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Geometrie

Organised by
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Abstract. The program of this meeting covered a wide range of of recent developments in geometry such as geometric flows, metric geometry, positively and negatively curved manifolds and dynamics of quasiconformal maps.

Mathematics Subject Classification (2000): 53-xx.

Introduction by the Organisers

The official program consisted of 20 lectures and therefore left plenty of space for fruitful informal collaboration for the 44 participants. One emphasis with 9 talks at this meeting was on geometric flows. Exciting progress could be reported on

- existence results for mean curvature as well as for the Ricci flow with singular initial data,
- stability and convergence results for the Ricci flow and new invariant curvature conditions,
- an extension of Perelman's work to open 3-manifolds with positive scalar curvature, and an improved singularity analysis.

Another important theme was metric and Finsler geometry with five talks covering the following topics

- a survey on currents in metric spaces, and existence of a curvature tensor in a measured sense on Alexandrov spaces which are noncollapsed limits,
- existence of path isometries to the Euclidean space for a wide range of metric spaces,
- closed geodesics on Finsler manifolds and volume entropy of Hilbert's Finsler metrics on convex sets.

The other six talks covered other aspects of geometry including

- dynamics of topological holomorphic maps on 2-sphere and geodesics of the Weil-Petersson metric,
- a cohomogeneity one example of a positively curved manifold and Einstein / Ricci-soliton solvmanifolds,
- a discrete analogue of conformal equivalence and isoparametric hypersurfaces.

Several connections between the different areas became apparent during the workshop. For example, the initial value problem for the Ricci flow with singular initial data is closely linked to smoothing problems occuring in Alexandrov geometry.

Workshop: Geometrie

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Abstracts

Relative Expander Monotonicity and MCF with Singular Initial Data

Tom Ilmanen

Let M_0 be a hypersurface in $\mathbb{R}^{n+1}$ with isolated point singularities modeled on regular hypercones. Consider a mean curvature flow $(M_t)_{t\geq 0}$ with initial data M_0. We have the following questions:
(1) Does M_t evolve by self-similar expansion aymptotically near each singular point p?
(2) Is M_t smooth for a short time?
These question are answered affirmatively for $n \leq 6$ by the following theorems.

Theorem 1. *(all n) Any tangent flow $(P_t)_{t\geq 0}$ to $(M_t)_{t\geq 0}$ at p has the form*

$$P_t = \sqrt{t}\cdot P, \qquad t > 0.$$

Such a P is called an *expander* and solves the elliptic equation

$$\vec{H} - \frac{x^\perp}{2} = 0, \qquad x \in P,$$

which is the Euler-Lagrange equation of the functional

$$K(P) := \int_P e^{|x|^2/4}.$$

Theorem 2. *(all n) The flow $(M_t)_{t\geq 0}$ can be constructed so that the expander P minimizes K with respect to compact replacements.*

Theorem 3. *($n \leq 6$) The flows of Theorem 2 are smooth for $0 < t < \varepsilon = \varepsilon(M_0)$.*

The proofs employ a *relative forward monotonicity formula*

$$\frac{d}{dt}K^t(M_t \cap B_{R(t)}) = -\int_{M_t \cap B_{R(t)}} \phi \left|\vec{H} - \frac{x^\perp}{2}\right|^2 + o(1),$$

where

$$K^t(A) := \int_A \phi(x,t), \qquad \phi(x,t) := \frac{e^{|x|^2/4t}}{t^{n/2}},$$

and $R(t) \to 0$ is chosen suitably.

Currents in metric spaces

Urs Lang

An m-dimensional de Rham current in an open set $U \subset \mathbb{R}^n$ is a real-valued linear function on the space of compactly supported differential m-forms on U, continuous with respect to convergence of forms in a suitable C^∞-topology. In [1], L. Ambrosio and B. Kirchheim developed an elegant theory of currents with finite mass in complete metric spaces, employing $(m+1)$-tuples of real-valued Lipschitz functions in place of differential m-forms. The talk presented a variant of the theory that does not rely on a finite mass condition, exposed in detail in [3].

Throughout this note, X stands for a locally compact metric space. We denote by $\mathcal{D}^m(X)$ the set of $(m+1)$-tuples $(f, \pi_1, \ldots, \pi_m)$ of real-valued functions on X, where f is Lipschitz with compact support $\mathrm{spt}(f)$ and $\pi_1, \ldots, \pi_m$ are locally Lipschitz. The guiding principle is that if $X = U$ is an open subset of $\mathbb{R}^n$ and if $(f, \pi) = (f, \pi_1, \ldots, \pi_m) \in C^\infty_{\mathrm{c}}(U) \times [C^\infty(U)]^m$, then this tuple represents the form $f\, d\pi_1 \wedge \ldots \wedge d\pi_m$. (This correspondence is made rigorous in Theorem 2 below.) An m-dimensional *metric current* T in X is defined as an $(m+1)$-linear real-valued function on $\mathcal{D}^m(X)$, continuous with respect to convergence of tuples in a suitable topology involving locally uniform bounds on Lipschitz constants, and satisfying $T(f, \pi_1, \ldots, \pi_m) = 0$ whenever some π_i is constant on a neighborhood of $\mathrm{spt}(f)$. The vector space of m-dimensional metric currents in X is denoted by $\mathcal{D}_m(X)$. The defining conditions give rise to a set of further properties, corresponding to the usual rules of calculus for differential forms. Every $T \in \mathcal{D}_m(X)$ is alternating in the m last arguments and satisfies the following product rule: If $(f, \pi) \in \mathcal{D}^m(X)$, and if $g \colon X \to \mathbb{R}$ is locally Lipschitz, then

$$T(f, g\pi_1, \pi_2, \ldots, \pi_m) = T(fg, \pi_1, \ldots, \pi_m) + T(f\pi_1, g, \pi_2, \ldots, \pi_m).$$

Moreover, a chain rule holds, a special case of which states that if $(f, \pi) \in \mathcal{D}^m(X)$ and $g \in [C^{1,1}(\mathbb{R}^m)]^m$, i.e., all partial derivatives $D_k g_i$ are locally Lipschitz, then

$$T(f, g \circ \pi) = T(f \det((Dg) \circ \pi), \pi).$$

Every function $u \in L^1_{\mathrm{loc}}(U)$ on an open set $U \subset \mathbb{R}^m$ induces a metric current $[u] \in \mathcal{D}_m(U)$ satisfying

$$[u](f, g) = \int_U uf \det(Dg)\, dx$$

for all $(f, g) = (f, g_1, \ldots, g_m) \in \mathcal{D}^m(U)$. This corresponds to the integration of a simple m-form over U.

For every metric current T there is a smallest closed set in X, denoted by $\mathrm{spt}(T)$, such that $T(f, \pi)$ depends only on the restrictions of f and π to this set. (This would allow to define, more generally, currents with locally compact support in arbitrary metric spaces.) For a classical m-current $\bar{T}$, the boundary is defined so that $\partial\bar{T}(\phi) = \bar{T}(d\phi)$ for every compactly supported $(m-1)$-form ϕ.

Correspondingly, the *boundary* of a metric current $T \in \mathcal{D}_m(X)$ is the $(m-1)$-current $\partial T \in \mathcal{D}_{m-1}(X)$ satisfying

$$\partial T(f, \pi_1, \dots, \pi_{m-1}) = T(\sigma, f, \pi_1, \dots, \pi_{m-1})$$

for all $(f, \pi_1, \dots, \pi_{m-1}) \in \mathcal{D}^{m-1}(X)$ and for all σ such that $\sigma = 1$ on some neighborhood of $\operatorname{spt}(f)$. We have $\partial \circ \partial = 0$. Given $T \in \mathcal{D}_m(X)$, another locally compact metric space Y, and a locally Lipschitz map $F\colon X \to Y$ such that $F|_{\operatorname{spt}(T)}$ is proper, the *push-forward* $F_\# T \in \mathcal{D}_m(Y)$ is defined so that

$$F_\# T(f, \pi) = T(\tilde{f}, \pi \circ F)$$

whenever $(f, \pi) \in \mathcal{D}^m(Y)$ and $\tilde{f}\colon X \to \mathbb{R}$ is a compactly supported Lipschitz function that agrees with $f \circ F$ on $\operatorname{spt}(T)$.

Given a metric m-current T, we define its *mass* $\mathbf{M}_V(T)$ in an open set $V \subset X$ as the least number $M \in [0, \infty]$ such that

$$\sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) \le M$$

whenever Λ is a finite set, $(f_\lambda, \pi^\lambda) \in \mathcal{D}^m(X)$, $\pi_1^\lambda, \dots, \pi_m^\lambda$ are 1-Lipschitz, $\operatorname{spt}(f_\lambda) \subset V$, and $\sum_{\lambda \in \Lambda} |f_\lambda| \le 1$. There is a Borel regular outer measure $\|T\|$ on X such that $\|T\|(V) = \mathbf{M}_V(T)$ for all open sets $V \subset X$. If $\|T\|$ is locally finite, then $\|T\|$ is a Radon measure, and

$$T(f, \pi) \le \int_X |f| \, d\|T\|$$

whenever $(f, \pi) \in \mathcal{D}^m(X)$ and the restrictions of $\pi_1, \dots, \pi_m$ to $\operatorname{spt}(f)$ are 1-Lipschitz. This last inequality allows to extend T to all tuples (f, π) such that f is a bounded Borel function with compact support and $\pi_1, \dots, \pi_m$ are still locally Lipschitz. For a Borel set $B \subset X$, the restriction $T \lfloor B$ is then defined as the m-current satisfying

$$(T \lfloor B)(f, \pi) = T(\chi_B f, \pi)$$

for all $(f, \pi) \in \mathcal{D}^m(X)$, where χ_B is the characteristic function of B.

An m-current T is called *locally normal* if the quantity $\mathbf{N}_V(T) := \mathbf{M}_V(T) + \mathbf{M}_V(\partial T)$ ($\mathbf{N}_V(T) := \mathbf{M}_V(T)$ in case $m = 0$) is finite for every open set $V \Subset X$ (with compact closure). The following compactness theorem holds.

Theorem 1. *Suppose that $T_1, T_2, \ldots \in \mathcal{D}_m(X)$ are locally normal currents such that $\operatorname{spt}(T_k)$ is separable for every k and $\sup_k \mathbf{N}_V(T_k) < \infty$ for every open set $V \Subset X$. Then there is a subsequence $T_{k(1)}, T_{k(2)}, \ldots$ that converges weakly to some $T \in \mathcal{D}_m(X)$, i.e., $\lim_{i\to\infty} T_{k(i)}(f, \pi) = T(f, \pi)$ for every $(f, \pi) \in \mathcal{D}^m(X)$.*

Since $\mathbf{M}_V$ is lower semicontinuous with respect to weak convergence, and also $\partial T_{n(i)} \to \partial T$ weakly, the limit T is locally normal. In the context of classical currents, there is a similar compactness theorem for currents with locally finite mass. Such a result is not available for metric currents. For instance, if $u_1, u_2, \ldots$ is a suitable sequence of mollifiers on $\mathbb{R}^m$, the corresponding currents $[u_k] \in \mathcal{D}_m(\mathbb{R}^m)$ satisfy $\mathbf{M}([u_k]) = 1$ and $\operatorname{spt}([u_k]) \subset B(0, 1/k)$ for all k. No subsequence converges weakly to a current $\mathcal{D}_m(\mathbb{R}^m)$, for there is no metric m-current for $m \ge 1$ whose support is a point. This is the first indication that the class $\mathcal{D}_m(U)$ of metric currents in an open set $U \subset \mathbb{R}^n$ does not correspond exactly to the space of general classical (de Rham) currents $\mathcal{D}_m^{\mathrm{dR}}(U)$. However, the following comparison theorem shows, in particular, that the former constitutes a fairly large subclass of the latter, and that the correspondence is precise for locally normal currents. For $\bar{T} \in \mathcal{D}_m^{\mathrm{dR}}(U)$, $\|\bar{T}\|$ denotes its variation measure, and $\mathbf{F}_m^{\mathrm{loc}}(U)$ is the space of locally flat chains in U, as defined in [2].

