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Graph Theory

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Abstract. Highlights of this workshop on structural graph theory included new developments on graph and matroid minors, continuous structures arising as limits of finite graphs, and new approaches to higher graph connectivity via tree structures.

Mathematics Subject Classification (2000): 05Cxx.

Introduction by the Organisers

The aim of this workshop was to offer an exchange forum for the leading researchers from the various fields of structural graph theory. When preparing the invitations, we did not define any particular foci this time, but concentrated on people. All the same, some particularly active fields can be identified, and the workshop even brought some surprises as to what these are.

One area of strong recent activity are *graph limits*: properties of finite graphs are studied through a non-discrete limit objects they define. The idea behind this is that one limit object can encapsulate the typical features of the (finite) graphs with a given property, and methods from other areas of mathematics, both algebraic and analytical, can be brought to bear on them in a way alien to individual discrete structures. This has become an exciting new development for the study of dense graphs in the last few years. The approach is now beginning to be adapted to sparse graphs too, and connections are drawn to more traditional ways of forming limits of graphs, such as boundaries – compactifications or metric completions – of infinite graphs.

In a similar development on extremal graph theory's home turf, we now appear to have the definitive version of the sparse regularity lemma, announced and explained in a major talk at the workshop for the first time.

Another area with striking new results is matroid theory. The structure theory for the finite matroids representable over a given finite field is taking shape now, and the proof that these matroids are well-quasi-ordered as minors appears to be nearing completion. There is now a theory of infinite matroids that admits duality and is based on axiom systems much like the finite matroid axioms; this finally solved a problem of Rado of 1966.

A surprising recent development reflected by the workshop is that, 30 years after its beginnings and more than 10 years after the publication of most of the proof of the graph minor theorem, graph minor structure theory is finally coming of age, being taken up by other researchers at a level comparable to the original papers. Its central technical result, the structure theorem for an excluded minor, has several more mature versions now, partly with new and substantially simpler proofs (which are still difficult but becoming managable), and applications e.g. in computer science. The same is true for some of the more algorithmic parts of the theory.

In graph connectivity, there are interesting recent attempts to extend to higher connectivity Tutte's tree-decomposition of a graph into cycles and 3-connected components. The aim is to find, for any fixed integer k, a canonical set of nested k-separations that shapes the graph into a coarse tree structure made of $(k+1)$-connected components. This theory started as a tree-structure theorem separating the ends of an infinite graph but is now being applied to separate highly connected finite parts, rather than rays, in a possibly finite graph.

This graph theory week in Oberwolfach was perhaps the liveliest we have ever had. In addition to some excellent main talks it owed most of its spirit to numerous informal workshop organised spontaneously by the participants: we had 12 such gatherings in all, with mostly about 5–10 participants. It was in these workshops that trends such as those mentioned above could really be felt.

Workshop: Graph Theory

Table of Contents

Abstracts

Unfriendly partition for countable graphs without a subdivision of an infinite clique

Eli Berger

Let $G = (V, E)$ be a graph. A *partition* of G is a function $p : V \to \{1, 2\}$. For $i = 1, 2$, write $V_i^p = p^{-1}(i) = \{v \in V : p(v) = i$. For a vertex $v \in V$ and $i = 1, 2$, we write $N(v) = \{u : \{u, v\} \in E\}$ and $N_i^p(v) = N(v) \cap V_i^p$. We write $N_f^p(v) = N_{p(v)}^p(v)$ and $N_u^p(v) = N_{3-p(v)}^p(v)$ (The letters f and u stand for "friendly" and "unfriendly" respectively). If the identity of p is clear we write V_i, $N_i(v)$, $N_f(v)$ and $N_u(v)$ instead of V_i^p, $N_i^p(v)$, $N_f^p(v)$ and $N_u^p(v)$ respectively. We say that p is *unfriendly* to a vertex v if $|N_u^p(v)| \geq |N_f^p(v)|$. A partition which is unfriendly to all vertices is called an *unfriendly partition.* A set $S \subseteq V$ is called *satisfiable* if there exists a partition of G which is unfriendly to all members of S.

Clearly every finite graph has an unfriendly partition. Cowan and Emerson [3] conjectured that the same property holds for infinite graphs as well. A counterexample to this conjecture was constructed by Milner and Shelah [4], however this construction uses uncountably many vertices, leaving the countable case open. In the same article by Milner and Shelah [4], it was shown that every graph does have an unfriendly partition into three sets. Aharoni, Milner and Prikry [1] proved that every graph with finitely many vertices of infinite degree has an unfriendly partition. Bruhn, Diestel, Georgakopoulos and Sprüssel [2] proved that every rayless graph has an unfriendly partition.

In the talk the following is proved

Theorem 1. *If a countable graph contains no subdivision of the complete countable graph as a subgraph, then the graph has an unfriendly partition.*

The main two ingredients in the proof are the following theorems:

Theorem 2. [5] *If a graph G contains no subdivision of the complete countable graph as a subgraph, then there exists a tree T_G and a function $B : V(T_G) \to 2^{V(G)}$ such that*

(1) Every edge of G is contained in a set $B(v)$ for some $v \in V(T_G)$.
(2) For every vertex $x \in V(G)$ the set $B^{-1}(x) = \{v \in V(T_G) : x \in B(v)\}$ forms a subtree of T_G.
(3) Every set $B(v)$ is finite.
(4) For every ray $v_1, v_2, v_3, \ldots$ in T_G, the set $\bigcap_{i<\omega} \bigcup_{j:i<j<\omega} B(v_i)$ is finite.

Theorem 3. [1] *If a set S is satisfiable in a graph G and if F is a finite set of vertices of G then $S \cup F$ is also satisfiable.*

In the rest of this extended abstract, we assume the existence of a fixed tree T_G and a fixed function $B : V(T_G) \to 2^{V(G)}$ as in Theorem 2. For a subtree T of T_G, we write $B_T = \bigcup_{v \in V(T)} B(v)$ and we say that T is satisfiable if B_T is.

Let $\mathcal{S}$ be a set of subtrees of T_G. We say that $\mathcal{S}$ is *hereditary* if every subtree of a member of $\mathcal{S}$ is a member of $\mathcal{S}$. We define the *boundary* $\partial\mathcal{S}$ of $\mathcal{S}$ in the following way. A vertex $x \in V(G)$ is in $\partial\mathcal{S}$ if it has infinitely many neighbors in $B_{\bigcup \mathcal{S}}$ but finitely many neighbors in B_T for every $T \in \mathcal{S}$. A *graft* of $\mathcal{S}$ is a tree of the form

$$T_0 \cup \bigcup_{v \in V(T_0)} T_v$$

where $T_0 \in \mathcal{S}$ and for every $v \in V(T_0)$, the tree $T_v \in \mathcal{S}$ intersects T_0 only at v. If every graft of $\mathcal{S}$ is in $\mathcal{S}$ then we say that $\mathcal{S}$ is *graft closed*.

The main lemma in the proof of Theorem 1 is

Lemma 4. *Let $\mathcal{S}$ be a set of satisfiable subtrees of T_G, which is hereditary, graft closed and has a finite boundary. Then $\bigcup \mathcal{S}$ is satisfiable.*

A *distinction* is a function $d : V(G) \to 2^{V(G)}$ with $|d(x) \cap N(x)| = |N(x)|$ for every $x \in V(G)$. We say that a partition p satisfies a subtree T of T_G with distinction d if for every $x \in B_T$ the inequality $|N_f^p(x)| \leq |N_u^p(x) \cap d(x)|$. The following lemma helps proving lemma 4

Lemma 5. *Let $\mathcal{S}$ be a set of subtrees of T_G, which is hereditary and graft closed. Let d be a distinction and assume every element of $\mathcal{S}$ is satisfiable with distinction d. Let $x \in B_{\bigcup \mathcal{S}} \setminus \partial\mathcal{S}$. Then every element of $\mathcal{S}$ is satisfiable with distinction d', where d' is a distinction which is the same as d except to its value at x, which is equal to B_T for some $T \in \mathcal{S}$.*

Once Lemma 4 is proved, the steps in the proof of Theorem 1 are as follows:

(1) Every finite subtree of T_G is satisfiable.
(2) Every subtree of T_G, isomorphic to a tree obtained from a finite tree by replacing some of the leaves by rays, is satisfiable.
(3) Every subtree of T_G, not containing as a subgraph a subdivision of the complete infinite binary tree, is satisfiable.
(4) T_G is satisfiable.

