Mathematisches Forschungsinstitut Oberwolfach

Report No. 45/2008

Geometry and Arithmetic around Hypergeometric Functions

Organised by
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September 28th – October 4th, 2008

Abstract. Hypergeometric functions form a classical subject of Mathematics providing an interesting link between apparently quite different fields of research like differential equations, group theory, differential geometry, physics, computational mathematics, moduli spaces, arithmetic and algebraic geometry. The conference succeeded to give insights into new developements in many different directions and to encourage discussions and exchange of ideas between participants who would never had met otherwise or at other places than Oberwolfach.

Mathematics Subject Classification (2000): 33C, 11F, 11J, 14G, 14M.

Introduction by the Organisers

The workshop *Geometry and Arithmetic around Hypergeometric Functions*, organised by Gert Heckman (Nijmegen), Jürgen Wolfart (Frankfurt a.M.) and Masaaki Yoshida (Fukuoka) was held September 28th – October 4th, 2008. It was attended by more than 40 participants coming from more than 10 different countries and representing a remarkable variety of mathematical research directions centered around hypergeometric functions. The public of the conference was mixed not only with respect to its mathematical interests but also in age and experience: it ranged from doctoral students to excellent and well–known senior researchers.
The program concentrated on important new developements of the last years, let sufficient time for discussions in between the talks, and gave also promising younger people the possibility to present their work, partly in more informal evening seminars. Even the contributions given on Friday afternoon and evening were attended

by almost all participants, so we may assume that the program was greatly appreciated.

A long standing question about algebraicity and transcendence of special values of Gauss hypergeometric functions has been answered in recent years using work of Wüstholz, Edixhoven, Yafaev and Klingler about conjectures concerning special subvarieties of Shimura varieties. The conference had a special day with four featured talks of these authors about the subject and an informal evening seminar concerning generalizations of this question to higher dimensions.

A remarkable event on the classical number theoretic side of hypergeometric functions was Don Zagier's featured talk explaining a new link of complex differential equations to Hilbert and Teichmüller modular forms.

The arithmetic–geometric mean can be written in terms of the hypergeometric function – a classical fact. Matsumoto-Shiga reported about possibilities of its genralization.

Algebro–geometric aspects of the Painlevé equations and their generalizations were reported by van der Put and Tsuda. It is well known that these equations are related to the hypergeometric equations and its generalizations.

About the classical hypergeometric equation, Vidunas proved that there are still many things to be studied, and Sasaki–Yoshida presented differential geometric and singularity theoretic aspects of the hyperbolic Schwarz map.

The deep relation between hypergeometric differential equations and ball quotients originated with Schwarz, Picard, Terada, and Deligne-Mostow. New compactification techniques of such ball quotients (and more generally of so called Heegner divisor complements in such ball quotients) were reported by Heckman. These compactifications were introduced in full generality by Looijenga, inspired by the work of Heckman and Looijenga on the moduli space of rational elliptic surfaces. The geometric examples discussed in the lecture of Heckman were all related to groups generated by order four complex reflections (also called tetraflections) and go back to the work of Deligne-Mostow and of Kondo. Allcock presented a surprising conjecture about a ball quotient and the monster group. In this lecture the complex reflection groups were related to order three complex reflections (triflections in Conway's terminology), and in a sense the lecture by Heckman also served as a kind of introduction for Allcock's lecture. The conjecture by Allcock is truely amazing, and would in fact reveal a natural geometric surrounding (via generalized hypergeometric periods) for most of the sporadic groups. Holzapfel and his collaborators discussed towers of ball quotients and morphsims among these.

A vast generalization of the hypergeometric equation developed during the last twenty years is the GKZ (Gelfand–Kapranov–Zelevinsky) equation was another important subject of the meeting. Beukers presented a criterion for their solvability by algebraic functions, and Alicia Dickenstein reported about examples admitting rational solutions.

Workshop: Geometry and Arithmetic around Hypergeometric Functions

Table of Contents

Abstracts

Algebraic hypergeometric functions of GKZ-type

Frits Beukers

The classically known hypergeometric functions of Gauss (notation: ${}_2F_1$), its one-variable generalisations ${}_{p+1}F_p$ and the many variable generalisations, such as Appell's functions, the Lauricella functions and Horn series are all examples of the so-called A-hypergeometric functions, now called GKZ-hypergeometric functions, introduced by Gel'fand, Kapranov, Zelevinsky in [7].
Their definition begins with a finite subset $A \subset \mathbb{Z}^r$ consisting of N vectors $\mathbf{a_1}, \dots, \mathbf{a_N}$ forming a set of rank r and such that

i) The $\mathbb{Z}$-span of $\mathbf{a_1}, \dots, \mathbf{a_N}$ equals $\mathbb{Z}^r$.
ii) There exists a linear form h on $\mathbb{R}^r$ such that $h(\mathbf{a_i}) = \mathbf{1}$ for all i.

The latter condition ensures that we shall be working in the case of so-called Fuchsian systems. We are also given a vector of parameters $\alpha = (\alpha_1, \dots, \alpha_r)$ which could be chosen in $\mathbb{C}^r$, but we shall restrict to $\alpha \in \mathbb{R}^r$. The lattice $L \in \mathbb{Z}^N$ of relations consists of all $(l_1, \dots, l_N) \in \mathbb{Z}^N$ such that $\sum_{i=1}^N l_i \mathbf{a_i} = \mathbf{0}$.
The GKZ-hypergeometric equations are a set partial differential equations with independent variables $v_1, \dots, v_N$. This set consists of two groups. The first are the structure equations

$$\Box_{\mathbf{l}}\Phi := \prod_{l_i>0} \partial_i^{l_i} \Phi - \prod_{l_i<0} \partial_i^{|l_i|} \Phi = 0 \tag{GKZ1}$$

for all $\mathbf{l} = (\mathbf{l_1}, \dots, \mathbf{l_N}) \in \mathbf{L}$.
The second groups consists of the homogeneity equations.

$$Z(\mathbf{m})\mathbf{\Phi} := (\mathbf{m}(\mathbf{a_1})\mathbf{v_1}\partial_\mathbf{1} + \mathbf{m}(\mathbf{a_2})\mathbf{v_2}\partial_\mathbf{2} + \dots + \mathbf{m}(\mathbf{a_N})\mathbf{v_N}\partial_\mathbf{N} - \mathbf{m}(\alpha))\,\mathbf{\Phi} = \mathbf{0} \tag{GKZ2}$$

for all linear forms m on $\mathbb{R}^r$.
In general the GKZ-system is a holonomic system of dimension equal to the $r-1$-dimensional volume of the so-called A-polytope $Q(A)$, which is the convex hull of the endpoints of the $\mathbf{a_i}$. The volume-measure is normalised to 1 for a $r-1$-simplex of lattice-points in the plane $h(\mathbf{x}) = \mathbf{1}$ having no other latticepoints in its interior. In the first days of the theory of GKZ-systems there was some confusion as to what 'general' means, however this was clarified by A.Adolphson [1] and later completely in .
The positive cone generated by the vectors $\mathbf{a_i}$ is denoted by $C(A)$. This is a simplicial cone with a finite number of faces. We recall the following Theorem.

Theorem 1. *The GKZ-system is irreducible if and only if $\alpha + \mathbb{Z}^r$ contains no points in any face of $C(A)$.*

This Theorem is proved in [8] using perverse sheaves. It would be nice to have a more elementary proof however.
Let us now assume that $\alpha \in \mathbb{Q}^r$. We shall be interested in those irreducible GKZ-systems that have a complete set of solutions algebraic over $\mathbb{C}(v_1, \ldots, v_N)$. This question was first raised in the case of Gauss hypergeometric functions and the answer is provided by the famous list of H.A.Schwarz. In 1989 this list was extended to general one-variable ${}_{p+1}F_p$ by Beukers and Heckman [5]. For the several variable cases, a characterization for Appell-Lauricella F_D was provided by T.Sasaki [13] and Wolfart, Cohen [6]. The Appell F_2 and F_4 were classified by M.Kato [11], [12]. For general GKZ-systems such a classification is not known yet. We note that the existence of a complete set of algebraic solutions is equivalent to the monodromy group of the system being finite. In most of the cases so far extension of the Schwarz list was possible through a detailed knowledge of the monodromy group. For general GKZ-systems such a description is not (yet) available. Only special cases have been considered, for example in [14]. However, there is another appoach which uses the Grothendieck conjecture which roughly says that a Fuchsian holonomic system of linear differential equation defined over $\overline{\mathbb{Q}}$ has finite monodromy if and only if for almost all primes p the reduced system modulo p has a set of polynomial solutions of maximal rank.
In a recent paper F.Beukers [3] has turned the existence of a maximal polynomial solution set modulo p into a combinatorial criterion which, in many cases, can be checked quite easily.
In the case of one-variable hypergeometric functions this criterion comes down to the so-called *interlacing condition* discussed in [5]. In general the combinatorial criterion is slightly more complicated.
We assume that $\alpha \in \mathbb{R}^r$ and define $K_\alpha = (\alpha + \mathbb{Z}^r) \cap C(A)$. A point $\mathbf{p} \in \mathbf{K}_\alpha$ is called an *apexpoint* if $\mathbf{p} \notin \mathbf{q} + \mathbf{C(A)}$ for every $\mathbf{q} \in \mathbf{K}_\alpha$ with $\mathbf{q} \neq \mathbf{p}$. We call the number of apexpoints the *signature* of the polytope A and parameters α. Notation: $\sigma(A, \alpha)$. It can be shown that $\sigma(A, \alpha)$ is less than the volume of $Q(A)$ and we say that the signature is *maximal* if it equals the volume of $Q(A)$.
Using proved cases of the Grothendieck conjecture (by N.M.Katz in [10]) one can show the following result.

Theorem 2 (Beukers, 2007). *Let $\alpha \in \mathbb{Q}^r$ and suppose the GKZ-system is irreducible. Let D be the common denominator of the coordinates of α. Then the solution set of the GKZ-system consists of algebraic solutions (over $\mathbb{C}(v_1, \ldots, v_N)$) if and only if $\sigma(A, k\alpha)$ is maximal for all integers k with $1 \leq k < D$ and* $\gcd(k, D) = 1$.

As a simple application of this statement we have studied the Horn series

$$G_3(a, b, x, y) = \sum_{m \geq 0, n \geq 0} \frac{(a)_{2m-n}(b)_{2n-m}}{m!n!} x^m y^n.$$

with $a, b \in \mathbb{Q}$. We assume $a \not\equiv -2b (\text{mod } \mathbb{Z})$ and $b \not\equiv -2a (\text{mod } \mathbb{Z})$ to assure irreducibility. It is a simple exercise to show that the condition in Theorem 2 implies that the monodromy group of the Horn G_3 system is finite if and only if one of the following holds

a.) $a + b \in \mathbb{Z}$
b.) $a \equiv 1/2(\text{mod } \mathbb{Z}), b \equiv \pm 1/3(\text{mod } \mathbb{Z})$
c.) $a \equiv \pm 1/3(\text{mod } \mathbb{Z}), b \equiv 1/2(\text{mod } \mathbb{Z})$

Moreover, we have explicitly

$$G_3(a, 1-a, x, y) = f(x,y)^a \sqrt{\frac{g(x,y)}{\Delta}}$$

where

$$\Delta = 1 + 4x + 4y + 18xy - 27x^2y^2$$

and

$$xf^3 - y = f - f^2, \qquad g(g - 1 - 3x)^2 = x^2\Delta.$$

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Moduli spaces for linear differential equations and the Painlevé equations

Marius van der Put

This is joint work with Masa-Hiko Saito (university of Kobe). The Painlevé equations are related to certain families of connections on the complex projective line. A systematic search yields ten such families. For each one of them the 'monodromy space' is computed and turns out to be a family of affine cubic surfaces. These families are brought into relation with the Okamoto–Painlevé pairs.
The connections that are of interest here can be written in the form $\frac{d}{dz}+A(z)$ where $A(z)$ is a 2×2-matrix with rational entries and trace 0. The number of elements $\#S$ of the singular locus $S\subset \mathbb{P}^1_{\mathbb{C}}$ is given and for each $s\in S$, the Katz invariant $r(s)$ is prescribed. This gives rise to a 'naive' space of connections $\mathcal{M}$. The data also define a monodromy space $\mathcal{R}$ which is build out of the usual monodromy, links and Stokes matrices. This is an affine variety over $\mathbb{C}$. There is an analytic Riemann–Hilbert map $RH:\mathcal{M}\to\mathcal{R}$ which associates to each connection its monodromy data. The surjectivity of the Riemann–Hilbert morphism $\mathcal{M}\to\mathcal{R}$ is proven and connects with recent work of Bolibruch, Malek and Mitschi (Expo. Math 24(3), 2006, pp 235-272).
The fibres of RH are (by definition) the isomonodromic families. The morphism RH forgets the position of the points S and the coefficients of the (generalized) eigenvalues at the irregular singular points. The list of ten families is deduced from the requirement that the fibers of RH have dimension 1.
For each of the ten cases the monodromy space $\mathcal{R}$ is provided with a natural morphism $par:\mathcal{R}\to\mathcal{P}$ to a space of parameters. The theory of the Okamoto–Painlevé pairs predicts that the fibres of par have a certain Painlevé pair as analytic resolution. The monodromy spaces $\mathcal{R}\to\mathcal{P}$ are computed with the help of precise information given by multisummation and invariant theory. It turns out that $\mathcal{R}\to\mathcal{P}$ is a family of affine cubic surfaces with three lines at infinity. The singularities of the fibers of par are due to reducibility and resonance. The preimages of these singularities of the cubic surface in the Okamoto–Painlevé pair consists of a number of curves, called Riccati curves. The latter correspond to Riccati solutions of the corresponding Painlevé equation. The Okamoto–Painlevé pairs are classified by extended Dynkin diagrams and their Riccati curves are known. This enables to identify the Okamoto-Painlevé pair, the Dynkin diagram and the type of Painlevé equation for each of the ten families. The first of the ten families is the classical one. Six of the remaining 9 are known by Garnier, Fuchs, Jimbo, Miwa, Ueno, Flaschka, Newell et all. The 3 new ones were also recently found by Ohyama and Okumura. We give now the table of the ten cases and the list of the momodromy spaces $\mathcal{R}\to\mathcal{P}$.

Dynkin	Painlevé eqn	$r(0)$	$r(1)$	$r(\infty)$	$r(t)$	$\dim \mathcal{P}$
$\tilde{D}_4$	PVI	0	0	0	0	4
$\tilde{D}_5$	PV	0	0	1	-	3
$\tilde{D}_6$	PIII(D6)	0	0	1/2	-	2
$\tilde{D}_6$	PIII(D6)	1	-	1	-	2
$\tilde{D}_7$	PIII(D7)	1/2	-	1	-	1
$\tilde{D}_8$	PIII(D8)	1/2	-	1/2	-	0
$\tilde{E}_6$	PIV	0	-	2	-	2
$\tilde{E}_7$	PII	0	-	3/2	-	1
$\tilde{E}_7$	PII	-	-	3	-	1
$\tilde{E}_8$	PI	-	-	5/2	-	0

PVI $x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - s_1x_1 - s_2x_2 - s_3x_3 + s_4 = 0,\ s_1, \ldots, s_4 \in \mathbb{C}$.

PV $x_1x_2x_3 + x_1^2 + x_2^2 - (s_1 + s_2s_3)x_1 - (s_2 + s_1s_3)x_2 - s_3x_3 + s_3^2 + s_1s_2s_3 + 1 = 0$ with $s_1, s_2 \in \mathbb{C},\ s_3 \in \mathbb{C}^*$.

PIII(D6) $x_1x_2x_3 - x_1^2 - x_2^2 + s_0x_1 + s_1x_2 - 1 = 0$ with $s_0, s_1 \in \mathbb{C}$.

PIII(D6) $x_1x_2x_3 + x_1^2 + x_2^2 + (1 + \alpha\beta)x_1 + (\alpha + \beta)x_2 + \alpha\beta = 0$ with $\alpha, \beta \in \mathbb{C}^*$.

PIII(D7) $x_1x_2x_3 + x_1^2 + x_2^2 + \alpha x_1 + x_2 = 0$ with $\alpha \in \mathbb{C}^*$.

PIII(D8) $x_1x_2x_3 + x_1^2 - x_2^2 - 1 = 0$.

PIV $x_1x_2x_3 + x_1^2 - (s_2^2 + s_1s_2)x_1 - s_2^2x_2 - s_2^2x_3 + s_2^2 + s_1s_2^3$ with $s_1 \in \mathbb{C},\ s_2 \in \mathbb{C}^*$.

PII $x_1x_2x_3 + x_1 - x_2 + x_3 + s_1 = 0$, with $s_1 \in \mathbb{C}$.

PII $x_1x_2x_3 + x_1 + x_2 + \alpha x_3 + \alpha + 1 = 0$ with $\alpha \in \mathbb{C}^*$.

PI $x_1x_2x_3 + x_1 + x_2 + 1 = 0$.

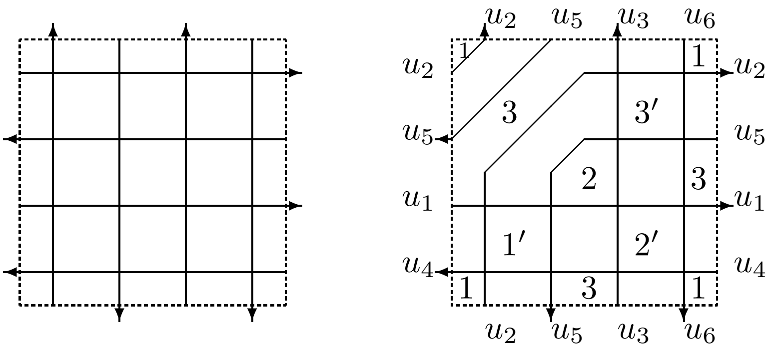
Apart from PVI, the list seems to be new. That we find families of cubic surfaces with 3 lines at ∞ is, at present, a mystery. Finally, for each of the ten cases the isomonodromic families are studied and by using Lax pairs the corresponding Pianlevé equation is computed explicitly.

Fast computation of principal $\mathcal{A}$-determinants by means of dimer models; an introduction through three concrete examples.

Jan Stienstra

$\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ is a set of N vectors in $\mathbb{Z}^{N-2}$ which spans $\mathbb{Z}^{N-2}$ over $\mathbb{Z}$ and for which there exists a linear map $h : \mathbb{Z}^{N-2} \to \mathbb{Z}$ such that $h(\mathcal{A}) = \{1\}$. The *principal $\mathcal{A}$-determinant* was defined by Gelfand, Kapranov, Zelevinsky [1]. It describes the singularities of the associated GKZ $\mathcal{A}$-hypergeometric system of differential equations. It also gives the locus in $(u_1, \ldots, u_N)$-space where the zero-set of the $(N-2)$-variable Laurent polynomial $\sum_{j=1}^{N} u_j \mathbf{x}^{\mathbf{a}_j}$ has singularities. According to GKZ it is a linear combination of monomials in the variables $u_1, \ldots, u_N$ with exponents being the integer points in the so-called *secondary polygon*. GKZ gave the coefficients for the vertices of the secondary polygon. We want all coefficients. In [3] it was proven that the principal $\mathcal{A}$-determinant equals the determinant of (an appropriate form of) the *Kasteleyn matrix* of a *dimer model*. D. Gulotta describes in [2] an efficient algorithm for constructing this dimer model from $\mathcal{A}$, or rather from a $2 \times N$-matrix $B = (b_{ij})$ of which the rows form a basis for the lattice of relations $\{(\ell_1, \ldots, \ell_N) \in \mathbb{Z}^N \mid \ell_1\mathbf{a}_1 + \ldots + \ell_N\mathbf{a}_N = \mathbf{0}\}$. Let me now restrict to concrete examples.

