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Convex Geometry and its Applications

Organised by
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November 29th – December 5th, 2009

Abstract. The geometry of convex domains in Euclidean space plays a central role in several branches of mathematics: functional and harmonic analysis, the theory of PDE, linear programming and, increasingly, in the study of other algorithms in computer science. High-dimensional geometry, both the discrete and convex branches of it, has experienced a striking series of developments in the past 10 years. Several examples were presented at this meeting, for example the work of Rudelson et al. on conjunction matrices and their relation to confidential data analysis, that of Litvak et al. on remote sensing and a series of results by Nazarov and Ryabogin et al. on Mahler's conjecture for the volume product of domains and their polars.

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Introduction by the Organisers

The meeting *Convex Geometry and its Applications* organised by Keith Ball, Martin Henk and Monika Ludwig, was held from November 29 to December 5, 2009. The meeting was attended by some 50 participants working in all areas of high-dimensional geometry. The program involved 10 plenary lectures of one hour's duration and about 20 shorter lectures. Some highlights of the program were as follows.

Alexander Barvinok gave an extremely intriguing talk about new approximate formulae for the volume of (or number of lattice points inside) a body determined by linear programming constraints. These formulae are based on a surprising use of the central limit theorem and its refinements, and are accurate under quite weak conditions on the constraints. For example, the formulae are correct apart from a constant factor for transportation polytopes: an astonishing degree of accuracy in high-dimensional spaces.

Mark Rudelson gave a very clear talk about the amount of noise that must be added to a contingency table of confidential attributes, before the release of statistics based on the table, in order to ensure the privacy of the individuals represented. The problem is to add the minimum amount of noise that will make reconstruction of the table impossible from the statistics that are made public. This minimum amount of noise is determined by the least singular value of the conjunction matrix (formed by entry-wise multiplication of the rows of the contingency matrix). Rudelson and his collaborators were able to employ an array of machinery concerning smallest singular values, developed mainly by participants at this meeting (especially Rudelson himself).

In a related vein, Alexander Litvak spoke about estimates for the largest and smallest singular values for random matrices with columns that are independent but whose entries are not. This considerably weakens the conditions under which such estimates have been found and makes it possible to answer completely, a question of Kannan, Lovász and Simonovits on the empirical sample size needed to estimate the covariance matrix of a domain. The results also extend the range in which the remote sensing method of Candes, Donoho and Tao can be applied.

There has been a sudden upsurge in interest in the famous conjecture of Mahler that the product of the volumes of a domain and its polar should be minimised by the simplex (or the cube/cross-polytope pair in the symmetric case). Franck Barthe explained the surprisingly delicate proof of his result with Matthieu Fradelizi: that the conjecture holds for domains having many symmetries. Dmitry Ryabogin explained his recent joint work showing that the cube is a local minimiser for the volume product, among symmetric domains. He also gave an impromptu evening lecture on Nazarov's complex-analytic proof of the approximate Mahler conjecture proved by Bourgain and Milman.

Guillaume Aubrun presented his very elegant proof of the recent result of Aldaz on the unboundedness (as a function of dimension) of the weak 1-1 norm of the maximal operator for high-dimensional cubes. Aubrun's proof uses accurate probabilistic tools for counting lattice points in high-dimensional cubes and yields a stronger lower bound than the original proof.

There were several excellent talks by young researchers. Luis Rademacher presented his solution to a problem that had become well-known from the work of Bárány, Vu, Reitzner and others: is it true that if K and L are convex domains with $K \subset L$, then a random simplex in K (a simplex with corners chosen independently at random from K) has smaller expected volume than a random simplex in L? Bizarrely, the answer is no and this helps to explain the difficulty in estimating volumes. Eugenia Saorín presented her joint solution of a problem going back to Hadwiger on the differentiability of extensions of the classical quermassintegrals. Hadwiger originally asked the question only in dimension 3 but the characterisation given here extends to higher dimensions. David Alonso-Gutiérrez spoke about his joint work on the slicing conjecture for domains with few vertices giving a simplified proof of the result of Junge in this direction and establishing a bound independent of dimension for domains whose number of vertices is proportional to dimension. Gergely Ambrus discussed the polarisation problem, which arises from the study of polynomials on normed spaces, and his remarkable solution of the 2-dimensional case of the *strong* polarisation problem, using Blaschke products. There was widespread view that the (relatively) new arrangements to support young visitors to Oberwolfach are paying off handsomely.

Workshop: Convex Geometry and its Applications

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Abstracts

Gaussian formulas for volumes and the number of integer points in polytopes

Alexander Barvinok
(joint work with J.A. Hartigan)

Let $P \subset \mathbb{R}^n$ be a polytope and let $\mathbb{Z}^n \subset \mathbb{R}^n$ be the standard integer lattice. We are interested in the following two questions:

How to compute or estimate the volume $\mathrm{vol}(P)$ of P?

How to compute or estimate the number $|P \cap \mathbb{Z}^n|$ of integer points in P?

We assume that the polytope P is defined as the intersection of a d-dimensional affine subspace with the non-negative orthant $\mathbb{R}^n_+$. Using coordinates, we define P by a system of linear equations $Ax = b$ and inequalities $x \geq 0$. Here A is a $d \times n$ matrix of rank d, $A = (a_{ij})$, and b is a d-vector.

0.1. **The Gaussian formula for volumes.** Let

$$f(x) = n + \sum_{i=1}^{n} \ln \xi_i \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n)\,.$$

Then $f(x)$ is a strictly concave function on $\mathbb{R}^n_+$ and hence attains its maximum on P at a unique point $z = (\zeta_1, \ldots, \zeta_n)$. The maximum point z can be efficiently computed by interior point methods, see [7].

Let us compute $d \times d$ matrices, $C = (c_{ij})$ and $Q = (q_{ij})$ by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} \quad \text{and} \quad q_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} \zeta_k^2.$$

We approximate the volume of P:

$$\mathrm{vol}(P) \approx \frac{e^{f(z)} \sqrt{\det C}}{(2\pi)^{d/2} \sqrt{\det Q}}. \tag{1}$$

0.2. **The Gaussian formula for the number of integer points.** Here we assume, additionally, that A is an integer matrix, b is an integer vector and that the columns of A generate integer lattice $\mathbb{Z}^d$. Let

$$g(x) = \sum_{i=1}^{n} (\xi_i + 1) \ln (\xi_i + 1) - \xi_i \ln \xi_i \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n)\,.$$

Then g is a strictly concave on $\mathbb{R}^n_+$ and attains its maximum on P at a unique point $z = (\zeta_1, \ldots, \zeta_n)$, which can be efficiently computed. We compute a $d \times d$ matrix $Q = (q_{ij})$ by

$$q_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} \left(\zeta_k^2 + \zeta_k\right)$$

and approximate the number of integer points in P:

$$|P \cap \mathbb{Z}^n| \approx \frac{e^{g(z)}}{(2\pi)^{d/2}\sqrt{\det Q}}. \tag{2}$$

0.3. **Examples.** The following examples were computed by De Loera [6]. Formula (2) overestimates the true number of 4×4 non-negative integer matrices with the row sums 220, 215, 93, and 64 and the column sums 108, 286, 71, and 127 by about 6%. In this case, the polytope P is defined in the 16-dimensional space by a system of 7 equations (the row and column sums are not independent). If, instead, we apply formula (2) to estimate the number of $3 \times 3 \times 3$ non-negative integer arrays with the sums $[31, 22, 87], [50, 13, 77], [42, 87, 11]$ along the coordinate hyperplanes, the relative error drops to about 0.185%. Here the polytope P is defined in the 27-dimensional space by a system of 7 equations.

Obviously, there are cases where formula (1) and especially (2) produce estimates which are very far from the truth. General theorems stating sufficient conditions for formulas (1) and (2) to hold asymptotically, as well as applications to multi-way transportation polytopes (multi-index arrays of non-negative numbers with prescribed sums along coordinate affine hyperplanes) are given in [2]. Curiously, for ordinary transportation polytopes (polytopes of non-negative matrices with prescribed rows and column sums) formulas (1) and (2) hold up to a constant correction factor. For example, for the polytope of doubly stochastic matrices the approximation (1) should be multiplied by $e^{1/3}$, as follows from [5]. Correction factors in formula (2) in the case of all row sums being equal and all column sums being equal are found from the asymptotic formulas of [4], while in the case of general row and column sums the correction factors are computed in [3]. Those correction factors represent, essentially, the *Edgeworth correction* to the Gaussian distribution.

We present below some intuition behind formulas (1) and (2), which also explains the name "Gaussian".

Recall that a random variable is *exponential* if its density is ae^{-ax} for $x \geq 0$, where $a > 0$ is a constant, and 0 for $x < 0$.

Theorem 1. *Let $x_1, \ldots, x_n$ be independent exponential random variables such that*

$$\mathbf{E}\, x_i = \zeta_i \quad \textit{for} \quad i = 1, \ldots, n,$$

where $z = (\zeta_1, \ldots, \zeta_n)$ is the solution of the optimization problem of Section 0.1. Then the density of random vector $X = (x_1, \ldots, x_n)$ is constant on P and equal to $e^{-f(z)}$ at every point of P.

Denoting $Y = AX$ we obtain a d-dimensional random vector such that

$$\mathrm{vol}(P) = e^{f(z)}\sqrt{\det C} \cdot (\text{the density of } Y \text{ at } b)\,.$$

We observe that $\mathbf{E}\, Y = b$, that the covariance matrix of Y is matrix Q of Section 0.1 and that Y is a weighted sum of n independent random d-vectors. We obtain formula (1) if we assume, in the spirit of a Local Central Limit Theorem, that Y is approximately Gaussian.

Recall that a random variable x is *geometric* if $\mathbf{P}\,(x = k) = pq^k$ for $k = 0, 1, 2, \ldots$ and some positive $p + q = 1$.

Theorem 2. *Let $x_1, \ldots, x_n$ be independent geometric random variables such that*

$$\mathbf{E}\, x_i = \zeta_i \quad \textit{for} \quad i = 1, \ldots, n,$$

where $z = (\zeta_1, \ldots, \zeta_n)$ is the solution of the optimization problem of Section 0.2. Then the probability mass function of vector $X = (x_1, \ldots, x_n)$ is constant on $P \cap \mathbb{Z}^n$ and equal to $e^{-g(z)}$ at every integer point of P.

Denoting $Y = AX$ we obtain a random d-dimensional random vector such that

$$|P \cap \mathbb{Z}^n| = e^{g(z)} \mathbf{P}\,(Y = b).$$

We observe that $\mathbf{E}\, Y = b$, that the covariance matrix of Y is matrix Q of Section 0.2 and that Y is a weighted sum of n independent random d-vectors. We obtain formula (2) if we assume, in the spirit of a Local Central Limit Theorem, that Y is approximately Gaussian.

We observe that function $f(x)$ of Section 0.1 is the maximum possible entropy of a probability distribution on $\mathbb{R}^n_+$ with the expectation x and that the corresponding maximum entropy distribution is necessarily the product of independent exponential distributions of the coordinates of x. Similarly, function $g(x)$ of Section 0.2 is the maximum possible entropy of a probability distribution on $\mathbb{Z}^n_+$ with the expectation x and the corresponding maximum entropy distribution is necessarily the product of independent geometric distributions of the coordinates of x. In [2] it is shown how to apply the maximum entropy principle to obtain similar Gaussian approximation formulas in related situations, such as to count 0-1 vectors in a given polytope P.

Finally, we remark that for a wider class of polytopes one can obtain [1] less precise *large deviation approximation formulas*

$$\ln \mathrm{vol}(P) \approx f(z) \quad \text{and} \quad \ln |P \cap \mathbb{Z}^n| \approx g(z).$$

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Stability of some inequalities related to the Prékopa-Leindler inequality

Károly J. Böröczky

[1] This is a truncated version of my talk, and discusses stability versions of the Brunn-Minkowski, Prékopa-Leindler and the Blaschke-Santaló inequalities. We work in $\mathbb{R}^n$. K and M denote convex bodies (compact convex sets with non-empty interior), $|K|$ and $|M|$ denote their volumes, and o denotes the origin. Let γ always denote a constant depending only on the dimension n.

The "mother of all geometric inequalities", is the Brunn-Minkowski inequality from the end of the 19th century (see [8], [16], [21]).

THEOREM 1 (Brunn-Minkowski). *If $|K| = |M| = 1$ and $\lambda \in (0, 1)$, then*

$$|\lambda K + (1 - \lambda)M)| \geq 1,$$

with equality if and only if K and M are translates.

Already H. Minkowski himself provided some stability version in the plane. In any dimension, the stability version in terms of "homothetic distance" is known (see [15]).

THEOREM 2 (Diskant). *If $|K| = |M| = 1$, o is the centroid of K and M, and $\lambda \in (0, \frac{1}{2}]$, then for $h = \min\{\ln t : t^{-1}K \subset M \subset tK\} \leq n$, we have*

$$|\lambda K + (1 - \lambda)M)| \geq 1 + \gamma \cdot \lambda^{\frac{n}{2}} \cdot h^n.$$

Since the Brunn-Minkowski inequality is about volume, it is more natural to have a stability version in terms of volume. The estimate of optimal order is provided in [11] and [12].

THEOREM 3 (Figalli, Maggi, Pratelli). *If $|K| = |M| = 1$, o is the centroid of K and M, and $\lambda \in (0, \frac{1}{2}]$, then*

$$|\lambda K + (1 - \lambda)M)| \geq 1 + \gamma \cdot \lambda \cdot |K \Delta M|^2.$$

Now the Brunn-Minkowski inequality is intimately connected to the Prékopa-Leindler inequality (see [4], [7], [14]).

THEOREM 4 (Prékopa-Leindler). *If $\lambda \in (0, 1)$, and m, f, g are non-negative integrable functions on $\mathbb{R}^n$ satisfying $m(\lambda x + (1 - \lambda)y)) \geq f(x)^\lambda g(y)^{1-\lambda}$ for $x, y \in \mathbb{R}^n$, and $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$, then*

$$\int_{\mathbb{R}^n} m \geq 1.$$

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Equality if and only if f, g and m are translates of the same log-concave function up to a set of measure zero.

For a possible stability version (see Conjecture 1 and Theorem 5), we fix $\lambda \in (0,1)$, and non-negative integrable functions functions f, g, m on $\mathbb{R}^n$ such that f, g are probability distributions with zero mean, and $m(\lambda x+(1-\lambda)y)) \geq f(x)^\lambda g(y)^{1-\lambda}$ for $x, y \in \mathbb{R}^n$. In other words, we assume $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$ and $\int_{\mathbb{R}^n} xf(x)\,dx = \int_{\mathbb{R}^n} xg(x)\,dx = 0$. A natural notion of distance is the truncated L_1-metric

$$\delta_1(f,g) = \min\left\{1, \int_{\mathbb{R}^n} |f-g|\right\}.$$

Conjecture 1. $\displaystyle\int_{\mathbb{R}^n} m \geq 1 + \gamma \cdot \lambda \cdot \delta_1(f,g)^2.$

If $n = 1$, probably $\int_{\mathbb{R}} m \geq 1 + \gamma \cdot \delta_1(f,g)$. The following partial results are proved in [5] and [6].

THEOREM 5 (Ball, Böröczky)**.** *If m is log-concave, then*

$$\begin{aligned}
\int_{\mathbb{R}} m &\geq 1 + \gamma \cdot \lambda^4 \cdot \delta_1(f,g)^4 \quad \text{if } n = 1,\\
\int_{\mathbb{R}^n} m &\geq 1 + \gamma \cdot \lambda^8 \cdot \delta_1(f,g)^8 \quad \text{if } n \geq 2 \text{ and } f, g \text{ are even.}
\end{aligned}$$

As [2] observed, the Prékopa-Leindler inequality is in turn connected to the Blaschke-Santaló inequality (see [19] and [20] for "modern treatment", and [18] for relations to other geometric inequalities). For this, if $o \in \mathrm{int}K$, then the polar of K is $K^o = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \forall y \in K\}$. In addition B_2, B_∞, and T denote an L_2-ball, and L_∞ ball, and a simplex, whose centroids are the origin.

THEOREM 6 (Blaschke-Santaló)**.** *If o is the centroid of K, then*

$$|K| \cdot |K^o| \leq |B_2|^2,$$

with equality if and only if K is an ellipsoid.

[6] and [9] verify stability versions in terms of the Banach-Mazur distance $\delta_{\mathrm{BM}}(K,M) = \min\{\ln\lambda : K-x \subset \Phi(M-y) \subset \lambda(K-x) \text{ for } \Phi \in \mathrm{GL}(n), x, y \in \mathbb{R}^n\}$.

THEOREM 7 (Böröczky)**.** $|K| \cdot |K^o| \leq (1 - \gamma \cdot \delta_{\mathrm{BM}}(K, B^n)^{5n}) \cdot |B^n|^2$

Here the optimal exponent is probably $(n+1)/2$ instead of $5n$. The subsequent papers [3], [1], [13] and [17] proved the following functional form of the Blaschke-Santaló inequality.

THEOREM 8 (Ball, Artstein-Klartag-Milman, Fradelizi-Meyer, Lehec)**.** *For any measurable $f : \mathbb{R}^n \to \mathbb{R}_+$ with positive integral there exists $z \in \mathbb{R}^n$ such that if measurable $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ with positive integrals satisfy*

$$f(x)g(y) \leq \varrho(\langle x-z, y-z\rangle)^2$$

for every $x, y \in \mathbb{R}^n$ *with* $\langle x - z, y - z \rangle > 0$*, then*

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(x)\,dx \le \left(\int_{\mathbb{R}^n} \varrho(|x|^2)\,dx \right)^2.$$

The equality case is known (see [13] and [17]), and [10] even provides a weak stability version for log-concave functions in terms of $\delta_1(f, g)$. We note that the inequality is partially based on the Prékopa-Leindler inequality, and the argument in [10] uses the known stability version of the one dimensional Prékopa-Leindler inequality.

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How to make quermassintegrals differentiable: solving a problem by Hadwiger

Eugenia Saorín Gómez
(joint work with M. A. Hernández Cifre)

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets in the Euclidean space $\mathbb{R}^n$. The subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior is denoted by $\mathcal{K}^n_0$. Let B_n be the n-dimensional unit ball and $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$. We denote by $\mathrm{V}(M)$ the volume (n-dimensional Lebesgue measure) of a set $M \subset \mathbb{R}^n$ and by $\mathrm{cl}\, M$ its closure.

For two convex bodies $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$ and a non-negative real number λ, the *outer parallel body of K (relative to E) at distance* λ is the Minkowski sum $K + \lambda E$. On the other hand, for $0 \leq \lambda \leq \mathrm{r}(K;E)$ the *inner parallel body of K (relative to E) at distance* λ is the set

$$K \sim \lambda E = \{x \in \mathbb{R}^n : \lambda E + x \subset K\},$$

where the *relative inradius* $\mathrm{r}(K;E)$ of K with respect to E is defined by

$$\mathrm{r}(K;E) = \sup\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\,E \subset K\}.$$

If $E = B_n$, $\mathrm{r}(K;B_n) = \mathrm{r}(K)$ is the classical inradius. Notice that $K \sim \mathrm{r}(K;E)E$ is the set of (relative) incenters of K, usually called *kernel* of K with respect to E. The dimension of $\ker(K;E)$ is strictly less than n. We will write K_λ to denote the (relative) inner/outer parallel bodies of K, i.e.,

$$K_\lambda := \begin{cases} K \sim |\lambda|\, E & \text{for } -\mathrm{r}(K;E) \leq \lambda \leq 0, \\ K + \lambda E & \text{for } 0 \leq \lambda < \infty. \end{cases} \tag{1}$$

The so called *relative Steiner formula* states that the volume of the outer parallel body $K + \lambda E$ is a polynomial of degree n in $\lambda \geq 0$,

$$\mathrm{V}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^i.$$

The coefficients $\mathrm{W}_i(K;E)$ are called *relative quermassintegrals* of K and, in particular, $\mathrm{W}_0(K;E) = \mathrm{V}(K)$ and $\mathrm{W}_n(K;E) = \mathrm{V}(E)$. In [3] the following definition is introduced.