Theorem 2. *Let $U \subset \mathbb{R}^n$ be an open set, $n \geq 1$. For every $m \geq 0$, there exists an injective linear map $C_m \colon \mathcal{D}_m(U) \to \mathcal{D}_m^{\mathrm{dR}}(U)$ such that*

$$C_m(T)(f\, dg_1 \wedge \ldots \wedge dg_m) = T(f, g_1, \ldots, g_m)$$

for all $(f, g_1, \ldots, g_m) \in C_c^\infty(U) \times [C^\infty(U)]^m$. The following properties hold:

(1) *For $m \geq 1$, $\partial \circ C_m = C_{m-1} \circ \partial$.*
(2) *For all $T \in \mathcal{D}_m(U)$, $\|T\| \leq \|C_m(T)\| \leq \binom{n}{m} \|T\|$.*
(3) *The restriction of C_m to the space of metric locally normal currents is an isomorphism onto the space of classical locally normal currents.*
(4) *The image of C_m contains $\mathbf{F}_m^{\mathrm{loc}}(U)$.*

It is an open problem whether or not $C_m(\mathcal{D}_m(U)) = \mathbf{F}_m^{\mathrm{loc}}(U)$.

We call a current $T \in \mathcal{D}_m(X)$ *locally rectifiable* if the measure $\|T\|$ is locally finite and concentrated on some countably m-rectifiable set E (the union of countably many Lipschitz images of subsets of $\mathbb{R}^m$), and if T satisfies the following integrality condition: Whenever $B \subset X$ is a Borel set with compact closure and $\pi \colon X \to \mathbb{R}^m$ is Lipschitz, then $\pi_\#(T \lfloor B) = [u_{B,\pi}]$ for some $u_{B,\pi} \in L^1(\mathbb{R}^m, \mathbb{Z})$. These properties also ensure that $\mathrm{spt}(T)$ is separable and that $\|T\|$ is absolutely continuous with respect to m-dimensional Hausdorff measure. As in the classical theory, T is called a *locally integral current* if both T and ∂T are locally rectifiable (every such T is locally normal). The following boundary rectifiability theorem yields a simpler criterion.

Theorem 3. *If $T \in \mathcal{D}_m(X)$ is locally rectifiable, $m \geq 1$, and if $\|\partial T\|$ is locally finite, then ∂T is locally rectifiable, so that T is a locally integral current.*

The compactness theorem for locally integral currents is now valid in arbitrary locally compact metric spaces.

Theorem 4. *Suppose that $T_1, T_2, \ldots \in \mathcal{D}_m(X)$ are locally integral currents such that $\sup_k \mathbf{N}_V(T_k) < \infty$ for every open set $V \Subset X$. Then there is a subsequence $T_{k(1)}, T_{k(2)}, \ldots$ that converges weakly to some locally integral current $T \in \mathcal{D}_m(X)$.*

This result allows to solve various generalized Plateau problems. An application to the asymptotic geometry of nonpositively curved metric spaces has been obtained in joint work (in preparation) of B. Kleiner and the author.

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Ricci flow in higher dimensions

Christoph Böhm
(joint work with Burkhard Wilking)

On a compact smooth n-dimensional manifold M^n the Ricci flow is the geometric evolution equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) \ , \quad g(0) = g_0$$

for a one-parameter family $g(t)_{t\in I}$ of Riemannian metrics on M^n.

The curvature operator R_p of a Riemannian manifold (M^n, g) at a point p is a selfadjoint endomorphism of the second exterior product $\Lambda^2 T_pM^n$. If the curvature operator of an initial metric g_0 is at any point "almost" a positive multiple of the identity then the (volume) normalized Ricci flow will converge to a metric of constant positive sectional curvature (see e.g. [4], [5], [1], [2]). As a consequence the underlying manifold must be a spherical space form.

We are interested in curvature conditions C which are invariant under Ricci flow. That is, if the curvature operator of an initial metric satisfies C (at any point), then also the curvature operators of the evolved metrics $g(t)$ will do so. On the one hand the curvature condition C should be so "large" that the (volume) normalized Ricci flow doesn't necessarily converge to a metric of constant positive sectional curvature but may develop singularities. On the other hand C should be "small" enough to be able to "classify" these singularities. For instance positive isotropic curvature is an invariant curvature condition due to independent work of Nguyen [8] and Brendle and Schoen [2] – the four-dimensional case is due to Hamilton [6]. The final hope is to introduce Ricci flow with surgery for initial metrics satisfying C as done by Perelman in dimension three for arbitrary initial metrics.

In dimensions above three the curvature operator of a Riemannian manifold (M^n, g) is not an arbitrary element of the vector space $S^2(\Lambda^2 T_pM^n)$ of selfadjoint endomorphisms but satisfies the first (linear) Bianchi identity. The space $S^2_B(\Lambda^2 T_pM^n)$ of algebraic curvature operators can be decomposed under the natural action of the orthogonal group $O(T_pM^n)$ into three irreducible submoduls:

$$S^2_B(\Lambda^2 T_pM^n) \quad = \quad \langle \mathrm{Id} \rangle \oplus \langle \mathrm{Ric}_0 \rangle \oplus \langle \mathrm{W} \rangle \, .$$

Using this we write $R = R_{\mathrm{Id}} + R_{\mathrm{Ric}_0} + R_{\mathrm{W}}$. (Recall that $2\mathrm{tr}R = 2\mathrm{tr}R_{\mathrm{Id}}$ equals to the scalar curvature of R.)

We view any closed, convex, $O(n)$-invariant subset C of $S^2_B(\Lambda^2 T_pM^n)$ as a curvature condition. By Hamiltons maximum principle, C defines a Ricci flow invariant curvature condition provided that it is invariant under the ODE

$$\frac{d}{dt} R(t) = R^2(t) + R^{\#}(t)$$

where $R^{\#}(t)$ denotes a Lie-theoretically defined square of $R(t)$ (see e.g. [1]).

For $d > 0$ let

$$C_d := \{R \in S^2_B(\Lambda^2 T_p M^n) \mid \mathrm{scal}(R) > d \cdot \|R_{\mathrm{W}}\|\}\,.$$

Moreover, let

$$d(n) := \sqrt{2(n-2)(n-1)}\,.$$

Theorem 1. *There exists $n_0 \in \mathbb{N}$, such that in all dimensions $n \geq n_0$ the following holds true:*

(1) *If n is even then $\overline{C_{d(n)}}$ is invariant under the Ricci flow.*
(2) *If n is odd then $\overline{C_{d(n)+\epsilon(n)}}$ is invariant under the Ricci flow for $\epsilon(n) \in [-\epsilon_1(n), \epsilon_2(n)]$.*

The explicitly known numbers $\epsilon_1(n), \epsilon_2(n)$ are positive and will converge to zero for $n \to \infty$. We conjecture that $\overline{C_d}$ being invariant under the Ricci flow implies that d must be one of the above stated ones. Moreover, we conjecture that the above theorem holds true for $n_0 = 12$.

Corollary 1. *There exists $n_0 \in \mathbb{N}$, such that in all dimensions $n \geq n_0$ the following holds true: Let (M^n, g) be a simply connected Einstein manifold. Then:*

(1) *If $n = 2m$ is even and $R \in \overline{C_{d(n)}}$ then either (M^n, g) is isometric to (S^n, g_{stand}) or to $(S^m \times S^m, g_{stand})$.*
(2) *If $n = 2m+1$ is odd and $R \in \overline{C_{d(n)-\epsilon_1(n)}}$ then either (M^n, g) is isometric to (S^n, g_{stand}) or to $(S^m \times S^{m+1}, g_{stand})$.*

We turn to the class $\mathcal{M}_{d(n)}$ of closed manifolds M^n which admit a Riemannian metric g with $R \in C_{d(n)}$. It is not hard to see that spherical space form bundles $F^k \to M^n \to B^{n-k}$ belong to $\mathcal{M}_{d(n)}$ provided that $k > n-k$ and that the structure group of the bundle is contained in the isometry group of the fibre. This shows already that $\mathcal{M}_{d(n)}$ is much larger then the class of spherical space forms. The next theorem shows that the class $\mathcal{M}_{d(n)}$ of manifolds has also very nice topological properties:

Theorem 2. *The class $\mathcal{M}_{d(n)}$ of closed manifolds is invariant under surgery of codimension $> \frac{n}{2} + 1$.*

Recall that the class of closed manifolds which admit Riemannian metrics of positive scalar curvature, that is $R \in C_0$, is invariant under surgery of codimension ≥ 3 due to Gromov and Lawson [3] and Schoen and Yau [9]. Also by the work of Micallef and Wang [7] the space of closed manifolds M^n which admit a Riemannian metric of positive isotropic curvature is invariant under surgery of codimension $\geq n$, that is one can form connected sums.

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Conformal equivalence of triangulated surfaces

Ulrich Pinkall
(joint work with Peter Schröder, Boris Springborn)

A metric l on a combinatorial triangulated surface M assigns to every edge between adjacent vertices i, j a positive number

$$l_{ij} = e^{\lambda_{ij}/2} \tag{1}$$

such that the triangle inequalities hold for each triangle. Two metrics on the same combinatorial surface are called *conformally equivalent* if there is a function u on the vertex set for which

$$\begin{aligned} \tilde{l}_{ij} &= e^{(u_i+u_j)/2} l_{ij} \\ \tilde{\lambda}_{ij} &= \lambda_{ij} + u_i + u_j \end{aligned} \tag{2}$$

It is not difficult to see that l and $\tilde{l}$ are conformally equivalent if and only if for each interior edge (i, j) the cross ratios

$$cr_{ij} = \frac{l_{ih} l_{jk}}{l_{ik} l_{jh}} \tag{3}$$

coincide:

$$\tilde{cr}_{ij} = cr_{ij} \tag{4}$$

A conformal structure on a combinatorial surface M (a conformal equivalence class of metrics) is therefore described by an assignment of $cr_{ij} > 0$ to each edge such that for each vertex i

$$\prod cr_{ij} = 1$$

The dimension of the moduli space of conformal structures on a compact triangulated surface of genus g with vertex set V turns out to be $2|V| + 6g - 6$, which is the same as the dimension of the Teichmüller space of compact Riemann surfaces of genus g with $|V|$ punctures.

Another strong indication of the adequacy of the proposed notion of conformal equivalence is the fact that for triangulated surfaces embedded in $\mathbb{R}^n$ Möbius transformations g of the ambient space induce conformal changes of the metric. Since it suffices to check this for the case that g is the inversion in the unit sphere, this follows from

$$\left|\frac{p}{|p|^2} - \frac{q}{|q|^2}\right| = \frac{1}{|p|}\cdot\frac{1}{|q|}\,|p-q| \tag{5}$$

An important task is to find within a conformal class a metric with prescribed cone angles at each vertex. (The cone angle α_i at a vertex i is the sum of the angles at i of all triangles adjacent to i.) This demand results in a set of highly nonlinear equations for the conformal factors u_i. Fortunately, these equations can be rephrased as a variational problem: A function u on the vertex set solves these equations if and only if u is a critical point of

$$\begin{aligned} E(u) = \sum_{t_{ijk}\in T} \quad & \tilde{\alpha}^i_{jk}\lambda_{jk} + \tilde{\alpha}^j_{ki}\lambda_{ki} + \tilde{\alpha}^k_{ij}\lambda_{ij} - \pi(u_i+u_j+u_k) \\ & + 2\big(\mathrm{L}(\tilde{\alpha}^i_{jk}) + \mathrm{L}(\alpha^j_{ki}) + \mathrm{L}(\tilde{\alpha}^k_{ij})\ \big) \\ + \sum_{v_i\in V}\alpha_i u_i & \end{aligned} \tag{6}$$

It turns out that E is a strictly convex function of u, but due to the triangle inequalities the domain of definition of E is not convex. However, under mild conditions on the cone angles α_i it turns out that E can be extended to a proper convex function on the whole of $\mathbb{R}^{|V|}$.

This allows us to conclude that there indeed exists a minimum u of E on the extended domain. If the triangle inequalities are satisfied for u, then u is unique and provides a conformal metric with the desired cone angles.

The above yields an efficient numerical algorithm for producing conformally equivalent metrics with prescribed cone angles. The case of outstanding practical importance for texture mapping in computer graphics is when all but a few cone angles are 2π, in which case (after inserting suitable cuts) we get conformal maps into the euclidean plane.