Appell's hypergeometric function F_1:
$\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$. We present Gulotta's algorithm for this example in two pictures. These should be viewed on a torus which is drawn here as a square with opposite sides identified:



Start in the left hand picture with $\sum_{j=1}^{N} |b_{1j}|$ horizontal lines alternatingly oriented $\rightarrow$ resp. $\leftarrow$ and $\sum_{j=1}^{N} |b_{2j}|$ vertical lines alternatingly oriented $\uparrow$ resp. $\downarrow$. The reader may try to guess how the other pictures are created from the starting picture, or look up the algorithm in [2]. The final situation shown in the right-hand picture consists of six oriented zigzag paths with winding numbers on the torus given by the columns of matrix B; $u_1, u_2, u_3, u_4, u_5, u_6$ are variables corresponding with the columns of B and with the zigzags. The connected components of the complement of this zigzag configuration are 2-cells of which the boundary is either oriented or unoriented. There are three cells with positive (resp. negative) oriented boundary which we label 1, 2, 3 (resp. 1′, 2′, 3′). The *dimer model* mentioned in the title "is" this configuration of 2-cells with oriented boundaries. As in [3] we collect the information on the zigzags, their intersection points and intersection numbers and the cells with oriented boundary in the following 3×3-matrix, which is a form of the *Kasteleyn matrix* of the dimer model.

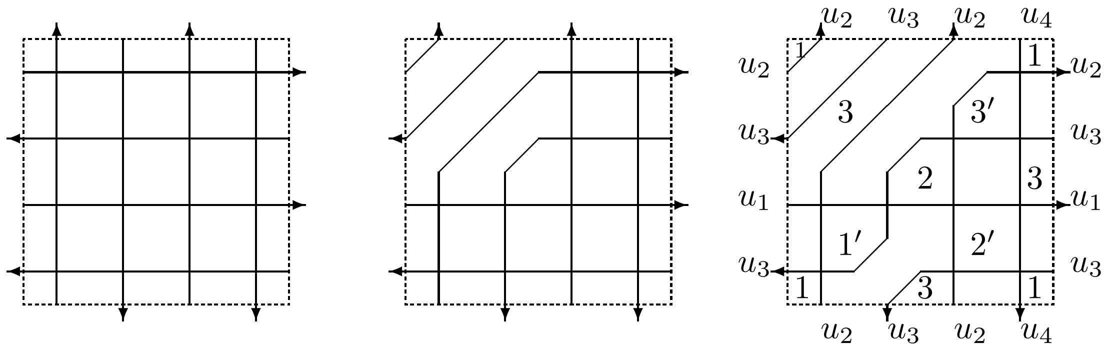
$$\mathcal{K}(u_1, u_2, u_3, u_4, u_5, u_6) = \begin{bmatrix} u_2u_4 & u_1u_5 & u_1u_2 + u_4u_5 \\ u_4u_6 & u_1u_3 & u_1u_6 + u_3u_4 \\ u_2u_6 & u_3u_5 & u_2u_3 + u_5u_6 \end{bmatrix};$$

the matrix entries in this matrix correspond to the intersection points of the zigzags: an entry $p\,u_ku_m$ in position (i, j) means that it is an intersection point of zigzags u_k and u_m, that it is a vertex of the oriented 2-cells i' and j and that p is the number of intersection points of the zigzags u_k and u_m. According to [3] the sought for principal A-determinant for this example is

$$\det(u_1u_2u_3u_4u_5u_6\,\mathcal{K}(u_1^{-1},u_2^{-1},u_3^{-1},u_4^{-1},u_5^{-1},u_6^{-1})) = \\ = u_1u_2u_3u_4u_5u_6(u_1u_2-u_4u_5)(u_2u_3-u_5u_6)(u_1u_6-u_3u_4)\,.$$

This result is written in the GKZ formalism. In order to connect it with Appell's F_1 one must set $x=u_1u_2u_4^{-1}u_5^{-1}$ and $y=u_2u_3u_5^{-1}u_6^{-1}$ (cf. with the rows of B).

Horn's hypergeometric function G_3:
$\mathcal{A}=\begin{bmatrix}1&1&1&1\\0&2&1&3\end{bmatrix}$, $B=\begin{bmatrix}1&1&-2&0\\0&2&-1&-1\end{bmatrix}$. Gulotta's algorithm for Horn's G_3 in three pictures, the first two of which are the same as for Appell's F_1:



The final situation shown in the right-hand picture consists of four oriented zigzag paths with winding numbers on the torus given by the columns of matrix B; u_1,u_2,u_3,u_4 are variables corresponding with the columns of B and with the zigzags. In the complement of this zigzag configuration there are three cells with positive (resp. negative) oriented boundary which we label 1, 2, 3 (resp. $1'$, $2'$, $3'$). As before we collect the information on the zigzags, their intersection points and intersection numbers and the cells with oriented boundary in the following 3×3-matrix, which is a form of the *Kasteleyn matrix* of the dimer model.

$$\mathcal{K}(u_1,u_2,u_3,u_4) = \begin{bmatrix} 3\,u_2u_3 & 1\,u_1u_3 & 2\,u_1u_2 \\ 2\,u_3u_4 & 2\,u_1u_2 & 1\,u_1u_4+3\,u_2u_3 \\ 1\,u_2u_4 & 3\,u_2u_3 & 2\,u_3u_4 \end{bmatrix}.$$

According to [3] the sought for principal A-determinant for this example is

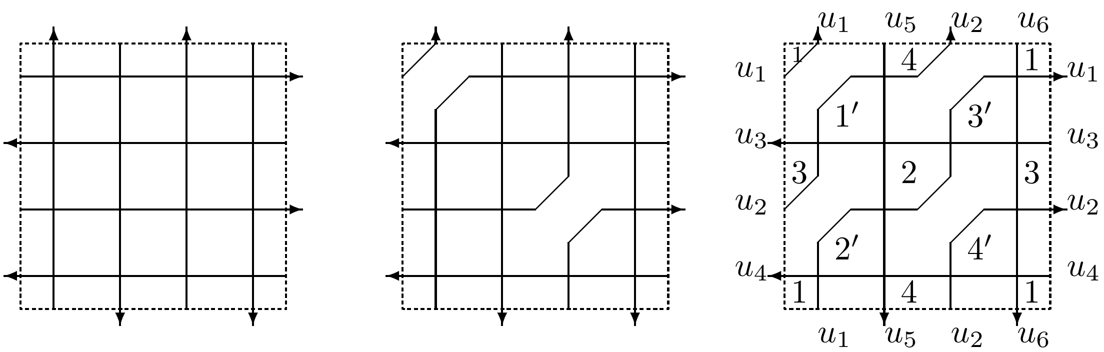
$$\det(u_1u_2u_3u_4\,\mathcal{K}(u_1^{-1},u_2^{-1},u_3^{-1},u_4^{-1})) = \\ = 18\,u_1^2u_2u_3u_4^2 + u_1u_2^2u_3^2u_4 - 4\,u_1u_3^3u_4 - 4\,u_1^2u_2^3u_4 - 27\,u_1^3u_4^3\,.$$

This becomes 0 exactly when the polynomial $u_1+u_3\,x+u_2\,x^2+u_4\,x^3$ does not have three distinctinct non-zero roots.

Appell's hypergeometric function F_4:

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

Gulotta's algorithm for this example in three pictures:



The right-hand picture leads to the matrix

$$\mathcal{K}(u_1, u_2, u_3, u_4, u_5, u_6) \ = \begin{bmatrix} 0 & u_3u_5 & u_2u_3 & u_2u_5 \\ u_1u_4 & u_1u_5 & 0 & u_4u_5 \\ u_1u_6 & u_1u_3 & u_3u_6 & 0 \\ u_4u_6 & 0 & u_2u_6 & u_2u_4 \end{bmatrix}.$$

We leave the computation of the principal $\mathcal{A}$-determinant as an exercise.

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Non-symplectic automorphisms on K3 surfaces

Michela Artebani

(joint work with Alessandra Sarti and Shingo Taki)

The aim of this talk is to present a work in progress on the classification of non-symplectic automorphisms of prime order on K3 surfaces and to show how ball quotients arise as moduli spaces of these surfaces.

A K3 surface X is a simply connected compact complex surface admitting a nowhere vanishing holomorphic 2-form ω_X. Examples of such surfaces are smooth quartic surfaces in $\mathbb{P}^3$, double covers of the projective plane branched along smooth sextic curves and Kummer surfaces. An interesting property of K3 surfaces is that their cohomology group $H^2(X,\mathbb{Z})$, equipped with the cup product, is always isometric to the lattice

$$L_{K3} = U \oplus U \oplus U \oplus E_8 \oplus E_8.$$

Many geometric features of K3 surfaces can be expressed in terms of arithmetic properties of two sublattices of L_{K3}, the Picard lattice and the transcendental lattice:

$$\mathrm{Pic}(X) = \{x \in L_{K3} : (x, \omega_X) = 0\} \qquad T(X) = \mathrm{Pic}(X)^{\perp}.$$

If G is a group acting on a K3 surface, then there is a natural exact sequence

$$1 \to G_0 \to G \xrightarrow{\alpha} \mathbb{Z}_m \to 1,$$

where $\mathbb{Z}_m = \langle \zeta_m \rangle$ is the group of m-th roots of unity and α is defined by

$$\sigma^*(\omega_X) = \alpha(\sigma)\, \omega_X\,, \quad \sigma \in G.$$

The group G is called *symplectic* if $G = G_0$ and *non-symplectic* if $G_0 = \{0\}$. In [6] V.V. Nikulin proved that the transcendental lattice of X is a free module over $\mathbb{Z}[\zeta_m]$. This easily implies that a non-symplectic group is cyclic and its order, if prime, is at most 19. Non-symplectic involutions have been classified in [5], while order three automorphisms have been first investigated in [1] and [Tsu05].
The action of a non-symplectic automorphism σ of prime order p on a K3 surface can be described by looking at the properties of the fixed sets:

$$\mathrm{Fix}(X) = \{x \in X : \sigma(x) = x\} \qquad L^{\sigma} = \{v \in L_{K3} : \sigma^*(v) = v\}.$$

The fixed locus of σ is known to be the disjoint union of points and smooth curves, such that there is at most one curve of genus $g > 0$ if $p > 2$. We denote by n the number of fixed points, by k the number of fixed rational curves and by g the maximum genus of a fixed curve. The lattice L^{σ} is p-elementary (i.e. $(L^{\sigma})^*/L^{\sigma} \cong \mathbb{Z}_p^a$) hence, according to a theorem by Rudakov-Shafarevich in [7], its isometry class is identified by the pair (m, a), where $m = (22 - \mathrm{rk}(L^{\sigma}))/(p-1)$.
In [1] we determined the relations between the pairs of invariants (n, k) and (m, a) for $p = 3$. This allowed us to classify $\mathrm{Fix}(\sigma)$ and L^{σ}.

Theorem. *Let σ be a non-symplectic automorphism of order 3 on a K3 surface. Then $\mathrm{Fix}(\sigma)$ and L^{σ} are described in the following table. Moreover, the action of σ^* on L_{K3} only depends on the pair (n, k) (up to conjugacy).*

Let T^{σ} be the orthogonal complement of L^{σ} in L_{K3}. As a consequence of the last statement in the theorem and of the Torelli-type theorem, the moduli space of K3 surfaces carrying a non-symplectic automorphism of order 3 with fixed locus of type (n, k) is an open dense subset in the quotient $\mathcal{M}_{n,k} = B_{n,k}/\Gamma_{n,k}$, where

$$B_{n,k} = \{\omega \in \mathbb{P}(T^{\sigma} \otimes \mathbb{C}) : \sigma^*(\omega) = \zeta_3 \omega,\ (\omega, \bar{\omega}) > 0\}$$

is isomorphic to a $(m-1)$-dimensional complex ball and

$$\Gamma_{n,k} = \{\gamma \in \mathrm{O}(T^{\sigma}) : \gamma \circ \sigma^* = \sigma^* \circ \gamma\}.$$

The maximal irreducible components of the moduli space are the ball quotients

$$\mathcal{M}_{0,1} \quad \mathcal{M}_{0,2} \quad \mathcal{M}_{3,0}.$$

n	k	g	L^σ
0	1	4	$U(3)$
	2	5	U
1	1	3	$U(3) \oplus A_2$
	2	4	$U \oplus A_2$
2	1	2	$U(3) \oplus A_2^2$
	2	3	$U \oplus A_2^2$
3	0		$U(3) \oplus E_6^*(3)$
	1	1	$U(3) \oplus A_2^3$
	2	2	$U \oplus A_2^3$
	3	3	$U \oplus E_6$
4	1	0	$U(3) \oplus A_2^4$
	2	1	$U \oplus A_2^4$
	3	2	$U \oplus E_6 \oplus A_2$
	4	3	$U \oplus E_8$

n	k	g	L^σ
5	2	0	$U \oplus A_2^5$
	3	1	$U \oplus A_2^2 \oplus E_6$
	4	2	$U \oplus E_8 \oplus A_2$
6	3	0	$U \oplus E_6 \oplus A_2^3$
	4	1	$U \oplus E_6^2$
7	4	0	$U \oplus E_6 \oplus E_6 \oplus A_2$
	5	1	$U \oplus E_6 \oplus E_8$
8	5	0	$U \oplus E_6 \oplus E_8 \oplus A_2$
	6	1	$U \oplus E_8 \oplus E_8$
9	6	0	$U \oplus E_8 \oplus E_8 \oplus A_2$

In [3] S. Kondō proved that $\mathcal{M}_{0,1}$ is birational to the moduli space of curves of genus 4 and $\mathcal{M}_{0,2}$ to the moduli space of 12 points in $\mathbb{P}^1$. Moreover, he showed that the groups $\Gamma_{0,1}$ and $\Gamma_{0,2}$ are commensurable with the monodromy group of a hypergeometric differential equation appearing in Deligne-Mostow's list (no.1, [4]).
In the joint work [2] we extend this ideas to classify non-symplectic automorphisms of prime order $p > 3$.

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Subvarieties of Shimura varieties

Bas Edixhoven

The aim of this lecture is to explain what the main theorem of the article [2] says, and to give an introduction to the formalism of Shimura varieties that is used in this and the next two lectures.
The main motivation for the specific form of the main theorem of [2] comes from an application to Wolfart's work [3] on algebraicity of values of hyper-geometric functions at algebraic arguments. So, in a way, one can say that this article was written especially for this conference.
The following conjecture is named after Yves André and Frans Oort; the terminology used in its statement and in the statement of the theorem following it will be explained in the first 45 minutes of the lecture.

Conjecture 1 (André–Oort). *Let (G, X) be a Shimura datum, $K \subset G(\mathbb{A}_f)$ a compact open subgroup, and $Z \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ a closed algebraic subvariety that contains a Zariski dense set of special points. Then Z is special, i.e., of Hodge type.*

The main theorem of [2] is the following.

Theorem 2. *Let (G, X) be a Shimura datum, $K \subset G(\mathbb{A}_f)$ a compact open subgroup, and $Z \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ a closed algebraic curve that contains an infinite set Σ of special points. Let V be a finite dimensional faithful representation $\underline{\text{of } G,}$ and, for all $h \in X$, let V_h be the corresponding $\mathbb{Q}$-Hodge structure. For $x = \overline{(h, g)}$ in $\mathrm{Sh}_k(G, X)(\mathbb{C})$ let $[V_x]$ be the isomorphism class of V_h. Assume that all $[V_x]$, for x ranging through Σ, are equal. Then Z is special, i.e., of Hodge type.*

As explained in the previous lecture by Gisbert Wüstholz, Theorem 2 fills a gap in Wolfart's proof that if a certain hyper-geometric function has algebraic values at infinitely many algebraic arguments, then the monodromy group of that function is arithmetic. It is important to note that, in Wolfart's situation, the algebraic points at which the algebraic values are taken are known to correspond to abelian varieties with complex multiplication of a fixed type.
Wolfart's arguments involve certain explicit families of abelian varieties with certain types of endomorphisms, so one could think that the use of Shimura varieties in this lecture should be limited to the moduli spaces of precisely these kinds of abelian varieties. But there are at least two good reasons even in this case to use the general terminology of Shimura varieties. The first reason is that we get more flexibility: one can reduce the problem directly to the *smallest* Shimura variety containing Z, and one can reduce to the case $G = G^{\mathrm{ad}}$ (which, in terms of moduli interpretations, is more complicated than one would like). The second reason is that using the general formalism, as established by Deligne in [1], makes the situation actually a lot simpler (although more technical, maybe); the essential data are all encoded in the pair (G, X).

Let us now begin our explanation of what the statement of Theorem 2 above means. In this statement, G is a (connected) reductive linear algebraic group over $\mathbb{Q}$. We give two examples: $\mathbf{GL}_{2,\mathbb{Q}}$ and $\mathrm{GSp}_{2g,\mathbb{Q}}$ (symplectic similitudes).
The symbol X in the pair (G,X) is a $G(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{S},G_{\mathbb{R}})$, the set of morphisms of algebraic groups over $\mathbb{R}$. Here $\mathbb{S}=\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G_m}$. Hence, for every $\mathbb{R}$-algebra A, one has $\mathbb{S}(A)=(\mathbb{C}\otimes_{\mathbb{R}}A)^\times=\{(\begin{smallmatrix} a & -b\\ b & a\end{smallmatrix})\in\mathbf{GL}_2(A)\}$.
In the example $G=\mathbf{GL}_{2,\mathbb{Q}}$ one can take

$$X=\mathcal{H}^\pm=\mathbb{P}^1(\mathbb{C})-\mathbb{P}^1(\mathbb{R})=\mathbf{GL}_2(\mathbb{R})/\mathbb{S}(\mathbb{R}).$$

The importance of $\mathbb{S}$ comes from its property that, for V a finite dimensional $\mathbb{R}$-vector space, to give a Hodge structure on V is the same as giving it an action by $\mathbb{S}$.
The pair (G,X) must satisfy the following three properties:

a.) $\forall h\in X$ the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ is of type $\{(-1,1),(0,0),(1,-1)\}$;
b.) for all $h\in X$: $\mathrm{inn}_{h(i)}$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$, that is, the group $\{g\in G^{\mathrm{ad}}(\mathbb{C})\mid h(i)\overline{g}h(i)^{-1}=g\}$ is compact;
c.) write $G^{\mathrm{ad}}=\prod_i G_i$ with G_i simple, then for all i and all $h\in X$, the induced morphism $\mathbb{S}\to G_{i,\mathbb{R}}$ is not trivial.

These conditions assure: X has a unique complex structure such that every representation V of $G_{\mathbb{R}}$ gives a variation of $\mathbb{R}$-HS on X; the connected components of X are hermitian symmetric domains (notation: X^+); $\pi_0(X)$ is finite, and $G(\mathbb{Q})$ acts transitively on it.
At this point, we know what (G,X) and V_h are.
The adèles. We let $\mathbb{A}_{\mathrm{f}}:=\prod'_p\mathbb{Q}_p=\mathbb{Q}\otimes\hat{\mathbb{Z}}$, and $\mathbb{A}:=\mathbb{A}_{\mathrm{f}}\times\mathbb{R}$; both are topological $\mathbb{Q}$-algebras, $\hat{\mathbb{Z}}$ is open in $\mathbb{A}_{\mathrm{f}}$ and has its own profinite topology. Then K in Theorem 2 is a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$ (the topology on $G(\mathbb{A}_{\mathrm{f}})$ is obtained by embedding G as a *closed* subvariety of an affine N-space over $\mathbb{Q}$, and then restricting the topology of $\mathbb{A}_{\mathrm{f}}^N$). In the example $G=\mathbf{GL}_{2,\mathbb{Q}}$ one can take a maximal compact subgroup $\mathbf{GL}_2(\hat{\mathbb{Z}})$, or, more generally, for $n\in\mathbb{Z}$ non-zero, $\ker(\mathbf{GL}_2(\hat{\mathbb{Z}})\to\mathbf{GL}_2(\mathbb{Z}/n\mathbb{Z}))$.
For (G,X) and K as above, one then defines:

$$\mathrm{Sh}_K(G,X)(\mathbb{C}):=G(\mathbb{Q})\backslash(X\times G(\mathbb{A}_{\mathrm{f}})/K).$$

The set $G(\mathbb{Q})\backslash(\pi_0(X)\times G(\mathbb{A}_{\mathrm{f}})/K)$ is finite; choosing representatives $([X^+],g_i)$ shows that $\mathrm{Sh}_K(G,X)(\mathbb{C})=\coprod_i\Gamma_i\backslash X^+$, with the $\Gamma_i=G(\mathbb{Q})_{[X^+]}\cap g_iKg_i^{-1}$ arithmetic subgroups of $G(\mathbb{Q})$. Baily and Borel have shown that $\mathrm{Sh}_K(G,X)(\mathbb{C})$ is, (log) canonically, the complex analytic variety associated to a quasi-projective complex algebraic variety $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$. For example, $\mathrm{Sh}_{\mathrm{GSp}_{2g}(\hat{\mathbb{Z}})}(\mathrm{GSp}_{2g},\mathcal{H}_g^\pm)_{\mathbb{C}}$ is the moduli space $A_{g,1,\mathbb{C}}$ for complex principally polarised abelian varieties of dimension g.
Varying K gives a projective system of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$, with finite transition morphisms. One defines $\mathrm{Sh}(G,X)_{\mathbb{C}}$ as the projective limit of this system; it is a scheme, not locally of finite type, but say pro-algebraic. The reason to consider this limit is that $G(\mathbb{A}_{\mathrm{f}})$ acts on it, and one has, for $K\subset G(\mathbb{A}_{\mathrm{f}})$ open compact: $\mathrm{Sh}_K(G,X)_{\mathbb{C}}=(\mathrm{Sh}(G,X)_{\mathbb{C}})/K$. The $G(\mathbb{A}_{\mathrm{f}})$-action also gives a nice description of

Hecke correspondences. For K and K' open compact, and for $g \in G(\mathbb{A}_f)$, one has a correspondence $T_g\colon \mathrm{Sh}_K(G,X)_{\mathbb{C}} \leftarrow \mathrm{Sh}_{K\cap gK'g^{-1}}(G,X)_{\mathbb{C}} \to \mathrm{Sh}_{K'}(G,X)_{\mathbb{C}}$, induced by the action of g on $\mathrm{Sh}(G,X)_{\mathbb{C}}$.