Definition 1. *Let $E \in \mathcal{K}^n_0$. A convex body $K \in \mathcal{K}^n$* belongs to the class $\mathcal{R}_p$*, $0 \leq p \leq n-1$, if for all $0 \leq i \leq p$ and for $-\mathrm{r}(K;E) \leq \lambda < \infty$ it holds*

$${}'\mathrm{W}_i(\lambda) = \mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda).$$

Here ${}'\mathrm{W}_i$ and W_i' denote, respectively, the left and right derivatives of the function $\mathrm{W}_i(\lambda) := \mathrm{W}_i(K_\lambda;E)$. It is a natural definition, since from the concavity of the family (1) and the general Brunn-Minkowski theorem for relative quermassintegrals (see e.g. [8, p. 339]), we get ${}'\mathrm{W}_i(\lambda) \geq \mathrm{W}_i'(\lambda) \geq (n-i)\mathrm{W}_{i+1}(\lambda)$ for $i = 0, \ldots, n-1$. Moreover, it holds (see e.g. [1]) that the volume is always differentiable and $\mathrm{V}'(\lambda) = n\mathrm{W}_1(\lambda)$. Last property implies that $\mathcal{R}_0 = \mathcal{K}^n$ and moreover,

$\mathcal{R}_{i+1} \subset \mathcal{R}_i$ strictly, $i = 0, \ldots, n-2$. The problem of determining the convex bodies belonging to the class $\mathcal{R}_p$ was originally posed by Hadwiger in dimension 3 and for $E = B_3$ (see [2, §23]). In [3] the general n-dimensional problem is studied; in particular, it is shown that the smallest class is given by

$$\mathcal{R}_{n-1} = \bigl\{K = L + \lambda E : L \in \mathcal{K}^n,\ \dim L \le n-1,\ \lambda \ge 0\bigr\}.$$

We have determined the convex bodies lying in the class $\mathcal{R}_{n-2}$, which solves the original Hadwiger problem: to classify the convex bodies in $\mathbb{R}^3$ depending on the differentiability of their quermassintegrals.

In order to state the result we need some further definitions. As usual in the literature, we write $h(K,u) = \sup\bigl\{\langle x, u\rangle : x \in K\bigr\}$, $u \in \mathbb{R}^n$, to denote the *support function* of $K \in \mathcal{K}^n$. On the other hand, a vector $u \in \mathbb{S}^{n-1}$ is an *r-extreme normal vector* of K, $0 \le r \le n-1$, if we cannot write $u = u_1 + \cdots + u_{r+2}$, with u_i linearly independent normal vectors at one and the same boundary point of K. We denote the set of r-extreme normal vectors of K by $\mathcal{U}_r(K)$. Then, the (relative) *form body* of a convex body $K \in \mathcal{K}_0^n$ with respect to $E \in \mathcal{K}_0^n$, denoted by K^*, is defined as

$$K^* = \bigcap_{u \in \mathcal{U}_0(K)} \bigl\{x : \langle x, u\rangle \le h(E,u)\bigr\}.$$

Finally, a convex body $K \in \mathcal{K}^n$ containing $E \in \mathcal{K}_0^n$ is called a *p-tangential body* of E, $p \in \{0, \ldots, n-1\}$, if each $(n-p-1)$-extreme support plane of K supports E [8, pp. 75–76]. Here a support plane is said to be p-extreme if its outer normal vector is a p-extreme normal vector of K.

In [5] we prove the announced characterization theorem for the class $\mathcal{R}_{n-2}$.

Theorem 1. *Let $E \in \mathcal{K}_0^n$ be regular and strictly convex. The only sets in $\mathcal{R}_{n-2}$ are 1-tangential bodies of convex bodies lying in $\mathcal{R}_{n-1}$ which satisfy*

$$\mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*) \tag{2}$$

for $-\mathrm{r} \le \lambda \le 0$, and their outer parallel bodies.

In dimension $n = 3$, with $E = B_3$ and with the original notation by Hadwiger for the classes $\mathcal{R}_p$, namely, $\mathcal{R}_\beta = \mathcal{R}_1$ and $\mathcal{R}_\gamma = \mathcal{R}_2$, the result is stated as follows.

Corollary 1. *The only sets in $\mathcal{R}_\gamma$ are outer parallel bodies of k-dimensional convex bodies, for $0 \le k \le 2$, i.e., $\mathcal{R}_\gamma = \bigl\{K = L + \lambda B_3 : L \in \mathcal{K}^3,\ \dim L \le 2,\ \lambda \ge 0\bigr\}$.*

The only sets in $\mathcal{R}_\beta$ are 1-tangential bodies of convex bodies lying in $\mathcal{R}_\gamma$ which satisfy condition (2) *for $-\mathrm{r} \le \lambda \le 0$, and their outer parallel bodies.*

In order to prove the characterization of $\mathcal{R}_{n-2}$, several lemmas concerning the geometry of tangential bodies satisfying condition (2), and in particular of form bodies, are needed. The main steps in the proof consist of showing that condition (2) is necessary for a convex body lying in the class $\mathcal{R}_{n-2}$, and the following characterization theorem.

Theorem 2 ([4])**.** *Let $K \in \mathcal{K}_0^n$. Then $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \le \lambda \le 0$ if and only if K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying* (2) *for all $-\mathrm{r} \le \lambda \le 0$.*

Then the final crucial steps are to show that every body lying in $\mathcal{R}_{n-2}$ can be decomposed as $K = K_\lambda + |\lambda|\, K^*$, for every $-\mathrm{r} \leq \lambda \leq 0$, and that the only tangential bodies lying in $\mathcal{R}_{n-2}$ are 1-tangential bodies.

We can ask if there is some "geometry" behind condition (2), i.e., how does a convex body $K \in \mathcal{R}_{n-2}$ look like? It is a 1-tangential body of an outer parallel body of a (strictly) lower dimensional convex body. But any 1-tangential body is not valid: the additional points which determine the set when constructing the convex hull with $K_{-\mathrm{r}} + \mathrm{r}E$ cannot lie anywhere. For instance, if $\dim K_{-\mathrm{r}} = 1$ and $E = B_2$, i.e., if $K_{-\mathrm{r}} + \mathrm{r}B_2 := S$ is a sausage, then those points should lie in the (infinite) cylinder containing S with 2-dimensional spherical cross section $\mathrm{r}B_2$; otherwise the kernel $K_{-\mathrm{r}}$ would not be a summand of K and, moreover, 1-extreme normal vectors would appear when taking $K_{-\mathrm{r}} + \mathrm{r}K^*$, contradicting condition (2).

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On the monotonicity of the expected volume of a random simplex

Luis Rademacher

Let a random simplex in a d-dimensional convex body be the convex hull of $d+1$ random points from the body. We study the following question: As a function of the convex body, is the expected volume of a random simplex monotone non-decreasing under inclusion? We show that this holds if d is 1 or 2, and does not hold if $d \geq 4$. We also prove similar results for the second moment of the volume of a random simplex and the determinant of the covariance matrix of a convex body.

In [8], Mark Meckes asked whether for any pair of convex bodies $K, L \subseteq \mathbb{R}^d$, $K \subseteq L$ implies

$$\mathbb{E}_{X_0,\dots,X_d \in K} \operatorname{vol}\operatorname{conv} X_0, \dots, X_d \leq \mathbb{E}_{X_0,\dots,X_d \in L} \operatorname{vol}\operatorname{conv} X_0, \dots, X_d.$$

His "strong conjecture" claims that this holds. He also stated the following "weak conjecture": there exists $c > 0$ such that $K \subseteq L$ implies

$$\mathbb{E}_{X_0,\ldots,X_d \in K} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_d \leq c^d \, \mathbb{E}_{X_0,\ldots,X_d \in L} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_d.$$

Clearly, the strong conjecture implies the weak conjecture. Also, Matthias Reitzner in [11] asked whether $K \subseteq L$ implies

$$\mathbb{E}_{X_0,\ldots,X_n \in K} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_n \leq \mathbb{E}_{X_0,\ldots,X_n \in L} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_n$$

for arbitrary n.

While these are natural questions in the understanding of random polytopes, one of their main motivations comes from their connection with the slicing conjecture (also known as hyperplane conjecture): All d-dimensional convex bodies of volume 1 have a hyperplane section of $(d-1)$-dimensional volume at least a universal positive constant. Meckes's weak conjecture is equivalent to the slicing conjecture.

In this work we show that Meckes's strong conjecture has a negative answer if $d \geq 4$ and a positive answer if d is 1 or 2. More precisely, we show:

Theorem 1 (random simplex)**.** *If d is 1 or 2, and K, L are two d-dimensional convex bodies, then $K \subseteq L$ implies*

$$\mathbb{E}_{X_0,\ldots,X_d \in K} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_d \leq \mathbb{E}_{X_0,\ldots,X_d \in L} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_d.$$

If $d \geq 4$, then there exist two convex bodies $K \subseteq L \subseteq \mathbb{R}^d$ such that

$$\mathbb{E}_{X_0,\ldots,X_d \in K} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_d > \mathbb{E}_{X_0,\ldots,X_d \in L} \operatorname{vol} \operatorname{conv} X_0, \ldots, X_d.$$

For the case $d = 3$, numerical integration suggests that the same counterexample used for $d \geq 4$ works for $d = 3$. Certain approximations used in those integrals in the proof for higher d fail to give a proof for $d = 3$, while an exact evaluation of the integrals looks somewhat involved and is left as an open question.

From the proof of Theorem 1 one can infer the following counterexample: In d dimensions, let L be the convex hull of a half-ball (say, the unit ball with the constraint $x_1 \geq 0$) and a point at distance $\epsilon > 0$ from the center of the ball (say, the point $(-\epsilon, 0, \ldots, 0)$). That is, L is the union of a half-ball and a cone. Let K be L with the tip of the cone truncated at distance $\delta > 0$ (say, $K = L \cap \{x : x_1 \geq -\epsilon + \delta\}$). Then the proof of Theorem 1 shows that the pair K, L is a counterexample to the monotonicity for $d \geq 4$ and ϵ, δ sufficiently small. Numerical integration suggests the same for $d = 3$.

The intuition for our answer to Meckes's question came from our solution to another simpler but related question asked by Santosh Vempala: is the determinant of the covariance matrix of a convex body monotone under inclusion? (This was also motivated by the slicing conjecture.) Here we show:

Theorem 2 (determinant of covariance)**.** *If d is 1 or 2 and K, L are two d-dimensional convex bodies, then $K \subseteq L$ implies* $\det A(K) \leq \det A(L)$*. If $d \geq 3$, then there exist two convex bodies $K \subseteq L \subseteq \mathbb{R}^d$ such that* $\det A(K) > \det A(L)$*.*

($A(K)$ is the covariance matrix of the uniform distribution on K.)

The high level idea of the proof of Theorem 2 is the following: To understand the monotonicity it is enough to compute and understand the derivative of $\det A(\cdot)$ as one intersects the convex body with a moving halfspace. We then find conditions under which this derivative has always the right sign. In the proof of Theorem 2 it is shown that understanding such a derivative is enough.

In view of the following formula valid for any d-dimensional convex body K with centroid $\mu(K)$:

$$\begin{aligned}\det A(K) &= d!\,\mathbb{E}_{X_i \in K}\big((\operatorname{vol}\operatorname{conv}\mu(K), X_1, \dots, X_d)^2\big)\\ &= \frac{d!}{d+1}\,\mathbb{E}_{X_i \in K}\big((\operatorname{vol}\operatorname{conv}X_0, X_1, \dots, X_d)^2\big),\end{aligned}$$

one would think that if a pair of convex bodies is a good example that the monotonicity of $\det A(\cdot)$ does not hold, then it could also be such an example for the functional

$$K \mapsto \mathbb{E}_{X_i \in K}\big((\operatorname{vol}\operatorname{conv}\mu(K), X_1, \dots, X_d)^2\big)$$

or even

$$K \mapsto \mathbb{E}_{X_i \in K}\big((\operatorname{vol}\operatorname{conv}X_0, X_1, \dots, X_d)\big).$$

Given these similarities, it should be no surprise that techniques and examples similar to those for $\det A(\cdot)$ also work for the expected volume of a random simplex.

For the proof of Theorem 1 we use a special case of Crofton's theorem[1] [13, Chapter 5], [7, Chapter 2]. Crofton's theorem has been formalized at least twice, once with differential geometry [1] and another time with conditional probability [4]. It is likely that using either of these two versions one could prove Theorem 1 in a simpler but less elementary way.

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Iteration of intersection body operator

Artem Zvavitch
(joint work with A. Fish, F. Nazarov, D. Ryabogin)

The notion of an *intersection body of a star body* was introduced by E. Lutwak: K is called the intersection body of L if the radial function of K in every direction is equal to the $(d-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction:

$$\rho_K(\xi) = \mathrm{vol}_{d-1}(L \cap \xi^\perp), \ \forall \xi \in S^{d-1},$$

where $\rho_K(\xi) = \sup\{a : a\xi \in K\}$ is the radial function of the body K and $\xi^\perp = \{x \in \mathbb{R}^d : (x,\xi) = 0\}$ is the central hyperplane perpendicular to the vector ξ. Using the formula for the volume in polar coordinates in $\xi^\perp$, we derive the following analytic definition of an *intersection body of a star body*: K is the intersection body of L if

$$\rho_K(\xi) = \frac{1}{d-1}\mathcal{R}\rho_L^{d-1}(\xi) := \frac{1}{d-1}\int\limits_{S^{d-1}\cap\xi^\perp} \rho_L^{d-1}(\theta)d\theta.$$

Here $\mathcal{R}$ stands for the spherical Radon transform. The notion of intersection bodies turned to be extremely natural and useful in Convex Geometry and Geometric Tomography.

Let us denote by $\mathcal{I}L$ the intersection body of a body L. Let $\mathcal{S}_d$ be the set of all star-shaped origin symmetric bodies in $\mathbb{R}^d$ endowed with the Banach-Mazur distance

$$d_{BM}(K,L) = \mathcal{I}nf\{b/a : \exists\ T \in GL(d) \text{ such that } aK \subseteq TL \subseteq bK\}.$$

We note that $\mathcal{I}(TL) = |\det T|(T^*)^{-1}(\mathcal{I}L),$ for all $T \in GL(d)$, hence the action of $\mathcal{I}$ on $\mathcal{S}_d$ is well defined, and $d_{BM}(\mathcal{I}(TK),\mathcal{I}(TL)) = d_{BM}(\mathcal{I}K,\mathcal{I}L)$.

The action of $\mathcal{I}$ on $\mathcal{S}_2$ is quite simple; since $\mathcal{I}L$ is just L rotated by $\pi/2$ and stretched 2 times, we have $\mathcal{I}L = L$ in $\mathcal{S}_2$, so every point of $\mathcal{S}_2$ is a fixed point of $\mathcal{I}$.

Let B_d be the unit Euclidean ball. We have

$$\rho_{\mathcal{I}(B_d)}(\xi) = \mathrm{vol}_{d-1}(B_d \cap \xi^\perp) = \mathrm{vol}_{d-1}(B_{d-1}).$$

Thus, B_d is a fixed point of $\mathcal{I}$ in $\mathcal{S}_d$.

Question: *Do there exist any other fixed or periodic points of $\mathcal{I}$ in $\mathcal{S}_d$, $d \geq 3$?*

In this talk we discussed that there are no such points in a small neighborhood of the ball B_d. This will immediately follow from the following

Theorem:

$$\mathcal{I}^m L \xrightarrow{\mathcal{S}_d} B_d \text{ as } m \to \infty,$$

for all L sufficiently close to B_d in Banach-Mazur distance.

We would like to note that a similar question for projection bodies is much better understood. It is quite easy to observe that the projection body of a cube is again (a dilation of) a cube. W. Weil described the polytopes that are stable under the projection body operation. Still the general question of the description of all fixed points remains open.

Unique determination of convex polytopes by non-central sections

Vlad Yaskin

A well-known classical result in geometric tomography states that origin-symmetric convex bodies are uniquely determined by the volumes of their central sections. That is, if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H), \tag{1}$$

then

$$K = L,$$

see, for example, [4, Corollary 7.2.7]. Note that this result does not hold without the symmetry assumption.

There are many results in the literature that deal with modifications of this theorem. Of particular interest are questions of unique determination of convex bodies that are not necessarily symmetric. For example, it is shown independently by Falconer [2] and Gardner [3] that any convex body is uniquely determined by the volumes of hyperplane sections through any two points in the interior of the body. A result of Groemer [5] says that convex bodies are uniquely determined by half-sections.

In [1] Barker and Larman ask the following question. Let K and L be convex bodies in $\mathbb{R}^n$ containing a sphere of radius t in their interiors. Suppose that condition (1) holds for every hyperplane H tangent to the sphere. Does this mean that $K = L$?

The problem is still open. Several partial results are obtained by the authors of this problem in [1]. They show that in $\mathbb{R}^2$ the uniqueness holds if one of the bodies is a Euclidean disk centered at the origin. In $\mathbb{R}^n$ they prove that the answer to this conjecture is affirmative if hyperplanes are replaced by planes of a larger codimension. However, the answer to the original question is still unknown, even in dimension 2.

In [7] we affirmatively solve the problem for convex polytopes. Namely, we prove the following.

Theorem 1. *Let P and Q be convex polytopes in $\mathbb{R}^n$ containing a sphere of radius t in their interiors. If*

$$\mathrm{vol}_{n-1}(P \cap H) = \mathrm{vol}_{n-1}(Q \cap H)$$

for every hyperplane H tangent to the sphere, then

$$P = Q.$$

Note that the case $n = 2$ of the latter theorem was recently settled by Xiong, Ma and Cheung [6].

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First steps in quaternionic integral geometry

Andreas Bernig

Consider a Euclidean vector space V. A *convex valuation* is a map μ on the space of convex bodies in V which is finitely additive in the sense that

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

whenever $K, L, K \cup L$ are convex bodies.

Let G be a subgroup of $SO(V)$. The space Val^G of translation invariant, G-invariant continuous convex valuations on V has finite dimension if and only if G acts transitively on the unit sphere. Groups with this property are classified:

$$SO(n), U(n), SU(n), Sp(n), Sp(n) \cdot U(1), Sp(n) \cdot Sp(1), G_2, Spin(7), Spin(9).$$

For $G = SO(n)$, Hadwiger's theorem describes Val^G. The cases $G = U(n)$ and $G = SU(n)$ were studied in [1, 2, 6, 5, 4]. For the exceptional groups G_2 and $Spin(7)$, a Hadwiger-type theorem was shown in [3]. For the quaternionic groups, a Hadwiger-type theorem was previously unknown.

Let $V = \mathbb{H}^n$ be a quaternionic vector space. Each of the groups

$$G = Sp(n),\ Sp(n) \cdot U(1),\ Sp(n) \cdot Sp(1)$$

acts naturally on V, and this action is transitive on the unit sphere. Hence $\dim \mathrm{Val}_k^G$ is finite, where $0 \leq k \leq 4n$ is the degree of homogeneity (recall that μ is k-homogeneous if $\mu(tK) = t^k \mu(K)$ for all $t \geq 0$). We can describe these dimensions explicitly. It turns out that they behave in a rather irregular way. We do not know of any geometrically meaningful basis of Val_k^G.

