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Wave Motion

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Abstract. This workshop was devoted to recent progress in the mathematical study of water waves, with special emphasis on nonlinear phenomena. Both aspects related to the governing equations (free boundary Euler equations) as well as aspects related to various model equations were of interest.

Mathematics Subject Classification (2000): 35, 76, 79.

Introduction by the Organisers

The workshop **Wave Motion** that took place in the period February 9–13, 2009 was dedicated to the study of nonlinear wave phenomena. Waves lie at the forefront of modern applied mathematics and theoretical physics. The study of wave phenomena leads to a variety of involved mathematical issues, such as partial differential equations, functional analysis, harmonic analysis, dynamical systems, bifurcation theory. Fluids have been a rich source of deep mathematical theories for over 200 years. The conference focused on four very active areas involving fluids:

- water waves with vorticity,
- stability theory of fluids,
- mathematical aspects of edge waves,
- current aspects of integrable systems and solitons.

The programme of the workshop consisted in 17 talks, presented by international experts in nonlinear waves coming from Austria, China, England, France, Germany, Ireland, Italy, Norway, Sweden, U.S.A., and by three discussion sessions on the topics “Modelling of water waves”, “Waves of large amplitude”, “Integrable shallow water equations”. Moreover, several doctoral and post-doctoral fellows participated in the workshop and did benefit from the unique academic atmosphere at the Oberwolfach Institute. The organisers gratefully acknowledge the support of two younger scientists by the Leibniz Association within the grant “Oberwolfach Leibniz Graduate Students”.

The proceedings of the workshop will appear as a special issue of the journal *Wave Motion* .

Workshop: Wave Motion

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Abstracts

On two-dimensional steady edge waves

Mats Ehrnström
(joint work with Joachim Escher and Bogdan-Vasile Matioc)

Edge waves are regular, essentially two-dimensional waves propagating along the beach, and vanishing fast in the direction perpendicular to the shoreline. They reside on a semi-infinite domain in the direction away from land. In [3] the following model for steady two-dimensional edge waves was proposed:

$$\Delta\psi + \partial_x |\nabla\psi|^2 = 0 \qquad \text{in} \qquad \Omega := \mathbb{R} \times (-\infty, 0). \tag{1}$$

Here $\nabla\psi$ is a normalized velocity field, and the Cartesian coordinates are chosen so that x denotes the position along the shoreline, and y measures the position in the direction perpendicular to the beach. Since equation (1) is invariant under the scaling $(x, y, \psi) \mapsto (\lambda x, |\lambda| y, \psi/\lambda)$, for periodic waves there is no loss of generality in restricting attention to the period 2π. To this aim we set $\Sigma := \mathbb{S} \times (-\infty, 0)$. We then have the following result [1], showing that small shore-line profiles can be extended to rapidly decaying waves in the whole fluid domain.

Theorem 1 (Periodic waves) *There exists $K > 0$, such that for given boundary data $\psi(x, 0) = f(x)$ with $\|f\|_{C^{2+\alpha}(\mathbb{S})} \leq K$, the problem* (1) *has a solution with $\sup_{x\in\mathbb{R}} \psi(x, y) \to 0$ as $y \to -\infty$,*

$$\|\psi\|_{C^{2+\alpha}(\Sigma)} \leq 1/4, \qquad \textit{and} \qquad \|\psi\|_{H^1(\Sigma)} \leq 2\|f\|_{H^1(\mathbb{S})}.$$

The solution ψ is unique within $BUC^{2+\alpha}(\Sigma)$, and if f is odd, then ψ vanishes exponentially fast as $y \to -\infty$, with $\psi(\cdot, y)$ being odd for all $y \leq 0$.

The equation (1) is quasi-linear, of changing type, and defined on a an unbounded domain. For existence, we therefore first have to establish *a priori* properties of solutions. It turns out that the nonlinear part of (1) carries much structure, and for solutions with a small gradient, it yields decay properties in the seaward direction. By combining periodicity and elementary integration techniques with maximum principles and classical elliptic estimates, we obtains several necessary features of solutions. In particular, we find a class of solutions which admits uniform estimates in a sequence of bounded domains whose limit fill out the half-plane Ω. The class of solutions found all reside within the elliptic domain of equation (1).

For the existence of solitary waves, there is the further question of the infinite extension also in the x-direction. In our approach, we consider solitary (i.e. localized) waves as limits of periodic waves, and one then needs to find uniform estimates for some class of shoreline data with growing period. For this we use Sobolev estimates, which rely on the particular form of the nonlinearity in (1). The following extension of Theorem 1 is obtained [3].

Theorem 2 (Solitary waves) *There exists $L > 0$, such that for $f \in BUC^{3+\alpha}(\mathbb{R}) \cap H^2(\mathbb{R})$ with $\|f\|_{C^{3+\alpha}(\mathbb{R})} < L$, there is a solution ψ of* (1) *with $\psi(\cdot, 0) = f$,*

$$\|\psi\|_{C^{3+\alpha}(\Omega)} \leq 1/4, \qquad \|\nabla\psi\|_{H^1(\Omega)} \leq 27\|f\|_{H^2(\mathbb{R})},$$

and for any $n \in \mathbb{N}$, we have $\max_{y\in[-n,0]} \psi(x, y) \to 0$ as $|x| \to \infty$.

The solutions which correspond to odd shoreline data are also odd, and additionally they vanish as $y \to -\infty$. It also turns that any solution which is everywhere even, has to be of the form $\lambda \cos(cx) \exp(cy)$. Such a strong property is not be expected for a general disturbance of the Laplace equation, and it points to the specific form of the nonlinearity in (1).

All taken together, equation (1) seems to encompass very well the qualitative behaviour of steady edge-waves. It allows for desirable well-posedness results, and for a large class of solutions it imposes a wave-like structure.

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Dispersive properties of surface water waves

Vera Mikyoung Hur

(joint work with Hans Christianson and Gigliola Staffilani)

The *water-wave problem* in its simplest form concerns the two-dimensional dynamics of an incompressible inviscid liquid of infinite depth and the wave motion on its one-dimensional surface, under gravity and surface tension. The *moving* interface is given as a nonself-intersecting parametrized curve. The liquid occupies the domain below the interface, where the motion is described by the Euler equations with gravity and the flow is irrotational. The kinematic and dynamic boundary conditions hold at the interface. The motion at great depths is assumed to be almost at rest, and the interface is to be asymptotically flat.

Since the works by Wu [2, 3] the initial value problem associated to, more generally, a class of the Euler equations with moving boundary has been well studied via the *energy method*. While the method successfully yields local well-posedness, nonetheless, it does not provide further information about solutions, other than that they remain as smooth as their initial states. The present investigation is the *dispersive* aspect of surface water waves. Specifically, the main result establishes the local smoothing effect for the water-wave problem with surface tension. It contrasts markedly to what can be said from the energy method alone.

The analysis is based on the formulation of the water-wave problem with surface tension as a second-order in time nonlinear dispersive equation

$$(1) \qquad \partial_t^2 u - \tfrac{1}{2} S H \partial_\alpha^3 u + g H \partial_\alpha u = -2u\partial_t\partial_\alpha u - u^2 \partial_\alpha^2 u + R(u, \partial_t u).$$

Here, u is related to the tangential velocity at the interface; $t \in \mathbb{R}_+$ is the temporal variable and $\alpha \in \mathbb{R}$ is the arclength parametrization of the interface. The Hilbert transform H may be defined via the Fourier transform as $\widehat{Hf}(\xi) = -i\mathrm{sgn}(\xi)\widehat{f}(\xi)$. The remainder R is of lower order compared to $2u\partial_t\partial_\alpha u$ and $u^2\partial_\alpha^2 u$ in the sense that $\|R(u, \partial_t u)\|_{H^s} \leq C(\|u\|_{H^{s+1}}, \|\partial_t u\|_{H^s})$ for $s \geq 1$. Here and elsewhere, H^s means the Sobolev space of order s in the variable $\alpha \in \mathbb{R}$.

Theorem 1 (Main-theorem) *Let $S > 0$ and $g \geq 0$. For $s > 2 + 1/2$ the initial value problem of (1) with $u(0,\alpha) = u_0(\alpha)$ and $\partial_t u(0,\alpha) = u_1(\alpha)$, where $(u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-3/2}(\mathbb{R})$, is locally well-posed on $t \in [0, T_0]$ for some $T_0 > 0$ and $(u(t), \partial_t u(t)) \in C([0,T]; H^s(\mathbb{R}) \times H^{s-3/2}(\mathbb{R}))$. Moreover, if $s \geq s_0 > 1$ is sufficiently large, then for $0 < T < T_0$ sufficiently small, the inequality*

$$(2) \qquad \int_0^T \int_{-\infty}^{\infty} |\langle\alpha\rangle^{-\rho} D_\alpha^{s+1/4} u(t,\alpha)|^2 \, d\alpha dt \leq C(T, \|u_0\|_{H^s}, \|u_1\|_{H^{s-3/2}})$$

holds, where $\rho \geq 3$. Here, $\langle\alpha\rangle = (1+\alpha^2)^{1/2}$ and $D_\alpha = -i\partial_\alpha$.

The proof of (2) is motivated by the local smoothing effect of the linear part of (1). When $S > 0$ the solution to the initial value problem

$$\partial_t^2 u - \tfrac{1}{2} S H \partial_\alpha^3 u + g H \partial_\alpha u = 0, \qquad u(0,\alpha) = u_0(\alpha) \quad \text{and} \quad \partial_t u(0,\alpha) = u_1(\alpha)$$

possesses the estimate

$$\sup_{\alpha\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |D_\alpha^{1/4} u(t,\alpha)|^2 dt \right)^{1/2} \leq C(\|u_0\|_{L^2_\alpha(\mathbb{R})} + \|u_1\|_{H_\alpha^{-3/2}(\mathbb{R})}),$$

and the solution to the corresponding inhomogeneous equation

$$\partial_t^2 v - \tfrac{1}{2} S H \partial_\alpha^3 v + g H \partial_\alpha v = R(t,\alpha). \qquad v(0,\alpha) = 0 = \partial_t v(0,\alpha)$$

possesses the estimate

$$\sup_{\alpha\in\mathbb{R}} \left(\int_{-\infty}^{\infty} |D_\alpha^2 v(t,\alpha)|^2 dt \right)^{1/2} \leq C \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t,\alpha)|^2 dt \right)^{1/2} d\alpha.$$

The main difficulty of the proof is that the dispersive property of (1) is too weak to control the nonlinearity. The smoothing effect of the linear part of (1) can treat up to 2 derivatives in the nonlinearity. However, the worst nonlinear term $u\partial_t\partial_\alpha u$ in (1) contains $2 + 1/2$ derivatives.