Definition 3. *Let (G,X) be a Shimura datum, and $K \subset G(\mathbb{A}_f)$ a compact open subgroup. A closed subvariety S of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ is called* special*, or* of Hodge type*, if there exists a Shimura datum (G',X'), a morphism $f\colon G' \to G$ such that for each $h \in X'$ one has $f \circ h \in X$, and an element $g \in G(\mathbb{A}_f)$, such that S is an irreducible component of the image of:*

$$\mathrm{Sh}(G',X')_{\mathbb{C}} \xrightarrow{f} \mathrm{Sh}(G,X)_{\mathbb{C}} \xrightarrow{\cdot g} \mathrm{Sh}(G,X)_{\mathbb{C}} \xrightarrow{q} \mathrm{Sh}_K(G,X)_{\mathbb{C}},$$

where q denotes the quotient morphism. A point s in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ is called special *if it is a zero-dimensional special subvariety.*

For example, the special points in $A_{g,1,\mathbb{C}}$ are precisely the points that correspond to abelian varieties of CM-type.

At this point of the talk, the statement of Theorem 2 makes sense. However, the notion of Mumford-Tate group associated to a $\mathbb{Q}$-HS helps to understand it better. For (G,X) a Shimura datum and $h \in X$, one lets $\mathbf{MT}(h)$ be the smallest sub algebraic group H of G such that h factors through $H_{\mathbb{R}}$; these $\mathbf{MT}(h)$ are reductive. For P in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ one then has an algebraic $\mathbb{Q}$-group $\mathbf{MT}(P)$ together with a $G(\mathbb{Q})$-conjugacy class of embeddings in G. A point P is then special if and only if $\mathbf{MT}(P)$ is commutative, i.e., a torus. For Z a closed subvariety of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ there is then a *generic* Mumford-Tate group $\mathbf{MT}(Z)$ with the property that for all $P \in Z$ one has $\mathbf{MT}(P) \subset \mathbf{MT}(Z)$, with equality outside a countable union of proper closed subvarieties of Z; the points P where equality occurs are called *Hodge generic*. This $\mathbf{MT}(Z)$ gives us a description of the smallest special subvariety of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ containing Z.

We can now give a very short description of the proof of Theorem 2. Let S be the connected component of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ that contains Z. One first replaces the ambient Shimura variety by the smallest one containing Z, i.e., Z is then Hodge generic. The second step is to replace G by G^{ad}. Then comes the difficult step, where one invokes the help of Hecke and Galois. One manages to produce a Hecke correspondence T_q on S such that all $T_q + T_{q^{-1}}$-orbits in S are dense (for the Archimedean topology), and such that $T_q Z = Z = T_{q^{-1}} Z$. Then, of course, $Z = S$ and Z is special. In order to get $Z \subset T_q Z$ one uses the Galois orbits of elements in Σ: one can arrange for $Z \cap T_q Z$ containing such a Galois orbit that is larger than the degree allows if the intersection would be proper.

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Galois orbits and equidistribution

Andrei Yafaev
(joint work with Emmanuel Ullmo)

In this talk we present an *alternative* between Galois-theoretic and ergodic-theoretic properties of subvarieties of Shimura varieties which yields a strategy for proving the André-Oort conjecture. In a subsequent talk, Bruno Klingler explains how this alternative is used to prove the conjecture. The statement of the André-0ort conjecture is as follows :

Conjecture 1 (André-Oort). *Let S be a Shimura variety and Σ a set of special points in S. The components of the Zariski closure of Σ are special subvarieties.*

Let S be as in the statement and let F be a number field over which S admits a canonical model. The proof is based on the following alternative : let Z_n be a sequence of special subvarieties of S. Assume the Generalised Riemann Hypothesis (GRH) for CM fields. Then *one* of the following occurs :

a.) The degrees of Galois orbits $\deg(Gal(\overline{F}/F)) \cdot Z_n$ go to infinity. Here the degree is taken with respect to the Baily-Borel line bundle on S.
b.) The sequence Z_n is equidistributed in the following sense : let μ_n be the sequence of probability measures canonically attached to Z_n. There exists a special subvariety Z and a subsequence n_k such that μ_{n_k} weakly converges to μ_Z (probability measure canonically attached to Z) and $Z_{n_k} \subset Z$ for all k large enough.

Note that if the second case of the alternative occurs, then the components of Zariski closure of the union of the Z_n are special. This alternative is actually effective. We give effective lower bounds for the degrees of Galois orbits.

Theorem 1.1 (Lower bounds for degrees of Galois orbits). *Assume the GRH for CM fields. Fix a Shimura datum (G, X) and a compact open subgroup K of $G(\mathbb{A}_f)$. We assume that G is adjoint and K is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$. Let F be a number field over which $Sh_K(G, X)$ admits a canonical model. There exists a real number B such that the following holds. Let (H, X_H) be a sub-Shimura datum of (G, X) such that H is the generic Mumford-Tate group on X_H. Let K_H be $H(\mathbb{A}_f) \cap K$. Let T be the connected centre of H. We suppose that T is non-trivial. We let K_T^m and K_T be the maximal compact open subgroup of $T(\mathbb{A}_f)$ and $T(\mathbb{A}_f) \cap K$ respectively. These are also products and we let $i(T)$ be the number of primes p such that $K_{T,p}^m \neq K_{T,p}$ and L_T the splitting field of T. Then for every geometric component V of the image of $Sh_{K_H}(H, X_H)$ in $Sh_K(G, X)$, and for any positive integer N,*

$$\deg(Gal(\overline{E}/E).V) \geq c_N B^{i(T)} \cdot |K_T^m/K_T| \cdot (\log(|\mathrm{disc}(L_T)|))^N.$$

for a real constant c_N depending only on N.

We introduce the notion of T-special subvarieties as follows:

Definition 1.2. *Let T be a torus such that $T(\mathbb{R})$ is compact. A T-sub-Shimura datum (H, X_H) of (G, X) is a sub-Shimura datum such that H^{der} is non trivial and T is the connected centre of the generic Mumford-Tate group of X_H.*

We show that sequences of T-special subvarieties are equidistributed in the sense explained above. This is a consequence of the equidistribution result of Clozel and Ullmo which in turn is a consequence of Ratner's theorem. We next prove the following result, which is the alternative explained above :

Theorem 1.3. *Assume the GRH for CM fields. Let A be an integer. There exists a finite set $\{T_1, \dots, T_r\}$ of $\mathbb{Q}$-tori of G with the following property. Let Z be a special subvariety of S such that $Gal(\overline{F}/F) \cdot Z$ has degree at most A. Then Z is a T_i-special subvariety for some $i \in \{1, \dots, r\}$.*

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The André-Oort conjecture

Bruno Klingler
(joint work with Andrei Yafaev)

This talk, the last in a series of four, was devoted to sketching the proof of the following result, up to now the most general result on the André-Oort conjecture :

Theorem 1 (Klingler-Yafaev)**.** *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$ and let Σ be a set of special subvarieties in $\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$. We make* **one** *of the following assumptions :*

- *a.) Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*
- *b.) Assume that there exists a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that with respect to this representation, the generic Mumford-Tate groups $\mathcal{H}_V$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class as V ranges through Σ.*

Then every irreducible component of the Zariski closure of $\bigcup_{V \in \Sigma} V$ in $\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$ is a special subvariety.

The case where the V's in Σ are of dimension zero is the (conditional) André-Oort conjecture.

Let's recall some notations and results obtained by Ullmo and Yafaev in [3].

Let V be a special subvariety of the Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$. We denote by $\mathcal{H}_V \subset \mathbf{G}$ the generic Mumford-Tate group of V and $(\mathcal{H}_V, X_V)$ the Shimura subdatum of $(\mathbf{G}, X)$ defining V, $E_{\mathcal{H}_V}$ the reflex field of $(\mathcal{H}_V, X_{\mathcal{H}_V})$, $\mathcal{T}_V$ the torus connected center of $\mathcal{H}_V$ (we will say that V is non-strongly special if the torus $\lambda(\mathcal{T}_V)$ is non-trivial, where $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ is the adjoint morphism), $K^{\mathrm{m}}_{\mathcal{T}_V}$ the maximal compact open subgroup of $\mathcal{T}_V(\mathbb{A}_\mathrm{f})$, $K_{\mathcal{T}_V}$ the compact open subgroup

$\mathcal{T}_V(\mathbb{A}_f) \cap K \subset K^{m}_{\mathcal{T}_V}$, $i(\mathcal{T}_V)$ the number of primes p such that $K^{m}_{\mathcal{T}_{V,p}} \neq K_{\mathcal{T}_V,p}$, $\mathbf{C}_V$ the torus $\mathcal{H}_V/\mathcal{H}_V^{\mathrm{der}}$ isogenous to $\mathcal{T}_V$, $d_{\mathcal{T}_V}$ the absolute value of the discriminant of the splitting field L_V of $\mathbf{C}_V$, n_V the absolute degree of L_V, $\beta_V := \log|d_{\mathcal{T}_V}|$ (in particular $\beta_V = 0$ if V is strongly special).
Ullmo and Yafaev proved a lower bound on the degree of Galois orbits of non-strongly special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ (with respect of the natural line bundle L_K on the Baily-Borel compactification of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$) :

Theorem 2 (Ullmo-Yafaev)**.** *Let $(\mathbf{G}, X)$ be a Shimura datum. Let $K = \prod_{p \text{ premier}} K_p$ be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$.*
Assume the GRH for CM fields. Let N be a positive integer. There exist real numbers $B(N) > 0$ and $C(N) > 0$ such that the following holds.
Let $(\mathcal{H}, X_{\mathcal{H}})$ be a Shimura subdatum of $(\mathbf{G}, X)$ and V be a non-strongly special subvariety of $\mathrm{Sh}_{K_{\mathcal{H}}}(\mathcal{H}, X_{\mathcal{H}})_{\mathbb{C}}$, where $K_{\mathcal{H}} \subset \mathcal{H}(\mathbb{A}_f)$ denotes the compact open subgroup $K \cap \mathcal{H}(\mathbb{A}_f)$. Then the following inequality holds :

$$(0.1) \qquad \deg_{L_{K_{\mathcal{H}}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_{\mathcal{H}_V}) \cdot V) > C(N) \cdot B(N)^{i(\mathcal{T}_V)} \cdot |K^{m}_{\mathcal{T}_V}/K_{\mathcal{T}_V}| \cdot \beta_V^N \ ,$$

Furthermore, if one considers only the subvarieties V such that the associated tori $\mathcal{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.

If $V \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is strongly special we define $\alpha_V(N) = 0$.

Let Z be an irreducible subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a Zariski-dense subset $\bigcup_{n \in \mathbb{N}} W_n$, where each W_n, $n \in \mathbb{N}$, is a special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Let's say that a subsequence $(W_{n_k})_{k \in \mathbb{N}}$ of $(W_n)_{n \in \mathbb{N}}$ is admissible if $\bigcup_{k \in \mathbb{N}} W_{n_k}$ is still Zariski-dense in Z. Using theorem 2, Ullmo and Yafaev proved that if the sequence $(\alpha_{V_n}(N)\beta_{V_n})_{n \in \mathbb{N}}$ is bounded then the tori $\lambda(\mathcal{T}_{W_n})$, $n \in \mathbb{N}$, are all equal (after possibly taking an admissible subsequence). Recall that each W_n defines a canonical probability measure μ_{W_n} on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$, supported on W_n, associated to the locally Hermitian structure of W_n. One then easily proves, generalizing Clozel and Ullmo [1], that the sequence of probability measures μ_{W_n}, $n \in \mathbb{N}$, weakly converges to the probability measure μ_W of some special subvariety W (after possibly an admissible extraction). Moreover for n large, W_n is contained in V. One easily concludes in this case that $Z = W$, thus Z is special and we are done in this case.

Thus we are reduced in the proof of theorem 1 to the case where (after any possible admissible extraction) the function $(\alpha_V(N)\beta_V)_{V \in \Sigma}$ is unbounded. Theorem 1 is then a consequence of the following :

Theorem 3 (Klingler-Yafaev)**.** *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Let F be a number field containing the reflex field $E(\mathbf{G}, X)$.*
Let Z be a Hodge-generic F-irreducible F-subvariety of the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set $\mathbf{\Sigma}$, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$ and

such that for some positive integer N and for any admissible modification Σ' of Σ the set $\{\alpha_V(N)\beta_V,\ V \in \Sigma'\}$ is unbounded.
We make **one** *of the following assumptions :*

a.) Assume the Generalized Riemann Hypothesis (GRH) for CM fields.
b.) Assume that there is a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that with respect to this representation, the centers $\mathcal{T}_V$ of the generic Mumford-Tate groups $\mathbf{MT}(V)$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class as V ranges through Σ.

After possibly replacing Σ by a modification, for every V in Σ there exists a special subvariety V' such that $V \subsetneq V' \subset Z$.

The proof of theorem 3 is divided in 4 steps :

Step 1. We first prove the following geometric criterion for constructing a higher dimensional special subvariety V' in Z :

Theorem 4. *Let $(\mathbf{G}, X)$ be a Shimura datum, $K = \prod_{p \text{ prime}} K_p \subset \mathbf{G}(\mathbb{A}_\mathrm{f})$ an open compact subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$. We assume there exists a prime p_0 such that the compact open subgroup $K_{p_0} \subset \mathbf{G}(\mathbb{Q}_{p_0})$ is neat. Let F be a number field containing the field of definition of $S_K(\mathbf{G}, X)_\mathbb{C}$.*
Let V be a non-strongly special subvariety of $S_K(\mathbf{G}, X)_\mathbb{C}$ contained in a Hodge-generic F-irreducible F-subvariety Z of $S_K(\mathbf{G}, X)_\mathbb{C}$.
Let l be a prime number splitting $\mathcal{T}_V$ and m an element of $\mathcal{T}_V(\mathbb{Q}_l)$. We assume that the compact open subgroup K is of the form $K = K^l \cdot K_l$, where K^l is a compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f}^l)$ and K_l is a compact open subgroup of $\mathbf{G}(\mathbb{Q}_l)$.
Suppose that Z satisfies the conditions

(1) *$Z \subset T_m Z$.*
(2) *for every k_1 and k_2 in K_l the image of $k_1 m k_2$ in $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ generates an unbounded subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$*

Then Z contains a special subvariety V' containing V properly.

Step 2. We exhibit some Hecke correspondences candidate for satisfying theorem 4 :

Theorem 5. *Let $(\mathbf{G}, X)$ be a Shimura datum, $K \subset \mathbf{G}(\mathbb{A}_\mathrm{f})$ a neat open compact subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f})$ and F a number field containing the field of definition of $S_K(\mathbf{G}, X)_\mathbb{C}$. There is a positive integer k such that the following holds.*
Let V be a non-strongly special subvariety contained in a Hodge-generic F-irreducible F-subvariety Z of $S_K(\mathbf{G}, X)_\mathbb{C}$.
Let l be a prime number splitting $\mathcal{T}_V$ and m an element of $\mathcal{T}_V(\mathbb{Q}_l)$. We assume that the compact open subgroup K is of the form $K = K^l \cdot K_l$, where K^l is a compact open subgroup of $\mathbf{G}(\mathbb{A}_\mathrm{f}^l)$ and K_l is a compact open subgroup of $\mathbf{G}(\mathbb{Q}_l)$ contained in an Iwahori subgroup I_l of $\mathbf{G}(\mathbb{Q}_l)$ (c.f. next paragraph) in good position with respect to $\mathcal{T}_V$.
Then there exists an element $m \in \mathcal{T}_V(\mathbb{Q}_l)$ satisfying the following conditions :

(1) $\mathrm{Gal}(\overline{F}/F) \cdot V \subset Z \cap T_m Z$.

(2) *For every* $k_1, k_2 \in K_l$ *the image of* $k_1 m k_2$ *in* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ *generates an unbounded subgroup of* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$.
(3) $[K_l : K_l \cap m K_l m^{-1}] < l^k$.

Step 3. We use theorem 2, theorem 4, theorem 5 to show (under one of the assumption of theorem 1) that the existence of a prime number l satisfying certain conditions forces a subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a non-strongly special subvariety V to contain a special subvariety V' containing V properly.

Theorem 6. *Assume the GRH.*
Let $(\mathbf{G}, X)$ *be a Shimura datum,* $K = \prod_{p \text{ prime}} K_p$ *a neat compact open subgroup of* $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$*, and* F *a number field containing the reflex field* $E(\mathbf{G}, X)$*. Let* N *be a positive integer,* k *the constant defined in theorem 5, and* f *any integer such that for any prime* l *the index of an Iwahori subgroup* I_l *in a maximal compact open subgroup containing* K_l *is at most* l^f*.*
Let $V \subset S_K(\mathbf{G}, X)_{\mathbb{C}}$ *be a non-strongly special subvariety. Let* l *be a prime splitting* $\mathcal{T}_V$ *such that* K_l *is contained in a special maximal compact subgroup* $K_l^{\max}$ *of* $\mathbf{G}(\mathbb{Q}_l)$ *in good position with respect to* $\mathcal{T}_V$*.*
Let Z *be a Hodge-generic* F*-irreducible* F*-subvariety* $S_K(\mathbf{G}, X)_{\mathbb{C}}$ *containing* V *and satisfying*

$$l^{(k+2f)\cdot 2^{a(r+1)}} \cdot (\deg_{L_K} Z)^{2^{a(r)}} < C(N)\alpha_V(N)\beta_V^N \quad , \tag{0.2}$$

where r *denotes* $\dim Z - \dim V$ *and* $a : \mathbb{N} \longrightarrow \mathbb{N}$ *is the function defined by* $a(n) = \frac{n(n+1)}{2}$.
Then Z *contains a special subvariety* V' *that contains* V *properly.*
Moreover if one considers only the subvarieties V *such that the associated tori* $\mathcal{T}_V$ *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class, then the assumption of the GRH can be dropped.*

Step 4. Finally we use the effective Cebotarev theorem under GRH (or the usual Cebotarev theorem if we are under the second assumption of theorem 1) to exhibit a prime l satisfying theorem 6 and thus conclude the proof of theorem 3.

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Hypergeometric systems and the Cohen–Macaulay property

Laura Felicia Matusevich
(joint work with Ezra Miller and Uli Walther)

A-hypergeometric systems, or GKZ systems, were introduced by Gelfand, Graev, Kapranov and Zelevinsky in the late 1980s. The importance of this notion lies in the fact that it explicitly connects the classical theory of hypergeometric functions to the rich combinatorial and algebro-geometric theory of toric varieties. The abundance of tools provided by this connection has fueled many hypergeometric advances. Perhaps more surprisingly, hypergeometric intuition has already lead to new discoveries in combinatorial commutative algebra.