We finally describe an approach that avoids problems with triangle inequalities by eliminating the fixed combinatorics from the picture:

Let the initial metric triangulation define a flat metric on a compact 2-manifold M with boundary that has finitely many cone singularities. Then choose a Delaunay triangulation of M (interiors of circumcircles contain no other vertices). This triangulation then yields lengths l_{ij} and cross ratios cr_{ij} as decribed above. Now let each triangle inherit from its circumcircle the metric of an ideal hyperbolic triangle (viewed in the Klein model). The given crossratios allow to glue all these triangles together to obtain a complete hyperbolic metric on $M - V$ with cusps at the vertices.

Definition 1. *Two flat metrics with cone points* $v_1, \ldots, v_n$ *on a compact 2-manifold are called* conformally equivalent *if the corresponding complete hyperbolic metrics on* $M - \{v_1, \ldots, v_n\}$ *with cusps at* $v_1, \ldots, v_n$ *are isometric.*

Using a theorem of Rivin [1] we then can prove the following uniformization result.

Theorem 1. *Every flat metric on* $S^2 - \{v_1, \ldots, v_n\}$ *with cone points at* $v_1, \ldots, v_n$ *is conformally equivalent to the boundary of a convex polyhedron in* $\mathbb{R}^3$ *with vertices on* S^2 *(unique up to Moebius transformations)*

References

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Ricci flow with surgery on open 3-manifolds and positive scalar curvature

Sylvain Maillot
(joint work with Laurent Bessières, Gérard Besson)

All 3-manifolds considered here are *smooth, orientable, and without boundary.* A Riemannian metric is said to have *bounded geometry* if it has bounded sectional curvature, and injectivity radius bounded away from zero.

Theorem 1. *Let* M *be a (possibly noncompact)* 3*-manifold. If* M *admits a complete metric* g_0 *of bounded geometry and scalar curvature* ≥ 1*, then* M *is a (possibly infinite) connected sum of spherical space forms and copies of* $S^2 \times S^1$.

When M is compact, this was proven by Perelman [4], completing earlier work of Schoen-Yau [5] and Gromov-Lawson [1]. Our proof is based on a generalization of Perelman's work.

The talk focused on the special case where M is open and irreducible. In this case, the conclusion simply asserts that M is diffeomorphic to $\mathbb{R}^3$, and is a consequence from the following result:

Theorem 2. *Assume that* M *is open and irreducible. If* M *is not diffeomorphic to* $\mathbb{R}^3$*, then for every* $T > 0$*, for every complete metric of bounded geometry* g_0 *on* M*, there exists a* 1*-parameter family* $\{g(t)\}_{t\in[0,T]}$ *of complete Riemannian metrics of bounded geometry on* M*, such that* $g(0) = g_0$*, and satisfying the following two conditions:*

(1) *The evolution is by Ricci flow except for a finite number of values of* t*, called* singular times*, where the evolution is discontinuous;*

(2) *if t_0 is a singular time, then $g(t)$ is continuous from the left, and admits a right-limit $g_+(t_0)$, whose infimal scalar curvature is greater than or equal to that of $g(t_0)$.*

It is a well-known fact, following from the maximum principle, that any complete Ricci flow of bounded curvature satisfies an *a priori* lower scalar curvature bound, which implies that the solution blows up before time $3/2$ if the initial condition has scalar curvature ≥ 1. Condition (2) therefore implies that if g_0 has scalar curvature ≥ 1, then any family of metrics produced by Theorem 2 satisfies the same bound, hence cannot exist on $[0, 3/2]$. This shows that Theorem 1 in the open, irreducible case follows from Theorem 2.

The proof of Theorem 2 relies heavily on previous work of Hamilton and Perelman on the Ricci flow. In extending Perelman's proof to the noncompact setting, one faces several new difficulties: first, the volume is infinite, so Perelman's argument to rule out accumulation of surgeries breaks down; second, there may be a time T such that Ricci flow $g(t)$ is defined and has bounded curvature for all $t < T$, but as $t \to T$, $g(t)$ converges in the $\mathcal{C}^\infty_{\mathrm{loc}}$-topology to a metric of unbounded curvature.

To overcome those difficulties, we use a surgery procedure different from Perelman's, where surgery is done before the singular time rather that at the singular time (compare [2, 3].)

In order to prove Theorem 1 in the general case, one needs a more general notion, where the manifold (and not only the metric) may evolve with time. At surgery times, surgery may break off connected sums, and some components whose topology has been recognized are thrown away. The lower scalar curvature bound ensures *finite extinction*, i.e. for some finite time t_0 we are left with the empty set. One can then reconstruct the topology of M.

Remark. Through conversations shortly before and during the meeting, it gradually became clear to the author that Theorem 1 can be improved in two distinct ways. Firstly, the conclusion can be strengthened by imposing that there are finitely many summands up to diffeomorphism (excluding, e.g. a connected sum of an infinite sequence of pairwise non-homeomorphic Lens spaces.) This is very nice, since the converse of this strengthened version is easily shown to hold.

Secondly, the conclusion of Theorem 1 (in its original weak form) still holds under the weaker hypothesis that the universal cover of (M, g_0) has bounded geometry. This is because surgery can be done equivariantly (cf. J. Dinkelbach's lecture in these proceedings.)

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Complex dynamics and quasiconformal geometry

Mario Bonk
(joint work with Daniel Meyer)

A continuous map $f : S^2 \to S^2$ on a topological 2-sphere S^2 is called *topologically holomorphic* if near each point $p \in S^2$ it can be written as $z \mapsto z^k$ near 0 with $k \in \mathbb{N}$ in suitable local coordinates in source and target. The point p is called a *critical point* of such a map f if $k \geq 2$, that is, if f is not locally injective near p. The *postcritical set* P_f of f is the union of all orbits of critical points under forward-iteration of f. So

$$P_f := \bigcup_{n=1}^{\infty} \{f^n(p) : p \in S^2 \text{ critical point of } f\},$$

where f^n denotes the n-th iterate of f. If P_f is a finite set, then f is called *postcritically finite*. Thurston investigated such maps and found a necessary and sufficient condition when a postcritically finite map is "equivalent" (in a suitable sense) to a rational map on the Riemann sphere [2].

For "expanding" postcritically finite maps there is an interesting relation to Cannon's conjecture about Gromov hyperbolic groups whose boundary at infinity is a topological 2-sphere [1]. In my talk I reported about some recent developments in this area.

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Ricci soliton solvmanifolds

Jorge Lauret

Let M be a differentiable manifold. The question of whether there is a 'best' Riemannian metric on M is intriguing. A great deal of deep results in Riemannian geometry have been motivated, and even inspired, by this single natural question. For several good reasons, an Einstein metric is a good candidate, if not the best, at least a very distinguished one (see [1, Chapter 0]). The Einstein condition is very subtle, even when restricted to almost any subclass of metrics on M one may like. It is too strong to allow general existence results, and sometimes even just to find a single example, and at the same time, it is too weak to get obstructions or classification results.

The latest fashion generalization of Einstein metrics, although they were introduced by R. Hamilton more than twenty years ago, is the notion of *Ricci soliton*:

$$\mathrm{ric}_g = cg + L_X g, \qquad \text{for some } c \in \mathbb{R}, \quad X \in \chi(M), \tag{1}$$

where $L_X g$ is the usual Lie derivative of g in the direction of the field X. A more intuitive equivalent condition to (1) is that ric_g is tangent at g to the space of all metrics which are homothetic to g (i.e. isometric up to a constant scalar multiple). Recall that Einstein means ric_g tangent to $\mathbb{R}_{>0}g$. Ricci solitons correspond to solutions of the Ricci flow

$$\tfrac{\mathrm{d}}{\mathrm{d}t} g(t) = -2\,\mathrm{ric}_{g(t)},$$

that evolves self similarly, that is, only by scaling and the action by diffeomorphisms, and often arise as limits of dilations of singularities of the Ricci flow. We refer to [2] and the references therein for further information on the Hamilton-Perelman theory of Ricci flow and Ricci solitons.

In the homogeneous case, it is known that only expanding (i.e $c < 0$) non-Einstein Ricci solitons are allowed. Examples can be obtained by considering a left invariant metric on a Lie group with Ricci operator $\mathrm{Ric} = cI + D$ for some $c \in \mathbb{R}$ and some derivation D of the Lie algebra (see [5]). It is easy to see that this neat algebraic condition is also necessary if the Lie algebra is completely solvable (i.e. the eigenvalues of each $\mathrm{ad}\, X$ are all real numbers), and it is proved in [9] that the same holds for any Lie group. To this day, mostly nilpotent Lie groups admitting one of these special metrics had been found, which are called *nilsolitons* and are intimately related to Einstein solvmanifolds (see the survey [8]).

Let S be a solvmanifold, that is, a simply connected solvable Lie group endowed with a left invariant Riemannian metric. S will be often identified with its metric Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$, where $\mathfrak{s}$ is the Lie algebra of S and $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathfrak{s}$ which determines the metric. We consider the orthogonal decomposition

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n},$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$ (i.e. maximal nilpotent ideal). The *mean curvature vector* of S is the only element $H \in \mathfrak{a}$ such that $\langle H, A\rangle = \mathrm{tr}\,\mathrm{ad}\, A$ for any $A \in \mathfrak{a}$. If B denotes the symmetric map defined by the Killing form of $\mathfrak{s}$ relative to $\langle\cdot,\cdot\rangle$ (i.e. $\langle BX, X\rangle = \mathrm{tr}\,(\mathrm{ad}\, X)^2$ for all $X \in \mathfrak{s}$) then $B(\mathfrak{a}) \subset \mathfrak{a}$ and $B|_{\mathfrak{n}} = 0$. The Ricci operator Ric of S is given by (see for instance [1, 7.38]):

$$\mathrm{Ric} = R - \tfrac{1}{2}B - S(\mathrm{ad}\, H), \tag{2}$$

where $S(\mathrm{ad}\, H) = \frac{1}{2}(\mathrm{ad}\, H + (\mathrm{ad}\, H)^t)$ is the symmetric part of $\mathrm{ad}\, H$ and R is the symmetric operator defined by

$$\langle RX, X\rangle = -\tfrac{1}{2}\sum \langle [X, X_i], X_j\rangle^2 + \tfrac{1}{4}\sum \langle [X_i, X_j], X\rangle^2, \qquad \forall X \in \mathfrak{s}, \tag{3}$$

where $\{X_i\}$ is any orthonormal basis of $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$.

Remark. If $\mathfrak{s}$ is nilpotent then $\operatorname{Ric} = R$ and hence the scalar curvature is simply given by $\operatorname{tr}\operatorname{Ric} = -\frac{1}{4}||[\cdot,\cdot]||^2$.

It follows from [6, Propositions 3.5, 4.2] that this anonymous tensor R in the formula of the Ricci operator satisfies

$$m([\cdot,\cdot]) = \tfrac{4}{||[\cdot,\cdot]||^2}R, \tag{4}$$

where $m : \Lambda^2\mathfrak{s}^* \otimes \mathfrak{s} \longrightarrow \operatorname{sym}(\mathfrak{s})$ is the moment map for the natural action of $\operatorname{GL}(\mathfrak{s})$ on $\Lambda^2\mathfrak{s}^* \otimes \mathfrak{s}$. In other words, R may be alternatively defined as follows:

$$\operatorname{tr} RE = \tfrac{1}{4}\langle \pi(E)[\cdot,\cdot],[\cdot,\cdot]\rangle, \qquad \forall E \in \operatorname{End}(\mathfrak{s}), \tag{5}$$

where we are considering the Lie bracket $[\cdot,\cdot]$ of $\mathfrak{s}$ as a vector in $\Lambda^2\mathfrak{s}^* \otimes \mathfrak{s}$ and π is the corresponding representation of $\mathfrak{gl}(\mathfrak{s})$ on $\Lambda^2\mathfrak{s}^* \otimes \mathfrak{s}$.