This difficulty is overcome by viewing (1) as

$$\partial_t^2 u - \tfrac{1}{2} S H \partial_\alpha^3 u + g H \partial_\alpha u + 2u\partial_t\partial_\alpha u + u^2\partial_\alpha^2 u = R(u, \partial_t u).$$

That means, $2u\partial_t\partial_\alpha u$ and $u^2\partial_\alpha^2 u$ are considered as "linear" components of the equation, but with variable coefficients which happen to depend on the solution itself. The chief effort of the proof is then to establish the local smoothing effect for the variable-coefficient linear operator $\partial_t^2 - \frac{1}{2} S H \partial_\alpha^3 + g H \partial_\alpha + 2V(t,\alpha)\partial_\alpha\partial_t +$

$V^2(t,\alpha)\partial_\alpha^2$. Our approach is based on the construction of an approximate solution ("parametrix") for high frequencies. The proof combines the energy method with techniques of pseudodifferential operators and Fourier integral operators, propagation of singularities. The detail is contained in [1].

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Two component integrable systems modelling shallow water waves

Rossen I. Ivanov

The aim of this talk is to describe the derivation of shallow water model equations for the *constant vorticity* case and to demonstrate how these equations can be related to two integrable systems: a two component integrable generalization of the Camassa-Holm equation and the Kaup - Boussinesq system.

The motion of inviscid fluid is described by Euler's equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \mathbf{g}, \qquad \nabla\cdot\mathbf{v} = 0,$$

where ρ is a constant density, $\mathbf{v}(x,y,z,t)$ is the velocity of the fluid at the point (x,y,z) at the time t, P is the pressure in the fluid, $\mathbf{g}=(0,0,-g)$ is the constant Earth's gravity acceleration.

We consider a motion of a shallow water over a flat bottom, which is located at $z=0$. We assume that the motion is in the x-direction, and that the physical variables do not depend on y. Let h be the mean level of the water and let $\eta(x,t)$ describes the shape of the water surface, i.e. the deviation from the average level. The pressure is $P = P_A + \rho g(h-z) + p(x,z,t)$, where P_A is the constant atmospheric pressure, and p is a pressure variable, measuring the deviation from the hydrostatic pressure distribution.

On the surface $z=h+\eta$, $P=P_A$ and therefore $p=\eta\rho g$. Taking $\mathbf{v}\equiv(u,0,w)$ we can write the kinematic condition on the surface as (e.g. following [1]) $w=\eta_t+u\eta_x$ on $z=h+\eta$. Finally, there is no horizontal velocity at the bottom, thus $w=0$ on $z=0$.

Let us introduce now dimensionless parameters $\varepsilon=a/h$ and $\delta=h/\lambda$, where a is the typical amplitude of the wave and λ is the typical wavelength of the wave. Now we can introduce dimensionless quantities, according to the magnitude of the physical quantities, see [1, 2] for details: $x\to\lambda x$, $z\to zh$, $t\to\frac{\lambda}{\sqrt{gh}}t$, $\eta\to a\eta$, $u\to\varepsilon\sqrt{gh}u$, $w\to\varepsilon\delta\sqrt{gh}w$, $p\to\varepsilon\rho gh$.

Now let us notice that there is an exact solution of the governing equations of the form $u=\tilde{U}(z)$, $0\le z\le h$, $w\equiv 0$, $p\equiv 0$, $\eta\equiv 0$. This solution represents an arbitrary underlying 'shear' flow. In the presence of a shear flow the horizontal velocity of the fluid will be $\tilde{U}(z) + u$. The scaling for such solution is clearly $u \to \sqrt{gh}(\tilde{U}(z) + \varepsilon u)$, and the scaling for the other variables is as before. The system of equations is (the prime denotes derivative with respect to z):

$$\begin{aligned} u_t + \tilde{U}u_x + w\tilde{U}' + \varepsilon(uu_x + wu_z) &= -p_x, \\ \delta^2(w_t + \tilde{U}w_x + \varepsilon(uw_x + ww_z)) &= -p_z, \\ u_x + w_z &= 0, \\ w = \eta_t + (\tilde{U} + \varepsilon u)\eta_x, \qquad p = \eta, \qquad \text{on} \quad z &= 1 + \varepsilon\eta, \\ w = 0 \qquad \text{on} \quad z &= 0. \end{aligned}$$

The simplest nontrivial case is a linear shear, $\tilde{U}(z) = Az$, where A is a constant. We choose $A > 0$, so that the underlying flow is propagating in the positive direction of the x-coordinate.

The vorticity is $\omega = (U + u)_z - w_x$ or in terms of the rescaled variables, $\omega = A + \varepsilon(u_z - \delta^2 w_x)$. We are looking for a solution with constant vorticity $\omega = A$, and therefore we require that $u_z - \delta^2 w_x = 0$. Together with the equation $u_x + w_z = 0$ it gives

$$u = u_0 - \delta^2 \frac{z^2}{2} u_{0xx} + \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2), \quad w = -zu_{0x} + \delta^2 \frac{z^3}{6} u_{0xxx} + \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2),$$

where $u_0(x, t)$ is the leading order approximation for u.

With these expressions we obtain the following from the condition on the surface, ignoring terms of order $\mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2)$:

$$\eta_t + A\eta_x + \left[(1 + \varepsilon\eta)u_0 + \varepsilon \frac{A}{2}\eta^2\right]_x - \delta^2 \frac{1}{6} u_{0xxx} = 0 \tag{1}$$

From the second of the Euler's equations and the condition on the surface we have $p = \eta - \delta^2 \left[\frac{1-z^2}{2} u_{0xt} + \frac{1-z^3}{3} A u_{0xx}\right]$, then the first of the Euler's equations gives (Note that there is no z-dependence!)

$$\left(u_0 - \delta^2 \frac{1}{2} u_{0xx}\right)_t + \varepsilon u_0 u_{0x} + \eta_x - \delta^2 \frac{A}{3} u_{0xxx} = 0. \tag{2}$$

The linearised equations are

$$u_{0t} + \eta_x = 0, \qquad \eta_t + A\eta_x + u_{0x} = 0, \tag{3}$$

giving $\eta_{tt} + A\eta_{tx} - \eta_{xx} = 0$. This linear equation has a travelling wave solution $\eta = \eta(x - ct)$ with a velocity c satisfying $c^2 - Ac - 1 = 0$, or

$$c = \frac{1}{2}\left(A \pm \sqrt{4 + A^2}\right).$$

If there is no shear ($A = 0$), then $c = \pm 1$. In general, there is one positive and one negative solution, representing left and right running waves. Suppose that we have only one of these waves, then $\eta = cu_0 + \mathcal{O}(\varepsilon, \delta^2)$ - e.g. from (3).

By introduction of a new variable $\rho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\gamma u_{0xx}$, where

$$\alpha = \frac{1}{3(1+c^2)} + \frac{2c^2}{3(1+c^2)}\Big(1+\frac{Ac}{2}\Big), \quad \beta = \frac{1-(3+c^2)(1+\frac{Ac}{2})}{3(1+c^2)}\alpha, \quad \gamma = \frac{\alpha}{6(c-A)},$$

and a change of variables (rescaling) $u_0 \to \frac{1}{\alpha\varepsilon}u_0$, $x \to \frac{\delta}{\sqrt{B}}\, x$, $t \to \frac{\delta}{\sqrt{B}}t$ where $B = \frac{1}{2} + \frac{1}{6(c-A)}\Big(A - \frac{1}{c-A}\Big)$ the equations (1), (2) transform into the system

$$(4) \quad m_t + Am_x - Au_{0x} + 2mu_{0x} + u_0 m_x + \rho\rho_x = 0, \qquad m = u_0 - u_{0xx}$$

$$(5) \quad \rho_t + A\rho_x + (\rho u_0)_x = 0,$$

Before the rescaling we had $\alpha\varepsilon\eta = \rho - 1 - \varepsilon^2\beta c^2 u_0^2 - \varepsilon\delta^2\gamma u_{0xx}$. Since in the leading order $\eta = cu_0$ the rescaling of η is $\eta \to \frac{1}{\alpha\varepsilon}\eta$. Thus in terms of the rescaled variables $\eta = \rho - 1 - \frac{\beta c^2}{\alpha^2}u_0^2 - B\frac{\gamma}{\alpha}u_{0xx}$.

The system (4), (5) is an integrable 2-component Camassa-Holm system that appears in [3], generalizing the famous Camassa-Holm equation [4]. The Lax representation for this system is (ζ is a spectral parameter)

$$\Psi_{xx} = \Big(-\zeta^2\rho^2 + \zeta(m - \frac{A}{2}) + \frac{1}{4}\Big)\Psi,$$

$$\Psi_t = \Big(\frac{1}{2\zeta} - u_0 - A\Big)\Psi_x + \frac{1}{2}u_{0x}\Psi.$$

An alternative derivation for the case of zero vorticity, based on the Green-Naghdi equations is reported in [5].

