

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 46/2009

DOI: 10.4171/OWR/2009/46

## Mathematical Aspects of General Relativity

Organised by  
Piotr T. Chruściel, Tours/Oxford  
James Isenberg, Eugene  
Alan D. Rendall, Golm

October 11th – October 17th, 2009

ABSTRACT. Mathematical relativity, the subject of this conference, has recently become more and more devoted to the theory of nonlinear evolution equations, with global questions becoming ever more accessible. This is reflected by the fact that more than half of the talks given were concerned with the global dynamics of solutions of evolution equations related more or less directly to the Einstein equations of general relativity. Progress was reported in understanding subjects such as black holes, gravitational radiation, cosmology and the relation of general relativity to Newtonian gravitational theory.

*Mathematics Subject Classification (2000):* 83Cxx.

### Introduction by the Organisers

The participants of this conference included a good mixture of established workers in the area of mathematical relativity, promising young researchers in the field and experts in related subjects.

At a previous Oberwolfach conference with the same title in 2006 there were two talks related to the question of the stability of black holes. Since then that subject has developed rapidly and this development, including the most recent results, was described in the talk of Dafermos. A new influential approach to this problem using harmonic analysis was presented in the talk of Tataru. A technique with a stonger input from differential geometry was the subject of the presentation of Blue. Holzegel talked about work related to black hole stability in the context of a negative cosmological constant. Other aspects of the theory of black holes were also discussed at the conference. Alexakis described progress on uniqueness

theorems for black holes. Rein presented results on the formation of black holes due to collapse of matter. Hennig's talk was concerned with the question of whether the spin of a pair of black holes can balance their gravitational attraction.

The issue of the stability of black holes and other solutions of the Einstein equations belongs to the global theory of hyperbolic equations. Other talks in that area, concerning Einstein's equations or other nonlinear wave equations, were given by Struwe (analytical) and Bizon (numerical and heuristic). An influential model in this area is the work of Christodoulou and Klainerman on the nonlinear stability of Minkowski space. In her talk Bieri explained how she has been able to extend this result in various directions. A related topic in cosmology is the so-called 'cosmic no hair theorem'. The work reported in the talk of Speck indicates that, assuming certain hypotheses, this has now finally attained the status of a theorem, having been a conjecture for many years. Smulevici presented results of the dynamics of cosmological models with symmetry including matter and a positive cosmological constant. Heinzle explained ways in which the choice of a matter model influences the dynamics of cosmological models.

The topic of the Newtonian limit and post-Newtonian approximations is a subject of great physical importance which had been resistant to mathematical progress. In his presentation Oliynyk explained how he has been able to overcome some of the major difficulties in this subject. Beig talked about some solutions in scalar gravity which provide a simple model related to gravitational radiation. Corvino discussed the construction of initial data for the motion of many bodies. Miao gave a lucid exposition of topics related to the Brown-York quasilocal mass. Maxwell described his recent studies which show that, at least for certain simple families of initial data, the conformal methods appear to have difficulties in dealing with non constant mean curvature solutions of the Einstein constraints. Parabolic equations were represented by the talk of Mazzeo on the subject of the Ricci flow on open surfaces.

The number of talks at the conference was limited so as to leave plenty of time for discussions. On one evening an informal discussion session was arranged where anyone who wished could give a ten-minute account of a subject of their choice. This was an opportunity to share ideas not represented in the main talks.

**Workshop: Mathematical Aspects of General Relativity****Table of Contents**

Mihalis Dafermos (joint with I. Rodnianski)	
<i>The black hole stability problem</i> .....	2589
Daniel Tataru	
<i>Decay estimates for the wave equation on asymptotically flat space-times</i>	2594
Pieter Blue	
<i>Hidden symmetries and decay for the wave equation on the Kerr spacetime</i> .....	2597
Gustav Holzegel	
<i>The massive wave equation on slowly rotating Kerr-AdS spacetimes</i> ....	2599
Spyros Alexakis (joint with A. Ionescu and S. Klainerman)	
<i>A black hole uniqueness theorem</i> .....	2602
Gerhard Rein (joint with Håkan Andréasson, Markus Kunze)	
<i>Gravitational collapse for the Einstein-Vlasov system</i> .....	2604
Jörg Hennig (joint with Gernot Neugebauer)	
<i>Non-existence of stationary two-black-hole configurations</i> .....	2607
Michael Struwe	
<i>Global well-posedness of the Cauchy problem for a super-critical nonlinear wave equation in 2 space dimensions</i> .....	2610
Piotr Bizoń	
<i>Late-time tails of self-gravitating waves</i> .....	2612
Lydia Bieri	
<i>Null Asymptotics of Solutions of the Einstein-Maxwell Equations in General Relativity and Gravitational Radiation</i> .....	2614
Jared Speck (joint with Igor Rodnianski)	
<i>A Wave Coordinate Approach to the Stability of the Irrotational Euler-Einstein System with a Positive Cosmological Constant</i> .....	2616
Jacques Smulevici	
<i>Structure of singularities in cosmological spacetimes with symmetry</i> ....	2620
J. Mark Heinzle (joint with Simone Calogero)	
<i>Matter matters – On the dynamics of spatially homogeneous cosmological models with anisotropic matter</i> .....	2623
Todd A. Oliynyk	
<i>Cosmological post-Newtonian expansions</i> .....	2626

---

Robert Beig (joint with Bernd G.Schmidt)	
<i>Helical solutions in scalar gravity</i> .....	2629
Justin Corvino (joint with Piotr T. Chruściel and James Isenberg)	
<i>Construction of N-body initial data sets in general relativity</i> .....	2630
Pengzi Miao	
<i>On geometric problems related to Brown-York and Liu-Yau quasilocal mass</i> .....	2633
David Maxwell	
<i>A model problem for conformal parameterizations of the Einstein constraint equations</i> .....	2635
Rafe Mazzeo	
<i>Ricci flow on open surfaces</i> .....	2637

## Abstracts

### The black hole stability problem

MIHALIS DAFERMOS

(joint work with I. Rodnianski)

The nonlinear stability of the Kerr family of black holes is one of the great unsolved problems of classical general relativity. This talk surveyed recent advances in the underlying linear theory which may be useful for this conjecture's eventual resolution.

A rough formulation of the non-linear stability conjecture is given below (see [11] for an introduction to the relevant concepts):

**Conjecture** (Non-linear stability of Kerr). *Let  $(\Sigma, \bar{g}, K)$  denote an initial data set for the Einstein vacuum equations  $R_{\mu\nu} = 0$ , the data set assumed sufficiently close to a Kerr initial data set with parameters  $a_i, M_i$ ,<sup>1</sup> and let  $(\mathcal{M}, g)$  denote its maximal development. Then  $\mathcal{M}$  possesses a complete future null infinity  $\mathcal{I}^+$ , which can be realised as an ideal boundary of spacetime, where  $\mathcal{M} \setminus J^-(\mathcal{I}^+) \neq \emptyset$ , and such that  $g$  restricted to  $J^-(\mathcal{I}^+)$  asymptotically approaches at a sufficiently fast rate towards the future another Kerr metric with parameters  $a_f, M_f$ , where  $a_f$  and  $M_f$  are close to  $a_i, M_i$ , respectively.*

The reader should compare this conjecture with the stability of Minkowski space, first proven in 1993 by Christodoulou–Klainerman [5]. See also [3, 14]. As with the statement of stability of Minkowski space, the above conjecture is a statement of *asymptotic stability*. In view of the supercriticality of the Einstein equations, one does not expect to be able to prove weaker forms of stability which do not include as part of their statement asymptotic stability with quantitative, sufficiently fast rates of approach to another Kerr solution. As compared with the problem of stability of Minkowski space, the non-linear stability problem of Kerr has two additional new features which are already clear from its formulation: (i) It is now not an individual solution but a two parameter family which one must show is asymptotically stable. One must thus have a way of picking the final parameters  $a_f, M_f$ . Moreover, (ii) it is not the entirety of the maximal development  $(\mathcal{M}, g)$  which is conjectured to be stable, but the region  $J^-(\mathcal{I}^+)$ , the so-called domain of outer communications, and this region is defined teleologically, i.e. one cannot identify it a priori from the data. The non-emptiness of the set  $\mathcal{M} \setminus J^-(\mathcal{I}^+)$  is precisely the statement that  $(\mathcal{M}, g)$  contains a black hole; these black hole regions are in fact thought to be unstable deep in their interior, in accordance with another great conjecture of classical relativity, so-called *strong cosmic censorship*. See [6, 11].<sup>2</sup>

<sup>1</sup>A Kerr initial data set is defined by the induced geometry of a complete asymptotically flat slice  $S$  (with 2 ends) of a Kerr spacetime.

<sup>2</sup>In the model of [6], the black hole interior is, however, shown to be stable near the event horizon, on the basis of the *red-shift* effect, discussed below, in fact, stable until one approaches

The relation of boundedness and decay of linear equations to non-linear stability results is well known (see again [5]). Understanding linear equations in a sufficiently robust setting, indeed, any understanding whatsoever in the case of Kerr black holes, was only accomplished recently. The state of the art concerning scalar waves is given by the two theorems stated below together with the general argument of [12] mentioned here briefly.

The first theorem is a general boundedness theorem for solutions of the wave equation  $\square_g \psi = 0$  on axisymmetric stationary spacetimes sufficiently close to Schwarzschild.

**Theorem 1** ([10]). *Let  $\mathcal{R}$  denote the underlying manifold with stratified boundary given by the intersection of the closure of a domain of outer communications of a Schwarzschild spacetime  $(\mathcal{M}, g_M)$  with the future of a Cauchy surface  $\Sigma$ , where the latter is chosen to intersect the horizon  $\mathcal{H}$  to the future of the sphere of bifurcation. Let  $g$  be a Lorentzian metric on  $\mathcal{R}$  such that*

- (1)  *$g$  is  $C^1$  close to  $g_M$  in a weighted sense (see [10]).*
- (2) *The Schwarzschild Killing fields  $T = \partial_t$  and  $\Phi = \partial_\phi$  are Killing with respect to  $g$ .*
- (3) *The boundary  $\mathcal{H}$  of  $\mathcal{R}$  is null with respect to  $g$ , and  $T$  and  $\Phi$  span the null generator of  $\mathcal{H}$ .*

*Then the following is true: Let  $\phi_\tau$  denote the 1-parameter group of isometries generated by  $T$ , and let  $\Sigma_\tau = \phi_\tau(\Sigma)$ , let  $N$  be a globally timelike  $\phi_\tau$ -invariant vector field on  $\mathcal{R}$  such that  $N = T$  in a neighborhood of spacelike infinity. Then there exists a constant  $B$  depending only on  $M$  such that for all solutions  $\psi$  of  $\square_g \psi = 0$ , the bound*

$$(1) \quad \int_{\Sigma_\tau} J_\mu^N[\psi] n_{\Sigma_\tau}^\mu \leq B \int_\Sigma J_\mu^N[\psi] n_\Sigma^\mu$$

*holds, where  $J^N[\psi]$  denotes the energy current associated to  $N$ , and  $n_{\Sigma_\tau}^\mu$  denotes the normal.*

*The bound (1) also holds when  $\psi$  is replaced by  $N^m \psi$  for any positive integer  $m$ , and, together with elliptic estimates, this implies the uniform boundedness of suitable Sobolev norms on  $\Sigma_\tau$  of arbitrary order without degeneration on the horizon. This yields in particular the uniform pointwise estimate*

$$|\psi|^2 \leq B \sup_\Sigma |\psi|^2 + B \int_\Sigma (J_\mu^N[\psi] + J_\mu^N[N\psi]) n_\Sigma^\mu$$

*in  $\mathcal{R}$ .*

The above theorem applies in particular to Kerr and Kerr-Newman for  $|a| \ll M$ ,  $|Q| \ll M$ , but its domain of validity is of course much larger. In particular, the  $C^1$  assumptions on the metric mean that the result does not depend on details of

---

very near to what will be a Cauchy horizon, where there is an opposite, *blue-shift* effect. This red-shift property is crucial for showing that a so-called apparent horizon forms, also contained in the region of stability. This fact may be useful in non-linear stability proofs in the absence of symmetry.

geodesic flow. Philosophically, this point is very important. Let us note moreover that the assumption that  $T, \Phi$  are Killing can be replaced by the assumption that their deformation tensors suitably decay.

The second element of the ‘state of the art’ proves a weak form of decay, but is restricted to spacetimes which are exactly Kerr:

**Theorem 2.** (*[11]*) *Let  $\mathcal{R}$  be as above and let  $g_{M,a}$  denote the Kerr metric defined on  $\mathcal{R}$ , with  $|a| \ll M$ . Let  $\tilde{\Sigma}$  be a spacelike hypersurface terminating at either spatial infinity or  $\mathcal{I}^+$ , let  $\tilde{\Sigma}_\tau$  be as before, and let  $\mathcal{R}_\tau = J^+(\tilde{\Sigma}_\tau) \cap \mathcal{R}$ . Then there exists a nonnegative  $\phi_t$ -invariant function  $\chi$  vanishing in a neighborhood of  $r = 3M$  and a constant  $B$  such that for all solutions  $\psi$  of  $\square_{g_{M,a}}\psi = 0$ , then*

$$(2) \quad \int_{\mathcal{R}_\tau} \chi J_\mu^N[\psi] N^\mu \leq B \int_{\tilde{\Sigma}_\tau} J_\mu^N[\psi] n_{\tilde{\Sigma}_\tau}^\mu,$$

where both integrals are with respect to the induced 4 and 3-dimensional volume forms, respectively.

The setting of the above theorem can in fact be made more general; one can replace  $g_{M,a}$  with metrics  $g$  which asymptote sufficiently fast (with respect to  $\tau$ ) to exactly the Kerr metric  $g_{M,a}$ .

A result similar to Theorem 2 has independently been obtained by [18], and a slightly weaker result by [2]. We note that the above statement (2) is not the sharpest form of the result as, in particular, not all derivatives of  $\psi$  degenerate, and more precise information can be said about the behaviour of  $\chi$  at infinity.

The statement of the above theorem can in fact be retrieved for the general subextremal case  $|a| < M$  in forthcoming work of ours.

Theorem 2 can easily be seen to imply Theorem 1 when the statement of the latter is restricted to exactly Kerr spacetimes. As we shall see below, however, the proof of Theorem 2 makes essential use of properties of geodesic flow of the Kerr spacetime. These properties ‘live’ at a higher level of differentiability than the assumptions of Theorem 1. This reflects the fact that the physical basis for the argument of Theorem 1 is in fact entirely different and of a much more general nature.

Theorem 2 is a statement of “integrated local energy decay”. We shall say less here about the last element of the ‘state of the art’ which is essentially the statement that, *given the results of Theorems 1 and Theorem 2*, together with “good behaviour” of  $g$  at null infinity, one can obtain quantitative decay bounds of energy through a foliation  $\tilde{\Sigma}_\tau$  terminating on null infinity, as well as pointwise decay. This latter statement is in fact a much more general type of result which is best discussed elsewhere. See our recent [12]. This improves earlier understanding based on constructions of conformal Morawetz vector fields (see [2, 11] for Kerr, and for the Schwarzschild case, see [4, 7]). See also Luk [15]. Both these original methods and the method of [12] are very robust; in particular they do not require exact stationarity of the metric (in fact, the method of [12] is perhaps more robust than the traditional method for Minkowski space in that it uses neither multipliers

nor commutators with weights in  $t!$ ), and thus could be useful in a nonlinear setting.

This is not the place to comment at length at the methods of proof. Both Theorems 1 and 2 are proven by exploiting vector field multiplier currents and vector field commutators which reflect the geometrically interesting aspects of the spacetime. This is important, because only such methods have proven sufficiently robust to handle non-linear problems such as the stability of Minkowski space. The interesting aspects of spacetime which must be captured here, briefly, are (i) the redshift effect, (ii) superradiance, and (iii) trapping.

The celebrated redshift effect (i) in the context of what we now understand as black holes was first discussed by Oppenheimer-Snyder [17]. This effect already occurs in Schwarzschild, but is in fact very general; a localised version of the effect depends in fact only on the positivity of surface gravity of the Killing horizon. It turns out that the red-shift effect can be completely captured by the positivity properties an energy identity associated to an appropriately chosen vector field  $N$ , when restricted to a neighborhood of the horizon. See [7, 8] for the original constructions and [11] for a generalisation. In particular, this requires no frequency analysis. The positivity properties of  $J^N$  are preserved by commutation in a transversal direction (see [10, 11]). This use of the redshift effect plays an important role in the proof of both of the above theorems, as well as [12].

The problem of superradiance (ii) is not present in the Schwarzschild case, but is present for all Kerr's with  $a \neq 0$ . Superradiance is just the property that, since  $T = \partial_t$  is not timelike in the domain of outer communication, the conserved current  $J^T$  associated to  $T$  is no longer positive definite on spacelike hypersurfaces, and thus fails to control the solutions, which certainly can radiate more energy to infinity than their initial  $J^T$ -flux. Under the assumptions of Theorem 1, one can show, using a Fourier-based decomposition of the solution into its 'superradiant' and 'non-superradiant' parts, that *the superradiant part is not trapped* (see the discussion of trapping below). Using also the red-shift, one can then construct a non-degenerate energy current which roughly speaking shows that the superradiant part of the solution disperses, and, together with the usual conserved energy current applied to the non-superradiant part of the solution, this shows that the total solution (i.e. the sum of the two) is bounded. Defining the decomposition is technically challenging because it relates to the Fourier transform in time, whereas the solution is not known a priori to be even bounded in time. Nonetheless, this difficulty can be overcome. See [10] for details.

The issue of trapping (iii) concerns the presence of null geodesics which neither cross the event horizon, nor escape to null infinity. This is a familiar issue from the study of the wave equation outside of obstacles in Euclidean space. In particular, it is known that integrated decay estimates of the form (2) must degenerate near trapping, and decay bounds for the energy flux through the foliation  $\tilde{\Sigma}_\tau$  must 'lose derivatives'. As with the other two difficulties, trapping can also be 'captured' through suitable energy currents. In a different language, the study of such currents in the Schwarzschild case was initiated in [13]. See [7–9, 11, 16]. As

was shown in [1], these constructions cannot be immediately generalised to Kerr. This has to do with the fact that the dynamics of geodesic flow in a neighborhood of trapped geodesics cannot be well understood by projecting to physical space. In particular, in contrast to the Schwarzschild case, it is no longer the case in Kerr that future-trapped geodesics asymptote to a codimension-1 hypersurface of physical space. There are now three methods for these constructions: the original independent constructions [11] and [18] which use a frequency localised construction, and the more recent [2], which accomplishes frequency localisation by cleverly commuting with higher order operators. All three arguments make fundamental use of essentially the same property, namely *the complete integrability of geodesic flow*. See above for a comparison with the general boundedness statement (Theorem 1) which is independent of the properties of geodesics. We refer the reader to the original papers to get an idea of the difficulties involved.

## REFERENCES

- [1] S. Alinhac *Energy multipliers for perturbations of Schwarzschild metric* Comm. Math. Phys. **288** (2009), no. 1, 199–224
- [2] L. Andersson and P. Blue *Hidden symmetries and decay for the wave equation on the Kerr spacetime*, arXiv:0908.2265
- [3] L. Bieri *An extension of the stability theorem of the Minkowski space in general relativity*, Thesis, ETH, 2007
- [4] P. Blue and J. Sterbenz *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space* Comm. Math. Phys. **268** (2006), no. 2, 481–504
- [5] D. Christodoulou and S. Klainerman *The global nonlinear stability of the Minkowski space* Princeton University Press, 1993
- [6] M. Dafermos *The interior of charged black holes and the problem of uniqueness in general relativity* Comm. Pure Appl. Math. **58** (2005), 445–504
- [7] M. Dafermos and I. Rodnianski *The redshift effect and radiation decay on black hole spacetimes* Comm. Pure Appl. Math. **52** (2009), 859–919.
- [8] M. Dafermos and I. Rodnianski *The wave equation on Schwarzschild-de Sitter spacetimes*, arXiv:0709.2766v1 [gr-qc]
- [9] M. Dafermos and I. Rodnianski *A note on energy currents and decay for the wave equation on a Schwarzschild background*, arXiv:0710.0171v1 [math.AP]
- [10] M. Dafermos and I. Rodnianski *A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds*, arXiv:0850.4309v1 [gr-qc]
- [11] M. Dafermos and I. Rodnianski *Lectures on black holes and linear waves*, arXiv:0811.0354 [gr-qc]
- [12] M. Dafermos and I. Rodnianski *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*, in press, Proceedings of the Int. Cong. Math. Phys., Prague 2009, 418–429
- [13] I. Laba and A. Soffer *Global existence and scattering for the nonlinear Schrödinger equation on Schwarzschild manifolds* Helv. Phys. Acta **72** (1999), no. 4, 272–294
- [14] H. Lindblad and I. Rodnianski *The global stability of Minkowski space in harmonic gauge* to appear, Ann. of Math.
- [15] J. Luk *Improved decay for solutions to the linear wave equation on a Schwarzschild black hole*, arXiv:0906.5588
- [16] J. Marzuola, J. Metcalfe, D. Tataru, M. Tohaneanu *Strichartz estimates on Schwarzschild black hole backgrounds* arXiv:0802.3942v3
- [17] J. R. Oppenheimer and H. Snyder *On continued gravitational contraction* Phys. Rev. **56** (1939), 455–459

- [18] D. Tataru and M. Tohaneanu *Local energy estimate on Kerr black hole backgrounds*, arXiv:0810.5766

## Decay estimates for the wave equation on asymptotically flat space-times

DANIEL TATARU

The aim of the talk was to provide an overview of recent and ongoing work concerning global in time decay properties for the wave equation on asymptotically flat space-times. Parts of this work are joint with the following collaborators: Jeremy Marzuola, Jason Metcalfe and Mihai Tohaneanu. Some of this research was motivated by problems in general relativity concerning the decay properties for the wave equation on Schwarzschild and Kerr backgrounds. Partly for this reason, all the results are presented in  $3 + 1$  space dimensions; for similar results in other dimensions one can consult the references.

We consider decay estimates for the forward wave equation

$$(1) \quad (\square_g + V)u = f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1$$

For the metric  $g$  and the potential  $V$  we consider two cases:

**Case A:**  $g$  is a smooth Lorenzian metric in  $\mathbb{R} \times \mathbb{R}^3$ , with the following properties:

(i) The level sets  $t = \text{const}$  are space-like, and (ii)  $g$  is asymptotically flat, i.e.  $g = m + o_r(1)$  and  $V = o_r(r^{-2})$  as  $|x| = r \rightarrow \infty$ , where  $m$  stands for the Minkowski metric.