If we fix the degree and let the dimension n go to infinity (actually $n \geq k$ is enough), then the following formulas are obtained:

Theorem 1. *As formal power series,*

$$\sum_{k=0}^{\infty} \dim \mathrm{Val}_k^{Sp(\infty)} x^k = \frac{x^4 - 3x^3 + 6x^2 - 3x + 1}{(1-x)^7(1+x)^3}$$

$$\sum_{k=0}^{\infty} \dim \mathrm{Val}_k^{Sp(\infty)\cdot U(1)} x^k = \frac{x^6 - 2x^5 + 2x^4 + 2x^2 - 2x + 1}{(x^2+1)(x^2+x+1)(1+x)^2(1-x)^6}$$

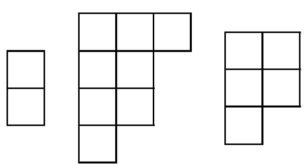
$$\sum_{k=0}^{\infty} \dim \mathrm{Val}_k^{Sp(\infty)\cdot Sp(1)} x^k = \frac{x^5 + 2x^4 + x^3 + 1}{(x^2+1)(x^2+x+1)(1+x)^2(1-x)^4}.$$

Here we list the first few values of these asymptotic dimensions.

k	$\dim \mathrm{Val}_k^{Sp(\infty)}$	$\dim \mathrm{Val}_k^{Sp(\infty)\cdot U(1)}$	$\dim \mathrm{Val}_k^{Sp(\infty)\cdot Sp(1)}$
0	1	1	1
1	1	1	1
2	7	3	2
3	14	6	4
4	42	14	8
5	84	24	11
6	182	44	17
7	330	72	24
8	603	117	34
9	1001	177	44
10	1645	265	58

Our second theorem applies to all dimensions, but the resulting formula is of a more combinatorial nature.

First recall that a *Young diagram* $(\lambda_1, \ldots, \lambda_j)$ is an arrangement of a finite number of boxes into rows with $\lambda_1, \ldots, \lambda_j$ boxes. Below are the Young diagrams $(1,1), (3,2,2,1)$ and $(2,2,1)$:



If the number of boxes in each row is even, we call the Young diagram *even*. The number of rows is the *depth* of λ, the number of boxes is its *weight*.

A *filling* of a Young diagram by numbers $\{1, 2, \ldots, m\}$ is given by putting one of these numbers in each box such that in each row, the numbers are weakly increasing, and in each column, the numbers are strictly increasing. Some fillings of the above diagrams with $m = 4$:

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2} 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 4\\\cline{1-1}\end{array}$$

Given a filling T, we denote by x^T the monomial $x_1^{i_1} \ldots x_m^{i_m}$, where i_k is the number of times k appears in the filling. The *Schur polynomial* of a Young diagram λ is defined as

$$s_\lambda(x_1, \ldots, x_m) = \sum_T x^T$$

where T ranges over all fillings.

As an example, consider

$$\lambda = \begin{array}{|c|c|}\hline &\\\hline \\\cline{1-1}\end{array}$$

which has weight 3 and depth 2. The possible fillings with entries from $\{1, 2, 3\}$ are given by

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array},$$

and hence

$$\begin{aligned} s_\lambda(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 \\ &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2 x_1 x_2 x_3 \end{aligned}$$

The Schur polynomials s_λ, as λ ranges over all Young diagrams, give a basis for the vector space of symmetric functions in the variables $x_1, \ldots, x_m$.

We define polynomials by

$$E(x) = \sum_\lambda s_\lambda(x, x),$$

where the sum is over all even Young diagrams λ of depth ≤ 2 with $\lambda_1 \leq 2n$. and

$$F_m(x) := \sum_\lambda s_\lambda(x, x, 1, 1)$$

where the sum is over all even Young diagrams λ of depth ≤ 4 with $\lambda_1 \leq 2n - 2$ and $|\lambda| = 2m$.

Theorem 2. *The dimension of the space* $\mathrm{Val}_k^{Sp(n)}$ *of* k*-homogeneous,* $Sp(n)$*-invariant, translation invariant continuous valuations on* $\mathbb{H}^n$ *satisfies*

$$\sum_{k=0}^{4n} \dim \mathrm{Val}_k^{Sp(n)} x^k = E(x) - F_{2n}(x) - (1+x)^2 F_{2n-1}(x) + x(1+3x+x^2)F_{2n-2}(x).$$

Similar formulas exist for the groups $Sp(n) \cdot U(1)$ and $Sp(n) \cdot Sp(1)$.

As an example, for $n \leq 5$ we obtain the following dimensions:

n	$\dim \mathrm{Val}_k^{Sp(n)}, k = 0, \ldots, 4n$
1	$1, 1, 6, 1, 1$
2	$1, 1, 7, 13, 29, 13, 7, 1, 1$
3	$1, 1, 7, 14, 41, 71, 111, 71, 41, 14, 7, 1, 1$
4	$1, 1, 7, 14, 42, 83, 169, 259, 344, 259, 169, 83, 42, 14, 7, 1, 1$
5	$1, 1, 7, 14, 42, 84, 181, 317, 532, 742, 903, 742, 532, 317, 181, 84, 42, 14, 7, 1, 1$

n	$\dim \mathrm{Val}_k^{Sp(n)U(1)}, k = 0, \ldots, 4n$
1	$1, 1, 2, 1, 1$
2	$1, 1, 3, 5, 9, 5, 3, 1, 1$
3	$1, 1, 3, 6, 13, 19, 25, 19, 13, 6, 3, 1, 1$
4	$1, 1, 3, 6, 14, 23, 39, 53, 64, 53, 39, 23, 14, 6, 3, 1, 1$
5	$1, 1, 3, 6, 14, 24, 43, 67, 98, 124, 141, 124, 98, 67, 43, 24, 14, 6, 3, 1, 1$

n	$\dim \mathrm{Val}_k^{Sp(n)Sp(1)}, k = 0, \ldots, 4n$
1	$1, 1, 1, 1, 1$
2	$1, 1, 2, 3, 5, 3, 2, 1, 1$
3	$1, 1, 2, 4, 7, 8, 9, 8, 7, 4, 2, 1, 1$
4	$1, 1, 2, 4, 8, 10, 14, 16, 18, 16, 14, 10, 8, 4, 2, 1, 1$
5	$1, 1, 2, 4, 8, 11, 16, 21, 26, 28, 30, 28, 26, 21, 16, 11, 8, 4, 2, 1, 1$

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Positive definite functions and stable random vectors

Alexander Koldobsky

In 1930's, Levy [Le] proved that, for any subspace $(\mathbb{R}^n, \|\cdot\|)$ of L_p with $0 < p \le 2$, the function $\exp(-\|\cdot\|^p)$ is positive definite on $\mathbb{R}^n$ and is the characteristic functional of a random vector $X = (X_1, ..., X_n)$ having a remarkable property - all linear combinations of the coordinates are identically distributed up to a constant. This result gave a start to the theory of stable processes.

The search for other examples of random vectors with the same property started immediately after Levy's discovery. In 1938 Schoenberg [S1] posed the problem of finding the exponents $0 < p \le 2$ for which the function $\exp(-\|\cdot\|_q^p)$ is positive definite on $\mathbb{R}^n$, where $\|x\|_q = (|x_1|^q + ... + |x_n|^q)^{1/q}$ is the norm of the space ℓ_q^n with $2 < q \le \infty$. This problem had been open for more than fifty years and was solved for $q = \infty$ in 1989 by Misiewicz [M2], and for $2 < q < \infty$ in 1991 in [K1]. The answers turned out to be the same in both cases: the function $\exp(-\|\cdot\|_q^p)$ is not positive definite for any $p \in (0, 2]$ if $n \ge 3$, and for $n = 2$ the function is positive definite if and only if $0 < p \le 1$.

The general situation was described by Eaton [E]. A random vector $X = (X_1, ..., X_n)$ is said to be an *n-dimensional version* of a random variable Y if there exists a function $\gamma : \mathbb{R}^n \to \mathbb{R}$, called the *standard* of X, such that $\gamma(a) > 0$ for every $a \in \mathbb{R}^n,\ a \ne 0$, and for every $a \in \mathbb{R}^n$ the random variables

$$\sum_{i=1}^n a_i X_i \quad \text{and} \quad \gamma(a)Y \tag{1}$$

are identically distributed. It is easy to see that $\gamma = \|\cdot\|_K$ is the Minkowski functional of some origin symmetric star body K in $\mathbb{R}^n$.

The main problem that has been studied by many people is to

Problem 1. *Characterize all n-dimensional versions and, in particular, find all functions γ that can appear as the standard of an n-dimensional version.*

It is easily seen that a random vector is an n-dimensional version with standard $\|\cdot\|_K$ if and only if the characteristic functional of this vector has the form $f(\|\cdot\|_K)$, where f is a non-constant continuous function on $[0, \infty)$. Let $\Phi(K)$ be the class of continuous functions $f : [0, \infty) \to \mathbb{R}$ for which $f(\|\cdot\|_K)$ is a positive definite function on $\mathbb{R}^n$. In view of Bochner's theorem, Problem 1 admits an equivalent formulation:

Problem 2. *Characterize the classes $\Phi(K)$ and, in particular, find all star bodies K for which the classes $\Phi(K)$ are non-trivial, i.e. contain non-constant functions.*

The classes $\Phi(K)$ have been studied by a number of authors: Schoenberg [S2], Bretagnolle, Dacunha-Castelle and Krivine [BDK], Cambanis, Keener and Simons [CKS], Richards [R], Gneiting [G], Aharoni, Maurey and Mityagin [AMM]. Misiewicz [M2] proved that for $n \ge 3$ the classes $\Phi(\ell_\infty^n)$ are trivial, and Lisitsky [Li1] and Zastavnyi [Z1], [Z2] showed the same for the classes $\Phi(\ell_q^n),\ q > 2,\ n \ge 3$.

In all these results the classes $\Phi(K)$ appear to be non-trivial only if K is the unit ball of a subspace of L_p with $0 < p \leq 2$. An old conjecture, explicitly formulated for the first time by Misiewicz [M1], is that the class $\Phi(K)$ can be non-trivial only in this case. A slightly weaker conjecture was formulated by Lisitsky [Li2]: if the class $\Phi(K)$ is non-trivial, then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. The concept of embedding in L_0 was introduced and studied in [KKYY]: a space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if there exist a finite Borel measure μ on the sphere S^{n-1} and a constant $C \in \mathbb{R}$ so that, for every $x \in \mathbb{R}^n$,

$$\ln \|x\|_K = \int_{S^{n-1}} \ln |(x, \xi)| \, d\mu(\xi) + C.$$

In this talk we prove the conjecture of Lisitsky (see [K2] for details):

Theorem 1. *([K2]) Let K be an origin symmetric star body in $\mathbb{R}^n$, $n \geq 2$ and suppose that there exists an even non-constant continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\|\cdot\|_K)$ is a positive definite function on $\mathbb{R}^n$. Then the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.*

Equivalently, if a function γ is the standard of an n-dimensional version of a random variable, then there exists an origin symmetric star body K in $\mathbb{R}^n$ such that $\gamma = \|\cdot\|_K$ and the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.

Besides being almost optimal as a characterization of the standards, this result also significantly generalizes the solution to Schoenberg's problem, as there are many examples of spaces that do not embed in L_0. For example, if a normed space contains the q-sum of two normed spaces, with $q > 2$ and dimension of one of the summands greater than 2, then no function of the norm of this space can be positive definite.

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Random conjunction matrices

Mark Rudelson

(joint work with Shiva Kasiviswanathan, Adam Smith, Jonathan Ullman)

The current work is a part of a joint research [2] on minimal distortion necessary for private release of statistical averages of data bases. This problem is reduced to finding the lower bound for the smallest singular value of a random matrix with correlated rows. The least singular value of random matrices with *independent* entries has been extensively studied, and a significant progress was achieved in recent years. Much less is known in the situation, when the entries of the matrix are interdependent. One type of such interdependency is analyzed in a new work Adamczak et al. [1]. Another type arises from random conjunction matrices, which originate in privacy analysis. To obtain a lower bound for the last singular value of such matrix, the authors develop a new approach relying on geometric functional analysis and convex geometry methods.

The goal of private data analysis is to provide global statistical properties of a data set of sensitive information, while protecting the privacy of the individuals, whose records the data set contains. There is a vast body of work on this subject in statistics and computer science. However, until recently, most schemes proposed in literature lacked rigor.

We consider a problem of releasing a contingency table of a large data base. A data base in our setting is a $d \times n$ $\{0,1\}$ matrix, whose columns represent the records of n individuals. Each individual has d binary attributes. For any set of $k < d$ attributes we release the percentage of the individuals which have all k attributes. The list of these percentages forms the marginal, or contingency table, which is the method of choice for government agencies releasing statistical summaries of categorical data.

To analyze the contingency table, we generate a new matrix. Namely, let D be the original $d \times n$ data base. For each subset $I = \{i_1, \dots, i_k\} \subset \{1, \dots, d\}$ we generate a vector $a_I \in \mathbb{R}^n$ which is the entry-wise product of the rows $d_{i_1}, \dots, d_{i_k}$ of the data base D. These vectors form a new $\binom{d}{k} \times n$ matrix A, which is called the conjunction matrix of D. The name comes from the fact that the rows of A can be viewed as conjunctions of the rows of D. The entries of the contingency table are percentages of ones in each row of the conjunction matrix.

Assume that the data base contains $d-1$ publicly available attributes, and one private attribute for each individual. The case when the number of private attributes is more than one, can be easily reduced to this. If the contingency table is released precisely, one can form the conjunction matrix, and after solving a linear system, reconstruct the vector of private attributes. To avoid this breach of privacy, the contingency table should be released with some noise. However, adding a large amount of noise lowers the value of the the contingency tables for statistical analysis. Therefore, we are interested in estimating the minimal amount of noise, which is necessary to ensure privacy. It can be shown that this amount is defined by the *least singular value* of the conjunction matrix. Since we want to construct a bound which is valid for most data bases, we consider a random matrix D and strive to obtain a bound, which is valid with overwhelming probability.

For matrices with independent entries such bound in full generality has been recently obtained in [4]. It is shown that the least singular value of a $d \times n$ random matrix with centered subgaussian entries of variance 1 satisfies the inequality

$$s_n(A) \geq c(\sqrt{d} - \sqrt{n-1})$$

with probability at least $1 - e^{-cn}$. For $d \gg n$ this boils down to $s_n(A) \geq c\sqrt{d}$. The entries of the conjunction matrix are obviously not independent. Nevertheless, numerical experiments show that the least singular value of a random conjunction matrix behaves like for a matrix with independent entries, as long as the number of rows is significantly bigger than the number of columns. Our main theorem provides a rigorous confirmation of the results of these experiments.

Define the iterated logarithm function by induction. For $t \geq 1$ $\log_{(1)} t := \log t$. If $n > 1$, set $\log_{(l)} t := \max(\log_{(l-1)} t, 1)$. For simplicity, we formulate the result for 2-conjunction matrices.

Theorem 1. *Let d, n and l be natural numbers such that*

$$d \leq n \leq \frac{c' d^2}{\log_{(l)} d}.$$

and let D be an $d \times n$ matrix with independent entries taking values 0 *and* 1 *with probability* 1/2. *Let A be the $\binom{d}{2} \times n$ matrix whose rows $a_{\{i,j\}}$ are the entry-wise products of the rows d_i and d_j of D. Then there exists a constant $c < 1$ such that*

$$\mathbb{P}\left(s_n(A) \leq c^l d\right) \leq e^{-cd},$$

provided that d is big enough ($d \geq C(l)$).

The least singular value of the matrix A is the minimum of $\|Ax\|$, over x in the unit sphere. An important tool in the proof is the *small ball probability*, which is the probability that $\|Ax\|$ is small for a *fixed* vector x. The small ball probability depends on the direction of the vector x. To obtain the uniform estimate of $\|Ax\|$, we decompose the sphere into many pieces, and for each piece construct an epsilon-net tailored according to the small ball probability. The estimate is then extended to the whole unit sphere by approximation. A prototype of this scheme was introduced in [3], and became since then one of major tools in random matrices literature. However, its realization for random conjunction matrices is more difficult, and requires new tools of probabilistic and geometric nature. This is due to the fact that the rows of such matrix are correlated, which makes the methods based on independence unavailable.

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On the behavior of random matrices with independent columns

Alexander E. Litvak
(joint work with R. Adamczak, O. Guédon, A. Pajor, N. Tomczak-Jaegermann)

In this talk we discuss behavior of several parameters of a random $n \times N$ matrix A, whose columns are independent random vectors in $\mathbf{R}^n$ satisfying some natural conditions. In particular, we obtain bounds for the operator norm $\|A : \ell_2^N \to \ell_2^n\|$; i.e. the for the largest singular value of A; for the smallest singular value; for the norm of A on the set of all m-sparse vectors (i.e. vectors having at most m nonzero coordinates), which is denoted by A_m. Our estimates hold with overwhelming probability, that is, with the probability tending to one as the dimension grows to infinity. In particular, we obtain that for isotropic log-concave i.i.d. random vectors X_i's

$$\mathrm{Prob}\left(\exists m \le N \,:\, A_m \ge C\left(\sqrt{n} + \sqrt{m}\log\frac{2N}{m}\right)\right) \le \exp\left(-c\sqrt{n}\right),$$

where c and C are absolute positive constants. Note here that $A_N = \|A\|$.

We apply our results to solve several problems. First, we provide asymptotically sharp answer to the question posed by R. Kannan, L. Lovász, M. Simonovits: *Let K be an isotropic convex body in $\mathbf{R}^n$. Given $\varepsilon > 0$, how many independent points X_i uniformly distributed on K are needed for the empirical covariance matrix to approximate the identity up to ε with overwhelming probability?* Namely, we show that it is enough to take $N \approx C(\varepsilon)n$ vectors. Then we turn to applications to compressed sensing and convex geometry. We investigate RIP (Restricted Isometry Property) of random matrices with independent columns and show that the matrix A, considered above, satisfies RIP. Thus, as was shown in works of E. Candes and T. Tao, and D. L. Donoho, such a matrix can be used to solve exact reconstruction process of m-sparse vectors via ℓ_1 minimization as well as to construct neighborly polytopes.

The results mentioned in the talk are published in papers listed below.

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On affine invariants for smooth convex bodies

Alina Stancu

We construct one-parameter families of smooth convex bodies with everywhere positive Gauss curvature and whose interiors contain the origin. Each one of these families is in fact a short-time solution to an affine flow on the boundaries of the above convex bodies in the sense that the flow commutes with any origin-preserving affine transformation. Consequently, any derivative of a global affine invariant associated to a smooth convex body with respect to this parameter will result into another affine invariant of the body. Most of these invariants are new and a precise analytic description can be given even if it can be quite complex. Moreover, we also obtain in this way a geometric interpretation of known affine invariants like the affine surface area and the p-affine surface areas.

We will illustrate here some particular directions of interest in the study of these invariants. Let $\mathcal{K}_0^+$ be the set of smooth convex bodies in $\mathbb{R}^n$ containing the origin in their interior and having everywhere positive Gauss curvature.

Let $K \in \mathcal{K}_0^+$ defined by the immersion $X : \mathbb{S}^{n-1} \to \mathbb{R}^n$ and impose on it the following deformations along its affine normal vector $\mathcal{N}$:

$$\frac{\partial X(u,t)}{\partial t} = k_0^{\frac{n(p-1)}{(n+1)(n+p)}}(u,t)\,\mathcal{N}(u,t), \text{ if } \quad \frac{p}{n+p} > 0 \tag{1}$$

$(p \neq n)$ and

$$\frac{\partial X(u,t)}{\partial t} = -k_0^{\frac{n(p-1)}{(n+1)(n+p)}}(u,t)\;\mathcal{N}(u,t), \quad \text{if } \frac{p}{n+p} < 0, \tag{2}$$

where $k_0(u,t) = \dfrac{k(u,t)}{(X(u,t)\cdot u)^{n+1}}$ is the centro-affine curvature, k is the Gauss curvature.