I would like to thank Ron Aharoni, Peter Komjath, Philipp Sprüssel, Paul Seymour and Robin Thomas for stimulating discussion. In particular, I would like to thank Agelos Georgakopoulos for pointing out the need for Lemma 5 and Reinhard Diestel for convincing me that this problem is within reach.

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A clique obstruction for holes in claw-free graphs

Henning Bruhn
(joint work with Akira Saito)

Given two non-adjacent vertices x and y in a graph G, what is an obvious obstruction for the existence of a hole (an induced cycle of length ≥ 4) through x and y? Clearly, a clique that separates x and y. Ideally, we would like to prove that such a clique is the only obstruction:

(1) *there is a hole through x and y if and only if there does not exist any clique that separates x and y.*

If G is the line graph of a graph H then an easy application of Menger's theorem to H shows that the statement is true. On the other hand, (1) is false in general; an example may be found in Figure 1 on the left. This is not at all surprising as Bienstock [1] (see also Corrigendum [2]) proved that the following problem is NP-complete, so that one should not expect a simple necessary and sufficient obstruction.

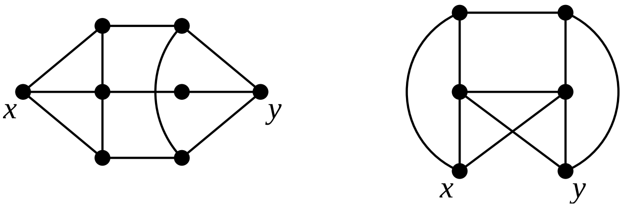

Figure 1. No clique separating x from y and no hole through x and y either

Hole-through-two-Vertices. *Given a graph G and two non-adjacent vertices x, y, check whether there is a hole through x and y.*

Restricted to claw-free graphs, however, the problem becomes solvable in polynomial time as demonstrated by Lévêque, Lin, Maffray and Trotignon [5]. Thus, there is hope for (1) to extend to claw-free graphs, and indeed this is our main result:

Theorem 1. [3] *Let G be a claw-free graph, and let x and y be two non-adjacent vertices without common neighbours. Then, there exists a hole through x and y if and only if no clique separates x and y.*

We remark that the exclusion of common neighbours of x and y is necessary, see the right graph in Figure 1. However, it is easy to modify the theorem so that common neighbours may be admitted.

The structure theorem allows us to pursue an indirect approach to Hole-through-two-Vertices: Instead of for a hole we can search for a separating clique. This can be done quite efficiently, thanks to Tarjan's clique decomposition algorithm [6], so that we get an improvement over the $O(n^4)$-algorithm of Lévêque et al.

Theorem 2. [3] *Let a claw-free graph G and two non-adjacent vertices x and y be given. If G has n vertices and m edges then it can be checked in $O(mn)$-time whether there is a hole containing x and y.*

We derive two further applications from Theorem 1. The first concerns the Three-in-a-Tree problem introduced by Chudnovsky and Seymour:

Three-in-a-Tree. *Given a graph G and three vertices x, y, z decide whether there exists an induced subtree of G containing x, y, z.*

Chudnovsky and Seymour show that Three-in-a-Tree can be solved in $O(|V(G)|^4)$-time for an arbitrary graph G. In a claw-free graph every induced tree is a path, so that in order to solve Three-in-a-Tree we need to check whether there is an induced *path* through three given vertices. The following theorem describes a necessary and sufficient obstruction:

Theorem 3. [3] *Let x, y, z be three vertices in a claw-free graph G. Then exactly one of the following two statements holds:*

(i) *There is an induced x–z path through y.*
(ii) *There is a clique other than $\{y\}$ that separates $\{x, z\}$ from $\{y\}$, or $N(x) \setminus \{y\}$ separates y from z, or $N(z) \setminus \{y\}$ separates x from y.*

Again we can use Tarjan's algorithm to check the conditions of Theorem 3. This results in an algorithm with running time $O(mn)$ for Three-in-a-Tree if the input is restricted to claw-free graphs.

The second application of Theorem 1 is an induced version of Menger's theorem for two paths. Given a graph G, let us call two subgraphs or vertex sets S, T *non-touching* if S and T are disjoint and if there does not exist any edge with one endvertex in S and the other in T.

Theorem 4. [3] *Let X, Y be two non-touching vertex sets of cardinality 2 in a claw-free graph G. Then exactly one of the following statements holds:*

(i) *There are two non-touching X–Y paths.*
(ii) *There exists a clique separating X from Y in G; or there exists $z \in X \cup Y$ so that X is separated from Y by $N(z)$.*

We remark that the theorem becomes false if X and Y are allowed to touch. Figure 2 shows a claw-free graph with X and Y touching where neither (i) nor (ii) is satisfied.

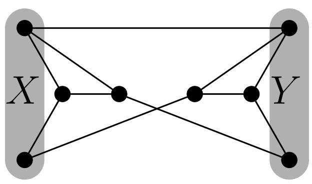

Figure 2. Theorem 4 may fail if X and Y touch

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Large cliques and stable sets excluding paths and antipaths

Maria Chudnovsky
(joint work with Yori Zwols)

We say that a graph H has the *Erdős-Hajnal property* if there exists $\varepsilon(H) > 0$ such that every graph on n vertices that does not have H as an induced subgraph contains either a clique or a stable set of size at least $n^{\varepsilon(H)}$. Clearly, if H has the property, then so does H^c. Erdős and Hajnal [3] conjectured that all graphs have the property. It is known to be true for every graph H with $|V(H)| \leq 4$. In [1], it was shown that if two graphs H_1 and H_2 have the Erdős-Hajnal property, then so does the graph constructed from H_1 by replacing a vertex $x \in V(H_1)$ by H_2 and making H_2 complete to the neighbors of x in H_1 and anticomplete to the non-neighbors of x in H_1 (this operation is known as the substitution operation). Moreover, it was shown in [2] that the triangle with two disjoint pendant edges (this graph is known as the *bull*) has the property. This leaves the four-edge-path P_4 and the cycle C_5 of length five as the remaining open cases for graphs on at most 5 vertices. Here we deal with the case where H is a four-edge path, where, in addition, we exclude the complement of a five-edge path. Let $\mathcal{G}$ be the class of all graphs that do not have an induced subgraph isomorphic to the four-edge-path or the complement of a five-edge-path. We prove that

Theorem 1. *Every graph $G \in \mathcal{G}$ has a clique or a stable set of size at least $|V(G)|^{1/6}$.*

. For a graph G, let $\omega(G)$ denote the size of the largest clique in G and let $\chi(G)$ denote the chromatic number of G. G is called *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G. We say that a function $g : V(G) \to \mathcal{R}^+$ is a *covering function for G* if $\sum_{p \in V(P)} g(p) \leq 1$ for every perfect induced subgraph P of G. For $\eta \geq 1$, we say that a graph G is η-narrow if $\sum_{v \in V(G)} g^\eta(v) \leq 1$ for every covering function g. It was shown in [2] that bull-free graphs are 2-narrow. We take a similar approach and prove that

Theorem 2. *Every graph in $\mathcal{G}$ is 3-narrow.*

This result suffices for proving Theorem 1, because of the following result:

Theorem 3. *Let G be a η-narrow graph. Then G has a clique or stable set of size at least* $|V(G)|^{1/2\eta}$.

Proof. Let $\mathcal{P}$ be the set of all perfect induced subgraphs of G. Let $K = \max_{P \in \mathcal{P}} |V(P)|$. Consider the function $g : V(G) \to \mathcal{R}^+$ with $g(v) = 1/K$ for all $v \in V(G)$. Clearly, $\sum_{v \in V(P)} g(v) \leq 1$ for all $P \in \mathcal{P}$. Therefore, since G is η-narrow, it follows that g satisfies

$$1 \geq \sum_{v \in V(G)} g(v)^\eta = \frac{|V(G)|}{K^\eta}.$$

Equivalently, we have $K \geq |V(G)|^{\frac{1}{\eta}}$. Thus, G has a perfect induced subgraph H with $|V(H)| \geq |V(G)|^{\frac{1}{\eta}}$. Since H is a perfect graph, H satisfies $|V(H)| \leq \chi(H)\alpha(H) = \omega(H)\alpha(H)$ and hence $\max(\omega(H), \alpha(H)) \geq \sqrt{|V(H)|} \geq |V(G)|^{1/2\eta}$. Therefore, H has a clique or stable set of size at least $|V(G)|^{1/2\eta}$. Since H is an induced subgraph of G, G has a clique or stable set of size at least $|V(G)|^{1/2\eta}$. This proves Theorem 3. □

Notice that the proof of Theorem 3 also shows that a graph G is 1-narrow if and only if G is perfect.