**Case B:**  $g$  is a smooth Lorenzian metric in an exterior domain  $\mathbb{R} \times \mathbb{R}^3 \setminus B(0, R_0)$  which satisfies (i) and (ii) above, as well as (iii) the lateral boundary  $\mathbb{R} \times \partial B(0, R_0)$  is outgoing space-like.

The second case is modeled after the Schwarzschild and Kerr metrics, which satisfy the above conditions in suitable advanced time coordinates. There the parameter  $R_0$  is chosen so that  $0 < R_0 < 2M$  in the case of the Schwarzschild metric, respectively  $r^- < R_0 < r^+$  in the case of Kerr, so that the exterior of the  $R_0$  ball contains a neighbourhood of the event horizon. There are several types of estimates which are of interest:

**I. Energy estimates**, where the aim is to obtain uniform in time bounds,

$$\|\nabla_{x,t} u\|_{L^\infty L^2} \lesssim \|\nabla_{x,t} u(0)\|_{L^2} + \|f\|_{L_t^1 L_x^2}$$

**II. Local energy decay**, i.e. integrated energy decay in compact spatial regions:

$$\|\nabla_{x,t} u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE} \lesssim \|\nabla_{x,t} u(0)\|_{L^2} + \|f\|_{LE^*}$$

where the dual  $LE$  and  $LE^*$  norms are defined using the partition of the space-time  $\mathbb{R}^+ \times \mathbb{R}^3$  into dyadic regions  $A_m = \{\langle r \rangle \approx 2^m\}$

$$\|v\|_{LE} = \sup_m \|\langle r \rangle^{-\frac{1}{2}} v\|_{L_{x,t}^2(A_m)}, \quad \|f\|_{LE^*} = \sum_m \|\langle r \rangle^{\frac{1}{2}} f\|_{L_{x,t}^2(A_m)}$$

**III. Strichartz estimates**, i.e. global space-time integrated decay:

$$\| |D_x|^s \nabla u \|_{L_t^p L_x^q} \lesssim \|\nabla_{x,t} u(0)\|_{L^2} + \|f\|_{L_t^1 L_x^2}, \quad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad \frac{1}{p} + \frac{3}{q} = \frac{3}{2} - s$$

**IV. Pointwise decay** for the homogeneous problem ( $f = 0$ ):

$$(2) \quad |u(t, x)| \lesssim \frac{\|u_0\|_{Z^{m+1,1}} + \|u_1\|_{Z^{m,2}}}{\langle t + |x| \rangle \langle t - |x| \rangle^2}, \quad |\partial_t u(t, x)| \lesssim \frac{\|u_0\|_{Z^{m+1,1}} + \|u_1\|_{Z^{m,2}}}{\langle t + |x| \rangle \langle t - |x| \rangle^3}$$

where  $Z^{m,k}$  represent the fixed time weighted norms

$$\|u\|_{Z^{m,k}} = \sum_{j \leq m} \|\langle r \rangle^{j+k} \nabla_x^j u\|_{LE^*}$$

Local energy estimates were first obtained in work of Morawetz [15], and have a long history up to the present. Of interest here is the following result:

**Theorem 1** ( [14]). *Case A: If  $g - m$  is small then (LE) holds.*

There is no general result available for large perturbations of the Minkowski space time. What we hope to prove (and is known in special cases) is

**Theorem 2** (in progress). *Case A: If (i)  $g, V$  are stationary, (ii)  $g$  nontrapping and (iii)  $\square_g + V$  admits no eigenmode  $\lambda$  with  $\Re \lambda \geq 0$  and no zero resonance then (LE) holds.*

The key result in [14], which we do not describe here, is the construction of an outgoing parametrix for  $\square_g + V$  outside a compact spatial region. Using this, we were able to show that local energy decay governs the Strichartz estimates.

**Theorem 3** ( [14]). *Case A: If (LE) holds then (SE) holds.*

In Case B, one can sometimes establish directly uniform energy bounds. This is the case for the Schwarzschild and Kerr space-times:

**Theorem 4** ( [6], [5], see also [8], [20]). *Case B: (EE) holds for Schwarzschild and Kerr space-times with small angular momentum  $|a| \ll M$ , as well as for a class of small perturbations thereof.*

However, one does not expect the full local energy decay estimates as above since trapping will necessarily occur. Nevertheless, a weaker form of decay can still occur when the trapped null geodesics are hyperbolic:

**Theorem 5** ( [10], [20], see also [3], [6], [1], [4], [2]). *Case B: A weaker form of (LE) holds for Schwarzschild and Kerr space-times with small angular momentum  $|a| \ll M$ .*

Combining this with the approach in [14] for the dynamics near infinity, one obtains:

**Theorem 6** ( [10], [22]). *Case B: (SE) hold for Schwarzschild and Kerr space-times with small angular momentum  $|a| \ll M$  for nonsharp indices,  $1/p + 1/q < 1/2$ .*

Finally, we come to the pointwise decay estimates, where we have

**Theorem 7** ([21]). *Case A,B: Suppose that the metric  $g$  and the potential  $V$  satisfy*

$$g = m + O_{radial}(r^{-1}) + o(r^{-1}), \quad g = m + O_{radial}(r^{-3}) + o(r^{-3})$$

*If (EE) and a weak form of (LE) hold then the pointwise decay estimates (PD).*

Such a result was conjectured by Price [16] in the case of the Schwarzschild space-time and proved by [18] when  $g = m$  and  $V$  is small. By Theorems 4,5 this result implies Price's conjecture for both the Schwarzschild and Kerr space-times with small angular momentum  $|a| \ll M$ . Partial results in these two cases have been previously proved in [3], [6], [7], [11] [4], [12], [17].

#### REFERENCES

- [1] Serge Alinhac. On the Morawetz–Keel–Smith–Sogge inequality for the wave equation on a curved background. *Publ. Res. Inst. Math. Sci.*, 42(3):705–720, 2006.
- [2] Lars Andersson and Pieter Blue. Hidden symmetries and decay for the wave equation on the Kerr spacetime. arXiv:0908.2265.
- [3] Pieter Blue and Jacob Sterbenz. Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space. *Comm. Math. Phys.*, 268(2):481–504, 2006.
- [4] Mihalis Dafermos and Igor Rodnianski. Lectures on black holes and linear waves. arXiv:0811.0354.
- [5] Mihalis Dafermos and Igor Rodnianski. A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. arXiv:0805.4309.
- [6] Mihalis Dafermos and Igor Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Comm. Pure Appl. Math.*, 62(7):859–919, 2009.
- [7] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau. Decay of solutions of the wave equation in the Kerr geometry. *Comm. Math. Phys.*, 264(2):465–503, 2006.
- [8] Felix Finster and Joel Smoller. A time-independent energy estimate for outgoing scalar waves in the Kerr geometry. *J. Hyperbolic Differ. Equ.*, 5(1):221–255, 2008.
- [9] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, London, 1973. Cambridge Monographs on Mathematical Physics, No. 1.
- [10] Daniel Tataru, Jeremy Marzuola, Jason Metcalfe and Mihai Tohaneanu. Strichartz estimates on Schwarzschild black hole backgrounds. arXiv:0802.3942.
- [11] Johann Kronthaler. Decay rates for spherical scalar waves in the Schwarzschild geometry. preprint, arXiv:0709.3703.
- [12] Jonathan Luk. Improved decay for solutions to the linear wave equation on a schwarzschild black hole. arXiv:0906.5588.
- [13] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.*, 255(6):1497–1553, 2008.
- [14] Jason Metcalfe and Daniel Tataru. Global parametrices and dispersive estimates for variable coefficient wave equations. arXiv:0707.1191, 2007.
- [15] Cathleen S. Morawetz. Exponential decay of solutions of the wave equation. *Comm. Pure Appl. Math.*, 19:439–444, 1966.
- [16] Richard H. Price. Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. *Phys. Rev. D (3)*, 5:2419–2438, 1972.
- [17] Avy Soffer, Roland Donniger, Wilhelm Schlag. A proof of price's law on Schwarzschild black hole manifolds for all angular momenta. arXiv:0908.4292.
- [18] Nikodem Szapak, Piotr Bizoń, Tadeusz Chmaj, and Andrzej Rostworowski. Linear and nonlinear tails. II. Exact decay rates and spherical symmetry. *J. Hyperbolic Differ. Equ.*, 6(1):107–125, 2009.

- [19] Daniel Tataru. Parametrix and dispersive estimates for Schrödinger operators with variable coefficients. *Amer. J. Math.*, 130(3):571–634, 2008.
- [20] Daniel Tataru and Mihai Tohaneanu. Local energy estimates on Kerr black hole backgrounds. arXiv:0810.5766.
- [21] Daniel Tataru. Local decay of waves on asymptotically flat stationary space-times. arXiv:0910.5290
- [22] Mihai Tohaneanu. Strichartz estimates on Kerr black hole backgrounds. preprint.

## Hidden symmetries and decay for the wave equation on the Kerr spacetime

PIETER BLUE

We study the wave equation

$$(1) \quad \nabla^\alpha \nabla_\alpha \psi = 0$$

in the exterior region of the Kerr spacetime, which is described in Boyer-Lindquist coordinates by  $(t, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times S^2$ , where  $r_+ = M + \sqrt{M^2 - a^2}$  and where  $M$  and  $aM$  are the mass and angular momentum of the Kerr black hole.

The Kerr spacetime has two symmetries, time translation and rotation about the axis of symmetry, and these generate the Killing vector fields  $\partial_t$  and  $\partial_\phi$ . In addition, there is Carter’s Killing 2-tensor which is said to generate a “hidden symmetry” and allows us to define a second-order operator  $Q$ .

In work with L. Andersson [1], we have shown

**Theorem 1.** *There are positive constants  $\bar{a}$  and  $C'$ , and a positive quadratic form  $\|\cdot\|^2(t)$  defined on each hypersurface of constant  $t$  such that if  $|a| < \bar{a}$  then, for parameters  $r_1$  and  $r_2$  satisfying  $r_+ < r_1 < r_2 < \infty$ , there is a positive constant  $C$  for which all solutions  $\psi$  of the wave equation satisfy the estimate that  $\forall t > 0, r \in [r_1, r_2], (\theta, \phi) \in S^2$ :*

$$|\psi(t, r, \theta, \phi)| \leq Ct^{-1+C'|a|} \|\psi\|(0).$$

The norm  $\|\psi\|(t)$  is bounded if  $\psi$  is both smooth and has a decay rate, for itself and its first nine derivatives with respect to the Boyer-Lindquist coordinates, of  $r^{-5/2+\delta}$  for any positive  $\delta$ . We also have proved the corresponding decay rates for  $\psi$  as  $r \rightarrow r_+$  and  $r \rightarrow \infty$ .

Previously, separation of variables was used to expand solutions as  $\psi(t, r, \theta, \phi) = \sum_{L_z} \int \sum_n e^{iL_z\phi} e^{i\omega t} R_{L_z, \omega, n}(r) Y_{L_z, \omega, n}(\theta) d\omega$  and to obtain decay results of the form  $\lim_{t \rightarrow \infty} \psi(t, r, \theta, \phi) \rightarrow 0$  when there are only finitely many  $L_z$  in the expansion of  $\psi$  [7].

Another approach was used to prove decay estimates in the subcase of the Schwarzschild spacetime, where  $a = 0$ , using the vector-field method, which essentially involves only integration by parts and choices of vector fields. For the wave equation, decay rate estimates were proved [3], proved with less regularity loss [2], and proved independently with stronger estimates near the event horizon [6]. This work built on earlier work for a model problem [8]. These results for the wave equation did not require a restriction to finitely many modes.

In 2008, uniform boundedness was proven for the wave equation on slowly rotating Kerr spacetimes [5, 9]. A result similar to theorem 1 was also announced and a detailed outline of the proof was given in [4]. Shortly before that uniform boundedness results were proved in [5, 9]. These results combined vector field methods with Fourier analytic or microlocal techniques. This does not use the complete separability of solutions, but does use the possibility of expansion in  $e^{i\omega t} e^{iL_z \phi}$ , which follows from the Kerr symmetries of time translation and rotation about the axis of symmetry.

In applying the vector field method to the wave equation on the Kerr spacetime, there are three major obstacles

- (1) there is no globally timelike vector field  $T$ ,
- (2) there is not the full set of rotational symmetries  $\Theta_i$ , and
- (3) the set of orbiting geodesics fills a four-dimensional volume.

It is well known that timelike Killing vector fields generate positive conserved energies, but that the Kerr spacetime has no timelike Killing vector field. The vector fields  $\partial_t$  and  $\partial_t + \omega_H \partial_\phi$ , where  $\omega_H = a/(r_+^2 + a^2)$ , are both Killing. The first generates the time translation symmetry, and the second generates the null tangents to the event horizon. For sufficiently large  $r$ , the vector field  $\partial_t$  is timelike, and, for  $r$  sufficiently close to  $r_+$ , the vector field  $\partial_t + \omega_H \partial_\phi$  is timelike. When  $|a|$  is sufficiently small, these far and near regions overlap, so that it is possible to construct a globally timelike Killing vector,  $\mathbf{T}_\chi$ , by smoothly transitioning from one vector field to the other in the region where both are timelike. The region where  $\mathbf{T}_\chi$  fails to be timelike covers only a compact set of  $r$  values. Thus, to prove that the associated energy is bounded, it is sufficient to control the spacetime integral of  $|\partial_r \psi| |\partial_\phi \psi|$  in this region.

The standard way to go from estimates for the energy of a solution to estimates for the solution itself is to consider not just the energy of  $\psi$  but also derivatives of  $\psi$  which also satisfy the wave equation. If  $\psi$  is a solution of the wave equation, then so is  $S\psi$  for any  $S$  in the algebra of symmetries generated by

$$\begin{aligned} \oplus_{n=0}^{\infty} \mathbb{S}_n, \quad \mathbb{S}_n = \{ \partial_t^{n_t} \partial_\phi^{n_\phi} Q^{n_Q} \mid n_t, n_\phi, n_Q \in \mathbb{N}, n_t + n_\phi + 2n_Q = n \}, \\ Q = (\sin \theta)^{-1} \partial_\theta \sin \theta \partial_\theta + \sin^2 \theta \partial_\phi^2 + a^2 \sin^2 \theta \partial_t^2. \end{aligned}$$

Thus, if  $E_{\mathbf{T}_\chi, \mathbf{1}}[\psi]$  is the energy of  $\psi$ , defined using  $\mathbf{T}_\chi$ , then we can define  $E_{\mathbf{T}_\chi, \mathbf{3}}[\psi]$  to be the sum of the energies  $E_{\mathbf{T}_\chi, \mathbf{1}}[S\psi]$  with  $S \in \oplus_{n=0}^2 \mathbb{S}_n$ . Since the  $S\psi$  also satisfy the wave equation, this growth of the third-order energy will be controlled by the spacetime integral of third-derivatives of  $\psi$  in a fixed range of  $r$  values.

Spacetime integrals of this form can be controlled by Morawetz estimates, which already played an important role in the proof of decay rate estimates in the Schwarzschild case. Such estimates require the construction of a vector field which is adapted to point away from the orbiting geodesics. In the Schwarzschild case, where the orbiting geodesics are constrained to the three-dimensional hypersurface  $r = 3M$ , such a vector field can be chosen by taking  $\mathbf{A} = \mathcal{F}(\mathbf{r}) \partial_{\mathbf{r}}$  with  $\mathcal{F}$  a smooth, increasing function which changes sign at  $r = 3M$ .

Unfortunately, obstacle 3, in conjunction with certain nonvanishing- derivative and smoothness conditions required for the Morawetz estimate, prevents the construction of a vector field  $A$  which points away from the photon orbits, since there is no single value at which the weight  $\mathcal{F}(r)$  should change sign. By replacing  $\mathcal{F}(r)$  by  $\mathcal{F}^{ab}(r)S_a S_b$  where  $\mathbb{S}_2 = \{S_a\}$ , we have been able to construct a Morawetz estimate without using Fourier transforms in  $t$ . At one point in the proof, we still require that there are sufficiently small  $\epsilon$  and sufficiently large  $N$  such that  $\int_{S^2} (\psi - N\partial_\phi^2 \psi)^2 d\theta d\phi > \epsilon \int_{S^2} (\psi^2 + (\partial_\phi^2 \psi)^2) d\theta d\phi$ , which follows from the discrete spectrum of  $\partial_\phi^2$ .

## REFERENCES

- [1] Pieter Blue and Lars Andersson. Hidden symmetries and decay for the wave equation on the Kerr spacetime, 2009. arXiv:0908.2265v1.
- [2] Pieter Blue and Avy Soffer. Phase space analysis on some black hole manifolds. *J. Funct. Anal.*, 256(1):1–90, 2009.
- [3] Pieter Blue and Jacob Sterbenz. Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space. *Comm. Math. Phys.*, 268(2):481–504, 2006.
- [4] Mihalis Dafermos and Igor Rodnianski. Lectures on black holes and linear waves, 2008. arXiv.org:0811.0354.
- [5] Mihalis Dafermos and Igor Rodnianski. A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds, 2008.
- [6] Mihalis Dafermos and Igor Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Communications on Pure and Applied Mathematics*, 62, 2009.
- [7] Felix Finster, Niky Kamran, Joel Smoller, and Shing-Tung Yau. Decay of solutions of the wave equation in the Kerr geometry. *Comm. Math. Phys.*, 264(2):465–503, 2006.
- [8] Izabella Laba and Avy Soffer. Global existence and scattering for the nonlinear Schrödinger equation on Schwarzschild manifolds. *Helv. Phys. Acta*, 72(4):274–294, 1999.
- [9] Daniel Tataru and Mihai Tohaneanu. Local energy estimate on Kerr black hole backgrounds, 2008.

## The massive wave equation on slowly rotating Kerr-AdS spacetimes

GUSTAV HOLZEGEL

We study the massive linear wave equation

$$(1) \quad \square_g \psi - \alpha \frac{\Lambda}{3} \psi = 0$$

for  $\alpha < \frac{9}{4}$  on asymptotically anti-de Sitter (AdS) black hole backgrounds. That is to say that, in particular,  $g$  satisfies

$$(2) \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

with  $\Lambda < 0$ , and decays suitably to the AdS metric towards null-infinity. Note that, as we use signature  $(-, +, +, +)$ , the range  $0 < \alpha < \frac{9}{4}$  corresponds to negative mass. The choice  $\alpha = 2$  has particular relevance as it produces the conformally invariant wave equation in (1). Finally, the bound  $\alpha < \frac{9}{4}$  is known as the Breitenlohner-Freedman bound [4].

Asymptotically AdS spacetimes are not globally hyperbolic. To make equation (1) well-posed on these backgrounds, one has to impose, in addition to initial data on a spacelike hypersurface  $\Sigma$ , boundary conditions at null infinity ( $\mathcal{I}$ ). Constructing an appropriate coordinate system  $(t, r, \theta, \phi)$  near  $\mathcal{I}$ , and exploiting the asymptotic timelike Killing field  $\partial_t$  and its associated energy, one establishes the following well-posedness statement:

**Theorem 1.** *Fix an asymptotically AdS spacetime whose causal geometry is similar to either pure AdS or Kerr-AdS. Then, for  $\alpha < \frac{9}{4}$  and suitably regular data on a spacelike hypersurface, there exists a unique solution  $\psi$  of (1) with the property that its associated energy flux (arising from  $\partial_t$ ) through null-infinity vanishes.*

It is noteworthy that the solution is expected to be non-unique, if one imposes only  $\psi = 0$  at  $\mathcal{I}$ . This type of non-uniqueness has been shown explicitly in the pure AdS case in [3, 4]. The point here is that membership of the solution in the energy space (as arising from  $\partial_t$ ) imposes decay conditions on  $\psi$  which are much stronger than just  $\psi = 0$  at  $\mathcal{I}$ .

We are interested in the global behavior of solutions produced by *Theorem 1* on black hole spacetimes. For the Schwarzschild-AdS case the metric – in regular coordinates including the event horizon – reads (setting  $\Lambda = -\frac{3}{l^2}$ )

$$g = - \left( 1 - \frac{2M}{r} + \frac{r^2}{l^2} \right) (dt^*)^2 + \frac{4M}{r \left( 1 + \frac{r^2}{l^2} \right)} dt^* dr + \frac{1 + \frac{2M}{r} + \frac{r^2}{l^2}}{\left( 1 + \frac{r^2}{l^2} \right)^2} dr^2 + r^2 d\omega_2$$

and we prove

**Theorem 2.** *Fix a Schwarzschild-anti de Sitter spacetime  $(\mathcal{M}, g_{M>0,\Lambda})$  and  $\Sigma_0 = \Sigma_{\tau_0}$  a slice of constant  $t^* = \tau_0$  in  $\mathcal{D} = \overline{J^+(\mathcal{I}) \cap J^-(\mathcal{I})}$ . Let  $\alpha < \frac{9}{4}$  and  $\psi$  be a solution arising from *Theorem 1*. If*

$$(3) \quad \sum_{k=0}^n \int_{\Sigma_0} \left( \frac{1}{r^2} (\partial_{t^*} \Omega^k \psi)^2 + r^2 (\partial_r \Omega^k \psi)^2 + |\nabla \Omega^k \psi|^2 \right) r^2 dr d\omega < \infty$$

then

$$(4) \quad \sum_{k=0}^n \int_{\Sigma_\tau} \left( \frac{1}{r^2} (\partial_{t^*} \Omega^k \psi)^2 + r^2 (\partial_r \Omega^k \psi)^2 + |\nabla \Omega^k \psi|^2 \right) r^2 dr d\omega \leq C \left[ \sum_{k=0}^n \int_{\Sigma_0} \left( \frac{1}{r^2} (\partial_{t^*} \Omega^k \psi)^2 + r^2 (\partial_r \Omega^k \psi)^2 + |\nabla \Omega^k \psi|^2 \right) r^2 dr d\omega \right].$$

for a constant  $C$  which just depends on  $M, l$  and  $\alpha$ . Here  $\Sigma_\tau$  denotes any constant  $t^*$  slice to the future of  $\Sigma_0$  and restricted to  $r \geq r_{\text{hoz}}$ .

In *Theorem 2* we commute with a basis of angular Killing vectorfields  $\Omega_i$ . Using Sobolev embedding on  $S^2$  we obtain a pointwise bound for  $\psi$  on the entire black hole exterior.

The proof of *Theorem 2* uses vectorfield multipliers and commutators and has two essential ingredients. Firstly, in the range  $0 < \alpha < \frac{9}{4}$  the energy momentum

tensor associated with  $\psi$  does not satisfy the dominant energy condition due to the zeroth order term. However, it turns out that in the energy identity arising from the Killing field  $T = \partial_{t^*}$ , one can absorb the zeroth order term by the  $r$ -derivative term using a weighted Hardy inequality. Hence the energy is still positive in an *integrated* sense.