The following more explicit formula for the Ricci operator of S follows from a straightforward computation by using (2) and (3):

$$\begin{aligned}
\langle \operatorname{Ric} A, A\rangle &= -\tfrac{1}{2}\sum ||[A,A_i]||^2 - \operatorname{tr} S(\operatorname{ad} A|_{\mathfrak{n}})^2,\\
\langle \operatorname{Ric} A, X\rangle &= -\tfrac{1}{2}\sum\langle [A,A_i],[X,A_i]\rangle - \tfrac{1}{2}\operatorname{tr}(\operatorname{ad} A|_{\mathfrak{n}})^t \operatorname{ad} X|_{\mathfrak{n}}\\
&\quad -\tfrac{1}{2}\langle [H,A],X\rangle, \qquad (6)\\
\langle \operatorname{Ric} X, X\rangle &= \tfrac{1}{4}\sum\langle [A_i,A_j],X\rangle^2 + \tfrac{1}{2}\sum\langle [\operatorname{ad} A_i|_{\mathfrak{n}},(\operatorname{ad} A_i|_{\mathfrak{n}})^t](X),X\rangle\\
&\quad -\tfrac{1}{2}\sum\langle [X,X_i],X_j\rangle^2 + \tfrac{1}{4}\sum\langle [X_i,X_j],X\rangle^2 - \langle [H,X],X\rangle,
\end{aligned}$$

for all $A \in \mathfrak{a}$ and $X \in \mathfrak{n}$, where $\{A_i\}$, $\{X_i\}$, are any orthonormal basis of $\mathfrak{a}$ and $\mathfrak{n}$, respectively. It is now clear from (6) that the simplification for Ric is really substantial under the assumptions $[\mathfrak{a},\mathfrak{a}] = 0$ (i.e. S *standard*) and $\operatorname{ad} A$ symmetric for all $A \in \mathfrak{a}$. This gives rise to the following natural construction of *solsolitons* (i.e. Ricci soliton solvmanifolds) starting from a nilsoliton.

Proposition 1. *Let $(\mathfrak{n},\langle\cdot,\cdot\rangle_1)$ be a nilsoliton, say with Ricci operator $\operatorname{Ric}_1 = cI + D_1$, $c < 0$, $D_1 \in \operatorname{Der}(\mathfrak{n})$, and consider $\mathfrak{a}$ any abelian Lie algebra of symmetric derivations of $(\mathfrak{n},\langle\cdot,\cdot\rangle_1)$. Then the solvmanifold S with Lie algebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ (semidirect product) and inner product given by*

$$\langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}} = \langle\cdot,\cdot\rangle_1, \qquad \langle\mathfrak{a},\mathfrak{n}\rangle = 0, \qquad \langle A,A\rangle = -\tfrac{1}{c}\operatorname{tr} A^2 \qquad \forall A \in \mathfrak{a},$$

is a solsoliton with $\operatorname{Ric} = cI + D$, where D is a derivation of $\mathfrak{s}$ defined by $D|_{\mathfrak{a}} = 0$, $D|_{\mathfrak{n}} = D_1 - \operatorname{ad} H|_{\mathfrak{n}}$, H the mean curvature vector of S. Furthermore, S is Einstein if and only if $D_1 \in \mathfrak{a}$.

The aim of this talk was to show that this very simple procedure actually yields all solsolitons up to isometry.

Theorem 1. [9] *Let S be a solvmanifold with metric Lie algebra $(\mathfrak{s},\langle\cdot,\cdot\rangle)$ and consider the orthogonal decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$. Then $\operatorname{Ric} = cI + D$ for some $c < 0$, $D \in \operatorname{Der}(\mathfrak{s})$ (i.e. S is a solsoliton) if and only if the following conditions hold:*

(i) $(\mathfrak{n}, \langle\cdot,\cdot\rangle|_{\mathfrak{n}\times\mathfrak{n}})$ *is a nilsoliton with Ricci operator* $\operatorname{Ric}_1 = cI + D_1$*, for some* $D_1 \in \operatorname{Der}(\mathfrak{n})$.
(ii) $[\mathfrak{a}, \mathfrak{a}] = 0$ *(i.e. S standard).*
(iii) $(\operatorname{ad} A)^t \in \operatorname{Der}(\mathfrak{s})$ *(or equivalently,* $[\operatorname{ad} A, (\operatorname{ad} A)^t] = 0$*) for all* $A \in \mathfrak{a}$.
(iv) $\langle A, A\rangle = -\frac{1}{c}\operatorname{tr} S(\operatorname{ad} A)^2$ *for all* $A \in \mathfrak{a}$.

By using [3, Proposition 2.5], we conclude that any solsoliton is isometric to one which is constructed as in the proposition above. In this way, we obtain structural results for solsolitons which are identical to those proved by J. Heber in [3] for Einstein solvmanifolds.

The proof of the theorem uses tools from geometric invariant theory based on the relationship with the moment map given in (4), mainly a GL-invariant stratification for $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, which was defined in [7] by adapting to this context the construction given in [4, Section 12] for reductive group representations over an algebraically closed field (see [8, Section 7] for a kind overview).

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Ricci flow of 3-manifolds with maximal volume growth, and curvature bounded from below

Miles Simon

In this talk we consider smooth complete Riemannian manifolds (M, g_0) with no boundary satisfying:

(a) $\sup_M |\operatorname{Riem}(g)| < \infty$
(b) $\operatorname{Ricci}(g) \geq -2$
(c) $\operatorname{vol}(B_1(x)) \geq v_0 > 0$ for all $x \in M$.

We show that a Ricci flow of such a Riemannian manifold exists for a short time interval $[0, T)$ where $T = T(v_0) > 0$. Note that T does depend on the value of the upper bound for the curvature (as it does in the case of Shi [1]).

Theorem 1. *Let* (M, g_0) *be a three (or two) manifold satisfying (a),(b),(c) above. Then there exists a* $T = T(v_0) > 0$ *and* $K = K(v_0) > 0$ *and a solution* $(M, g(t))_{t\in[0,T)}$ *to Ricci-flow satisfying*

$$\sup_M |\mathrm{Riem}(g(t))| \leq \frac{K^2}{t} \quad \forall \ \ t \in (0,T) \tag{1}$$

$$\mathrm{Ricci}(g(t)) \geq -K^2 \quad \forall \ \ t \in (0,T), \tag{2}$$

$$\mathrm{vol}(B_1(x,t)) \geq \frac{v_0}{2} > 0 \ \ \forall x \in M, \ \forall \ \ t \in (0,T). \tag{3}$$

What is happening? The diffusion term in the evolution of the curvature is competing (and winning) against the reaction term:

$$\frac{\partial}{\partial t} R = \Delta R + 2|\mathrm{Ricci}|^2.$$

As an application of this theorem we obtain (NOTATION: $d(g)(\cdot,\cdot)$ is the metric on M induced by (M,g)):

Corollary 1. *Let* $(M_i, \overset{i}{g}_0)$ *be a sequence of three (or two) manifolds satisfying the conditions (a),(b), and (c) above, and let* $(X, d, x) = \lim_{i\to\infty}(M_i, d(\overset{i}{g}_0), x_i)$ *be a pointed Gromov-Hausdorff limit of this sequence. Let* $(M_i, \overset{i}{g}(t))_{t\in[0,T)}$ *be the solutions to Ricci-flow coming from the theorem above. Then there exists a Hamilton limit solution* $(M, g(t), x)_{t\in(0,T)} := \lim_{i\to\infty}(M_i, \overset{i}{g}(t), x_i)_{t\in(0,T)}$ *satisfying* (1), (2),(3) *and* $(M, d(g(t)), p) \to (X, d, p)$ *in the Gromov-Hausdorff sense as* $t \to 0$ *(for any* $p \in M$*).*

Remark 1. One may think of this as a Ricci flow of the possibly singular space (X, d, p).

Remark 2. Gromov's Compactness theorem guarantees that after taking a subsequence (X, d, x) always exists.

Remark 3. If $(M^3, \overset{i}{g}_0)$ have $\mathrm{Ricci}(\overset{i}{g}_0) \geq -\varepsilon(i)$ where $\varepsilon(i) \to 0$ as $i \to \infty$, then we can show that the limit solution has $\mathrm{Ricci}(g(t)) \geq 0$ for all $t > 0$.

Remark 4. In earlier work [2], we required the condition of Remark 3.

Remark 5. It is possible to replace Ricci by sec in everything above.

Remark 6. A result (at the moment) for dimensions $n > 3$ is: replace $\mathrm{Ricci}(g_0) \geq -1$ in the theorem (corollary) by $\mathcal{R}(g_0) \geq 0$. Then the theorem (corollary) holds with $\mathcal{R}(g(t)) \geq 0$ for all $t \in (0,T)$.

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Weak convergence of curvature tensor

Nina Lebedeva
(joint work with Anton Petrunin)

Let M_n be a sequence of m-dimensional manifolds with curvature ≥ -1, Assume $M_n \xrightarrow{\text{GH}} A$ without collapse, i.e. $\dim A = m$.

For a point p in manifold or Alexandrov space let us denote the smoothed distance function by

$$\widetilde{\operatorname{dist}}_{B_r(p)} = \fint_{B_r(p)} \operatorname{dist}_x dx.$$

If $p \in A$ we can construct a lifting of the function $\widetilde{\operatorname{dist}}_{B_r(p)}$, namely we can choose sequences $p_n \in M_n$, such that $p_n \to p$ and take function $\widetilde{\operatorname{dist}}_{B_r(p_n)}$.

We will describe curvature of Riemannian manifold using co-sectional curvature. Namely given a simple bi-vector $x_1 \wedge x_2$ one can consider its dual $(n-2)$-vector $v_1 \wedge v_2 \wedge \cdots \wedge v_{n-2}$, i.e. such that $\langle v_i, x_j \rangle = 0$ and

$$x_1 \wedge x_2 \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_{n-2} = |x \wedge y|^2 \omega$$

where ω is the volume form on M.

Then define $\tilde{R}_M(v_1 \wedge \cdots \wedge v_{m-2}) \stackrel{\text{def}}{=} R_M(x_1 \wedge x_2)$.

Theorem 1. *Let M_n be a sequence of m-dimensional manifolds with curvature ≥ -1. Assume $M_n \xrightarrow{\text{GH}} A$ without collapse, i.e. $\dim A = m$. Let $\{p^k\}_{k=1}^{m-2} \in A$ be a set of points and choose approximating sets $p_n^i \in M_n$ such that $p_n^k \to p^k$ as $n \to \infty$. We denote gradients of corresponding smoothed distance functions by $v_n^k = \nabla \widetilde{\operatorname{dist}}_{B_r(p_n^k)}$. Then $\tilde{R}_{M_n}(v_n^1 \wedge \cdots \wedge v_n^{m-2})$ — the Riemann curvature of M_n — weakly converges to a locally finite sign-measure $\tilde{R}_A(v^1 \wedge \cdots \wedge v^{m-2})$.*

Corollary 1. *Assume A is an Alexandrov m-space which admits smoothing, i.e. there is a sequence of Riemannian m-manifolds M_n with uniform lower curvature bound such that $M_n \to A$. Then one can define a measure valued tensor field $\tilde{R}_A$ on A independent on the smoothing sequence.*

Corollary 2. *Let $\mathcal{R}(T)$ denotes the space of all algebraic curvature tensors on the space T. Assume $W \subset \mathcal{R}(T)$ is a convex subset which is invariant with respect to rotations of the space T and such that for any curvature tensor in W, its minimal sectional curvature is at least -1.*

Assume M_n is a sequence of Riemannian m-manifolds such that the curvature tensor at each point belongs to W. If $M_n \xrightarrow{\text{GH}} M$ without collapse, and M is a Riemannian manifold then its curvature tensor at each point belongs to W.

For example, let M_n be a sequence of Riemannian manifolds with non-negative curvature operator. If it converges to a Riemannian manifold M, then M also has non-negative curvature operator.