In this note, we explore how a fundamental notion of commutative algebra, namely Cohen–Macaulaynes (applied to the coordinate ring of an affine toric variety), plays a crucial role in the A-hypergeometric setting. Our results are contained in the article [1].

Let $A = (a_{ij})$ be an integer $d \times n$ matrix of rank $d \leq n$, whose columns form a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. Assume also that the columns $a_1, \ldots, a_n$ of A span a *pointed cone*, that is, a cone that contains no lines. Note that this is satisfied if all the columns of A lie in a hyperplane away from the origin. This latter condition will characterize hypergeometric systems of differential equations having *regular singularities*, and so we call a matrix A satisfying it *regular*.

The matrix A induces a $\mathbb{Z}^d$-grading, called the A-*grading*, in the Weyl algebra D_n of linear partial differential operators in $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_n$ via $\deg(\partial_i) = a_i$, $\deg(x_i) = -a_i$. Denote $\theta_i = x_i\partial_i$ and define the *Euler operators*:

$$E_i = \sum_{j=1}^{n} a_{ij}\theta_j \ ; \quad i = 1, \ldots, d.$$

The most important A-graded ideal in the polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_n] = \mathbb{C}[\partial]$ (which is a commutative subring of D_n) is the *toric ideal*

$$I_A = \langle \partial^u - \partial^v \mid u, v, \in \mathbb{N}^n, Au = Av \rangle \subseteq \mathbb{C}[\partial],$$

whose zero set in $\mathbb{C}^n$ is an *affine toric variety*. (If the matrix A is regular, this zero set is the cone over a projective toric variety in $\mathbb{P}^{n-1}$.) The ring $\mathbb{C}[\partial]/I_A$ is a d-dimensional domain.

The main object in this note is the A-*hypergeometric system* associated to a parameter vector $\beta \in \mathbb{C}^d$:

$$H_A(\beta) = I_A + \langle E - \beta \rangle \subseteq D_n$$

where $E - \beta$ denotes the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$.

Any left D_n-ideal can be thought of as a system of partial differential equations; its *holonomic rank* is the dimension of the space of germs of holomorphic solutions at a generic nonsingular point. One of the earliest results about A-hypergeometric systems, proved by Gelfand, Kapranov and Zelevinsky in the case that A is regular, and by Adolphson in the general case, concerns their holonomic rank. We define the *volume* of A to be $\mathrm{vol}(A)$, the volume of the convex hull of $\{0, a_1, \dots, a_n\} \subset \mathbb{Z}^d$, normalized so that the unit simplex has volume 1.

Theorem 3. *The holonomic rank of an A-hypergeometric system,* rank $(H_A(\beta))$ *is greater than or equal to* $\mathrm{vol}(A)$ *for all parameters $\beta \in \mathbb{C}^d$, and equality holds for generic β. If the ring $\mathbb{C}[\partial]/I_A$ is Cohen–Macaulay, then* rank $(H_A(\beta)) = \mathrm{vol}(A)$ *for all $\beta \in \mathbb{C}^d$.*

We recall the definition of a Cohen–Macaulay ring; this notion has many equivalent formulations, we choose here the most relevant in our context.

Definition 4. Let $R = \mathbb{C}[\partial]/I$ of dimension d, where I is an ideal of $\mathbb{C}[\partial]$ that is homogeneous for some grading, and let $\mathfrak{m} = \langle \partial_1, \dots, \partial_n \rangle$ be the graded maximal ideal. For $f \in R$, $R[f^{-1}]$ is the localization of R at f. Consider the following Čech complex

$$0 \to \bigoplus_i R[\partial_i^{-1}] \to \cdots \to \bigoplus_{i_1 < \cdots < i_k} R[(\partial_{i_1} \cdots \partial_{i_k})^{-1}] \to \cdots \to R[(\partial_1 \cdots \partial_n)^{-1}] \to 0$$

The *local cohomology* of R at the maximal ideal $\mathfrak{m}$ is the cohomology of the Čech complex above. If the nontop local cohomology of R at $\mathfrak{m}$ vanishes, that is, if $\oplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(R) = 0$, then R is a *Cohen–Macaulay ring.*

Note that the local cohomology modules of the ring $\mathbb{C}[\partial]/I_A$ inherit the A-grading from I_A, therefore we may consider the following set of *true degrees* of local cohomology:

$$\mathrm{tdeg}(\oplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(\mathbb{C}[\partial]/I_A)) = \{\alpha \in \mathbb{C}^d \mid \big(\oplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(\mathbb{C}[\partial]/I_A)\big)_\alpha \neq 0\} \subset \mathbb{Z}^d.$$

The Zariski closure of this set in $\mathbb{C}^d$ is by definition the set of *quasidegrees* of local cohomology, denoted by $\mathrm{qdeg}(\oplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(\mathbb{C}[\partial]/I_A))$.

The following theorem is our main result.

Theorem 5. $\{\beta \mid \mathrm{rank}\ (H_A(\beta)) > \mathrm{vol}(A)\} = \mathrm{qdeg}(\oplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(\mathbb{C}[\partial]/I_A))$.

An immediate corollary of this result is that rank $(H_A(\beta))$ is constant as a function of β if and only if $\mathbb{C}[\partial]/I_A$ is Cohen–Macaulay. Thus, Theorem 5 refines the first part of Theorem 3 and provides a converse for the second.

The proof of Theorem 5 does not involve an explicit construction of solution spaces, or a calculation of the rank. Rather, the idea is to detect rank jumps using homological methods. The situation is entirely analogous to a familiar one in algebraic geometry: if one has a family over a base, then under mild assumptions, fiber dimension is upper semicontinuous on the base, and one can tell where fiber dimension jumps from the nonvanishing of the homology of an adequately chosen Koszul complex.

Applying a fundamental result of Kashiwara, we can express holonomic rank in an algebraic manner. In our case, the formula is

$$\operatorname{rank}(H_A(\beta)) = \dim_{\mathbb{C}(x)} \mathbb{C}(x) \otimes_{\mathbb{C}[x]} (D_n/H_A(\beta)),$$

and since $\mathbb{C}(x) \otimes_{\mathbb{C}[x]} (D_n/H_A(\beta))$ is a commutative object (a finitely generated $\mathbb{C}(x)[\beta]$-module, where the parameters β are now thought of as indeterminates), homological tools from algebraic geometry become available to us.

Once we have a homological way of detecting rank jumps, we need to connect it to the local cohomology of $\mathbb{C}[\partial]/I_A$. Our main tool is a Koszul-like complex, called the *Euler–Koszul complex*, based on the sequence $E - \beta$. If M is an A-graded $\mathbb{C}[\partial]$-module, its Euler–Koszul homology is denoted by $\mathcal{H}_\bullet(E-\beta; M)$. A hint of where the quasidegrees arise comes from the following fact: $\beta \notin \operatorname{qdeg}(M)$ if and only if $\mathcal{H}_0(E-\beta; M) = 0$, if and only if $\mathcal{H}_i(E-\beta; M) = 0$ for all i. (This requires some reasonable assumptions on the A-graded module M).

A recent development is that there are now combinatorial formulas to determine rank $(H_A(\beta))$, due to Okuyama for $d = 3$, and Berkesch for general d. However, the precise relation between rank and local cohomology remains elusive. Furthermore, we are far from understanding how the solutions of $H_A(\beta)$ vary in terms of the parameters. These are among the many exciting open problems in the study of A-hypergeometric functions and differential equations.

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Rational bivariate hypergeometric functions

Alicia Dickenstein

(joint work with Eduardo Cattani, Fernando Rodríguez Villegas)

This talk described ongoing work with Eduardo Cattani and Fernando Rodríguez Villegas on the structure of all codimension-two lattice configurations A which admit a *stable rational A-hypergeometric function*, and the corresponding translation to classical bivariate hypergeometric series.

Following Gel′fand, Kapranov and Zelevinsky [10, 11] we associate to a configuration $A = \{a_1, \ldots, a_n\} \subset \mathbf{Z}^d$ of lattice points spanning $\mathbf{Z}^d$ (which we encode as the columns of a $d \times n$ integer matrix, also called A) and a vector $\beta \in \mathbf{C}^d$, a left ideal in the Weyl algebra in n variables $D_n := \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ as follows.

Definition 6. The *A-hypergeometric system* with exponent β is the left ideal $H_A(\beta)$ in the Weyl algebra D_n generated by the *toric operators* $\partial^u - \partial^v$, for all $u, v \in \mathbb{N}^n$ such that $Au = Av$, and the *Euler operators* $\sum_{j=1}^n a_{ij} z_j \partial_j - \beta_i$ for $i = 1, \ldots, d$.

A local holomorphic function $F(x_1, \dots, x_n)$ is *A-hypergeometric of degree β* if it is annihilated by $H_A(\beta)$. We say that F is *stable* if it is not annihilated by any partial derivative. The codimension of A is the rank of its integer kernel $\mathbf{L}$, which we can assume –without loss of generality– to be equal to $m := n - d$. The A-hypergeometric systems are homogeneous (and complete) versions of classical hypergeometric systems in m variables [9, 8]. The "translation" goes as follows. Let $B \in \mathbf{Z}^{n\times(m)}$ be a matrix whose colums span the integer kernel $\mathbf{L}$ of A, and also call $B = \{b_1, \dots, b_n\} \subset \mathbf{Z}^m$ the *Gale dual* lattice configuration given by the rows of B. Consider the surjective open map

$$x^{\mathcal{B}} : (\mathbf{C}^*)^n \to (\mathbf{C}^*)^m, \quad x \mapsto (\prod_{j=1}^{n} x_j^{b_{j1}}, \dots, \prod_{j=1}^{n} x_j^{b_{jm}}).$$

Let $U \subseteq (\mathbf{C}^*)^n$, $V = x^{\mathcal{B}}(U)$ be simply connected open sets and denote by $y = (y_1, \dots, y_m)$ the coordinates in V, $(\theta_y)_i = y_i \frac{\partial}{\partial_{y_i}}, i = 1, \dots, m$. Given a holomorphic function $\psi \in \mathcal{O}(V)$, call $\varphi = x^c \psi(x^{\mathcal{B}})$, where $c \in \mathbf{C}^n$. Then, $\big(\sum_{j=1}^n a_{kj} x_j \partial_{x_j}\big)(\varphi) = (A \cdot c)_k \varphi$, for $k = 1, \dots d$. Moreover, for any $u = \mathcal{B} \cdot \lambda \in \mathbf{L}$, we have that $T_u(\varphi) = 0$ if and only if $H_u(\psi) = 0$, where T_u and H_u denote the following differential operators in n and m variables respectively:

$$T_u = \prod_{u_j>0} \left(\frac{\partial}{\partial_{x_j}}\right)^{u_j} - \prod_{u_j<0} \left(\frac{\partial}{\partial_{x_j}}\right)^{-u_j}$$

$$H_u = \prod_{u_j>0} \prod_{l=0}^{u_j-1} (b_j \cdot \theta_y + c_j - l) - y^\lambda \prod_{u_j<0} \prod_{l=0}^{|u_j|-1} (b_j \cdot \theta_y + c_j - l).$$

The ideal $H_A(\beta)$ is always holonomic. We will assume that the points in A lie in a hyperplane off the origin. In this case, A is called regular and by a result of Hotta, $H_A(\beta)$ has regular singularities. The singular locus of the hypergeometric D_n-module $D_n/H_A(\beta)$ equals the zero locus of the principal A-determinant E_A [10, 12]. The main irreducible factor of E_A is the sparse discriminant D_A of the configuration A; the remaining irreducible factors are the sparse discriminants corresponding to the facial subsets of A. A regular configuration A is called gkz-rational if it admits a stable rational A-hypergeometric function for some integer homogeneity β. This means that discriminant D_A depends on all the variables and there exists a rational hypergeometric function F with poles on the discriminant locus $\{D_A = 0\}$.

Conjecture 1.3 in [5] asserts that the only gkz-rational configurations are those affinely equivalent to an *essential Cayley configuration*. Besides the general case and the cases $d = 2, 3, n - 1$ proved in that paper, the case of codimension-two configurations in dimension four was set in [4]. We are now able to show that the conjecture holds for all codimension two configurations.

Definition 7. A codimension-two lattice configuration A is called Cayley essential if the number of points $n = d + 2$ is of the form $2r + 3$, and there exist lattice configurations A_i, $i = 1, \dots, r+1$, of $\mathbf{Z}^r$ such that

$$A = e_1 \times A_1 \cup \dots e_{r+1} \times A_{r+1}.$$

Moreover, all but one of the A_i's contain two points and the remaining one contains three points, and the affine span of any proper subset $\{A_{i_1}, \dots, A_{i_{r'}}\}(r' \leq r)$, is of dimension r'. Equivalently, the vectors in the Gale configuration $b_1, \dots, b_{2r+3}$ consist of r pairs of vectors $\{b_i, -b_i\}$ and a triple of vectors spanning $\mathbf{Z}^2$ and adding up to zero.

The translation of our main result to classical hypergeometric functions in two variables essentially asserts that the only infinite Horn hypergeometric series in the first quadrant which are expansions of rational functions have the form $P(\theta_y)(f_s)$ with $s = (s_1, s_2) \in \mathbf{Z}^2_{>0}$, where f_s is the hypergeometric function

$$f_s(x) := \sum_{n_1, n_2 \geq 0} \frac{(s_1 n_1 + s_2 n_2)!}{(s_1 n_1)!(s_2 n_2)!} x_1^{n_1} x_2^{n_2},$$

and P is a bivariate polynomial which is a product of linear forms with integer coefficients.
For an essential Cayley configuration A, the sparse discriminant D_A agrees with the resultant of the essential family of configurations $A_1, \dots, A_{r+1}$[12], since a generic sparse polynomial f with support A decomposes as

$$f(y_1, \dots, y_{r+1}, t_1, \dots, t_r) = y_1 f_1(t) + \dots + y_{r+1} f_{r+1}(t), \tag{0.1}$$

where f_i is a generic polynomial with exponents in A_i for all $i = 1, \dots, r+1$. The conditions in the previous definition ensure that for any $i = 1, \dots, r+1$, the zero set in the torus $(\mathbf{C}^*)^n$ of polynomials $\{f_j, j = 1, \dots, r+1, j \neq i\}$ with generic coefficients is non empty.
The hypothesis that the Cayley configuration is essential ensures that D_A depends on all the variables. Moreover, as shown in [5] essential Cayley configurations are gkz-rational. This is proved by exhibiting an explicit rational A-hypergeometric function constructed as a toric residue in the sense of [3, 7], associated with the configurations $A_1, \dots, A_{r+1}$ and an integer parameter vector β. Conjecture 5.9 in [5] asserts that a suitable derivative of any stable rational hypergeometric function is a constant multiple of an explicit toric residue We checked this statement for binomial residues in [6]. We now prove this conjecture for all codimension two configurations A.
Besides our previous results on the structure of gkz-rational configurations, our main tools in this work are the following. The *diagonals* of a rational bivariate power series f define algebraic one-variable functions [13]. When f is a Horn hypergeometric series, these univariate functions are classical hypergeometric functions. We reduce the study of these one-variable functions to those studied by Beukers-Heckmann [1] (see also [2, 14, 15]). In order to show that any sufficiently high derivative of a Laurent A-hypergeometric series is a toric residue, we need to make precise the support of Laurent series expansions of hypergeometric rational functions.

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Complex hyperbolic geometry and the monster simple group (conjectural)

Daniel Allcock

I explained the ideas and coincidences which led me to conjecture in [2] that a group closely related to the monster simple group is got from the orbifold fundamental group of a certain 13-dimensional complex-analytic variety by adjoining a certain relation.

Conway conjectured [6] that a group he called the bimonster is generated by 16 involutions satisfying certain braid and commutation relations (the ones specified by the Y_{555} diagram), together with one extra relation $w^{10} = 1$. The bimonster is $(M \times M)$:2, where M is the monster simple group. (Conway was working with the bimonster rather than the monster because it made working with a subgroup $\frac{1}{2}(S_5 \times S_{12})$ of M more convenient.) Ivanov [8] and Norton [9] proved this. I found the Y_{555} diagram appearing in my work in complex hyperbolic reflection groups [1], so naturally I wondered if there was a connection. For me, it appeared because one of my reflection groups contains 16 triflections (order 3 complex reflections) satisfying exactly the same commutation and braid relations.

How can one compare two groups, similar except with generators of different orders? One way is to find a larger group, with generators of infinite order, of which both groups are quotients. My reflection group approach suggested such a group. Call my group $\mathbf{G}$; it acts on the complex 13-ball B^{13}. Write $\mathcal{H}$ for the union of the mirrors (fixed-point sets) of the triflections, and define $X = B^{13}/\mathbf{G}$ and $X_0 = (B^{13} - \mathcal{H})/\mathbf{G}$. Essentially by construction, B^{13} is the covering space of X which is universal among all those having ramification of degree 3 along $\Delta = \mathcal{H}/\mathbf{G}$ and no other ramification. A way to express this is that $\mathbf{G}$ is got from $\pi_1(X_0)$ by demanding that a loop around Δ have order 3. (Remark: π_1 here means orbifold π_1.) If we instead demand that such a loop has order 2, then we get a group which satisfies all the relations of the bimonster, except perhaps the w^{10} relation. So I conjectured that this quotient actually is the bimonster.

Implicit in the last few sentences is the fact that $\pi_1(X_0)$ has 16 generators that satisfy the braid and commutator relations of the Y_{555} diagram. One may find the generators by picking a suitable point p of the ball and taking certain paths based at p. Each of these travels toward one of 16 nearby mirrors, travels 1/3 of the way around it, and then travels backwards along the translate of the first part of the path. Basak has recently established [5] that these loops do indeed satisfy the braid and commutation relations. It remains open whether they generate $\pi_1(X_0)$, and we don't know what other relations might be present in $\pi_1(X_0)$. It is known that the 16 triflections in $\mathbf{G}$ do generate $\mathbf{G}$; see [4] and [3].

In fact, Basak found that there are 26 mirrors closest to p, so it's natural to adjoin the 10 extra generators to our 16, and it turns out that these 26 satisfy the braid and commutation relations of the incidence graph of the points and lines of $P^2(\mathcal{F}_3)$. Exactly the same thing happens in the bimonster! Conway's 16 involutions extend to 26, satisfying these same commutation and braid relations.

A model for the whole conjecture is the largest of the Deligne-Mostow ball quotients [7], which uniformizes the moduli space of unordered 12-tuples in $\mathbf{C}P^1$. One replaces B^{13} by B^9, our $\mathbf{G}$ by a discrete subgroup $\mathbf{G}^{DM}$ of $U(9,1)$ generated by triflections, and defines $X^{DM}, X_0^{DM}, \mathcal{H}^{DM}, \Delta^{DM}$ as above. Then $\pi_1(X_0^{DM})$ is the spherical braid group on 12 strands, and a loop around Δ^{DM} is one of the standard generators. Killing its cube reduces $\pi_1(X_0^{DM})$ to $\mathbf{G}^{DM}$, while killing its square reduces it to S_{12}. In fact, this example is embedded in our situation, and the S_{12} corresponds to the factor of the $S_{12} \times S_5$ mentioned at the beginning.

This also suggests that X may be a moduli space of some sort of algebra-geometric objects. Whatever that type of objects is, it would have some sort of notion of marking, for which the monodromy group on markings would be the bimonster. The analogy in the Deligne-Mostow case is that an unordered 12-tuple admits a notion of marking for which the monodromy group is S_{12}—which it certainly does, namely an ordering of the points. Another suggestive moduli connection is that the 10-dimensional subvariety of X corresponding to the Y_{551} diagram is the moduli space of cubic threefolds.