The second ingredient is a redshift vectorfield  $Y$ , which was constructed by Dafermos and Rodnianski for any non-degenerate Killing horizon [2]. It turns out that the energy arising from  $T$  plus the redshift vectorfield suffice to prove boundedness (cf. [1] for the asymptotically flat context).

We next generalize *Theorem 2* to the case of Kerr-AdS. The additional difficulty caused by the ergosphere can be resolved in a much simpler fashion than in the asymptotically flat case: For sufficiently small angular momentum, there exists a globally causal Killing field  $K$  on the black hole exterior [5]. Using  $K$  instead of  $T$  one proves (again with the help of the redshift) that the non-degenerate energy can be controlled from the data for all times (i.e. the analogue of (4) for  $k = 0$ ). For higher derivatives a problem arises since one can no-longer commute (trivially) with angular momentum operators. To resolve this problem we again apply the results of [2], suitably adapted to the AdS setting: Commuting with  $T$  (or  $K$ ) and with the redshift vectorfield near the horizon is sufficient to gain control over all derivatives using elliptic estimates. In brief this is possible because the worst term in the commutation with the redshift vectorfield has a good sign. We obtain

**Theorem 3.** *Fix  $\alpha < \frac{9}{4}$ , a mass  $M$  and a negative cosmological constant  $\Lambda$ . Then there exists a parameter  $\tilde{a} > 0$  (depending on  $M$  and  $\Lambda$ ) such that for any fixed Kerr-AdS spacetime with parameters  $(M, |a| < \tilde{a}, \Lambda)$ , solutions of the massive wave equation as arising from *Theorem 1* remain uniformly bounded on the black hole exterior.*

We finally remark that one can generalize *Theorem 3* even further to include backgrounds which are sufficiently close to the Kerr-AdS spacetimes and admit a global causal Killing field which is null on the horizon.

## REFERENCES

- [1] Dafermos, Mihalis and Rodnianski, Igor: *A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds*, arXiv:0805.4309 (2009).
- [2] Dafermos, Mihalis and Rodnianski, Igor: *Lectures on black holes and linear waves*, arXiv:0811.0354 (2008).
- [3] Alain Bachelot: *The Dirac System on the Anti-de Sitter Universe*, Commun. Math. Phys. 283, 127-167 (2008).
- [4] Breitenlohner, Peter and Freedman, Daniel Z.: *Stability in Gauged Extended Supergravity*, Ann. Phys. 144, 249 (1982).
- [5] Hawking, S. W. and Reall, H. S.: *Charged and rotating AdS black holes and their CFT duals*, Phys. Rev. D61,024014 (2000).

## A black hole uniqueness theorem

SPYROS ALEXAKIS

(joint work with A. Ionescu and S. Klainerman)

This report covers recent work of the author, in collaboration with A. Ionescu and S. Klainerman on the black hole uniqueness question, [2, 3].

This question is motivated by the expectation<sup>1</sup> that general, dynamic black hole exteriors should settle down asymptotically to non-radiative, stationary solutions; this is projected due to radiation of matter and gravitational energy into the black hole region and to infinity. One therefore wishes to understand the possible stationary black hole solutions, in the hope that the general dynamic black hole solutions should asymptotically resemble such a state.

Hawking and Ellis asserted, based on prior work of Carter and Robinson [4, 12], that the Kerr family of solutions are the only possible sufficiently regular *stationary* black hole solutions with a simply connected non-degenerate bifurcate event horizon. They proved this assertion, imposing the a priori assumption of *real-analyticity* on the space time metric. We review the Hawking argument, since it will set a blueprint for our result:

*The classical black hole uniqueness proof:* Hawking noted that given any stationary black hole exterior  $(M_{ext}, \mathbf{g})$ , stationarity implies the existence of a *jet* of a *second, rotational* Killing field  $\mathbf{Z}$  on the event horizon. The assumption of real-analyticity enabled him to conclude that this formal jet extends to an *actual* rotational Killing field  $\mathbf{Z}$  in the entire exterior region  $M_{ext}$ .<sup>2</sup> Thus the space-time  $(M_{ext}, \mathbf{g})$  is not only stationary but also axis-symmetric. The conclusion then follows from the work of Carter and Robinson who proved that a black hole exterior which is *both* stationary *and* axisymmetric must necessarily be isometric to a Kerr solution. (*Remark:* The original ingenious arguments in [4, 12] have since been simplified and cast in new light by numerous authors; I wish to make particular reference to the work of Mazur and Weinstein [11, 13]. The recent work of Chusciel-Costa [5] provides a definite account of all elements in the proof and fills in gaps in the prior literature).

One unsatisfactory point with the above approach is its reliance on the a priori assumption of real-analyticity. Indeed, this assumption is highly un-natural, given that stationary (non-static) black hole exteriors must necessarily possess an *ergo-region*, where the Killing field  $\mathbf{T}$  is space-like. There is *a priori* no reason whatsoever for the metric to be real-analytic there. Our theorem addresses this short-coming; we feel that the result in [2, 3] provides the strongest evidence to-date on the validity of the black hole uniqueness conjecture:<sup>3</sup>

**Theorem 1** (A., Ionescu, Klainerman, [2,3]). *Let  $(M_{ext}, \mathbf{g})$  be a vacuum, stationary, globally hyperbolic black hole exterior, with a bifurcate event horizon  $\mathcal{H}^+ \cup \mathcal{H}^-$ ;*

---

<sup>1</sup>Formulated in [7].

<sup>2</sup>A rigorous proof of this fact was later given by Friedrich-Racz-Wald in [6].

<sup>3</sup>We provide a simplified formulation of the main theorem. We refer the reader to [2] for the proper formulation.

denote the stationary Killing field by  $\mathbf{T}$ . Assume that the Mars-Simon tensor  $\mathcal{S}_{ijkl}$  satisfies a smallness condition of the form  $|(1 - \sigma)\mathcal{S}(\mathbf{T}, T^a, T^b, T^c)| \leq \epsilon$  for some sufficiently small  $\epsilon$ .<sup>4</sup> Then under suitable regularity assumptions (on the behaviour at space-like infinity and at the event horizon), described in [2] we prove that  $(M_{ext}, \mathbf{g})$  must be isometric to a member of the Kerr family of solutions.

*Remark:* The Mars-Simon tensor  $\mathcal{S}_{ijkl}$  is a natural tensor defined on any stationary vacuum space-time, see [9, 10]. One of its remarkable features is that its vanishing is a local characterization of the Kerr solutions. Thus, the smallness condition we impose can be interpreted as a “closeness” condition to the Kerr family of solutions.

**Ideas in the proof:** The proof of the above relies on two main components. The first is a general unique continuation theorem for the vacuum Einstein equations. This is used to derive that for vacuum Einstein metrics, a Killing field can be extended across any hypersurface  $\mathcal{H}$  provided  $\mathcal{H}$  is *pseudo-convex*, in an appropriate sense.<sup>5</sup> In fact, based on Carleman estimates in [8], we show in that the rotational Killing field  $\mathbf{Z}$  can be extended to a neighborhood of the event horizon, *without* any smallness assumption on  $\mathcal{S}$ . The second component is the observation that the smallness assumption on  $\mathcal{S}$  implies the existence of a foliation of  $M_{ext}$  be a family of hypersurfaces which *do* satisfy the (appropriate) notion of pseudo-convexity. Specifically:

*First step, [2]:* Such a unique continuation result was originally derived in [1]. The proof in [2] provides a substantial simplification of the ideas in [1]. The argument relies on coupling the (differentiated) Einstein equations<sup>6</sup> to a system of second order ODE’s, which fix the gauge choice through a moving frame construction.<sup>7</sup> Unique continuation for the Einstein metric then follows by combining this system of equations with Carleman estimates applied to the wave operator.

*Second Step, [3]:* The required foliation is explicitly constructed by using the level sets of  $Re(\frac{1}{1-\sigma})$ . We then prove that these level level sets are *conditionally pseudo-convex*. This notion is the *correct notion* of pseudo-convexity in this setting: Ionescu-Klainerman showed [8] that in *stationary space-times*, Carleman estimates (for the wave operator coupled with an ODE in the Killing direction) hold across a given hypersurface  $\mathcal{H}$ , provided *conditional* pseudo-convexity (as opposed to the *full* notion of pseudo-convexity) holds across  $\mathcal{H}$ . Here we define a smooth hypersurface  $\mathcal{H}$  to which  $\mathbf{T}$  is tangent to be *conditionally pseudo-convex* (in a given direction) when  $\mathcal{H}$  is convex relative *only to null geodesics which are normal to  $\mathbf{T}$* .

<sup>4</sup>Here  $\sigma$  is the (complexified) Ernst potential for the Killing field  $\mathbf{T}$ .  $\mathcal{S}_{ijkl}$  is the Mars-Simon tensor and  $T^0, T^1, T^2, T^3$  stands for an orthonormal frame defined on the Cauchy surface in  $M_{ext}$ .

<sup>5</sup>The classical notion of *pseudo-convexity* was introduced by Hörmander. In this setting it means that  $\mathcal{H}$  should be convex (in a specific direction) *only with respect to null geodesics*.

<sup>6</sup>We use the well-known fact that the Einstein equations imply a non-linear wave equation on the space-time curvature tensor.

<sup>7</sup>The argument in relied on fixing the gauge through a canonical choice of “double Fermi” coordinates. This gave rise to a second-order ODE, which caused difficulties and complications in the proof.

## REFERENCES

- [1] S. Alexakis, *Unique continuation for the vacuum Einstein equations*, arXiv:0902.1131.
- [2] S. Alexakis, A. D. Ionescu, S. Klainerman, *Hawking's local rigidity theorem without analyticity*, arXiv:0902.1173.
- [3] S. Alexakis, A. D. Ionescu, S. Klainerman, *Uniqueness of smooth stationary black holes in vacuum: small perturbations of the Kerr spaces*, arXiv:0904.0982.
- [4] B. Carter, *Black hole equilibrium states*, Black holes/Les astres occlus (École d'Été Phys. Théor., Les Houches, 1972), pp. 57–214.
- [5] P. T. Chrusciel and J. L. Costa, *On uniqueness of stationary vacuum black holes*, Preprint (2008).
- [6] H. Friedrich, I. Racz, R. Wald, *On the rigidity theorem for spacetimes with a stationary event horizon or a compact Cauchy horizon*, Commun. Math. Phys. **204**, 691707 (1999).
- [7] S. Hawking, G. F. R. Ellis, *The large-scale structure of space-time*, Cambridge Monographs on Mathematical Physics, 1975.
- [8] A. D. Ionescu and S. Klainerman, *On the uniqueness of smooth, stationary black holes in vacuum*, Preprint (2007).
- [9] M. Mars, *A spacetime characterization of the Kerr metric*, Classical Quantum Gravity **16** (1999), 2507–2523.
- [10] M. Mars, *Uniqueness properties of the Kerr metric*, Classical Quantum Gravity **17** (2000), 3353–3373.
- [11] P.O. Mazur *Proof of Uniqueness for the Kerr-Newman Black Hole Solution*. J. Phys A: Math Gen., **15** (1982) 3173-3180.
- [12] D.C. Robinson, *Uniqueness of the Kerr black hole*, Phys. Rev. Lett. **34** (1975), 905-906.
- [13] G. Weinstein, *On rotating black holes in equilibrium in general relativity*, Comm. Pure Appl. Math, **43** (1990), 903-948.

## Gravitational collapse for the Einstein-Vlasov system

GERHARD REIN

(joint work with Håkan Andréasson, Markus Kunze)

An important open problem in general relativity is the validity of the weak cosmic censorship conjecture which says that generic asymptotically flat initial data for the Einstein-matter equations have a maximal future development possessing a complete future null infinity. An answer to this problem in full generality is beyond reach of present mathematical techniques, but under the assumption of spherical symmetry Christodoulou showed that the conjecture holds if “matter” is described as a massless scalar field, while it is violated if matter is described as dust, i.e., an ideal fluid with pressure zero, cf. [6] and the references there.

In my talk I argued that a collisionless gas as described by the Vlasov equation provides a suitable matter model for investigating the weak cosmic censorship conjecture, and I presented recent results in that direction.

The Einstein-Vlasov system describes in the framework of general relativity an ensemble of particles, say the stars in a galaxy or a globular cluster, which interact only through the gravitational field which they create collectively. If  $f = f(x^\alpha, p_\beta) \geq 0$  denotes the number density of the particles on the co-tangent bundle of the spacetime manifold and  $g_{\alpha\beta}$  the Lorentz metric with induced Einstein

tensor  $G_{\alpha\beta}$ , then the system takes the form

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

$$p^\alpha \partial_{x^\alpha} f - \frac{1}{2} \partial_{x^\alpha} g^{\beta\gamma} p_\beta p_\gamma \partial_{p_\alpha} f = 0,$$

$$T_{\alpha\beta} = |g|^{-1/2} \int p_\alpha p_\beta f \frac{dp_0 dp_1 dp_2 dp_3}{m};$$

$|g|$  denotes the modulus of the determinant of the metric and  $m$  the rest mass of the particle with coordinates  $(x^\alpha, p_\beta)$ . Usually we restrict ourselves to the case that  $m = 1$  for all particles. We emphasize that in our analysis  $f$  is a (smooth) distribution *function* and not a possibly singular measure; by a suitable choice of the latter type dust would become a special case of the Vlasov matter model. The Newtonian limit of the Einstein-Vlasov system, the so-called Vlasov-Poisson system, is frequently used as a model in astrophysics.

A distinguishing feature of the latter system is that sufficiently regular initial data launch smooth, global-in-time solutions [7, 9, 10]. Possible singularities in the solutions of the Einstein-Vlasov system must therefore have some relativistic origin. In addition, for a given, sufficiently smooth metric the Vlasov equation simply says that the particles in the ensemble move along timelike geodesics so this matter model does not produce singularities by itself. These two features make the Vlasov equation a particularly suitable matter model when investigating gravitational collapse, the formation of possible spacetime singularities, and the cosmic censorship conjecture. On the other hand the dynamics of the system is rich already in spherical symmetry and qualitatively different types of solution behavior are possible: Small data launch geodesically complete solutions where the matter disperses [11], the system possesses an abundance of steady states [12, 13], in numerical simulations one finds both stable and unstable such steady states, and the perturbation of stable ones seems to lead to periodic oscillations while the perturbation of unstable ones can lead to gravitational collapse [3]. Numerical investigations of critical gravitational collapse for the Einstein-Vlasov system have so far always lead to so-called type I behavior [3, 8, 15], in support of the conjecture that weak cosmic censorship holds for this system.

In order to analyze solutions of the asymptotically flat and spherically symmetric Einstein-Vlasov system and possible gravitational collapse the choice of coordinates is important. A first choice are Schwarzschild coordinates where the metric takes the form

$$ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

From [14] it is known that if a solution in these coordinates develops a singularity the first one must be at the center. Hence solutions exist globally on domains like

$$D := \{(t, r) \mid r \geq \gamma(t)\}$$

the boundary curve of which is a radially outgoing null geodesic:

$$\dot{\gamma} = e^{\mu(t,\gamma) - \lambda(t,\gamma)}, \quad \gamma(0) = r_0 > 0.$$

In [1] explicit conditions on the initial data are specified such that the behavior of the solutions on  $D$  can be analyzed in sufficient detail to conclude that a black hole forms, the spacetime has a complete future null infinity in the sense of [6], and hence weak cosmic censorship holds, cf. also [2]. The basic structure of the data is that some matter, which can be represented by a steady state, is inside  $\{r < r_0\}$  while a shell of particles which move inward sufficiently fast is situated at radii larger than but close to the Schwarzschild radius of the total ensemble.

A drawback of Schwarzschild coordinates in such an analysis is that they cannot cover regions of spacetime containing trapped surfaces. Related to this is the fact that they do not seem to completely cover the generator of the event horizon; at least it has not been possible to show that they do. For the data under consideration a necessary condition for the completeness of the event horizon is that

$$\lim_{t \rightarrow \infty} m(t, 2M) = M$$

where  $m = \frac{r}{2}(1 - e^{-2\lambda})$  is the quasi-local ADM mass and  $M = m(t, \infty)$  the ADM mass of the solution. The latter asymptotic behavior of the matter has been established in [4].

In [5] the system was analyzed in Eddington-Finkelstein coordinates where the metric takes the form

$$ds^2 = -a(v, r) b^2(v, r) dv^2 + 2b(v, r) dv dr + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

Here  $v$  is an advanced null coordinate, and  $b$  must be strictly positive but  $a$  can change sign. Surfaces of constant  $v$  and  $r$  are trapped if  $a(v, r) < 0$ . Explicit conditions on initial data with  $a > 0$  are specified which guarantee that a trapped surface forms. The evolution of the solutions is analyzed after the formation of a trapped surface, and it is shown that the generator of the event horizon is future complete. The structure of the initial data is similar to what was explained above. Finally, by studying a situation where a shell of Vlasov matter is sent into a black hole solutions are found where apparent horizon and event horizon do not coincide. In textbooks the latter phenomenon is usually illustrated by Vaidya spacetimes where so-called null dust is used as an ad-hoc matter model.

We conjecture that for the class of data investigated in [5] a curvature singularity arises at the center, and strong cosmic censorship holds. Numerical simulations support this conjecture.

#### REFERENCES

- [1] H. Andréasson, M. Kunze, G. Rein, *The formation of black holes in spherically symmetric gravitational collapse*, preprint 2008, arXiv: 0706.3787.
- [2] H. Andréasson, M. Kunze, G. Rein, *Gravitational collapse and the formation of black holes for the spherically symmetric Einstein-Vlasov system*, Quarterly of Appl. Math., to appear, arXiv: 0812.1645.
- [3] H. Andréasson, G. Rein, *A numerical investigation of the stability of steady states and critical phenomena for the spherically symmetric Einstein-Vlasov system*, Class. Quantum Grav. **23** (2006), 3659–3677.

- [4] H. Andréasson, G. Rein, *The asymptotic behaviour in Schwarzschild time of Vlasov matter in spherically symmetric gravitational collapse*, Math. Proc. Camb. Phil. Soc., to appear, arXiv: 0902.4097.
- [5] H. Andréasson, G. Rein, *Formation of trapped surfaces for the spherically symmetric Einstein-Vlasov system*, preprint 2009.
- [6] D. Christodoulou, *On the global initial value problem and the issue of singularities*, Class. Quantum Gravity **16** (1999), A23–A35.
- [7] P.-L. Lions, B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system*, Invent. Math. **105** (1991), 415–430.
- [8] I. Olabarrieta, M. W. Choptuik, *Critical phenomena at the threshold of black hole formation for collisionless matter in spherical symmetry*, Phys. Rev. D. **65** (2002), 024007.
- [9] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data*, J. Differential Equations **95** (1992), 281–303.
- [10] G. Rein, *Collisionless Kinetic Equations from Astrophysics—The Vlasov-Poisson System*, Handbook of Differential Equations, Evolutionary Equations. Vol. 3. Eds. C. M. Dafermos and E. Feireisl, Elsevier (2007), 383–476.
- [11] G. Rein, A. D. Rendall, *Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data*, Comm. Math. Phys. **150** (1992), 561–583.
- [12] G. Rein, A. D. Rendall, *Smooth static solutions of the spherically symmetric Vlasov-Einstein system*, Ann. de l’Inst. H. Poincaré, Physique Théorique **59** (1993), 383–397.
- [13] G. Rein, A. D. Rendall, *Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics*, Math. Proc. Camb. Phil. Soc. **128** (2000), 363–380.
- [14] G. Rein, A. D. Rendall, J. Schaeffer, *A regularity theorem for solutions of the spherically symmetric Vlasov-Einstein system*, Comm. Math. Phys. **168** (1995), 467–478.
- [15] G. Rein, A. D. Rendall, J. Schaeffer, *Critical collapse of collisionless matter—a numerical investigation*, Phys. Rev. D **58** (1998), 044007.

## Non-existence of stationary two-black-hole configurations

JÖRG HENNIG

(joint work with Gernot Neugebauer)

This talk is meant to contribute to the present discussion about the existence or non-existence of stationary equilibrium configurations consisting of separate bodies at rest. In Newtonian theory it is a classical result that there exists no static  $n$ -body configuration (with bodies separated by a plane and with  $n > 1$ ). Recently, a similar statement was shown in the context of General relativity: Beig and Schoen [1] were able to prove a non-existence theorem for a reflectionally symmetric *static*  $n$ -body configuration.

Our intention is to involve the interaction of the angular momenta of rotating bodies (“spin-spin interaction”) which could generate repulsive effects compensating the omnipresent mass attraction. An interesting result in this direction is a generalization of the above mentioned theorem by Beig and Schoen that proves the non-existence of symmetric two-body configuration with *anti-aligned* spins, cf. [2]. However, motivated from post-Newtonian expansions one would expect *repulsive* spin-spin effects for bodies with *aligned* spins.

As a characteristic example for a stationary configuration with separated bodies we investigate the possibility of equilibrium between two aligned rotating axisymmetric black holes. We will present and review a chain of old and new arguments which finally forbid this equilibrium situation.

Interestingly, there exists an exact solution to the Einstein equations that was extensively discussed as a good candidate for two black holes in equilibrium [4, 6–13] — the *double-Kerr-NUT solution*, first investigated by Kramer and Neugebauer [9]. This solution is a particular case of a more general solution which was constructed by applying a  $N$ -fold *Bäcklund transformation*<sup>1</sup> to an arbitrary seed solution [14]. The double-Kerr-NUT solution can be obtained as the special case of a two-fold ( $N = 2$ ) Bäcklund transformation applied to Minkowski spacetime [15]. Since a single Bäcklund transformation generates the Kerr-NUT solution that contains, by a special choice of its parameters, the Kerr black hole solutions and since Bäcklund transformations act as a “nonlinear superposition principle”, the double-Kerr-NUT solution was considered to be a good candidate for the solution of the two-horizon problem.

However, there was no argument requiring that this particular solution be the *only* candidate. In this talk we will remove this objection and show that the discussion of a boundary value problem for two separate horizons necessarily leads to (a subclass of) the double-Kerr-NUT family of solution. For that purpose, we utilize another soliton method — the *inverse scattering method*. Hereby, an associated linear problem is analyzed, whose integrability conditions are equivalent to the non-linear field equations in axisymmetry and stationarity, see [14]. Details of the solution of the boundary value problem can be found in [16, 17].

In order to investigate whether the double-Kerr-NUT solution really describes the desired equilibrium between two black holes, we test physical inequalities which have to be satisfied for reasonable black hole spacetimes. To this end we study equilibrium configurations containing sub-extremal black holes (defined by existence of trapped surfaces in every sufficiently small interior neighborhood of the event horizon, cf. [3]) and degenerate black holes (defined by vanishing surface gravity  $\kappa$ ). In particular, we analyze the following configurations:

- (1) *Configurations with two sub-extremal black holes:*

As shown in [5], every sub-extremal black hole satisfies the inequality  $8\pi|J| < A$  between angular momentum  $J$  and horizon area  $A$ . However, the explicit formulae for angular momenta and horizon areas of the two objects described by the double-Kerr-NUT solution show that at least one of these objects violates this inequality. Hence we conclude that an equilibrium between two sub-extremal black holes is impossible, see [17].