In order to prove Theorem 2, we prove the following structural result:

Theorem 4. *For every $G \in \mathcal{G}$, either*

- *G contains no induced cycle of length six, and for every $v \in V(G)$, either $G|N(v)$ or $G \setminus N(v)$ is perfect, or*
- *G contains an induced cycle of length six, but for every $v \in V(G)$, $G|N(v)$ contains no induced cycle of length six, or*
- *G admits a certain decomposition, called a* quasi-homogeneous set decomposition.

In order to use Theorem 4, we observe the following two facts:

Theorem 5. *Let G be a graph, and let η be a positive integer. If G admits a quasi-homogeneous set decomposition, and every induced subgraph of G is η-narrow, then G is η narrow.*

Theorem 6. *Let G be a graph, and let η be a positive integer. If for every $v \in V(G)$, either $G|N(v)$ or $G \setminus N(v)$ is η-narrow, then G is $\eta + 1$-narrow.*

We can now prove Theorem 2. Let us first show that if $G \in \mathcal{G}$ has no induced cycle of length six, then G is 2-narrow. Since G contains no induced cycle of length six, either the first, or the third outcome of Theorem 4 holds for G. If the first one holds, then, since every perfect graph is 1-narrow, Theorem 6 implies that G is 2-narrow. If the third outcome holds, the result follows by induction, using Theorem 5. This proves that if $G \in \mathcal{G}$ has no induced cycle of length six, then G is 2-narrow.

Now we may assume that $G \in \mathcal{G}$ and G contains an induced six cycle. Then either the second, or the third outcome of Theorem 4 holds for G. If the second one holds, then, by the previous claim and by 6, it follows that G is 3-narrow. If the third outcome holds, the result follows by induction as before, using Theorem 5. This proves 2.

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Bidimensionality

Erik Demaine

(joint work with MohammadTaghi Hajiaghayi and others)

Bidimensionality theory is an approach to algorithmic graph minor theory. This theory provides general tools for designing fast (constructive, often subexponential) fixed-parameter algorithms, and approximation algorithms (often PTASs), for a wide variety of NP-hard graph problems in graphs excluding a fixed minor. For example, some of the most general algorithms for feedback vertex set and connected dominating set are based on bidimensionality. Another approach is "deletion and contraction decompositions", which split any graph excluding a fixed minor into a bounded number of small-treewidth graphs. For example, this approach has led to some of the most general algorithms for graph coloring and the Traveling Salesman Problem on graphs. I will describe these and other approaches to efficient algorithms through graph minors.

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Infinite matroids with duality

Reinhard Diestel

(joint work with H. Bruhn, M. Kriesell, P. Wollan)

Infinite matroids are usually defined like finite ones, with the following additional axiom:

(I4) An infinite set is independent as soon as all its finite subsets are.

We shall call such set systems *finitary matroids*. Circuits in finitary matroids are clearly finite.

Regrettably, the additional axiom (I4) spoils duality: finitary matroids do not normally have duals that are also finitary matroids. For example, the cocircuits of an infinite uniform matroid of rank k would be the sets missing exactly $k-1$

points; since these sets are infinite, they cannot be the circuits of another finitary matroid. Similarly, every bond of an infinite graph would be a circuit in any dual of its cycle matroid, but these sets can be infinite and hence will not be the circuits of a finitary matroid.

Motivated by observable cycle-bond duality in infinite graphs that clearly should, but could not, be described in matroid terms, we developed axioms for infinite matroids that are not finitary, and which have duals. This solves a problem of Rado of 1966 [3, Problem P531].

Adapting the usual finite matroid axioms to infinite structures one faces two problems: to avoid the use of cardinalities, and to deal with limits. For example, consider two independent sets I_1, I_2 in a finite matroid. How can we translate the assumption, made in the third of the standard independence axioms, that $|I_1| < |I_2|$? If $I_1 \subseteq I_2$, this is equivalent (for finite sets) to $I_1 \subsetneq I_2$, and we can use the latter statement instead. But if $I_1 \not\subseteq I_2$, the only way to designate I_1 as 'smaller' and I_2 as 'larger' is to assume that I_2 is maximal among all the independent sets while I_1 is not—a much stronger statement that fails to capture size differences among non-maximal independent sets. Nevertheless, it turns out that this distinction will be enough.

As concerns limits, we need both that every independent set extends to a basis (so that there can be an equivalent set of basis axioms, in which independent sets are defined as subsets of bases), and that every dependent set contains a circuit (so that there can be an equivalent set of circuit axioms, in which independent sets are defined as the sets not containing a circuit). It turns out that we have to require one of these as an additional axiom, but the other will then follow.

In order to be able to refer to the infinite extension axiom from within all contexts, independently of the definition of independence in that context, we state it first in more general terms, without reference to independence. Let E be our ground set, and let $\mathcal{I} \subseteq 2^E$ be a collection of subsets of E. The following statement describes a possible property of $\mathcal{I}$:

(M) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{ I' \in \mathcal{I} \mid I \subseteq I' \subseteq X \}$ has a maximal element.

We are now ready to state our axiom systems. They imply each other in the usual way [1]. For finite sets they default to the usual finite matroid axioms. The matroids they describe have duals, defined as usual, with contraction and deletion as dual operations.

Independence axioms. Let $\mathcal{I} \subseteq 2^E$, and write $\mathcal{I}^{\max}$ for the set of its maximal elements.

(I1) $\emptyset \in \mathcal{I}$.
(I2) $\mathcal{I}$ is closed under taking subsets.
(I3) For all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$ there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.
(IM) $\mathcal{I}$ satisfies (M).

Basis axioms. Let $\mathcal{B} \subseteq 2^E$.

(B1) $\mathcal{B} \neq \emptyset$.
(B2) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there is an element y of $B_2 \setminus B_1$ such that $(B_1 - x) + y \in \mathcal{B}$.
(BM) The set $\mathcal{I}$ of all $\mathcal{B}$-*independent* sets satisfies (M). These are the subsets of elements of $\mathcal{B}$.

Closure axioms. Let $\mathrm{cl}\colon 2^E \to 2^E$ be a function.

(CL1) For all $X \subseteq E$ we have $X \subseteq \mathrm{cl}(X)$.
(CL2) For all $X \subseteq Y \subseteq E$ we have $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.
(CL3) For all $X \subseteq E$ we have $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.
(CL4) For all $Z \subseteq E$ and $x, y \in E$, if $y \in \mathrm{cl}(Z + x) \setminus \mathrm{cl}(Z)$ then $x \in \mathrm{cl}(Z + y)$.
(CLM) The set $\mathcal{I}$ of all cl-*independent* sets satisfies (M). These are the sets $I \subseteq E$ such that $x \notin \mathrm{cl}(I - x)$ for all $x \in I$.

Circuit axioms. Let $\mathcal{C} \subseteq 2^E$.

(C1) $\emptyset \notin \mathcal{C}$.
(C2) No element of $\mathcal{C}$ is a subset of another.
(C3) Whenever $X \subseteq C \in \mathcal{C}$ and $(C_x \mid x \in X)$ is a family of elements of $\mathcal{C}$ such that $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \setminus \big(\bigcup_{x \in X} C_x\big)$ there exists an element $C' \in \mathcal{C}$ such that $z \in C' \subseteq \big(C \cup \bigcup_{x \in X} C_x\big) \setminus X$.
(CM) The set $\mathcal{I}$ of all $\mathcal{C}$-*independent* sets satisfies (M). These are the sets $I \subseteq E$ such that $C \nsubseteq I$ for all $C \in \mathcal{C}$.

Since finitary matroids are also matroids in our sense, they now have duals. These are not normally finitary. Duals of finitary matroids thus form a large class of examples of our matroids. There are also some natural 'primary' examples, both graphic and algebraic [1]. Further examples can be derived from Higgs [2], whose theory of 'B-matroids' describes the same structures as our axioms do.