There are some more consistency checks on the conjecture, notably that $w^{20} = 1$ in $\mathbf{G}$; please refer to [2], [4] and [5] for more details.
I close with one thing that is not-well-enough known: in the setting of complex triflection groups, the A_4 Dynkin diagram should always make one pay attention. The reason is that it plays the same role as E_8 does in the usual setting of Coxeter groups. If you take 4 triflections satisfying the braid and commutation relations of the A_4 diagram, write G for the group generated and α for a "root" defining one of the triflections, then the G-translates of α span a copy of the E_8 lattice (equipped with a module structure over $\mathbf{Z}[\sqrt[3]{1}]$). You can see three A_4's in the Y_{555} diagram, and two in the $Y_{550} = A_{11}$ diagram, the latter being the one relevant to $\mathbf{G}^{DM}$.

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A geometric approach to tropical Weyl group actions and q-Painlevé equations

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(joint work with Tomoyuki Takenawa)

The aim of this work is to develop the theory of birational representation of Weyl groups associated with algebraic varieties and to apply it in the study of (discrete) Painlevé equations; see [TT06].
At the beginning of the twentieth century, it was discovered by Coble and Kantor, and later by Du Val, that certain types of Cremona transformations act on the configuration space of point sets. Let $X_{m,n}$ be the configuration space of n points in general position in the projective space $\mathbb{P}^{m-1}$. Then, the Weyl group $W(T_{2,m,n-m})$ corresponding to the Dynkin diagram $T_{2,m,n-m}$ (see Figure 1) acts birationally on $X_{m,n}$ and is generated by permutations of n points and the standard Cremona transformation with respect to each m points. An algebro-geometric and modern

interpretation of this theory is due to Dolgachev and Ortland [DO88]; they showed that the Cremona action of $W(T_{2,m,n-m})$ induces a *pseudo isomorphism*, i.e., an isomorphism except for subvarieties of codimension two or higher, between varieties blown-up from $\mathbb{P}^{m-1}$ at generic n points, which they call *generalized Del Pezzo varieties*. It is worth mentioning that if $(m,n)=(3,9)$, the affine Weyl group of type $E_8^{(1)}$ appears and its lattice part gives rise to an important discrete dynamical system, i.e., the elliptic-difference Painlevé equation [Sak01]; see also [ORG01, KMNOY03].

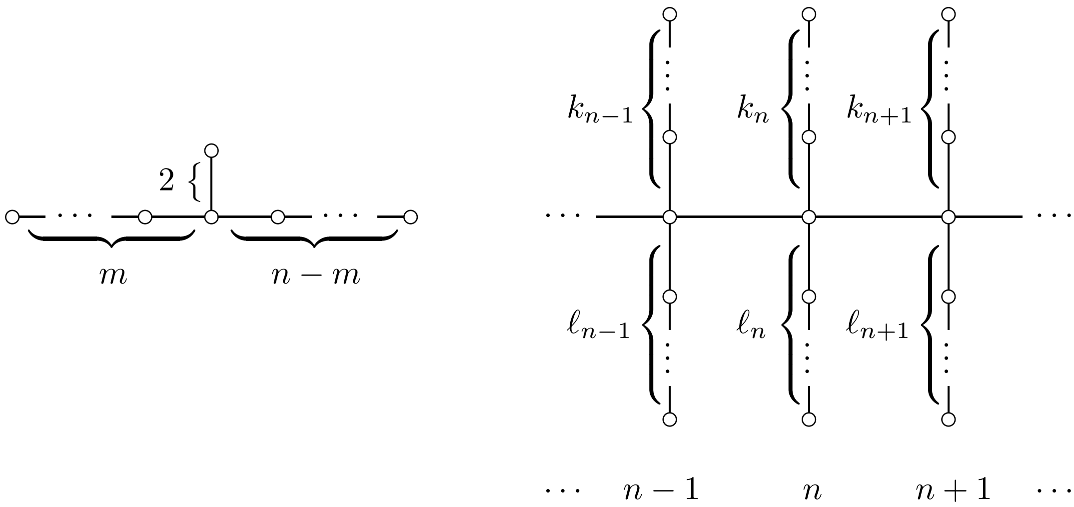

FIGURE 1. Dynkin diagrams $T_{2,m,n-m}$ and $T_{\boldsymbol{\ell}}^{\boldsymbol{k}}$

In this work, starting from a certain rational variety blown-up from $(\mathbb{P}^1)^N$ along appropriate subvarieties that are not only point sets, we construct, as a group of pseudo isomorphisms, a birational representation of Weyl groups corresponding to the Dynkin diagram $T_{\boldsymbol{\ell}}^{\boldsymbol{k}}$. Here $T_{\boldsymbol{\ell}}^{\boldsymbol{k}}$ refers to the tree given in Figure 1, specified by a pair of sequences $\boldsymbol{k}=(k_1,\dots,k_N), \boldsymbol{\ell}=(\ell_1,\dots,\ell_N)\in(\mathbb{Z}_{>0})^N$. It is remarkable that $T_{\boldsymbol{\ell}}^{\boldsymbol{k}}$ includes all of the simply-laced affine cases $A_n^{(1)}$, $D_n^{(1)}$ and $E_n^{(1)}$; for example if $N=3$ and $\boldsymbol{k}=\boldsymbol{\ell}=(1,2,1)$, we have $T_{(1,2,1)}^{(1,2,1)}=D_4^{(1)}$. In an affine case, our construction yields (higher order) q-difference Painlevé equations and their algebraic degree grows quadratically. This representation of Weyl groups is *tropical*, i.e., given in terms of subtraction-free birational mappings and possesses a geometric framework of τ-functions.

We begin with blowing-up $(\mathbb{P}^1)^N$ along certain subvarieties of codimension three. Let X be the rational variety thus obtained. These X's constitute a family. Applying a cohomological technique, we construct the root and coroot lattices of type $T_{\boldsymbol{\ell}}^{\boldsymbol{k}}$ included in the Néron-Severi bilattice $N(X)\simeq(H^2(X,\mathbb{Z}),H_2(X,\mathbb{Z}))$. The associated Weyl group $W=W(T_{\boldsymbol{\ell}}^{\boldsymbol{k}})$ acts on $N(X)$ as isometries. We see that this linear action of W on $N(X)$ leads to a birational representation of W

on the family of varieties itself as a group of pseudo isomorphisms. An element of W naturally induces an appropriate permutation among the set of exceptional divisors on X, as similar to the classical topic: 27 lines (or exceptional curves) on a cubic surface and a Weyl group of type E_6. For the purpose of describing the action of W at the level of defining polynomials of exceptional divisors, we introduce τ-functions. One important advantage of our τ-functions is that we can trace the resulting value for any $w \in W$ by using only the defining polynomials of suitable divisors, although it is generally difficult to compute iterations of rational mappings. In particular, our representation in an affine case provides a discrete dynamical system arising from the lattice part of the affine Weyl group. Such a discrete dynamical system is equipped with a set of commuting discrete time evolutions and its algebraic degree grows in the quadratic order. And it is regarded as a (higher order) q-difference Painlevé equation. Moreover, from a soliton-theoretic point of view, we show some interesting relationships between τ-functions and the *character polynomials* appearing in representation theory of classical groups, i.e., the Schur functions or the universal characters; cf. [Tsu05].

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Discrete analogues of Ramanujan's series for $1/\pi$

Wadim Zudilin

There is a nice example of interrelation between the theories of hypergeometric series, elliptic functions and modular forms; it was for a long time a major method of computing the number π. In 1914, S. Ramanujan [7] reported a list of 17 series for $1/\pi$, most producing rapidly converging (rational) approximations to the number. Some entries from Ramanujan's list are

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3}(4n+1)\cdot(-1)^n = \frac{2}{\pi}, \tag{0.1}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^5}(20n+3)\frac{(-1)^n}{2^{2n}} = \frac{8}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3}(21460n+1123)\cdot\frac{(-1)^n}{882^{2n+1}} = \frac{4}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3}(26390n+1103)\cdot\frac{1}{99^{4n+2}} = \frac{1}{2\pi\sqrt{2}}$$

(in fact, the first formula was proved by G. Bauer [1] already in 1859, much earlier than Ramunujan's birth), and since that time many different proofs and generalizations of the formulae have appeared [9], like Guillera's series [3], [4]

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3(\frac{1}{4})_n(\frac{3}{4})_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}} = \frac{32}{\pi^2}$$

for $1/\pi^2$. Here

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1) & \text{for } n \ge 1, \\ 1 & \text{for } n = 0, \end{cases}$$

denotes the Pochhammer symbol (the rising factorial).

Fix the letter p for primes. It was observed by L. Van Hamme [8] that the Ramanujan–Bauer formula (0.1) and some others admit very nice conjectural p-analogues; namely, he conjectured that

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^3}{n!^3}(4n+1)(-1)^n \equiv \left(\frac{-1}{p}\right)p \pmod{p^3} \quad \text{for } p > 2, \tag{0.2}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre (quadratic residue) symbol. Van Hamme himself was able to prove the congruence (0.2) modulo p (not p^3); recently E. Mortenson [6] gave a proof for (0.2). As far as we know, no progress has been made towards proving the other supercongruences conjectured in [8]. (A congruence modulo a prime p is said to be a supercongruence if it happens to hold modulo some higher power of p.) It seems that *all* known Ramanujan-type formulae for $1/\pi$ and their generalizations admit similar p-analogues. In particular, an easy verification shows the validity of the following supercongruences up to $p < 1000$:

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(20n+3)\frac{(-1)^n}{2^{2n}} \equiv 3\left(\frac{-1}{p}\right)p \pmod{p^3} \quad \text{for } p > 2, \tag{0.3}$$

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(21460n+1123)\frac{(-1)^n}{882^{2n}} \equiv 1123\left(\frac{-1}{p}\right)p \pmod{p^3} \quad \text{for } p > 7,$$

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(26390n+1103)\frac{1}{99^{4n}}\equiv 1103\left(\frac{-2}{p}\right)p \pmod{p^3} \quad \text{for } p>11,$$

(0.4)

$$\sum_{n=0}^{p-1}\frac{(\frac12)_n^3(\frac14)_n(\frac34)_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}}\equiv 3p^2 \pmod{p^5} \quad \text{for } p>2.$$

Unfortunately, all these congruences remain conjectures, except the already mentioned example (0.2) and, as we explain in our talk, the congruences (0.3) and (0.4). We do not dispose of a suitable general method to prove our experimentally discovered supercongruences. But at least in the three cases we can use the so-called WZ-method [2], [3], [4] which results in very simple proofs of (0.2), (0.3) and (0.4).

We wonder whether there exists a deep general theory behind all these Ramanujan-type supercongruences, like the theory of overconvergent p-adic modular forms, which could replace the theory of modular forms used in the proofs of Ramanujan's formulas for $1/\pi$.

We also wonder whether the methods of the classical theory of hypergeometric transformations can be applied to prove the supercongruences mentioned above. We have at least two successful examples of their application: D. McCarthy and R. Osburn [5] use them to prove a different supercongruence conjectured in [8], and E. Mortenson [6] gives a similar proof of the supercongruence (0.2) (whose proof is essentially different from ours and more 'hypergeometrically' involved).

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Hypergeometric Functions and Mean Iterations

Keiji Matsumoto

It is known that the Gauss hypergeometric function $F(\alpha,\beta,\gamma;z)$ satisfies the following quadratic transformation formula

$$(0.1)\qquad (1+z)^{2\alpha}F(\alpha,\alpha-\beta+\frac{1}{2},\beta+\frac{1}{2};z^2)=F(\alpha,\beta,2\beta;\frac{4z}{(1+z)^2}).$$

As analogies of this formula, we give some transformation formulas for Lauricella's hypergeometric function F_D of k-variables defined by

$$F_D(\alpha,\beta,\gamma;z)=\sum_{n_1,\dots,n_k\geq 0}^{\infty}\frac{(\alpha,\sum_{j=1}^k n_j)\prod_{j=1}^k(\beta_j,n_j)}{(\gamma,\sum_{j=1}^k n_j)\prod_{j=1}^k(1,n_j)}\prod_{j=1}^k z_j^{n_j},$$

where $\beta=(\beta_1,\dots,\beta_k)$, $\gamma\neq 0,-1,-2,\dots,$ and $z=(z_1,\dots,z_k)$ satisfies $|z_j|<1$ $(j=1,\dots,k)$.

Theorem 8 ([6]). *For the F_D of 2-variables, we have*

$$\begin{aligned}&\left(\frac{1+z_1+z_2}{3}\right)^c F_D(\frac{c}{3},\frac{c+1}{6},\frac{c+1}{6},\frac{c+1}{2};1-z_1^3,1-z_2^3)\\ =\;& F_D(\frac{c}{3},\frac{c+1}{6},\frac{c+1}{6},\frac{c+5}{6};z_1',z_2'),\end{aligned}$$

where $z=(z_1,z_2)$ is near to $(1,1)$, $(\frac{1+z_1+z_2}{3})^c=1$ at $(1,1)$, and

$$(z_1',z_2')=((\frac{1+\omega z_1+\omega^2 z_2}{1+z_1+z_2})^3,(\frac{1+\omega^2 z_1+\omega z_2}{1+z_1+z_2})^3),\quad \omega=\frac{-1+\sqrt{-3}}{2}.$$

Remark 9. This formula for $c=1$ is given in [4] by K. Koike and H. Shiga. Another transformation formula for F_D of 2-variables is given in [5]. It is extended in [6].

Theorem 10 ([8]). *For the F_D of 2-variables, we have*

$$\begin{aligned}&(z_1z_2)^{\frac{1-c}{2}}\left(\frac{z_1+z_2}{2}\right)^c F_D\left(\frac{3+c}{4},\frac{1+c}{4},\frac{1+c}{4},\frac{3+3c}{4};1-z_1^2,1-z_2^2\right)\\ =\;& F_D\left(c,\frac{1+c}{4},\frac{1+c}{4},\frac{3+3c}{4};1-\frac{z_1(1+z_2)}{z_1+z_2},1-\frac{z_2(1+z_1)}{z_1+z_2}\right),\end{aligned}$$

where (z_1,z_2) is near to $(1,1)$, $(z_1z_2)^{\frac{1-c}{2}}=\left(\frac{z_1+z_2}{2}\right)^c=1$ at $(z_1,z_2)=(1,1)$.

Theorem 11 ([6]). *For the F_D of 3-variables, we have*

$$\begin{aligned}&\left(\frac{1+z_1+z_2+z_3}{4}\right)^{\frac{c}{2}} F_D(\frac{c}{4},\frac{c+2}{12},\frac{c+2}{12},\frac{c+2}{12},\frac{c+2}{3};1-z_1^2,1-z_2^2,1-z_3^2)\\ =\;& F_D(\frac{c}{4},\frac{c+2}{12},\frac{c+2}{12},\frac{c+2}{12},\frac{c+5}{6};z_1',z_2',z_3'),\end{aligned}$$

where (z_1, z_2, z_3) *is near to* $(1,1,1)$, $(\frac{1+z_1+z_2+z_3}{4})^{c/2} = 1$ *at* $(1,1,1)$, *and*

$$(z_1', z_2', z_3') = ((\frac{1-z_1-z_2+z_3}{1+z_1+z_2+z_3})^2, (\frac{1-z_1+z_2-z_3}{1+z_1+z_2+z_3})^2, (\frac{1+z_1-z_2-z_3}{1+z_1+z_2+z_3})^2).$$

We apply these formulas to expressions of common limits of some multiple sequences given by mean iterations. Let $\mathbb{R}_+^*$ be the multiplicative group of positive real numbers. A mean m of $x_1, \ldots, x_k \in (\mathbb{R}_+^*)^k$ is a continuous function satisfying

$$\min(x_1, \ldots, x_k) \leq m(x_1, \ldots, x_k) \leq \max(x_1, \ldots, x_k),$$
$$m(t \cdot x_1, \ldots, t \cdot x_k) = t \cdot m(x_1, \ldots, x_k), \quad \text{for any } t \in \mathbb{R}_+^*.$$

For $x = (x_1, \ldots, x_k) \in (\mathbb{R}_+^*)^k$ and k-means $m_1, \ldots, m_k$, put $\mathbf{x}[0] = x$ and define $\mathbf{x}[1]$, $\mathbf{x}[2], \ldots, \mathbf{x}[n], \ldots,$ as the iteration of these means:

$$\mathbf{x}[n+1] = (x_1[n+1], \ldots, x_k[n+1]) = \mathbf{m}(\mathbf{x}[n]) = (m_1(\mathbf{x}[n]), \ldots, m_k(\mathbf{x}[n])) \in (\mathbb{R}_+^*)^k.$$

In this way, we have a k-ple sequence $\{\mathbf{x}[n]\}_{n\in\mathbb{N}}$. It is known that this k-ple sequence $\{\mathbf{x}[n]\}_{n\in\mathbb{N}}$ has a common limit under some mild conditions for means.

Fact 1 (Invariance Principle in [2])**.** Suppose that $\mathbf{x}[n]$ has a common limit $\mu(x)$. Then it is the unique mean characterized by

(i) μ is continuous,
(ii) $\mu(t, \ldots, t) = t$ for any $t \in \mathbb{R}_+^*$,
(iii) $\mu(m_1(x), \ldots, m_k(x)) = \mu(x)$ for any $x \in (\mathbb{R}_+^*)^k$.

The arithmetic-geometric mean $\mu_1(x_1, x_2)$ is defined as the common limit of the double sequence with initial (x_1, x_2) given by the iteration of the arithmetic mean $m_1(x_1, x_2) = \frac{x_1+x_2}{2}$ and the geometric mean $m_2(x_1, x_2) = \sqrt{x_1x_2}$. Invariance Principle and the transformation formula (0.1) for $\alpha = \beta = 1/2$ imply the expression

$$\mu_1(x_1, x_2) = \frac{x_1}{F(\frac{1}{2}, \frac{1}{2}, 1; 1 - \frac{x_2^2}{x_1^2})}.$$

We give some analogies of this expression. Put $x = (x_1, x_2, x_3)$ and define three functions as

$$m_1(x) = \frac{x_1+x_2+x_3}{3}, \; m_2(x) = \sqrt[3]{m_1(x)^3 - \ell_2(x)^3}, \; m_3(x) = \sqrt[3]{m_1(x)^3 - \ell_3(x)^3},$$

where

$$\ell_2(x) = \frac{x_1 + \omega x_2 + \omega^2 x_3}{3}, \quad \ell_3(x) = \frac{x_1 + \omega^2 x_2 + \omega x_3}{3},$$

and we select a suitable branch for each cubic root. Give a triple sequence $\{\mathbf{x}[n]\}_{n\in\mathbb{N}}$ with $\mathbf{x}[0] = x$ by the iteration of m_1, m_2 and m_3. Though $m_2(x)$ and $m_3(x)$ take complex values for a real triple $\mathbf{x}[0] = x$, $\mathbf{x}[2]$ becomes a real triple again.

Theorem 12 ([4])**.** *The triple sequence* $\{\mathbf{x}[n]\}_{n\in\mathbb{N}}$ *has a common limit* $\mu_2(x)$*, which can be expressed by*

$$\mu_2(x_1, x_2, x_3) = \frac{x_1}{F_D(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1 - (\frac{x_2}{x_1})^3, 1 - (\frac{x_3}{x_1})^3)}.$$

Put $x = (x_1, x_2, x_3)$ and define three means as

$$m_1(x)=\frac{\sqrt{x_1}(\sqrt{x_2}+\sqrt{x_3})}{2},\ m_2(x)=\frac{\sqrt{x_2}(\sqrt{x_3}+\sqrt{x_1})}{2},\ m_3(x)=\frac{\sqrt{x_3}(\sqrt{x_1}+\sqrt{x_2})}{2}.$$

Give a triple sequence $\{\mathbf{x}[n]\}_{n\in\mathbb{N}}$ with $\mathbf{x}[0] = x$ by the iteration of these means.

Theorem 13 ([8]). *The triple sequence* $\{\mathbf{x}[n]\}_{n\in\mathbb{N}}$ *has a common limit* $\mu_3(x)$*, which can be expressed by*

$$\mu_3(x_1, x_2, x_3) = \frac{x_1}{F_D(1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - \frac{x_2}{x_1}, 1 - \frac{x_3}{x_1})}.$$

Put $x = (x_1, x_2, x_3, x_4)$ and define four means as

$$m_1(x) = \frac{x_1 + x_2 + x_3 + x_4}{4}, \qquad m_2(x) = \frac{\sqrt{(x_1 + x_4)(x_2 + x_3)}}{2},$$
$$m_3(x) = \frac{\sqrt{(x_1 + x_3)(x_2 + x_4)}}{2}, \quad m_4(x) = \frac{\sqrt{(x_1 + x_2)(x_3 + x_4)}}{2}.$$

Give a quadruple sequence $\{\mathbf{x}[n]\}_{n\in\mathbb{N}}$ with $\mathbf{x}[0] = x$ by the iteration of these means.