- (2) *Configurations with two degenerate black holes:*

In this case, the ADM mass  $M$  of the double-Kerr-NUT solution turns out to be negative — a contradiction to the positive mass theorem. Therefore, also this equilibrium configuration cannot exist.

---

<sup>1</sup>The Bäcklund transformation is a particular method from soliton theory that creates new solutions to nonlinear equations from a previously known one.

- (3) *Configurations with one degenerate and one sub-extremal black hole:*  
 Together with the inequality  $8\pi|J| < A$  for the sub-extremal black hole, the ADM mass of the spacetime can be estimated. The result is again a negative total mass,  $M < 0$ , i.e. also this configuration cannot be in equilibrium.

Therefore, we arrive at the conclusion that physically reasonable two-black-hole equilibrium configurations (containing sub-extremal or degenerate black holes) do not exist.

## REFERENCES

- [1] R. Beig and R. M. Schoen, *On static n-body configurations in relativity*, Class. Quantum Grav. **26** (2009), 075014.
- [2] R. Beig, G. W. Gibbons and R. M. Schoen, *Gravitating opposites attract*, Class. Quantum Grav. **26** (2009), 225013.
- [3] I. Booth and S. Fairhurst, *Extremality conditions for isolated and dynamical horizons*, Phys. Rev. D **77** (2008), 084005.
- [4] W. Dietz G. Hoenselaers, *Two mass solution of Einstein's vacuum equations: The double Kerr solution*, Ann. Phys. **165** (1985), 319.
- [5] J. Hennig, M. Ansorg and C. Cederbaum, *A universal inequality between the angular momentum and horizon area for axisymmetric and stationary black holes with surrounding matter*, Class. Quantum Grav. **25** (2008), 162002.
- [6] C. Hoenselaers and W. Dietz, Talk given at the GR10 meeting, Padova (1983).
- [7] C. Hoenselaers, *Remarks on the Double-Kerr-Solution*, Prog. Theor. Phys. **72** (1984), 761.
- [8] M. Kihara and A. Tomimatsu, *Some properties of the symmetry axis in a superposition of two Kerr solutions*, Prog. Theor. Phys. **67** (1982), 349.
- [9] D. Kramer and G. Neugebauer, *The superposition of two Kerr solutions*, Phys. Lett. A **75** (1980), 259.
- [10] D. Kramer, *Two Kerr-NUT constituents in equilibrium*, Gen. Relativ. Gravit. **18** (1986), 497.
- [11] G. Krenzer, *Schwarze Löcher als Randwertprobleme der axisymmetrisch-stationären Einstein-Gleichungen*, PhD Thesis, University of Jena (2000).
- [12] V. S. Manko, E. Ruiz and J. D. Sanabria-Gómez, *Extended multi-soliton solutions of the Einstein field equations: II. Two comments on the existence of equilibrium states*, Class. Quantum Grav. **17** (2000), 3881.
- [13] V. S. Manko and E. Ruiz, *Exact solution of the double-Kerr equilibrium problem*, Class. Quantum Grav. **18** (2001), L11.
- [14] G. Neugebauer, *A general integral of the axially symmetric stationary Einstein equations*, J. Phys. A **13** (1980), L19.
- [15] G. Neugebauer, *Recursive calculation of axially symmetric stationary Einstein fields*, J. Phys. A **13** (1980), 1737.
- [16] G. Neugebauer and R. Meinel, *Progress in relativistic gravitational theory using the inverse scattering method*, J. Math. Phys. **44** (2003), 3407.
- [17] G. Neugebauer and J. Hennig, *Non-existence of stationary two-black-hole configurations*, Gen. Relativ. Gravit. **41** (2009), 2113.

**Global well-posedness of the Cauchy problem for a super-critical  
nonlinear wave equation in 2 space dimensions**

MICHAEL STRUWE

In [2] Ibrahim, Majdoub, and Masmoudi demonstrated that the initial value problem for the equation

$$(1) \quad u_{tt} - \Delta u + ue^{u^2} = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2$$

is well-posed for smooth Cauchy data

$$(2) \quad (u, u_t)|_{t=0} = (u_0, u_1)$$

with initial energy

$$(3) \quad E(u(0)) = \int_{\mathbb{R}^2} e(u(0)) dx \leq 2\pi,$$

where

$$(4) \quad e(u) = \frac{1}{2}(|u_t|^2 + |\nabla u|^2 + e^{u^2} - 1).$$

Equation (1) is related to the critical Sobolev embedding in 2 space dimensions. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ . Recall the Moser-Trudinger inequality

$$(5) \quad \sup_{u \in H_0^1(\Omega); \|\nabla u\|_{L^2(\Omega)}^2 \leq 1} \int_{\Omega} e^{4\pi u^2} dx < \infty;$$

see [6], [12]. The exponent  $\alpha = 4\pi$  is critical for this Orlicz space embedding in the sense that for any  $\alpha > 4\pi$  there holds

$$(6) \quad \sup_{u \in H_0^1(\Omega); \|\nabla u\|_{L^2(\Omega)}^2 \leq 1} \int_{\Omega} e^{\alpha u^2} dx = \infty.$$

On account of the obvious scaling property

$$(7) \quad \sup_{u \in H_0^1(\Omega); \|\nabla u\|_{L^2(\Omega)}^2 = 1} \int_{\Omega} e^{\alpha u^2} dx = \sup_{u \in H_0^1(\Omega); \|\nabla u\|_{L^2(\Omega)}^2 = \alpha} \int_{\Omega} e^{u^2} dx$$

and in view of (5), (6) the Cauchy problem for (1) with initial energy  $E(u(0)) < 2\pi$  then may be regarded as “*sub-critical*”, while the cases  $E(u(0)) = 2\pi$  or  $E(u(0)) > 2\pi$  may be termed “*critical*” or “*super-critical*”, respectively.

The work [2] of Ibrahim, Majdoub, and Masmoudi thus shows that the Cauchy problem for equation (1) is well-posed in the sub-critical and critical regimes, as one might conjecture in view of the known results for nonlinear wave equations

$$(8) \quad u_{tt} - \Delta u + u|u|^{p-2} = 0 \text{ on } \mathbb{R} \times \mathbb{R}^n$$

in  $n \geq 3$  space dimensions where well-posedness was shown to hold for  $p \leq \frac{2n}{n-2}$ ; see for instance the Notes in [7] for a brief survey and references.

It therefore may seem quite surprising that in the case of equation (1) the restriction (3) on the size of the initial data is unnecessary, at least in the case of radially symmetric data. Indeed, in [10] we establish the following result.

**Theorem 1.** *For any radially symmetric data  $(u_0, u_1) = (u_0(|x|), u_1(|x|)) \in C^\infty(\mathbb{R}^2)$  there exists a unique, smooth solution  $u = u(t, |x|)$  to the Cauchy problem (1), (2), defined for all time.*

Our method can also be applied to obtain global regularity of radially symmetric solutions to nonlinear wave equations (8) in higher dimensions. In particular, when  $n = 3$  we obtain a new proof of global well-posedness of (8) in the critical case  $p = 6$ , first established in [9]. However, it is not clear how the method might be adapted to the case when  $p > 6$ . With some luck, it might be possible to deal with logarithmically super-critical problems as treated in recent work of Tao [11].

On the other hand the recent results [1], [3] of Ibrahim, Jrad, Majdoub, and Masmoudi show that the local solution of the Cauchy problem (1), (2) in general does not depend continuously on the initial data in the energy norm when  $E(u(0)) > 4\pi$ . In this respect then equation (1) is similar to nonlinear wave equations (8) with super-critical nonlinearities in dimensions  $n \geq 3$ .

It is therefore not clear whether Theorem 1 may be extended also to non-symmetric data, even though it seems that a singularity would most likely appear in the radially symmetric case. Perhaps the recent work [4] of Ibrahim, Majdoub, Masmoudi, and Nakanishi on the scattering behavior of solutions to (1) with Cauchy data satisfying (3), or the references [1], [3], and [5] can help provide further intuition for this problem and with regard to the issue of well-posedness and ill-posedness of super-critical wave equations in general.

#### REFERENCES

- [1] S. Ibrahim, R. Jrad: *On well and ill posedness of an energy critical 2d Klein-Gordon equation in bounded domains*, preprint (2009).
- [2] S. Ibrahim, M. Majdoub, N. Masmoudi: *Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity*, Comm. Pure Appl. Math. 59 (2006), no. 11, 1639–1658.
- [3] S. Ibrahim, M. Majdoub, N. Masmoudi: *Well and ill-posedness issues for energy supercritical waves*, preprint (2009).
- [4] S. Ibrahim, M. Majdoub, N. Masmoudi, K. Nakanishi: *Scattering for the 2D energy critical wave equation*, Duke Math. J. (to appear).
- [5] G. Lebeau: *Non linear optic and supercritical wave equation*, Bull. Soc. Roy. Sci. Liège 70 (2001), no. 4-6, 267–306.
- [6] J. Moser: *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.
- [7] J. Shatah, M. Struwe: *Geometric wave equations*, Courant Lecture Notes 2, New York University (1998), 2nd edition: AMS (2000).
- [8] J. Shatah, A.S. Tahvildar-Zadeh: *Regularity of harmonic maps from the Minkowski space into rotationally symmetric manifolds*, Comm. Pure Appl. Math. 45 (1992), 947–971.
- [9] M. Struwe: *Globally regular solutions to the  $u^5$  Klein-Gordon equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 15 (1988), no. 3, 495–513 (1989).
- [10] M. Struwe: *Global well-posedness of the Cauchy problem for a super-critical nonlinear wave equation in 2 space dimensions*, preprint (2009).
- [11] T. Tao: *Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data*, preprint (2006), arXiv:math/0606145v1.
- [12] N.S. Trudinger: *On embeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.

## Late-time tails of self-gravitating waves

PIOTR BIZOŃ

The work presented in this talk is part of a long-time joint project with Tadek Chmaj and Andrzej Rostworowski aimed at the detailed quantitative description of the process of relaxation to equilibrium for nonlinear wave equations defined on spatially unbounded manifolds. By equilibrium we mean here a stable stationary solution, like a soliton, a black hole, or just a flat space. The convergence to these solutions occurs through a mechanism of radiating an excess energy to infinity. For a large class of physically interesting systems the late stages of this process are universal: for intermediate times the convergence has the form of exponentially damped oscillations (called quasinormal modes) and asymptotically it has the form of polynomial decay (called a tail). This very last stage of the relaxation process, the tail, is the subject of my talk. The key point which I wish to emphasize here is that, in general, the tail is a strictly nonlinear phenomenon. Before showing this for Einstein's equations, let me illustrate this point with a simple example of a radial wave equation with a potential and power nonlinearity

$$(1) \quad \phi_{tt} - \phi_{rr} - \frac{2}{r}\phi_r + V(r)\phi \pm |\phi|^{p-1}\phi = 0.$$

For positive small potential  $V(r) \sim r^{-\alpha}$  as  $r \rightarrow \infty$  ( $\alpha > 2$ ) and  $p > 1 + \sqrt{2}$  it is well known that solutions starting from small, smooth, compactly supported initial data exist globally in time and scatter. It is not so well known that the asymptotic behavior of solutions for  $t \rightarrow \infty$  ( $r = \text{const}$ ) is [1]

$$(2) \quad \phi(t, r) \sim t^{-\gamma}, \quad \gamma = \min\{\alpha, p - 1\}.$$

It follows from (2) that linearization yields the sharp decay rate only if  $p > \alpha + 1$ . Otherwise, the late-time behavior is inherently nonlinear, a fact frequently overlooked in physics literature.

Now, I will show that a similar situation arises for late-time tails of self-gravitating matter fields. First, let us consider the spherically symmetric Einstein-massless scalar field system

$$(3) \quad G_{\alpha\beta} = 8\pi \left( \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\nabla_\mu \phi \nabla^\mu \phi) \right), \quad g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0.$$

This toy-model of gravitational collapse has been intensively studied in the past leading to valuable insights about the validity of the weak cosmic censorship and no-hair conjectures. In particular, Christodoulou proved that there are two generic endstates of evolution: Minkowski spacetime for small initial data [2] and Schwarzschild black hole for large initial data [3]. For dispersive solutions Christodoulou showed that  $\phi(t, r) \leq Ct^{-3}$ , while for collapsing solutions Dafermos and Rodnianski showed that  $\phi(t, r) \leq Ct^{-3+\epsilon}$  [4]. The first reliable numerical simulations of the late-time asymptotics of this relaxation process have been done by Gundlach, Price, and Pullin (GPP) [5]. They found that, regardless of the end-state of evolution, the scalar field develops a tail which falls off as  $t^{-3}$  near timelike infinity (for compactly supported initial data). Since this decay rate coincides with

the decay rate for the linear wave equation on the Schwarzschild background (so called Price's law [6]), GPP suggested that the predictions of linearized theory might apply in a nonlinear regime. It turns out that this interpretation is too naive and in actual fact the above coincidence of decay rates is accidental. To substantiate this claim in the case of small initial data  $(\phi, \phi_t)_{t=0} = \varepsilon(f(r), g(r))$  we computed the late-time tails (both the decay rate and the amplitude) in  $d = 2L + 3$  space dimensions using nonlinear perturbation expansion [7]. The result, obtained by elementary methods, reads

$$(4) \quad \phi(t, r) \sim \varepsilon^3 \frac{\Gamma_L}{t^{3L+3}},$$

where the coefficient  $\Gamma_L$  is determined explicitly in terms of functions  $(f(r), g(r))$ . For small values of  $\varepsilon$  this third-order approximation is in excellent agreement with the results of numerical integration of the full Einstein-scalar field equations.

As mentioned above, for  $L = 0$  (that is,  $d = 3$ ) the decay rate of the tail (4) agrees with Price's tail. However, this agreement is lost in higher dimensions ( $L \geq 1$ ) where Price's tail behaves as  $t^{-(6L+4)}$  [8]. Although the formula (4) was derived only for small dispersive solutions, the same mechanism is at work for collapsing solutions. In this case the tail has two components: the linear one coming from the backscattering on the black hole potential and the nonlinear one. For example, in six spacetime dimensions for small perturbations of Schwarzschild we have  $\phi(t, r) \sim A\varepsilon/t^{10} + B\varepsilon^3/t^6$  (where  $A, B$  are constants depending on initial data), hence we observe the crossover from the linear to the nonlinear tail at time  $t \sim \varepsilon^{1/2}$ .

In [9] we studied the analogous problem for wave maps which are a natural geometric generalization of the wave equation for the massless scalar field. This generalization seems interesting because in the so called equivariant case the homotopy index  $\ell$  of the map plays the role similar to the multipole index for spherical harmonics. However, in contrast to the decomposition of a scalar field into spherical harmonics, which makes sense only at the linearized level, it is consistent to study nonlinear evolution for the wave map within a fixed equivariance class. In this sense  $\ell$ -equivariant self-gravitating wave maps can serve as a poor man's toy-model of non-spherical collapse. The  $\ell = 0$  case reduces to the spherically symmetric massless scalar field described above. For  $\ell \geq 1$  we showed that for small compactly supported initial data the late-time tail of the self-gravitating  $\ell$ -equivariant wave map decays as  $t^{-(2\ell+2)}$  at timelike infinity. Note that this decay is by power slower than Price's tail  $t^{-(2\ell+3)}$  for higher multipoles, which can be viewed as another example of the inapplicability of linearized theory in the analysis of radiative relaxation processes.

I would like to emphasize that our results by no means diminish the importance of a recent flurry of results on late-time asymptotics of the linear wave equation on a fixed background (cf. talks by Mihalis Defermos, Peter Blue, and Daniel Tataru). Good understanding of the linear problem is a necessary first step in iteration-type proofs of nonlinear stability of the background spacetime. Our work shows only

that in general is not correct to draw conclusions about the late-time asymptotics of the nonlinear problem on the basis of linearization.

#### REFERENCES

- [1] W.A. Strauss and K. Tsutaya, *Discrete Contin. Dynam. Systems* **3**, 175 (1997)
- [2] D. Christodoulou, *Commun. Math. Phys.* **105**, 337 (1986).
- [3] D. Christodoulou, *Comm. Pure Appl. Math.* **44**, 339 (1991).
- [4] M. Dafermos and I. Rodnianski, *Invent. Math.* **162**, 381 (2005).
- [5] C. Gundlach, R. Price, and J. Pullin, *Phys. Rev. D* **49**, 890 (1994)
- [6] R.H. Price, *Phys. Rev. D* **5**, 2419 (1972).
- [7] P. Bizoń, T. Chmaj, and A. Rostworowski, *Class. Quantum Grav.* **26**, 175006 (2009)
- [8] P. Bizoń, T. Chmaj, and A. Rostworowski, *Phys. Rev. D* **76**, 124035 (2007).
- [9] P. Bizoń, T. Chmaj, A. Rostworowski, and S. Zając, *Class. Quantum Grav.* **26**, 225015 (2009)

### **Null Asymptotics of Solutions of the Einstein-Maxwell Equations in General Relativity and Gravitational Radiation**

LYDIA BIERI

A major goal of mathematical General Relativity (GR) and astrophysics is to precisely describe and finally observe gravitational radiation, one of the predictions of GR. In order to do so, one has to study the null asymptotical limits of the spacetimes for typical sources. Among the latter we find binary neutron stars and binary black hole mergers. In these processes typically mass and momenta are radiated away in form of gravitational waves. Among the pioneering papers, we find [3], [4], [5], [6], [10].

D. Christodoulou showed that every gravitational-wave burst has a nonlinear memory [8]. Before that such an effect had already been studied in linearized theory [5], [6], [10], but it was so small, that for typical sources it would be negligible. In [8] Christodoulou proved that this effect is much bigger in the nonlinear theory, that is the nonlinear memory effect can in principle be observed. The insights of this work are based on the precise description of null infinity obtained by D. Christodoulou and S. Klainerman in [9] (see also [7]). Among the many pioneering results they derived the Bondi mass loss formula. This is all in the regime of the Einstein vacuum equations. N. Zipser studied the Einstein-Maxwell equations and computed limits along the lines of Christodoulou and Klainerman for this case [11], [12]. She derived a Bondi mass formula in the EM case. In this talk, we discuss the null asymptotics for spacetimes solving the Einstein-Maxwell (EM) equations, compute the radiated energy and derive limits at null infinity and compare them with the Einstein vacuum (EV) case. Here, we rely on the methods introduced in the works of Christodoulou and Klainerman [9], Bieri [1], [2] and Zipser [11], [12].

The nonlinear memory effect, that is the permanent displacement of the test masses of a laser interferometer detector is governed by

$$\Sigma^+ - \Sigma^- = \frac{1}{2} \int_{-\infty}^{\infty} \Xi(u) du$$

See [8] for its derivation. Here,  $\Sigma$  denotes the asymptotic shear of outgoing null hypersurfaces  $C_u$  and  $\Sigma^+$  and  $\Sigma^-$  are limits of  $\Sigma$  as  $u$  tends to  $+\infty$  and  $-\infty$ , respectively. And  $\Xi$  is the (weighted) limit of the trace-free part of the conjugate null second fundamental form of a closed spacelike surface  $S$  in spacetime.

Whereas in the EV case [9], [8] the total energy  $\frac{F}{4\pi}$  radiated to infinity in a given direction per unit solid angle is obtained from

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} |\Xi|^2 du ,$$

in the EM case  $F$  takes the form

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} \left( |\Xi|^2 + \frac{1}{2} |A_F|^2 \right) du .$$

with  $A_F$  denoting a component of the electromagnetic field. We investigate the nonlinear memory effect in the presence of an electromagnetic field.

As for the gravitational wave experiment, considering the Jacobi equation, we show that a component of the electromagnetic field comes in but only at lower order. We derive the formulae for the EM case and compare them with the EV situation.

#### REFERENCES

- [1] L. Bieri. *An Extension of the Stability Theorem of the Minkowski Space in General Relativity*. ETH Zurich, Ph.D. thesis. **17178**. Zurich. (2007).
- [2] L. Bieri. *Extensions of the Stability Theorem of the Minkowski Space in General Relativity. Solutions of the Einstein Vacuum Equations*. AMS-IP. **45**. Studies in Advanced Mathematics. Cambridge. MA. (2009).
- [3] H. Bondi. *Nature*. **186**. (1960). 535.
- [4] H. Bondi. M.G.J. van Burg, A.W.K. Metzner. *Proc.Roy.Soc. London A* **269**. (1962). 21-48.
- [5] V.B. Braginsky, L.P. Grishchuk. *Zh.Eksp.Teor.Fiz* **89**. (1985). 744. [*Sov.Phys. JETP* **62**. (1986). 427.]
- [6] V.B. Braginsky, K.S. Thorne. *Nature*. (London) **327**. (1987). 123.
- [7] D. Christodoulou. *The stability of Minkowski spacetime*. Surveys in Diff. Geom. VI. Int. Press. Boston. MA. (1999). 365-385.
- [8] D. Christodoulou. *Nonlinear Nature of Gravitation and Gravitational-Wave Experiments*. *Phys.Rev.Letters*. **67**. (1991). no.12. 1486-1489.
- [9] D. Christodoulou, S. Klainerman. *The global nonlinear stability of the Minkowski space*. Princeton Math.Series **41**. Princeton University Press. Princeton. NJ. (1993).
- [10] Ya.B. Zel'dovich. A.G. Polnarev. *Astron.Zh.* **51**. (1974). 30. [*Sov.Astron.* **18**. (1974). 17.]
- [11] N. Zipser. *The Global Nonlinear Stability of the Trivial Solution of the Einstein-Maxwell Equations*. Ph.D. thesis. Harvard Univ. Cambridge MA. (2000).

- [12] N. Zipser. *Extensions of the Stability Theorem of the Minkowski Space in General Relativity. - Solutions of the Einstein-Maxwell Equations.* AMS-IP. **45**. Studies in Advanced Mathematics. Cambridge, MA. (2009).