Another integrable system matching the water waves asymptotic equations to the first order of the small parameters ε, δ is the Kaup - Boussinesq system. We describe briefly its derivation. Introducing $V = u - \delta^2(\frac{1}{2} - \frac{A}{3c})u_{xx}$ the equation (2) can be written as $V_t + \varepsilon VV_x + \eta_x = 0$. Equation (1) in the first order in ε, δ is

$$\eta_t + \Big[A\eta + (1+\varepsilon\eta)u_0 + \varepsilon\frac{A}{2}\eta^2\Big]_x - \delta^2\frac{1}{6}u_{0xxx} = 0$$

and with a shift $\eta \to \eta - \frac{1}{\varepsilon}$ it becomes

$$\eta_t + \varepsilon(1+\frac{Ac}{2})(\eta u_0)_x - \delta^2\frac{1}{6}u_{0xxx} = 0 \quad \text{or} \quad \eta_t + \varepsilon\frac{1+c^2}{2}(\eta V)_x - \delta^2\frac{1}{6}V_{xxx} = 0.$$

Further rescaling leads to the Kaup - Boussinesq system

$$V_t + VV_x + \eta_x = 0, \qquad \eta_t - \frac{1}{4}V_{xxx} + \frac{1+c^2}{2}(\eta V)_x = 0,$$

which is integrable iff $A = 0$ ($c^2 = 1$) with a Lax pair [6]

$$\Psi_{xx} = -\Big((\zeta - \frac{1}{2}V)^2 - \eta\Big)\Psi, \qquad \Psi_t = -(\zeta + \frac{1}{2}V)\Psi_x + \frac{1}{4}V_x\Psi.$$

It is interesting to investigate further which specific properties of the original governing equations are preserved in the 'integrable' approximate models. For example the 2-component Camassa-Holm system for certain initial data admits breaking waves solutions [5].

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Using parameter asymptotics to find approximate solutions for periodic waves over finite-depth, constant-vorticity flows

Robin Johnson

We present some preliminary (formal) results that provide asymptotic descriptions for the flow field (specifically, the streamlines) and for the structure of stagnation points, if they arise. This work was prompted by the recent successes in producing rigorous statements about the nature of periodic waves with vorticity (see [1]), and some corresponding numerical simulations of these solutions (see [2]).

The model that we take is that of classical water waves: an inviscid, incompressible fluid with constant pressure at the free surface (no surface tension) and a constant undisturbed depth. The flow is steady and so, in a frame moving with the wave, the problem in terms of the Euler equation (suitably non-dimensionalised) becomes:

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p,\ \nabla\cdot\mathbf{u} = 0 \quad \text{with} \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right),\ \mathbf{u} = (u, v),$$

where p is measured relative to the hydrostatic pressure distribution, and

$$p = h(x)\ \&\ v = uh_x \text{ on } y = h(x) \text{ and } v = 0 \text{ on } y = -d_0 = \text{constant}.$$

(Here, u replaces the actual horizontal velocity component, less the wave speed, i.e. u replaces the familiar $u - c$, so our $u \leq 0$, with equality at stagnation.) Equivalently, at the surface, we could us a Bernoulli condition:

$$u^2 + v^2 + 2(h + d_0) = Q = \text{constant (the total head) on } y = h(x).$$

The vorticity is $v_x - u_y = \gamma$, and so $\nabla^2\psi = -\gamma(\psi)$, where ψ is the stream function; the total mass flux is $\int_{-d_0}^{h(x)} u(x,y)\,dy = -p_0 (< 0)$, a constant, and we can choose $\psi = 0$ on $y = h(x)$ (so that $\psi = p_0$ on $y = -d_0$). We shall describe some results in the cases , where is a constant, the upper sign therefore giving positive vorticity (and the lower, negative).

Following the work previously mentioned, the problem is conveniently transformed according to the Dubreil-Jacotin formulation, where we define $D(x, \psi) =$

$y + d_0$, so that $\partial/\partial y \equiv (1/D_\psi)\partial/\partial\psi$ and $\partial/\partial x \to (D_x/D_y)\partial/\partial\psi$ (and then $u = 1/D_\psi$, $v = D_x/D_\psi$). The resulting problem becomes

$$D_{\psi\psi} + D_\psi^2 D_{xx} + D_x^2 D_{\psi\psi} - 2D_\psi D_x D_{x\psi} = \pm\omega D_\psi^3,$$

with

$$1 + (2D - Q)D_\psi^2 + D_x^2 = 0 \text{ on } \psi = 0 \quad \text{and} \quad D = 0 \text{ on } \psi = p_0\,(> 0);$$

we seek solutions periodic in x (period $2L$) in $R = (-L, L) \times (0, p_0)$. However, before we proceed, it is convenient to write this version of the problem in a suitably normalised form: we define $D = d/\sqrt{\omega}$ and $Q = \omega q$, to give

$$(1) \quad \begin{cases} d_{\psi\psi} \pm d_\psi^3 = \omega^{-1}(2d_\psi d_x d_{\psi x} - d_\psi^2 d_{xx} - d_x^2 d_{\psi\psi}) & \\ \text{with} \quad 1 - (q - 2\omega^{-3/2}d)d_\psi^2 + \omega^{-1}d_x^2 = 0 & \text{on} \quad \psi = 0 \\ \text{and} \quad d = 0 & \text{on} \quad \psi = p_0. \end{cases}$$

This is a particularly useful form on two counts: firstly, the terms on the left in the PDE are the only ones that contribute to the uniform-flow solution and, secondly, the limiting case of $\omega \to \infty$ follows directly. The essential idea behind our results is to perturb the uniform flows (defined by (1), either by introducing a wave of arbitrary (but small) amplitude, or simply by allowing $\omega \to \infty$ in (1); we start with the former.

The procedure is altogether routine; we let δ measure the amplitude of the wave, and impose 2π-periodicity (because this was the choice in the work quoted earlier). For positive vorticity, we find that

$$(2) \quad \begin{cases} d \sim \sqrt{b - 2\psi} - \sqrt{b - 2p_0} + \dfrac{\delta\cos(x)}{\sqrt{b - 2\psi}}\sinh\left[\dfrac{1}{\sqrt{\omega}}(\sqrt{b - 2\psi} - \sqrt{b - 2p_0})\right], \; \delta \to 0 \\ \text{where } b \geq 2p_0 \text{ satisfies } \tanh\left[\dfrac{1}{\sqrt{\omega}}(\sqrt{b} - \sqrt{b - 2p_0})\right] = \dfrac{b}{\sqrt{b\omega} + \omega^{-1}} \end{cases}$$

and solutions of this form exist only for $0 < \omega/p_0 \leq k$, $k \approx 7.009$ (which corresponds to $b = 2p_0$.) However, the asymptotic solution described in (2) is not uniformly valid for b close to $2p_0$, when ψ is also close to p_0; the breakdown occurs where $b - 2p_0 = O(\delta^2)$, $\psi - p_0 = O(\delta^2)$. In this case, with $b = 2p_0 + \lambda^2\delta^2$ and $\psi = p_0 - \delta^2\psi$, a solution that maches to (2) (when that is evaluated for this new b, and for $\psi - p_0 = O(1)$) is

$$b \sim \sqrt{2\psi + (\lambda - \omega^{-1/2}\cos x)^2} - (\lambda - \omega^{-1/2}\cos x), \quad \delta \to 0,$$

and this predicts a stagnation point at $\psi = 0$ (i.e. on the bed), under the crest, if $\lambda = 1/\sqrt{\omega}$.

For this solution, we find that $q = 2p_0 + 2\omega^{-3/2}(\sqrt{2p_0 + \delta^2\lambda^2} - \delta\lambda)$; comparison with the numerical results is then possible (but we cannot claim good numerical agreement without a more complete analysis- we are simply attempting to capture the essential features of this problem).

The corresponding problem for negative vorticity yields

$$d \sim \sqrt{b+2p_0} - \sqrt{b+2\psi} + \frac{\delta\cos(x)}{\sqrt{b+2\psi}}\sinh\left[\frac{1}{\sqrt{\omega}}(\sqrt{b+2p_0} - \sqrt{b+2\psi})\right],\ \delta \to 0,$$

where $\tanh\left[\frac{2}{\sqrt{\omega}}(\sqrt{b+2p_0} - \sqrt{b})\right] = \frac{b}{\omega^{-1}-\sqrt{b\omega}}$, and this has solutions for $0 < \omega < \infty$ (and in particular $b \sim cp_0^2/\omega$ as $\omega \to \infty$, $c \sim 1.61$, $b \sim 1/\omega^3$ as $\omega \to \infty$). However, when we follow the procedure adopted in the previous case, which near stagnation would require $b = O(\delta)$ with $\psi = O(\delta)$, we find that there is no appropriate solution. This is because the limit here is $b \to 0$ at ω fixed, and the perturbation solution (above) suggests that we require $\omega \to \infty$ as $b \to 0$. With this result in mind, we examine the problem of letting $\omega \to \infty$, in the case of negative vorticity.

With a little care, and judicious choices of the parameters, we find that (for $\delta \to 0$)

$$\begin{aligned} d \sim & \sqrt{2p_0 + \delta A(x)} - \sqrt{2\psi + \delta A(x)} \\ & - \frac{\delta}{4\omega}\sinh\left(\sqrt{2p_0} - \sqrt{2\psi + \delta A} + \frac{2(\psi - p_0)}{3\sqrt{2p_0}}\right)A'',\ \delta \to 0, \end{aligned}$$

and then with $\delta A(x) = \omega^{-3}(1+\omega^{-5/2}A_0(x))$ (so a stagnation point is not possible, although we shall be close to stagnation as $\omega \to \infty$), we find that

$$\frac{1}{3}\sqrt{2p_0}A_0'' = Q - \frac{1}{4}A_0^2 \tag{3}$$

where $q \sim -2\omega^{-3}\sqrt{2p_0 + \omega^{-3}} - \omega^{-3} - Q\omega^{-8}$. Equation (3) possesses an appropriate solution in terms of Jacobian elliptic functions; all of this information can then be used to produce representations of the streamlines, fairly close to stagnation, for suitable q. The corresponding problem ($\omega \to \infty$) for positive vorticity does not appear to lead to a corresponding solution (presumably because of the non-existence of small-amplitude waves for $\omega/p_0 > k$).

These results demonstrate, albeit in only a formal and preliminary way, that the use of suitable parameter expansions, with due attention to break down and matching, may provide some useful insights. In particular, we have attempted to capture some of the details of the mechanisms involved in these interesting and important flows.

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Asymptotic models for internal waves

David Lannes

We are interested in the asymptotic description of internal waves at the interface of two fluids. The system we consider here, when it is at rest, consists of a homogeneous fluid of depth d_1 and density ρ_1 lying over another homogeneous fluid of depth d_2 and density $\rho_2 > \rho_1$. The bottom on which both fluids rest is presumed to be horizontal and featureless while the top of fluid 1 is restricted by the rigid lid assumption, which is to say, the top is viewed as an impenetrable, bounding surface.