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Generalization of the reduced distance in the Ricci flow - monotonicity and applications

Jörg Enders

We consider the evolution of a family of complete oriented connected smooth Riemannian manifolds $(M^n, g(t))$, $t \in [0,T)$, with bounded curvature, by the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2Ric_{g(t)},$$

as first introduced by Hamilton [3]. In dimensions $n \geq 3$, this evolution generally develops singularities and unlike in the case of strong curvature positivity assumptions does not limit to a constant curvature metric (after renormalization). One of Perelman's main analytical contributions [9] to the study of those singularities is the monotone quantity of the reduced volume: For any $p \in M^n$ and $0 < t_0$ less than the first singular time T, the reduced distance l_{p,t_0} arises from a space-time version of Riemannian geometry adapted to the Ricci flow. It is a locally Lipschitz function on $M^n \times [0, t_0]$ and satisfies important partial differential inequalities. Those imply that the reduced volume based at (p, t_0)

$$\tilde{V}_{p,t_0}(\bar{t}) := \int_{M^n} \big(4\pi(t_0 - \bar{t})\big)^{-\frac{n}{2}} e^{-l_{p,t_0}(q,\bar{t})}\, dvol_{g(\bar{t})}(q)$$

is monotone nondecreasing in $\bar{t}$ along the Ricci flow on $[0, t_0]$. Moreover, $\tilde{V}_{p,t_0}(\bar{t})$ is constant in $\bar{t}$ if and only if $(M^n, g(t))$ is isometric to Euclidean space with the flat (non-evolving) metric.

We report on some of our work in [2], which is partially motivated by the following: Huisken's monotonicity formula for the mean curvature flow [5] and Perelman's entropy for compact Ricci flows give rise to general self-similar solutions in the equality case of the monotonicity. In case of the Ricci flow, those are gradient shrinking solitons, i.e. Riemannian manifolds (M^n, g) with a potential function $f : M^n \to \mathbb{R}$ satisfying

$$Ric_g + \nabla^g \nabla^g f - \frac{1}{2} g = 0.$$

These determine a Ricci flow (gradient shrinking soliton in canonical form) on (∞, T), which only evolves by shrinking in scale and changing by diffeomorphism, and were conjectured by Hamilton [4] to model singularities of type I.

We extend the reduced distance and volume to the singular time T. To do this, we introduce a new (mild) curvature bound, which is more general than the type I assumption: Let $T < \infty$. A complete $n-$dimensional Ricci flow $(M^n, g(t))$ on

$[0, T)$ is said to be of *type A* if there exist $C > 0$ and $r \in [1, \frac{3}{2})$ such that for all $t \in [0, T)$

$$\sup_{M^n} |Rm_{g(t)}|_{g(t)} \leq \frac{C}{(T-t)^r}.$$

We can then prove the following: Let $(M^n, g(t))$ be a complete $n-$dimensional Ricci flow on $[0, T)$ of type A. Also let $p \in M^n$ and $t_i \nearrow T$. Then there exists a locally Lipschitz *reduced distance based at singular time* $l_{p,T} : M^n \times (0, T) \to \mathbb{R}$, which is a subsequential limit

$$l_{p,t_i} \xrightarrow{C^0_{loc}(M^n \times (0,T))} l_{p,T}$$

and for all $(q, \bar{t}) \in M^n \times (0, T)$ satisfies the partial differential inequality

$$-\frac{\partial}{\partial \bar{t}} l_{p,T}(q, \bar{t}) - \Delta_{g(\bar{t})} l_{p,T}(q, \bar{t}) + |\nabla^{g(\bar{t})} l_{p,T}(q, \bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})}(q) + \frac{n}{2(T-\bar{t})} \geq 0$$

in the sense of distributions. (See also [7] for a similar result obtained independently under the type I assumption.)

For $\bar{t} \in (0, T)$, we now define a *reduced volume based at singular time* (p, T) by

$$\tilde{V}_{p,T}(\bar{t}) := \int_{M^n} \left((4\pi(T-\bar{t})\right)^{-\frac{n}{2}} e^{-l_{p,T}(q,\bar{t})} dvol_{g(\bar{t})}(q),$$

and show that any reduced volume based at singular time is monotone nondecreasing in $\bar{t}$. This uses the differential inequality in the theorem. Moreover, $\tilde{V}_{p,T}(\bar{t}) \leq 1$ for all $\bar{t} \in (0, T)$, and if $\tilde{V}_{p,T}(\bar{t})$ is constant in $\bar{t}$ then $(M^n, g(t), l_{p,T}(\cdot, t))$ is a gradient shrinking soliton in canonical form (which is normalized for all t). While for compact (normalized) gradient shrinking solitons $f = l_{p,T}$, it is interesting to study the relationship between f and $l_{p,T}$ in the noncompact case.

The reduced volume based at singular time can be used to analyze singularities in arbitrary dimensions: One can prove Hamilton's conjecture that rescaling limits around type I singularities are gradient shrinking solitons (see [7] or [2]). This has been previously shown in the special case of compact rescaling limits in [11] using the monotonicity of Perelman's entropy, whose scaling properties similarly imply that it is constant on the limit flow. The conjecture implies that the study of Ricci flow in higher dimensions requires the classification of gradient shrinking solitons (see e.g. [7], [10], [8], [6]).

Moreover, for a type A Ricci flow on $[0, T)$, any $(p, t_0) \in M^n \times [0, T]$, and a reduced distance l_{p,t_0}, we define *a density of* (p, t_0) *in the Ricci flow* $(M^n, g(t))$ by

$$\theta_{p,t_0} = \lim_{\bar{t} \nearrow t_0} \tilde{V}_{p,t_0}(\bar{t}) \in (0, 1].$$

In the special case of gradient shrinking solitons with quadratic curvature decay, this was discussed in [1]. We can prove the following: Let $(M^n, g(t))$ be a maximal type I Ricci flow on $[0, T)$ with singular set Σ. If $p \in M^n \backslash \Sigma$, then $\theta_{p,T} = 1$. We expect regularity theorems to hold for this quantity.

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A new type of a positively curved manifold

Karsten Grove

(joint work with Luigi Verdiani, Wolfgang Ziller)

Spaces of positive curvature play a special role in geometry. Although the class of manifolds with positive (sectional) curvature is expected to be relatively small, so far there are only a few known obstructions. In particular, for simply connected manifolds these are: (1) The Gromoll-Meyer theorem stating that a complete noncompact manifold with positive curvature is diffeomorphic to Euclidean space, (2) the Betti number theorem of Gromov which asserts that the homology of a compact manifold with non-negative sectional curvature has an a priory bound on the number of generators depending only on the dimension, and (3) a result of Lichnerovich implying that a spin manifold with positive scalar curvature has trivial $\hat{A}$ genus.

One way to gain further insight is to construct and analyze examples. This is quite difficult and has been achieved only a few times. Aside from the classical rank one symmetric spaces, i.e., the spheres and the projective spaces with their canonical metrics, and the recently proposed deformation of the so-called Gromoll-Meyer sphere [PW], examples were only found in the 60's by Berger [Be], in the 70's by Wallach [Wa] and by Aloff and Wallach [AW], in the 80's by Eschenburg [E1, E2], and in the 90's by Bazaikin [Ba]. The examples by Berger, Wallach and Aloff-Wallach were shown, by Wallach in even dimensions [Wa] and by Berard-Bergery [BB] in odd dimensions, to constitute a classification of simply connected homogeneous manifolds of positive curvature, whereas the examples due to Eschenburg and Bazaikin typically are non-homogeneous, even up to homotopy. All of these examples can be obtained as quotients of compact Lie groups G with a biinvariant metric by a free isometric "two sided" action of a subgroup $H \subset G\times G$.

Despite the fact that most of the known manifolds with positive curvature can be described as the total space of a bundle over another positively curved base space, as long as one also allows orbifold fibrations [FZ], there are no general methods for constructing examples in this manner. In [CDR] a necessary and sufficient condition was given for a *connection metric* on the total space of a bundle to have positive curvature when the metric on the fiber is shrunk sufficiently. This also applies in the setting of orbifolds. Since the projection map is a Riemannian submersion, it is of course built into this condition that the curvature of the base is positive. In the special case where the metric on the base is self dual Einstein, the general Chaves-Derdziński-Rigas condition reduces simply to having positive curvature on the base. This was used by Dearricott in [De1, De2] to construct new examples of metrics with positive curvature. These metrics, however, were metrics on some of the Eschenburg spaces already known to carry metrics of positive curvature.

In general, the attempt to classify positively curved manifolds with large isometry groups provides a systematic framework in the search for new examples, see [Gr],[Wi]. Such an attempt was carried out in [V1, V2] and in [GWZ] in the situation where the isometry group is assumed to act by *cohomogeneity one*, i.e., when the orbit space is one dimensional, or equivalently the principal orbits have codimension one. In addition to the normal homogeneous manifolds of positive curvature and a subset among the Eschenburg and Bazaikin spaces which admit a cohomogeneity one action, two infinite families, P_k, Q_k and one exceptional manifold R, all of dimension seven (with ineffective actions of $S^3 \times S^3$), appeared as new candidates (see [GWZ] and the survey [Zi]). Here P_1 is the 7-sphere and Q_1 is the normal homogeneous positively curved Aloff-Wallach space. It is a curious fact, that the infinite families admit a different description: They are the two-fold universal covers, $P_k \to H_{2k-1}$ and $Q_k \to H_{2k}$ of the frame bundle H_ℓ of self-dual 2-forms associated to the self dual Einstein orbifolds O_ℓ constructed by Hitchin in [Hi]. As such, these manifolds come with natural 3-Sasakian metrics. Unfortunately the curvature of the Hitchin metrics are positive only for $O_1 = S^4$ and for $O_2 = CP^2/Z_2$ [Zi]. This description of the manifolds also means that P_k and Q_k are S^3 principal bundles over S^4, in the sense of orbifolds. It is thus natural to consider connection metrics on the total spaces of these bundles. In this language our main theorem can be stated as:

The manifold P_2 admits a cohomogeneity one connection metric with positive curvature.

The importance of working in the orbifold category is also reflected by the fact that a connection metric on a smooth S^3 bundle over S^4 has positive curvature only in the case of the Hopf bundle, where the total space is S^7, [DR].

Since P_2 is 2-connected with $\pi_3 = Z_2$ (see [GWZ]) this is indeed a new example, since the only other known 2-connected positively curved 7-manifolds are S^7 and the Berger space $B^7 = SO(5)/SO(3)$ with $\pi_3(B^7) = Z_{10}$ [Be]. Furthermore, from [KZ],[To], it follows that the only homogeneous space or biquotient with the same cohomology groups as P_2 is $T_1(S^4)$. We do not know though whether it is diffeomorphic to it.

In the manuscript [De3], Dearricott has offered a construction of a positively curved metric on P_2. His method is to make a conformal change of the Hitchin metric on the base, keep the Hitchin principal connection and use the CDR condition for this special case, i.e., a condition on the Hitchin metric and the conformal change exclusively. In the same manuscript Dearricott also offers a proof that his method will not work for any of the other candidates. No estimates have been provided in [De3] in support of the delicate computer assisted evidence that the metric has positive curvature.

We now outline the proof that our example has positive curvature. As mentioned above, the Hitchin metrics on O_ℓ do not have positive curvature when $\ell \geq 3$. However, on O_3 this metric has positive curvature on a large region and only relatively small negative curvature , see Figure 8 in [Zi]. This may suggest that it might be possible to make a small change of the Hitchin metric on O_3 with positive curvature, choose a connection close to the Hitchin connection, and get positive curvature on the total space after shrinking the metric on the fiber sufficiently. We use this idea only as a guide in our choice of metric and connection. Our metric on the base, and the principal connection, are explicitly given by polynomials. For this we divide the interval on which the metric is defined into three subintervals, two close to the singular orbits, and a larger one in the middle. Near the singular orbits we find functions consisting of polynomials of degree at most 3. In the middle we glue with the unique polynomials of degree 5 such that the resulting metric on the manifold is C^2. It is then obvious that any smooth C^2 perturbation will have positive curvature as well.

To prove that our metric has positive curvature (on each piece), we use a method due to Thorpe [Th1, Th2] as implemented in higher dimensions by Püttmann [Pü]. Specifically, rather than working only with the curvature operator, this means that we seek to add a suitable (invariant) 4-form, not affecting the sectional curvature, so as to make the modified operator positive definite when the fiber metric is shrunk sufficiently. To prove positive definiteness, given our choices, boils down to checking that specific polynomials with integer coefficients have no zeroes on a particular closed interval. To prove this, we use Sturm's theorem, which counts real zeroes of such polynomials by computing the gcd of the polynomial and its derivative (i.e. applying the Euclidean algorithm).