## A Wave Coordinate Approach to the Stability of the Irrotational Euler-Einstein System with a Positive Cosmological Constant

JARED SPECK

(joint work with Igor Rodnianski)

### 1. INTRODUCTION

The irrotational Euler-Einstein system models the evolution of a dynamic spacetime<sup>1</sup>  $(\mathcal{M}, g)$  containing a perfect fluid with vanishing vorticity. In this article, we endow this system with a positive cosmological constant  $\Lambda$  and consider the equation of state  $p = c_s^2 \rho$ , where  $p \geq 0$  is the fluid *pressure*,  $\rho \geq 0$  is the *proper energy density*, and the non-negative constant  $c_s$  is the *speed of sound*. Under these assumptions, the irrotational Euler-Einstein system comprises the equations<sup>2</sup>

$$(1) \quad R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}^{(scalar)},$$

$$(2) \quad D_\alpha(\sigma^s D^\alpha \Phi) = 0,$$

where  $g_{\mu\nu}$  is the *spacetime metric*,  $R_{\mu\alpha\nu}{}^\beta$  is the *Riemann curvature tensor*<sup>3</sup>,  $R_{\mu\nu} \stackrel{\text{def}}{=} R_{\mu\alpha\nu}{}^\alpha$  is the *Ricci curvature tensor*,  $R = g^{\alpha\beta}R_{\alpha\beta}$  is the *scalar curvature*,  $\Phi$  is the *fluid potential*<sup>4</sup>,  $T_{\mu\nu}^{(scalar)} = 2\sigma^s(\partial_\mu\Phi)(\partial_\nu\Phi) + g_{\mu\nu}(1+s)^{-1}\sigma^{s+1}$  is the energy-momentum tensor of the irrotational fluid,  $\sigma = -g^{\alpha\beta}(\partial_\alpha\Phi)(\partial_\beta\Phi) \geq 0$  is the *enthalpy per particle*, and  $s = (1 - c_s^2)/(2c_s^2)$ . The fundamental unknowns are  $(\mathcal{M}, g, \partial\Phi)$ , while the pressure and proper energy density can be expressed as  $p = \frac{1}{s+1}\sigma^{s+1}$ ,  $\rho = \frac{2s+1}{s+1}\sigma^{s+1}$ . We remark that since  $\Phi$  itself never directly enters into the equations, we may consider  $\partial\Phi$ , the spacetime gradient of  $\Phi$ , to be the unknown.

<sup>1</sup>By spacetime, we mean a 4-dimensional manifold  $\mathcal{M}$  together with a Lorentzian metric  $g_{\mu\nu}$  on  $\mathcal{M}$  of signature  $(-, +, +, +)$ .

<sup>2</sup>See [3] for details concerning the fluid potential formulation of the irrotational relativistic Euler equations.

<sup>3</sup>In this article, Greek indices  $\alpha, \beta, \dots$  take on the values 0, 1, 2, 3, while Latin indices  $a, b, \dots$  take on the values 1, 2, 3. Pairs of adjacent, repeated indices, with one raised and one lowered, are summed (from 0 to 3 if they are Greek, and from 1 to 3 if they are Latin). We raise and lower indices with the spacetime metric  $g_{\mu\nu}$  and its inverse  $g^{\mu\nu}$ . In our wave coordinate systems, which are introduced in Section 3,  $x^0 = t \in \mathbb{R}$  will denote the “time” coordinate, and  $(x^1, x^2, x^3)$  will denote the standard “spatial” coordinates on  $\mathbb{T}^3 \stackrel{\text{def}}{=} [-\pi, \pi]^3$ .  $D$  denotes the covariant derivative induced by  $g_{\mu\nu}$ , and  $\partial_\mu = \frac{\partial}{\partial x^\mu}$ .

<sup>4</sup>In [3], Christodoulou refers to  $\Phi$  as the *wave function*.

In a future article, we will prove the future global nonlinear stability of a family of Friedmann-Lemaître-Robertson-Walker (FLRW) type “background” solutions to the system (1) - (2) under the assumption  $0 < c_s < \sqrt{1/3}$ . In this article, we provide an introduction to some of the main ideas of our proof.

### 2. BACKGROUND SOLUTIONS

We now introduce the family of FLRW type solutions which, under the assumption  $0 < c_s < \sqrt{1/3}$ , will be shown to be future stable in a later publication. Using an ODE ansatz (see e.g. Chapter 5 of [12]), one can show that the system (1) - (2) has the following background solutions on the manifold<sup>5</sup>  $\mathcal{M} = (-\infty, \infty) \times \mathbb{T}^3$ , where  $\mathbb{T}^3 \stackrel{\text{def}}{=} [-\pi, \pi]^3$  with the ends identified:

$$(3) \quad \tilde{g}(t) = -dt^2 + a(t)^2 \sum_{a=1}^3 (dx^a)^2, \quad \partial_t \tilde{\Phi}(t) = \bar{\Psi} e^{-W\Omega(t)}, \quad \partial_j \tilde{\Phi}(t) = 0,$$

where  $a(t) \sim e^{(\sqrt{\frac{\Lambda}{3}})t}$  is the solution to  $\dot{a} = a\sqrt{\frac{\Lambda}{3} + \frac{\bar{\rho}}{3a^3(1+c_s^2)}}$  with  $a(0) > 0$ , the constant  $\bar{\rho} > 0$  is the initial proper energy density of the fluid,  $\bar{\Psi} = \left(\bar{\rho} \frac{s+1}{2s+1}\right)^{1/(2s+2)}$ ,  $\Omega(t) \stackrel{\text{def}}{=} \ln(a(t))$ , and  $W \stackrel{\text{def}}{=} \frac{3}{2s+1} = 3c_s^2$ .

### 3. THE INITIAL VALUE PROBLEM FORMULATION AND THE NOTION OF STABILITY

In order to discuss the notion of stability, we need to introduce an initial value problem formulation of the Einstein equations. Although the existence of such a formulation is well understood by now, it remains a decidedly subtle issue. In particular, because of their diffeomorphism invariance, the hyperbolic nature of the Einstein equations does not become apparent until one makes a gauge choice. This difficulty was first resolved in the seminal work [2] by Choquet-Bruhat, who showed the existence of a *wave coordinate* system in which  $\Gamma^\mu \stackrel{\text{def}}{=} g^{\alpha\beta} \Gamma_{\alpha\beta}^\mu \equiv 0$ . In our analysis of the irrotational Euler-Einstein system, we will use a version of the wave coordinate condition that is similar to the one used by Ringström in [11]. That is, we work in a coordinate system in which

$$(4) \quad \Gamma^\mu \equiv \tilde{\Gamma}^\mu = 3\omega\delta_0^\mu,$$

where the  $\tilde{\Gamma}^\mu = 3\omega\delta_0^\mu$  are the contracted Christoffel symbols of the background solution metric, and  $\omega(t) = \frac{d}{dt}\Omega(t)$ . As discussed in Section 5, in our future publication, we will use identities valid in a wave coordinate system to derive a modified hyperbolic system that (in a wave coordinate system) is equivalent to (1) - (2), and that features energy-dissipating terms.

---

<sup>5</sup>Technically, the term “FLRW” is usually reserved for a class of solutions that have spatial slices diffeomorphic to  $\mathbb{S}^3$ ,  $\mathbb{R}^3$ , or hyperbolic space (see [12]).

The initial data for (1) - (2) consist of a 3-dimensional manifold  $\Sigma$  together with the following fields on  $\Sigma$  : a Riemannian metric  $\bar{g}$ , a covariant two-tensor<sup>6</sup>  $\bar{K}$ , and the functions  $\bar{\partial}\bar{\Phi}, \bar{\Psi}$ , which are the tangential and normal derivatives along  $\Sigma$  of the fluid potential  $\Phi$ . The initial value problem for the Einstein equations is overdetermined; the data are subject to the *Gauss* and *Codazzi* constraints<sup>7</sup>:

$$(5) \quad \bar{R} - \bar{K}_{ab}\bar{K}^{ab} + (\bar{g}^{ab}\bar{K}_{ab})^2 = 2T_{00}^{(scalar)}|_{\Sigma},$$

$$(6) \quad \bar{D}^a\bar{K}_{aj} - \bar{g}^{ab}\bar{D}_j\bar{K}_{ab} = T_{0j}^{(scalar)}|_{\Sigma},$$

where  $\bar{R}$  is the scalar curvature of  $\bar{g}$ , and  $\bar{D}$  is the covariant derivative induced by  $\bar{g}$ .

Our goal in our future paper is the following: to show that all sufficiently small perturbations of the data corresponding to the background solutions of Section 2 that satisfy the Gauss and Codazzi constraints launch solutions to the irrotational Euler-Einstein system with the following properties: **i)** in our wave coordinate system, the solutions exist for  $(t, x^1, x^2, x^3) \in [0, \infty) \times \mathbb{T}^3$ , **ii)** the spacetimes are future causally geodesically complete, and **iii)** the solutions converge in some sense as  $t \rightarrow \infty$ . Because of the inclusion of property **iii)**, this type of stability is called *asymptotic stability*. The fact that we will prove asymptotic stability is intimately connected to the fact that our proof will rely upon energy estimates and decay estimates. In the next two sections, we will see why such estimates are available for the modified system.

#### 4. THE MODEL PROBLEM

As a model problem, we consider the inhomogeneous wave equation  $g^{\alpha\beta}D_{\alpha}D_{\beta}v = F$  for the metric  $g = -dt^2 + e^{2t}\sum_{a=1}^3(dx^a)^2$  on  $[0, \infty) \times \mathbb{T}^3$ . The contracted Christoffel symbols of  $g$  are  $\Gamma^{\mu} = 3\delta_0^{\mu}$ , which implies that relative to this coordinate system, the wave equation can be written as follows:

$$(7) \quad -\partial_t^2 v + e^{-2t}\delta^{ab}\partial_a\partial_b v = 3(\partial_t v)^2 + F.$$

To estimate solutions to (7), we define the energy  $E^2(t) = \frac{1}{2}\int_{\mathbb{T}^3}(\partial_t v)^2 + e^{-2t}\delta^{ab}(\partial_a v)(\partial_b v)d^3x$ , and after an application of integration by parts and the Cauchy-Schwarz inequality, we find that

$$(8) \quad \frac{d}{dt}E \leq -E + \|F\|_{L^2}.$$

From (8), it is clear that sufficient estimates of  $\|F\|_{L^2}$  in terms of  $E$  (for example,  $\|F\|_{L^2} \leq CE$ , where  $C < 1$  is a constant) will lead to energy decay.

<sup>6</sup>The tensor  $\bar{K}$  is the *second fundamental form*.

<sup>7</sup>In the constraint equations, we raise and lower indices with the metric  $\bar{g}_{ab}$  and its inverse  $\bar{g}^{ab}$ .

## 5. THE MODIFIED IRROTATIONAL EULER-EINSTEIN SYSTEM

In our future publication, we will make use of algebraic identities that are valid in our wave coordinate system (see (4) for our wave coordinate condition) to construct a modified version of the irrotational Euler-Einstein system. The modified system has several key features. First, it comprises quasilinear wave equations and therefore is of hyperbolic character. Second, following Rinström [11], we will add gauge terms<sup>8</sup> to the system in order to produce an energy-dissipating effect that is analogous to the effect created by the  $3(\partial_t v)^2$  term on the right-hand side of the model equation (7). It is exactly these dissipation-inducing terms that will play a key role in our global existence argument.

## 6. COMPARISON WITH PREVIOUS WORK

The main precursor to our work is Ringström's article [11]. Using a wave coordinate system similar to the one in (4), he showed the future global nonlinear stability of a large class of solutions to the Einstein-non-linear scalar field system<sup>9</sup> featuring accelerated expansion. Moreover, he stated that one of his main motivations for producing the work [11] was that the wave coordinate framework is easy to adapt to handle various matter models. Our work can be viewed as an example of the robustness of his methods.

Next, we remark that the behavior of the fluid equation (2) on exponentially expanding backgrounds is very different than it is in flat spacetime. More specifically, Christodoulou's monograph [3] shows that on the Minkowski space background, shock singularities can form in solutions to the irrotational fluid equation arising from data that are arbitrarily close to that of a uniform quiet fluid state. Our original intuition for our work was that rapid spacetime expansion should smooth out the fluid and discourage the formation of shocks.

Finally, we note that Brauer, Rendall, and Reula have shown [1] a Newtonian analogue of our main result. More specifically, they studied Newtonian cosmological models<sup>10</sup> with a positive cosmological constant and with perfect fluid sources under the equation of state  $p = C\rho^\gamma$ , where  $\rho \geq 0$  is the density, and  $\gamma > 1$ . They showed that small perturbations of a uniform quiet fluid state of constant positive density lead to a global solution. It is of particular interest to note that they do not require the fluid to be irrotational. This suggests that our main result can be extended to allow for (small) non-vanishing vorticity. We will address this issue in another forthcoming article.

---

<sup>8</sup>That is, terms that are equal to 0 in wave coordinates.

<sup>9</sup>Although Ringström set  $\Lambda = 0$ , his scalar field  $\Phi$  was a solution to  $g^{\alpha\beta} D_\alpha D_\beta \Phi = V'(\Phi)$ , where  $V(0) > 0$ ,  $V'(0) = 0$ , and  $V''(0) > 0$ . In effect, the nonlinearity  $V(\Phi)$  emulates the presence of a positive cosmological constant.

<sup>10</sup>Their models were based on Newton-Cartan theory, which is a slight generalization of ordinary Newtonian gravitational theory that can be endowed with a highly geometric interpretation.

## REFERENCES

- [1] U. BRAUER, A. RENDALL, AND O. REULA, *The cosmic no-hair theorem and the non-linear stability of homogeneous Newtonian cosmological models*, Classical Quantum Gravity, 11 (1994), pp. 2283–2296.
- [2] Y. F. (CHOQUET)-BRUHAT, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta Mathematica, 88 (1952), pp. 141–225.
- [3] D. CHRISTODOULOU, *The formation of shocks in 3-dimensional fluids*, European Mathematical Society, Zürich, Switzerland, 2007.
- [4] ———, *Mathematical problems of general relativity. I*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.
- [5] D. CHRISTODOULOU AND S. KLAINERMAN, *The global nonlinear stability of the Minkowski space*, vol. 41 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993.
- [6] T. DE DONDER, *La gravifique einsteinienne*, Gauthier-Villars, Paris, 1921.
- [7] H. FRIEDRICH, *On the existence of  $n$ -geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure*, Comm. Math. Phys., 107 (1986), pp. 587–609.
- [8] ———, *On the global existence and the asymptotic behavior of solutions to the Einstein-Maxwell-Yang-Mills equations*, J. Differential Geom., 34 (1991), pp. 275–345.
- [9] H. LINDBLAD AND I. RODNIANSKI, *The global stability of the Minkowski space-time in harmonic gauge*, 2004.
- [10] H. LINDBLAD AND I. RODNIANSKI, *Global existence for the Einstein vacuum equations in wave coordinates*, Comm. Math. Phys., 256 (2005), pp. 43–110.
- [11] H. RINGSTRÖM, *Future stability of the Einstein-non-linear scalar field system*, Invent. Math., 173 (2008), pp. 123–208.
- [12] R. M. WALD, *General relativity*, University of Chicago Press, Chicago, IL, 1984.

## Structure of singularities in cosmological spacetimes with symmetry

JACQUES SMULEVICI

The study of the global Cauchy problem constitutes one of the main areas of research in mathematical relativity and is one of the most natural problems to investigate in view of the hyperbolicity of the Einstein equations and of the theorems concerning the local Cauchy problem<sup>1</sup>.

The results presented here are concerned with solutions of the vacuum Einstein equations or of the Einstein-Vlasov system<sup>2</sup>, arising from initial data of arbitrarily large size, but enjoying certain symmetries<sup>3</sup>, so as to reduce the equations for the geometrical part to certain systems of  $1 + 1$  wave equations.

The classes of symmetry addressed are the so-called  $T^2$ -symmetric and *surface-symmetric* spacetimes. The  $T^2$ -symmetric spacetimes constitute a class of solutions arising from initial data with spatial topology  $T^3$  and admitting a torus action. They contain as special subcases the  $T^3$ -Gowdy spacetimes and the polarized  $T^2$ -symmetric spacetimes. The surface-symmetric spacetimes constitute a class of solutions arising from initial data where the initial Riemannian 3-manifold

<sup>1</sup>See for instance [18] for a detailed exposition of the Cauchy problem in relativity.

<sup>2</sup>See [21] for an introduction to the Einstein-Vlasov system.

<sup>3</sup>As well as high regularity. See [23] for a precise description of the classes of initial data.

is given by a product  $S^1 \times \mathcal{S}$ , where  $\mathcal{S}$  is a compact 2-surface of constant curvature  $k$  and such that the rest of the initial data is invariant under the local isometries of  $\mathcal{S}$ . By rescaling,  $k$  may be taken as being  $-1, 0$  or  $+1$  and the different cases are known as hyperbolic, plane<sup>4</sup> or spherical symmetry.

In the case of  $T^2$ -symmetric or  $k \leq 0$  surface-symmetric spacetimes, the local geometry of the solution possesses the particular property that, unless the spacetime is flat, the symmetry orbits are either trapped or antitrapped [5, 19, 20]. If we denote by  $t$  the area of the symmetry orbits, this means that the gradient of  $t$  is everywhere timelike and that  $t$  may be used as a time coordinate. For the vacuum  $T^2$ -symmetric case with zero cosmological constant ( $\Lambda = 0$ ), the existence of a global areal foliation where  $t$  takes value in  $(t_0, \infty)$  with  $t_0 \geq 0$  was proven in [13]. The proof was then extended to the Vlasov case [11, 12] and to the case with  $\Lambda > 0$  [10]. Similarly, the existence of a global areal foliation for the surface-symmetric case with  $k = -1$ ,  $\Lambda = 0$  and Vlasov matter<sup>5</sup> was proven in [9] and extended to the case with  $\Lambda > 0$  in [7, 8].

It was already realized in [6] that in the vacuum  $T^3$ -Gowdy case with  $\Lambda = 0$ , one has  $t_0 = 0$  unless the spacetime is flat. The natural question arose: *Is  $t_0 = 0$  generically for all the possible cases?* This question is equivalent to the question of global existence on  $(0, \infty) \times S^1$  for the solutions of the reduced system of equations obtained by writing the Einstein equations in areal coordinates. The proofs that  $t_0 = 0$  generically for  $T^2$ -symmetric spacetimes with  $\Lambda = 0$ , in the vacuum or with Vlasov matter, were given in [4] and [3]. It has also been proven that  $t_0 = 0$  in the case of plane symmetric initial data with  $\Lambda = 0$  and Vlasov matter as well as in the case of plane or hyperbolic symmetric initial data with  $\Lambda \geq 0$  and Vlasov matter under an extra small data assumption [1, 2].

One motivation for the study of the value of  $t_0$  was the expectation that, in the cases where  $t_0 = 0$ , the curvature should in general blow up as  $t$  goes to 0, thus providing a proof of inextendibility (and thus of the strong cosmic censorship conjecture) for these cases. We refer to the introduction of [23] for a more detailed exposition of the relations between the values of  $t_0$  and the strong cosmic censorship conjecture.

In [23], the problem of the past asymptotic value of  $t$  for the remaining cases was resolved. More precisely, we proved the following global existence theorems:

**Theorem 1.** *Let  $(\mathcal{M}, g, f)$  be the maximal development of  $T^2$ -symmetric initial data with Vlasov matter and  $\Lambda \geq 0$ . Suppose that the Vlasov field  $f$  does not vanish identically. Then  $(\mathcal{M}, g)$  admits a global foliation by areal coordinates with the time coordinate  $t$  taking all values in  $(0, \infty)$ , i.e.  $t_0 = 0$ .*

Thus the presence of Vlasov matter forbids  $t_0 > 0$ . In the vacuum case, we know that non-flat solutions with  $t_0 > 0$  exist (see appendix E in [22]) which already indicates that this case is more difficult. We proved the following:

<sup>4</sup>Note that the plane symmetric case is a special case of  $T^3$ -Gowdy polarized solutions.

<sup>5</sup>Note that, in the surface-symmetric case, a result analogous to Birkhoff's theorem applies, by which we mean that these spacetimes have no dynamical degrees of freedom in the vacuum.

**Theorem 2.** *Let  $(\mathcal{M}, g)$  be the maximal Cauchy development of vacuum  $T^2$ -symmetric initial data with  $\Lambda > 0$  and suppose that the spacetime is not polarized. Then  $(\mathcal{M}, g)$  admits a global foliation by areal coordinates with the time coordinate  $t$  taking all values in  $(0, \infty)$ , i.e.  $t_0 = 0$ .*

Finally, a result analogous to Theorem 1 holds in the hyperbolic symmetric case:

**Theorem 3.** *Let  $(\mathcal{M}, g, f)$  be the maximal development of  $k = -1$  surface-symmetric initial data with Vlasov matter and  $\Lambda \geq 0$ . Suppose that the Vlasov field  $f$  does not vanish identically. Then  $(\mathcal{M}, g)$  admits a global foliation by areal coordinates with the time coordinate  $t$  taking all values in  $(0, \infty)$ , i.e.  $t_0 = 0$ .*

The proofs of these three theorems may be found in [23]. Let us just mention that the proofs of Theorems 1 and 3 follow the approaches developed in [3, 4], while, in the case of Theorem 3, a key *a priori* estimate no longer holds and forced us to introduce a new strategy. Key features of the proof of Theorem 3 include: a new blow up criterion, a hierarchisation of the equations exploiting the assumption of non-polarization, new *a priori* estimates for the geometry of the quotient of the spacetime by the orbits of symmetry and refined null cone estimates.

#### REFERENCES

- [1] S. B. TCHAPNDA, *Structure of solutions near the initial singularity for the surface-symmetric Einstein-Vlasov system*, Classical Quantum Gravity, 21 (2004), pp. 5333–5346.
- [2] G. REIN, *Cosmological solutions of the Vlasov-Einstein system with spherical, plane, and hyperbolic symmetry*, Math. Proc. Cambridge Philos. Soc. 119 (1996), p. 739.
- [3] M. WEAVER, *On the area of the symmetry orbits in  $T^2$ -symmetric spacetimes with Vlasov matter*, Classical Quantum Gravity, 21 (2004), pp. 1079–1097.
- [4] J. ISENBERG AND M. WEAVER, *On the area of the symmetry orbits in  $T^2$ -symmetric spacetimes*, Classical Quantum Gravity, 20 (2003), pp. 3783–3796.
- [5] P. T. CHRUSCIEL, *On space-times with  $U(1) \times U(1)$  symmetric compact Cauchy surfaces*, Ann. Physics, 202 (1990), pp. 100–150.
- [6] V. MONCRIEF, *Global properties of Gowdy spacetimes with  $T^3 \times \mathbb{R}$  topology*, Ann. Physics, 132 (1981), pp. 87–107.
- [7] S. B. TCHAPNDA AND N. NOUTCHEGUEME, *The surface-symmetric Einstein-Vlasov system with cosmological constant*, Math. Proc. Cambridge Philos. Soc. 138 (2005), pp. 541–553.
- [8] S. B. TCHAPNDA AND A. D. RENDALL, *Global existence and asymptotic behaviour in the future for the Einstein-Vlasov system with positive cosmological constant*, Classical Quantum Gravity, 20 (2003), pp. 3037–3049.
- [9] H. ANDRÉASSON, G. REIN, AND A. D. RENDALL, *On the Einstein-Vlasov system with hyperbolic symmetry*, Math. Proc. Cambridge Philos. Soc. 134 (2003), pp. 529–549.
- [10] A. CLAUSEN AND J. ISENBERG, *Areal foliation and asymptotically velocity-term dominated behavior in  $T^2$ -symmetric space-times with positive cosmological constant*, J. Math. Phys. 48 (2007), p. 082501.
- [11] H. ANDRÉASSON, A. D. RENDALL, AND M. WEAVER, *Existence of CMC and constant areal time foliations in  $T^2$ -symmetric spacetimes with Vlasov matter*, Comm. Partial Differential Equations, 29, no. 1-2 (2004), pp. 237–262.
- [12] H. ANDRÉASSON, *Global Foliations of Matter Spacetimes with Gowdy Symmetry*, Comm. Math. Phys. 206 (1999), pp. 337–365.
- [13] B. K. BERGER, J. ISENBERG, P. T. CHRUSCIEL, AND V. MONCRIEF, *Global foliations of vacuum spacetimes with  $T^2$  isometry*, Ann. Physics, 260 (1997), pp. 117–148.