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Triangle removal lemma

Jacob Fox

Szemerédi's regularity lemma [17] is one of the most powerful tools in graph theory. It was introduced by Szemerédi in his celebrated proof [16] of the Erdős-Turán conjecture on long arithmetic progressions in dense subsets of the integers. Roughly speaking, it says that every large graph can be partitioned into a small number of parts such that the bipartite subgraph between almost every pair of parts is random-like. This structure is quite useful for counting the number of copies of some fixed subgraph.

To properly state the regularity lemma requires some terminology. The edge density $d(X, Y)$ between two subsets of vertices of a graph G is the fraction of pairs $(x, y) \in X \times Y$ that are edges of G. A pair (X, Y) of vertex sets is called *ϵ-regular* if for all $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$, we have $|d(X', Y') - d(X, Y)| < \epsilon$. A partition $V = V_1 \cup \ldots \cup V_k$ is called equitable if $||V_i| - |V_j|| \leq 1$ for all i and j. The regularity lemma states that for each $\epsilon > 0$, there is a positive integer $M(\epsilon)$ such that the vertices of any graph G can be equitably partitioned $V(G) = V_1 \cup \ldots \cup V_k$ into $k \leq M(\epsilon)$ parts where all but at most ϵk^2 of the pairs (V_i, V_j) are ϵ-regular. For more background on the regularity lemma, see the excellent survey by Komlós and Simonovits [11].

In the regularity lemma, $M(\epsilon)$ can be taken to be a tower of twos of height proportional to ϵ^{-5}. On the other hand, Gowers [7] proved a lower bound on $M(\epsilon)$ which is a tower of twos of height proportional to $\epsilon^{-1/16}$, thus demonstrating that $M(\epsilon)$ is inherently large as a function of ϵ^{-1}. Unfortunately, this implies that the bounds obtained by applications of the regularity lemma are usually quite poor. It remains an important problem to determine if new proofs giving better quantitative estimates for certain applications of the regularity lemma exist (see, e.g., [9]).

One of the most interesting applications of the regularity lemma is the triangle removal lemma, proved by Ruzsa and Szemerédi [13]. It says that for each $\epsilon > 0$ there is $\delta > 0$ such that every graph on n vertices with at most δn^3 triangles can be made triangle-free by removing at most ϵn^2 edges. The triangle removal lemma has many applications in graph theory, additive combinatorics, discrete geometry, and theoretical computer science.

An elegant application of the removal lemma was found by Solymosi [15]. He showed that the lemma can be applied to give a short proof of the corner theorem of Ajtai and Szeméredi [1], which states that for each $\epsilon > 0$ there is $N(\epsilon)$ such that for $N \geq N(\epsilon)$, any subset S of the $N \times N$ grid with $|S| \geq \epsilon N^2$ contains the vertices (x, y), $(x + d, y)$, $(x, y + d)$ of an isoscles right triangle. It is easy to show that the corners theorem implies Roth's theorem [12] that every subset of the integers of positive upper density contain a three-term arithmetic progression. Erdős and Graham and also Gowers [8], [9], [10] asked for good quantitative estimates for the corners theorem. Improving on several earlier papers, the current best known upper bound on $N(\epsilon)$, due to Shkredov [14], is double-exponential in a polynomial in ϵ^{-1}. The proof extends ideas from Gowers' proof of Szemerédi's theorem [10].

The corners theorem also follows from a very special case of the triangle removal lemma, known as the diamond-free theorem. This says that every graph on n vertices in which each edge is in precisely one triangle has $o(n^2)$ edges. The first problem in Gowers' [8] list of unsolved problems is to provide a new proof of the diamond-free theorem giving a better quantitative estimate than that provided by the regularity lemma. Variants of this question were asked by Erdős and Rothschild [6] and Tao and Trevisan [18].

The diamond-free theorem appears in many different guises in extremal graph theory [5], [13]. One such example is the equivalent induced matching theorem [13]. A *matching* M in a graph is a subgraph in which each vertex has degree one. A matching M is an *induced matching* if there are no other edges of the graph between the vertices of M. The induced matching theorem states that every bipartite graph H on n vertices which is the union of n induced matchings has $o(n^2)$ edges. The key idea in the proof that the diamond-free theorem and the induced matching theorem are equivalent is that, in a graph in which each edge is in precisely one triangle, the neighborhood of each vertex is an induced matching.

Alon [2] asked to improve the bound in the triangle removal lemma. We answer the questions discussed by Erdős [6], Alon [2], Gowers [8], and Tao [18] by giving a new proof of the triangle removal lemma which does not use the regularity lemma and gives a much better quantitative bound.

Theorem 1. *If δ^{-1} is a tower of twos of height $200\log\epsilon^{-1}$, then every graph G on n vertices with at most δn^3 triangles can be made triangle-free by removing ϵn^2 edges.*

For comparison, the regularity proof gives a bound that is a tower of twos of height polynomial in ϵ^{-1}.

We next sketch our proof for the special case of the induced matching theorem and contrast it with the proof of the regularity lemma. The *size* of a matching is the number of edges in it. It is straightforward to show that the induced matching theorem is equivalent to the following statement. For fixed $\epsilon > 0$, every bipartite graph $H = (U, V, E)$ on n vertices which can be edge-partitioned into a collection $\mathcal{M}$ of induced matchings each with size at least ϵn has $o(n^2)$ edges. Let $e(H)$ denote the number of edges of H and $\alpha = \frac{e(H)}{|U||V|}$ denote the edge density of H.

Define the *relative density* of a matching M of size m in a bipartite graph $H = (U, V, E)$ to be $\frac{m}{\sqrt{|U||V|}}$. At each stage of our proof, we will find large vertex subsets $U' \subset U$ and $V' \subset V$ such that a large proportion of matchings in $\mathcal{M}$ have many of their edges between U' and V'. More precisely, the relative density of these matchings restricted to U' and V' is at least a factor 1.1 larger than its original relative density. We restrict to U' and V' each of the matchings whose relative density increases by a factor at least 1.1. We let $\mathcal{M}'$ denote the resulting collection of matchings, and H' denote the bipartite subgraph whose edges are those contained in the matchings in $\mathcal{M}'$. The proof shows that the inverse of the edge density of H' is at most exponential in the inverse of the edge density α of

H. This process of taking bipartite subgraphs must stop after $O(\log \epsilon^{-1})$ steps as the relative density cannot be more than 1.

At each stage of the proof of the regularity lemma, we have a partition $V(H) = V_1 \cup \ldots \cup V_k$ of the vertex set into parts which differ in cardinality by at most 1. Let $p_i = |V_i|/|V(H)|$. The *mean density square* with respect to the partition is $\sum_{1 \le i,j \le k} p_i p_j d(U_i, V_j)^2$. If the partition does not satisfy the conclusion of the regularity lemma, then the partition can be refined such that the mean density square increases by $\Omega(\epsilon^5)$ while the number of parts is at most exponential in k. This process must stop after $O(\epsilon^{-5})$ steps as the mean density square cannot be more than 1.

Let us summarize the two approaches for comparison. At each new stage of the regularity proof, the mean density square is $\Omega(\epsilon^{-5})$ larger than at the previous stage, and the number of parts in the partition increases at most exponentially. At each stage of our new proof, we find a subgraph which can be edge-partitioned into induced matchings such that the relative density of the matchings increases by a factor 1.1, and the inverse of the edge density increases at most exponentially.

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The k edge-disjoint paths problem in digraphs with bounded independence number

Alexandra Fradkin
(joint work with Paul Seymour)

In [2], Fortune, Hopcroft, and Wyllie showed that the following algorithmic problem (k-EDP) is NP-complete with $k = 2$:

k Edge-Disjoint Paths (k-EDP)
Instance: A digraph G, and k pairs $(s_1, t_1), \dots, (s_k, t_k)$ of vertices of G.
Question: Do there exist directed paths $P_1, \dots, P_k$ of G, mutually edge-disjoint, such that P_i is from s_i to t_i for $i = 1, \dots, k$?

On the other hand, in [3], Robertson and Seymour showed that the analogous problem for undirected graphs can be solved in polynomial time for any fixed natural number k. A natural question to ask is, for which restricted classes of digraphs is the problem also polynomial?