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Finsler metrics and closed geodesics

Hans-Bert Rademacher

In this talk results about the existence and stability of closed geodesics of non reversible Finsler metrics are discussed. Lusternik and Fet proved in 1951 the existence of a closed geodesic on a compact and simply-connected manifold carrying a Finsler metric. In contrast to the reversible case resp. the Riemannian case there are examples of non reversible Finsler metrics on spheres S^{2n} resp. S^{2n-1} carrying only $2n$ geometrically distinct closed geodesics. These metrics were first introduced by Katok, they are bumpy and have constant flag curvature. For a bumpy metric with only finitely many geometrically distinct closed geodesics an algebraic relation between the average indices of the closed geodesics is shown in [4]. As a consequence one obtains that a bumpy metric on S^2 carries at least two geometrically distinct closed geodesics. If there are only finitely many closed geodesics then there are at least two elliptic ones. Bangert and Long prove in [2] that for *every* non-reversible Finsler metric on S^2 there are two closed geodesics. In higher dimensions there is the following result independently shown by Duan and Long [1] and by the author [6]:

Theorem 1. *A bumpy and non-reversible Finsler metric on S^n with $n \geq 3$ carries at least two geometrically distinct closed geodesics.*

In [7] it is shown that a bumpy Finsler metric on the complex projective plane $\mathbb{C}P^2$ carries at least two geometrically distinct closed geodesics.
Using a lower bound for the length of a closed geodesic one can prove the existence of closed geodesics on positively curved manifolds and give an upper bound for their lengths. For example one obtains:

Theorem 2. [5] *Let an n-dimensional sphere S^n carry a Finsler metric with reversibility λ and flag curvature K satisfying $(\lambda/(\lambda+1))^2 < K \leq 1$. Then there exist $n/2-1$ geometrically closed geodesics of length $< n\pi$. One of these closed geodesics is non-hyperbolic.*

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Quarter pinched flag curvature and Ricci flow of 4-manifolds

Huy Nguyen
(joint work with Ben Andrews)

My research centres on the construction of invariant curvature cones and the Ricci flow. In my thesis and together in a paper with my supervisor, Dr. Ben Andrews, we develop a new technique to construct sets of curvature operators that are preserved by the Ricci flow. This technique is based on the maximum principle for geometric evolution equations. The idea is as follows. We consider a set of curvature operators defined by an inequality of a curvature function of the orthonormal frame bundle, $F(R_{iijk}) \geq 0$. Examples of such functions are linear combinations of sectional curvature. To show that such sets are preserved, by the advanced maximum principle for tensors, it suffices to show that the ODE associated to the nonlinearity of the Ricci flow,

$$\frac{d}{dt}F(R) = F(R)^2 + F(R)^{\#},$$

preserves the set. We note here that the nonlinearity is quadratic in the curvature. Furthermore, to show that the curvature cone is preserved, we need only to show that the ODE preserves the set at the boundary, that is where $F(R_{ijkl}) = 0$. However, F is a function of the orthonormal frame bundle, and as it takes a minimum at the boundary, we may differentiate the equation with respect to derivatives in $O(n)$. Consequently, the first order derivatives are zero and the matrix of second order derivatives is non-negative. Using the differential equality, we simplify the curvature evolution equation. To show that the evolution equations preserves the curvature cone, it remains to use the matrix of second derivatives and control the remaining terms in the nonlinearity. This part of the proof has additional subtleties, the matrix of second derivatives has entries whose terms are linear in curvature, whereas the nonlinearity is quadratic. Using generalized determinants we are able to overcome this problem. Using this technique, we are able to prove the following theorem.

Theorem 1 ([AN07])**.** *Let M be a compact 4-manifold, and g_0 a Riemannian metric on M which has λ-pinched flag curvatures, with $\lambda > 1/4$. Then M is diffeomorphic to a spherical space form.*

The condition quarter-pinched flag curvature is explained as follows. Let (M, g) be a compact Riemannian 4-manifold, with curvature tensor R. We suppose that M has positive sectional curvatures and that for every $x \in M$ and every orthonormal basis $\{e_1, \ldots, e_4\}$ for T_xM, we have

$$R(e_2, e_1, e_2, e_1) \geq \lambda R(e_3, e_1, e_3, e_1). \qquad (1)$$

To put this in a more geometric way, for each e_1 in T_xM there is an associated bilinear form R_{e_1} on the orthogonal subspace, the flag curvature in direction e_1, defined by $R_{e_1}(v, v) = R(e_1, v, e_1, v)$. The condition (1) says precisely that the ratio of any two eigenvalues of R_{e_1} is bounded below by λ. That is, each of the flag curvatures of M is λ-pinched. The technique is first applied to show that strictly 1/4-flag pinching is preserved by Ricci flow and furthermore a lower bound on the pinching quantity is preserved. Next, we show that the ratio sectional curvature approaches unity at the scalar curvature blows up, this constitutes a pinching estimate. Finally using a compactness argument, we show that the manifold admits a metric with constant positive sectional curvature, hence is a spherical space form. We note here that if we allow weak pinching that is $\lambda \geq 1/4$, then M is diffeomorphic to a spherical space form, $\mathbb{S}/\Gamma$, or isometric to $\mathbb{C}P^2$. We note here this technique was also used to show that non-negative isotropic curvature is preserved by the Ricci flow. This was done in the authors thesis [Ngu07] and in the paper [BS07], where it is the crucial step in proving the quarter-pinched diffeomorphism sphere theorem.

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Ricci flow on almost flat manifolds

Galina Guzhvina

A compact Riemannian manifold M^n is called ε-flat if its curvature is bounded in terms of the diameter as follows:

$$|K| \leq \varepsilon \cdot d(M)^{-2},$$

where K denotes the sectional curvature and $d(M)$ the diameter of M. If one scales an $\varepsilon-$flat metric it remains $\varepsilon-$flat.
By almost flat we mean that the manifold carries ε-flat metrics for arbitrary $\varepsilon > 0$. The (unnormalized) Ricci flow is the geometric evolution equation in which one starts with a smooth Riemannian manifold (M^n, g_0) and evolves its metric by the equation:

$$\frac{\partial}{\partial t} g = -2ric_g, \tag{1}$$

where ric_g denotes the Ricci tensor of the metric g.
The present talk studies how the Ricci flow acts on almost flat manifolds. We show that on a sufficiently flat Riemannian manifold (M, g_0) the Ricci flow exists for all $t \in [0, \infty)$, $\lim_{t\to\infty} |K|_{g(t)} \cdot d(M, g(t))^2 = 0$ as $g(t)$ evolves along (1), moreover, if $\pi_1(M, g_0)$ is Abelian, $g(t)$ converges along the Ricci flow to a flat metric. More precisely, we establish the following result:

Main Theorem (Ricci Flow on Almost Flat Manifolds.) *In any dimension n there exists an $\varepsilon(n) > 0$ such that for any $\varepsilon \leq \varepsilon(n)$ an ε-flat Riemannian manifold (M^n, g) has the following properties:*
(i) the solution $g(t)$ to the Ricci flow (1)

$$\frac{\partial g}{\partial t} = -2ric_g, \qquad g(0) = g,$$

exists for all $t \in [0, \infty)$, (ii) along the flow (1) one has

$$\lim_{t \to \infty} |K|_{g_t} \cdot d^2(M, g_t) = 0$$

(iii) $g(t)$ converges (in the C^∞-sense) to a flat metric along the flow (1), if and only if the fundamental group of M is (almost) Abelian (= Abelian up to a subgroup of finite index).

Fundamental results concerning the algebraic structure of almost flat manifolds were obtained by Gromov at the end of 70's.
In fact, Gromov [1] showed that each nilmanifold (= compact quotient of a nilpotent Lie group) is almost flat. It means that almost flat manifolds which do not carry flat metrics exist and occur rather naturally. Moreover, he showed that nilmanifolds are, up to finite quotients, the only almost flat manifolds.

The above-mentioned results establish a close connection between the ε-flat manifolds with ε sufficiently small and nilmanifolds. In addition, we can show that sequences of universal covers of ε_k-flat manifolds with $\varepsilon_k \to 0$ subconverge to nilmanifolds thus giving a kind of motivation to consider first the Ricci flow on nilmanifolds.
Interesting observations concerning the behaviour of the Ricci flow on almost flat manifolds were made by J.Lauret (cf. [3]). From [3] follows (implicitly) the next property of the Ricci flow:
On a nilpotent Lie group Ricci flow (1) is a gradient flow of the functional $F = trRic^2$.
To obtain the necessary estimates it often makes sense to consider instead of (1) the normalised Ricci flow (2):

$$\frac{\partial g}{\partial t} = -2ric_g - 2\|ric_g\|_g^2 g, \tag{2}$$

where $\|ric_g\|_g^2 = trRic_g^2$ and we normalise the scalar curvature $sc(g_0) = -1$.
Lauret [3] also showed that the critical points of flow (2) are Ricci nilsolitons.
A Ricci nilsoliton is a special solution of the Ricci flow on a nilpotent Lie group which moves along the equation by a one-parameter group of automorphisms of N. If $\varphi_t = \exp(-\frac{t}{2}D)$, $D \in Der(\eta)$ then $\frac{\partial}{\partial t}|_0 \varphi_t^\star g = g(-D\cdot, \cdot)$.
Operators of the type D on nilmanifolds were studied in great detail by J. Heber [2]. In particular, he established the following property of their eigenvalues:

For some positive multiple the operator D has all the eigenvalues integer and positive.
This allows us to make the following important observation:
Every nilsoliton strongly contracts the metric. In other words, there exists a constant $\lambda > 0$, such that, if (N, g) is a Ricci nilsoliton, then along the flow (2), for any $t \geq 0$, $h > 0$, holds $g(h+t) < e^{-\lambda h} g(t)$, where g is considered as a symmetric operator on η. Now let U be a neighbourhood of all solitons such that on any manifold in U the corresponding left-invariant metric contracts along the flow (2). More precisely, there exists a $\lambda > 0$ such that for any $(N, g) \in U$, as long as $g(t) \in U$, $\forall t > 0, \forall h > 0$, holds: $g(t+h) < e^{-\frac{\lambda}{2} h} g(t)$ along the flow (2). Such a neighbourhood exists, as follows from the theory of the continuous dependence of solutions of ODE's on the initial data.
This permits us to obtain the following important result:
Choose a neighbourhood of the critical set as above. There exists a constant C such that for any nilmanifold (N, g) along the normalised Ricci flow flow (2) the measure of the set $I := \{t : (N, g(t)) \notin U\}$ is less or equal then C.
So we can conclude that no matter where we start with the flow, the metric will be non-contracting only on a set of finite measure. It can also be shown that the curvature declines along the Ricci flow on a nilmanifold. Since the nilmanifolds can be considered as "limit" spaces for almost flat manifolds the same will be true for the manifolds flat enough as well.
Now we consider manifolds with the (almost) Abelian fundamental group. The proof of iii) will be based on three inequalities.
a) Shi estimates. Let $(M^n, g_{ij}(x))$ be a compact manifold with its curvature tensor R_{ijkl} satisfying $|R_{ijkl}|^2 \leq k_0$ on M, $0 < k_0 < \infty$. Then there exists a constant $T(n, k_0) > 0$, s. t. the Ricci flow has a smooth solution $g_{ij}(x) > 0$ on M for a short time $0 \leq t \leq T(n, k_0)$ and satisfies the following estimates: for any integer $m \geq 0$ there exist constants $c(m, n)$, depending only on m and n such that

$$\|\nabla^m R_{ijkl}(x,t)\|^2 \leq \frac{c(m,n) \cdot k_0}{t^m} \tag{3}$$

for any $t \in [0, T]$.
b) For any n-dimensional Riemannian manifold flat enough there exists a constant $c(n)$ (depending only on the dimension) such that

$$\|R\|^{3/2} \leq c(n) \|\nabla R\|. \tag{4}$$

This inequality we prove using the standard blow-up argument.
c) In any dimension n there exist constants $c(n), \varepsilon(n)$ such that for any $\varepsilon(n)$-flat Riemannian manifold with the almost Abelian fundamental group holds:

$$\|\nabla R\| \leq c(n) \cdot d(M, g_t)(\|\nabla^2 R\| + \|R\|^2), \tag{5}$$

The last inequality is valid only in the almost Abelian case, and proof of it is intricate. Basically, we proof the fact that the translation of tensors along the geodesic loops of such manifolds are periodical modulo ε. The proof thereof uses the techniques developed by Gromov (cf. [1]).