- [14] Y. CHOQUET-BRUHAT, *Theorem of Uniqueness and Local Stability for Liouville-Einstein Equations*, J. Math. Phys. 11 (1970), pp. 3238–3243.
- [15] Y. CHOQUET-BRUHAT, *Problème de Cauchy pour le système intégral différentiel d'Einstein-Liouville*, Ann. Inst. Fourier, 21 (1971), pp. 181–201.
- [16] Y. CHOQUET-BRUHAT AND R. GEROCH, *Global aspects of the Cauchy problem in general relativity*, Comm. Math. Phys. 14 (1969), pp. 329–335.
- [17] Y. CHOQUET-BRUHAT, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta Math. 88 (1952), pp. 141–225.
- [18] H. Ringström. *The Cauchy problem in general relativity*. Lectures in mathematics and physics. European Mathematical Society (EMS), Zürich, 2009.
- [19] A. D. RENDALL, *Crushing singularities in spacetimes with spherical, plane and hyperbolic symmetry*, Classical Quantum Gravity, 12 (1995), pp. 1517–1533.
- [20] A. D. RENDALL, *Existence of constant mean curvature foliations in spacetimes with two-dimensional local symmetry*, Comm. Math. Phys. 189 (1997), pp. 145–164.
- [21] A. D. Rendall *The Einstein-Vlasov system* In: P. T. Chruściel and H. Friedrich [eds.], *The Einstein equations and the large scale behavior of gravitational fields*, 231–250, Birkhäuser, Basel, 2004
- [22] J. SMULEVICI, *Strong cosmic censorship for  $T^2$ -symmetric spacetimes with cosmological constant and matter*, Ann. Henri Poincaré, 9 (2008), pp. 1425–1453.
- [23] J. SMULEVICI, *On the area of the symmetry orbits of cosmological spacetimes with toroidal or hyperbolic symmetry*, arXiv:0904.0806, 70 pages.

## Matter matters – On the dynamics of spatially homogeneous cosmological models with anisotropic matter

J. MARK HEINZLE

(joint work with Simone Calogero)

A spatially homogeneous cosmological model is a solution  $(M, {}^4\mathbf{g})$  of the Einstein equations that admits an isometry group whose orbits are spacelike hypersurfaces that foliate the spacetime  $M$ . The prime examples are the Bianchi models, which are of the form  $M = I \times G$ , where  $I$  is an interval of the real numbers, and  $G$  a 3-dimensional Lie group; the Lorentzian metric  ${}^4\mathbf{g}$  has the warped product structure

$$(1) \quad {}^4\mathbf{g} = -dt^2 + g(t),$$

where, for each time  $t$ , the (Riemannian) metric  $g(t)$  is a left-invariant metric on  $G$ . By  $g_{ij}(t)$  we mean the components of  $g(t)$  w.r.t. a time-independent left-invariant frame.

The dynamics of these cosmological models, i.e., the time-evolution of  $g(t)$ , is known to depend crucially on the choice of Lie group  $G$  (the ‘Bianchi type’). In comparison, little is known about the influence of the choice of matter model, the main question being whether the results on the dynamics of vacuum and orthogonal perfect fluid models (where the matter field is a perfect fluid whose four-velocity is aligned with  $\partial_t$ ) are robust under a change of matter model. Previous work [1–3] indicates that this is not to be expected. The work [4] is a first step towards a systematic analysis of the problem—in the present report we focus on two particular results.

Of particular interest are models of Bianchi type IX, for which  $G = S^3$  ('closed cosmologies'). Under rather general conditions these models are expected to expand from an initial singularity ('big bang'), reach a maximum volume, and then recontract and recollapse to a 'big crunch' [5]. The conditions are the energy conditions imposed on the matter, in particular the strong energy condition which makes gravity 'an attractive force'. A proof of this 'closed-universe-recollapse conjecture' is available under the more restrictive condition of a non-negative average pressure [6]. However, as we show in [4] there exist anisotropic matter models such that closed-universe recollapse fails in general: We consider locally rotationally symmetric models of Bianchi type IX and show the existence of matter models with

$$(2) \quad w = \frac{p}{\rho} \in \left(-\frac{1}{3}, \frac{1-\sqrt{3}}{3}\right), \quad \beta \in (\beta_-, \beta_+) \subset \left(-\frac{1}{2}, 0\right),$$

such that there exists an open set of initial data for which the associated solutions do not recollapse but expand for all times. In (2),  $\rho$  is the energy density and  $p$  the average pressure (i.e., the average of the principal pressures) of the matter. The parameter  $\beta$  measures the degree of anisotropy; it is constructed from the minimum/maximum of the principal pressures and  $\rho$ . (For perfect fluids we have  $\beta = 0$ , elastic materials are characterized by positive values of  $\beta$ ; collisionless (Vlasov) matter has  $\beta = 1$ ; finally, magnetic fields are related to models with  $\beta = -2$ . The matter model satisfying (2) does not correspond to any of these explicit models.)

The main reason to study models of Bianchi type IX lies in their conjectured role as building blocks for generic spacelike singularities: The asymptotic dynamics of type IX models toward the (initial) singularity is expected to be representative of the asymptotic dynamics of spacetimes without symmetries that form spacelike singularities [7] (see, however, [8]). In this context, it is also expected that 'matter does not matter', which means that the dynamics of solutions of the Einstein-matter equations is asymptotically the same as the dynamics of vacuum solutions. In [4] we show that this is not true in general already for locally rotationally symmetric models. While, in the vacuum and (non-stiff) perfect fluid case, typical solutions are asymptotic to the Taub solution

$$(3) \quad g_{11} = a t^2, \quad g_{22} = g_{33} = b, \quad (a, b = \text{const}),$$

solutions with collisionless matter, i.e., Vlasov-Einstein, exhibit an oscillatory behavior towards the initial singularity; the effects of the matter field cannot be neglected, i.e., certain matter models 'matter'.

The methods used in [4] to obtain these results are methods from the theory of dynamical systems. Since the metric is (1), the Einstein equations reduce to a system of ordinary differential equations. The main difficulty is to regularize these equations in order to obtain a dynamical system that is defined on a relatively compact state space. Once this has been achieved, the analysis is largely built on the construction of monotone functions and application of the monotonicity principle to determine the  $\alpha/\omega$ -limit sets of solutions. We refer to Figure 1

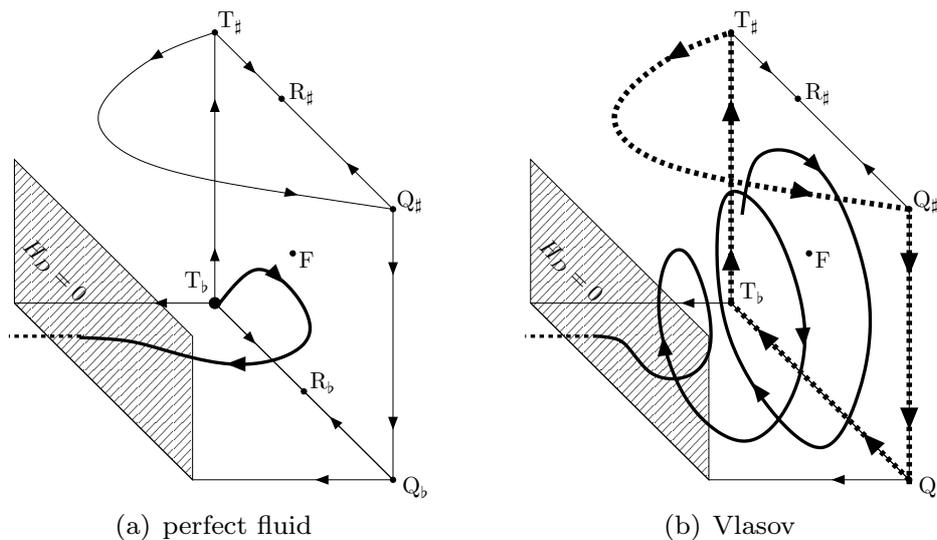


FIGURE 1. Phase portraits of Bianchi type IX solutions in the dynamical systems formulation. In the perfect fluid case, the  $\alpha$ -limit of the solution is the fixed point  $T_b$ , which represents the Taub solution (flat Kasner solution). In the Vlasov case, the  $\alpha$ -limit set is a heteroclinic cycle; the asymptotic behavior is thus oscillatory. Matter ‘matters’.

for a depiction of the state space and the dynamics in the perfect fluid and the collisionless matter case.

#### REFERENCES

- [1] V.G. LeBlanc, *Asymptotic states of magnetic Bianchi I cosmologies*, *Class. Quant. Grav.* **14** (1997), 2281–2301.
- [2] J.M. Heinzle, C. Uggla, *Dynamics of the spatially homogeneous Bianchi type I Einstein-Vlasov equations*, *Class. Quant. Grav.* **23** (2006), 3463–3490.
- [3] S. Calogero, J.M. Heinzle, *Dynamics of Bianchi Type I elastic spacetimes*, *Class. Quant. Grav.* **24** (2007), 5173–5200.
- [4] S. Calogero, J.M. Heinzle, *Bianchi cosmologies with anisotropic matter: Locally rotationally symmetric models*, Preprint <http://arxiv.org/abs/0911.0667> (2009).
- [5] J.D. Barrow, G.J. Galloway, and F.J. Tipler, *The closed-universe recollapse conjecture*, *Mon. Not. R. Astron. Soc.* **223** (1986), 835–844.
- [6] X.f. Lin, R.M. Wald, *Proof of the closed-universe-recollapse conjecture for diagonal Bianchi type IX cosmologies*, *Phys. Rev. D* **40** (1989), 3280–3286.
- [7] V.A. Belinskii, I.M. Khalatnikov, and E.M. Lifshitz, *Oscillatory approach to a singular point in the relativistic cosmology*, *Adv. Phys.* **19** (1970), 525–573.
- [8] J.M. Heinzle, C. Uggla, *Mixmaster: Fact and Belief*, *Class. Quant. Grav.* **26** (2009), 075015.

## Cosmological post-Newtonian expansions

TODD A. OLIYNYK

By generalizing Newtonian gravity to the cosmological setting [6], it appears that Newtonian theory can accurately describe gravity on all scales except in regions near compact neutron stars or black holes [1, 3]. Moreover, by including post-Newtonian corrections, relativistic effects such as perihelion shifts and gravitational lensing can be taken into account. Since the arguments of [1, 3] assume the existence of the Newtonian limit and the validity of post-Newtonian expansions, the conclusions of the articles [1, 3] provide considerable motivation to mathematically justify the Newtonian limit and post-Newtonian expansions.

The difficulty in justifying the Newtonian limit and post-Newtonian expansions arises from the singular nature of the limit

$$\epsilon = \frac{v_T}{c} \searrow 0$$

for the Einstein equations<sup>1</sup>. In the limit  $\epsilon \searrow 0$ , the Einstein equations degenerate from a hyperbolic system for  $\epsilon > 0$  into a scalar elliptic equation at  $\epsilon = 0$ . It is this transition from a hyperbolic system for the spacetime metric to a scalar elliptic equation for the Newtonian potential that is the source of the analytical difficulties in justifying the Newtonian limit and post-Newtonian expansions.

A particularly important class of matter for cosmological studies is the perfect fluid. We recall that gravitating perfect fluids are governed by the Einstein-Euler equations

$$G^{ij} = \frac{8\pi G}{c^4} T^{ij} \quad \text{and} \quad \nabla_i T^{ij} = 0,$$

where

$$T^{ij} = (\rho + c^{-2}p)v^i v^j + p(\rho)g^{ij},$$

with  $\rho$  the fluid density,  $p = p(\rho)$  the fluid pressure,  $v^i$  the fluid four-velocity normalized by  $v^i v_i = -c^2$ ,  $c$  the speed of light, and  $G$  the Newtonian gravitational constant. By suitably rescaling, these equations can be written as

$$(1) \quad G^{ij} = 2\epsilon^4 T^{ij} \quad \text{and} \quad \nabla_i T^{ij} = 0,$$

where

$$T^{ij} = (\rho + \epsilon^2 p)v^i v^j + pg^{ij} \quad \text{and} \quad v^i v_i = -\frac{1}{\epsilon^2}.$$

For cosmological spacetimes of the form  $M = [0, T) \times \mathbb{T}^3$ , the appropriate limit equations satisfied by solutions of (1) in the limit  $\epsilon \searrow 0$  are [7, 8]:

$$(2) \quad \partial_t \rho = -w^I \partial_I \rho - \rho \partial_I w^I - \frac{3}{2} \frac{\tilde{a}'}{\tilde{a}} \rho,$$

$$(3) \quad \partial_t w^J = -w^I \partial_I w^J - \frac{1}{\tilde{a}\rho} \partial^J p(\rho) - \frac{\tilde{a}'}{\tilde{a}} w^J + g^J,$$

$$(4) \quad \Delta \Phi = 4\tilde{a}(\rho - \tilde{\mu}),$$

---

<sup>1</sup>Here,  $c$  is the speed of light and  $v_T$  is a typical speed associated with the gravitating matter.

where

$$\begin{aligned} \tilde{\mu}(t) &= \int_{\mathbb{T}^3} \tilde{\rho}(t) d^3x, \\ \tilde{a}(t) &= \exp \left( \int_0^t \left( \frac{8}{3} \int_{\mathbb{T}^3} \rho(s) d^3x \right)^{\frac{1}{2}} ds \right), \\ g^J &= -\frac{1}{\tilde{a}} \left( \frac{3}{2} \frac{\tilde{a}'}{\tilde{a}} \frac{1}{\tilde{\mu}} \int_{\mathbb{T}^3} \rho w^J d^3x + \frac{1}{4} \partial^J \Phi \right), \\ \Delta &= \delta^{IJ} \partial_I \partial_J, \quad \partial^J = \delta^{IJ} \partial_I, \end{aligned}$$

and  $\langle \cdot | \cdot \rangle_{L^2}$  is the standard  $L^2$  inner-product on  $\mathbb{T}^3$ , i.e.

$$\langle \psi_1 | \psi_2 \rangle_{L^2} = \int_{[0,1]^3} \psi_1(x) \psi_2(x) d^3x.$$

We refer to these as the *cosmological Poisson-Euler equations* and note that these equations agree with the Newton-Cartan field equations for a gravitating fluid formulated in adapted coordinates [4, 6].

For purposes of interpretation, it is often useful to introduce Galilei coordinates [4, 6]. This is done as follows: suppose  $\{\rho(t, x), w^I(t, x), \Phi(t, x)\}$  is a solution of the cosmological Poisson-Euler equations (2)-(3) on  $M = [0, T) \times \mathbb{T}^3$ . Then, letting  $\tilde{M} = [0, T) \times \mathbb{R}^3$  denote the covering space, we define a diffeomorphism on  $\tilde{M}$  by

$$\psi : \tilde{M} \longrightarrow \tilde{M} : (t, x) \longmapsto (t, x/\sqrt{\tilde{a}(t)}).$$

Lifting the cosmological Poisson-Euler equations to  $\tilde{M}$ , and then pulling back by  $\psi$  shows that

$$\begin{aligned} \hat{\rho}(t, x) &= \rho(t, x/\sqrt{\tilde{a}(t)}), \\ \hat{w}^J(t, x) &= \sqrt{\tilde{a}(t)} w^J(t, x/\sqrt{\tilde{a}(t)}) + \frac{1}{2} \frac{\tilde{a}'(t)}{\tilde{a}(t)} x^J, \\ \hat{\rho}(t, x) &= \Phi(t, x/\sqrt{\tilde{a}(t)}), \end{aligned}$$

satisfy

$$(5) \quad \partial_t \hat{\rho} = -\hat{w}^I \partial_I \hat{\rho} - \hat{\rho} \partial_I \hat{w}^I,$$

$$(6) \quad \partial_t \hat{w}^J = -\hat{w}^I \partial_I \hat{w}^J - \frac{1}{\hat{\rho}} \partial^J p(\hat{\rho}) + \hat{g}^J,$$

$$(7) \quad \Delta \hat{\Phi} = 4(\hat{\rho} - \tilde{\mu}),$$

where

$$\hat{g}^J = -\frac{3}{2} \frac{\tilde{a}'}{\tilde{a}^{3/2}} \frac{1}{\tilde{\mu}} \int_{\mathbb{T}^3} \rho w^J d^3x - \frac{1}{4} \partial^J \hat{\Phi} - \frac{\tilde{\mu}}{3} x^J.$$

A Newtonian potential can be defined by

$$\check{\Phi} = \frac{\hat{\Phi}}{4} + \frac{\tilde{\mu}}{6} \delta_{IJ} x^I x^J + \frac{4}{\tilde{a}' \tilde{a}^{7/2}} \delta_{IJ} x^I \int_{\mathbb{T}^3} \tilde{\rho}_0 \tilde{w}_0^J d^3x.$$

This potential satisfies the usual Poisson equation

$$(8) \quad \Delta \check{\Phi} = \hat{\rho}$$

while the acceleration due to gravity  $\hat{g}^J$  takes the familiar form

$$(9) \quad \hat{g}^J = -\partial^J \check{\Phi}.$$

Together, equations (5), (6), (8), and (9) show that solutions to the cosmological Poisson-Euler equations determine solutions to the standard Poisson-Euler equations on the covering space  $\tilde{M}$ .

The main result of the articles [7, 8] is rigorously justify the Newtonian limit for the Einstein-Euler equations and to establish the existence of solutions that admit post-Newtonian expansions to arbitrary order. More specifically, we prove the existence of a large class of one-parameter families of solutions to (1) defined for  $0 < \epsilon < \epsilon_0$  that

- (i) exist on a common piece of spacetime of the form  $M = [0, T) \times \mathbb{T}^3$ ,
- (ii) converge as  $\epsilon \searrow 0$  to solutions of the cosmological Poisson-Euler equations (2)-(4) of Newtonian gravity, and
- (iii) are differentiable in  $\epsilon$  to any prescribed order  $\ell \in \mathbb{N}$ .

Properties (i)-(iii) guarantee that these one parameter families of solutions to the Einstein-Euler equations have valid Newtonian limits and admit a post-Newtonian expansions to order  $\ell/2$ .

In light of the significant and well-known difficulties that are encountered at the formal (and rigorous) level in trying to develop post-Newtonian expansions on asymptotically flat spacetimes beyond the order 2.5 [5], it is somewhat surprising that these difficulties are absent in the cosmological setting. On asymptotically flat spacetimes, the problems that occur in the higher order post-Newtonian expansions are often attributed to the reaction of gravitational radiation with itself and matter. The analysis contained in the article [7, 8] shows that this is not the complete story as the these effects are also present in the cosmological setting, but do not cause similar difficulties

## REFERENCES

- [1] J. Hwang and H. Noh, *Newtonian versus relativistic nonlinear cosmology*, Gen. Rel. Grav. **38** (2006), 703-710.
- [2] J. Hwang, H. Noh, and D. Puetzfeld, *Cosmological nonlinear hydrodynamics with post-Newtonian corrections*, JCAP **3** (2008), 10.
- [3] A. Ishibashi and R.M. Wald, *Can the acceleration of our universe be explained by the effects of inhomogeneities?*, Class. Quant. Grav. **23** (2006), 235-250.
- [4] H.P. Künzle *Covariant Newtonian limit of Lorentz space-times*, Gen. Rel. Grav. **7** (1976), 445-457.
- [5] A.D. Rendall, *On the definition of post-Newtonian approximations*, Proc. R. Soc. Lond. A **438** (1992), 341-360.
- [6] C. Rüede and N. Straumann, *On Newton-Cartan cosmology*, Helv.Phys.Acta **70** (1997), 318-335.
- [7] T.A Oliynyk, *Cosmological post-Newtonian expansions to arbitrary order*, Commun. Math. Phys. (accepted) [arXiv:0908.2836]

- [8] T.A Oliynyk, *A rigorous formulation of the cosmological Newtonian limit*, preprint [arXiv:0908.4454].

## Helical solutions in scalar gravity

ROBERT BEIG

(joint work with Bernd G.Schmidt)

An important step in understanding the solution space of a field theory consists in the study of solutions with continuous symmetries. In the case of general relativity, and for asymptotically flat solutions with timelike ADM four-momentum, the possible solutions have been classified in the paper [BC]. Namely it was shown that only possible are: (a) stationary symmetry, (b) axial symmetry, in particular spherical symmetry, (c) helical symmetry, and, finally, combinations of the above. By helical symmetry (called stationary-rotating symmetry in [BC]) one means the existence of a Killing vector  $\xi$  which at spatial infinity approaches a vector field which is a linear combination of a time translation and a rotation, i.e.  $\xi = \partial_t + \Omega \partial_\phi$  and where neither of them by itself is a Killing vector. The cases (a) and (b) are well understood and of course there are plenty of examples. Concerning (c) nothing is known.

Some known examples in other theories are the following. In the Newtonian 1-body problem one has the Jacobi ellipsoids which describe triaxial ellipsoids consisting of ideal fluid in steady rotation about one of their axes. For 2 bodies there are of course 2 point particles in a circular orbit around their center of mass: for extended bodies consisting of ideally elastic material these have been constructed [BS]. In electromagnetism one has the Schild solution, describing two charged point-particles interacting via the half-retarded plus half-advanced field on circular orbits, and the analogue for a special relativistic, scalar theory of gravity, and for an arbitrary number of point particles, has been constructed in [BHS]. These solutions can be described by saying that the radiation emitted by the particles is exactly balanced by incoming radiation. The behavior of the field at spatial infinity is of the form  $\sim \frac{e^{i\Omega t}}{r}$  and thus incompatible with the total energy being finite.

It was the aim of the work [BS1] to construct similar solutions where the point particles are replaced by elastic bodies. For simplicity we treated the case of one body in steady rotation: if this body is not axially symmetric and-or rotation is not about its axis of symmetry of the body, the resulting solution has merely helical symmetry.

