integrator is affected by the employed strategy of the large trader. Bank and Baum [1] have chosen the Kunita integral for this purpose. Under the crucial assumption that there exists a universal martingale measure simultaneously for all primitive price processes P^{ϑ} (which in fact then turns out to be a martingale measure for all possible value processes corresponding to admissible strategies) they prove, amongst others, the absence of arbitrage for the large trader. Despite the different approaches chosen, there are a couple of similar findings in Çetin, Jarrow and Protter [2], and Bank and Baum [1]. Let us here just single out that the authors of both papers agree in that the large trader should use ‘tame’ strategies, i.e., continuous strategies of finite variation. Block trades as well as highly fluctuating strategies are disadvantageous since they induce transaction costs for the large investor.

The main point of the present work is to extend the results of Bank and Baum [1] by relaxing the assumption of a universal martingale measure. Our main assumption is closer to the one used in the discrete time framework of Jarrow [7]: There are no arbitrage opportunities in the market as long as the large trader employs only elementary buy-and-hold strategies. This corresponds to requiring the existence of a martingale measure Q^{ϑ} for each price process P^{ϑ} on the supply curve. For every buy-and-hold strategy θ of the large trader we then construct a martingale measure Q^{θ} for the associated value process by concatenating the martingale measures corresponding to each position which is constant over some period of time. The main issue, however, is how to extend this concatenation procedure in case the large trader employs a dynamic trading strategy. It is here where the very general stochastic integral of Carmona and Nualart [3] seems to be 'tailor-made' to resolve this problem. The key is that this integral provides a link to the semimartingale topology $\mathcal{S}$: If we approximate a dynamic strategy by buy-and-hold strategies uniformly in probability, then the associated value processes converge in $\mathcal{S}$. This continuity property allows infinitesimally 'to glue' together the individual martingale measures which govern the dynamics of the price process for just one point in time. It results that each bounded semimartingale strategy θ induces a martingale measure Q^θ for the value process, from which in turn it follows easily that the large investor has no arbitrage opportunities. Let us mention, though, that the problem of (super-)replication of claims under our assumptions is still very much an open question.

We then study the following utility maximization problem: We want to solve

$$\sup_{\theta\in\Theta} E\left[u\left(V_T\left(\theta\right)\right)\right]. \tag{1}$$

Here u is some utility function defined on the whole real line which is bounded from above with $u(0) = 0$. The exponential utility function would be one prime example. $V_T\left(\theta\right)$ is the terminal wealth associated to the large trader's strategy θ, and Θ some space of admissible integrands. Firstly, we choose $\Theta = \Theta_K$ which contains all tame strategies which are absolutely continuous with respect to Lebesgue measure, with derivative a.s. bounded by K. This choice of the strategy space is motivated by the arbitrage considerations in the first part of the paper. By an Arzelà-Ascoli argument, one gets readily the existence of an optimal strategy within Θ_K. We study conditions under which either of the two following scenarios takes place: 1) the expected utility increases with K – this is the case if the impact of the investor on the drift is so large that she tries to buy as many shares as possible; 2) there is a unique optimal strategy for all K large enough. In the latter stable regime it turns out that in the presence of a large trader who invests in an optimal way, a small investor would choose the same optimal strategy, but in contrast to Bank and Baum [1] she could sometimes achieve a higher expected utility. These findings are illustrated in an illiquid Bachelier model.

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A large investor trading at market indifference prices

Peter Bank
(joint work with Dmitry Kramkov)

We develop a financial market model where several market makers fill orders which are dynamically generated by a large investor. The market makers are characterized by their random endowments forming a Pareto-optimal allocation and their attitudes towards risk as described by some utility functions with bounded absolute risk aversion. They quote prices for a finite number of financial securities whose payoff at maturity is specified as some random variable. Quotes are given according to what we call market indifference prices. From the large investor's point of view, these are the best prices at which his orders can be filled without any of the market makers being worse off in terms of expected utility after a transaction than before it. For simple strategies of the large investor, we show that this indifference principle in fact induces a unique cash balance process and a unique sequence of Pareto-optimal allocations which accommodate the investor's orders. In a Brownian setting, we use the Clark-Ocone-Haussmann formula to show how to pass from simple to general predictable strategies. A key observation is that, given the investor's strategy, it suffices to keep track of the market makers' process of indirect utilities which, due to our indifference principle, turns out to evolve as a multi-dimensional martingale and can actually be described as solution to a system of SDEs. This observation is also used to prove the absence of arbitrage opportunities for the large investor and to construct hedging strategies in certain special cases.