We describe here the strategy developed in [1] to describe qualitatively the motion of the interface. Namely, following the procedure introduced in [2, 3, 4], we rewrite the full system as a system of two evolution equations posed on $\mathbb{R}^d$, where $d = 1$ or 2 depending upon whether a one- or two-dimensional model is being contemplated. The reformulated system, which depends only upon the spatial variable on the interface, involves two non-local operators, a Dirichlet-to-Neumann operator $G[\zeta]$, and what we term an "interface operator" $\mathbf{H}[\zeta]$, defined precisely below. Of course the operator $\mathbf{H}[\zeta]$ does not appear in the theory of surface waves, and this is an interesting new aspect of the internal wave theory.

A rigorously justified asymptotic expansion of the non-local operators with respect to dimensionless small parameters is then mounted. We consider both the "weakly nonlinear" case and the "fully nonlinear" situation and cover a variety of scaling regimes. For the considered scaling regimes, these expansions then lead to an asymptotic evolution system. In each case a family of asymptotic models may then be inferred by using the "BBM trick" and suitable changes of the dependent variables. This analysis recovers most of the systems which have been introduced in the literature and also some interesting new ones. For instance, in certain of the two-dimensional regimes, a non-local operator appears whose analog is not present in any of the one-dimensional cases.

All the systems derived are proved to be consistent with the full Euler system. In rough terms, this means that any solution of the latter solves any of the asymptotic systems up to a small error.

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A novel integrable generalization of the nonlinear Schrödinger equation

Jonatan Lenells

I present joint work with Prof A. S. Fokas. We consider the following integrable generalization of the nonlinear Schrödinger (NLS) equation, which was first derived in [A. S. Fokas, *Phys.* D **87** (1995), 145–150] using bi-Hamiltonian methods:

$$iu_t - \nu u_{tx} + \gamma u_{xx} + \rho |u|^2 (u + i\nu u_x) = 0, \qquad x \in \mathbb{R}, \, t > 0, \tag{1}$$

where ν, γ, ρ are real parameters and $u(x,t)$ is a complex-valued function. Equation eqrefJL is related to NLS in the same way that the Camassa-Holm equation is related to KdV—note that equation (1) reduces to NLS when $\nu = 0$. In our study of (1) we: (a) Provide a physical derivation in the context of fiber optics; (b) Establish a relation with the first negative member of the derivative nonlinear Schrödinger (DNLS) hierarchy; (c) Derive a Lax pair; (d) Implement the Inverse Scattering Transform formalism; (e) Consider the initial-boundary value (IBV) problem formulated on the half-line; (f) Analyze solitons and traveling-wave solutions.

Invariant manifolds of Euler equations

Zhiwu Lin

(joint work with Chongchun Zeng)

Consider the Euler equation

$$\text{(E)} \qquad \begin{cases} D_t v \triangleq v_t + v \cdot \nabla v = -\nabla p & \text{in } \Omega \\ \nabla \cdot v = 0 & \text{in } \Omega \\ v.N = 0 & \text{on } \partial\Omega \end{cases},$$

where Ω is a bounded domain in $\mathbf{R}^2$ or $\mathbf{R}^3$, N is the outer normal of $\partial\Omega$. The function space for (E) is

$$X \triangleq \{ w \in H^k(\Omega, \mathbf{R}^n) \mid \nabla \cdot w = 0 \text{ in } \Omega, \; w \cdot N = 0 \text{ on } \partial\Omega \},$$

with $n = 2$ or 3, $k > n/2+1$. The Euler equation has infinitely many steady states. Example 1: shear flows $v_0 = (U(y), 0)$ in 2D or $(U(y,z), 0, 0)$ in 3D. Example 2: in 2D, $v_0 = \nabla^\perp \psi_0$, where the stream function ψ_0 satisfies $\Delta\psi_0 = F(\psi_0)$ in Ω for some function F and ψ_0 takes constant on $\partial\Omega$. The stability of these steady states is an important problem in fluid mechanics. The linearized Euler operator L around v_0 had been studied quite a lot, for both the unstable essential spectrum (i.e. [3] [5] [6]) and discrete spectrum (i.e. [2] [7] [9]). The passing of nonlinear instability from linear instability was also proved for the 2D Euler equation ([1] [4] [8] [11]). The invariant manifolds are important to describe the local dynamics near an unstable steady flow v_0. However, their existence has been open for Euler equations, due to two main difficulties: First, the nonlinear term $v \cdot \nabla v$ contains the loss of derivatives. Second, L has no smoothing effects! In the proof of nonlinear instability, some techniques such as a nonlinear bootstrap were developed to overcome the difficulty of the loss of derivative. However, these tricks strongly use the property of solutions of Euler equation and cannot be used in constructing invariant manifolds.

Recently, with Chongchun Zeng, we obtained the following result.

Theorem 1 (L & Zeng, [10]) *The C^1 stable and unstable manifolds exist uniquely in the phase space $X \subset H^k(\Omega)$ with $k > \frac{n}{2} + 1$, under the assumptions that $v_0 \in H^{k+4}(\Omega)$ and*

1. $\exists$ closed invariant subspaces E^{cs} and finite dimensional E^u of L, s. t.

$$\begin{aligned} X &= \{w \in H^k(\Omega, \mathbf{R}^n) \mid \nabla \cdot w = 0 \text{ in } \Omega, w \cdot N = 0 \text{ on } \partial\Omega\} \\ &= E^{cs} \oplus E^u. \end{aligned}$$

2. $\exists \lambda_u > \lambda_{cs} \geq 0$, $C > 0$ s. t. for $t \geq 0$,

$$|e^{-tL}|_{L(E^u)} \leq Ce^{-\lambda_u t}, \quad |e^{tL}|_{L(E^{cs})} \leq Ce^{\lambda_{cs} t}.$$

3. $4(k+2)\mu < \lambda_u - \lambda_{cs}$ where μ is the Liapunov exponent of v_0.

In our proof, we use a combination of Lagrangian and Eulerian approaches to overcome the difficulties mentioned above.

Examples for applying above theorem include:

1. Unstable shear flows between rigid walls

$$\Omega = S^1\left(2\pi/\alpha\right) \times (0, \pi), \quad v_0 = (U\left(y\right), 0),$$

2. Unstable shear flows in a torus

$$\Omega = S^1\left(2\pi/\alpha\right) \times S^1, \quad v_0 = (U\left(y\right), 0),$$

3. Unstable rotating flows in an annulus

$$\Omega = \{a < r < b\}, \quad v_0 = \rho\left(r\right) \vec{e}_\theta,$$

4. Unstable 3D shear flows

$$\Omega = S^1\left(2\pi/\alpha\right) \times (0, \pi) \times S^1, \quad v_0 = (U\left(y, z\right), 0, 0),$$

For all above examples, the Liapunov exponent $\mu = 0$ and the assumptions of the Theorem can be verified if there exists an unstable discrete eigenvalue.

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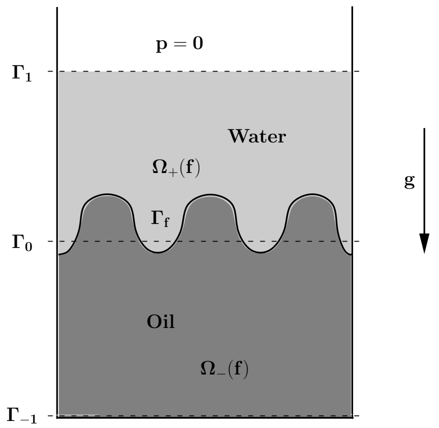
Fingering patterns for the Muskat problem

Bogdan-Vasile Matioc
(joint work with Joachim Escher)

The Muskat problem, proposed 1934 (see [4]), describes the evolution of the interface between two immiscible fluids in a porous medium or Hele-Shaw cell. The one-phase version of the problem, in which one of the fluids has zero viscosity is exactly the Hele-Shaw problem. We consider the Muskat problem for two fluids in a two-dimensional vertical Hele-Shaw cell or a porous medium (see the adjacent figure).



The laterals and the bottom of the cell are assumed to be impermeable and the pressure is normalized to be 0 on the boundary component Γ_1. It is very common in literature to presuppose that the fluids are separated by a sharp interface which moves along with the fluids.

In the unstable case, when the less dense fluid lies on the bottom of the cell, we prove using a bifurcation argument, existence of non-trivial, finger-shaped stationary solutions which are obtained as analytic bifurcation branches from a trivial flat equilibrium. It is also shown that these equilibria are a priori smooth. The local bifurcation branches can be actually defined globally. Under some reasonable assumptions it is also shown that the fingering patterns disappear when the surface tension coefficient $\gamma \to \infty$.

Since the bottom an the the laterals of the cell are impermeable we can prove that the volume of each fluid is preserved by the flow. From the conservation of mass we derive the following free boundary problem describing the stationary states of the Muskat problem:

$$(1)\qquad \begin{cases} \Delta u_\pm & = & 0 & \text{in} & \Omega_\pm(f), \\ u_+ & = & g(\rho_- + \rho_+) + g(\rho_- - \rho_+)f & \text{on} & \Gamma_1, \\ \partial_\nu u_- & = & 0 & \text{on} & \Gamma_{-1}, \\ u_+ - u_- & = & \gamma\kappa_f & \text{on} & \Gamma_f, \\ \partial_\nu u_\pm & = & 0 & \text{on} & \Gamma_f. \end{cases}$$

The suffix + refers to the upper fluid and the suffix – to the lower and g denotes the gravity constant. In this case the fluids occupy the domain $\Omega := \mathbb{S}^1 \times (-1, 1)$,

$$\Omega_-(f) := \{(x, y) \in \Omega \,:\, -1 < y < f(x)\}, \quad \Omega_+(f) := \{(x, y) \in \Omega \,:\, f(x) < y < 1\}.$$

and ν denotes the outward unit normal to the boundary $\partial\Omega_-(f)$. The function $f \in C^{4+\alpha}(\mathbb{S}^1)$ is to be determined such that $\|f\|_{C^{4+\alpha}(\mathbb{S}^1)} < 1$ and $\int_{\mathbb{S}^1} f\,dx = 0$ (if we presuppose that the cell contains equal volumes of both fluids).