Theorem 14 ([3]). *The quadruple sequence* $\{\mathbf{x}[n]\}_{n\in\mathbb{N}}$ *has a common limit* $\mu_4(x)$*, which can be expressed by*

$$\mu_4(x_1, x_2, x_3, x_4) = \frac{x_1}{F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1 - (\frac{x_2}{x_1})^2, 1 - (\frac{x_3}{x_1})^2, 1 - (\frac{x_4}{x_1})^2)^2}.$$

In 1876, C.W. Borchardt considered in [1] the quadruple sequence by the iteration of four means

$$m_1(x) = \frac{x_1 + x_2 + x_3 + x_4}{4}, \quad m_2(x) = \frac{\sqrt{x_1x_4} + \sqrt{x_2x_3}}{2},$$
$$m_3(x) = \frac{\sqrt{x_1x_3} + \sqrt{x_2x_4}}{2}, \qquad m_4(x) = \frac{\sqrt{x_1x_2} + \sqrt{x_3x_4}}{2}.$$

These means are induced by the 2τ-formulas of theta constants defined on the Siegel upper half space of degree 2. By Thomae's formula, the common limit μ'_4 of this quadruple sequence can be expressed in terms of period integrals of a hyperelliptic curve C corresponding to τ. In [7], we define a generalized arithmetic-geometric mean μ'_{2^g} of 2^g terms by 2τ-formulas of theta constants on the Siegel upper half space of degree g. We express it in terms of period integrals of a hyperelliptic curve C under some conditions for initial terms.

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A variant of Jacobi type formula for Picard curves

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(joint work with Keiji Matsumoto)

1. Introduction

Start from the family of elliptic curves

$$E(\lambda) : w^2 = z(z-1)(z-\lambda) \quad \lambda(\lambda-1) \neq 0.$$

For a real parameter λ in the interval $(0, 1)$, take the ratio of the periods

$$\tau = \int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \Bigg/ \int_{-\infty}^0 \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \quad \text{with } \frac{\tau}{i} > 0.$$

We have the theta representation of the λ-invariant

$$\lambda(\tau) = \frac{\vartheta_{01}^4(\tau)}{\vartheta_{00}^4(\tau)}. \tag{1.1}$$

Here, ϑ_{jk} indicates the Jacobi theta constant

$$\vartheta_{jk}(\tau) = \sum_{n\in\mathbb{Z}} \exp[\pi i(n+\frac{j}{2})^2\tau + 2\pi i(n+\frac{j}{2})\frac{k}{2}] \quad \text{for } \tau \in \boldsymbol{H} = \{\text{Im } \tau > 0\}.$$

The classical theorem of arithmetic geometric mean by Gauss says

$$\frac{1}{M(1,x)} = F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-x^2\right). \tag{1.2}$$

Here $M(a, b)$ means the arithmetic geometric mean with the initial positive values a, b, and $F(\alpha, \beta, \gamma; x)$ indicates the Gauss hypergeometric function. Recall the Jacobi formula relating the elliptic integral and the theta constant:

Theorem 2. (see [J] p.235) *Under the relation (1.1) we have*

$$\vartheta_{00}^2(\tau) = F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda\right) = \frac{1}{\pi}\int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-(1-\lambda))}}. \tag{2.1}$$

Using the duplication formula

$$\begin{cases}\vartheta_{00}^2(2\tau) = \frac{1}{2}\left(\vartheta_0^2(\tau) + \vartheta_{01}^2(\tau)\right), \\ \vartheta_{01}^2(2\tau) = \vartheta_{00}(\tau)\vartheta_{01}(\tau),\end{cases}$$

and by putting $x = \vartheta_{01}^2(\tau)/\vartheta_{00}^2(\tau)$, we can derive the Gauss AGM theorem (1.2) from the above Jacobi formula. In fact, we have

$$M(\vartheta_{00}^2(\tau), \vartheta_{01}^2(\tau)) = \lim_{n\to\infty} \frac{1}{2}\left(\vartheta_{00}^2(2^n\tau) + \vartheta_{01}^2(2^n\tau)\right) = \lim_{\tau\to i\infty} \vartheta_{00}^2(\tau) = 1.$$

So we have $\vartheta_{00}^2(\tau)\, M(1, x) = 1$.

In this article we show a variant of this Jacobi formula for the Picard curves (3.1). The Jacobi formula shows that the modular form $\vartheta_{00}^4(\tau)$ with respect to the principal congruence subgroup $\Gamma(2)$ of $PSL(2,\mathbb{Z})$ has an expression by the Gauss hypergeometric function $F(\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda)$ of the algebraic parameter λ via the inverse of the period map (1.1) for the family of elliptic curves $E(\lambda)$.

Our result is a two dimensional exact analogy of this context. We use the Picard curves with two algebraic parameters λ_1, λ_2. The inverse of the period map is given by (4.1). Our modular form $\vartheta_0^3(u, v)$ is defined on a two dimensional complex ball $\mathscr{D} = \{2\mathrm{Re}\ \mathrm{v} + |\mathrm{u}|^2 < 1\}$, that can be realized as a Shimura variety in the Siegel upper half space of degree 3 by the map (3.4). It is expressed in terms of the Appell hepergeometric function $F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2)$.

As a consequence of our main theorem, we can give a new proof of the three terms AGM theorem already discovered in [K-S] (Theorem 2.2). Still, as a byproduct we show a one variable variant of the Jacobi formula (Theorem 8) for the Borweins curves (7.1).

3. Jacobi type formula for the Picard curves

3.1. **The Picard modular form revisited.** We express the Picard curve with the projective parameters:

$$C(\xi) : y^3 = x(x-\xi_0)(x-\xi_1)(x-\xi_2), \tag{3.1}$$

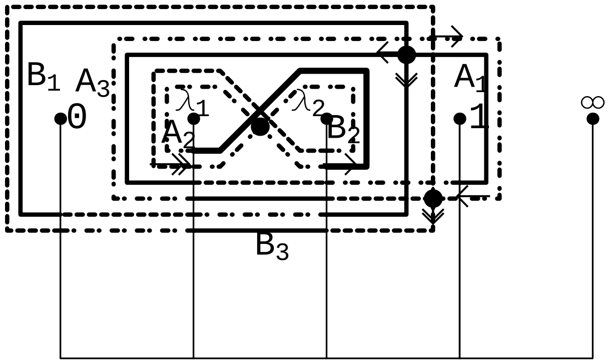
where

$$\xi \in \Xi = \{[\xi_0 : \xi_1 : \xi_2] \in \mathbb{P}^2(\mathbb{C}) : \xi_0\xi_1\xi_2(\xi_0-\xi_1)(\xi_1-\xi_2)(\xi_2-\xi_0) \neq 0\}.$$

It is a curve of genus three, The Jacobian variety $Jac(C(\xi))$ of $C(\xi)$ has a generalized complex multiplication by $\sqrt{-3}$ of type $(2, 1)$. In fact we have a basis of holomorphic differentials

$$\varphi = \varphi_1 = \frac{dz}{w},\ \varphi_2 = \frac{dz}{w^2},\ \varphi_3 = \frac{zdz}{w^2}.$$

Put $\lambda_1 = \xi_1/\xi_0, \lambda_2 = \xi_2/\xi_0$. And we assume $0 < \lambda_1 < \lambda_2 < 1$. Under this condition, we choose the following basis of $H_1(C,\mathbb{Z})$ already used in [S]. Here we put cut lines starting from branch points in the lower half z-plane to get simply connected sheets. The real line (resp. the dotted line, the chained line) indicates a path on the first sheet (resp. the second sheet, the third sheet).



homology basis

Setting $\rho(z,w)=(z,\omega w)$, we have

$$B_3=\rho(B_1),\ A_3=-\rho^2(A_1),\ B_2=-\rho^2(A_2),$$

here ω stands for $\exp[2\pi i/3]$. We have $A_iB_j=\delta_{ij}$. Put

$$\eta_0=\int_{A_1}\varphi,\ \eta_1=-\int_{B_3}\varphi,\ \eta_2=\int_{A_2}\varphi. \tag{3.2}$$

By the analytic continuation, they are multivalued analytic functions on the (λ_1,λ_2)-space $\mathbb{P}^2(\mathbb{C})$. It holds for $i=2,3$

$$\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}\int_{A_1}\varphi_1\\ -\omega^2\int_{B_1}\varphi_1\\ \int_{A_2}\varphi_1\end{pmatrix}=\begin{pmatrix}-\omega^2\int_{A_3}\varphi_i\\ -\int_{B_3}\varphi_i\\ -\omega^2\int_{B_2}\varphi_i\end{pmatrix},\ \begin{pmatrix}\int_{A_1}\varphi_i\\ -\omega\int_{B_1}\varphi_i\\ \int_{A_2}\varphi_i\end{pmatrix}=\begin{pmatrix}-\omega\int_{A_3}\varphi_i\\ -\int_{B_3}\varphi_i\\ -\omega\int_{B_2}\varphi_i\end{pmatrix} \tag{3.3}$$

Set

$$\Omega_1=(\int_{A_j}\varphi_i),\ \Omega_2=(\int_{B_j}\varphi_i),\quad (1\le i,j\le 3).$$

The normalized period matrix of $C(\xi)$ is given by $\Omega=\Omega_1^{-1}\Omega_2$. By the relations of periods (3.3) together with the symmetricity ${}^t\Omega=\Omega$, we can rewrite

$$\Omega=\Omega_1^{-1}\Omega_2=\begin{pmatrix}\frac{u^2+2\omega^2v}{1-\omega} & \omega^2u & \frac{\omega u^2-\omega^2v}{1-\omega}\\ \omega^2u & -\omega^2 & u\\ \frac{\omega u^2-\omega^2v}{1-\omega} & u & \frac{\omega^2u^2+2\omega^2v}{1-\omega}\end{pmatrix}, \tag{3.4}$$

here we put $u=\frac{\eta_2}{\eta_0},\ v=\frac{\eta_1}{\eta_0}$. So we set $\Omega=\Omega(u,v)$. The Riemann period relation $\mathrm{Im}\ \Omega>0$ induces the inequality $2\mathrm{Re}\ (\mathrm{v})+|\mathrm{u}|^2<0$. We set

$$\mathscr{D}=\{\eta=[\eta_0:\eta_1:\eta_2]\in\mathbb{P}^2:{}^t\eta H\overline{\eta}<0\}=\{(u,v)\in\mathbb{C}^2:2\mathrm{Re}\ (\mathrm{v})+|\mathrm{u}|^2<0\},$$

here we put $H=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$. We define our period map $\Phi:\Xi\to\mathscr{D}$ by

$$\Phi(\lambda_1,\lambda_2)=[\eta_0,\eta_1,\eta_2].$$

Set the Picard modular group

$$\Gamma=\{g\in\mathrm{GL}_3(\mathbb{Z}[\omega]):{}^t\overline{g}Hg=H\},$$

and set $\mathbf{G}(\sqrt{-3}) = \{g \in \mathbf{G} : g \equiv I_3(\mathrm{mod}\ \sqrt{-3})\}$.

The element $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in \Gamma$ acts on $\mathscr{D}$ by

$$g(u,v) = \left(\frac{p_3 + q_3 v + r_3 u}{p_1 + q_1 v + r_1 u}, \frac{p_2 + q_2 v + r_2 u}{p_1 + q_1 v + r_1 u} \right). \tag{3.5}$$

Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ be in $\mathbb{Q}^3$. Set the Riemann theta constant

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega) = \sum_{n \in \mathbb{Z}^3} \exp[\pi i (n+a)\Omega^t (n+a) + 2\pi i (n+a)^t b],$$

here Ω is a variable on the Siegel upper half space of degree 3. We use the following Riemann theta constants and their Fourier expansions (see [S], p.327, also [K-S] formula (1.3)):

$$\vartheta_k(u,v) = \vartheta \begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix} (\Omega(u,v)) = \sum_{\mu \in \mathbb{Z}[\omega]} \omega^{2k\mathrm{tr}\mu} H(\mu u) q^{N(\mu)} \tag{3.6}$$

with an index $k \in \mathbb{Z}$, where $\mathrm{tr}\mu = \mu + \bar{\mu}$, $\mathrm{N}(\mu) = \mu\bar{\mu}$ and

$$H(u) = \exp[\frac{\pi}{\sqrt{3}} u^2] \vartheta \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix} (u, -\omega^2), \quad q = \exp[\frac{2\pi}{\sqrt{3}} v].$$

Apparently it holds $\vartheta_k(u,v) = \vartheta_{k+3}(u,v)$, so k runs over $\{0,1,2\} = \mathbb{Z}/3\mathbb{Z}$.
The following properties are already established.

Fact 2. (i) ([P], [D-M], [T], [S] p.349) The period map Φ induces a biholomorphic isomorphism from the ξ-space $\mathbb{P}^2$ to the Satake compactification $\overline{\mathscr{D}/\Gamma(\sqrt{-3})}$ of $\mathscr{D}/\Gamma(\sqrt{-3})$.

(ii) ([S] p.327) The map $\Lambda : \mathscr{D} \longrightarrow \mathbb{P}^2$ defined by

$$\Lambda([\eta_0, \eta_1, \eta_2]) = [\vartheta_0(u,v)^3, \vartheta_1(u,v)^3, \vartheta_2(u,v)^3] \tag{3.7}$$

gives the inverse of the period map Φ.

(iii) ([S] p.329) The projective group $\mathbf{G}(\sqrt{-3})/\{1, \omega, \omega^2\}$ is generated by

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ \omega - \omega^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ \omega - 1 & 1 & \omega - 1 \\ 1 - \omega^2 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 1 & \omega - \omega^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & \omega - 1 & \omega - 1 \\ 0 & 1 & 0 \\ 0 & 1 - \omega & 1 \end{pmatrix}.$$

Let G denote the group generated by $g_1, \ldots, g_5$.

(iv) ([S] p.346) We have the automorphic property:

$$\vartheta_k(g(u,v))^3 = (p_1 + q_1 v + r_1 u)^3 \, \vartheta_k(u,v)^3 \tag{3.8}$$

for $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in G$.

4. Main Theorem

Under the relation

$$(\lambda_1, \lambda_2) = \left(\frac{\vartheta_1(u,v)^3}{\vartheta_0(u,v)^3}, \frac{\vartheta_2(u,v)^3}{\vartheta_0(u,v)^3} \right) \tag{4.1}$$

stated in Fact 2 , we have the following our main theorem:

Theorem 5.

$$\vartheta_0(u,v) = C_0\, F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2),$$
$$C_0 = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (-\omega^2),$$

here $F_1(a,b,b',c;\lambda_1,\lambda_2)$ indicates the Appell hypergeometric function

$$F_1(a,b,b',c;\lambda_1,\lambda_2) = \sum_{m,n\geq 0} \frac{(a,m+n)(b,m)(b',n)}{(c,m+n)m!n!} \lambda_1^m \lambda_2^n \tag{5.1}$$

with

$$(a,n) = \begin{cases} a(a+1)\cdots(a+n-1) & n>0 \\ 1 & n=0. \end{cases}$$

Corollary 5.1. *We have*

$$\vartheta_i(u,v)^3 = C_0^3\, \lambda_i \left(F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2) \right)^3, \quad (i=1,2). \tag{5.2}$$

Remark 5.2. *According to some classical literature (also in* [M-T-Y]*), it holds*

$$C_0 = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (-\omega^2) = \frac{3^{3/8}}{2\pi} \exp(\frac{5\pi\sqrt{-1}}{72}) \Gamma(\frac{1}{3})^{3/2}. \tag{5.3}$$

In [K-S], a new three terms arithmetic geometric mean $M_3(a,b,c)$ is introduced. For three positive numbers a,b,c, set a new triple (a',b',c') with $a' = \frac{1}{3}(a+b+c)$, $b'^3 + c'^3 = \frac{1}{3}(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2)$, $b'^3 - c'^3 = \frac{1}{3\sqrt{-3}}(a-b)(b-c)(c-a)$. Define a AGM process by

$$(a',b',c') = \psi(a,b,c).$$

We can take a nice choice of the cubic roots for b', c' so that $\psi^2(a,b,c)$ becomes to be a triple of positive numbers again. Thus, we get a unique positive number

$$M_3(a,b,c) := \lim_{n\to\infty} \psi^n(a,b,c).$$

As a consequence of Main Theorem we obtain a new proof of the three terms AGM theorem in [K-S] (p.134 Theorem 2.2) :

Corollary 5.3.

$$\frac{1}{M_3(1,x,y)} = F_1(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;1-x^3,1-y^3), \quad (|x|<1,|y|<1).$$

Observing the following isogeny formula (see [K-S] Theorem 1.1 p.132)

$$\begin{cases} \vartheta_0(\sqrt{-3}u,3v) = \frac{1}{3}(\vartheta_0+\vartheta_1+\vartheta_2) \\ \vartheta_1^3(\sqrt{-3}u,3v)+\vartheta_2^3(\sqrt{-3}u,3v) = \frac{1}{3}(\vartheta_0^2\vartheta_1+\vartheta_1^2\vartheta_2+\vartheta_2^2\vartheta_0+\vartheta_0\vartheta_1^2+\vartheta_1\vartheta_2^2+\vartheta_2\vartheta_0^2) \\ \vartheta_1^3(\sqrt{-3}u,3v)-\vartheta_2^3(\sqrt{-3}u,3v) = \frac{1}{3\sqrt{-3}}(\vartheta_0-\vartheta_1)(\vartheta_1-\vartheta_2)(\vartheta_2-\vartheta_0) \end{cases}$$

we obtain the above corollary by the exactly analogous argument in the introduction.

6. Proof of the Main Theorem

Note that $\vartheta_k(u,v)$ $(k=0,1,2)$ is holomorphic on $\mathscr{D}$. The period $\eta_0=\int_{A_1}(x(x-1)(x-\lambda_1)(x-\lambda_2))^{-1/3}dx$ is a single valued holomorphic function on $\mathscr{D}$ via the relation (4.1). And η_0 has only zeros possibly on $\Phi(\{\xi_0=0\})$.

For an element $g=\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in G$, there are actions on $(u,v)\in\mathscr{D}$ given by (3.5) and on the triple (η_0,η_1,η_2) by

$$\begin{aligned} g(\eta) &= (g(\eta_0),g(\eta_1),g(\eta_2)) \\ &= (p_1\eta_0+q_1\eta_1+r_1\eta_2, p_2\eta_0+q_2\eta_1+r_2\eta_2, p_3\eta_0+q_3\eta_1+r_3\eta_2) \end{aligned}$$

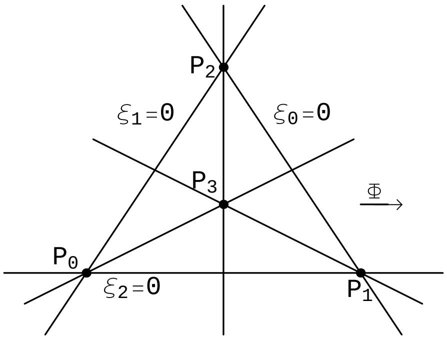
Note that we don't have the ambiguity of the choice of the projective group in G. From Fact 2 (iii) we have

$$\frac{\vartheta_0(g(u,v))^3}{g(\eta_0)^3} = \frac{\vartheta_0^3(\mathrm{id}(u,v))}{\mathrm{id}(\eta_0)^3}.$$

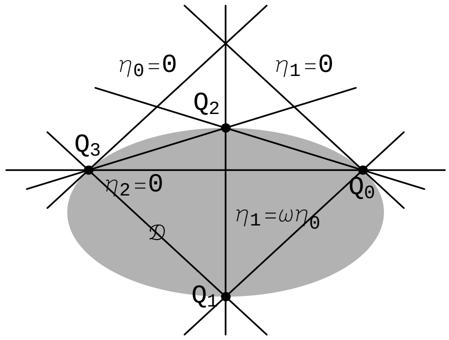
So $\vartheta_0(u,v)^3/\eta_0^3$ is invariant under the action of G, namely it is a single valued rational function on $\overline{\mathscr{D}/G}$, and that is also identified as a function on the ξ space $\mathbb{P}^2$ via the isomorphism Φ. By (3.6) we have

$$(1:\lambda_1:\lambda_2) = (\xi_0:\xi_1:\xi_2) = \left(\frac{\vartheta_0(u,v)^3}{\eta_0^3} : \frac{\vartheta_1(u,v)^3}{\eta_0^3} : \frac{\vartheta_2(u,v)^3}{\eta_0^3}\right).$$

The compactification $\overline{\mathscr{D}/G}$ is obtained by attaching 4 points corresponding to $P_0=[\xi_0,\xi_1,\xi_2]=[1,0,0], P_1=[0,1,0], P_2=[0,0,1], P_3=[1,1,1]$ to $\mathscr{D}/G$. Put $Q_i=\Phi(P_i), \quad (i=0,1,2,3)$.