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Stability of Euclidean space under Ricci flow

Felix Schulze

(joint work with Oliver C. Schnürer, Miles Simon)

In this work, [1], we study the evolution of a family of complete non-compact Riemannian manifolds $(\mathbb{R}^n, g(t))$ under Ricci flow,

$$\frac{d}{dt} g_{ij} = -2\,\mathrm{Ric}_{ij} \tag{1}$$

where g_0 is a given initial metric on $\mathbb{R}^n$. We aim at studying the long-term behavior as $t \to \infty$ of solutions to (1) for initial metrics g_0, which are C^0-close to the standard Euclidean metric h.

For analytic reasons, it is convenient to study the Ricci harmonic map heat flow which is a variant of the Ricci-DeTurck flow

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i & \text{in } \mathbb{R}^n \times [0,\infty), \\ g_{ij}(\cdot,t) \to (g_0)_{ij}(\cdot) & \text{in } C^0_{loc}(\mathbb{R}^n) \text{ as } t \searrow 0, \end{cases} \tag{2}$$

where $V_i = g_{ik}\left({}^g\Gamma^k_{rs} - {}^h\Gamma^k_{rs}\right) g^{rs}$, which gives for a flat metric h: $V_i = g_{ik}\Gamma^k_{rs} g^{rs}$. The flow (2) and the Ricci flow (1) are equivalent up to diffeomorphisms.

Let g_1 and g_2 denote two Riemannian metrics on a given manifold. We say that g_1 is ε-close to g_2, if

$$(1+\varepsilon)^{-1} g_2 \le g_1 \le (1+\varepsilon)\, g_2\,.$$

We can show, using techniques from [2] and [3], that if our initial metric g_0 is a continuous Riemannian metric on $\mathbb{R}^n$, which is ε_0-close to the standard metric $h = \delta$, for ε_0 sufficiently small, depending only on n, then there exists a smooth solution $(g(t))_{t\in(0,\infty)}$ to (2). This solution remains ε'-close to h and if in addition g_0 is $\varepsilon(r)$-close to h on $\mathbb{R}^n \setminus B_r(0)$ with $\varepsilon(r) \to 0$ as $r \to \infty$, then

$$g(t) \to h \quad \text{in } C^\infty \text{ as } t \to \infty.$$

Note that we do not have to impose any decay rates to obtain this convergence. If (λ_i) are the eigenvalues of (g_{ij}) with respect to (h_{ij}), we work with the geometric quantities $\phi_m = \sum_{i=1}^n \frac{1}{\lambda_i^m}$ and $\psi_m = \sum_{i=1}^n \lambda_i^m$. To prove convergence back to the Euclidean metric we consider the following combination of the above quantities: Note that

$$\phi_m + \psi_m - 2n = \sum_{i=1}^{n} \left(\frac{1}{\lambda_i^m} + \lambda_i^m - 2 \right) = \sum_{i=1}^{n} \frac{1}{\lambda_i^m} \left(\lambda_i^m - 1 \right)^2 \geq 0 \tag{3}$$

is non-negative and vanishes precisely when $\lambda_i = 1$ for all $i \in \{1, \ldots, n\}$. Now if in addition, for ε' small enough, we have that g_0 is smooth and initially for $1 \leq p < n/2$,

$$\int\limits_{\mathbb{R}^n} (\phi_m + \psi_m - 2n)^p < \infty,$$

which is equivalent to the condition that $\|g_0 - h\|_{L^{2p}(\mathbb{R}^n)} < \infty$, then we can relate solutions to (2) to solutions of (1): There exists a smooth family $(\phi_t)_{t \geq 0}$ of diffeomorphisms of $\mathbb{R}^n$, $\phi_0 = \mathrm{id}_{\mathbb{R}^n}$, such that for $\tilde{g}(t) := \phi_t^* g(t)$ the family $(\tilde{g}(t))_{t \geq 0}$ is a smooth solution to (1) satisfying

$$\tilde{g}(t) \to (\phi_\infty)^* h \quad \text{in } C^\infty \left(\mathbb{R}^n \right) \text{ as } t \to \infty$$

for some smooth diffeomorphism ϕ_∞ of $\mathbb{R}^n$ with $\phi_t \to \phi_\infty$ in $C^\infty (\mathbb{R}^n, \mathbb{R}^n)$ as $t \to \infty$.

The idea in the proof of the results described above is to measure the perturbation of the metric $g(t)$ from the Euclidean metric h in terms of the quantity (3). The key estimate is to show essentially that $\int (\phi_m + \psi_m - 2n)$ is non-increasing in time. Here we integrate over the manifold at a fixed time. We have interior estimates for the gradient of the metric evolving under DeTurck flow, which become better for larger times. This implies that, if at a large time t, one eigenvalue $\lambda_i(x,t)$ differs significantly from one, then on a controlled spatial neighborhood this is true as well. Thus we obtain an arbitrarily large contribution to $\int (\phi_m + \psi_m - 2n)$, contradicting the fact that it is monotone in time. Thus $\lambda_i(x,t) \to 1$, uniformly in x as $t \to \infty$.

In two space dimensions, Ricci flow is a conformal flow given by the evolution equation

$$\frac{d}{dt} g = -Rg,$$

where R is the scalar curvature. In this situation, we obtain that if $g_0 = e^{-u_0} h$, where $h = \delta$ is the standard Euclidean metric, and $u_0 \in C^0(\mathbb{R}^2)$ such that

$$\sup_{\mathbb{R}^2 \setminus B_r(0)} |u_0| \to 0 \quad \text{as } r \to \infty,$$

then there exists a smooth solution $(g(t))_{t \in (0,\infty)}$, $g(t) = e^{-u(\cdot,t)} h$, to Ricci flow such that $u(\cdot,t) \to u_0$ in C^0_{loc} as $t \searrow 0$. Furthermore, as $t \to \infty$, we show that $u(\cdot,t) \to 0$ in $C^\infty(\mathbb{R}^2)$.

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Path isometries to Euclidean space

Anton Petrunin

I consider length-metric spaces which admit path isometries to the Euclidean m-space. The main result roughly states that the class of these spaces coincides with the class of inverse limits of m-dimensional polyhedral spaces.

Geodesics of the Weil-Petersson metric

Ursula Hamenstädt

The *Teichmüller space* $\mathcal{T}(S)$ of a closed oriented surface S of genus $g \geq 2$ is the quotient of the space of all hyperbolic metrics on S under the group of diffeomorphisms of S which are isotopic to the identity. The Teichmüller space admits a natural complex structure. With this structure, $\mathcal{T}(S)$ is biholomorphic to a bounded domain in $\mathbb{C}^{3g-3}$.

The *mapping class group* $\mathcal{M}(S)$ of isotopy classes of orientation preserving diffeomorphisms of S naturally acts on $\mathcal{T}(S)$. This action preserves the complex structure. In fact, $\mathcal{M}(S)$ is precisely the group of all biholomorphic automorphism of $\mathcal{T}(S)$. The action of $\mathcal{M}(S)$ on $\mathcal{T}(S)$ is properly discontinuous, but neither free nor cocompact.

There are several natural and interesting $\mathcal{M}(S)$-invariant metrics on $\mathcal{T}(S)$. The best known of these metrics is the Kobayashi metric (which is better known as the *Teichmüller metric*). This metric is an $\mathcal{M}(S)$-invariant complete Finsler metric with the additional property that any two points in $\mathcal{T}(S)$ can be connected by a unique geodesic.

The *Weil-Petersson metric* is an $\mathcal{M}(S)$-invariant Kähler metric g_{WP} on $\mathcal{T}(S)$. Its sectional curvature is negative, but it is neither bounded from above by a negative constant, nor is it bounded from below. Moreover, the Weil-Petersson metric is not complete. Nevertheless, any two points in $\mathcal{T}(S)$ can be connected by a unique geodesic which depends smoothly on its endpoints. The completion of $\mathcal{T}(S)$ with respect to the Weil-Petersson metric is a CAT(0)-space which is not locally compact.

For some $\epsilon > 0$, the *ϵ-thick part* $\mathcal{T}(S)_\epsilon$ of $\mathcal{T}(S)$ consists of all hyperbolic metrics whose *systole*, i.e. whose shortest length of a simple closed geodesic, is at least ϵ. The mapping class group preserves $\mathcal{T}(S)_\epsilon$ and acts on it cocompactly. We explain the following result [4] which was independently obtained by Jeff Brock, Howard Masur and Yair Minsky [3].

Theorem 1. *For every $\epsilon > 0$ there is a number $\delta > 0$ with the following property.*

(1) *Let* $\gamma : [a,b] \to \mathcal{T}(S)_\epsilon$ *be a Teichmüller geodesic. Then the Weil-Petersson geodesic with the same endpoints is contained in* $\mathcal{T}(S)_\delta$.
(2) *Let* $\gamma : [a,b] \to \mathcal{T}(S)_\epsilon$ *be a Weil-Petersson geodesic. Then the Teichmüller geodesic with the same endpoints is contained in* $\mathcal{T}(S)_\delta$.

As an application, one can investigate surface bundles M with fibre S and base a closed surface S' of genus $g' \geq 2$ whose fundamental group is word hyperbolic (such bundles are conjectured not to exist). For each such bundle $M \to S'$ there is an exact sequence

$$0 \to \pi_1(S) \to \pi_1(B) \to \pi_1(S') \to 0$$

and hence a monodromy homomorphism $\rho : \pi_1(S') \to \mathcal{M}(S)$.

A *quasi-convex* subset of $(\mathcal{T}(S), g_{WP})$ is a set $A \subset \mathcal{T}(S)$ with the property that there is a number $c > 0$ such that the Weil-Petersson geodesic connecting any two points in A is contained in the c-neighborhood of A. We obtain.

Theorem 2. *The surface bundle* $M \to S'$ *is hyperbolic if and only if an orbit for the action of* $\rho(\pi_1(S'))$ *on* $\mathcal{T}(S)$ *is quasi-convex for the Weil-Petersson metric.*

As an application, one obtains a new proof of a result of Bowditch [1]: For each fixed surface S' there are only finitely many homeomorphism types of S-bundles over S' with word hyperbolic fundamental group.

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Equivariant Ricci flow with surgery and finite group actions on geometric 3–manifolds

Jonathan Dinkelbach
(joint work with Bernhard Leeb)

Given a smooth finite group action on a closed geometric manifold, Thurston raised the question whether the geometric structure can always be chosen to be compatible with the action [Thu82, Question 6.2]. In other words, does there exist an invariant complete locally homogeneous metric on the manifold?

In dimension two this follows from the geometrization of 2–orbifolds; and also in dimension 3 this is known to be true in a lot of cases: A result of Meeks and Scott [MS86] covers the cases where the geometry is of type $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{R}^3$, $\widetilde{SL}(2,\mathbb{R})$, *Nil* or *Sol*. For free actions (and any geometry), it is a consequence of Perelman's Geometrization Theorem [Per03a], and for non-free, orientation-preserving actions it follows from the Orbifold Theorem of Boileau, Leeb and Porti [BLP05].

In this talk I discussed the following result for spherical, hyperbolic and $S^2 \times \mathbb{R}$–geometry, see [DL08], [Din08]:

Theorem. *Let M be a closed geometric 3–manifold, such that the model geometry is one of S^3, $S^2 \times \mathbb{R}$ or $\mathbb{H}^3$. Then for any smooth finite group action $\rho\colon G \curvearrowright M$ there exists a ρ–invariant complete locally homogeneous metric on M.*

In the spherical and hyperbolic case, geometric structures are unique in the sense that if g_1 and g_2 are two spherical (respectively hyperbolic) metrics on M, then (M, g_1) and (M, g_2) are isometric. Thus, the Theorem implies the following reformulation:

Corollary. *Let $\rho\colon G \curvearrowright M$ be a smooth finite group action on a spherical or hyperbolic closed 3–manifold. Then the action ρ is conjugate to an isometric action.*

The idea of our approach is to apply an equivariant version of Perelman's Ricci flow with surgery: It suffices to consider the case that M is orientable (otherwise we pass to its orientable double cover and the corresponding $G \times \mathbb{Z}_2$–action). Since the group action is finite, we can equip M with a ρ–invariant initial metric g_0. With respect to this metric the action is isometric, but of course in general g_0 is not locally homogeneous. If one now runs the Ricci flow, its uniqueness guarantees that the action remains isometric as long as the flow is defined. We are done if the Ricci flow converges to a complete, locally homogeneous metric as in the case of positive Ricci curvature [Ham82].