In which financial markets do mutual fund theorems hold true?

Mihai Sîrbu
(joint work with Walter Schachermayer, Erik Taflin)

We consider a financial market, on a finite time interval $[0,T]$, with one risk-free asset S^0 called the bond (or better, money market account) and d risky assets called stocks. We choose S^0 as numéraire (which means we normalize $S^0 = 1$) and denote by $S^1, \dots, S^d$ the prices of the risky assets measured in units of S^0. The price process of the stocks $S = (S^i)_{1 \le i \le d}$ is assumed to be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ satisfies the *usual conditions* (right continuous and saturated) and $\mathcal{F} = \mathcal{F}_T$. A portfolio is defined as a pair (x, H), where the constant x represents the initial capital and $H = (H^i)_{1 \le i \le d}$ is a $(\mathcal{F}_t)_{0 \le t \le T}$-predictable and S-integrable process in the vector integration sense. The wealth process $X = (X_t)_{0\leq t\leq T}$ of the portfolio evolves in time as the stochastic integral of H with respect to S, i.e.,

$$X_t = x + \int_0^t (H_u, dS_u), \quad 0 \leq t \leq T. \tag{1}$$

For each $x > 0$ we denote by $\mathcal{X}(x)$ the family of wealth processes $X = (X_t)_{0\leq t\leq T}$ with nonnegative capital at any instant and with initial value equal to x, i.e.,

$$\mathcal{X}(x) = \{X \geq 0 : \ X \text{ is defined by (1)}\}. \tag{2}$$

The preferences of an investor are modeled by a utility function $U : (0, \infty) \to \mathbb{R}$ which is strictly increasing, strictly concave, continuously differentiable on $(0, \infty)$ and satisfies the Inada conditions $\lim_{x\to 0} U'(x) = \infty$, $\lim_{x\to\infty} U'(x) = 0$. We assume that the investor maximizes his/her *utility from terminal wealth*, generating an *indirect utility function*

$$u(x) = \sup_{X\in\mathcal{X}(x)} \mathbb{E}[U(X_T)], \ x > 0. \tag{3}$$

The optimization problem (3) is extensively studied in the literature, and, under some suitable technical conditions (which we assume to hold true), admits a unique optimal wealth process $X(x, U)$. We denote by N the optimal wealth process for the logarithmic maximizer with initial endowment 1, i.e., $N = X(1, \ln)$ (usually called numéraire portfolio in the literature), and call a *mutual fund* any wealth process $M \in \mathcal{X}(1)$.

Definition 1. Consider a financial market S and let $\mathcal{U}$ be a family of utility functions. We say that the financial market S satisfies the mutual fund theorem (MFT) with respect to $\mathcal{U}$ if there exists a mutual fund M such that for each $U \in \mathcal{U}$ and $x > 0$ there exists a a real-valued $(\mathcal{F}_t)_{0\leq t\leq T}$-predictable and M-integrable process $k = k(x, U)$ such that

$$\hat{X}_t(x, U) = x + \int_0^t k_u dM_u, \ 0 \leq t \leq T. \tag{4}$$

The process M is then called a mutual fund for the market S and the class of utility functions $\mathcal{U}$.

Using the above definition, and denoting by $\mathcal{A}(S)$ the class of all utility functions for which the optimization problem (3) is well-posed, we can state the main results of the paper:

Theorem 1. Consider a semimartingale financial market S. If there exists a mutual fund M such that each bounded European (i.e., path-independent) option f with maturity T and written on the numéraire portfolio N can be replicated by trading only in the mutual fund M, then the financial model S satisfies the mutual fund theorem with respect to the class $\mathcal{A}(S)$.