$\xi-$ space



$\eta-$ space

$\vartheta_k^3(u,v)$ is zero on the divisor $\Phi(\{\xi_k = 0\})$, and does not vanish at the point corresponding to a nondegenerate Picard curve. This is due to the argument in [S] p.316. If $\vartheta_0^3(u,v)$ has a zero outside $\Phi(\{\xi_0 = 0\})$, it should be a common zero of all $\vartheta_k^3(u,v)$'s. Q_3 is given by

$$[\eta_0, \eta_1, \eta_2] = [0,1,0] = \lim_{u\to 0, v\to -\infty} (u,v).$$

According to the Fourier expansion in (3.6) we have

$$\lim_{(u,v)\to(0,-\infty)} \vartheta_k(u,v) = H(0) = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (0, -\omega^2) \neq 0 \quad \text{for } k = 0,1,2. \tag{6.1}$$

So, there is no common divisor of $\vartheta_0^3(u,v), \vartheta_1^3(u,v), \vartheta_2^3(u,v)$ on $\Phi(\{\xi_i = \xi_j\})$ $(i \neq j)$.

Set $G' \subset \Gamma$ be the group generated by G and $\delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma$. δ induces the permutation of Q_0 and Q_3, and it is coming from the symplectic transformation

$$M_\delta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that acts on the basis $(B_1, B_2, B_3, A_1, A_2, A_3)$ from left. Here we note that $\vartheta_0^3(u,v)$ is a modular form with respect to the modular group in the sense that we have

$$\vartheta_0^3(g(u,v)) = (p_1 + q_1 v + r_1 u)^3 \vartheta_0^3(u,v) \text{ for } (u,v) \in \mathscr{D}, g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in G',$$

and that $\vartheta_1^3(u,v)$ and $\vartheta_2^3(u,v)$ are not the case. This assertion is derived by calculating the action of M_δ on the characteristics of ϑ_k. $Q_0 = \Phi(P_0)$ is given by $[\eta_0, \eta_1, \eta_2] = [1,0,0]$, namely $(u,v) = (0,0)$. Then we have $\delta(Q_3) = Q_0$. Thus, we have $\lim_{(u,v)\to(0,0)} v^3 \vartheta_0^3(u,v) = \vartheta_0^3(Q_3) = H(0)^3 \neq 0$. Hence, $\Phi(\{\xi_1 = 0\})$ and

$\Phi(\{\xi_2 = 0\})$ are not contained in the divisor of $\vartheta_0^3(u,v)$. Namely, the support of the divisor of $\vartheta_0^3(u,v)$ is exactly $\Phi(\{\xi_0 = 0\})$. This is contained in the support of η_0. As we already mentioned, ϑ_0^3/η_0^3 is a rational function on the $\xi-$ space $\mathbb{P}^2$. Thus, we have

$$\frac{\vartheta_0(u,v)^3}{\eta_0^3} = \alpha$$

for some constant α. Now let us determine the value α. Suppose $0 < \lambda_1 < \lambda_2 < 1$. We have

$$\eta_0 = \int_{A_1} (z(z-1)(z-\lambda_1)(z-\lambda_2))^{-1/3} dz = c_1 \int_0^{-\infty} (z(z-1)(z-\lambda_1)(z-\lambda_2))^{-1/3} dz$$

with some constant c_1. Now recall the integral representation

$$F_1(a,b,b',c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^\infty z^{b+b'-c}(z-1)^{c-a-1}(z-x)^{-b}(z-y)^{-b'} dz.$$

By changing the variable $z = 1 - z'$ we have

$$\eta_0 = c_2 F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2)$$

with some constant c_2. Hence we have

$$\vartheta_0(u,v) = \beta F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2)$$

with some constant β. If we put $\xi = P_3 = [1,1,1]$, it corresponds to $(\xi_0 : \xi_1 : \xi_2) = (1:1:1)$. So the right hand side is equal to $\beta F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 0, 0) = \beta$. Recall again that $\Phi(P_3) = \lim_{u\to 0, v\to -\infty}(u,v)$. By using (6.1) we have $\beta = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (0, -\omega^2)$. Thus, we obtained the required equality.

q.e.d.

7. Degeneration to the case of Borweins' case

As a degenerate case $\lambda = \lambda_1 = \lambda_2$, we obtain the Jacobi type formula for the Borweins curves (see [B-B], [K-S] p.141)

$$w^3 = z(z-1)(z-\lambda)^2. \tag{7.1}$$

Set

$$\begin{cases} \theta_0(\tau) = \sum_{\mu\in\mathbb{Z}[\omega]} q^{N(\mu)} = \sum_{m,n\in\mathbb{Z}} \left(e^{2\pi i\tau/3}\right)^{m^2-mn+n^2}, \\ \theta_1(\tau) = \sum_{\mu\in\mathbb{Z}[\omega]} e^{2\pi i Tr(\mu)/3} q^{N(\mu)} = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m+n)/3} \left(e^{2\pi i\tau/3}\right)^{m^2-mn+n^2}, \\ q = \exp[2\pi i\tau/3], N(\mu) = \mu\overline{\mu}, Tr(\mu) = \mu + \overline{\mu}. \end{cases}$$

Putting $u = 0, \tau = -i\sqrt{3}v$ in (4.1) we have the expression of $\lambda = \lambda_1 = \lambda_2$:

$$\lambda = \frac{\theta_1(\tau)^3}{\theta_0(\tau)^3}. \tag{7.2}$$

Theorem 8. (Borweins [B-B] p.695) *Under the relation (7.2), we have*

$$\theta_0(\tau) = F\left(\frac{1}{3}, \frac{2}{3}, 1; 1-\lambda\right).$$

This equality is obtained just as the case $\lambda = \lambda_1 = \lambda_2$, namely the case $u = 0$, in the main theorem.

Remark 8.1. *Borweins have shown this theorem by using their AGM theorem. But we proved it directly with modular arguments. So our theorem induces their AGM theorem also. This is the context already discussed in the proof of Cor 5.3.*

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Spectral transfer morphisms for affine Hecke algebras

Eric Opdam

In this talk we introduce the notion of a spectral transfer morphism between affine Hecke algebras. This notion relates to the role affine Hecke algebras play in the harmonic analysis of p-adic reductive groups. Admittedly, the subject of this talk is not immediately related to the main topic of the meeting, hypergeometric functions. It could be mentioned though that affine Hecke algebras play a dominant role in the theory of the so-called hypergeometric functions associated to root systems, a theory in which hypergeometric functions are viewed as generalizations of elementary zonal spherical functions of both real and p-adic reductive groups (Macdonald-Cherednik theory, see e.g. [2], [4], [7], [9]).
The results on spectral transfer morphisms presented in this talk have not yet been published. A publication with complete proofs is in preparation.

1. Spectral transfer morphisms and spectral correspondences

By an affine Hecke algebra we mean in this talk the "deformation" $\mathcal{H}$ of the group algebra $\Lambda[W]$ of an (extended) affine Weyl group W with coefficients in the algebra $\Lambda := \mathbb{C}[v^{\pm 1}]$ defined by keeping the homogeneous relations between the simple generators of W untouched, but replacing the relations $s_i^2 = 1$ for the simple reflections s_i of W by

$$(N_i - v^{m_i})(N_i + v^{-m_i}) = 0 \qquad (1.1)$$

for certain integers m_i satisfying $m_i = m_j$ if s_i and s_j are conjugate in W. The algebra $\mathcal{H}$ is free over Λ with Λ-basis N_w indexed by $w \in W$. The algebra $\mathcal{H}$ comes equipped with a conjugate linear involution $*$ defined by $N_w^* = N_{w^{-1}}$ and $v^* = v$. Finally we assume given an element $d \in \mathbf{M}$, the subgroup of the multiplicative group $\mathbf{K}^\times$ of the quotient field of Λ generated by $q = v^2$, by $\mathbf{Q}^\times$, and by the cyclotomic polynomials in q, satisfying $d(q^{-1}) = \pm d(q)$ and $d(\mathbf{q}) > 0$ if $\mathbf{q} > 1$.
Then for all $\mathbf{v} > 1$ the linear function τ defined on the specialization $\mathcal{H}_\mathbf{v}$ of $\mathcal{H}$ at $\mathbf{v} > 1$ by $\tau(N_w) = d(\mathbf{v}^2)\delta_{w,e}$ is a positive trace. In particular this gives rise to a pre-C^*-algebra structure on the fibre $\mathcal{H}_\mathbf{v}$ [10]. We denote by $\mathfrak{C}_\mathbf{v}$ the C^*-algebra completion of $\mathcal{H}_\mathbf{v}$. Its irreducible spectrum is denoted by $\Sigma_\mathbf{v}$ and is referred to as the tempered spectrum of $\mathcal{H}_\mathbf{v}$.
Since $\mathcal{H}$ is an algebra over Λ it is clear that the irreducible spectrum of $\mathcal{H}$ is fibred over $\mathbb{C}^\times$, the maximal spectrum of Λ. This fibration is quite complicated. If we restrict to the subset $\mathbf{v} > 1$ the situation becomes simpler. For example, the irreducible spectra of $\mathcal{H}_\mathbf{v}$ and $\mathcal{H}_{\mathbf{v}'}$ are in natural bijection via "scaling maps" if $\mathbf{v}, \mathbf{v}' > 1$ [10]. These bijections induce homeomorphisms of the corresponding tempered spectra $\Sigma_\mathbf{v}$ and $\Sigma_{\mathbf{v}'}$ [13]. One may construct a trivial bundle $\mathfrak{C}$ of C^*-algebras over $\mathbb{R}_{>1}$, with fibre $\mathfrak{C}_\mathbf{v}$ [13]. Its irreducible spectrum is denoted by Σ, and comes equipped with a (trivial) fibration $\Sigma \to \mathbb{R}_{>1}$ with fibre $\Sigma_\mathbf{v}$ at $\mathbf{v} > 1$.
All relevant structures and maps below are relative with respect to this fibration over $\mathbb{R}_{>1}$, but we hide this in the exposition. The spectral decomposition of the $\mathbb{R}_{>1}$-family of positive trace(s) τ defines a $\mathbb{R}_{>1}$-family of positive measure(s) on the fibers $\Sigma_\mathbf{v}$ of Σ [10], to which we refer as the *Plancherel measure* ν_{Pl} *of* $\mathcal{H}$.
An affine Hecke algebra $\mathcal{H}$ is of finite type over its center $\mathcal{Z}$. Hence there exists a finite map p_S from the irreducible spectrum of $\mathcal{H}$ to the maximal spectrum S of its center $\mathcal{Z}$. The restriction p_S^{temp} of p_S to Σ is continuous if we equip the complex variety S with the analytic topology, which is what we will do from now on. It is known that there exists an open, dense subset $\Sigma' \subset \Sigma$ such that p_S^{temp} restricted to Σ' is a finite covering map onto its image [10, Theorem 4.39].
We will not give the precise definition of the notion of spectral transfer morphisms here, since this would require a much more serious discussion of the structure of affine Hecke algebras. In order to define spectral transfer morphisms one first of all needs to remark that the structure of an affine Hecke algebra $\mathcal{H}$ is determined by a certain rational function μ on S (the Harish-Chandra μ-function). A spectral transfer morphism $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is then, loosely speaking, given by a morphism $\phi_S : S_1 \to S_2$ compatible in a specific way with the μ-functions μ_1 and

μ_2. We remark that such a morphism ϕ is not (in general) given by an algebra homomorphism.
The main property of spectral transfer morphisms is the following result. Its proof is based on [10, Theorem 4.43].

Theorem 15. *Let $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ be a spectral transfer morphism. Consider the correspondence between Σ_1 and Σ_2 given by the fibered product Σ_{12} of Σ_1 and Σ_2 according to the diagram:*

$$\begin{array}{ccc} \Sigma_{12} & \xrightarrow{p_1} & \Sigma_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \phi_S \circ p_{S_1}^{temp}} \\ \Sigma_2 & \xrightarrow[p_{S_2}^{temp}]{} & S_2 \end{array} \tag{1.2}$$

If $C \subset \Sigma_{12}$ is a component then $C_i := p_i(C) \subset \Sigma_i$ is a component ($i = 1, 2$). Moreover there exists a positive measure ν on C and $r_i \in \mathbb{Q}_+$ such that

$$(p_i)_*(\nu) = r_i \nu_{Pl,i}|_{C_i} \tag{1.3}$$

for $i = 1, 2$.

2. Tempered depth-zero representations of split groups

Let k be non-archimedean local field and let $\mathbf{G}$ be a connected split simple algebraic group of adjoint type over k. Let K be a maximal unramified extension of k, and let $G = \mathbf{G}(K)$ be the group of K-rational points of $\mathbf{G}$. Let $B \subset G$ be an Iwahori subgroup. We choose a topological generator F of $\mathrm{Gal}(K/k)$, then $G^F = \mathbf{G}(k)$ is a locally compact group whose Haar measure we normalize by requiring that $\mathrm{vol}(B^F) = 1$. Let $\mathbf{q}$ be the cardinality of the residue field of k, and let $\mathbf{v}$ be the positive square root of $\mathbf{q}$.
The correspondences of spectral transfer morphisms can be used to put the irreducible tempered depth-zero representations of the group G^F in packets which are parametrized by points of the (maximal) spectrum $S_{\mathbf{v}}^B$ of the center of the Iwahori Hecke algebra $\mathcal{H}_{\mathbf{v}}^B := \mathcal{H}(G^F, B^F)$ of the pair (G^F, B^F), and which are compatible with the Plancherel measure of G^F.
The category of depth-zero representations of G^F is a certain finite product $R(G^F)_{unr}$ of "Bernstein blocks" of the category $R(G^F)$ of smooth representations of G^F. We denote by $\mathcal{B}(G^F)_{unr}$ the corresponding finite set of components of the Bernstein variety of G^F. Thus we have by definition

$$R(G^F)_{unr} := \prod_{s \in \mathcal{B}(G^F)_{unr}} R(G^F)_s \tag{2.1}$$

It follows from (independent) results of Lusztig (see [6]) and Morris [8] that every component $s \in \mathcal{B}(G^F)_{unr}$ admits an "s-type" whose associated convolution algebra $\mathcal{H}_{\mathbf{v}}(s)$ is an affine Hecke algebra. Therefore we have an equivalence of categories

(the "Bernstein functor")

$$\beta : R(G^F)_{unr} \to \prod_{s\in\mathcal{B}(G^F)_{unr}} \mathcal{H}_{\mathbf{v}}(s) - \mathrm{mod} \tag{2.2}$$

We remark that the finite set $\mathcal{B}(G^F)_{unr}$ depends on G, not on G^F. This equivalence of categories is compatible with the harmonic analysis on G^F [1], [3] in that there exist elements $d(s) \in \mathbf{M}$ such that the Plancherel measure of $\mathcal{H}_{\mathbf{v}}(s)$ corresponds, via the equivalence, with the restriction of the Plancherel measure of G^F to the component $\mathrm{Irr}(G^F)_s^{temp}$ of the tempered dual of G^F.
The structure of the category whose objects are affine Hecke algebras of the form $\mathcal{H}(s)$ with $s \in \mathcal{B}(G^F)_{unr}$ and whose morphisms are spectral transfer morphisms is very simple. It can be described explicitly in terms of generators and relations in each case. This case-by-case analysis leads to the following remarkable result.

Theorem 3. *Let $G^\vee$ be the Langlands dual group of G^F. There exists an essentially unique $Z(G^\vee)$-equivariant spectral transfer morphism*

$$\phi^G : \prod_{s\in\mathcal{B}(G^F)_{unr}} \mathcal{H}(s) \to \mathcal{H}^B \tag{3.1}$$

to the Iwahori Hecke algebra $\mathcal{H}^B$.

Let $T^\vee \subset G^\vee$ be a maximal torus of $G^\vee$. Then $W_0\backslash T^\vee$ can be identified with the maximal spectrum $S_{\mathbf{v}}^B$ of the center $\mathcal{Z}_{\mathbf{v}}^B$ of the Iwahori Hecke algebra $\mathcal{H}_{\mathbf{v}}^B$. By the Kazhdan-Lusztig correspondence [5] the set of central characters $\xi \in S_{\mathbf{v}}^B$ supporting discrete series representations of $\mathcal{H}_{\mathbf{v}}^B$ is in bijection with the set of $G^\vee$-orbits of unramified discrete local Langlands data $\Phi : \widehat{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ (Φ is called discrete if $C_{G^\vee}(\mathrm{Im}(\Phi))$ is finite).
The following result can be checked case-by-case. It is in agreement with a conjecture of Reeder [12] on formal degrees and L-packets (also see the classification [6]) and extends results of [12] on exceptional groups to the general case. The proof is based on the above in combination with the classification of discrete series characters of affine Hecke algebras with unequal parameters [11].

Theorem 4. *Let ϕ^G be a $Z(G^\vee)$-equivariant spectral transfer morphism as in Theorem 3. Let Φ be an unramified discrete local Langlands datum for G^F, with corresponding central character $\xi = \xi_\Phi \in S_{\mathbf{v}}^B$ of $\mathcal{H}_{\mathbf{v}}^B$ under the Kazhdan-Lusztig correspondence. Then the set*

$$\Pi_\Phi = [\beta]^{-1}\Big(\bigsqcup_{\xi_s\in S_{s,\mathbf{v}}:\phi_S^G(\xi_s)=\xi} (p_{S_s}^{temp})^{-1}(\xi_s)\Big) \tag{4.1}$$

consists of $|\mathrm{Irr}(C_{G^\vee}(\mathrm{Im}(\Phi))/Z(G^\vee))|$ equivalence classes of irreducible depth-zero discrete series representations of G^F with the property that the ratio of the formal degrees of any two members of Π_Φ is a rational number independent of $\mathbf{q}$.

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Arithmetic and Geometry around a Shimura Quartic

Rolf-Peter Holzapfel, Maria Petkova

We consider and construct (finite and infinite) towers of Picard modular surfaces with trivial (t.m. rational) function fields but non-trivial discriminants, geometrically known as orbital (branch) divisors, of the involved coverings. It is convenient to regard them as special cases of Galois-Reflection Towers, which will be defined in arbitrary dimensions. We prove that (finitely many) reflections generate the Picard modular groups defining such a tower. We use this knowledge for explicit algebraic-geometric classifications of the Baily-Borel compactifications of tower members and explicit description of the corresponding Picard modular groups by means of reflections. Finally, we turn to dimension 1 considering arithmetic subdiscs of the 2-ball and their algebraic image curves. On this way we get an explicit tower of Shimura curves embedded in the constructed surface tower. Via reductions mod p one gets important towers of coding theory, whose members can be explicitly determined step by step.