Let $\overline{u} := g(\rho_- + \rho_+)$. For fixed γ, the triple $(f, u_+, u_-) = (0, \overline{u}, \overline{u})$ is the unique flat solution for the problem (1). For appropriate γ we prove that there are also non-flat stationary solutions. To this scope, we reduce the system (1) into an equation in the Banach space $C^{1+\alpha}_{0,e}(\mathbb{S}^1)$:

$$(2)\qquad \Phi(\gamma, f) = 0, \quad (\gamma, f) \in \mathbb{R}_{>0} \times \mathcal{V}_{0,e},$$

where Φ is a nonlinear and analytic operator and

$$\mathcal{V}_{0,e} := \{f \in C^{4+\alpha}_{0,e}(\mathbb{S}^1) \,:\, \|f\|_0 < 1\}.$$

The Banach space $C^{m+\alpha}_{0,e}(\mathbb{S}^1)$, $m \in \mathbb{N}$, is the subspace of $C^{m+\alpha}(\mathbb{S}^1)$ which contains only of even functions with integral mean 0. Given $l \in \mathbb{N}$, let

$$(3)\qquad \overline{\gamma}_l := \frac{g(\rho_+ - \rho_-)}{l^2 \cosh(l)}.$$

Assume that $\rho_- < \rho_+$ and let $l \geq 1$ be fixed. Using the bifurcation theorem from simple eigenvalues due to Crandall-Rabinowitz (see [2]) we obtain in [3] that the pair $(\overline{\gamma}_l, 0)$, with $\overline{\gamma}_l$ defined by relation (3), is a bifurcation point of the flat solution $(\gamma, 0, \overline{u}, \overline{u})$. More precisely, there exists $\delta > 0$ and a real analytical function $(\gamma_l, f_l) : (-\delta, \delta) \to \mathbb{R}_{>0} \times \mathcal{V}_{0,e}$ such that $(\gamma_l(0), f_l(0)) = (\overline{\gamma}_l, 0)$, $(\gamma_l(\varepsilon), f_l(\varepsilon))$ is a solution of the free boundary problem (2) for all $\varepsilon \in (-\delta, \delta)$ and $f_l'(0) = \cos(lx)$. Letting

$$\Sigma_l^+ := \{(\gamma(\varepsilon), f(\varepsilon)) \,:\, \varepsilon \in (0, \delta))\},$$

we may use a global bifurcation result presented in [1] to obtain that there exists a continuous curve $\Sigma_l \subset \mathbb{R}_{>0} \times C^\infty(\mathbb{S})$ which consists only of solutions of (2) and which extends the local bifurcation branch Σ_l^+ globally.

Moreover, if we presuppose that the continuous curve Σ_l satisfies:

(a) $\lim_{\varepsilon \to \infty} \gamma_l(\varepsilon) = \infty$,
(b) $\sup_{\varepsilon \geq 0} \|f_l(\varepsilon)\|_0 < 1$,
(c) $\sup_{\varepsilon \geq 0} \|f_l(\varepsilon)\|_{C^{4+\alpha}(\mathbb{S}^1)} < \infty$,

then we have that $\lim_{\varepsilon\to\infty} f_l(\varepsilon) = 0$ in $C^{4+\alpha}(\mathbb{S}^1)$, i.e. the stationary fingers flatten out as $\gamma \to \infty$.

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Stability of multi antipeakon-peakons profile

Luc Molinet

(joint work with Khaled El Dika)

Abstract. The Camassa-Holm equation possesses well-known peaked solitary waves that are called peakons. Their orbital stability has been established by Constantin and Strauss in [5]. In [9] we proved the stability of trains of peakons. Here, we continue this study by presenting a result on the stability of ordered trains of anti-peakons and peakons.

The Camassa-Holm equation (C-H),

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}^2, \tag{1}$$

can be derived as a model for the propagation of unidirectional shalow water waves over a flat bottom by writing the Green-Naghdi equations in Lie-Poisson Hamiltonian form and then making an asymptotic expansion which keeps the Hamiltonian structure ([2]). It was also found independently by Dai [7] as a model for nonlinear waves in cylindrical hyperelastic rods and was, in fact, first discovered by the method of recursive operator by Fokas and Fuchsteiner [10] as an example of bi-Hamiltonian equation.

(C-H) is completely integrable (see [2],[3]). It possesses among others the following invariants

$$E(v) = \int_{\mathbb{R}} v^2(x) + v_x^2(x)\,dx \text{ and } F(v) = \int_{\mathbb{R}} v^3(x) + v(x)v_x^2(x)\,dx \tag{2}$$

and can be written in Hamiltonian form as

$$\partial_t E'(u) = -\partial_x F'(u) \quad . \tag{3}$$

It possesses peaked solitary waves that are given by

$$u(t,x) = \varphi_c(x-ct) = c\varphi(x-ct) = ce^{|x-ct|},\ c \in \mathbb{R}.$$

They are called peakon if $c > 0$ and antipeakon if $c < 0$. Note that (C-H) has to be rewriten as

$$u_t + uu_x + (1-\partial_x^2)^{-1}\partial_x(u^2 + u_x^2/2) = 0\ . \tag{4}$$

to give a meaning to these solutions. Their stability seems not to enter the general framework developed for instance in [1], [11]. However, Constantin and Strauss [5] succeeded in proving their orbital stability by a direct approach. In [9] we combined the general strategy initiated in [12](note that due to the reasons mentioned above, the general method of [12] is not directly applicable here) and a monotonicity result proved in [9] with localized versions of the estimates established in [5] to derive the stability of the trains of peakons. In this work we pursue this study by proving the stability of trains of anti-peakons and peakons. The main new ingredient is a monotonicity result on the part of the functional $E(\cdot) - \delta F(\cdot)$, $\delta \geq 0$, at the right of a localized solution traveling to the right.

Before stating the main result we have to introduce the function space where we will define the flow of the equation. For I a finite or infinite interval of $\mathbb{R}$, we denote by $Y(I)$ the function space[1]

$$Y(I) := \Big\{ u \in C(I; H^1(\mathbb{R})) \cap L^\infty(I; W^{1,1}(\mathbb{R})),\ u_x \in L^\infty(I; BV(\mathbb{R})) \Big\}. \tag{5}$$

We are now ready to state our main result.

Theorem 1 *Let be given N non vanishing velocities $c_1 < .. < c_{q-1} < 0 < c_q < .. < c_N$. There exist γ_0, $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that if $u \in Y([0,T[)$, with $0 < T \leq \infty$, is a solution of (C-H) satisfying*

$$\|u_0 - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - z_j^0)\|_{H^1} \leq \varepsilon^2 \tag{6}$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geq L$, with $L > L_0$, then there exist $x_1(t), .., x_N(t)$ such that

$$\sup_{[0,T[} \|u(t,\cdot) - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - x_j(t))\|_{H^1} \leq A(\sqrt{\varepsilon} + L^{-1/8}) \tag{7}$$

Moreover there exists C^1-functions $\tilde{x}_1, .., \tilde{x}_N$ such that, $\forall j \in \{1, .., N\}$,

$$|x_j(t) - \tilde{x}_j(t)| = O(1) \text{ and } \frac{d}{dt}\tilde{x}_j = c_j + O(\varepsilon^{1/4}) + O(L^{-1}) \quad \forall t \in [0, T[\,. \tag{8}$$

Remark 1 *Accorging to [13], if $u_0 \in H^1(\mathbb{R})$ satisfies $y_0 := u_0 - u_{0,xx} \in \mathcal{M}$ with $rmsuppy_0^- \subset]-\infty, x_0]$ and $\mathrm{supp} y_0^+ \subset [x_0, +\infty[$ for some $x_0 \in \mathbb{R}$ (Note that trains of antipeakons-peakons satisfy this properties as soon as they are well-ordered), then the corresponding solution u belongs to $Y(]0,T[)$ for all $T > 0$. Therefore for such initial data that satisfy (6), (7)-(8) hold for all positive times.*

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(Generalized) Breather solutions in periodic media

Guido Schneider
(joint work with Vincent Lescarret, Carsten Blank, Martina Chirilus-Bruckner, Christopher Chong)

We present some results about breather solutions and generalized breather solutions for a nonlinear wave equation of the form

$$s(x)\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - q(x)u(x,t) + \gamma r(x)u^3(x,t), \tag{1}$$

where $\gamma = \pm 1$, $x \in \mathbb{R}$, $t \in \mathbb{R}$, $u(x,t) \in \mathbb{R}$ and a-periodic coefficients s, q and r, i.e.,

$$s(x) = s(x+a), \quad q(x) = q(x+a), \quad \text{and} \quad r(x) = r(x+a),$$

where w.l.o.g. in the following $a = 1$. Breather solutions are spatially localized, $2\pi/\omega$-time periodic solutions of finite energy. In [2] it has been proved

Theorem 1 *There is a coefficient function $s = s(x)$ and a constant $\mu_0 \in \mathbb{R}$ such that either for $\gamma = 1$ or $\gamma = -1$ and either for $q(x) = \mu_0 - \varepsilon^2$ or $q(x) = \mu_0 + \varepsilon^2$ the following holds. Under the validity of some nondegeneracy condition there exist an $\varepsilon_0 > 0$ and a $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ Equation (1) possesses breather solutions with period $2\pi/\omega$ for an $\omega > 0$, i.e., there are solutions $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of (1) which satisfy*

$$\begin{aligned} \lim_{|x|\to\infty} u(x,t) &= 0, \qquad \forall t \in \mathbb{R}, \\ u(x,t) &= u(x, t + 2\pi/\omega), \qquad \forall x, t \in \mathbb{R}. \end{aligned} \tag{2}$$

The breather solutions are given in lowest order by

$$\sup_{x,t\in\mathbb{R}} \left| u(x,t) - \left(\varepsilon\eta_1 \mathrm{sech}(\varepsilon\eta_2 x) w_n(x) e^{i\pi x} e^{i\omega t} + c.c.\right) \right| \leq C\varepsilon^2, \tag{3}$$

with constants η_1, η_2 *and a* 1*-periodic function* w_n*. The solution is* C^∞ *w.r.t.* t*, but only* C^0 *w.r.t.* x*.*