In [1], Bang-Jensen showed that there exists a polynomial-time algorithm to solve 2-EDP in tournaments. We generalize Bang-Jensen's result in two ways: we exhibit a polynomial time algorithm to solve k-EDP (for all natural numbers k) in digraphs with bounded independence number (where bounded means at most α for some fixed constant α).

A digraph has "cutwidth" at most k if its vertex set can be ordered such that there are at most k edges leaving from each initial interval. The key to our algorithm is the following fact: for a digraph with bounded independence number, either its cutwidth is bounded or it contains a large "widget", a small set of vertices ordered in a circle such that each is joined to the next by many vertex-disjoint paths, all of length at most three.

If the cutwidth of a digraph is bounded, then we can solve k-EDP directly by using dynamic programming. If, on the other hand, a digraph contains a widget, as described above, then we can identify two vertices of the widget and not change the outcome of the problem (i.e., the paths will exist after the identification if and only if they exist before). The "algorithm," then, is to keep identifying pairs of vertices until the cutwidth becomes bounded by a predetermined function of k and then use dynamic programming. (We note that identifying vertices may create parallel edges, but if we ever have more than k edges between two vertices we can delete edges until there are exactly k and not change the outcome of the problem).

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Binary matroid minors

Jim Geelen
(joint work with Bert Gerards and Geoff Whittle)

Since 1999, we have been working on extending the Graph Minors Project of Neil Robertson and Paul Seymour to matroids. Unfortunately, matroids are too general, indeed, there are infinite antichains of real-representable rank-3 matroids and testing for a given two-element minor is NP-hard. Therefore we restrict our attention to matroids representable over a finite field. The main conjectures are:

1. WQO Conjecture: *For any finite field F, any infinite set of F-representable matroids contains two matroids, one isomorphic to a minor of the other.*

2. Minor-testing Conjecture: *For any finite field $\boldsymbol{F}$ and $\boldsymbol{F}$-representable matroid N, there is a polynomial-time algorithm for testing whether or not an $\boldsymbol{F}$-represented matroid contains an N-minor.*

3. Rota's Conjecture: *For any finite field $\boldsymbol{F}$, there are, up to isomorphism, only finitely many excluded minors for the class of $\boldsymbol{F}$-representable matroids.*

We are following the approach of Robertson and Seymour; we are first trying to prove structural results on minor-closed families of $\boldsymbol{F}$-representable matroids and intend to obtain the above conjectures as corollaries. In 2008 we obtained such results for the class of binary matroids and, in 2009, we used these structural results to prove the WQO Conjecture and the Minor-Testing Conjecture for binary matroids. (Rota's Conjecture was already known for binary matroids.) The structural results essentially say that the members of any minor-closed class of binary matroids can be obtained by piecing graphs together. We are currently working on extending the structural results to other finite fields and have obtained several significant partial results.

Hyperbolic graphs, fractal boundaries, and graph limits

Agelos Georgakopoulos

In a seminal paper [5] Gromov introduced the notion of a hyperbolic graph and defined a finitely generated group to be hyperbolic if its Cayley graphs are hyperbolic. This notion, and the related construction of the *hyperbolic boundary*, has had a tremendous impact on group theory and other fields, starting with the work of Gromov [5] and developed further by many researchers [7]; see also [2]. Here we will concentrate on hyperbolic graphs from the point of view of graph theory. We will discuss how a sequence of finite graphs can give rise to an infinite hyperbolic graph, whose *boundary* can be thought of as a 'limit' of the sequence.

Let G be an infinite, locally finite graph. A *geodetic triangle* in G is a subgraph consisting of three vertices and a shortest path between each two of these vertices; these paths are the *sides* of the geodetic triangle. A geodetic triangle T is *δ-thin* if for every side S of T and every vertex v of S, the distance, in G, between v and the union of the other two sides of T is at most δ. We say that G is *δ-hyperbolic* if every geodetic triangle of G is δ-thin, and we say that G is *hyperbolic* if there is a $\delta \in \mathbb{N}$ such that G is δ-hyperbolic. See [8] for some equivalent definitions.

For example, every tree is 0–hyperbolic. Other examples of hyperbolic graphs include all tessellations of the hyperbolic plane.

Although hyperbolicity of a graph is a simple and rather local property, it implies deeper and more global properties. One of the most striking ones is the behaviour of geodesics: given a hyperbolic graph G and a vertex $v \in V(G)$, it is possible to fix an upper bound $M \in \mathbb{N}$ such that for every two 1-way infinite geodesics R, L starting at v one of the two following possibilities must hold. Either R, L are *parallel* to each other, that is, R is contained in the cylinder $\{u \in V(G) \mid d(u, L) < M\}$ of radius M around L (and vice versa), or R, L *diverge exponentially*; see [8] for details and a proof.

Hyperbolic graphs yield much of their importance from the *hyperbolic compactification*: this is a natural way to compactify a hyperbolic graph by adding a boundary to which the geodesics of the graph converge. This *hyperbolic boundary* is defined as the set of equivalence classes of 1-way infinite geodesics starting at a fixed vertex v where two such geodesics are equivalent if they are parallel. This set is endowed with a metric in which, intuitively, two classes of geodesics are close if they have representatives with long common initial subpaths. In the (hyperbolic) graph of Figure 1 for example, the boundary is homeomorphic to the real unit interval. A different approach for defining the hyperbolic boundary and its metric, based on an assignment of lengths to the edges of the graph, is explained in [3].

The variety of spaces that can be obtained as the boundary of some hyperbolic graph is impressive:

Theorem 1 ([5]). *Every compact metric space is isometric to the hyperbolic boundary of some hyperbolic graph.*

In order to prove this assertion, one starts with a sequence $(G_i)_{i\in\mathbb{N}}$ of finite graphs, which we will call the *horizontal levels*, that approximate the compact

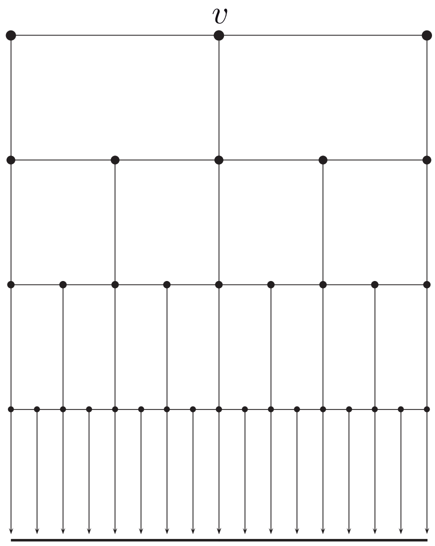

FIGURE 1. A hyperbolic graph and its boundary, which is in this case homeomorphic to the real unit interval.

metric space X, and joins all horizontal levels together into a single hyperbolic graph G by adding *perpendicular edges* that form a depth-first spanning tree of G. One does so in a manner that guarantees that G is hyperbolic and its boundary is isometric to X. For example, in the graph of Figure 1 the horizontal paths can be thought of approximations of the real unit interval, the boundary of that graph. Similarly, Figure 2 shows how to construct a hyperbolic graph whose boundary is the Sierpinski gasket [6].

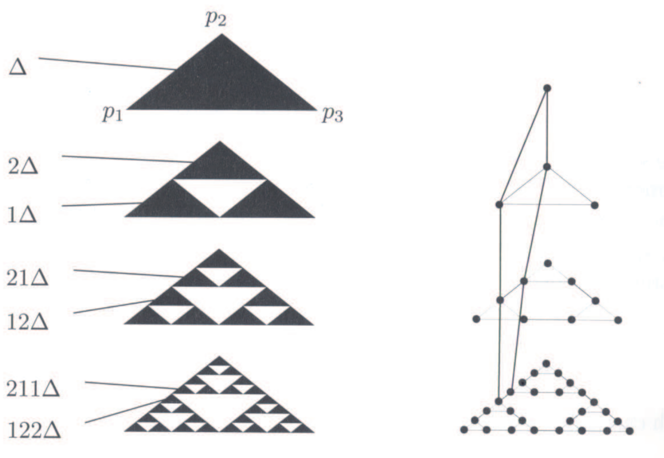

FIGURE 2. A hyperbolic graph whose boundary is the Sierpinski gasket. (Figure reproduced from [6].)