The studied model is given by the two fields  $(V, f)$  on Minkowski space  $M = (\mathbb{R}^4, \eta_{\mu\nu})$ , where  $V$  is a scalar and  $f$ , the elastic configuration, is a map from  $M$  into a domain  $\mathcal{B} \subset \mathbb{R}^3$ , called body or material manifold. The action  $S$  is of the form

$$(1) \quad S = \frac{1}{2} \int \eta^{\mu\nu} (\partial_\mu V)(\partial_\nu V) d^4x + 4\pi G \int \rho(1 + V) d^4x$$

where  $\rho$  is a function of the principal invariants of the linear map

$$H^A{}_B = \eta^{\mu\nu} (\partial_\mu f^A) (\partial_\nu f^C) \delta_{BC} \quad (\text{with } A, B, C = 1, 2, 3)$$

subject to certain regularity and constitutional assumptions. Our result states that the following: suppose the axis of rotation goes through the centre of mass of the body in its stress-free state and this axis coincides with one of the axes of inertia. Then there exists, for small enough values of  $G$  and  $\Omega$ , a helical solution to the equations given by the action (1) close to the trivial solution where  $V$  is zero and the elastic configuration is stress-free. Our result, which holds modulo a plausible conjecture concerning the differentiability in a certain Banach space of the self-field term in the elastic equation, uses methods similar to those in the papers [ABS1], [ABS].

#### REFERENCES

- [ABS1] Andersson, L., Beig, R. and Schmidt, B.G. (2008) *Static self-gravitating elastic bodies in Einstein gravity*, Communications on Pure and Applied Mathematics **61**, 988-1023
- [ABS] Andersson, L., Beig, R., and Schmidt, B.G. (2009) *Rotating elastic bodies in Einstein gravity*, Communications on Pure and Applied Mathematics, to appear, [arXiv:0811.0932 [gr-qc]].
- [BC] Beig, R. and Chruściel, P.T. (1997) *The isometry groups of asymptotically flat, asymptotically empty space-times with timelike ADM four-momentum*, Commun.Math.Phys. **188** 585–597, [arXiv:gr-qc/9610034].
- [BHS] Beig, R., Heinzle, J.M., and Schmidt, B.G. (2007) *Helically symmetric N-particle solutions in scalar gravity*, Phys.Rev.Lett. **98** 121102[4 pages] [arXiv:gr-qc/0612066]
- [BS] Beig, R. and Schmidt, B. G. (2008) *Celestial mechanics of elastic bodies*, Math. Z. **258** 381–394.
- [BS1] Beig, R. and Schmidt, B.G. (2009), *Helical Solutions in Scalar Gravity*, Gen.Rel.Grav. **41** 2031-2043, [arXiv:0905.2437 [gr-qc]]
- [S] Schild, A. (1963), *Electromagnetic Two-Body Problem*, Phys.Rev. **131** 2762-2766

### Construction of $N$ -body initial data sets in general relativity

JUSTIN CORVINO

(joint work with Piotr T. Chruściel and James Isenberg)

INTRODUCTION. Gluing methods have long been used for connected-sum and de-singularization constructions in geometry. Over the past decade, gluing techniques have been implemented to construct interesting solutions to the Einstein constraint equations. Isenberg, Mazzeo and Pollack developed connected-sum constructions using the conformal method [11]; around the same time, we used gluing ideas to study asymptotics of solutions to the time-symmetric constraints, and introduced a method to localize gluing [7]. This method was extended to a study of the full constraint operator in joint work with R. Schoen [10], as well as by Chruściel and Delay [4]. There have been many interesting applications of these methods, which include localized gluing constructions for scalar curvature [5], as well as constructions of solutions to the constraints (which provide initial data for the Einstein equation) with the following features: asymptotically Euclidean (AE) initial data

that evolves into asymptotically simple space-times in the sense of Penrose [3, 9]; and AE initial data with multiple apparent horizons [3, 6, 8]. In the present work we construct solutions of the Einstein constraints to model  $N$ -body systems.

**THE MAIN THEOREM.** We model isolated gravitational systems by AE initial data  $(M, g, K)$  for the Einstein equation, where  $M$  is a three-manifold, and  $(g, K)$  will be the induced first and second fundamental forms of  $M$  inside the space-time determined by the development of the space-like Cauchy data  $(M, g, K)$ . We let  $\Phi(g, K) = (R(g) - \|K\|^2 + (\text{tr}_g(K))^2, \text{div}_g(K) - d(\text{tr}_g(K)))$  be the constraint operator, so that the constraint equations are  $\Phi(g, K) = (16\pi\mu, 8\pi J)$ , where  $(\mu, J)$  is the energy-momentum density of the matter fields along  $M$ .

We specify a *body* to be a compact subset of an AE solution of the Einstein constraint equations. The solutions (hence the bodies) may be vacuum (pure gravity), or they may contain compactly supported  $(\mu, J)$  from physical fields. Our construction takes  $N$  such bodies and produces an AE solution to the constraint equations that contains the  $N$  bodies isometrically, and has an AE end in which the bodies interact. We remark that if the original AE solutions containing the bodies each has the topology of  $\mathbb{R}^3$ , then the resulting  $N$ -body data has the same topology. The proof uses the method of [4, 10], based on a construction of Chruściel and Delay [3]. We assume the given AE solutions each have well defined Poincaré charges (given by certain flux integrals at infinity), i.e. the ADM energy-momentum, the center-of-mass and angular momentum.

**Theorem 1.** *For each  $k = 1, \dots, N$ , let  $(M_k, g^k, K^k)$  be a three-dimensional connected AE initial data set, and let  $E_k \subset M_k$  be a vacuum AE end with well defined global Poincaré charges. Let  $U_k \subset M_k$  be pre-compact. Then for each  $\eta > 0$ , there is a solution  $(M, g_\eta, K_\eta)$  of the constraints containing a region  $U$  isometric to the disjoint union  $\bigcup_{k=1}^N (U_k, g^k, K^k)$ , such that  $M$  is connected, with one AE end isometric to a space-like slice of a Kerr metric with ADM energy-momentum four-vector  $(m(g_\eta), \vec{p}_\eta)$  satisfying  $\left| m(g_\eta) - \sum_{k=1}^N m_k \right| < \eta$ , and  $\left\| \vec{p}_\eta - \sum_{k=1}^N \vec{p}_k \right\| < \eta$ , where  $(m_k, \vec{p}_k)$  is the ADM energy-momentum four-vector of  $E_k$ .*

**SKETCH OF THE PROOF.** We sketch the proof here; details can be found in [1, 2]. Let  $\{|x| \geq \epsilon^{-1}\} \subset E_k \cap (M_k \setminus U_k)$ , for each  $k$ . We re-scale the AE solutions in which the bodies  $U_k$  are located, pulling back re-scaled data on  $\{\epsilon^{-1} \leq |x| \leq 2\epsilon^{-1}\}$  onto a fixed annulus. Consider a Euclidean ball containing points  $c_1, \dots, c_N$ , so that  $\sum_{k=1}^N m_k c_k = 0$ . We glue the standard Minkowski data  $(g, K) = (\dot{g}, 0)$  on the ball to the re-scaled data from each  $E_k$ , on annuli centered at each  $c_k$ , respectively, and to data from a space-like slice in Kerr in an annular region along the outer boundary of the ball. For suitable scalings and suitably chosen Kerr data (possibly boosted), we will have an *approximate solution* of the vacuum constraint equation  $\Phi = 0$  on a region  $\Omega$ , which is a ball with smaller balls around each

$c_k$  deleted. We note that we make better approximate solutions with scalings of larger annuli (smaller  $\epsilon$ ), in which case the data on  $\Omega$  approaches the Minkowski data. The linearization  $\mathring{L}$  of  $\Phi$  at the Minkowski data has formal adjoint given by  $\mathring{L}^*(f, X) = (-\Delta f \mathring{g} + \text{Hess}(f), -\frac{1}{2}\mathcal{L}_X \mathring{g})$ , where  $\mathcal{L}$  is the Lie derivative, and the operators are taken at the Euclidean metric. This operator has ten-dimensional kernel  $K_*$  given by  $\text{span}\{1, x^1, x^2, x^3\} \oplus \{\text{Euclidean Killing fields}\}$ . Thus if  $\Pi$  is a (weighted)  $L^2$ -projection onto  $K_*^\perp$  on  $\Omega$ , then  $\Pi \circ \Phi$  has surjective linearization at  $(\mathring{g}, 0)$ . What we can then solve by perturbation theory (implicit function theorem) is  $\Pi \circ \Phi(g, K) = 0$ . Using the fact that  $\mathring{L}^*$  is overdetermined-elliptic, we can solve in appropriate weighted spaces so that our solution  $(g, K)$  smoothly agrees with our glued data along the boundary  $\partial\Omega$  [4, 10]. Moreover,  $\Phi(g, K) = 0$  if and only if the projection of  $\Phi(g, K)$  onto  $K_*$  is zero. We compute the projections onto the standard basis of  $K_*$ , divided by the scale parameter  $\epsilon$ , to obtain the following (up to constant scale factors and  $O(\epsilon)$  error terms): the projection onto the constant direction yields the difference between the mass of the exterior Kerr and the sum of the masses of the bodies (i.e of the  $E_k$ ); the projection onto the  $x^j$  directions yields the mass-times-center parameter of the exterior Kerr (since the center-of-mass of the bodies has been normalized to zero); the projection onto the translation Killing fields yields the difference between the linear momentum of the exterior Kerr, and the sum of the linear momenta of the bodies; the projection onto the rotational Killing fields yields the difference between the angular momentum of the exterior Kerr and the sum of the angular momenta of the bodies, *minus* the total orbital angular momentum  $\sum_{k=1}^N c_k \times \vec{p}_k$ . For  $\epsilon$  small enough, we can arrange  $\Phi(g, K) = 0$  by choosing the parameters of the exterior Kerr appropriately. We now re-scale to restore the original metric on each  $U_k$ ; we note that doing so produces a re-scaling of the original center-of-mass configuration, and that for  $\epsilon$  small enough, the desired energy-momentum four-vector estimate will hold.

#### REFERENCES

- [1] Chruściel, P.T., Corvino, J., Isenberg, J.: Construction of  $N$ -body time-symmetric initial data sets in general relativity. Preprint: arXiv:0909.1101
- [2] Chruściel, P.T., Corvino, J., Isenberg, J.: Construction of  $N$ -body initial data sets in general relativity. *In preparation*
- [3] Chruściel, P., Delay, E.: Existence of non-trivial, vacuum, asymptotically simple spacetimes. *Classical Quantum Gravity* **19**, L71-79 (2002)
- [4] Chruściel, P.T., Delay, E.: On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications. *Mém. Soc. Math. Fr. (N.S.)* **94** (2003)
- [5] Chruściel, P.T., Isenberg, J., Pollack, D.: Initial data engineering. *Comm. Math. Phys.* **257**, no. 1, 29-42 (2005)
- [6] Chruściel, P.T., Mazzeo, R.: On “many-black-hole” vacuum spacetimes. *Classical Quantum Gravity*. **20**, no. 4, 729-754 (2003)
- [7] Corvino, J.: Scalar curvature deformation and a gluing construction for the Einstein constraint equations. *Comm. Math. Phys.* **214**, 137-189 (2000)
- [8] Corvino, J.: A note on asymptotically flat metrics on  $\mathbb{R}^3$  which are scalar-flat and admit minimal spheres. *Proc. Amer. Math. Soc.* **133**, no. 12, 3669-3678 (2005)

- [9] Corvino, J.: On the existence and stability of the Penrose compactification. *Ann. Henri Poincaré* **8**, 597-620 (2007)
- [10] Corvino, J., Schoen, R.M.: On the asymptotics of the vacuum Einstein constraint equations. *Jour. Diff. Geom.* **73**, no. 2, 185-217 (2006)
- [11] Isenberg, J., Mazzeo, R., Pollack, D.: Gluing and wormholes for the Einstein constraint equations. *Comm. Math. Phys.* **231**, no. 3, 529-568 (2002)

**On geometric problems related to Brown-York and Liu-Yau quasilocal mass**

PENGZI MIAO

In [11], using the Riemannian positive mass theorem [10,12], Shi and Tam proved the following remarkable result on the boundary behavior of compact manifolds with nonnegative scalar curvature:

**Theorem 1.** (Shi-Tam, 02) *Given an  $n$ -dimensional ( $n \geq 3$ ), compact, Riemannian spin manifold  $(\Omega^n, g)$  with boundary and with nonnegative scalar curvature, suppose its boundary  $\Sigma$  is isometric to some strictly convex hypersurface  $\Sigma_0 \subset \mathbb{R}^n$ . Let  $H, H_0$  be the mean curvature of  $\Sigma$  in  $(\Omega, g)$ ,  $\Sigma_0$  in  $\mathbb{R}^n$  respectively. If  $H > 0$ , then*

$$(0.1) \quad \int_{\Sigma} H \, d\sigma \leq \int_{\Sigma_0} H_0 \, d\sigma.$$

Moreover, the equality holds if and only if  $(\Omega, g)$  is isometric to a domain in  $\mathbb{R}^n$ .

A key ingredient in the proof of Theorem 1 in [11] is a monotonicity property of the integral  $\int_{\Sigma} (H_0 - H) \, d\sigma$  along a particular foliations  $\{\Sigma_t\}$  of a specially constructed asymptotically flat extension of  $(\Omega, g)$ . Recently, Shi, Tam and I [7] have discovered a general derivative formula, governing the evolution of  $\int_{\Sigma} (H_0 - H) \, d\sigma$  along an arbitrary geometric foliation of any given ambient space.

**Theorem 2.** (Miao-Shi-Tam, 09) *Let  $\{\Sigma_t\}$  be a smooth family of closed hypersurfaces evolving in an ambient manifold  $(M^n, g)$  according to an equation  $\frac{\partial F}{\partial t} = \eta\nu$ , where  $\nu$  is a unit vector field normal to  $\Sigma_t$  and  $\eta$  is the speed of evolution. Suppose  $\Sigma_t$  embeds isometrically to a hypersurface  $\Sigma_t^0$  in  $\mathbb{R}^n$  and  $\{\Sigma_t^0\}$  evolves smoothly in  $\mathbb{R}^n$ . Then*

$$(0.2) \quad \frac{d}{dt} \int_{\Sigma_t} (H_0 - H) \, d\sigma = \frac{1}{2} \int_{\Sigma_t} (|A_0 - A|^2 - |H_0 - H|^2 + R) \eta \, d\sigma,$$

where  $A_0$  and  $H_0$  are the second fundamental form and the mean curvature of  $\Sigma_t^0$  in  $\mathbb{R}^n$ ,  $A$  and  $H$  are the second fundamental form and the mean curvature of  $\Sigma_t$  in  $(M, g)$ , and  $R$  is the scalar curvature of  $(M, g)$ .

In Theorem 2, if  $\{\Sigma_t^0\} \subset \mathbb{R}^n$  evolves with a normal speed, say  $\frac{\partial F^0}{\partial t} = \eta^0\nu^0$ , where  $\nu^0$  is the outward unit normal to  $\Sigma_t^0$  and  $\eta^0$  is the speed, then (0.2)

directly implies

$$(0.3) \quad \frac{d}{dt} \int_{\Sigma_t} (H_0 - H) d\sigma = \frac{1}{2} \int_{\Sigma} \left[ - \left( 1 - \frac{\eta^0}{\eta} \right)^2 R_t + R \right] \eta d\sigma,$$

where  $R_t$  is the scalar curvature of each leaf  $\Sigma_t^0$  (or  $\Sigma_t$ ). In particular, if

$$(0.4) \quad R_t \geq 0, \quad R \leq 0, \quad \text{and} \quad \eta \geq 0,$$

then  $\int_{\Sigma_t} (H_0 - H) d\sigma$  is monotone non-increasing. This suggests that the key monotonicity property of  $\int_{\Sigma_t} (H_0 - H) d\sigma$  used by Shi and Tam [11] indeed can be generalized to *an arbitrary geometric foliation*  $\{\Sigma_t^0\}$  of  $\mathbb{R}^n$ , as long as each leaf  $\Sigma_t^0$  has nonnegative scalar curvature.

In [3], using the above observation, together with a result of Gerhardt [4] and Urbas [9] on evolving star-shaped surfaces into spheres, Eichmair, Wang and I obtained the following extension of Theorem 1.

**Theorem 3.** (Eichmair-Miao-Wang, 09) *The conclusion of Theorem 1 remains valid if the assumption that  $\Sigma$  embeds as a strictly convex hypersurface in  $\mathbb{R}^n$  is relaxed to the requirement that  $\Sigma$  has positive scalar curvature and is isometric to a mean-convex, star-shaped hypersurface in  $\mathbb{R}^n$ . Moreover, the spin assumption in Theorem 1 is not necessary if the dimension  $n$  satisfies  $3 \leq n \leq 7$ .*

Suppose  $\Sigma$  is a closed, connected, spacelike 2-surface in a spacetime  $N$ . Suppose  $\Sigma$  has positive Gaussian curvature and the mean curvature vector  $H$  of  $\Sigma$  in  $N$  is spacelike. We recall that the Liu-Yau quasi-local mass of  $\Sigma$  [5, 6] is given by

$$(0.5) \quad \mathbf{m}_{\text{LY}}(\Sigma) = \frac{1}{8\pi} \int_{\Sigma} (k_0 - |H|) d\sigma,$$

where  $k_0$  is the (positive) mean curvature of the isometric embedding of  $\Sigma$  into the Euclidean space  $\mathbb{R}^3$ ,  $|H|$  is the length of  $H$  in  $N$  and  $d\sigma$  is the volume form on  $\Sigma$ . When  $\Sigma$  bounds a compact, time-symmetric, hypersurface  $\Omega$  in  $N$ , the Liu-Yau mass reduces to the Brown-York mass [1, 2] which is

$$(0.6) \quad \mathbf{m}_{\text{BY}}(\Sigma) = \frac{1}{8\pi} \int_{\Sigma} (k_0 - k_{\Omega}) d\sigma,$$

where  $k_{\Omega}$  is the mean curvature of  $\Sigma$  in  $\Omega$ . Using the result of Shi and Tam 1, Liu and Yau proved that  $\mathbf{m}_{\text{LY}}(\Sigma) \geq 0$  and  $\mathbf{m}_{\text{LY}}(\Sigma) = 0$  only if  $N$  is flat along  $\Sigma$ . In [8], Ó Murchadha, Szabados and Tod constructed examples of 2-surfaces  $\Sigma$  in  $\mathbb{R}^{3,1}$  such that  $\mathbf{m}_{\text{LY}}(\Sigma) > 0$ . In a recent work [7] with Shi and Tam, using the techniques of *maximal surfaces*, we have shown

**Theorem 4.** (Miao-Shi-Tam, 09) *Let  $\Sigma$  be an arbitrary, closed, connected, spacelike 2-surface in  $\mathbb{R}^{3,1}$ . Suppose  $\Sigma$  spans a compact spacelike hypersurface in  $\mathbb{R}^{3,1}$ . If  $\Sigma$  has positive Gaussian curvature and has spacelike mean curvature vector, then  $\mathbf{m}_{\text{LY}}(\Sigma) > 0$  unless  $\Sigma$  lies on a hyperplane in  $\mathbb{R}^{3,1}$ .*

## REFERENCES

- [1] Brown, J. David and York, Jr., James W. Quasilocal energy in general relativity. In *Mathematical aspects of classical field theory (Seattle, WA, 1991)*, volume 132 of *Contemp. Math.*, pages 129–142. Amer. Math. Soc., Providence, RI, 1992.
- [2] Brown, J. David and York, Jr., James W. Quasilocal energy and conserved charges derived from the gravitational action. *Phys. Rev. D (3)*, 47(4):1407–1419,1993.
- [3] Eichmair, M., Miao, P., and Wang, X. *Boundary effect on compact manifolds with nonnegative scalar curvature - a generalization of a theorem of Shi and Tam*, arXiv:0911.0377.
- [4] Gerhardt, C., *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom., **32** (1990), 299-314.
- [5] Liu, C.-C.M. and Yau, S.-T., *Positivity of quasilocal mass*, Phys. Rev.Lett. 90(2003) No. 23, 231102
- [6] Liu, C.-C.M. and Yau, S.-T., *Positivity of quasilocal mass II*, J. Amer. Math. Soc. 19(2006) No. 1.181-204.
- [7] Miao, P., Shi, Y.-G. and Tam, L.-F., *On geometric problems related to Brown-York and Liu-Yau quasilocal mass*, arXiv: 0906.5451
- [8] N. Ó. Murchadha, L.B. Szabados, and K.P. Tod, *Comment on " Positivity of quasilocal mass"* Phys. Rev. Letter92(2004), 259001.
- [9] Urbas, J., *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. 205(1990), No. 3., 355–372.
- [10] Schoen, R. and Yau, S.-T., *On the proof of the positive mass conjecture in general relativity*, Commun. Math. Phys. **65** (1979), 45–76.
- [11] Shi, Y.-G. and Tam, L.-F., *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, J. Differential Geom. **62** (2002), 79–125.
- [12] Witten, E., *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), 381–402.

## A model problem for conformal parameterizations of the Einstein constraint equations

DAVID MAXWELL

Initial data for the Cauchy problem of general relativity consist of a Riemannian manifold and a second fundamental form that satisfy a system of nonlinear PDEs known as the Einstein constraint equations. It would be desirable to find a parameterization of all solutions of these equations on a given manifold, and hence a description of all possible initial data. The conformal method of Lichnerowicz and Choquet-Bruhat and York provides an elegant and complete solution to the problem of constructing all constant-mean curvature (CMC) solutions. For example, on compact manifolds the solutions of the Einstein constraint equations are effectively parameterized by selection of conformal data consisting of a conformal class for the metric, a so-called transverse-traceless tensor, and a (constant) mean curvature. The conformal method can also be used to generate initial data with non-constant mean curvatures, but little is known in this case, especially if the mean curvature is far-from CMC.

Until recently, virtually all results for the conformal method only applied to near-CMC initial data. The first construction using the conformal method of a family of initial data with arbitrarily specified mean curvature was given by Holst,

Nagy, and Tsogtgrel in [HNT08]. Although this result represents a breakthrough for the conformal method, it has a number of important limitations:

- The near-CMC hypothesis is replaced by a smallness assumption on the transverse-traceless tensor (i.e. a small-TT hypothesis).
- It is not known if small-TT conformal data determine a unique solution.
- The construction only works on Yamabe-positive compact manifolds.
- The construction requires non-vanishing matter fields.

It was subsequently shown in [Ma09] that the construction could be extended to vacuum initial data, but the other restrictions remain. These results are compatible with the possibility that a large set of conformal data lead to no solutions or multiple solutions; from the point of view of parameterizing the full set of solutions one would like to show that this does not occur.