Look at the globe with drawn equator and two meridians, all orthogonal to each other. These three circles describe the norm-1 curve configuration on a special Picard modular surface visualizing an octahedral group action. The six intersection

points are the cusp singularities of the surface. The notions will be explained now in a general setting of arbitrary dimension.
Let V be the space $\mathbb{C}^{n+1}$ endowed with hermitian metric $< .,. >$ of signature $(n, 1)$. Explicitly we will work with the diagonal representation. For $v \in V$ we call $n(v) =< v, v >$ the norm of v. The space of all vectors with negative (positive) norms is denoted by $V^-(V^+)$. The image $\mathbb{P}V^-$ of V^- in $\mathbb{P}V = \mathbb{P}^n$ is the hyperball denoted by $\mathbb{B}^n$. The unitary group $\mathbb{U}((n, 1), \mathbb{C})$ acts transitively on it. Now let K be an imaginary quadratic number field, $\mathcal{O}_K$ its ring of integers.
The arithmetic subgroup $\Gamma_K = \mathbb{U}((n, 1), \mathcal{O}_K)$ is called the *full Picard modular group*. Each subgroup Γ of finite index is a *Picard modular group*.
The ball quotients $\Gamma\backslash\mathbb{B}^n$ are quasiprojective. They have a minimal algebraic compactification $\widehat{\Gamma\backslash\mathbb{B}^n}$ constructed by Baily and Borel in [2]. The authors proved that these compactifications are normal projective complex varieties. The Picard modular groups of fixed K act also on the hermitian $\mathcal{O}_K$-lattice $\Lambda = (\mathcal{O}_K)^{n+1}$.
Let $\mathfrak{a} \in \Lambda$ be a primitive positive vector and $\mathfrak{a}^\perp$ its orthogonal complement in V. It is a hermitian subspace of V of signature $(n-1, 1)$. The intersection $\mathbb{D}_\mathfrak{a} := \mathbb{P}\mathfrak{a}^\perp \cap \mathbb{B}^n$ is isomorphic to $\mathbb{B}^{n-1}$. We call it an *arithmetic subball* of $\mathbb{B}^n$.
Take all elements of Γ acting on $\mathbb{D}_\mathfrak{a}$: $\Gamma_\mathfrak{a} := \{\gamma \in \Gamma; \gamma(\mathbb{D}_\mathfrak{a}) = \mathbb{D}_\mathfrak{a}\}$. This is an arithmetic group. The image $p(\mathbb{D}_\mathfrak{a})$ along the quotient projection $p : \mathbb{B}^n \to \Gamma\backslash\mathbb{B}^n$ is an algebraic subvariety $H_\mathfrak{a}$ of $\Gamma\backslash\mathbb{B}^n$ of codimension 1. The algebraic subvarieties $H_\mathfrak{a}$ are called *arithmetic hypersurfaces* of the Picard modular variety $\Gamma\backslash\mathbb{B}^n$. The same notion is used for the compactifications. The *norm* $n(H_\mathfrak{a})$ of $H_\mathfrak{a}$ is defined as $n(\mathfrak{a})$.
An element $\sigma \in \Gamma$ is called a Γ-*reflection*, iff it has an n-dimensional eigenspace $V_\mathfrak{a} \subset V$ of eigenvalue 1 and a positive eigenvector $\mathfrak{a} = \mathfrak{a}(\sigma)$ of other eigenvalue. The latter can be chosen primitive in Λ. The eigenspace $V_\mathfrak{a}$ is then nothing else but $\mathfrak{a}^\perp$, and σ acts identically on the arithmetic subball $\mathbb{D}_\mathfrak{a} = \mathbb{P}V_\mathfrak{a} \cap \mathbb{B}^n$ of $\mathbb{B}^n$. We call such $\mathbb{D}_\mathfrak{a}$ a Γ-*reflection subball* of $\mathbb{B}^n$. The hypersurface $H_\mathfrak{a}$ of the primitive eigenvector $\mathfrak{a} = \mathfrak{a}(\sigma)$ of a Γ-reflection σ is called a Γ-*reflection hypersurface*.
Let $\ldots\Gamma_{i+1} \subset \Gamma_i \subset \cdots \subset \Gamma_1 \subset \Gamma$, (1), be a (finite or infinite) normal series of subgroups of finite index of the Picard modular group Γ. We call it a Γ-*reflection series*, if Γ_i is generated by Γ_{i+1} and finitely many reflections for each in (1) occurring pair $(i + 1, i)$. The corresponding Galois tower of finite Galois coverings $\cdots \to \Gamma_{i+1}\backslash\mathbb{B}^n \to \Gamma_i\backslash\mathbb{B}^n \to \cdots \to \Gamma_1\backslash\mathbb{B}^n$, (2), with the normal factors Γ_i/Γ_{i+1} as Galois groups, is then called a *Galois-reflection tower*. In this case each normal factor is generated by a coset of Γ-reflections.
Theorem: Let $\Gamma\backslash\mathbb{B}^n$ be simply-connected and smooth. Then Γ is generated by finitely many Γ-reflections.
Corollary: Let $\Gamma' \subset \Gamma_N \subset \cdots \subset \Gamma_1$, $(1')$, be a normal series of Picard modular subgroups of the ball lattice Γ, and $\Gamma'\backslash\mathbb{B}^n \to \Gamma_N\backslash\mathbb{B}^n \to \cdots \to \Gamma_1\backslash\mathbb{B}^n$, $(2')$, the corresponding Galois tower of Picard modular varieties. If the varieties $\Gamma_i\backslash\mathbb{B}^n$, $i = 1, ..., N$, are smooth and simply-connected, then $(1')$ is a Galois-reflection series with Galois-reflection tower $(2')$.

Example: Uludag constructed in [9] the first (and only until now) infinite Galois-reflection tower in dimension > 1. It consists (compactified) of orbital projective planes $\mathbb{P}^2$. The successive Galois coverings $\mathbb{P}^2 \to \mathbb{P}^2$ have $K4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein's Vierergruppe) as Galois groups. The first member is the orbital $\mathbb{P}^2$ with Apollonius (branch divisor) configuration, see Holzapfel [4], first appearance by Yoshida [6]. In [4] we proved that the congruence subgroup $\Gamma_1 = \Gamma(1-i)$ is the uniformizing ball lattice, where $\Gamma = SU((2,1),\mathbb{Z}[i])$.
We use Galois-reflection towers step by step for explicit descriptions of the uniformizing ball lattices, if the orbital Picard modular surfaces are explicitly known and vice versa. The main goal is the first algebraic-geometric classification of the Picard modular surface of a natural congruence subgroup:
Proposition: A (singular) model of $\Gamma(2)\backslash\mathbb{B}^2$ is the space quartic $U^2T^2 - X^4 - Y^4 - T^4 + 2X^2Y^2 + 2X^2T^2 + 2Y^2T^2 = 0$ (in $\mathbb{P}^3$ with coordinates $(u:x:y:t)$). It has $\mathbb{P}^1 \times \mathbb{P}^1$ as smooth model. Blowing up suitable six points of $\mathbb{P}^1 \times \mathbb{P}^1$ we get the minimal desingularisation $\overline{(\widehat{\Gamma(2)\backslash\mathbb{B}^2})}$ of $\widehat{\Gamma(2)\backslash\mathbb{B}^2}$ (resolution of cusp singularities).
For the proof we climb step by step through a Galois-reflection diagram supported by the Uludag tower:

$$\begin{array}{ccccc} & \widehat{\Gamma(2)\backslash\mathbb{B}} & \rightarrow & \mathbb{P}^1\times\mathbb{P}^1 & \\ & | & & | & (T) \\ \cdots\ Uludag's\ tower\ \cdots\mathbb{P}^2 \rightarrow & \mathbb{P}^2 & \rightarrow & \mathbb{P}^2 & \end{array}$$

where each arrow corresponds to a reflection. The geometric rectangle consists of ball quotients of the following Γ-reflection series (of inclusions)

$$\begin{array}{ccc} \Gamma(2) & \rightarrow & <\Gamma(2),\sigma,\sigma'> \\ \downarrow & & \downarrow \\ <\Gamma(2),\rho> & \rightarrow & \Gamma(1-i) = <\Gamma(2),\rho,\sigma,\sigma'> \end{array}$$

with explicit reflections ρ, σ, σ'.
First we know the second orbital $\mathbb{P}^2$ of the Uludag tower, whose orbital (branch) divisor is drawn in [9], supported on a Ceva-configuration with 7 lines, which one can already find by Hirzebruch [2], p. 81. Its uniformizing ball lattice is $<\Gamma(2),\rho>$. Knowing the branch divisor of the left vertical double cover we know that $\widehat{\Gamma(2)\backslash\mathbb{B}^2}$ is the normalization of $\mathbb{P}^2$ along the function field extension $\mathbb{C}(x,y,w)/\mathbb{C}(x,y)$ with $w = \sqrt{(x^4+y^4-2x^2y^2-2x^2-2y^2+1)}$.
This gives the quartic space equation. To get $\mathbb{P}^1 \times \mathbb{P}^1$ as model one goes through the right side of the rectangle. In the thesis [5] of Matsumoto one can find the orbital $\mathbb{P}^1 \times \mathbb{P}^1$, which is easily recognized as double cover of the orbital Apollonius plane. The normalization of a birational transform of Matsumoto's $\mathbb{P}^1 \times \mathbb{P}^1$ finishes the proof of the theorem. Moreover, knowing all orbital divisors, one can see that $\overline{\Gamma(2)\backslash\mathbb{B}^2}$ is the $K4$-cover of orbital del Pezzo surface No. 20 in the Hirzebruch's table [2], p. 201, with orbital divisor supported by 10 projective lines, p. 196 [2].
In the case $n = 2$ the subballs are $K-$linear discs, which define algebraic curves on the ball quotient surface $\Gamma\backslash\mathbb{B}^2$. The quotient curve $C = \Gamma\backslash\mathbb{D} \subset \Gamma\backslash\mathbb{B}^2$, the projection of the $K-$linear disc $\mathbb{D}$, is an algebraic curve, whose embedded model on the Picard modular surface $\Gamma\backslash\mathbb{B}^2$ is defined over $\overline{\mathbb{Q}}$ (the proof is based on an

article of Shiga, Wolfart [7]). The particular consideration of this case is strongly motivated by results from coding theory. It was shown by T. Zink [8] that towers, i.e. sequences of finite covers, of Shimura curves defined over $\mathbb{F}_{q^2}$, are asymptotically optimal. They correspond to sequences of codes with good parameters. In [3] N. Elkies defines a construction for towers ($X_n \to X_{n-1} \to \cdots \to X_0$) of Shimura curves, which is based only on the first two curves X_0, X_1. In general it is quite difficult to compute equations for X_0 and X_1. The aim of our work is to find a way to obtain equations for X_0 and X_1, in the case where they are Shimura curves from $K-$ linear discs on Picard modular surfaces. For $K = \mathbb{Q}(i)$ and $\Gamma = SU((2,1), \mathcal{O}_K)(1-i)$ the curves of norm 1 and 2 have been completely described [4]. Studying systematically the curves of small norm, the next step is to try to compute an equation for a norm 3 Shimura curve. This is possible and we obtain a plane quadric defined by $Sh : 16xy + 4xz + 4yz - 3z^2 = 0$. We recognize the Shimura curve X_1 of Elkis' tower on the quartic space model of $\Gamma(2)\backslash\mathbb{B}^2$. It has the plane *Shimura quartic* model $16x^2y^2 + 4x^2z^2 + 4y^2z^2 - 3z^4 = 0$, which is an elliptic curve with j-invariant 2048/3.
To obtain the globe mentioned at the beginning one has to extend a little bit the Galois-reflection diagram (T).

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Tautological maps between Deligne-Mostow ball quotients

Martin Deraux

The main result of Deligne-Mostow theory can be interpreted as the construction of a complex hyperbolic structure on the moduli space of $n+3$ weighted points on $\mathbb{P}^1$ (see [DM86]). Here the weight μ_j of the j-th point x_j is assumed to be a rational number between 0 and 1. We assume moreover that $\sum \mu_j = 2$ and most importantly we require that the following integrality condition holds:

$$(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z} \cup \{\infty\} \quad \text{(INT)}$$

for all $i \neq j$. There is then a lattice $\Gamma_\mu \subset PU(n,1)$ such that the moduli space of $n+3$ points with weights $\mu = (\mu_1, \dots, \mu_{n+3})$ is precisely the quotient $\Gamma_\mu \setminus \mathbb{B}^n$.
If $n > 1$, there are only finitely many choices of the weights μ that satisfy condition INT (whereas there are infinitely many for $n = 1$, the 4-tuples satisfying the integrality condition being in correspondence with hyperbolic triangle groups).
The description of these ball quotient orbifolds as moduli spaces of points suggests a natural way to obtain maps between them, by considering either inclusions of moduli spaces, or forgetful maps (we call "tautological maps" either of the two kinds of maps mentioned above, coming either from inclusions of forgetful maps).
We concentrate mainly on the case of forgetful maps, and consider two sets of weights $\mu = (\mu_1, \dots, \mu_{m+3})$ and $\nu = (\nu_1, \dots, \nu_{n+3})$ with $m \geq n$, both sets of weights satisfying the integrality condition INT. Possibly after permuting the weights, we may think of the forgetful map as being defined by

$$(x_1, \dots, x_{m+3}) \mapsto (x_1, \dots, x_{n+3})$$

These are always well-defined on the level of open moduli spaces (i.e. moduli of pairwise distince points), but they don't necessarily extend to the relevant metric completion.
We consider the question of when such an extension exists and, if it does, we ask whether the extension is actually a map of orbifolds, i.e. whether it can be lifted to a map between suitable manifold covers (here one replaces the relevant lattices Γ_μ and Γ_ν by torsion-free subgroups of finite index).
There are very few forgetful maps that extend to completions, and even fewer that define maps of orbifolds. One way to summarize our results on forgetful maps between Deligne-Mostow ball quotients is the following:

a.) Most 2-dimensional examples have a forgetful map to a 1-dimensional example that is a map of orbifolds.
b.) A small number of forgetful maps from dimension 2 to dimension 2 are maps of orbifolds.
c.) There is precisely one forgetful map from a 3-dimensional example to a 1-dimensional example.
d.) No forgetful map from dimension m to dimension n is a map of orbifolds if $(m, n) \neq (1,1), (2,2), (2,1), (3,1)$.

Note that the statement remains the same if we work with a slightly larger class of groups, obtained by relaxing condition INT to Mostow's Σ-INT condition (see [Mos86]).
The maps corresponding to case (1) include the maps that were used in Livné's thesis, who obtained ball quotients that by construction fiber over certain Riemann surfaces (see [Liv81]). Some of the maps from case (2) were already known to Mostow, and their importance resides in the fact that they induce non-standard

homomorphisms of certain lattices in $PU(2,1)$, illustrating the failure of super-rigidity for that group (see [Mos80] or [Tol03]).
The map from case (3) seems not to have appeared anywhere in the literature. The existence of such a map should also be put in perspective with a recent result of Koziarz and Mok, that precludes the existence of *submersive* maps between ball quotients (see [KM08]). The maps obtained in this paper are indeed not submersive, and some explicit fibers are in fact singular divisors.
The fact that the forgetful construction does not produce any map from dimension m to dimension n when $m > n \geq 2$ can be thought of as giving some evidence for Siu's conjecture that no surjective holomorphic map should exist between ball quotients with that same restriction on the dimensions.

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A fake projective plane and related elliptic surfaces

Amir Džambić

In my talk I reported on my recent work on a construction of a fake projective plane and two elliptic surfaces covered by this fake plane using arithmetic ball quotients, which provides an alternative construction to the geometric approach of J.H. Keum (see [Keu]). The construction is based on the recent work of G. Prasad and S.K. Yeung [PY] as well as on papers of M-N. Ishida [Ish] and F. Kato [Kat]. Roughly skeched, the construction goes as follows:
Let $\zeta_7 = \exp(2\pi i/7)$, $L = \mathbb{Q}(\zeta_7)$, $K = \mathbb{Q}(\lambda)$ with $\lambda = \zeta_7 + \zeta_7^2 + \zeta_7^4 = \frac{-1+\sqrt{-7}}{2}$, let further $\sigma \in \mathrm{Gal}(L|\mathbb{Q})$ be defined by $\sigma : \zeta \mapsto \zeta^2$ and set $\alpha = \lambda/\bar{\lambda}$. Then the triple (L, σ, α) defines a cyclic division algebra $D = D(L, \sigma, \alpha)$ of degree 3, having an involution of second kind. Choosing an involution of second kind ι_b of signature $(2,1)$, i.e. an involution represented by by a hermitian matrix of that signature, one finds a maximal order $\mathcal{O}$ in D invariant under ι_b. The group $\Gamma = \Gamma(\mathcal{O}, \iota_b) := \{x \in \mathcal{O} \mid x \cdot \iota_b(x) = 1, Nrd(x) = 1\}$, where $Nrd(\ )$ denotes the reduced norm, is a cocompact arithmetic lattice in $\mathrm{SU}(2,1)$. Let $\mathbb{B}$ be the ball defined by the hermitian form induced from ι_b, let $\Gamma' = \Gamma(\lambda)$ be the principal congruence subgroup defined by λ. Then, using Prasad's volume formula in combination

with Hirzebruch's proportionality and a vanishing theorem, one deduces that the quotient $X_{\Gamma'} = \Gamma^{'}\backslash\mathbb{B}$ is a fake projective plane.
Since $\Gamma^{'}$ is a normal subgroup of index 7, it is natural to consider the orbifold covering $X_{\Gamma'} \longrightarrow X_\Gamma$ of degree 7. One shows: X_Γ has three cyclic singularities all of type $(7,3)$. The minimal resolution of these singularities $\tilde{X}_\Gamma$ is a minimal elliptic surface of Kodaira dimension one, whose geometric genus and the irregularity both vanish.
Going one step further, one can look at the quotient $X_{\Gamma''}$ of $\mathbb{B}$ by the normalizer of Γ in PU(2,1). Since $\Gamma^{''}$ contains Γ with index 3, we have an orbifold covering $X_\Gamma \longrightarrow X_{\Gamma''}$ of degree 3. Again one shows: $X_{\Gamma''}$ is singular, having 4 cyclic singularities of type $(7,3),(3,2),(3,2),(3,2)$ respectively. The minimal resolution of singularities $\tilde{X}_{\Gamma''}$ is a minimal elliptic surface of Kodaira dimension 1 with geometric genus and irregularity both 0.

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On geometric differential equations with differential Galois group G_2

Stefan Reiter

(joint work with M. Dettweiler)

The study of geometric differential equations has a long history. The most prominent examples are the Gauss hypergeometric differential equation and the generalized Gauss hypergeometric differential equation. Moreover these mentioned differential equations have the property that they are free from accessory parameters, i.e. they are uniquely determined by their local monodromy at $0,1,\infty$. The investigation of their monodromy groups was initiated by Schwarz for the Gauss hypergeometric differential equation and it was completed by Beukers and Heckman [1] for the generalized ones.
However due to N. Katz [7] it turned out that any irreducible differential equation being free from accessory parameters can be built up from a one dimensional differential equation by applying a finite sequence of middle convolution (Euler integral) and middle tensor operations (tensoring with one dimensional differential equation). Thus there exists a whole universe of such differential equations and it is still an open question which kind of monodromy groups/differential Galois groups can appear. We show using the convolution that there exist 7th order geometric differential equations whose differential Galois group are the exceptional group $G_2(\mathbb{C})$ using a constructive approach of the middle convolution, s. [2] and

[5]. These are the first examples of geometric differential equations with such a differential Galois group. So far only non geometric differential equations with differential Galois group $G_2(\mathbb{C})$ were known, s. [6]. Further we classify all such rigid tuples in dimension 7 having differential Galois group $G_2(\mathbb{C})$, s. [4]. As an example, s. [3], we show how one can construct the differential equation

$$Y' = \left(\frac{a_0}{2t} + \frac{a_1}{2(t-1)}\right) Y,$$

where $(a_0, a_1) =$

$$\left(\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & -2 \end{pmatrix}\right)$$

and the corresponding monodromy group generators A_0 and A_1 at the singularities 0 and 1, where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 4 & 4 & 1 \end{pmatrix}$$

Note that the Jordan forms of the monodromy matrices at $0, 1, \infty$ are

$$(-J(1)^4, J(1)^3), \quad (J(2), J(2), J(3)), \quad (J(7)),$$

where $J(k)$ denotes a Jordan block of length k and that $G_2(\mathbb{C}) \subseteq GL_7(\mathbb{C})$ is the only non classical group containing a regular unipotent element. Computing the local monodromy of the third exterior power Λ^3 of the representation one sees by applying the Scott formula that there exists a trivial subspace. This rules the possible classical groups out.

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