We denote by $K(t)$ the convex body represented by $X(u,t)$, where t is sufficiently small as to insure the short time existence of solutions. Then

Theorem 1. *For any $p \neq -n$,*

$$\Delta_p^m(K) := \left| \left(\frac{d^m}{dt^m} Vol(K(t)) \right)_{|t=0} \right|$$

are affine invariants of K having an explicit integral representation over ∂K.

In particular, one can see quite easily that $\Delta_p^1(K) = \Omega_p(K)$, where the p-affine surface area is defined as in [2] by

$$\Omega_p(K(t)) := \int_{\partial K} h^{\frac{n(1-p)}{n+p}} k^{\frac{p}{n+p}} \, d\mu_K.$$

A new affine invariant, $\Delta_p^2(K)$ is then used to derive novel affine inequalities. Among them, we present the following.

Proposition 1. *If $K \in \mathcal{K}_0^+$ and $K^\star$ denotes the polar of K with respect to the origin, then*

$$n^2 Vol(K^\star) Vol(K) \leq \Omega_{-2n}(K^\star)\Omega_{-2n}(K),$$

with equality if and only if K is an ellipsoid.

Other inequalities for the volume product were obtained under the additional assumption of p-ellipticity also due to Lutwak [2].

However, we prefer to end with the following result which we can actually obtain for convex bodies only in $\mathcal{C}_+^2$ by considering an extension of the method presented here.

Theorem 2. *For any smooth, concave $\phi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \phi(t)/t = 0$, there exists an affine invariant flow on $\mathcal{K}_0^+$ such that, $\forall K \in \mathcal{K}_0^+$:*

$$\Phi(K) = \lim_{\epsilon \searrow 0} \left| \frac{Vol(K(\epsilon)) - Vol(K(0))}{\epsilon} \right|,$$

where $K(0) = K$ and $K(\epsilon)$ is the solution to the flow at time ϵ.

The motivation of the latest result is based on Ludwig-Reitzner's complete characterization $SL(n)$-invariant, upper semicontinuous valuations $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ that vanish on polytopes as

$$\Phi(K) = \int_{\partial K} \phi(k_0) \, d\mu_{cK}, \quad \forall K \in \mathcal{K}_0^n,$$

where $\phi : [0,\infty) \to [0,\infty)$ concave, with $\lim_{t\to 0} \phi(t) = \lim_{t\to\infty} \phi(t)/t = 0$, [1].

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Minkowski Valuations

Franz E. Schuster

A function ϕ defined on convex bodies (convex, compact sets) in $\mathbb{R}^n$ and taking values in an Abelian semigroup is called a *valuation* if

$$\phi(K) + \phi(L) = \phi(K \cup L) + \phi(K \cap L)$$

whenever K, L and $K \cup L$ are convex. As a generalization of the notion of measure, valuations have always played a central role in geometry. A particularly exciting new development in the theory of valuations explores the strong connections between convex body valued valuations and the theory of geometric and analytic inequalities (see, e.g., [3, 4, 9, 10, 11]). The following examples should provide a first impression how the theory of valuations can shed new light on classical geometric inequalities and, at the same time, lead to important generalizations.

Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$, $n \geq 3$, endowed with the Hausdorff metric. A convex body $K \in \mathcal{K}^n$ is uniquely determined by its support function $h(K, u) = \max\{u \cdot x : x \in K\}$, for $u \in S^{n-1}$. For $i \in \{1, \ldots, n-1\}$, let $\mathrm{Gr}_{i,n}$ be the Grassmannian of i-dimensional subspaces in $\mathbb{R}^n$. The ith projection function $\mathrm{vol}_i(K|\cdot)$ of $K \in \mathcal{K}^n$ is the continuous function on $\mathrm{Gr}_{i,n}$ defined such that $\mathrm{vol}_i(K|E)$, for $E \in \mathrm{Gr}_{i,n}$, is the i-dimensional volume of the orthogonal projection of K onto E.

Definition *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Minkowski valuation if*

$$\Phi K + \Phi L = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K \cup L \in \mathcal{K}^n$ and addition on $\mathcal{K}^n$ is Minkowski addition.

Important examples of Minkowski valuations are the projection and the difference operator: The *projection body* ΠK of K is the convex body defined by

$$h(\Pi K, u) = \mathrm{vol}_{n-1}(K|u^\perp), \qquad u \in S^{n-1}.$$

The *difference body* $\mathrm{D}K$ of K is defined by

$$h(\mathrm{D}K, u) = \mathrm{vol}_1(K|u), \qquad u \in S^{n-1}.$$

Through the seminal work of Ludwig [2, 6, 7] classifications of Minkowski valuations have become the focus of increased attention. For example, Ludwig established characterizations of the projection and the difference operator as unique Minkowski valuations which are compatible with affine transformations of $\mathbb{R}^n$.

The celebrated *Blaschke–Santaló inequality* is by far the best known affine isoperimetric inequality: The product of the volumes of polar reciprocal convex bodies is maximized precisely by ellipsoids. In [11] Lutwak and Zhang obtained an important L_p version of the Blaschke–Santaló inequality. Their inequality includes as a limiting case the classical inequality for origin-symmetric bodies. For convex bodies which are not origin-symmetric this L_p extension yields a weaker inequality than the Blaschke–Santaló inequality. In a joint work with C. Haberl the author [4] established an L_p analog of the Blaschke–Santaló inequality that includes as a limiting case the classical inequality for all convex bodies. The discovery of this new L_p Blaschke–Santaló inequality was made possible only by an L_p extension of Ludwig's characterization of the projection operator established in [7].

Although a large part of the theory of convex body valued valuations deals with $\mathrm{SL}(n)$ intertwining operators, considerable effort has been invested in recent years to also classify all continuous rigid motion compatible Minkowski valuations. A Minkowski valuation $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called $\mathrm{O}(n)$ *equivariant* if $\Phi(\vartheta K) = \vartheta \Phi K$ for all $K \in \mathcal{K}^n$ and every $\vartheta \in \mathrm{O}(n)$. We denote by $\mathbf{MVal}$ the set of continuous translation invariant Minkowski valuations which are $\mathrm{O}(n)$ equivariant and we write $\mathbf{MVal}_i^{(+)}$ for its subset of all (even) Minkowski valuations of degree i. Here, a map Φ from $\mathcal{K}^n$ to $\mathcal{K}^n$ (or $\mathbb{R}$) is said to have *degree* i if $\Phi(\lambda K) = \lambda^i \Phi K$ for $K \in \mathcal{K}^n$ and $\lambda \geq 0$ and it is called *even* if $\Phi(-K) = \Phi K$ for $K \in \mathcal{K}^n$.

A description of Minkowski valuations in $\mathbf{MVal}_1$ was recently obtained by Kiderlen [5] (extending previous results by Schneider [12]). The following is a special case of [5, Theorem 1.3] for even valuations:

Theorem (Kiderlen [5]**).** *Suppose that* $\Phi \in \mathbf{MVal}_1^+$ *is smooth. Then there exists a unique smooth* $\mathrm{O}(n-1)$ *invariant even measure* μ *on* S^{n-1} *such that for every* $K \in \mathcal{K}^n$*,*

$$h(\Phi K, \cdot) = \mathrm{vol}_1(K|\cdot) * \mu. \tag{1}$$

The convolution in (1) is induced from $\mathrm{O}(n)$ by identifying S^{n-1} and $\mathrm{Gr}_{i,n}$ with the homogeneous spaces $\mathrm{O}(n)/\mathrm{O}(n-1)$ and $\mathrm{O}(n)/\mathrm{O}(i) \times \mathrm{O}(n-i)$, respectively. The notion of smooth translation invariant real-valued valuations was introduced by Alesker in [1]. In [14] this definition was extended to Minkowski valuations which are translation invariant and intertwine orthogonal transformations.

In 2001, Alesker [1] has given a complete description of continuous translation invariant valuations on convex bodies thereby confirming, in a much stronger form, a conjecture by McMullen. The proof of Alesker's landmark result, known as the Irreducibility Theorem, draws on new methods from representation theory and differential geometry. From applications of this deep result and the new techniques introduced by Alesker to the theory of valuations, the author [14], was able to establish a description of smooth Minkowski valuations in $\mathbf{MVal}_i^+$, $i \in \{1, \ldots, n-1\}$. This result provides a significant extension of the earlier work by Schneider [12], Kiderlen [5], and the author [13].

Theorem 1 ([14]). *Suppose that* $\Phi \in \mathbf{MVal}_i^+$ *is smooth. Then there exists a unique* $\mathrm{O}(i) \times \mathrm{O}(n-i)$ *invariant measure* μ *on* S^{n-1} *such that for every* $K \in \mathcal{K}^n$,

$$h(\Phi_i K, \cdot) = \mathrm{vol}_i(K|\cdot\,) * \mu.$$

Moreover, the author reduced in [14] the problem of describing all continuous translation invariant and $\mathrm{O}(n)$ equivariant even Minkowski valuations to the description of smooth ones (which is provided by Theorem 1).

Let $\Pi_i K$ denote the projection body of order i defined by

$$h(\Pi_i K, u) = V_i(K|u^\perp), \qquad u \in S^{n-1}.$$

In [8] Lutwak obtained an array of geometric inequalities for intrinsic volumes of projection bodies. A special case of [8, Theorem 6.2] is the following: If $K, L \in \mathcal{K}^n$ have non-empty interior and $i \in \{2, \ldots, n-1\}$, then

$$V_{i+1}(\Pi_i(K+L))^{1/i(i+1)} \geq V_{i+1}(\Pi_i K)^{1/i(i+1)} + V_{i+1}(\Pi_i L)^{1/i(i+1)}, \tag{2}$$

with equality if and only if K and L are homothetic.

The classical Brunn–Minkowski inequalities for intrinsic volumes are at the heart of the Brunn–Minkowski theory. As an application of Theorem 1 the author [14] obtained generalizations of both Lutwak's inequality (2) and the classical Brunn–Minkowski inequalities for the intrinsic volumes:

Theorem 2 ([14]). *Let* $\Phi_i \in \mathbf{MVal}_i^+$, *where* $i \in \{2, \ldots, n-1\}$. *If* $K, L \in \mathcal{K}^n$ *have non-empty interior, then*

$$V_{i+1}(\Phi_i(K+L))^{1/i(i+1)} \geq V_{i+1}(\Phi_i K)^{1/i(i+1)} + V_{i+1}(\Phi_i L)^{1/i(i+1)}.$$

If Φ_i *maps convex bodies with non-empty interiors to bodies with non-empty interiors, then equality holds if and only if* K *and* L *are homothetic.*

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Phase retrieval for characteristic functions of convex bodies and reconstruction from covariograms

Gabriele Bianchi and Richard J. Gardner*
(joint work with Markus Kiderlen)

* Joint extended abstract for two talks arising from the same paper. Gabriele Bianchi spoke on "Phase retrieval for characteristic functions of convex bodies" and Richard J. Gardner spoke on "Reconstruction from covariograms."

The *Phase Retrieval Problem* of Fourier analysis involves determining a function f on $\mathbb{R}^n$ from the modulus $|\widehat{f}\,|$ of its Fourier transform $\widehat{f}$. This problem arises naturally and frequently in various areas of science, such as X-ray crystallography, electron microscopy, optics, astronomy, and remote sensing, in which only the magnitude of the Fourier transform can be measured and the phase is lost. (Sometimes, as when reconstructing an object from its far-field diffraction pattern, it is the squared modulus $|\widehat{f}\,|^2$ that is directly measured.) Indeed, as Rosenblatt [6] remarks, the Phase Retrieval Problem "arises in all experimental uses of diffracted electromagnetic radiation for determining the intrinsic detailed structure of a diffracting object." It is no surprise, therefore, that the literature is vast; see, for example, [3] and [4], and the references given there.
Phase retrieval is fundamentally under-determined without additional constraints, which usually take the form of an a priori assumption that f has a particular support or distribution of values. An important example is when $f = 1_K$, the characteristic function of a convex body K in $\mathbb{R}^n$. In this setting, phase retrieval is very closely related to a geometric problem involving the *covariogram* of a convex body K in $\mathbb{R}^n$. This is the function g_K defined by

$$g_K(x) = V_n\left(K \cap (K + x)\right),$$

for $x \in \mathbb{R}^n$, where V_n denotes n-dimensional Lebesgue measure and $K + x$ is the translate of K by the vector x. It is also sometimes called the *set covariance* and is equal to the *autocorrelation* of 1_K, that is,

$$g_K = 1_K * 1_{-K},$$

where $*$ denotes convolution and $-K$ is the reflection of K in the origin. Taking Fourier transforms, we obtain the relation

$$\widehat{g_K} = \widehat{1_K}\,\widehat{1_{-K}} = \widehat{1_K}\,\overline{\widehat{1_K}} = \left|\widehat{1_K}\right|^2. \tag{1}$$

This connects the Phase Retrieval Problem, restricted to characteristic functions of convex bodies, to the problem of determining a convex body from its covariogram.

The covariogram was introduced by Matheron [5]. It has found application in fields such as stereology, geometric tomography, pattern recognition, image analysis, and mathematical morphology, where information about an unknown object is to be retrieved from chord length measurements. Baake and Grimm [1] explain how the problem of finding the atomic structure of a quasicrystal from its X-ray diffraction image involves recovering a subset of $\mathbb{R}^n$ called a window from its covariogram, and note that this window is in many cases a convex body. The covariogram has also played an increasingly important role in analytic convex geometry. For example, it was used by Rogers and Shephard in proving their famous difference body inequality, by Gardner and Zhang in the theory of radial mean bodies, and by Tsolomitis in his study of convolution bodies, which via the work of Schmuckenschläger and Werner allows a covariogram-based definition of the fundamental notion of affine surface area.

We effectively solve the following three problems. In each, K is a convex body in $\mathbb{R}^n$.

Problem 1 (Reconstruction from covariograms). Construct an approximation to K from a finite number of noisy (i.e., taken with error) measurements of g_K.

Problem 2 (Phase retrieval for characteristic functions of convex bodies: squared modulus). Construct an approximation to K (or, equivalently, to 1_K) from a finite number of noisy measurements of $|\widehat{1_K}|^2$.

Problem 3 (Phase retrieval for characteristic functions of convex bodies: modulus). Construct an approximation to K from a finite number of noisy measurements of $|\widehat{1_K}|$.

In order to discuss our results, we must first consider the corresponding uniqueness problems. In view of (1), these are equivalent, so we shall focus on the covariogram. It is easy to see that g_K is invariant under translations of K and reflection of K in the origin. Let $\mathcal{K}_o^n$ be the class of convex bodies in $\mathbb{R}^n$ and let $\mathcal{U}^n$ be the class of convex bodies in $\mathbb{R}^n$ that are determined, up to translation and reflection in the origin, by their covariograms. Let $\mathcal{P}^n$ be the class of n-dimensional convex polytopes in $\mathbb{R}^n$ and let $\mathcal{K}_s^n$ be the class of centrally symmetric convex bodies in $\mathbb{R}^n$. Recently Averkov and Bianchi proved that $\mathcal{U}^2 = \mathcal{K}_o^2$, confirming a 1986 conjecture of Matheron, and Bianchi proved that $\mathcal{P}^3 \subset \mathcal{U}^3$. It is easy to see that $\mathcal{K}_s^n \subset \mathcal{U}^n$. Goodey, Schneider, and Weil proved that most (in the sense of Baire category) convex bodies in $\mathbb{R}^n$ belong to $\mathcal{U}^n$. Nevertheless, Bianchi has constructed examples showing that $\mathcal{P}^n \not\subset \mathcal{U}^n$ for $n \geq 4$. It is still unknown whether $\mathcal{U}^3 = \mathcal{K}_o^3$.

None of the above uniqueness proofs provide a method for actually reconstructing a convex body from its covariogram. We are aware of only two papers dealing with the reconstruction problem: Schmitt [7] gives an explicit reconstruction procedure for a convex polygon when no pair of its edges are parallel, an assumption removed in an algorithm due to Benassi and D'Ercole [2]. In both these papers, all the exact values of the covariogram are supposed to be available.

In contrast, our first set of algorithms take as input only a finite number of values of the covariogram of an unknown convex body K_0. Moreover, these measurements are corrupted by errors, modeled by Gaussian noise of mean zero and a fixed variance. It is assumed that K_0 is determined by its covariogram, has its centroid at the origin, and is contained in a known bounded region of $\mathbb{R}^n$, which for convenience we take to be the unit cube $C_0^n = [-1/2, 1/2]^n$. We provide two different methods for reconstructing, for each suitable $k \in \mathbb{N}$, a convex polytope P_k that approximates K_0 or its reflection $-K_0$. Each method involves two algorithms, an initial algorithm that produces suitable outer unit normals to the facets of P_k, and a common main algorithm that goes on to actually construct P_k.

In the first method, the covariogram of K_0 is measured, multiple times, at the origin and at vectors $(1/k)u_i$, $i = 1, \dots, k$, where the u_i's are mutually nonparallel unit vectors that span $\mathbb{R}^n$. From these measurements, the initial Algorithm NoisyCovBlaschke constructs an o-symmetric convex polytope Q_k that approximates ∇K_0, the Blaschke body of K_0. The crucial property of ∇K_0 is that when K_0 is a convex polytope, each of its facets is parallel to some facet of ∇K_0. It follows that the outer unit normals to the facets of P_k can be taken to be among those of Q_k. Algorithm NoisyCovBlaschke utilizes the known fact that $-\partial g_{K_0}(tu)/\partial t$, evaluated at $t = 0$, equals the brightness function value $b_{K_0}(u)$, that is, the $(n-1)$-dimensional volume of the orthogonal projection of K_0 in the direction u. This connection allows most of the work to be done by a very efficient algorithm designed earlier by Gardner and Milanfar that reconstructs a o-symmetric convex body from finitely many noisy measurements of its brightness function.

The second method achieves the same goal with a quite different approach. This time the covariogram of K_0 is measured once at each point in a cubic array in $2C_0^n = [-1, 1]^n$ of side length $1/k$. From these measurements, the initial Algorithm NoisyCovDiff(φ) constructs an o-symmetric convex polytope Q_k that approximates $DK_0 = K_0 + (-K_0)$, the difference body of K_0. The set DK_0 has precisely the same property as ∇K_0, that when K_0 is a convex polytope, each of its facets is parallel to some facet of DK_0. Furthermore, DK_0 is just the support of g_{K_0}. The known property that $g_{K_0}^{1/n}$ is concave can therefore be combined with techniques from multiple regression. Algorithm NoisyCovDiff(φ) employs a Gasser-Müller type kernel estimator for g_{K_0}, with suitable kernel function φ, bandwidth, and threshold parameter.

The output Q_k of either initial algorithm forms part of the input to the main common Algorithm NoisyCovLSQ. The covariogram of K_0 is now measured again, once at each point in a cubic array in $2C_0^n = [-1, 1]^n$ of side length $1/k$. Using these measurements, Algorithm NoisyCovLSQ finds a convex polytope P_k, each of whose facets is parallel to some facet of Q_k, whose covariogram fits best the measurements in the least squares sense.

Much effort is spent in proving that these algorithms are strongly consistent. Whenever $K_0 \in \mathcal{U}^n$, we show that, almost surely,

$$\min\{\delta(K_0, P_k), \delta(-K_0, P_k)\} \to 0$$

as $k \to \infty$, where δ denotes Hausdorff distance. (If $K_0 \notin \mathcal{U}^n$, a rare situation in view of the uniqueness results discussed above, the algorithms still construct a sequence (P_k) whose accumulation points exist and have the same covariogram as K_0.) From a theoretical point of view, this completely solves Problem 1. Naturally, the consistency proof leans heavily on results and techniques from analytic convex geometry, as well as a suitable version of the Strong Law of Large Numbers.