The breather solutions are constructed with the use of spatial dynamics, center manifold theory, and bifurcation theory. Spatial dynamics means that (1) is written as an evolutionary system w.r.t. $x \in \mathbb{R}$ in the phase space of $2\pi/\omega$-time periodic functions, i.e., we consider

$$\begin{array}{rcl} \partial_x u(x,t) & = & v(x,t), \\ \partial_x v(x,t) & = & s(x)\partial_t^2 u(x,t) + q(x)u(x,t) - \gamma r(x)u(x,t)^3. \end{array} \tag{4}$$

Due to the periodicity of s, q, and r w.r.t. x the system is non-autonomous. Due to the symmetries we restrict ourselves to solutions which are odd w.r.t. t. If the linearization of the periodic spatial dynamics system (4) possesses two Floquet exponents with real part zero and if the rest of the Floquet spectrum is uniformly bounded away from the imaginary axis by using invariant manifold theory for periodic systems, the infinite-dimensional spatial dynamics system (4) can be reduced to a two-dimensional system on the center manifold which is associated with the two Floquet exponents with real part zero. For a given minimal period $2\pi/\omega$ the coefficients s and q have to be suitably chosen. By moving the two central Floquet exponents from the imaginary axis, bifurcating homoclinic solutions can be found in the reduced system using the reversibility of the reduced system. These homoclinic solutions of the spatial dynamics formulation (4) in the space of time-periodic solutions correspond to breather solutions in the original formulation (1). In order to have the reversibility of the spatial dynamics formulation, i.e. the invariance of (1) under $(x,u,v) \mapsto (-x,u,-v)$, the coefficient functions $s = s(x)$, $q = q(x)$, and $r = r(x)$ have to be even w.r.t. x.

It is well known that the solutions of the linearization are given by Bloch modes $e^{i\ell x} w_n(x) e^{i\omega_n(\ell)t}$ with $w_n(x) = w_n(x+1)$ and curves of eigenvalues $\ell \mapsto \omega_n(\ell)$ with $\omega_n(\ell) \in \mathbb{R}$ for $\ell \in [-\frac{\pi}{2}, \frac{\pi}{2})$ and $n \in \mathbb{Z}/\{0\}$. There are spectral gaps, i.e., the set $\{ \omega_n(\ell) \mid \ell \in [-\frac{\pi}{2}, \frac{\pi}{2}), n \in \mathbb{Z}/\{0\} \} \subset \mathbb{C}$ is not connected for periodic s and q. There is a one-to-one correspondence between the spectral pictures of the time evolutionary and space evolutionary system (1) and (4). When an integer multiple of the basic temporal wavenumber ω falls into a spectral gap of the time evolutionary system (1) then there are two Floquet exponents off the imaginary axis in the space evolutionary system (4). In the other case the Floquet exponents are on the imaginary axis. For smooth s the spectral gaps become smaller and smaller for larger m and, therefore, integer multiples $m\omega$ of ω in general do not fall into spectral gaps. If s is continuous then the spectral gaps close proportional to $1/m$ for $m \to \infty$. Therefore, for smooth coefficients we thus expect infinitely many Floquet exponents on the imaginary axis or at least arbitrarily close to the imaginary axis. Hence, in order to satisfy the spectral assumption the coefficient s has to be very irregular, i.e. at least some step function. An example of a function

s leading to $\mathcal{O}(1)$-spectral gaps around each value $(2n+1)\omega$ with $\omega = 13\pi/16$ and $n \in \mathbb{N}$ is

$$s(x) = \chi_{[0,6/13]} + 16\chi_{(6/13,7/13)} + \chi_{[7/13,1]}(x \bmod 1).$$

Since smooth q does not affect the asymptotics of the spectral gaps we can choose q to adjust two Floquet exponents on the imaginary axis without destroying the overall spectral picture. Since we have a cubic nonlinearity it is sufficient that only every second gap opens in the required way.

For homogeneous nonlinear wave equations there are no spectral gaps and so all eigenvalues of the spatial dynamics formulation lie on the imaginary axis and so up to rescaling the sine-Gordon equation is the only nonlinear wave equation in homogeneous medium which possesses breather solutions in NLS-form [4, 1]. In general, only solutions with small tails at infinity have been proven to exist [5, 6, 7]. Such solutions also exist for general periodic coefficients as has been shown in [8].

Theorem 2 *Let s, q and r be smooth 1-periodic, even functions. Assume that in the linearization of Equation (1) there is a band-gap which begins or ends at $\omega_{n_0}(0)$ and that for $|j| < N$ the integer multiples $j\omega_{n_0}(0)$ of the basic wave number hit no other band edge at $\ell = 0$.*

Then under the validity of some non-degeneracy condition there exist an $\varepsilon_0 > 0$ and a $C > 0$ such that either for $\gamma = 1$ or $\gamma = -1$ and either for $\omega^2 - \omega_{n_0}^2(0) = \varepsilon^2$ or $\omega^2 - \omega_{n_0}^2(0) = -\varepsilon^2$ the following holds. For all $\varepsilon \in (0, \varepsilon_0)$ Equation (1) possesses generalized modulating standing pulse solutions with period $1/\omega$, i.e., there are solutions $u : [-\varepsilon^{3-2N}, \varepsilon^{3-2N}] \times \mathbb{R} \to \mathbb{R}$ of (1) which satisfy

$$u(x,t) = u(-x,t), \quad u(x,t) = u\left(x, t + \frac{1}{\omega}\right)$$

and

$$\sup_{x \in [-\varepsilon^{3-2N}, \varepsilon^{3-2N}]} |u(x,t) - h(x,t)| \leq C\varepsilon^N,$$

where

$$\lim_{|x| \to \infty} h(x,t) = 0$$

and

$$\sup_{x,t \in \mathbb{R}} \left| h(x,t) - \left(\varepsilon\gamma_1 \operatorname{sech}(\varepsilon\gamma_2 x) w_{n_0}(0,x) e^{i\omega t} + c.c.\right)\right| \leq C\varepsilon^2 \tag{5}$$

with constants γ_1, γ_2 and w_{n_0} being a 1-periodic function.

Since we have a finite speed of propagation for (1), the solutions exist also for all $t \in [0, \varepsilon^{3-2N}]$, i.e., much longer than the $\mathcal{O}(1/\varepsilon^2)$-time scale guaranteed by [3].

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Hydroelastic waves–local and global theory

John Toland

We consider steady irrotational fluid motion beneath a heavy, hyperelastic membrane. There are two velocity parameters. The local theory considers a problem of bifurcation for a double eigenvalue and the existence of 3 sheets of solutions is established.

The global theory uses the direct method of the calculus on a functional of displacement and curvature of the membrane, and kinetic and potential energy of the fluid. Global properties of a Lagrangian are established.

On the existence of extreme waves and the Stokes conjecture with vorticity

Eugen Varvaruca
(joint work with Ovidiu Savin and Georg Weiss)

We study periodic travelling-wave solutions for the two-dimensional Euler equations describing the dynamics of an incompressible, inviscid, heavy fluid over a flat bottom and with a free surface. The corresponding mathematical problem is to find a domain Ω in the (X, Y)-plane, which lies above a horizontal line $\mathcal{B}_F := \{(X, F) : X \in \mathbb{R}\}$, where F is a constant, and below some a priori unknown curve $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is $2L$-periodic, and a function ψ in Ω which satisfies the following equations and boundary conditions:

$$\begin{aligned}
&\Delta\psi = -\gamma(\psi) \quad \text{in } \Omega,\\
&0 \le \psi \le B \quad \text{in } \Omega,\\
&\psi = B \quad \text{on } \mathcal{B}_F,\\
&\psi = 0 \quad \text{on } \mathcal{S},\\
&|\nabla\psi|^2 + 2gY = Q \quad \text{on } \mathcal{S},\\
&\psi(X + 2L, Y) = \psi(X, Y) \quad \text{for all } (X, Y) \in \Omega,
\end{aligned}$$

where B, g, L are given positive constants, $\gamma \in C^{1,\alpha}([0,B])$ is a given *vorticity function* and Q, F are parameters. By a vertical translation, we may always assume either that $Q = 0$ or that $F = 0$.

In most of what follows we consider *solutions of type (SMG)* (symmetric monotone graphs), for which $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where η is even, $\eta'(X) < 0$ on $(0, L)$, and $\psi_Y < 0$ in Ω, $\psi_X < 0$ in $\Omega \cap \{(X,Y) : 0 < X < L\}$. We are interested in the existence and properties of *extreme waves*, which are waves with *stagnation points* ($\nabla\psi = (0,0)$) on the free surface $\mathcal{S}$. At such points $\mathcal{S}$ need not be smooth, and we are interested in the shape of $\mathcal{S}$ close to such points. Note that for certain vorticity functions $\gamma : [0,B] \to \mathbb{R}$ there exist trivial extreme waves, whose free surface is a horizontal line all of whose points are stagnation points.

Until recently, most mathematical results on travelling water waves were restricted to the irrotational case, for which $\gamma \equiv 0$. A famous result in this theory is the *Stokes conjecture* (1880): the profile of any extreme wave has corners with included angle of 120° at stagnation points. The existence of extreme waves was proved by Toland (1978) and McLeod (1979), and the Stokes conjecture was proved by Amick, Fraenkel, and Toland (1982), and Plotnikov (1982).

The first global theory of waves with general vorticity $\gamma : [0,B] \to \mathbb{R}$ was given by Constantin and Strauss (2004). Many authors have since then contributed to this theory. Constantin and Strauss (2004) proved, under very general assumptions on γ, the existence of *almost extreme waves*: a sequence of waves of type (SMG) $\{(\mathcal{S}_j, \mathcal{B}_0, \psi^j, Q_j)\}_{j\geq 1}$ for which

$$\max_{\overline{\Omega}_j} \psi^j_Y \to 0 \text{ as } j \to \infty.$$

(Being of type (SMG), they satisfy $\psi^j_Y < 0$ everywhere in Ω_j.) Numerical evidence suggests that this sequence converges either to an extreme wave which satisfies the Stokes conjecture, or to a smooth wave with a stagnation point on the bottom directly below the crest.

Our main result on the existence of extreme waves deals with the case when the vorticity is everywhere nonpositive.