Theorem 1 has the following obvious consequence, which is a generalization of the classic results of Merton (see [2], [3]):

Corollary 1. If the numéraire portfolio process N defines a complete market with respect to its own filtration $(\mathcal{F}_t^N)_{0\leq t\leq T}$, then the financial model S satisfies the MFT with respect to the set of all utility functions $\mathcal{A}(S)$, and the numéraire portfolio N can be chosen as mutual fund.

It can be shown through an easy example of a one-dimensional market that there is no direct converse to Theorem 1. However, under a suitable assumption, which we call weak completeness, we can formulate a converse. The weak completeness assumption (**WC**) requires that each bounded European option on the numéraire portfolio N which expires at time T can be replicated by trading in the whole market S.

Theorem 2. If the semimartinale financial market S satisfies the MFT with respect to the class $\mathcal{A}(S)$ of all utility functions and also satisfies the weak completeness condition (**WC**), then all bounded European options on the numéraire portfolio N can be replicated by trading only in the mutual fund M.

As a by-product of our analysis, we also obtain a dual characterization of the condition (**WC**) in terms of the existence of a martingale measure which dominates stochastically in the second order all martingale measures. In addition, Theorems 1 and 2 can be easily generalized to "n-fund separation theorems", where the optimal investment strategies can be separated into n mutual funds $M^1, \ldots, M^n$, where $n \leq d$, instead of only one mutual fund M.

Our last main results is a "continuous process" analogue to the theorem which Cass and Stiglitz [1] obtained in discrete time:

Theorem 3. Consider a class $\mathcal{U}$ of utility functions. Assume that every complete two-dimensional Brownian financial market S satisfies MFT with respect to the class $\mathcal{U}$. Then the family $\mathcal{U}$ consists only of a single utility function U (modulo affine transformations) which is either

(1) $U(x) = \log(x),\ x > 0$ or
(2) $U(x) = \frac{x^\alpha}{\alpha}, x > 0$, for some $\alpha \in\ ]-\infty, 1[\ \setminus\{0\}$.

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Risk measures on Orlicz hearts

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(joint work with Tianhui Li)

Coherent risk measures were introduced in [1, 2] and extended to more general setups in [6, 7]. In [12, 13, 14, 15] the more general concepts of convex and monetary risk measures were established. In [1, 2] and the first part of [12], future financial positions are modelled by elements of the set $L(\Omega)$ of all real-valued functions on a finite sample space Ω, and a coherent, convex or monetary risk measure is a mapping $\rho : L(\Omega) \to \mathbb{R}$ satisfying certain properties. In this setting, and expressed in discounted units, the main structural results are that every monetary risk measure ρ can be written as

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{C}\}\,, \tag{1}$$

for the set of acceptable positions $\mathcal{C} := \{X \in L(\Omega) : \rho(X) \leq 0\}$, and every convex monetary risk measure has a convex dual representation of the form

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{D}} \left\{\mathrm{E}_{\mathbb{Q}}\left[-X\right] - \gamma(\mathbb{Q})\right\}\,, \tag{2}$$

where $\mathcal{D}$ denotes the set of all probability measures on Ω and γ is a function from $\mathcal{D}$ to $(-\infty, \infty]$. If ρ is coherent, then γ can be chosen so that it only takes the values 0 or ∞, and (2) reduces to

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathrm{E}_{\mathbb{Q}}\left[-X\right]\,, \tag{3}$$

for the set $\mathcal{Q} := \{\mathbb{Q} \in \mathcal{Q} : \gamma(\mathbb{Q}) = 0\}$.

The proof of (1) is elementary and easily generalizes to more general setups. On the other hand, the proofs of (2) and (3) are based on the separating hyperplane theorem and become more involved in more general frameworks. Also, the form of the representations can slightly change when the set of financial positions $\mathcal{X}$ is different from $L(\Omega)$. The cases of L^∞ and $\mathcal{L}^\infty$ over a general probability space have for instance been studied in [6, 7, 21, 12, 13, 14, 17, 20, 18]. Risk measures for different sets of unbounded random variables can, among others, be found in [6, 7, 8, 15, 16, 5, 3, 22, 23, 10, 4, 11, 19].