With algorithms for Problem 1 in hand, we move to Problem 2, assuming that K_0 is an unknown convex body satisfying the same conditions as before. The basic idea is simple enough: Use (1) and the measurements of $|\widehat{1_{K_0}}|^2$ at points in a suitable cubic array to approximate g_{K_0} via its Fourier series, and feed the resulting values into the algorithms for Problem 1. However, two major technical obstacles arise. The new estimates of g_{K_0} are corrupted by noise that now involves dependent random variables, and a new deterministic error appears as well. A substitute for the Strong Law of Large Numbers must be proved, and the deterministic error controlled using Fourier analysis and the fortunate fact that g_{K_0} is Lipschitz. In the end the basic idea works, assuming that for suitable $1/2 < \gamma < 1$, measurements of $|\widehat{1_{K_0}}|^2$ are taken at the points in $(1/k^\gamma)\mathbb{Z}^n$ contained in the cubic window $[-k^{1-\gamma}, k^{1-\gamma}]^n$, whose size increases with k at a rate depending on the parameter γ. With suitable restrictions on γ, the three resulting algorithms, Algorithm NoisyMod^{2}LSQ, Algorithm NoisyMod^{2}Blaschke, and Algorithm NoisyMod^{2}Diff(φ), are proved to be strongly consistent under the same hypotheses as for Problem 1.

Our final three algorithms, Algorithm NoisyModLSQ, Algorithm NoisyModBlaschke, and Algorithm NoisyModDiff(φ) cater for Problem 3. Again there is a basic simple idea, namely, to take two independent measurements at each of the points in the same cubic array as in the previous paragraph, multiply the two, and feed the resulting values into the algorithms for Problem 2. No serious extra technical difficulties arise, and we are able to prove that the three new algorithms are strongly consistent under the same hypotheses as for Problem 2. This provides a complete theoretical solution to the Phase Retrieval Problem for characteristic functions of convex bodies.

To summarize:

For Problem 1, use *either* Algorithm NoisyCovBlaschke *or* Algorithm NoisyCovDiff(φ) and then Algorithm NoisyCovLSQ.

For Problem 2, use *either* Algorithm NoisyMod^{2}Blaschke *or* Algorithm NoisyMod^{2}Diff(φ) and then Algorithm NoisyMod^{2}LSQ.

For Problem 3, use *either* Algorithm NoisyModBlaschke *or* Algorithm NoisyModDiff(φ) and then Algorithm NoisyModLSQ.

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On the isotropy constant of polytopes

David Alonso-Gutiérrez
(joint work with Jesús Bastero, Julio Bernués, Pawel Wolff)

A convex body $K \subseteq \mathbb{R}^n$ is said to be isotropic if it satisfies the following conditions:

- $|K| = 1$,
- $\int_K x dx = 0$ and
- $\forall \theta \in S^{n-1} \int_K \langle x, \theta\rangle^2 dx = L_K 2.$

This constant L_K, independent of the vector θ is called the isotropy constant of K. It is clear from the definition that if $U \in O(n)$ is an orthogonal map then UK is also isotropic with the same isotropy constant. It is very well known that for any convex body $K \subseteq \mathbb{R}^n$ there exists a unique $a \in \mathbb{R}^n$ and a unique (up to orthogonal transformations) linear map T such that $a + TK$ is isotropic.

This allows us to define the isotropy constant for any convex body as the isotropy constant of its isotropic image. It also admits the following definition, as a solution of a minimization problem:

$$nL_K 2 = \min\left\{\frac{1}{|TK|^{1+\frac{2}{n}}}\int_{a+TK} |x|^2 dx \,:\, a \in \mathbb{R}^n, T \in GL(n)\right\}.$$

It is very well known that among all the n dimensional convex bodies, the Euclidean ball is the one with the smallest isotropy constant, whose value is bounded from below by an absolute constant independent of the dimension

$$L_K \geq L_{B2_n} \geq c.$$

What is not known is if there exists an absolute constant bounding from above the isotropy constant of any convex body. This problem is known as the slicing problem.

Conjecture (Slicing problem) There exists an absolute constant C such that for any convex body

$$L_K \leq C$$

Since any convex body can be approximated by polytopes this conjecture is true for any convex body if and only if it is true for polytopes. Thus we study the isotropy constant of polytopes. We give a simple proof of the following Junge's result:

Theorem[J] Let $K \subset \mathbb{R}^n$ be a symmetric convex polytope with $2N$ vertices. Then

$$L_K \leq C \log N.$$

We also prove the following theorem, which gives a positive answer for the slicing problem for polytopes with few vertices:

Theorem[ABBW] Let $K \subseteq \mathbb{R}^n$ be a convex polytope with N vertices. Then

$$L_K \leq C\sqrt{\frac{N}{n}}.$$

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A convex body whose centroid and Santaló point are far apart.

Carsten Schütt

(joint work with Mathieu Meyer, Elisabeth Werner)

We are constructing convex bodies in $\mathbb{R}^n$ whose centroids and Santaló points are far apart.

Theorem 1. *There is an absolute constant $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is a convex body C in $\mathbb{R}^n$ with*

$$c \leq \frac{\| \mathrm{g}(C) - \mathrm{s}(C) \|}{\mathrm{vol}_1(\ell \cap C)} \;. \tag{1}$$

l is the line through $g(C)$ and $s(C)$ and $\| \cdot \|$ is the Euclidean norm.

The proof actually shows that we can asymptotically determine the constant c of the theorem.

$$\frac{\| \mathrm{g}(C) - \mathrm{s}(C) \|}{\mathrm{vol}_1(\ell \cap C)}$$

is asymptotically with respect to the dimension, greater than or equal to

$$\left(1 - \frac{1}{e}\right) \frac{\sqrt{e\pi} - 2}{\sqrt{e\pi} + \frac{2}{e-1}} = 0.142673...$$

For convex bodies K and L in $\mathbb{R}^n$ and real numbers $a > 0$ and $b > 0$, we construct a convex body M_n in $\mathbb{R}^{n+1}$

$$M_n = \mathrm{co}[(K, -a), (L, b)] = \{t(x, -a) + (1-t)(y, b) | x \in K, y \in L, 0 \leq t \leq 1\}. \tag{2}$$

Thus the body M_n is the convex hull of two n-dimensional faces K and L. As K we choose the Euclidean ball with volume 1 and for L we choose the cube of volume 1. The bodies we are using in Theorem 1 will be the polar bodies to M_n.

The Method of Voronoi

Peter M. Gruber

A classical criterion of Voronoi says that a positive definite quadratic form on $\mathbb{E}^d$ is extreme if and only if it is perfect and eutactic. Equivalently, a lattice packing of balls in $\mathbb{E}^d$ has locally maximum density if and only if it is perfect and eutactic. The idea of his proof was to identify quadratic forms on $\mathbb{E}^d$ with their coefficient vectors in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ and thus to transform the problem in $\mathbb{E}^d$ into a geometric problem in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ which is easier to solve. During the last 50 years this idea or modifications of it saw a series of applications in the following areas:

(i) lattice packings of balls and smooth convex bodies
(ii) lattice coverings of balls
(iii) Epstein zeta function
(iv) closed geodesics on the Riemannian manifolds of a Teichmüller space
(v) John type and minimum position problems

In this articel new results in the areas (i), (ii), (iii) and (v) will be presented.

1. Lattice Packings of Smooth Convex Bodies

The classical criterion of Voronoi for balls can be extended as to cover smooth convex bodies and more refined extremum properties. Withouth giving precise definitions, we state two pertinent results which should give the reader an idea of these extensions, where $\delta(C, L)$ is the density of the lattice packing of (a suitable multiple of the smooth convex body) C by the lattice L.

Theorem 1. *There is no lattice L such that $\delta(C, \cdot)$ is stationary at L. $\delta(C, \cdot)$ is semi-stationary at L if and only if L is semi-eutactic.*

Theorem 2. *$\delta(C, \cdot)$ is ultra extreme at L if and only if L is perfect and eutactic.*

Thus, in the special case where $C = B^d$ is a ball, each lattice packing of extreme density is already ultra extreme. Using these criteria for balls, one can determine the lattice packings of balls in $\mathbb{E}^2$ and $\mathbb{E}^3$ which have semi-stationary density:

$d = 2$: square and regular hexagonal lattices
$d = 3$: cubic primitive, cubic face centered, cubic body centered and special hexagonal primitive and tetragonal body centered lattices.

See, the author [5].

2. Lattice Coverings of Balls

The criterion of Voronoi for extreme lattice packings has counterparts for lattice coverings. Again, we state two results for balls without giving precise definitions. $\vartheta(B^d, L)$ is the density of the lattice covering of (a suitable multiple of) C by the lattice L.

Theorem 3. *$\vartheta(B^d, \cdot)$ is stationary at L if and only L is nano polyeutactic.*

Theorem 4. *$\vartheta(B^d, \cdot)$ is ultra maximum at L if and only if L is polyeutactic and polyperfect.*

The stationary lattices in $\mathbb{E}^2$, and $\mathbb{E}^3$ are the following:

$d = 2$: regular hexagonal lattices
$d = 3$: face centered cubic lattices.

See, the author [6].

3. Lattice Zeta Functions

Let C be a smooth convex body with center at o and let $\|\cdot\|_C$ be the corresponding norm on $\mathbb{E}^d$. The lattice zeta function $\zeta_C(L, s)$ is defined by

$$\zeta_C(L, s) = \sum_{l \in L \setminus \{o\}} \frac{1}{\|l\|_C^s} \text{ for } s > d.$$

A major problem is to determine for given $s > d$ or for all sufficiently large s the lattices L of determinant 1 which minimize ζ_C. We state two results.

Theorem 5. *$\zeta_C(\cdot, s)$ is stationary at L for given $s > d$ if and only if L is fully eutactic.*

Theorem 6. *$\zeta_C(\cdot, s)$ is quadratic minimum at L for all sufficiently large s if and only if L is perfect and each layer of L is strongly eutactic.*

See, the author [7].

4. John Type Results

Let C be o-symmetric. A pair $\langle E, \varrho E\rangle$ of o-centered ellipsoids is a minimum ellipsoidal shell of C if $E \subseteq C \subseteq \varrho E$ and $\varrho \geq 1$ is minimum. A counterpart of John's characterizations of maximum volume ellipsoids in C is the following result where for $u \in \mathbb{E}^d$ the tensor product $u \otimes u$ is the $d \times d$ matrix uu^T:

Theorem 7. *$\langle B^d, \varrho B^d\rangle$ is a minimum ellipsoidal shell of C if and only if the following hold: There are contact points $\pm u_1, \dots, \pm u_n \in B^d \cap \operatorname{bd} C$ and $\pm v_1, \dots, \pm v_l \in C \cap \operatorname{bd} \varrho B^d$ and reals $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_l > 0$ such that*

(i) $2 \leq k, l$ *and* $k + l \leq \frac{1}{2}d(d+1) + 1$,
(ii) $\sum \lambda_i - u_i \otimes u_i = \sum \mu_j v_j \otimes v_j$,
(iii) $\operatorname{lin}\{u_1, \dots, u_k\} = \operatorname{lin}\{v_1, \dots, v_l\}$.

See, the author [3].

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Low dimensional geometry of polytopes

Alain Pajor

(joint work with R. Adamczak, A. E. Litvak, N. Tomczak-Jaegermann)

Let $1 \leq m \leq n \leq N$ be integers and let $X_1, \ldots, X_N \in \mathbb{R}^n$. Denote by A the $n \times N$ matrix with $X_1, \ldots, X_N$ as columns and by $K(A) = K(X_1, \ldots, X_N)$ the convex hull of $\pm X_1, \ldots, \pm X_N$. Thus $K(A) = AB_1^N$ where B_1^N denotes the cross-polytope in $\mathbb{R}^N$. Recall that a centrally symmetric convex polytope is (centrally) m-neighborly if any set of $m+1$ vertices not including an antipodal pair, is the vertex set of a face.

A random vector $X \in \mathbb{R}^n$ is called isotropic if

$$\mathbb{E}\langle X, y\rangle = 0, \quad \mathbb{E}\,|\langle X, y\rangle|^2 = |y|^2 \quad \text{for all } y \in \mathbb{R}^n, \tag{1}$$

in other words, if X is centered and its covariance matrix is the identity.

A subset $K \subset \mathbb{R}^n$ is said to be isotropic if the random point X uniformly distributed in K is isotropic.

We show the following result

Theorem. *Let $1 \leq n \leq N$ be integers. Let $X_1, \ldots, X_N$ be independent isotropic vectors with log-concave densities. This is for instance the case if $X_1, \ldots, X_N$ are i.i.d. random vectors uniformly distributed on an isotropic convex body. Then, for any $N \leq \exp(cn^{1/2})$, with probability at least $1 - C\exp(-cn^{1/2})$, the polytope $K(A)$ is m-centrally-neighborly, for every m satisfying*

$$m \leq cn \big/ \log 2(CN/n)$$

where $C, c > 0$ are universal constants.

Towards an Orlicz Brunn-Minkowski theory: Orlicz projection bodies

Erwin Lutwak
(joint work with Deane Yang, Gaoyong Zhang)

The Brunn-Minkowski theory had its origin at the turn of the 19th into the 20th century, when Minkowski began his study of the volume of special combinations of convex bodies (which became known as *Minkowski combinations*). One of the core concepts that Minkowski introduced to the Brunn-Minkowski theory is that of *projection body*. Four decades ago a highly influential paper of Bolker showed how Minkowski's *projection operator*, its range (called the class of *zonoids*), and its polar were in fact objects of independent investigation in a number of disciplines.

Within the Brunn-Minkowski theory, the two classical inequalities that connect the volume of a convex body with that of its *polar projection body* are the Petty and Zhang projection inequalities. In retrospect, it is interesting to recall that these inequalities did not emerge out of Blaschke's school, and that it took seven decades from Minkowski's discovery of the projection operator, for the *Petty projection inequality* to appear. It took yet another two decades for the *Zhang projection inequality* to be discovered. Establishing the analogs of the Petty and Zhang projection inequalities for the projection operator (as opposed to the polar projection operator) are today major open problems within the field of convex geometric analysis.

Unlike the classical isoperimetric inequality, the Petty and Zhang projection inequalities are affine isoperimetric inequalities in that they are inequalities between a pair of geometric functionals whose ratio is invariant under affine transformations. The Petty projection inequality is not only stronger than (i.e., directly implies) the classical isoperimetric inequality, but it can be viewed as an optimal isoperimetric inequality. In the same manner that the classical isoperimetric inequality leads to (in fact is equivalent to) the Sobolev inequality, the Petty projection inequality has lead to the *Zhang-Sobolev inequality*, an analytic inequality that is stronger than (directly implies) the classical Sobolev inequality and yet is independent of any underlying Euclidean structure.

In the early 1960's, Firey introduced an L_p-extension of Minkowski's notion of combining convex bodies (now known as *Firey-Minkowski L_p-combinations*). Three decades later, it was shown that the study of the volume of these Firey-Minkowski L_p combinations leads to an embryonic L_p Brunn-Minkowski theory. This theory has expanded steadily.

An early achievement of the new L_p Brunn-Minkowski theory was the discovery of the L_p-analogue of the projection operator which in turn led Lutwak, Yang, and Zhang, and independently Campi and Gronchi, to establish the L_p Petty projection inequality. The new L_p inequality has found application in the field of analytic inequalities where it led Lutwak, Yang, and Zhang to establish an affine L_p Zhang-Sobolev inequalitiy and ultimately, Cianchi, Lutwak, Yang, and Zhang to establish affine Moser-Trudinger and affine Morrey-Sobolev inequalities.

Work of Ludwig showed that the previously studied L_p extension of Minkowski's projection operator is only one of a family of natural L_p extensions of their classical counterpart. Using this insight, Haberl and Schuster studied "asymmetric" L_p-analogs of the projection operator and obtained "asymmetric" L_p-analogs of the Petty projection inequality. For bodies that are not symmetric about the origin, the inequalities of Haberl and Schuster are stronger than the L_p Petty projection inequality. The operators considered by Haberl and Schuster are ideally suited for dealing with bodies that are not origin-symmetric.

The work of Haberl and Schuster together with recent work of Ludwig, and Ludwig and Reitzner, makes it apparent that the time is ripe for the next step in the evolution of the Brunn-Minkowski theory towards an Orlicz Brunn-Minkowski theory.

This talk presents the attempt of Lutwak, Yang, and Zhang to develop one of the elements of an Orlicz Brunn-Minkowski theory. Specifically, to define an Orlicz projection operator and to present an Orlicz analog of the classical Petty projection inequality. This new inequality has all its predecessors (including the Haberl-Schuster version) as special cases.

Affine analytic inequalities

Christoph Haberl

(joint work with Franz E. Schuster, Jie Xiao)

For $p \geq 1$ and $n \geq 2$, let $W^{1,p}(\mathbb{R}^n)$ denote the space of real-valued L^p functions on $\mathbb{R}^n$ with weak L^p partial derivatives. We use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^n$. For $f \in W^{1,p}(\mathbb{R}^n)$, we set

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx \right)^{1/p}.$$

The *symmetric decreasing rearrangement* $f^\star$ of a function f is defined as follows. For $f \in W^{1,p}(\mathbb{R}^n)$, denote by $\mu_f : [0,\infty) \to [0,\infty]$ the distribution function of the absolute value of f. The decreasing rearrangement $f^* : [0,\infty) \to [0,\infty]$ of f is defined to be zero for $s \geq \mu_f(0)$ and

$$f^*(s) = \sup\{t > 0 : \ \mu_f(t) > s\} \qquad \text{for} \qquad s < \mu_f(0).$$

Now, the symmetric decreasing rearrangement $f^\star : \mathbb{R}^n \to [0,\infty]$ is given by

$$f^\star(x) = f^*(\kappa_n |x|^n),$$

where $\kappa_n = \pi^{n/2}/\Gamma(1 + \frac{n}{2})$ denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$.

The classical Pólya–Szegö principle [6] states that the L^p norm of the gradient of a function on $\mathbb{R}^n$ does not increase under symmetric rearrangement. To be precise, if $f \in W^{1,p}(\mathbb{R}^n)$ for some $p \geq 1$, then $f^\star \in W^{1,p}(\mathbb{R}^n)$ and

$$\|\nabla f^\star\|_p \leq \|\nabla f\|_p. \tag{1}$$

Zhang [8], Lutwak, Yang, and Zhang [5] and Cianchi et al. [2] proved an affine version of the Pólya–Szegö principle: For every function $f \in W^{1,p}(\mathbb{R}^n)$

$$\mathcal{E}_p(f^\star) \leq \mathcal{E}_p(f). \tag{2}$$

Here, the L^p *affine energy* $\mathcal{E}_p(f)$ is defined by

$$\mathcal{E}_p(f) = c_{n,p} \left(\int_{S^{n-1}} \|\mathrm{D}_u f\|_p^{-n} \, du \right)^{-1/n}$$

where $c_{n,p} = (n\kappa_n)^{1/n} (\frac{n\kappa_n \kappa_{p-1}}{2\kappa_{n+p-2}})^{1/p}$ and $\mathrm{D}_u f$ is the directional derivative of f in direction u. Note that the normalizing constant $c_{n,p}$ is chosen such that

$$\mathcal{E}_p(f^\star) = \|\nabla f^\star\|_p.$$

We emphasize that $\mathcal{E}_p(f)$ is invariant under volume preserving affine transformations on $\mathbb{R}^n$. In contrast, $\|\nabla f\|_p$ is invariant only under rigid motions. It was shown in [5] that

$$\mathcal{E}_p(f) \leq \|\nabla f\|_p. \tag{3}$$

The last two relations immediately imply the remarkable fact that the affine inequality (2) is significantly stronger than its classical Euclidean counterpart (1).