Theorem 1 (Savin and Varvaruca (2009)) *Suppose that* $\gamma(r) \leq 0$ *for all* $r \in [0,B]$. *Let* $\{(\mathcal{S}_j, \mathcal{B}_0, \psi^j, Q_j)\}_{j\geq 1}$ *be a sequence of regular waves of type* (SMG) *such that*

$$\max_{\overline{\Omega}_j} \psi^j_Y \to 0 \text{ as } j \to \infty.$$

Then $\{(\mathcal{S}_j, \mathcal{B}_0, \psi^j, Q_j)\}_{j\geq 1}$ *'converges' along a subsequence to an extreme wave* $(\widetilde{\mathcal{S}}, \mathcal{B}_0, \widetilde{\psi}, \widetilde{Q})$ *with stagnation points at its crests. Moreover, the troughs of this extreme wave are not stagnation points and there are no stagnation points in the interior or on the bottom of the fluid domain.*

Theorem 1 is a consequence of some new a priori estimates obtained by the maximum principle, combined with Theorem 2 below which might be useful for proving the existence of extreme waves in more general situations.

Theorem 2 (Varvaruca (2009))*Let* $\{(\mathcal{S}_j, \mathcal{B}_0, \psi^j, Q_j)\}_{j\geq 1}$ *be a sequence of regular* *waves of type* (SMG)*. Suppose that*

$$\{Q_j\}_{j\geq 1} \text{ is bounded above.}$$

Then $\{(\mathcal{S}_j, \mathcal{B}_0, \psi^j, Q_j)\}_{j\geq 1}$ *'converges' along a subsequence towards a 'weak solution'* $(\widetilde{\mathcal{S}}, \mathcal{B}_0, \tilde{\psi}, \widetilde{Q})$ *of the water-wave problem. Let* $\widetilde{\Omega}$ *be the domain whose boundary consists of* $\widetilde{\mathcal{S}}$ *and* $\mathcal{B}_0$*. Then*

$$\tilde{\psi}_Y < 0 \text{ in } \widetilde{\Omega}.$$

If, in addition,

$$\max_{\overline{\Omega}_j} \psi^j_Y \to 0 \text{ as } j \to \infty,$$

then

$$\max_{\partial\Omega_j} \psi^j_Y \to 0 \text{ as } j \to \infty.$$

We now turn to the Stokes conjecture. For simplicity, we consider first the case in which the free boundary is monotone locally on each side of a stagnation point.

Theorem 3 (Weiss and Varvaruca (2009)) *Let* $(\mathcal{S}, \psi)$ *be an extreme wave, with* $Q = 0$*, such that the origin is a stagnation point. Suppose that* $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$*, where* $\eta : \mathbb{R} \to \mathbb{R}$ *is continuous, locally of bounded variation,* $\eta(0) = 0$ *and* η *is nondecreasing on* $[-A, 0]$ *and nonincreasing on* $[0, A]$ *for some* $A \in (0, L]$*. Suppose also that* $\psi_Y < 0$ *in* Ω*. Then*

$$\text{either } \lim_{X\to 0\pm} \frac{\eta(X)}{X} = \mp\frac{1}{\sqrt{3}} \text{ or } \lim_{X\to 0\pm} \frac{\eta(X)}{X} = 0.$$

Moreover, if $\gamma(r) \geq 0$ *for all* $r \in [0, \delta]$*, for some* $\delta \in (0, B]$*, then*

$$\lim_{X\to 0\pm} \frac{\eta(X)}{X} = \mp\frac{1}{\sqrt{3}}.$$

In the proof of Theorem 3, the behaviour close to the stagnation point of the curve $\mathcal{S}$ and the function ψ is studied by considering a *blow-up sequence*. Let $\{\varepsilon_j\}_{j\geq 1}$ be a sequence such $\varepsilon_j \searrow 0$ as $j \to \infty$, and let us consider the sequence $\{\psi^j\}_{j\geq 1}$ given by

$$\psi^j(X, Y) := \frac{1}{\varepsilon_j^{3/2}} \psi(\varepsilon_j X, \varepsilon_j Y).$$

It turns up that any weak limit $\tilde{\psi}$ of $\{\psi_j\}_{j\geq 1}$ along a subsequence satisfies a *limiting problem*: find a curve $\widetilde{\mathcal{S}} := \{(\tilde{u}(s), \tilde{v}(s)) : s \in \mathbb{R}\}$, where $s \mapsto (\tilde{u}(s), \tilde{v}(s))$ is injective on $\mathbb{R}$, $\tilde{u}(0) = 0$, $\tilde{v}(0) = 0$, $s \mapsto \tilde{u}(s)$ is nondecreasing on $\mathbb{R}$, $s \mapsto \tilde{v}(s)$ is nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$, $\lim_{s\to\pm\infty}(|\tilde{u}(s)| + |\tilde{v}(s)|) = \infty$, and a function $\tilde{\psi}$ in the unbounded domain $\widetilde{\Omega}$ below $\widetilde{\mathcal{S}}$, such that

$$\Delta\tilde{\psi} = 0 \quad \text{in } \widetilde{\Omega},$$
$$\tilde{\psi} = 0 \quad \text{on } \widetilde{\mathcal{S}},$$
$$|\nabla\tilde{\psi}|^2 + 2gY = 0 \quad \mathcal{H}^1\text{-almost everywhere on } \widetilde{\mathcal{S}}.$$

This has a trivial solution $(\widetilde{\mathcal{S}}_0, \tilde{\psi}_0)$, where $\widetilde{\mathcal{S}}_0 := \{(X, 0) : X \in \mathbb{R}\}$ and $\tilde{\psi}_0 \equiv 0$ in $\mathbb{R}^2_-$. Another solution, originally discovered by Stokes and nowadays called the *Stokes corner flow*, is the following: let $\widetilde{\mathcal{S}}^* := \{(X, \eta^*(X)) : X \in \mathbb{R}\}$, where

$$\eta^*(X) := -\frac{1}{\sqrt{3}}|X| \quad \text{for all } X \in \mathbb{R},$$

let $\widetilde{\Omega}^*$ be the domain below $\widetilde{\mathcal{S}}^*$, and let the function $\tilde{\psi}^*$ in $\widetilde{\Omega}^*$ be given, for all $(X, Y) \in \widetilde{\Omega}^*$, by

$$\tilde{\psi}^*(X, Y) := \frac{2}{3} g^{1/2} \operatorname{Im} \left(i(iZ)^{3/2} \right) \quad \text{where } Z = X + iY.$$

The key to the proof of Theorem 3 is the following uniqueness result.

Theorem 4 (Weiss and Varvaruca (2009)) *Any nontrivial solution $(\widetilde{\mathcal{S}}, \tilde{\psi})$ of the limiting problem which arises as a blow-up limit of a solution of the original problem is necessarily homogeneous of degree $3/2$, and therefore is the Stokes corner flow $(\widetilde{\mathcal{S}}^*, \tilde{\psi}^*)$.*

The proof uses a new ingredient, the *Monotonicity Formula*: the function

$$\begin{aligned}\Phi(r) := {} & r^{-3} \int_{B_r(0)} \left(|\nabla \psi|^2 - 2\Gamma(\psi) - 2gY\chi_{\{\psi>0\}} \right) d\mathcal{L}^2 \\ & - \frac{3}{2} r^{-4} \int_{\partial B_r(0)} \psi^2 d\mathcal{H}^1 + \int_0^r s^{-4} \int_{B_s(0)} \left(2\Gamma(\psi) - 3\gamma(\psi)\psi \right) d\mathcal{L}^2 \, ds\end{aligned}$$

satisfies, for almost every r sufficiently small,

$$\frac{d}{dr}\Phi(r) = r^{-3} \int_{\partial B_r(0)} 2 \left(\nabla\psi \cdot \nu - \frac{3}{2}\frac{\psi}{r} \right)^2 d\mathcal{H}^1.$$

(The function $\Gamma : [0, B] \to \mathbb{R}$ is defined by $\Gamma(t) := \int_0^t \gamma(s)\, ds$ for all $t \in [0, B]$.) The proof of this formula is by direct verification, using a Pohozaev-type identity. Let $\{\psi^j\}_{j\geq 1}$ be the blow-up sequence

$$\psi^j(X, Y) := \frac{1}{\varepsilon_j^{3/2}} \psi(\varepsilon_j X, \varepsilon_j Y).$$

It is immediate from the Monotonicity Formula that any weak limit $\tilde{\psi}$ of $\{\psi^j\}_{j\geq 1}$ along a subsequence is a function homogeneous of degree $3/2$, and hence it is either identically 0 or coincides with the Stokes corner flow.

A more general version of the Stokes conjecture answers a question raised by Shargorodsky and Toland: if no monotonicity assumption is made on the free boundary, can there be infinitely many stagnation points on a period of the wave?

Theorem 5 (Weiss and Varvaruca (2009)) *Let $(\mathcal{S}, \psi)$ be an extreme wave, with $Q = 0$. Suppose that $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is continuous and locally of bounded variation. Suppose also that $\psi_Y < 0$ in Ω. Let $(X_0, \eta(X_0))$*

be a stagnation point, i.e. $\eta(X_0) = 0$. *Then*

$$\text{either} \lim_{X\to X_0\pm} \frac{\eta(X)}{X - X_0} = \mp\frac{1}{\sqrt{3}} \text{ or } \lim_{X\to X_0\pm} \frac{\eta(X)}{X - X_0} = 0.$$

Moreover, if $\gamma(r) \geq 0$ *for all* $r \in [0, \delta]$, *for some* $\delta \in (0, B]$, *then*

$$\lim_{X\to X_0\pm} \frac{\eta(X)}{X - X_0} = \mp\frac{1}{\sqrt{3}}.$$

It is immediate that, in the situation of the second part of Theorem 5, any stagnation point is *isolated* and hence there can be at most finitely many stagnation points on a period of the wave.

Steady water waves with a critical layer

Erik Wahlén

This talk combines two classical subjects in fluid mechanics: critical layers and steady periodic water waves. The concept of a critical layer, consisting of a region with closed streamlines surrounding a stagnation point, has appeared in the study of many different types of fluids over the years. However, as far as we know, there is a lack of rigorous results concerning critical layers in water waves.