We study $(-\infty, \infty]$-valued coherent, convex and monetary risk measures on maximal subspaces of Orlicz classes. Following [9], we call such spaces Orlicz hearts. They include all L^p-spaces for $1 \leq p < \infty$ and allow for an elegant duality theory. We prove that every coherent or convex monetary risk measure on an Orlicz heart which is real-valued on a set with non-empty algebraic interior is automatically real-valued on the whole space and admits a robust representation of the form (3) or (2), respectively, such that the supremum is always attained. We also show that penalty functions of such risk measures have to satisfy a certain growth condition and that our risk measures are Luxemburg-norm Lipschitz-continuous in the coherent case and locally Luxemburg-norm Lipschitz-continuous in the convex monetary case. We then give general conditions for monetary risk measures to be Gâteaux-differentiable, strictly monotone with respect to almost sure inequality, strictly convex modulo translation, strictly convex modulo comonotonicity, or monotone with respect to different stochastic orders. The theoretical results are used to analyze various specific examples of risk measures. Some of them have appeared in earlier papers, others are new.

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Maturity-independent risk measures

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(joint work with Thaleia Zariphopoulou)

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbf{P})$ be a filtered probability space which satisfies the *usual conditions*, and let $\{S_t\}_{t\in[0,\infty)}$ be a d-dimensional locally bounded semimartingale.

A mapping $\rho : \mathcal{L} \to \mathbb{R}$, where $\mathcal{L} = \cup_{t\geq 0}\mathbb{L}^\infty(\mathcal{F}_t)$, is called a *maturity-independent risk measure* if it satisfies the following axioms:

(1) $\rho(X + m) = \rho(X) - m$,
(2) $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$,
(3) $X \leq Y$, a.s., implies $\rho(X) \geq \rho(Y)$, and
(4) $\rho(X + \int_0^t H_u\, dS_u) = \rho(X)$,

for all $X, Y \in \mathcal{L}$, $m \in \mathbb{R}$, $\alpha \in [0,1]$ and $H \in \mathcal{A}$, where $\mathcal{A}$ is the family of all predictable S-integrable processes $\{H_t\}_{t\in[0,\infty)}$ such that $\sup_{s\leq t} |\int_0^s H_u\, dS_u|$ is in $\mathbb{L}^\infty(\mathcal{F}_t)$ for all $t \geq 0$.

We show that in case there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that S is a $\mathbb{Q}$-martingale on $[0,\infty)$, the maturity-independent risk measures are naturally induced by (classical) convex measures on $\mathbb{L}^\infty(\mathcal{F})$. The more interesting case is when there exists a family $\{\mathbb{Q}^t\}_{t\geq 0}$ of probability measures equivalent to $\mathbb{P}$ such that S is a $\mathbb{Q}^t$-martingale on $[0,t]$, but no measure will do the job for the whole interval $[0,\infty)$. Most financial market models used in practice are of the latter type. In this case, we show that the upper hedging price is an extremal example of a maturity-independent risk measure.

A search for non-extremal examples of von Neumann-Morgenstern type leads naturally to the notion of *forward utility* introduced in [MZ03, MZ05, MZ06]. Indeed, we show that if ρ is given naturally as an indifference price for a utility function U (possibly depending on t and ω in addition to the wealth argument x), then U must be self-generating in the sense that

$$U(\omega, s, x) = \operatorname{ess\,sup}_H \mathbb{E}\left[U\left(\omega, t, x + \int_s^t H_u\, dS_u\right) \middle| \mathcal{F}_s \right], \text{ a.s.}$$

This result can be seen as an axiomatic underpinning of the notion of forward utility.

We show, further, by means of two counterexamples in different settings, that a naïve (and widely spread) choice of the exponential function for U – the entropic indifference price – may fail to produce a maturity-independent risk measure. In return, we provide a family of random utility functions of the form

$$U(\omega, t, x) = Z_t e^{x/Y_t + A_t}$$

which *do* give rise to a maturity-independent risk measure. Furthermore, a necessary and sufficient condition on the processes Z, Y and A is given for this to happen.

References

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