In [4], the *asymmetric L^p affine energy* $\mathcal{E}_p^+(f)$ of a function f was defined by

$$\mathcal{E}_p^+(f) = d_{n,p} \left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, du \right)^{-1/n}$$

where $d_{n,p} = 2^{1/p} c_{n,p}$ and $\mathrm{D}_u^+ f(x) = \max\{\mathrm{D}_u f(x), 0\}$ denotes the positive part of the directional derivative of f in direction u. The asymmetric L^p affine energy is again invariant under volume preserving affine transformations on $\mathbb{R}^n$. In [4], it is shown that for every function $f \in W^{1,p}(\mathbb{R}^n)$ the inequality

$$\mathcal{E}_p^+(f^\star) \leq \mathcal{E}_p^+(f) \tag{4}$$

holds. The affine energies $\mathcal{E}_p$ and $\mathcal{E}_p^+$ are related by

$$\mathcal{E}_p^+(f) \leq \mathcal{E}_p(f) \qquad \text{and} \qquad \mathcal{E}_p^+(f^\star) = \mathcal{E}_p(f^\star), \tag{5}$$

as was shown in [3]. Therefore, the asymmetric affine Pólya–Szegö inequality (4) is stronger than the symmetric one (2). In paricular, the asymmetric affine Pólya–Szegö inequality strengthens and directly implies the classical Pólya–Szegö inequality (1).

The asymmetric affine Pólya–Szegö inequality (4) gives rise to affine versions of several Sobolev inequalities. For example, for $1 < p < n$, it is shown in [3] (see also [4]) that

$$\|f\|_{\frac{np}{n-p}} \leq a_{n,p} \, \mathcal{E}_p^+(f), \tag{6}$$

where $a_{n,p}$ denotes the best constant in the sharp L^p Sobolev inequality

$$\|f\|_{\frac{np}{n-p}} \leq a_{n,p} \, \|\nabla f\|_p \tag{7}$$

due to Aubin [1] and Talenti [7]. From formulas (3) and (5) we infer that the affine version (6) is again stronger than its classical counterpart (7). In [4], affine versions of other Sobolev type inequalities including the Moser-Trudinger and the Morrey-Sobolev inequality are established.

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Mahler's conjecture for convex bodies with many symmetries

Franck Barthe
(joint work with Matthieu Fradelizi)

Mahler's conjecture predicts that among n-dimensional convex bodies of given volume, simplices minimize the volume of the polar body. The symmetric version of the conjecture asks whether cubes are minimal among convex bodies with a center of symmetry. It has been established for unconditional bodies by Saint-Raymond (a body is unconditional if it is invariant by the orthogonal symmetries with respect to all coordinate hyperplanes of some orthogonal basis). We confirm the conjecture for convex bodies having many hyperplane symmetries in the following sense: the intersection of the hyperplanes of symmetry is reduced to a point.

Asymptotic shape of a random polytope in a convex body

Nikos Dafnis
(joint work with A. Giannopoulos, A. Tsolomitis)

Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geq 1$ we consider the L_q–centroid body $Z_q(K)$ of K, defined by its support function:

$$h_{Z_q(K)}(x) = \|\langle \cdot, x\rangle\|_q := \left(\int_K |\langle y, x\rangle|^q dy\right)^{1/q}. \tag{1}$$

Our aim is to provide some precise quantitative information on the "asymptotic shape" of a random polytope $K_N = \mathrm{conv}\{x_1, \ldots, x_N\}$ spanned by N independent random points $x_1, \ldots, x_N$ uniformly distributed in K. Our approach is to compare K_N with the L_q-centroid body $Z_q(K)$ of K for $q \simeq \ln(N/n)$.

The origin of our work is in a study of the behavior of symmetric random ± 1–polytopes (by Giannopoulos and Hartzoulaki) and the subsequent sharper and more general approach by Litvak, Pajor, Rudelson, and Tomczak–Jaegermann: they worked in a more general setting which contains the previous Bernoulli model and the Gaussian model; let $K_{n,N}$ be the absolute convex hull of the rows of the random matrix $\Gamma_{n,N} = (\xi_{ij})_{1\leq i\leq N,\, 1\leq j\leq n}$, where ξ_{ij} are independent symmetric random variables satisfying certain conditions ($\|\xi_{ij}\|_{L^2} \geq 1$ and $\|\xi_{ij}\|_{L^{\psi_2}} \leq \rho$ for some $\rho \geq 1$, where $\|\cdot\|_{L^{\psi_2}}$ is the Orlicz norm corresponding to the function $\psi_2(t) = e^{t^2} - 1$). For this class of random polytopes they proved that, for every $0 < \beta < 1$,

$$K_{n,N} \supseteq c(\rho)\left(\sqrt{\beta \ln(N/n)}B_2^n \cap B_\infty^n\right) \tag{2}$$

with probability greater than $1 - \exp(-c_1 n^\beta N^{1-\beta}) - \exp(-c_2 N)$. The proof is based on a lower bound of the order of $\sqrt{N}$ for the smallest singular value of the random matrix $\Gamma_{n,N}$ with probability greater than $1 - \exp(-cN)$.

In a sense, both works correspond to the study of the size of a random polytope $K_N = \mathrm{conv}\{x_1, \ldots, x_N\}$ spanned by N independent random points $x_1, \ldots, x_N$ uniformly distributed in the unit cube $Q_n := [-1/2, 1/2]^n$. The connection of the estimate (2) with L_q–centroid bodies comes from the following observation. For $x \in \mathbb{R}^n$ and $t > 0$, define

$$K_{1,2}(x,t) := \inf\left\{\|u\|_1 + t\|x-u\|_2 : \, u \in \mathbb{R}^n\right\}. \tag{3}$$

For any $\alpha \geq 1$ define $C(\alpha) = \alpha B_2^n \cap B_\infty^n$. Then, $h_{C(\alpha)}(\theta) = K_{1,2}(\theta, \alpha)$ for every $\theta \in S^{n-1}$. On the other hand,

$$\|\langle \cdot, \theta\rangle\|_{L^q(Q_n)} \simeq \sum_{j\leq q} \theta_j^* + \sqrt{q}\left(\sum_{q<j\leq n} (\theta_j^*)^2\right)^{1/2} \tag{4}$$

for every $q \geq 1$. Then, Holmstedt's approximation formula for $K_{1,2}(x,t)$ shows that

$$C(\sqrt{q}) \simeq Z_q(Q_n) \tag{5}$$

where $Z_q(K)$ is the L_q-centroid body of K. This shows that (2) can be written in the form

$$K_{n,N} \supseteq c(\rho) Z_{\beta \ln(N/n)}(Q_n). \tag{6}$$

This observation leads us to consider a random polytope $K_N = \mathrm{conv}\{x_1, \ldots, x_N\}$ spanned by N independent random points $x_1, \ldots, x_N$ uniformly distributed in an isotropic convex body K and try to compare K_N with $Z_q(K)$ for a suitable value $q = q(N,n) \simeq \ln(N/n)$. Our first main result states that an analogue of (2) holds true in full generality.

Theorem 1. *Let $\beta \in (0, 1/2]$ and $\gamma > 1$. If*

$$N \geq N(\gamma, n) = c\gamma n, \tag{7}$$

where $c>0$ is an absolute constant, for every isotropic convex body K in $\mathbb{R}^n$ we have

$$K_N \supseteq c_1\, Z_q(K) \text{ for all } q \le c_2\beta \ln(N/n), \tag{8}$$

with probability greater than

$$1-\exp\left(-c_3 N^{1-\beta} n^{\beta}\right) - \mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^N\| \ge \gamma L_K \sqrt{N}), \tag{9}$$

where $\Gamma : \ell_2^n \to \ell_2^N$ is the random operator $\Gamma(y) = (\langle x_1, y\rangle, \dots \langle x_N, y\rangle)$ defined by the vertices $x_1, \dots, x_N$ of K_N.

It should be emphasized that a reverse inclusion of the form $K_N \subseteq c_4\, Z_q(K)$ cannot be expected with probability close to 1, unless q is of the order of n. This follows by a simple volume argument which makes use of the upper estimate of Paouris for the volume of $Z_q(K)$. However, one can easily see that K_N is "weakly sandwiched" between $Z_{q_i}(K)$ $(i=1,2)$, where $q_i \simeq \ln(N/n)$, in the following sense:

Proposition 2. *For every $\alpha > 1$ one has*

$$\mathbb{E}\left[\sigma(\theta : (h_{K_N}(\theta) \ge \alpha h_{Z_q(K)}(\theta))\right] \le N\alpha^{-q}. \tag{10}$$

This shows that if $q \ge c_5 \ln(N/n)$ then, for most $\theta \in S^{n-1}$, one has $h_{K_N}(\theta) \le c_6 h_{Z_q(K)}(\theta)$. It follows that several geometric parameters of K_N, e.g. the mean width, are controlled by the corresponding parameter of $Z_{[\ln(N/n)]}(K)$.

As an application, we discuss the volume radius of K_N: Let K be a convex body of volume 1 in $\mathbb{R}^n$. The question to estimate the expected volume radius

$$\mathbb{E}(K,N) = \int_K \cdots \int_K |\mathrm{conv}(x_1, \dots, x_N)|^{1/n} dx_N \cdots dx_1 \tag{11}$$

of K_N had been answered in the unconditional case; in this case,

$$\mathbb{E}(K,N) \simeq \min\left\{\frac{\sqrt{\ln(2N/n)}}{\sqrt{n}}, 1\right\} \tag{12}$$

for every $N \ge n+1$. Using a recent result of Paouris on the negative moments of the support function of $h_{Z_q(K)}$ we can give an answer to the question in full generality; for every convex body K and for the full range of values of N:

Theorem 2. *For every $N \le \exp(n)$, one has*

$$c_4 \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}} \le |K_N|^{1/n} \le c_5 L_K \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}} \tag{13}$$

with probability greater than $1-\frac{1}{N}$, where $c_4, c_5 > 0$ are absolute constants.

Relative entropy of cone measures and L_p centroid bodies

Elisabeth Werner
(joint work with Grigoris Paouris)

The starting point of our investigation was the study of the asymptotic behavior of the volume of L_p centroid bodies as p tends to infinity. This study resulted in the discovery of a new affine invariant, Ω_K. We then showed that the quantity Ω_K is the relative entropy of the cone measure of K and the cone measure of K°. Cone measures have been intensively studied in recent years (see e.g. works by Barthe, Guedon, Mendelson and Naor, Gromov and Milman, Naor and Romik and Schechtmann and Zinn. Finally, to our surprise, Ω_K appeared again naturally in a third way, namely as a limit of normalized L_p-affine surface areas. Thus, the invariant Ω_K introduces a novel idea -relative entropy- into the theory of convex bodies and links concepts from classical convex geometry like L_p centroid bodies and L_p-affine surface area with concepts from information theory. Such links have already been established. Guleryuz, Lutwak, Yang and Zhang use L_p Brunn Minkowski theory to develop certain entropy inequalities. Also, classical Brunn Minkowski theory is related to information theoretic concepts (see e.g. the works by Artstein-Avidan, Barthe, Ball and Naor).

The convex floating body K_δ of K [5] is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume δ from K.
It was shown by Milman and Pajor [4] that for "big" δ K_δ is isomorphic to the dual of the Binet ellipsoid from classical mechanics and consequently K_δ° is homothetic to the Binet ellipsoid. Lutwak and Zhang [2] generalized the notion of Binet ellipsoid and introduced the L_p centroid bodies: For a convex body K in $\mathbb{R}^n$ of volume 1 and $1 \leqslant p \leqslant \infty$, the L_p centroid body $Z_p(K)$ is this convex body that has support function

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{1/p}. \tag{1}$$

We generalize the result by Milman and Pajor and show that the floating body K_δ is - up to a universal constant - homothetic to the centroid body $Z_{\log \frac{1}{\delta}}(K)$:

Theorem 1. *Let K a symmetric convex body in $\mathbb{R}^n$ of volume 1. Let $\delta \in (0,1)$. Then*

$$c_1 Z_{\log \frac{1}{\delta}}(K) \subseteq K_\delta \subseteq c_2 Z_{\log \frac{1}{\delta}}(K),$$

where $c_1, c_2 > 0$ are universal constants.

L_p-affine surface area, an extension of classical affine surface area (the case $p = 1$), was introduced by Lutwak in the ground breaking paper [1] for $p > 1$ and for general p by Schütt and Werner [6].

From now on we will always assume that the centroid of a convex body K in $\mathbb{R}^n$ is at the origin. We write $K \in C^2_+$, if K has C^2 boundary with everywhere strictly positive Gaussian curvature κ_K. For real $p \neq -n$, we define the L_p-affine surface area $as_p(K)$ of K as in [1] ($p > 1$) and [6] ($p < 1, p \neq -n$) by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x), \tag{2}$$

provided the above integrals exist. $N_K(x)$ is the outer unit normal vector at x to ∂K, the boundary of K, and μ_K is the usual surface area measure on ∂K.

We use the L_p-affine surface area to define a new affine invariant:

$$\Omega_K = \lim_{p\to\infty} \left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p}. \tag{3}$$

This is a first way how Ω_K appears. We describe properties of this new invariant. E.g., we prove the following remarkable identity (4), which is the second way how Ω_K appears: It shows that the invariant Ω_K is the exponential of the relative entropy or Kullback-Leibler divergence D_{KL} of the cone measures cm_K and cm_{K° of K and K°.

$$\Omega_K^{1/n} = \frac{|K^\circ|}{|K|} \exp\Big(- D_{KL}(N_K N_{K^\circ}^{-1} cm_{\partial K^\circ} \| cm_{\partial K})\Big). \tag{4}$$

N_K^{-1} is the inverse of the Gauss map.

We show that an information inequality for the relative entropy of the cone measures implies an "information inequality" for convex bodies

$$\Omega_K \leqslant \left(\frac{|K|}{|K^\circ|}\right)^n$$

with equality if and only if K is an ellipsoid. Independently, we can derive this inequality from properties of the L_p-affine surface areas.

We also show that many other inequalities that hold for the affine invariant Ω_K, among them an isoperimetric inequality.

Theorem 1 states that the floating body K_δ is - up to a universal constant - homothetic to the centroid body $Z_{\log\frac{1}{\delta}}(K)$. This led us to investigate the L_p centroid bodies also in the context of affine surface area. Note the similarities in bahavior of the floating body and the L_p centroid body. Both "approximate" K as $\delta \to 0$ respectively $p \to \infty$: If K is symmetric and of volume 1, $Z_p(K) \to K$ as $p \to \infty$.

We found an amazing connection between the L_p centroid bodies and the new invariant Ω_K which is stated in the following theorem for convex bodies in C^2_+. A forthcoming paper will address general convex bodies.

Theorem 2. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1 that is in C^2_+. Then*

(i) $\lim_{p\to\infty} \frac{p}{\log p} \big(|Z_p^\circ(K)| - |K^\circ|\big) = \frac{n(n+1)}{2}\ |K^\circ|$.

$$\textit{(ii)} \qquad \lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p\ |K^\circ|\right) =$$
$$-\frac{1}{2}\int_{S^{n-1}} h_K(u)^{-n}\ \log\left(2^{n+1}\pi^{n-1}h_K(u)^{n+1}f_K(u)\right)d\sigma(u) =$$
$$\frac{1}{2}\int_{\partial K}\frac{\kappa(x)}{\langle x, N(x)\rangle^n}\ \log\left(\frac{\kappa(x)}{2^{n+1}\pi^{n-1}\langle x, N(x)\rangle^{n+1}}\right)d\mu(x)$$

In view of Theorem 1, the first part of the Theorem 2 came as a surprise to us because it reveals a different behaviour of the bodies K_δ and $Z_{\log\frac{1}{\delta}}(K)$ when $\delta \to 0$. Indeed, it was shown in [3] that $\lim_{\delta\to 0}\ c_n \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = as_{-n(n+2)}(K) = as_{-\frac{n}{n+2}}(K^\circ)$. where c_n is a constant that depends on n only.

Even more surprising is the second part of Theorem 2. We can show that

$$\lim_{p\to\infty}\frac{2p}{n}\left(\frac{(1-\frac{n(n+1)\log p}{2p})|Z_p^\circ(K)|}{|K^\circ|} - 1\right) = -\frac{1}{2}\log\frac{\Omega_K^{\frac{1}{n}}}{2^{n+1}\pi^{n-1}}.$$

This is the third way how Ω_K appears.

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Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture

Grigoris Paouris
(joint work with Nikos Dafnis)

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, center of mass at the origin, and its inertia matrix is a multiple of the identity. Equivalently, if there is a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1}. A well known open question (known as the Hyperplane conjecture or the slicing problem) is the following: There exists an absolute constant $C > 0$ such that $L_K \leqslant C$ for every convex body K. We refer to the article [7] of Milman and Pajor for background information about isotropic convex bodies. (see also [5]).

Bourgain proved in [2] that $L_K \leqslant c\sqrt[4]{n}\log n$ and, a few years ago, Klartag [6] obtained the estimate $L_K \leqslant c\sqrt[4]{n}$. The approach of Bourgain in [2] is to reduce the problem to the case of convex bodies that satisfy a ψ_2-estimate (with constant $\beta = O(\sqrt[4]{n})$). We say that K satisfies a ψ_2-estimate with constant β if

$$\|\langle \cdot, y\rangle\|_{\psi_2} \leq \beta \|\langle \cdot, y\rangle\|_2 \tag{1}$$

for all $y \in \mathbb{R}^n$. Bourgain proved in [3] that, if (1) holds true, then

$$L_K \leqslant C\beta \log \beta.$$

The purpose of this paper is to introduce a different method which leads to upper bounds for L_K. We prove that a positive answer to the hyperplane conjecture is equivalent to some very strong small probability estimates for the Euclidean norm on isotropic convex bodies; for $-n < p \leqslant \infty$, $p \neq 0$, we define

$$I_p(K) := \left(\int_K \|x\|_2^p dx \right)^{1/p}$$

and, for $\delta \geqslant 1$, we consider the parameter

$$q_{-c}(K, \delta) := \max\{p \geqslant 1 : I_2(K) \leq \delta I_{-p}(K)\}.$$

Then, the hyperplane conjecture is equivalent to the following statement:

> There exist absolute constants $C, \xi > 0$ such that, for every isotropic convex body K in $\mathbb{R}^n$,
>
> $$q_{-c}(K, \xi) \geqslant Cn.$$

The main idea in our approach is to start from an extremal isotropic convex body K in $\mathbb{R}^n$ with maximal isotropic constant $L_K \simeq L_n := \sup\{L_K : K \text{ is a convex body in } \mathbb{R}^n\}$. Building on ideas from the work [4] of Bourgain, Klartag and Milman, we construct a second isotropic convex body K_1 which is also extremal and, at the same time, is in α-regular M-position in the sense of Pisier (see [10]). Then, we use the fact that small ball probability estimates are closely related to estimates on covering numbers. This gives the estimate

$$L_{K_1} I_{-c\frac{n}{(2-\alpha)t^\alpha}}(K_1) \leqslant Ct\sqrt{n},$$

for $t \geqslant C(\alpha)$, where $c, C > 0$ are absolute constants. The construction of K_1 from K can be done inside any subclass of isotropic log-concave measures which is stable under the operations of taking marginals or products. This leads us to the definition of a *coherent* class of probability measures: a subclass $\mathcal{U}$ of the class of probability measures $\mathcal{P}$ is called coherent if it satisfies two conditions:

(1) If $\mu \in \mathcal{U}$ is supported on $\mathbb{R}^n$ then, for all $k \leqslant n$ and $F \in G_{n,k}$, $\pi_F(\mu) \in \mathcal{U}$.
(2) If $m \in \mathbb{N}$ and $\mu_i \in \mathcal{U}$, $i = 1, \ldots, m$, then $\mu_1 \otimes \cdots \otimes \mu_m \in \mathcal{U}$.

It should be noted that the class of isotropic convex bodies is not coherent. This is the reason for working with the more general class of log-concave measures. The basic tools that enable us to pass from one language to the other come from K. Ball's bodies (see [1]). In particular we show that the class of measures that satisfy (1) are coherent.