By the results of Perelman, one can also deal with singularities occurring during the flow. However, his Ricci flow with surgery is a-priori not equivariant. There are three main issues which need to be resolved:

First, one has to show that the surgery procedure can be done in an equivariant way. This is achieved by constructing equivariant surgery-necks, i. e. approximations by round cylinders such that the pulled-back action is isometric. This argument only concerns the neck-like part of the manifold: Here we can "average" a covering by an invariant family of (non-equivariant) necks in order to obtain an equivariant S^2–foliation which is close to the cylindrical product foliation.

Second, at a surgery time there might be components on which scalar curvature gets uniformly large, even though the metric does not converge to a geometric one (the curvature operator gets only almost non-negative). Those components are diffeomorphic to S^3, $\mathbb{R}P^3$, $\mathbb{R}P^3 \# \mathbb{R}P^3$ or $S^2 \times S^1$ and they are thrown away, so one needs to control the action restricted to them. It turns out that a component with radius (relative to scalar curvature) bounded below a certain bound C is *globally* approximated by a compact κ-solution with uniform lower sectional curvature bound, so we are in the controlled case. On the other hand, local models (κ- or standard-solutions) with radius greater than C have a neck-cap decomposition, i.e. are nicely covered by ϵ-necks and ϵ-caps. This decomposition carries over to the approximated manifold. Thus the action can be controlled by extending the invariant foliation on the neck-like part to an invariant (singular) foliation on the caps. This is the most difficult step of the proof and uses the compactness of the space of model solutions and a result of Grove and Karcher on $\mathcal{C}^1$-close actions [GK73].

Finally, having obtained that the action is standard on all components which get extinct, one needs to get back to the original manifold and action. Therefore, one has to relate the actions before and after a surgery. The result is that the original action is an equivariant connected sum of the action on a later time slice (which is possibly empty) and a standard action on a finite union of spherical space forms, $\mathbb{R}P^3 \# \mathbb{R}P^3$'s and $S^2 \times S^1$'s.

From this one can conclude the Theorem for spherical geometry: Then the manifold M is irreducible and has finite extinction time by [Per03b], [CM05]. Therefore, the original action is an equivariant connected sum of a standard action on M and a standard action on a finite union of 3-spheres which are attached along a tree. This equivariant connected sum can then be shown to be trivial.

In the case of $S^2 \times \mathbb{R}$-geometry one needs an extra argument since manifolds with this geometry are not irreducible, but the basic idea of the proof is similar. In case of hyperbolic geometry, one has in addition to consider the long-time behavior of the action on a component on which the metric converges (up to diffeomorphisms) to a hyperbolic one. There one applies Mostow-rigidity to conclude that the limit action is standard, and again by the result of Grove and Karcher also actions close to the limit (i.e. for large time) are standard.

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Volume entropy of Hilbert metrics

Andreas Bernig

(joint work with Gautier Berck, Constantin Vernicos)

Let K be a compact convex set with non-empty interior. The Hilbert metric on $int\, K$ is defined by

$$d(x,y) := \frac{1}{2} \log \frac{|ya|}{|xa|} \frac{|xb|}{|yb|},$$

where a, b are the intersection points of the line through x and y with the boundary of K (if $x = y$, then $d(x,y) := 0$).

The Hilbert metric is a particularly simple complete Finsler metric whose geodesics are straight lines. Since its definition only uses cross-ratios, the Hilbert metric is a projective invariant. In the particular case where K is an ellipsoid, the Hilbert geometry is isometric to the hyperbolic space.

The entropy of K is defined as

$$Ent\, K := \lim_{r \to \infty} \frac{\log vol\, B(o,r)}{r},$$

whenever the limit exists. The entropy does not depend on the particular choice of the base point $o \in int\, K$ nor on the particular choice of the volume (like Holmes-Thompson volume or Busemann volume). If the limit does not exist, one may define lower and upper entropies $\underline{Ent}, \overline{Ent}$ by replacing the limit in the definition by lim inf or lim sup.

Instead of taking the volume of balls, another natural choice is to study the volume growth of the metric spheres $S(o,r)$. One may define a (spherical) entropy by

$$Ent^s K := \lim_{r \to \infty} \frac{\log vol\, S(o,r)}{r},$$

whenever the limit exists. In general, one may define upper and lower spherical entropies $\overline{Ent}^s$ and $\underline{Ent}^s K$ by replacing the limit by a lim sup or lim inf.

Conjecture 1. *For any n-dimensional Hilbert geometry,*

$$\overline{Ent} K \leq n - 1.$$

Several particular cases of the conjecture were treated in the literature.

Theorem 1. (Colbois-Verovic [4]**)**
If K is C^2-smooth with strictly positive curvature, then the Hilbert metric of K is bi-Lipschitz to the hyperbolic metric and therefore $Ent\, K = n - 1$.

Theorem 2. (Colbois-Vernicos-Verovic [3]**)** *The Hilbert metric associated to a plane polygone is bi-Lipschitz to Euclidean plane. In particular, the entropy is 0.*

The following theorem is a spherical version of the theorem of Colbois-Verovic.

Theorem 3. (Borisenko-Olin [2]**)** *If K is of class C^3 and of dimension n, then $Ent^s = n - 1$.*

Definition 1. *The centro-projective area of K is*

$$\mathcal{A}_p(K) := \int_{\partial K} \frac{\sqrt{k}}{\langle n, p\rangle^{\frac{n-1}{2}}} \left(\frac{2a}{1+a}\right)^{\frac{n-1}{2}} dA,$$

where k is the Gauss curvature, n is the unit outward normal vector and a is the positive function defined by $-a(p)p \in \partial K$.

The centro-projective area is a projective invariant of K. Moreover, it is upper semicontinuous with respect to Hausdorff distance.

Theorem 4. *If K is $C^{1,1}$, then*

$$\lim_{r\to\infty} \frac{vol\, B(o,r)}{e^{(n-1)r}} = \frac{1}{2^{n-1}(n-1)} \mathcal{A}_p(K). \tag{1}$$

Moreover, $\mathcal{A}_p(K) > 0$ and therefore $Ent\, K = n-1$.

Theorem 5. *Let K be a two-dimensional convex body. Let d be the upper Minkowski dimension of $ex\, K$. Then the entropy of K is bounded by*

$$\overline{Ent\, K} \leq \frac{2}{3-d} \leq 1.$$

We give examples of two-dimensional convex bodies with entropy strictly between 0 and 1.

Theorem 6. *For each convex body K,*

$$\underline{Ent}^s K = \underline{Ent} K,$$
$$\overline{Ent}^s K = \overline{Ent} K.$$

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Isoparametric hypersurfaces (after S. Immervoll)

Linus Kramer

A closed connected hypersurface $M^n \subseteq S^{n+1}$ is called isoparametric if its principal curvatures are constant [2, 10, 4]. Let g denote the number of distinct principal curvatures.

Examples. a) $(g = 1)$ $S^n \subseteq S^{n+1}$, or any parallel hypersurface.
b) $(g = 2)$ $S^k(r) \times S^{n-k}(s)$, where $r^2 + s^2 = 1$.
c) $(g = 1, 2, 3, 4, 6)$ principal orbits of isometric actions of cohomogeneity 1. This includes a) and b) as special cases.

The examples of type c) were classified by Hsiang-Lawson [7]; they arise from isotropy representations of symmetric spaces of rank 2. Isoparametric hypersurfaces with $g \leq 3$ were classified by Cartan [2, 9]. A fundamental result due to Münzner [10] says that the number of distinct principal curvatures is $g \in \{1, 2, 3, 4, 6\}$. Furthermore, the hypersurface is the level set of a homogeneous polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree g satisfying the differential equations $|\nabla f(x)|^2 = g^2|x|^{2g-2}$ and $\Delta f(x) = g^2 \frac{m_2 - m_1}{2}|x|^{g-2}$. The numbers m_1, m_2 are the multiplicities of the principal curvatures (there are at most two different multiplicities). Isoparametric hypersurfaces with $g = 6$ and $m_1 = m_2 = 1$ were classified by Dorfmeister-Neher [5] and by Abresch; the remaining open case for $g = 6$ is $m_1 = m_2 = 2$ [1]. Ferus-Karcher-Münzner [6] constructed an infinite family of isoparametric hypersurfaces with $g = 4$ as follows.

Example. d) ($g = 4$). Let A be a Euclidean vector space with Clifford algebra $Cl(A)$ and orthonormal basis $a_0, \ldots, a_{m_1}$. Let V be an orthogonal $Cl(A)$-module (i.e. $|av| = |a||v|$ for $a \in Cl(A)$, $v \in V$) and put $f(x) = |x|^4 - 2\sum_i \langle a_i x, x\rangle$, with $\dim V = 2(m_1 + m_2 + 1)$. If $m_2 \geq 1$, then the regular level sets of f in the sphere are isoparametric.

There is some overlap between the cases c) and d). Stolz [11] determined the possible values m_1, m_2 for the multiplicities and showed that topologically, every isoparametric hypersurface with $g = 4$ looks like one of the examples in c) and d).

The following result was proved recently by Cecil-Chi-Jensen [3] and, independently and in a completely different way, by Immervoll [8].

Theorem. *Let M be an isoparametric hypersurface with $g = 4$ and multiplicities $m_1 \leq m_2$. If $m_2 \geq 2m_1 - 1$, then M is one of the examples d) arising from Clifford modules.*

In my talk I explained the main steps of Immervoll's proof [8]. Let S^2V denote the space of all selfadjoint operators on V. The starting point is to consider the selfadjoint operator $T : S^2V \otimes S^2V \to \mathbb{R}$ defined by $\langle T(x \otimes x), x \otimes x) = 3f(x)$ and its perturbation $\Phi_+ = -\frac{1}{4}(T - 2 \cdot \mathbf{1})$, restricted to the space $S^2_0 V$ of traceless selfadjoint operators. The following facts about Φ_+ are not difficult to prove. Let A_+ denote the set of all quadratic forms which vanish on the focal manifold $M_1 = f^{-1}(1) \cap S^{n+1}$.

- $\Phi_+|A_+ = (m_2 + 2)\mathbf{1}$.
- $Spec(\Phi) \subseteq [-(m_1 + 1), m_1 + 1] \cup \{m_2 + 2\}$.
- If $a, b \in A_+$, then $aba \in A_+$.
- If $m_2 \geq m_1$, then $A_+ = \ker(\Phi_+ - (m_2 + 2)\mathbf{1})$.
- If M is of type d) and if $m_2 \geq 2m_1 - 1$, then A_+ is precisely the image of A under $Cl(A) \to \mathrm{End}(V)$.

The main step is to show that the highest possible eigenvalue $m_2 + 2$ actually occurs in the spectrum of Φ_+. For $a_0 \in S^2_0 V$ let $a_{n+1} = \frac{1}{m_2+2}\Phi_+ a_n$. Immervoll shows that there exists a_0 such that the sequence $\langle a_n, p\otimes p - q\otimes q\rangle$ is bounded away from zero, for suitable points $p, q \in f^{-1}(-1) \cap S^{n+1}$, provided that $m_2 \geq 2m_1 - 1$.

It follows that $A_+ \neq 0$. Some further analysis then shows that $a^2 = |a|^2\mathbf{1}$ for all $a \in A_+$, so V is an a natural way a $Cl(A_+)$-module. This leads to the classification.

By Stolz' result [11] on the multiplicities, the condition $m_2 \geq 2m_1 - 1$ is almost always satisfied. The exceptions are the pairs $(m_1, m_2) = (2,2), (3,4), (4,5), (6,9), (7,8)$. The analysis of these cases is an ongoing joint project with S. Immervoll and S. Stolz.

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