In this talk we describe a family of highly symmetric conformal data that can be used to examine the solution theory of the conformal method for large transverse-traceless tensors and far-from CMC mean curvatures. The data are all specified on a conformally flat torus with the flat background metric, and are independent of all but one direction ( $x$ ). The mean curvatures in the family are of the form

$$\tau_t(x) = t + \lambda(x)$$

where  $\lambda$  is fixed, carefully chosen function describing fluctuations about a mean, and  $t$  is a constant that controls how close the mean curvature is to being CMC, with  $|t| \gg 1$  corresponding to the near-CMC case. The transverse-traceless tensors in the family are parameterized by two constants,  $\eta$  and  $\mu$ , that control the size of pieces of the tensor. Given parameters  $(t, \eta, \mu)$ , we seek a corresponding symmetric solution of the constraint equations.

Restricting our remarks to the case  $\mu = 0$ , we observe the following:

- If  $|t|$  is sufficiently large so that the mean curvature does not change sign, then there exists a corresponding symmetric solution of the constraint equations.
- If  $|t|$  is small enough so that the mean curvature changes sign, but  $t \neq 0$ , then there is a critical value  $\eta_0 > 0$ . If  $|\eta| > \eta_0$  there are no solutions with symmetry, and if  $0 < |\eta| < \eta_0$  there are at least two.
- When  $t = 0$  there are no solutions with symmetry unless  $\eta = 0$ , in which case there is a one-parameter family.

This is the first case where non-uniqueness for the standard, vacuum conformal method has been shown.

Intriguingly, we also find that for mean curvatures in the family with changing sign, the existence theory depends sensitively on the values of the constants involved in the nonlinear coupling of the conformal method. We show that these constants are balanced in such a way that any arbitrarily small adjustment to their values lead to one of two different existence theories.

Although the limitations of the conformal method described here only pertain to Yamabe-null manifolds, they suggest that the weaknesses of the Yamabe-positive results of [HNT08] and [Ma09] arise from real phenomena. At any rate, if the

Yamabe-positive results are to be extended to the Yamabe-null case, then a small-CMC condition will be necessary to obtain uniqueness, and it may be that solutions need not exist for large transverse-traceless tensors. Given the sensitivity of the existence theory of the model problem with respect to the coefficients in the equations, we expect it will be very difficult to obtain such an extension.

## REFERENCES

- [HNT08] M. Holst, G. Nagy, and G. Tsogtgerel, *Rough solutions of the Einstein constraint equations on closed manifolds without near-CMC conditions*, *Comm. Math. Phys.* **288** (2008), no. 2, 547–613.
- [Ma09] D. Maxwell, *A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature*, *Math. Res. Lett.* **16** (2009), no. 4, 627–645.

**Ricci flow on open surfaces**

RAFE MAZZEO

One natural, and surprisingly fairly open, area of investigation in the intensively studied field of Ricci flow is to study the behaviour of this evolution problem on various natural classes of complete, noncompact Riemannian manifolds, or on incomplete spaces with geometrically structured (e.g. stratified) singular sets. We survey some of the known results in these directions below, and then report on a handful of new results in both of these settings in the lowest dimensional case, i.e. for surfaces.

Since it is the focus here, we write down the relevant equation in two dimensions. For any metric  $g$  on a surface  $\Sigma$ , let  $R$  denote its scalar curvature (i.e. twice its Gauss curvature). The Ricci flow equation is then

$$(1) \quad \partial_t g(t) = (\rho - R)g, \quad g(0) = g_0,$$

where  $\rho$  is a normalizing constant. When  $\Sigma$  is compact, for example, then a suitable choice of  $\rho$  ensures that the area of  $(\Sigma, g(t))$  remains constant. Note that  $g(t)$  remains in the same conformal class as  $g_0$ ; this is in marked distinction to the higher dimensional case. Thus if we write  $g(t) = u(t)g_0$ , then (1) is equivalent to

$$(2) \quad \partial_t u = \Delta_{g_0} \log u - R_0 + \rho u, \quad u(0) = 1.$$

An important early result in the development of Ricci flow was obtained by Shi [13], who proved that if one starts at any initial metric which is complete and satisfies some quite general hypotheses of bounded geometry, then a solution of the flow equation exists for a short time. One shortfall of this theorem is that it does not address the fundamental issue of whether the flow preserves any fine asymptotic structure that the metric may have; indeed, it is not immediately apparent from his result whether the quasi-isometry type is preserved, although that was addressed later by Hamilton [4]. This is a key point in the discussion below. At around the same period, Wu [14] obtained some results about the long-time behaviour of Ricci flow for fairly general metrics on  $\mathbb{R}^2$ , cf. also the recent paper of Isenberg and Javaheri [6] which completes more of that story. Wu did

prove that the aperture at infinity, which is some sort of measure of asymptotic cone angle of the metric, remains constant. Certain initial metrics flow toward the flat metric on  $\mathbb{R}^2$ , but others flow toward more general objects, for example the cigar-shaped soliton metric.

In the past several years there have been several papers concerning the behaviour of Ricci flow on higher dimensional manifolds which are asymptotically Euclidean. Motivation for this comes from physics. We refer in particular to the work of Oleinik and Woolgar [11], Schnürer, Schulze and Simon [12] and most pertinently to our work, Dai and Ma [2]. The last paper works in the slightly more general setting of ALE (asymptotically locally Euclidean) spaces, and proves, in particular that in the AE setting, the ADM mass is preserved under the flow.

There are two main types of results one might hope to obtain concerning limiting behaviour of Ricci flow. The first addresses stability: if  $(M, \bar{g})$  is a canonical metric which is preserved under Ricci flow (i.e., either an Einstein metric or a soliton), and if  $g_0$  is any perturbation of  $\bar{g}$ , then stability means that the Ricci flow starting at  $g_0$  converges to  $\bar{g}$  as  $t \rightarrow \infty$ . For the second, we do not a priori assume the existence of a canonical metric, but produce it as the limit of the flow; this yields the dramatic results (e.g. the Poincaré conjecture and geometrization theorem in three dimensions). For the Ricci flow on surfaces, we are almost always studying problems of the first type since by the general uniformization theorem, there is always a complete constant Gauss curvature metric in any given conformal class (see [10] for a PDE proof of this fact). Note however that this canonical metric may not be quasi-isometric to the initial metric.

Here is a first example of this type of theorem, where everything works particularly simply.

**Theorem 1** (Ji-M-Sesum [9]). *Let  $(\Sigma, g_0)$  be a complete noncompact Riemannian surface where each end is asymptotic to a finite area hyperbolic cusp. Suppose that  $\chi(\Sigma) < 0$ , so that there exists a unique uniformizing metric  $\bar{g}$  conformal to  $g_0$  (in this case,  $\bar{g}$  is quasi-isometric to  $g_0$ ). Then if  $\rho = 4\pi\chi(\Sigma)/\text{Area}(\Sigma, g_0)$ , (1) admits a unique solution  $g(t)$  which exists for all  $t \geq 0$  and such that  $g(t) \rightarrow \bar{g}$ .*

For simplicity here we omit discussion of any regularity issues, as well as the precise rates of decay of the asymptotics along the end.

There are three main steps to proving this theorem. The first involves establishing a short-time existence result for (1) in the class of metrics with ends which are asymptotic to hyperbolic cusps. This requires demonstrating good mapping properties for the solution operator for the linearization of this equation at a metric of this type. The second is to obtain long-time existence, which is done using the notion of a potential function, i.e. a function satisfying  $\Delta\phi = R - \bar{R}$ , where  $\bar{R}$  is the average value of  $R$ , as in Hamilton's original paper [5]. More specifically, a simple maximum principle argument shows that  $R$  is uniformly bounded below for all  $t$ ; another maximum principle argument applied to the function  $Z := \Delta\phi + |\nabla\phi|^2$  gives an upper bound for  $R$ ; these two bounds can then be used to show that the conformal factor  $u$  in (2) is uniformly bounded on each bounded time interval  $[0, T)$ , and standard bootstrapping shows that the flow continues for all time. The

third and final step is to show that the metric converges to a hyperbolic one, and in this particular setting this follows very simply and almost exactly in the case of closed surfaces of negative Euler characteristic using the upper and lower estimates for  $R$ .

The somewhat novel part of this argument lies in the linear analysis needed to prove the existence of a potential function  $\phi$  for which  $\Delta\phi$  and  $|\nabla\phi|$  are both bounded (these are both necessary in order to apply Hamilton's comparison argument for  $Z$ ). The  $L^2$  spectral theory of the Laplace operator on surfaces with cusp ends is well-known and one could simply let  $\phi$  be the unique  $L^2$  solution which has average value 0. This does turn out to be the correct solution, but it is not at all clear that this has bounded gradient. Our proof involves an examination of the asymptotic expansion of  $\phi$  along each end and a slightly delicate computation (of a linear algebraic nature) involving the scattering matrix of the Laplacian.

The next case that has been studied is when  $(\Sigma, g_0)$  is a complete surface with asymptotically Euclidean, or slightly more generally, asymptotically conic ends. For technical reasons, we also assume that  $\chi(\Sigma) < 0$ . There is a bit of a surprise here, since just as in the previous result, the uniformizing metric is a hyperbolic metric of finite area, which has cusp ends and hence is not quasi-isometric to  $g_0$ . In work currently in progress, in collaboration with Isenberg and Sesum, we prove the following.

**Theorem 2.** *Let  $(\Sigma, g_0)$  be a complete surface with asymptotically conic ends, and with  $\chi(\Sigma) < 0$ . The Ricci flow admits a unique solution  $g(t)$ , which exists for all  $t \geq 0$ , and which is asymptotically conic for all  $t$ . There exists a uniform constant  $C > 0$  and for every compact set  $K \subset \Sigma$  a constant  $C_K > 0$  such that  $C_K(1+t)g_0 \leq g(t) \leq Cg(t)$  for all  $t \geq 0$ . Hence  $\tilde{g}(t) := t^{-1}g(t)$  is uniformly controlled on any compact set. This rescaled metric  $\tilde{g}(t)$  converges, as  $t \rightarrow \infty$ , to a complete hyperbolic metric.*

The first part of this theorem states that  $g(t)$  is 'inflating' at a rate proportional to  $t$ , but the lower bound is only uniform on compact sets. Hence if one rescales by  $1/t$ , then the geometry at infinity collapses as  $t \rightarrow \infty$ , although  $\tilde{g}(t)$  converges locally uniformly. The paper [7] contains further information about the nature of the incompleteness of the limiting metric. We note that current work of Dai and Wei also treats the Ricci flow for asymptotically Euclidean surfaces.

The steps in the proof of this result are much the same as before: one first proves the well-posedness of (1) within the class of asymptotically conic metrics, then produces a potential function and shows using the maximum principle that the flow exists for all time. This last step is significantly easier than in the asymptotically cusp case, but still requires some knowledge of the asymptotic behaviour of solutions of the Laplace equation on this class of surfaces. The linear upper bound is proved using a simple adaptation of an argument due to Aronson and Benilan. On the other hand, the locally uniform linear lower bound is more complicated and is proved somewhat indirectly. The completeness of the limiting metric follows from Hamilton's compactness theorem for solutions of Ricci flow.

These two theorems suggest that an interesting general problem would be to study the limiting behaviour of Ricci flow if one starts with any initial metric which has finite total curvature. This is a class of particularly interesting metrics. By classical results of Huber and Osserman, any such surface is conformal to punctured compact Riemann surfaces. The two results above are two of the easiest cases of this problem. Some other cases should not be so difficult, but this problem may be difficult in full generality.

We mention one other related recent result, by Albin, Aldana and Rochon [1]. If  $(\Sigma, g_0)$  is a surface where  $g_0$  is a conformally compact metric, i.e. one which is asymptotically hyperbolic and asymptotic to an infinite area hyperbolic ‘funnel’, then without any assumptions on  $\chi(\Sigma)$ , the Ricci flow exists for all time and converges to the unique complete infinite area hyperbolic metric in the conformal class of  $g_0$ . These authors also consider a certain regularized determinant of the Laplacian under this flow, both in this setting as well as in the asymptotically hyperbolic cusp one, and prove that it is monotone under this flow.

We conclude with a brief description of current work in progress, with Rubinstein and Sesum, which treats this problem in the setting of incomplete metrics with isolated conic singularities on a compact surface  $\Sigma$ . The first major difficulty here is to define the flow on this class of metrics in such a way that the conical singularities (as well as the values of the individual cone angles) are preserved under the flow. This is in contrast to recent work by Giesen and Topping [3], in which they prove the existence of a *smooth* family of metrics  $g(t)$  solving (1) and with  $g(0)$  an incomplete metric of a fairly general type. The existence of a suitable potential function requires some knowledge of the asymptotic behaviour of solutions of  $\Delta\phi = f$  at the conic points, and in order to use this potential function to obtain long-time existence one must prove that one can still apply similar maximum principle arguments, see [8].

The reason this problem is particularly interesting is that there does not always exist a constant curvature metric on  $\Sigma$  in the given conformal class and with prescribed cone angles. Indeed, assuming that all cone angles are less than  $2\pi$ , there is a simple linear condition on the cone angles, discovered originally by Troyanov, which is necessary and sufficient for the existence of a constant curvature metric. The goal of our work is to prove that in the cases where a constant curvature metric does exist, the flow converges to that metric, whereas in the remaining cases (still assuming that all cone angles are less than  $2\pi$ ) where no constant curvature metric exists, the flow still has a limit, which now is a Ricci soliton. If successful, this would provide a rich supply of new and interesting soliton metrics.

#### REFERENCES

- [1] P. Albin, C. Aldana and F. Rochon, *Ricci flow and the determinant of the Laplacian on noncompact surfaces*, arXiv:0909.0807
- [2] X. Dai and L. Ma, *Mass under the Ricci flow*, Comm. Math. Phys. **274** (2007) no. 1, 65-80.
- [3] G. Giesen and P. Topping, *Ricci flow of negatively curved incomplete surfaces*, arXiv:0906.3309

- 
- [4] R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry, International Press **2** (1995), 7-136.
  - [5] R. Hamilton, *The Ricci flow on surfaces*, Mathematics and General Relativity, Contemporary Mathematics **71** (1988), 237-261.
  - [6] J. Isenberg and Javaheri, *Convergence of Ricci flow on  $\mathbb{R}^2$  to flat space*, arXiv:0908.2264
  - [7] J. Isenberg, R. Mazzeo and N. Sesum, *The Ricci flow on asymptotically Euclidean surfaces*, in preparation.
  - [8] T. Jeffres, *A maximum principle for parabolic equations on manifolds with conic singularities*, Adv. in Geom. **5** (2005), 319-323.
  - [9] L. Ji, R. Mazzeo and N. Sesum, *Ricci flow on surfaces with cusps*, Math. Ann. **345** (2009) no. 4, 819-834.
  - [10] R. Mazzeo and M. Taylor, *Curvature and uniformization*, Israel J. Math. **130** (2002), 323-346
  - [11] T.A. Oleinyk and E. Woolgar, *Asymptotically flat Ricci flows*, Comm. Anal. Geom. **15** (2007), 535-568.
  - [12] O. Schnürer, F. Schulze and M. Simon, *Stability of Euclidean space under Ricci flow*, Comm. Anal. Geom. **16** (2008) no. 1, 127-158.
  - [13] Shi, W.-X., *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30** (1989), 223-301.
  - [14] L.F. Wu, *The Ricci flow on 2-orbifolds with positive curvature*, J. Diff. Geom. **33** (1991), 575-596.

## Participants

**Dr. Spiros Alexakis**

Department of Mathematics  
MIT  
Cambridge , MA 02139  
USA

**Dr. Paul Tyler Allen**

Lewis & Clark College  
Dept. of Mathematical Sciences  
MSC 110  
0615 S.W. Palatine Hill Road  
Portland OR 97219  
USA

**Prof. Dr. Michael T. Anderson**

Department of Mathematics  
Stony Brook University  
Stony Brook , NY 11794-3651  
USA

**Dr. Hakan Andreasson**

Department of Mathematics  
Chalmers University of Technology  
S-412 96 Göteborg

**Aneta Barbos**

MPI für Gravitationsphysik  
Albert-Einstein-Institut  
Am Mühlenberg 1  
14476 Golm

**Prof. Dr. Robert Bartnik**

School of Mathematical Sciences  
Monash University  
Clayton Victoria 3800  
AUSTRALIA

**Prof. Dr. Robert Beig**

Fakultät für Physik  
Universität Wien  
Boltzmannngasse 5  
A-1090 Wien

**Roger Bieli**

MPI für Gravitationsphysik  
Albert-Einstein-Institut  
Am Mühlenberg 1  
14476 Golm

**Dr. Lydia Bieri**

Department of Mathematics  
Harvard University  
One Oxford Street  
Cambridge , MA 02138  
USA

**Prof. Dr. Piotr Bizon**

Institute of Physics  
Jagiellonian University  
ul. Reymonta 4  
30-059 Krakow  
POLAND

**Pieter Blue**

School of Mathematics  
University of Edinburgh  
James Clerk Maxwell Bldg.  
King's Buildings, Mayfield Road  
GB-Edinburgh EH9 3JZ

**Prof. Dr. Piotr T. Chrusciel**

Departement de Mathematiques  
Faculte des Sciences  
Universite de Tours  
Parc de Grandmont  
F-37200 Tours

**Julien Cortier**

Universite Montpellier II  
Institut de Mathematiques et de  
Modelisation  
Place E. Bataillon  
F-34000 Montpellier

**Prof. Dr. Justin Corvino**

Department of Mathematics  
Lafayette College  
Easton , PA 18042-1781  
USA

**Dr. Mihalis Dafermos**

Dept. of Pure Mathematics and  
Mathematical Statistics  
University of Cambridge  
Wilberforce Road  
GB-Cambridge CB3 0WB

**Dr. Sergio Dain**

Fa. M. A. F.  
Universidad Nacional de Cordoba  
Medina Allende s/n  
5000 Cordoba  
ARGENTINA

**Prof. Dr. Erwann Delay**

Laboratoire d'Analyse non lineaire  
et Geometrie  
Universite d'Avignon  
33, rue Louis Pasteur  
F-84000 Avignon

**Prof. Dr. Bernold Fiedler**

Institut für Mathematik I (WE 1)  
Freie Universität Berlin  
Arnimallee 2-6  
14195 Berlin

**Prof. Dr. Felix Finster**

Naturwissenschaftliche Fakultät I  
Mathematik  
Universität Regensburg  
93040 Regensburg

**Dr. Helmut Friedrich**

MPI für Gravitationsphysik  
Albert-Einstein-Institut  
Am Mühlenberg 1  
14476 Golm

**Prof. Dr. Gregory Galloway**

Dept. of Mathematics and Computer  
Science  
University of Miami  
P.O. Box 248011  
Coral Gables , FL 33124  
USA

**Dr. Mark Heinzle**

Fakultät für Physik  
Universität Wien  
Boltzmanngasse 5  
A-1090 Wien

**Dr. Jörg Hennig**

MPI für Gravitationsphysik  
Albert-Einstein-Institut  
Am Mühlenberg 1  
14476 Golm

**Prof. Dr. Michael Holst**

Department of Mathematics  
University of California, San Diego  
9500 Gilman Drive  
La Jolla , CA 92093-0112  
USA

**Gustav Holzegel**

Department of Mathematics  
Princeton University  
609 Fine Hall  
Washington Road  
Princeton , NJ 08544  
USA

**Prof. Dr. Gerhard Huisken**

MPI für Gravitationsphysik  
Albert-Einstein-Institut  
Am Mühlenberg 1  
14476 Golm

**Prof. Dr. James Isenberg**

Department of Mathematics  
University of Oregon  
Eugene , OR 97403-5203  
USA

**David Klawonn**

MPI für Gravitationsphysik  
Albert-Einstein-Institut  
Am Mühlenberg 1  
14476 Golm

**Prof. Dr. Philippe G. LeFloch**

Laboratoire Jacques-Louis Lions  
Universite Paris 6  
4, Place Jussieu  
F-75252 Paris Cedex 05

**Prof. Dr. Marc Mars**

Dept. Fisica Fundamental  
Universidad de Salamanca  
E-37008 Salamanca

**Prof. Dr. David Maxwell**

Department of Mathematical Sciences  
University of Alaska Fairbanks  
PO Box 756660  
Fairbanks , AK 99775-6660  
USA

**Prof. Dr. Rafe Mazzeo**

Department of Mathematics  
Stanford University  
Stanford , CA 94305-2125  
USA

**Dr. Jan Metzger**

MPI für Gravitationsphysik  
Albert-Einstein-Institut  
Am Mühlenberg 1  
14476 Golm

**Prof. Dr. Pengzi Miao**

Dept. of Mathematics  
Monash University  
Clayton , Victoria 3168  
AUSTRALIA

**Dr. Todd Oliynyk**

School of Mathematical Sciences  
Monash University  
Clayton Victoria 3800  
AUSTRALIA

**Prof. Dr. Frank Pacard**

Centre de Mathematiques  
Faculte de Sciences et Technologie  
Universite Paris XII - Val de Marne  
61 Avenue du General de Gaulle  
F-94010 Creteil Cedex

**Prof. Dr. Gerhard Rein**

Mathematisches Institut  
Universität Bayreuth  
95440 Bayreuth

**Prof. Dr. Alan Rendall**

MPI für Gravitationsphysik  
Albert-Einstein-Institut  
Am Mühlenberg 1  
14476 Golm

**Prof. Dr. Bernd Schmidt**

MPI Potsdam  
Am Palais 10  
14469 Potsdam

**Jacques Smulevici**

Queens' College  
GB-Cambridge CB3 9ET

**Prof. Dr. Jared Speck**

Department of Mathematics  
Princeton University  
Fine Hall  
Washington Road  
Princeton , NJ 08544-1000  
USA

**Prof. Dr. Michael Struwe**

Departement Mathematik  
ETH-Zentrum  
Rämistr. 101  
CH-8092 Zürich

**Nikodem Szpak**

Institut für Physik der  
Universität Duisburg-Essen  
Lotharstr. 1  
47048 Duisburg

**Prof. Dr. Daniel Tataru**

Department of Mathematics  
University of California  
Berkeley , CA 94720-3840  
USA

**Prof. Dr. Gantumur Tsogtgerel**

Dept. of Mathematics and Statistics  
McGill University  
805, Sherbrooke Street West  
Montreal , P.Q. H3A 2K6  
CANADA

**Prof. Dr. Juan J. L. Velazquez**

Instituto Ciencias Matematicas (ICMAT)  
Universidad Complutense  
Plaza de Ciencias 3  
E-28040 Madrid

**Dr. Catherine Williams**

Department of Mathematics  
Stanford University  
Stanford , CA 94305-2125  
USA