We consider a two-dimensional, inviscid and incompressible fluid of constant density. The fluid is bounded below by a flat, impermeable bed $\{y = 0\}$ and above by a free surface $\{y = 1 + \eta\}$. Neglecting the effects of surface tension, the only restoring force is that of gravity. The fluid is governed by Euler's equations and we restrict ourselves to the case of constant vorticity. In a frame moving with the wave we then have the following formulation of the water wave problem:

$$\begin{aligned} &\Delta\psi = -\omega \quad \text{in} \quad 0 < y < 1 + \eta,\\ &\psi = 0 \quad \text{on} \quad y = 0,\\ &\psi = m_0 \quad \text{on} \quad y = 1 + \eta,\\ &\frac{1}{2}|\nabla\psi|^2 + \eta = Q \quad \text{on} \quad y = 1 + \eta, \end{aligned} \tag{1}$$

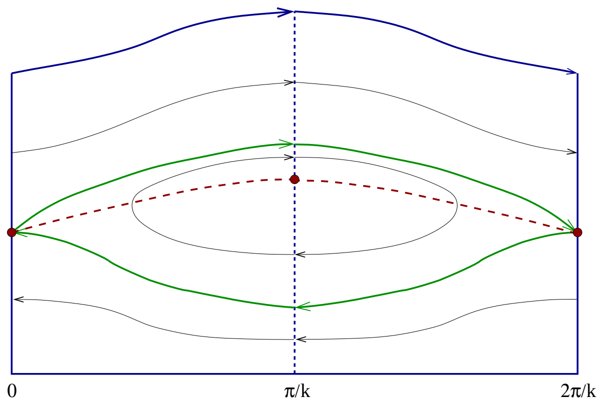
where ψ is a stream function, satisfying $\psi_x = -v$, $\psi_y = u - c$, (u, v) being the velocity field and c the wave speed, and where $\omega := v_x - u_y$ is the constant vorticity. The constant $m_0 = \int_0^{1+\eta(x)} (u(x, y) - c)\, dy$ is the relative mass flux, which is independent of x, and the constant Q is the total head. Note that (1) is a free boundary problem since the domain is a priori unknown. For any ω we have a family of trivial solutions (shear flows) $\eta = 0$, $\psi = -\omega y^2/2 + \lambda y$, with $\lambda \in \mathbb{R}$ arbitrary, for appropriately chosen m_0 and Q. By choosing $\lambda/\omega \in (0, 1)$, we can produce a trivial solution with internal stagnation.

Historically, the irrotational case $\omega = 0$ has received the most attention. In this case the stream function is harmonic and one can employ methods from complex analysis. In particular there is a hodograph transformation which fixes the domain.

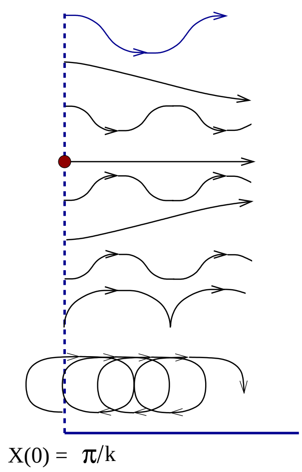
However, except for the case of a flat surface, irrotational waves cannot have interior stagnation points and hence cannot contain a critical layer.

In recent years there has been a lot of interest in water waves with vorticity, one of the highlights being the construction of a global continuum of steady periodic water waves by Constantin & Strauss [2] under the more general condition that $-\Delta\psi = \gamma(\psi)$, that is, the vorticity is functionally dependent on the stream function. These waves all have the property that $u < c$, so that critical layers are excluded. The main mathematical reason is that the transformation to a fixed domain used in [2] requires that ψ_y has constant sign.

By using a different change of variables, where the independent variable y is replaced by $y/(1+\eta)$, we can formulate problem (1) in a secure functional analytic setting, in which local bifurcation theory can be applied. This can be done even in the case when ψ_y changes sign. Choosing an appropriate wavelength, we can now produce small-amplitude waves which bifurcate from a trivial solution with internal stagnation [4]. These non-trivial waves will have a stagnation point surrounded by a critical layer (see the figure below). This work is inspired by a recent paper by Ehrnström and Villari [3], in which similar results were found for the water wave problem linearised about a shear flow with constant vorticity.



It is also possible to obtain a qualitative picture of the particle trajectories in the original physical frame. If the wave speed is chosen so that the horizontal velocity component u has zero average on the bed the following picture emerges.



The behaviour near the bottom, where the particles move in almost closed ellipses with a small forward drift, agrees with a recent investigation of the particle motion in irrotational Stokes waves [1]. However, farther up in the fluid there is no backward motion at all, in contrast to the irrotational case.

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Stratified steady water waves

Samuel Walsh

Stratification is a common feature of ocean waves, where the presence of salinity, in concert with an external gravitational force, can produce substantial heterogeneity in the fluid. The pronounced effects that may accompany even a moderate density variation have earned stratified flows a great deal of scholarly attention, particularly in the geophysics and oceanography communities. This talk will concern two-dimensional traveling periodic stratified water waves propagating over an impermeable flat bed and with a free surface at the interface with the atmosphere. We suppose these waves are subject to an external gravitational force and allow for surface tension with coefficient $\sigma \geq 0$.

By changing to a semi-Lagrangian coordinate scheme, we transform the domain into a fixed rectangle $R := \{(q,p) : 0 < q < L,\ p_0 < p < 0\}$, where L is the period and p_0 is the volumetric mass flux with respect to the pseudo-stream function. Reformulating the Yih-Long equation in these variables, we show that it suffices to solve the following problem: Find $(Q,h) \in \mathbb{R} \times C^{3+\alpha}(\overline{R})$, with h even and L-periodic in q and $h_p > 0$, satisfying

$$\tag{1} \begin{cases} (1+h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_q h_p h_{pq} & \\ \qquad -g(h-d(h))h_p^3\rho_p = -h_p^3\beta(-p) & p_0 < p < 0, \\ 1 + h_q^2 + h_p^2(2\sigma\kappa(h) + 2g\rho h - Q) = 0 & p = 0, \\ h = 0 & p = p_0, \end{cases}$$

where g is the gravitational constant, κ describes the mean curvature

$$\kappa(h) := -\frac{h_{qq}}{\left(1+h_q^2\right)^{3/2}},$$

and the nonlocal operator d is defined by

$$d(h) := \frac{1}{L}\int_0^L h(q,0)\,dq.$$

Here ρ and β are taken to be given. Roughly, they describe the variation of density and the variation of energy, respectively, as functions of the streamlines. These, along with the other given quantities are assumed to satisfy a Local Bifurcation Condition, which is both necessary and sufficient for our results. We also give an explicit size condition, along the lines of [1], that implies the LBC.

We construct a 1-parameter family of laminar flow solutions to (1), which we denote $\mathcal{T} := \{(H(\cdot;\lambda), Q(\lambda)) : \lambda \geq -2B_{\min}\}$. Analyzing the linearized problem along $\mathcal{T}$, we show that the LBC is equivalent to the existence of a generalized eigenvalue at some $(H(\cdot;\lambda^*), Q(\lambda^*))$. In fact, for $\sigma = 0$, or $\sigma > 0$ sufficiently large, we prove that this eigenvalue must be simple. With some additional argument then, the theory of Crandall and Rabinowitz [2] furnishes us with a C^1-curve of small amplitude solutions bifurcating from $\mathcal{T}$.

As is well-known for constant density capillary-gravity waves, however, when σ is positive but small the null space of the linearized operator may be two-dimensional (see, e.g., [4, 6].) This can result in many such curves emanating from a single point on $\mathcal{T}$. By means of a Lyapunov-Schmidt reduction, we provide a complete characterization of the bifurcation diagram at such points in certain regimes. In particular, if the eigenvalues are $0 < n_1 < n_2$, with $n_2/n_1 \neq 2$, then, under some additional size assumptions, we have that there exist precisely four C^1-curves of small amplitude solutions. These consist of two pitchfork bifurcation curves of $2\pi/n_1$- and $2\pi/n_2$-periodic solutions, respectively, as well as two pitchforks of so-called mixed solutions that lie (locally) within the interior of the span of the eigenfunctions.

Each solution curve is then continued globally by means of an alternative theorem in the spirit of Rabinowitz [5]. The major piece of machinery here is a variant of the classical Leray-Schauder degree due to Healey and Simpson [3]. In order to appeal to this theory, we must first prove various compactness properties of the problem. This is nontrivial, however, because of the presence of the nonlocal operator d in the interior equation of (1). To overcome this technical obstacle and obtain the sought-after *a priori* estimates, we introduce the method of "freezing" d. The basic idea is to replace d with a real number. Working in appropriately chosen function spaces, the resulting operator will then be uniformly elliptic with an oblique boundary condition if $\sigma = 0$, or Venttsel boundary data for $\sigma > 0$. The Schauder estimates (or their analogues for Venttsel-type problems) then imply the desired compactness properties are present for the frozen problem. With some care, we are able to leverage these estimates to derive similar results for the full equation and conclude global bifurcation.

Lastly, we prove a uniform regularity result that states, if the bifurcation curve is unbounded while remaining in a subset of the function space where the problem is uniformly elliptic and oblique, then it must be that, along some sequence $\{(h_n, Q_n)\}_{n=1}^{\infty}$ on the continuum, we have either (i) $|Q_n| \to \infty$, (ii) $\sup_{\overline{R}} |\partial_p h_n| \to \infty$, or, if $\sigma > 0$, potentially (iii) $\sup_T |\kappa(h_n)| \to \infty$. Possibility (iii) corresponds to the formation of a corner on the free surface. On the other hand, if either (i)–(ii) occur, or if the problem loses ellipticity or obliqueness, we show that somewhere in the fluid either the horizontal velocity is approaching $-\infty$, or a stagnation point is forming.

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Shock waves and wave breaking for the Degasperis-Procesi equation

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In the talk we first give a brief introduction of the Degasperis- Procesi equation. We then establish local well-posedness of the DP equation. We next present the precise blow-up scenario and show that the first blow-up of strong solution to the equation can occur only in the form of wave breaking and shock waves possibly appear afterwards. Moreover, we present several blow-up results and global existence results for strong solutions to the equation. Furthermore, we prove the existence and uniqueness of global "strong" weak solutions to the equation with certain initial profiles. We finally give an explicit example of weak solutions to the periodic DP equation, which may be considered as periodic shock waves.

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