Our main result is the following:

Theorem 1. *Let $\mathcal{U}$ be a coherent subclass of isotropic* log-*concave measures and let $n \geqslant 2$ and $\delta \geqslant 1$. Then,*

$$\sup_{\mu \in \mathcal{U}_{[n]}} f_\mu(0)^{\frac{1}{n}} \leqslant C\delta \sup_{\mu \in \mathcal{U}_{[n]}} \sqrt{\frac{n}{q_{-c}(\mu,\delta)} \log\left(\frac{en}{q_{-c}(\mu,\delta)}\right)},$$

where $C > 0$ is an absolute constant and $\mathcal{U}_{[n]}$ denotes the subclass of n-dimensional measures in $\mathcal{U}$.

The main results of [8] and [9] show that there exists a parameter $q_* := q_*(K)$ (related to the L_q–centroid bodies of K) with the following properties: (i) $q_*(K) \geqslant c\sqrt{n}$, (ii) $q_{-c}(K,\xi) \geqslant q_*(K)$ for some absolute constant $\xi \geqslant 1$, and hence, $I_2(K) \leqslant \xi I_{-q_*}(K)$. Moreover if K satisfies (1) one has that $q_*(K) \geqslant c\frac{n}{\beta^2}$, Combining these results we immediately deduce two facts:

(1) If a convex body K satisfies a ψ_2-estimate with constant β, then

$$L_K \leqslant C\beta\sqrt{\log\beta}.$$

(2) For every symmetric isotropic convex body K in $\mathbb{R}^n$,

$$L_K \leqslant C\sqrt[4]{n}\sqrt{\log n}.$$

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Recent Developments on the Polarization Problem

Gergely Ambrus
(joint work with Keith Ball)

The original polarization problem states the following.

Conjecture 1 (Polarization problem). *For any collection $u_1, \dots, u_n$ of unit vectors in a Hilbert space $\mathcal{H}$, there exists a unit vector $v \in \mathcal{H}$, such that*

$$\prod_{i=1}^{n} |\langle u_i, v\rangle| \geq n^{-n/2}. \tag{1}$$

The conjecture originates from the theory of Banach spaces as a variant of the polarization inequality. For complex Hilbert spaces, Arias-de-Reyna proved the statement in 1998 [2]. Later, K. Ball proved an even stronger result, the complex plank theorem [3]. However, for real Hilbert spaces, the conjecture is still open. Several estimates have been obtained by using various methods, for example: using the natural complexification of $\mathbb{R}^n$; inequalities about the eigenvalues, determinants and permanents of Gram matrices; geometric methods.

In our research, we devote our attention to a stronger conjecture.

Conjecture 2 (Strong polarization problem). *For any set $u_1, \dots, u_n$ of unit vectors in $\mathbb{R}^d$, there exists a unit vector $v \in \mathbb{R}^d$, such that*

$$\sum \frac{1}{\langle u_i, v\rangle^2} \leq n^2. \tag{2}$$

The AM-GM inequality immediately shows that this indeed implies the original polarization conjecture. One of the remarkable feature of this statement is the following, pointed out by K. Ball and P. Frenkel. It is conjectured that the only extremal vector system in the real polarization conjecture is the orthonormal system consisting of n unit vectors in $\mathbb{R}^n$. Therefore, if the number of vectors is larger than the dimension of X, we expect a stronger inequality to hold. The simplest example of this phenomenon is obtained when $X = \mathbb{R}^2$: If $(u_1, \dots, u_n)$ be a system of vectors on the unit circle, then, via the connection to the Chebyshev constant, the best constant turns out to be 2^{n-1}. This is obtained when the point set $(u_1, -u_1, \dots, u_n, -u_n)$ is equally distributed on the unit circle. The same example shows as well that the assertion of the affine plank theorem is essentially sharp, and nothing close to the estimate of the complex plank problem is true in the real setting.

Considering the real polarization problem, the picture is entirely different. As it turns out, the best constant obtained for systems of n vectors on the unit circle is the same as the one we get for the n-dimensional orthonormal system! Therefore, we "don't gain anything" by leaving the 2-dimensional space for $\mathbb{R}^n$, although, intuitively, one would think that in the latter it is possible to go "much farther away" from the orthogonal subspaces than in the plane. This rather remarkable geometric property was the first to suggest that the strong polarization problem is a good deal more natural than its original version, and in some sense it serves as the real analogue of the complex plank problem. We also show that there are extremal vector systems of any dimension up to n.

We present a complex analytic proof for the planar, $d = 2$ case of the strong polarization problem, which appeared in [1]. We start by transforming the function $\sum 1/\langle u_i, v\rangle^2$ to a complex rational function defined on the complex unit circle. If the set (u_i) is locally extremal with respect to Conjecture 2, then the resulting rational function oscillates in maximal order between 0 and a constant. A centered shift of such functions is called *equioscillating*; their characterisation appeared first in [4]. From this result we deduce that the rational function in question has a specific form. After some involved calculations, the comparison of the leading coefficients and the constant terms yields that for any locally extremal case, the sharp estimate in (2) is indeed n^2, and hence the result is true, moreover, every locally extremal set is extremal as well.

For the higher dimensional cases, Conjectures 1 and 2 can be transformed to purely geometric forms, which then can be attacked by convex geometric tools. In particular, using arguments similar to Fritz John's maximal ellipsoid result, we proved that the only non-degenerate locally extremal system for both conjectures is the orthonormal system. For the lower dimensional extremal cases, technical difficulties arise, and hence the polarization problems are still unsettled.

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Simplices in the Euclidean ball

Matthieu Fradelizi

(joint work with Grigoris Paouris, Carsten Schütt)

The starting point of this paper is the article [1], where it was shown that if all the extreme points of a convex body K in $\mathbb{R}^n$ have Euclidean norm greater than $r > 0$, then

$$\frac{1}{|K|}\int_K |x|_2^2 dx \geqslant \frac{r2}{9n} \tag{1}$$

where $|x|_2$ stands for the Euclidean norm of x and $|K|$ for the volume of K.

We improve here this inequality showing that the optimal constant is $\dfrac{r^2}{n+2}$, with equality for the regular simplex, with vertices on the Euclidean sphere of radius r. We also prove the same inequality under the different condition that K is in Löwner position. More generally, we investigate upper and lower bounds on the quantity

$$C_2(K) := \frac{1}{|K|}\int_K |x|_2^2 dx \ , \tag{2}$$

under various assumptions on the position of K. Some hypotheses on K are necessary because $C_2(K)$ is not homogeneous, one has $C_2(\lambda K) = \lambda^2 C_2(K)$.
Let $n \geqslant 2$. We denote by $\mathcal{K}^n$ the set of all convex bodies in $\mathbb{R}^n$, i.e. the set of compact convex sets with non empty interior and by Δ^n the regular simplex in $\mathbb{R}^n$ with vertices in S^{n-1}, the Euclidean unit sphere. For $K \in \mathcal{K}^n$, we denote by g_K, its centroid,

$$g_K = \frac{1}{|K|}\int_K x dx.$$

Under these notations we prove the following theorem.

Theorem 1. *Let $r > 0$, $K \in \mathcal{K}^n$ such that all its extreme points have Euclidean norm greater than r. Then*

$$C_2(K) := \frac{1}{|K|}\int_K |x|_2^2 dx \geqslant C_2(r\Delta^n) + \left(\frac{n+1}{n+2}\right)|g_K|_2 2 = \frac{r^2 + (n+1)|g_K|_2 2}{n+2} \ .$$

Moreover, if K is a polytope there is equality if and only if K is a simplex with its vertices on the Euclidean sphere of radius r.

In Theorem 1.1, for a general K, we don't have a characterization of the equality case because we deduce it by approximation from the case of polytopes. We conjecture that the equality case is still the same.

Notice that the condition imposed on K that all its extreme points have Euclidean norm greater than r is unusual. For example, if K has positive curvature, it is equivalent to either $K \supset rB_2^n$ or $K \cap rB_2^n = \emptyset$. Moreover, this hypothesis is not continuous with respect to the Hausdorff distance. Indeed, if we define $P = \mathrm{conv}\,(\Delta^n, x)$, where $x \notin \Delta^n$ is a point very close to the centroid of a facet of Δ^n then the distance of Δ^n and P is very small but the point x will be an extreme point of P of Euclidean norm close to $1/n$, *i.e.* much smaller than 1, the Euclidean norm of the vertices of Δ^n.

Other conditions on the position of K may be imposed. To state it, let us first recall the classical definitions of John and Löwner position. Let $K \in \mathcal{K}_n$. We say that K is in *John position* if the ellipsoid of maximal volume contained in K is B_2^n. We say that K is in *Löwner position* if the ellipsoid of minimal volume that contains K is B_2^n.

It was proved by Guédon in [2] (see also [3]) that if $K \in \mathcal{K}^n$ satisfies $g_K = 0$ and if $K \cap (-K)$ is in Löwner position (which is equivalent to say that B_2^n is the ellipsoid of minimal volume containing K and centered at the origin) then $C_2(K) \geq C_2(\Delta^n)$. Using the same ideas, we prove the following theorem.

Theorem 2. *Let K be a convex body in Löwner position. Then*

$$\frac{n}{n+2} = C_2(B_2^n) \geq C_2(K) \geq C_2(\Delta^n) + \frac{(n+1)2}{n(n+2)}|g_K|_2 2 = \frac{n+(n+1)2|g_K|_2 2}{n(n+2)}.$$

Moreover, if K is symmetric, then

$$\frac{n}{n+2} = C_2(B_2^n) \geq C_2(K) \geq C_2(B_1^n) = \frac{2n}{(n+1)(n+2)}.$$

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Maximal function for high-dimensional cubes

Guillaume Aubrun

Let $K \subset \mathbf{R}^n$ be a symmetric convex body. The maximal operator associated to K, denoted M_K, is defined for a positive Borel measure μ on $\mathbf{R}^n$ by

$$M_K\mu(x) = \sup_{r>0} \frac{\mu(x+rK)}{\mathrm{vol}(x+rK)} \qquad (x \in \mathbf{R}^n).$$

The operator M_K maps positive measures to positive functions taking possibly the value $+\infty$. Let $\Theta(K)$ be the best constant in the $L^1 \to$ weak L^1 inequality for M_K. That is, $\Theta(K)$ is the smallest constant C so that, for every positive measure μ and any $A \geq 0$,

$$\sup_{A>0} A \cdot \mathrm{vol}\{M_K\mu > A\} \leq C\mu(\mathbf{R}^n).$$

(a standard mollifying arguement shows that one can restrict oneself to absolutely continuous measures, i.e. functions in L^1). It turns out that $\Theta(K)$ is extremely hard to compute exactly. The only known value is for the one-dimensional case, where $\Theta([-1,1]) = \frac{11+\sqrt{61}}{12}$, a result by Melas [3]. Concerning higher dimensions, the following is known (K denotes a convex body in $\mathbf{R}^n$)

- A standard application of Vitali's covering lemma shows that $\Theta(K) \leq 3^n$.
- An argument by Stein and Strömberg [5] shows that $\Theta(K) \leq Cn\log n$ for some absolute constant C. This argument can be seen as a randomized version of Vitali's lemma (see [4]).
- For the Euclidean ball, it is known that $\Theta(B_2^n) \leq Cn$ for some absolute constant C [5]. This is the only case where an upper bound better that $Cn\log n$ is known.

A recent breakthrough was achieved by Aldaz [1] who proved that the sequence $\Theta([-1,1]^n)$ is unbounded. The goal of my talk was to present a simplified version of Aldaz's proof, using accurate probabilistic tools. This approach allows also to derive a lower bound $\Theta([-1,1]^n) \geq (\log(n))^{1-o(1)}$. Note that this lower bound is still far from the upper bound $O(n \log n)$.

Let us sketch our approach (we refer to [2] for details). We relate the problem to couting integer points in high-dimensional cubes. Denote $Q(x,r) = x + r[-1,1]^n$. For $A > 0$, let $E_A \subset [0,1]^n$ be the following set:

$$E_A = \{x \in [0,1]^n \text{ s.t. } \exists r > 0 \text{ s.t. } \#(Q(x,r) \cap \mathbf{Z}^n) \geq A \operatorname{vol} Q(x,r)\}.$$

The key point is the following

Proposition. For any $A > 0$, the set E_A occupies almost all $[0,1]^n$ when the dimension is large. That is, $\operatorname{vol}(E_A) = 1 - o(1)$ as n tends to infinity. More precisely, we have $\operatorname{vol}(E_{(\log n)^\eta}) = 1 - o(1)$ for any $\eta < 1$.

It is straightforward to deduce from this the lower estimates for $\Theta([-1,1]^n)$: one chooses as a measure the counting measure on a large box inside $\mathbf{Z}^n$ and uses the fact that for very large cubes, the number of integer points per unit volume is close to 1, so that large values of r can be discarded.

Let us give some examples to understand the structure of E_A

- The vertices of $[0,1]^n$ belong to E_A for any $A > 0$ (take $r = 0$).
- The point $(\frac{1}{2}, \dots, \frac{1}{2})$ belongs to E_{2^n} (take $r = 1/2$).
- One checks that the point $(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n})$ does not belong to E_{10}.

These examples suggest that belonging to E_A for some large A is related to the irregularity of distributions of coordinates. This is indeed the case, but it may be surprising to notice that one has to consider cubes of large radius (of order of $\sqrt{n}$ in $\mathbf{R}^n$).

Lemma 1. There is an absolute constant C so that the following holds for any $K \geq 1$. Let $x \in [0,1]^n$ and assume that, for some $t \in [\frac{CK^2}{n}, 1 - \frac{CK^2}{n}]$, the number of coordinates of x inside $[\frac{1-t}{2}, \frac{1+t}{2}]$ exceeds $nt + K\sqrt{nt(1-t)}$ (that is, exceeds the average by more than K standard deviations). Then $x \in E_A$ for $A = \exp(K^2/2.01)$.

We now use a fact from statistics: with large probability a random point of $[0,1]^n$ will satisfy the hypothesis of the lemma for *some* value of t when n becomes large. The can be shown using Donsker's theorem asserting that fluctations around the cumulative distribution functions are asymptotatically described by a Brownian bridge. The phenomenon can be made quantitative by invoking the law of the iterated logarithm.

Lemma 2. For any $c > 0$ the following holds. If $(X_i)_{1 \leq i \leq n}$ are i.i.d. random variables uniformly distributed on $[0,1]$, then with probability larger than $1 - o(1)$, there exists $t \in [\frac{c}{n}, 1 - \frac{c}{n}]$ so that

$$\#\left\{ i \text{ s.t. } X_i \in \left[\frac{1-t}{2}, \frac{1+t}{2}\right] \right\} \geq nt + \sqrt{1.99 \log\log n}\sqrt{nt(1-t)}.$$

Up to smaller order errors, we may apply Lemma 1 with $K = \sqrt{1.99 \log\log n}$, giving that most points from the cube belong to E_A for $A = \exp(K^2/2.01) \approx (\log n)^{0.99}$.

A very challenging open problem is the case of Euclidean balls instead of cubes: Is the sequence $\Theta(B_2^n)$ bounded ? I don't know whether a similar approach works. Here is a precise formulation:

Question. Denote $B(x,r) = x + rB_2^n$. For $A > 0$, let $\tilde{E}_A \subset [0,1]^n$ be the following set:

$$\tilde{E}_A = \{x \in [0,1]^n \text{ s.t. } \exists r > 0 \text{ s.t. } \#(B(x,r) \cap \mathbf{Z}^n) \geq A \operatorname{vol} B(x,r)\}.$$

Is it true that for any $A > 0$, $\operatorname{vol}(\tilde{E}_A) = 1 - o(1)$ when n tends to infinity ?

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A remark on the Mahler conjecture: local minimality of the unit cube

Dmitry Ryabogin
(joint work with F. Nazarov, F. Petrov, A. Zvavitch)

In 1939 Mahler [Ma] asked the following question. Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a convex origin-symmetric body and let

$$K^* := \{\xi \in \mathbb{R}^n : x \cdot \xi \leq 1 \ \forall x \in K\}$$

be its polar body. Define $\mathcal{P}(K) = \operatorname{vol}_n(K)\operatorname{vol}_n(K^*)$. Is it true that we always have

$$\mathcal{P}(K) \geq \mathcal{P}(B_\infty^n),$$

where $B_\infty^n = \{x \in \mathbb{R}^n : |x_i| \leq 1, 1 \leq i \leq n\}$?

Mahler himself proved in [Ma] that the answer is affirmative when $n = 2$. There are several other proofs of the two-dimensional result, see for example the proof of M. Meyer, [Me2], but the question is still open even in the three-dimensional case.

In the n-dimensional case, the conjecture has been verified for some special classes of bodies, namely, for bodies that are unit balls of Banach spaces with 1-unconditional bases, [SR], [R2], [Me1], and for zonoids, [R1], [GMR].

Bourgain and Milman [BM] (see also [Pi]) proved the inequality

$$\mathcal{P}(K)^{1/n} \geq c\mathcal{P}(B_\infty^n)^{1/n},$$

with some constant $c > 0$ independent of n. The best known constant $c = \pi/4$ is due to Kuperberg [Ku].

Note that the exact upper bound for $\mathcal{P}(K)$ is known:

$$\mathcal{P}(K) \leq \mathcal{P}(B_2^n),$$

where B_2^n is the n-dimensional Euclidean unit ball. This bound was proved by Santalo [Sa]. In [Pe] and [MeP] it was shown that the equality holds only if K is an ellipsoid.

Let $d_{BM}(K, L) = \inf\{b/a : \exists T \in GL(n) \text{ such that } aK \subseteq TL \subseteq bK\}$ be the Banach-Mazur multiplicative distance between bodies $K, L \subset \mathbb{R}^n$. In this talk I will present the following result.

Theorem. *Let $K \subset \mathbb{R}^n$ be an origin-symmetric convex body. Then*

$$\mathcal{P}(K) \geq \mathcal{P}(B_\infty^n),$$

provided that $d_{BM}(K, B_\infty^n) \leq 1 + \delta$, and $\delta = \delta(n) > 0$ is small enough. Moreover, the equality holds only if $d_{BM}(K, B_\infty^n) = 1$, i.e., if K is a parallelepiped.

The first difficulty in proving local minimality of the unit cube is that there are plenty of small perturbations with the same volume product, namely all close parallelepipeds. We overcome this difficulty by choosing a “canonical representative” in each class of affinely equivalent convex bodies. More precisely, we consider only the bodies K for which the unit cube is a parallelepiped of the least volume containing K. In addition to taking care of all close parallelepipeds, it allows us to fix $2n$ points on the boundary of K and K^* (the centers of the $(n-1)$-dimensional faces of B_∞^n). Our next step is to choose several additional points on the boundary of K and K^* and to construct two (not necessarily convex) polytopes $P \subset K$ and $Q \subset K^*$ such that

$$\mathrm{vol}_n(P)\mathrm{vol}_n(Q) \geq \mathcal{P}(B_\infty^n) - C\delta^2,$$

where δ is the least positive number for which $(1 - \delta)B_\infty^n \subset K$. We conclude that B_∞^n is a lower semi-stationary point for the volume product functional $\mathcal{P}$. This means that the perturbation of B_∞^n by δ in the Banach-Mazur distance may result in decreasing the product volume only by δ^2, i.e., in the second order rather than in the first. Our last step is to show that either K contains a point outside $(1 + c\delta)P$ or K^* contains a point outside $(1 + c\delta)Q$ for some small positive c. This allows us to conclude that $\mathcal{P}(K)$ exceeds $\mathrm{vol}_n(P)\mathrm{vol}_n(Q)$ by at least $c\delta$ and get the final estimate

$$\mathcal{P}(K) \geq \mathcal{P}(B_\infty^n) + c\delta - C\delta^2$$

from which the strict local minimality follows.