Hyperbolic graphs and their boundaries have been studied very thoroughly [7], but usually the graphs considered are Cayley graphs or otherwise closely related to some group. It was the main aim of my talk to argue that hyperbolic graphs can also be interesting in the absence of groups, and thus merit the attention of the graph theory community[1]. I mentioned two main reasons for this.

The first reason is that Theorem 1 provides a platform for proving topological results using graphs. Indeed, in [4] this theory is used in order to obtain a "graph-theoretical" characterization of path-connected continua, and this is applied to derive a graph-theoretical proof of the Hahn-Mazurkiewicz theorem. Interestingly, to achieve this, the problem of finding a 'space filling curve' in a locally connected metric space was reduced to finding well-behaved vertex-dominating walks in the finite graphs constituting the horizontal levels in the above construction. This suggests that (finite and infinite) graph theory might have applications in topology.

The second reason is that, starting with a sequence (G_i) of finite graphs, one could try to use constructions like the one of Figure 2 (see also [1] for further interesting examples) in order to obtain a hyperbolic boundary which can be thought of as the limit of the sequence (G_i); it would then be interesting to try to draw conclusions about the sequence by studying this limit. This approach seems to be more suited for sparse graphs (G_i).

Let me close with an informal problem.

Problem 2. *Find a way to construct infinite random hyperbolic graphs consisting of horizontal levels which are random finite graphs (of bounded degree?) joined by a perpendicular spanning tree (random or deterministic). What can you say about the random boundary of this graph? For example, is there a 'threshold' for its path-connectedness, and how does it relate to the threshold for connectedness of the horizontal levels?*

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Definable Graph Structure Theory, Isomorphism Testing, and Descriptive Complexity

Martin Grohe

Let $\mathcal{C}$ be a class of graphs with excluded minors. We prove that the graphs in $\mathcal{C}$ have "generic" treelike decompositions into pieces that admit a "generic" linear orders. Both the decompositions and the linear orders of the pieces are definable in a least fixed-point logic, an extension of first-order predicate logic by a mechanism formalising inductive definitions.

A consequence of this result is a simple combinatorial polynomial time isomorphism test for graphs in $\mathcal{C}$. Another consequence is a result in descriptive complexity theory: A property of graphs in $\mathcal{C}$ is decidable in polynomial time if and only if it is definable in least fixed-point logic with counting.

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Pairs of signed graphs with the same even cycles

Bertrand Guenin
(joint work with Irene Pivotto, Paul Wollan)

In this paper graphs can have multiple edges as well as loops. A *cycle* of G is a subgraph where all vertices have even degree (we view cycles as subset of edges). We denote by $\text{cycle}(G)$ the set of all cycles of G. Since the cycles of G correspond to the cycles of the *graphic matroid of* G, we identify $\text{cycle}(G)$ with that matroid. A *Whitney-flip* consists of decomposing a graph G along a two vertex cutset s and t into parts G_1 and G_2 and then recombining the two parts by identifying the vertex s (resp. t) of G_1 with vertex t (resp. s) of G_2 (rearranging blocks of a graphs is also viewed as a Whitney flip). Whitney [1] proved the following seminal result,

Theorem 1. *We have* $\text{cycle}(G) = \text{cycle}(G')$ *if and only if* G' *are* G' *are related by a sequence of Whitney-flips.*

A *signed graph* is a pair (G, Σ) where $\Sigma \subseteq E(G)$. A subset $B \subseteq E(G)$ is *even* (resp. *odd*) if $|B \cap \Sigma|$ is even (resp. *odd*). We denote by $\text{ecycle}(G, \Sigma)$ the set of all even cycles of (G, Σ). Since the even cycles of (G, Σ) correspond to the cycles of the *even cycle matroid of* (G, Σ), we identify $\text{ecycle}(G, \Sigma)$ with that matroid. Given two signed graphs (G, Σ) and (G', Σ') such that $\text{ecycle}(G, \Sigma) = \text{ecycle}(G', \Sigma')$ what is the relation between (G, Σ) and (G, Σ')? We shall provide two answers to this question.

Earlier result. Gerards and al [2] proved the following result,

Theorem 2. *Let* (G,Σ) *and* (G',Σ') *be signed graphs. Suppose that* $\text{ecycle}(G,\Sigma) = \text{ecycle}(G',\Sigma')$ *and that this matroid is graphic. Then* (G,Σ) *and* (G',Σ') *are related by a sequence of Whitney-flips, signature exchanges, and Lovász-flips.*

It remains to describe the terms *signature exchanges* and *Lovász-flips.*

We say that Σ' is a *signature* of (G,Σ) if $\text{ecycle}(G,\Sigma) = \text{ecycle}(G,\Sigma')$. It can be readily checked that Σ' is a signature of (G,Σ) if and only if $\Sigma' = \Sigma \triangle B$ where B is a cut of G. The operation that consists of replacing a signature of a signed graph by another signature is called a *signature exchange.*

Consider a signed graph (G,Σ). Suppose that there are vertices v_1, v_2 of G such that $\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2)$. We can construct a new signed graph (G',Σ) as follows:

(i) turn odd edges with ends v_1, v_2 into loops;
(ii) turn odd loops into edges with ends v_1, v_2;
(iii) replace end v_1 of odd edges by end v_2;
(iv) replace end v_2 of odd edges by end v_1.

Note we apply (iii) and (iv) to edges which are not in (i) or (ii). We say that (G',Σ) is obtained from (G,Σ) by a *Lovász-flip.* It can be readily checked that $\text{ecycle}(G,\Sigma) = \text{ecycle}(G',\Sigma)$.

A new result: isomorphism. We cannot omit the condition that $\text{ecycle}(G,\Sigma)$ be graphic in the statement of Theorem 2 as there are pairs of signed graphs which have the same even cycles but which are not related by Whitney-flips, Lovász-flips, or signature exchanges. The first main result of the paper is,

Theorem 3. *Let* (G,Σ) *and* (G',Σ') *be signed graphs. Suppose that* $\text{ecycle}(G,\Sigma) = \text{ecycle}(G',\Sigma')$ *and that this matroid is non-graphic and* 3*-connected. Then either* (G,Σ) *and* (G',Σ') *form a Shuffle pair, a Tilt pair, a Twist pair, a Nova pair or* (G,Σ) *and* (G',Σ') *are related by a sequence of Whitney-flips, signature exchanges, and Lovász-flips.*

The terms *Shuffle pair*, *Tilt pair*, *Twist pair*, and *Nova pair* are undefined. We shall only describe the first one.

Consider a signed graph (G,Σ) and let $\{a,b,c,d\} \subseteq V(G)$. Suppose that $E(G)$ can be partitioned into $X_1,\dots,X_4$ (not necessarily all non-empty) such that for all $i,j \in [4]$ where $i \neq j$, $V(G[X_i]) \cap V(G[X_j]) \subseteq \{a,b,c,d\}$. For all $i \in [4]$ denote by a_i (resp. b_i, c_i, d_i) the copy of vertex a (resp. b, c, d) of $G[X_i]$. Then construct G' by:

(1) identifying vertices a_1, b_2, c_3, d_4 to vertex say a';
(2) identifying vertices b_1, a_2, d_3, c_4 to vertex say b';
(3) identifying vertices d_1, c_2, b_3, a_4 to vertex say c';
(4) identifying vertices c_1, d_2, a_3, b_4 to vertex say d'.

Then (G,Σ) and (G',Σ') form a *Shuffle pair* if $\delta_{G'}(a')$ is a signature of (G,Σ) and $\delta_G(a)$ is a signature of (G',Σ').

A new result: equivalence classes. A *representation* of an even cycle matroid M is a signed graph (G, Σ) for which $M = \text{ecycle}(G, \Sigma)$. Theorem 2 says that any two representations of an even cycle matroid are related by a sequence of operations that preserve even cycles at each step. This describes how any two representations relate pairwise. The next result show that we can cover the set of all representations of an even cycle matroids by a small number of equivalence classes we call *bundles*. Moreover, the relation between any two representations in a bundle is much simpler than the relation between an arbitrary pair of representations of the even cycle matroid.

Consider a non-graphic 3-connected even cycle matroid M and let $\mathbf{B}$ be a set of representations of M. We call such a set $\mathbf{B}$ a *bundle*. We identify special bundle types (whose descriptions are omitted in this abstract). Let $\mathcal{T}$ denote the set of all such bundle types. The second main result of the paper is,

Theorem 4. *There exists a constant k such that for any 3-connected, non-graphic even-cycle matroid M, the set of all representations of M is included in at most k bundles each of which of a type $T \in \mathcal{T}$.*

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On characterising Vizing's edge colouring bound

Penny Haxell
(joint work with Jessica McDonald)

In this talk we consider edge colourings of multigraphs. For a multigraph G, we denote by $\Delta(G)$ the maximum degree of G, and by $\mu(G)$ the maximum edge multiplicity. The classical theorems of Shannon [12] and Vizing [15] give best possible bounds on the *chromatic index* $\chi'(G)$ of G, the minimum number of colours needed to colour the edges of G such that no two edges sharing a vertex have the same colour.

Theorem 1. [12] *(Shannon's Theorem) For every multigraph G we have $\chi'(G) \leq \lfloor \frac{3\Delta}{2} \rfloor$.*

Theorem 2. [15] *(Vizing's Theorem) For every multigraph G we have $\chi'(G) \leq \Delta + \mu$.*

Equality holds in Shannon's Theorem if and only if G contains a triangle with $\lfloor \frac{3\Delta}{2} \rfloor$ edges. This fact was proved by Vizing in his 1968 doctoral dissertation (see also [14]). However, the class of multigraphs for which Vizing's bound holds with equality is not nearly so easy to describe. Indeed, by the well-known theorem of Holyer [4], when $\mu = 1$ it is an NP-complete problem to determine if a graph G has chromatic index Δ or $\Delta + 1$. Various families of multigraphs for which $\chi' = \Delta + \mu$ were described by Scheide and Stiebitz in [10]. These facts raise the following question: is there a range of values of μ for which the set of multigraphs satisfying $\chi' = \Delta + \mu$ can be efficiently characterized?

Some necessary conditions for $\chi' = \Delta + \mu$ are known when $\mu \geq 2$. For example, Kierstead [6] proved that G must contain a triangle with at least 2μ edges, and McDonald [7] proved that if $G \neq \mu K_3$ then G must contain a specific 5-vertex multigraph with at least $4\mu - 2$ edges.

The following is a special case (restricted to our specific problem about Vizing's bound) of an old and notoriously difficult conjecture due to Goldberg [3] (1973) and Seymour [11] (1977).

Conjecture 3. [3] [11] *Let G b e a multigraph with $\mu \geq 2$. Then $\chi'(G) = \Delta + \mu$ if and only if there exists an odd subset $S \subseteq V(G)$ with $|S| \geq 3$, such that $|E[S]| > \frac{(|S|-1)}{2}(\Delta + \mu - 1)$.*

Note that the existence of such a set S clearly implies $\chi' = \Delta + \mu$, because the maximum size of a colour class in $G[S]$ is $\frac{|S|-1}{2}$ (since S is odd). Moreover, determining whether or not G has such an S is only as hard as checking membership in the matching polytope of G, and it follows from the work of Edmonds [2] that this can be done in polynomial time (see, for example, [1]). So, Conjecture 3 says that $\mu = 1$ is the only value of μ for which it is difficult to classify those multigraphs attaining equality in Vizing's bound. In this talk we prove that this conjecture is true provided μ is not too small with respect to Δ, namely that it is bounded below by a logarithmic function of Δ.

Theorem 4. *Let G be a multigraph with $\mu \geq \log_{5/4}(\Delta) + 1$. Then $\chi'(G) = \Delta + \mu$ if and only if there exists an odd subset $S \subseteq V(G)$ with $|S| \geq 3$, such that $|E[S]| > \frac{(|S|-1)}{2}(\Delta + \mu - 1)$.*

Another theorem relates more directly to the Seymour-Goldberg conjecture itself. The parameter $\rho(G)$ of a multigraph G is defined as follows.

$$\rho(G) := \max \left\{ \frac{2|E[S]|}{|S|-1} : S \subseteq V(G), |S| \geq 3 \text{ and odd} \right\}.$$

The Seymour-Goldberg conjecture in its general form states that for any multigraph G we have $\chi'(G) \leq \max\{\lceil \rho \rceil, \Delta + 1\}$. In spite of the large amount of work that has been done on this conjecture, it remains open in general (see e.g. [5]). Recently Scheide [8] (see also [9]) proved the following approximate version.

Theorem 5. [8] *For every multigraph G we have $\chi'(G) \leq \max\left\{\lceil \rho \rceil, \Delta + \sqrt{(\Delta - 1)/2}\right\}$.*

The bound in our second theorem gives additional information when μ is of smaller order than $\Delta / \log \Delta$. Here log denotes the natural logarithm.

Theorem 6. *Let G be a multigraph. Then $\chi'(G) \leq \max\left\{\lceil \rho \rceil, \Delta + 2\sqrt{\mu \log(\Delta)}\right\}$.*

We remark that in each of our theorems, the constants could be somewhat sharpened. We chose to state the results as above in order to keep the calculations very simple. Our methods are based on the method of Tashkinov trees [13], a sophisticated generalization of the method of alternating paths, developed from an earlier approach due to Kierstead [6].

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The complement of graphs with tree-width k and faithful orthogonal representations

Hein van der Holst

(joint work with John Sinkovic)

A faithful orthogonal representation of a graph $G = (V, E)$ in $\mathbb{R}^d$ is an assignment of a vector $\overrightarrow{v}$ to each vertex v of G such that for distinct vertices v, w, $\overrightarrow{v}$ and $\overrightarrow{w}$ are orthogonal if and only if v and w are nonadjacent. The smallest integer d such that G has a faithful orthogonal representation in $\mathbb{R}^d$ is called the minimum semidefinite rank of G, and is denoted by $\mathrm{mr}_+(G)$. Let G have n vertices. Denote by $\mathcal{S}_G$ the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} \neq 0$, $i \neq j$ if and only if i and j are adjacent. Clearly, $\mathrm{mr}_+(G)$ the minimum rank attained by any positive semidefinite matrix $\mathcal{S}_G$. The problem is to determine $\mathrm{mr}_+(G)$ for each graph G.

Graphs G with $\mathrm{mr}_+(G) \le 1$ are exactly those which are the disjoint union of a complete graph and some isolated vertices. Graphs G with $\mathrm{mr}_+(G) \le 2$ are exactly those graphs for which the complement is the join of a complete graph and a collection of complete bipartite graphs. Denote by $M_+(G)$ the maximum nullity attained by any positive semidefinite matrix in $\mathcal{S}_G$). Clearly, if G has n vertices, then $\mathrm{mr}_+(G)+M_+(G) = n$. Graphs G with $M_+(G) \le 1$ are exactly the trees (one direction follows from the work of Colin de Verdière [2]). Van der Holst [3] gave a characterization of those graphs G with $M_+(G) \le 2$. Beyond these values the speaker knows no other characterizations.

Certain properties of graphs may provide lower or upper bounds on $\mathrm{mr}_+(G)$. For example, a result of Lovász, Saks, and Schrijver [4, 5] says that a graph G is k-connected if and only if G has a general-position faithful orthogonal representation in $\mathbb{R}^{n-k}$. This implies that k-connected graphs G have $M_+(G) \ge k$.

Our result provides an upper bound on $\mathrm{mr}_+(G)$ in terms of the tree-width of the complement of the graph G. The tree-width can be defined as follows. A k-tree either is a complete graph on $k + 1$ vertices or can be obtained from a k-tree H with one vertex less by adding a new vertex and connecting it to all vertices of a k-clique in H. The tree-width of a graph G is the smallest integer $k \ge 0$ such that G is a subgraph of a k-tree. The result says that for a graph G with tree-width k, the complement $\overline{G}$ of G has $\mathrm{mr}_+(\overline{G}) \le k + 2$.

For a symmetric matrix A, the partial inertia of A is the pair (p, q), where p and q are the number of positive and negative eigenvalues of A, respectively. The inertia set of a graph is the set of all partial inertias of matrices in $\mathcal{S}_G$. The inertia set was introduced and studied by Barrett, Hall, and Loewy [1]. Notice that the inertia set of a graph G contains information about $\mathrm{mr}_+(G)$. Our result can be extended to: For each graph G whose complement has tree-width k, the inertia set of G includes $\{(p, q) \mid k + 2 \le p + q \le n\}$, where n is the order of G.