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The Nazarov proof of the Bourgain-Milman theorem

Dmitry Ryabogin

Recently Fedor Nazarov gave a new complex-analytic proof of the Bourgain-Milman estimate

$$v_n(K)v_n(K^\circ) \geq c^n \frac{4^n}{n!}$$

where K is an origin-symmetric bounded convex body in $\mathbb{R}^n$, $K^\circ = \{t \in \mathbb{R}^n : \langle x, t\rangle \leq 1\}$ is its polar body, v_n stands for the n-dimensional volume measure in $\mathbb{R}^n$, and $c > 0$ is a numeric constant (see [BM]).

The best value of c he could get on this way is $\left(\frac{\pi}{4}\right)^3$. The current record $c = \frac{\pi}{4}$ is due to Kuperberg [Ku].

In this talk I will present some ideas of Nazarov's work. Fedor starts with recasting the question into the language of Hilbert spaces of analytic functions of several complex variables. He uses the Paley-Wiener theorem, which asserts that the following two classes of functions are the same:

(1) The class of all entire finctions $f : \mathbb{C}^n \to \mathbb{C}$ of finite exponential type (i.e., satisfying the bound $f(z) \leq Ce^{C|z|}$ for all $z \in \mathbb{C}^n$ with some $C > 0$) such that their restriction to $\mathbb{R}^n$ belongs to L^2 and such that $|f(iy)| \leq Ce^{\rho_K(y)}$ with some $C > 0$ for all $y \in \mathbb{R}^n$ where $\rho_K(x) = \inf\{\beta > 0 : x \in \beta K\}$.

(2) The class of the Fourier transforms $f(z) = \int_{K^\circ} g(t)e^{-i\langle z,t\rangle}\, dv_n(t)$ of L^2-functions g supported on K°.

The class given by any of these conditions is denoted by $\mathrm{PW}(K)$. If $f \in \mathrm{PW}(K)$ is the Fourier transform of g, then, by Plancherel's formula,

$$\|f\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^n \|g\|^2_{L^2(K^\circ)}.$$

Now, using the Cauchy-Schwarz inequality, he has

$$|f(0)|^2 = \left|\int_{K^\circ} g\, dv_n\right|^2 \le v_n(K^\circ)\|g\|^2_{L^2(K^\circ)} = \frac{1}{(2\pi)^n} v_n(K^\circ)\|f\|^2_{L^2(\mathbb{R}^n)},$$

and one can observe that the equality sign is attained when $g = 1$ in K°. Thus,

$$v_n(K^\circ) = (2\pi)^n \sup_{f\in \mathrm{PW}(K)} |f(0)|^2 \cdot \|f\|^{-2}_{L^2(\mathbb{R}^n)}\,.$$

Since the quantity on the right does not include any metric characteristics of the polar body K° we see that the problem of *proving* a lower bound for $v_n(K^\circ)$ has been thus transformed into the problem of *finding an example* of an entire function $f \in \mathrm{PW}(K)$ that has not too small value at the origin and not too large $L^2(\mathbb{R}^n)$-norm.

The construction of fast decaying on $\mathbb{R}^n$ analytic functions of several complex variables is quite a non-trivial task by itself and Nazarov's approach would look rather hopeless if not for the remarkable theorem of Hörmander that allows one to conjure up such functions in Bergman type spaces $L^2(\mathbb{C}^n, e^{-\varphi}\, dv_{2n})$ with plurisubharmonic φ.

To use the Theorem of Hörmander Nazarov approximates the Paley-Wiener space by some weighted Bergman space with a Hörmander type weight and then he carries out the relevant computations at that space.

Finally Nazarov notes that there may be no ideal approximation of the Paley-Wiener space by a Bergman-Hörmander one and, in order to get the Mahler conjecture itself on this way, one would have to work directly with the Paley-Wiener space by either finding a good analogue of the Hörmander theorem allowing to control the Paley-Wiener norm of the solution, or by finding some novel way to construct decaying analytic functions of several variables.

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Jarník's convex lattice polygon for non-symmetric norms

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(joint work with Nathanael Enriquez)

What is the minimal perimeter L_n that a convex lattice polygon with n vertices can have? In 1926 Jarník [2] proved that $L_n = \frac{\sqrt{6\pi}}{9} n^{3/2} + O(n^{3/4})$. The target here is to extend this result to all, not necessarily symmetric, norms in the plane. As usual, such a norm is defined by a convex compact set $D \subset \mathcal{R}$ containing 0 in its interior, and the norm of $x \in \mathcal{R}$ is

$$||x|| = ||x||_D = \min\{t \geq 0 : x \in tD\}.$$

As usual let **Z2** be the lattice of integer points in $\mathcal{R}$, and write $\mathbf{P}_n$ ($n \geq 3$) for the set of all convex lattice n-gons in $\mathcal{R}$, that is, $P \in \mathbf{P}_n$ if $P = \text{conv}\{z_1, \dots, z_n\}$ where $z_1, \dots, z_n \in$ **Z2** are the vertices, in anticlockwise order, of P. The D-perimeter of P is defined by

$$\text{Per}\, P = \text{Per}_D P = \sum_{i=1}^{n} ||z_{i+1} - z_i||_D$$

where $z_{n+1} = z_1$ by convention. Define

$$L_n = L_n(D) = \min\{\text{Per}_D P : P \in \mathbf{P}_n\}$$

In the talk we describe the asymptotic behaviour of $L_n(D)$ for all norms. It is also shown that, after suitable scaling, the minimizing polygons have a limiting shape. To state the results we need some preparations.

Assume that the vertices of a minimizer $P_n \in \mathbf{P}_n$ are $z_1, \dots, z_n$ in anticlockwise order (which is the orientation giving the minimal D-perimeter). Then $E_n = \{z_2 - z_1, \dots, z_n - z_{n-1}, z_1 - z_n\}$ is the edge set of P_n. Define $C_n = \text{conv}\, E_n$. It is clear that E_n determines P_n uniquely (up to translation). It is also clear that E_n contains only primitive vectors, that is, $z = (x, y) \in E_n$ implies that x and y have no common divisor. Equally easy is to check that $L_n \leq L_{n+1}$.

We assume that $\text{Area}\, D = 1$. For convex sets $K, L \subset \mathcal{R}$, $\text{dist}\,(K, L)$ denotes their Hausdorff distance. Here come our main results.

Theorem 1. *Under the above conditions there is a unique convex set $C \subset \mathcal{R}$ such that* $\lim \text{dist}\,((\text{Area}\, C_n)^{-1/2} C_n, C) = 0$. *Moreover,* $\text{Area}\, C = 1$, $g(C) = 0$ *and* $\lim n^{-3/2} L_n(D)$ *exists and equals*

$$\alpha(D) = \frac{\pi}{\sqrt{6}} \int_C ||x|| dx.$$

Theorem 2. *There is a convex set $P \subset \mathcal{R}$ such that the following holds. Let P_n be an arbitrary sequence of minimizers for $L_n(D)$ translated so that* $\min\{x : (x, y) \in P_n\}$ *is reached at the origin. Then* $\lim \text{dist}\,(n^{-3/2} P_n, P) = 0$.

The uniqueness part of Theorem 1 is proved by considering the following variational problem, to be called $VP(r_0)$. Define F as the set of all positive continuous functions $r : [0, 2\pi] \to \mathbf{R}^+$ with $r(0) = r(2\pi)$. Such a function is the radial function of a *starshaped* set in $\mathcal{R}$; such a set contains the origin in its interior and the half-line starting at the origin in direction $u(t) = (\cos t, \sin t)$ intersects its boundary at a single point which is at distance $r(t)$ from the origin. Every convex compact set $K \subset \mathcal{R}$ with $0 \in \operatorname{int} K$ is, of course, starshaped. We denote by F^c the set of the radial function of such convex compact sets.

Let $r_0 \in \mathrm{F}^c$ be the radial function of D. Here comes the variational problem $VP(r_0)$. We seek a radial function $r \in \mathrm{F}$ that minimizes

$$\int_0^{2\pi} r3(t)/r_0(t)dt$$

$$\text{subject to} \quad \int_0^{2\pi} r3(t)\cos t dt = 0, \quad \int_0^{2\pi} r3(t)\sin t dt = 0,$$

$$\text{and} \quad \frac{1}{2}\int_0^{2\pi} r2(t)dt = 1.$$

Assume $r(t)$ is the radial function of a convex (or starshaped) compact set $K \subset \mathcal{R}$. Then the first condition says that the centre of gravity, $g(K)$, of K is at the origin, and the second condition says that $\operatorname{Area} K = 1$. The connection between L_n and $VP(r_0)$ is given by

Lemma 1. *Assume $K \subset \mathcal{R}$ starshaped with* $\operatorname{Area} K = 1$, $g(K) = 0$. *Then the radial function of K, r, is a feasible solution to $VP(r_0)$. Moreover, there is $Q_n \in \mathbf{P}_n$ (for every $n \geq 3$) with*

$$\lim n^{-3/2}\operatorname{Per} Q_n = \frac{\pi}{\sqrt{6}}\int_K \|z\|dz = \frac{\pi}{3\sqrt{6}}\int_0^{2\pi}\frac{r3(t)}{r_0(t)}dt.$$

The last identity follows from a simple integral transformation. Using the results concerning L_n we prove the following.

Theorem 3. *There is a unique solution $r \in \mathrm{F}$ to the variational problem. It is the radial function of the convex compact set C from Theorem 2. Moreover,*

$$\frac{1}{r} = \frac{a}{r_0} + b\cos t + c\sin t$$

with unique $a > 0$, $b, c \in \mathbf{R}$ that satisfy the constraints of $VP(r_0)$.

The main ingredient of the proof of the existence of C in Theorem 2 is the fact that almost all primitive points in C_n belong to E_n. More precisely the following holds.

Lemma 2. *For every $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon, D)$ such that for all $n \geq n_0$, every primitive point in $(1-\varepsilon)C_n$ belongs to E_n.*

Another important fact is the following:

Lemma 3. *There are positive numbers r and R (depending only on D) such that for all $n \geq 3$*

$$rB \subset (\operatorname{Area} C_n)^{-1/2} C_n \subset RB.$$

For the proof of the main theorems one establishes first that $\liminf n^{-3/2} L_n$ exists and equals a positive number, say α. The Blaschke selection theorem gives then a sequence of integers, $n_1 < n_2 < \dots$, such that $(\operatorname{Area} C_{n_k})^{-1/2} C_{n_k}$ tends to a fixed convex set C and, further, that $\lim n_k^{-3/2} L_{n_k} = \alpha$. One checks next that the radial function of C is an optimal solution to $VP(r_0)$ with $\frac{\pi}{\sqrt{6}} \int_c ||z|| dz = \alpha$. It follows follows from Lemma 1 then that $\lim n^{-3/2} L_n = \alpha$.

Once the existence of an optimal solution r to $VP(r_0)$ is established, the necessary conditions of optimality imply the equation in Theorem 3. Luckily, r is (almost) differentiable because it is the radial function of the convex set C. To prove uniqueness further ideas are needed. Details can be found in [1].

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Volume entropy of Hilbert geometries

Gautier Berck

(joint work with A. Bernig, C. Vernicos)

Hilbert geometries are particularly simple examples of metric spaces defined on the interior of a convex set. Their construction mimics the Klein model of the hyperbolic space which is recovered provided the convex set is an ellispoid. Since the distance function is defined using the cross-ratio only, projective maps between convex sets are isometries.

The volume entropy of those geometries is a projective invariant which was thought to be maximized by ellispoids. Together with A. Bernig and C. Vernicos, we proved this for the 2-dimensional case. The result for arbitrary dimensions was however obtained only under quite restrictive assumptions on the smoothness of the boundary of the convex set. To achieve this, we designed a new projective invariant of pointed convex bodies that is very similar to the centro-affine area.

Random embedding from ℓ_p^n into ℓ_1^N

Olivier Guédon
(joint work with Omer Friedland)

A famous result of Dvoretzky [1] tells that ℓ_2^n is uniformly represented in any infinite dimensional Banach space X, which means that for any $\varepsilon > 0$ and any integer n, $\ell_2^n \overset{1+\varepsilon}{\hookrightarrow} X$. For any $1 \le p < 2$, even if it is impossible to embed ℓ_p^n uniformly in any Banach space, Maurey and Pisier [11] proved that ℓ_p^n is uniformly represented in a Banach space X if and only if X is not of stable-type p. It is possible to quantify these results when X is of finite dimension. Milman [12] proved that if E is a normed space of dimension N, then for any $\varepsilon \in (0,1]$, $\ell_2^n \overset{1+\varepsilon}{\hookrightarrow} E$, where n depends only on ε and on a geometric parameter associated to E. If it is applied in the case of $E = \ell_1^N$, it tells that for any $\varepsilon > 0$, $\ell_2^n \overset{1+\varepsilon}{\hookrightarrow} \ell_1^N$, where $N = C(\varepsilon)n$ and $C(\varepsilon)$ is a function depending only on ε. Johnson and Schechtman [6] proved that for any $0 < r \le 1$ and $0 < r < p < 2$, for any $\varepsilon \in (0,1]$, $\ell_p^n \overset{1+\varepsilon}{\hookrightarrow} \ell_r^N$, where $N = C(\varepsilon)n$. Later, Pisier [14] gave a different proof and extended their result to the case of general finite dimensional normed space E of dimension N, proving that for any $\varepsilon > 0$, $\ell_p^n \overset{1+\varepsilon}{\hookrightarrow} E$, where n depends only on ε and on the stable-type constant of E. All these proofs are random, typically for Euclidean subspaces, it is possible to use matrices defined by Gaussian vectors, while for ℓ_p^n subspaces, the matrices are more complicated and defined by "approximating" p-stable vectors, for which there is no hope to get good properties of concentration around their mean. However, even if ε is taken the largest possible, say equal to 1, it is not possible to deduce the existence of an operator of rank say $[N/2]$ satisfying the desired property. In the Euclidean case, this is a theorem of Kashin [9] who proved that for any $\eta > 0$, for any integer n, $\ell_2^n \overset{C(\eta)}{\hookrightarrow} \ell_1^N$, where $N = [(1+\eta)n]$ and $C(\eta)$ depends only on η. This result was generalized to the case of normed spaces with bounded volume ratio by Szarek [18], Szarek and Tomczak-Jaegermann [19].

The main subject of the talk is to present a Kashin-type theorem for embedding from ℓ_p^n into ℓ_r^N, where $0 < r \le 1$, $0 < r < p < 2$, $N = (1+\eta)n$ and $\eta > 0$. In the case $r = 1$, Naor and Zvavitch [13] proved that for any $\eta \in (0,1)$ $\ell_p^n \overset{C(\log n,\eta)}{\hookrightarrow} \ell_1^{(1+\eta)n}$, where $C(\log n, \eta) = (c \log n)^{(1-\frac{1}{p})(1+\frac{1}{\eta})}$. It is important to note that they provide an explicit definition of a random operator, which satisfies the desired property. Slightly after, Johnson and Schechtman [7] proved the existence of an operator $T : \ell_p^n \to \ell_1^{(1+\eta)n}$, such that $\|T\| \|T^{-1}_{|\mathrm{Im} T}\| \le C(\eta)$. However, the proof depends heavily on the Elton [2] theorem, which is valid only in ℓ_1^N and doesn't give any explicit construction of the operator T.

Our result asserts that for any $0 < r < p < 2$, with $r \le 1$, $\ell_p^n \overset{C(\eta)}{\hookrightarrow} \ell_r^N$, where $N = [(1+\eta)n]$ and $C(\eta)$ depends only on p, r and η. Surprisingly, the random operator that satisfies the desired conclusion is already defined in Pisier [14], for the almost isometric result. It solves completely the question of extending the theorem of [6] to a Kashin-type setting.

The proof is based on a splitting of the unit sphere of ℓ_p^n into subsets. In the case $p = 2$, this idea appeared in [10, 17], and was deeply developed in [R, 4] to study the smallest singular value of some random operators. In any case, the main result concerns the study of a new type of small ball estimate. For a real random variable, it is common to use an inequality due to Esseen [3]. Several multi-dimensional versions of this result are known (see [5, 4, 4]). However, in our situation, none of them seemed to be adapted. Another type of multidimensional Esseen-type inequality is at the heart of our proof.

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A characterization of the mixed volume

Rolf Schneider
(joint work with Vitali Milman)

The *mixed volume* $V : (\mathcal{K}^n)^n \to \mathbb{R}$ is the unique symmetric function for which

$$|\lambda_1 K_1 + \cdots + \lambda_m K_m| = \sum_{1 \le i_1, \ldots, i_n \le m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n})$$

for $m \in \mathbb{N}$, $K_1, \ldots, K_m \in \mathcal{K}^n$, $\lambda_1, \ldots, \lambda_m \geq 0$; here $\mathcal{K}^n$ is the space of convex bodies in Euclidean space $\mathbb{R}^n$ and $|\cdot|$ denotes the volume. The mixed volume is a central notion of the Brunn–Minkowski theory of convex bodies; see [2], for example. While recent years have seen many new characterizations of the geometrically most significant valuations on convex bodies, it is surprising that no axiomatic approach to the mixed volume has been suggested. Motivated by an attempt to extend the mixed volume to log-concave measures, we have become interested in characterizing the mixed volume by some of its functional properties, not taking recourse to the notion of volume. The following properties of a function $F : (\mathcal{K}^n)^n \to \mathbb{R}$, which are shared by the mixed volume, are certainly good candidates for figuring in a characterization.

(P1) F is Minkowski additive in each variable.
(P2) F is increasing in each variable.
(P3) F is symmetric.

However, several different classes of functions $F : (\mathcal{K}^n)^n \to \mathbb{R}$ can be constructed which have these properties but are far from the mixed volume. What is missing, seems to be some assumption which ties the variables of the function closer together. In this direction, we have tested the following condition.

(P4) If the n-tuple $(K_1, \ldots, K_n)$ of convex bodies is degenerate, which means that there are no segments $S_i \subset K_i$, $i = 1, \ldots, n$, with linearly independent directions, then $F(K_1, \ldots, K_n) = 0$.

If we restrict ourselves to the space $\mathcal{K}^n_s$ of centrally symmetric convex bodies, then these conditions lead to a complete characterization of the mixed volume; it is even possible to omit the symmetry condition (P3).

Theorem 1. *If the function $F : (\mathcal{K}^n_s)^n \to \mathbb{R}$ satisfies* (P1), (P2), (P4), *then*

$$F(K_1, \ldots, K_n) = aV(K_1, \ldots, K_n) \quad \textit{for} \quad K_1, \ldots, K_n \in \mathcal{K}^n_s,$$

with a constant $a \geq 0$.

For non-symmetric bodies, this result cannot hold, because condition (P4) does not distinguish between a convex body K and its reflected image $-K$. In fact, if $F(K_1, \ldots, K_n)$ is defined as a linear combination, with constant coefficients, of $V(\pm K_1, \ldots, \pm K_n)$, then F satisfies (P1), (P2), (P4). One may conjecture that the converse is also true, but so far we can prove this only for $n = 2$.

Theorem 2. *If* $F : (\mathcal{K}^2)^2 \to \mathbb{R}$ *satisfies* (P1), (P2), (P4), *then*

$$F(A, B) = aV(A, \alpha B - (1 - \alpha)B)$$

for $A, B \in \mathcal{K}^2$*, with constants* $a \geq 0$ *and* $\alpha \in [0, 1]$*.*

For the proof of Theorem 2, we first employ a result of Firey [1] to construct a map $T : \mathcal{K}^2 \to \mathcal{K}^2$ with the property that $F(A, B) = V(A, TB)$ for $A, B \in \mathcal{K}^2$. From the properties of F we then deduce, with the aid of results of Weil [3], some properties of the map T, for example, that it is Minkowski additive and increasing. From Theorem 1 we conclude that on centrally symmetric convex bodies the map T acts as a homothety. An investigation of the effect of T on triangles finally yields the assertion.

Modified assumptions, which would yield higher-dimensional versions of Theorem 2, are still under investigation.

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