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Generic global rigidity of body-bar frameworks

Tibor Jordán
(joint work with Robert Connelly and Walter Whiteley)

Two frameworks $G(\mathbf{p})$ and $G(\mathbf{q})$ are *equivalent in* $\mathbb{R}^d$ if corresponding edge lengths are the same, where $\mathbf{p}$ and $\mathbf{q}$ are configurations in $\mathbb{R}^d$ corresponding to the vertices of a finite graph G. We say that $G(\mathbf{p})$ is *globally rigid in* $\mathbb{R}^d$ if when $G(\mathbf{q})$ in $\mathbb{R}^d$ is equivalent to $G(\mathbf{p})$, $\mathbf{q}$ is congruent to $\mathbf{p}$. The configurations $\mathbf{p}$ and $\mathbf{q}$ are *congruent* if there is a rigid congruence of $\mathbb{R}^d$ that takes $\mathbf{p}$ to $\mathbf{q}$.

A framework $G(\mathbf{p})$ is *rigid in* $\mathbb{R}^d$ if there is a neighborhood $U_\mathbf{p}$ in the space of configurations in $\mathbb{R}^d$ such that if $G(\mathbf{q})$ is equivalent to $G(\mathbf{p})$ and $\mathbf{q} \in U_\mathbf{p}$, then $\mathbf{q}$ is congruent to $\mathbf{p}$.

If one is given a particular configuration $\mathbf{p}$, by [11], determining global rigidity for any $d \geq 1$ is infeasible, and even for rigidity for $d \geq 2$ it seems unrealistic. A natural way to address this difficulty is to consider the case when the configuration $\mathbf{p}$ is *generic*, which means that all the coordinates of all the points of the configuration $\mathbf{p}$ are algebraically independent over the rational numbers. In other words, the only polynomial with integer coefficients that is satisfied by these coordinates is the 0 polynomial. This is something of an overkill, especially in the case of rigidity, since a reasonable finite set of polynomial equations, given by certain determinants, can be used in many instances. In the case of global rigidity, the equations that would determine the "bad" cases for global rigidity are much harder to determine.

With the concept of generic in mind, we define a graph G to be *generically rigid in* $\mathbb{R}^d$ if $G(\mathbf{p})$ is rigid at all generic configurations $\mathbf{p}$, and *generically globally rigid in* $\mathbb{R}^d$ if $G(\mathbf{p})$ is globally rigid at all generic configurations $\mathbf{p}$ [3, 4]. It is not obvious that global rigidity is a generic property, but recent results in [4, 7] prove that indeed global rigidity is a generic property for graphs in each dimension.

Two natural necessary conditions, observed by Hendrickson [8], for generic global rigidity in $\mathbb{R}^d$ are that the graph G be vertex $(d+1)$-connected, and that, for a generic configuration $\mathbf{p}$, $G(\mathbf{p})$ be *redundantly rigid*, which means that $G(\mathbf{p})$ is rigid and remains rigid after the removal of any edge.

For $d = 2$, Berg and Jordán [2] and Jackson and Jordán [9] confirm, using [4], that Hendrickson's necessary conditions are sufficient for generic global rigidity. For $d = 3$, Connelly [3] showed that the complete bipartite graph $K_{5,5}$ is generically redundantly rigid and vertex 5-connected, but not generically globally rigid, showing that Hendrickson's necessary conditions are not sufficient. Similar examples exist for all $d \geq 3$.

So it is natural to search for classes of graphs where generic global rigidity can be determined combinatorially in line with Hendrickson's necessary conditions, without recourse to matrix calculations for each graph, as in [4]. At a workshop at BIRS in 2008, two of the authors and Meera Sitharam conjectured that generic body-and-bar frameworks would be one such class. These consist of disjoint collections of vertices, grouped as *bodies*, where each body is joined to some of the other bodies by disjoint bars. Each body is assumed to be globally rigid in its own right, by insisting that each body have enough internal bars to ensure its own global rigidity. For a generic body-and-bar framework, all of the vertices of all of the bodies are generic. The connections between the bodies are recorded in a single multigraph H (without loops, but with multiple edges allowed), where each body is represented as a vertex in the multigraph. When we collect all the individual vertices of each body and their individual internal and external bars, we denote that graph by G_H. Note that any two bars joining a pair of bodies have disjoint vertices, making this a graph.

In [12, 13] it is shown that generic rigidity (and hence generic redundant rigidity) of body-and-bar frameworks in $\mathbb{R}^d$, for all $d \geq 1$, can be determined efficiently. The following is our main result.

Theorem 1. *A body-and-bar framework is generically globally rigid in* $\mathbb{R}^d$ *if and only if it is generically redundantly rigid in* $\mathbb{R}^d$.

For the proofs of previous results [2, 9], and for our main theorem here, we rely on several key techniques. In [4], a sufficient condition is given in terms of the rank of a stress matrix (to be defined later), that combines with (infinitesimal) rigidity at a generic point to imply generic global rigidity in any specific dimension (see also [5]). To apply this result, certain key inductive constructions have been shown to preserve both the maximal rank of the corresponding stress matrix, and the infinitesimal rigidity. It is also necessary that these inductive constructions generate all members of the class from a generically globally rigid seed (a minimal complete graph).

These results have significant theoretical interest as steps towards a full theory of generic global rigidity of arbitrary frameworks. There are also a wide range of applications for the algorithms that detect global rigidity, such as localization in wireless sensor networks [1, 10], molecular conformation [14], and stability of molecules.

We also note that by the results in [5], graphs G_H which are generically globally rigid in $\mathbb{R}^d$ are also generically globally rigid in spherical and hyperbolic d-space. $\mathbb{R}^d$ is the classical sample of a general class of metrics over which rigidity and generic global rigidity results are invariant.

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Disjoint trees and hypertrees

Tomáš Kaiser
(joint work with Petr Vrána)

The subject of this talk is a partial result on the following conjecture of Thomassen [5]:

Conjecture 1. *Every 4-connected line graph is hamiltonian.*

It is known [7] that the assertion of Conjecture 1 holds for 7-connected line graphs. A recent result of Kaiser and Vrána [2] improves this as follows:

Theorem 2. *Every 5-connected line graph of minimum degree at least 6 is hamiltonian.*

Using the claw-free closure developed by Ryjáček [4], Theorem 2 can be extended from line graphs to claw-free graphs.

The proof of Theorem 2 uses a result on spanning hypertrees in 4-edge-connected hypergraphs. A variant of the method used to find the spanning hypertrees also provides a short proof [1] of the characterization of graphs with k disjoint spanning trees of Tutte [6] and Nash-Williams [3]. In this talk, we describe the method and attempt to explain the main ideas of [2].

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Coloring $K_{3,k}$-minor-free graphs

Ken-ichi Kawarabayashi

$K_{3,k}$-minor-free graphs are a significant generalization of bounded-genus graphs. They also contain infinitely many minimal k-colorable graphs, i.e, k-color-critical graphs, for all $k \geq 4$. Motivated by this fact, we investigate excluded minors in non-5-colorable $K_{3,k}$-minor-free graphs. Specifically, we prove the following result.

There is a computable constant $f(k)$ such that every forbidden minor with respect to 5-colorability in $K_{3,k}$-minor-free graphs has at most $f(k)$ vertices.

Our proof of the above result implies the following algorithmic result, which is of independent interest.

For a graph G, there is an $O(n^2)$ algorithm to output one of the following:

(1) a 5-coloring of G, or
(2) a $K_{3,k}$-minor of G, or
(3) a minor R of G of order at most $f(k)$ ($f(k)$ comes from the above theorem) such that R does not have a $K_{3,k}$-minor nor is 5-colorable.

Let us emphasis that the chromatic number in our main result does NOT depend on k. This is a big contrast with the algorithmic result of Hadwiger's conjecture [1]. Note that testing 3-colorability of bounded genus graphs is NP-complete, and testing 4-colorability of them would require a significant generalization of the Four Color Theorem. Testing 5-colorability of bounded genus graphs can be done in polynomial time, as shown by Thomassen [2].

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Total fractional colorings of graphs with large girth

Daniel Král

(joint work with Tomáš Kaiser, František Kardoš, Andrew King, Jean-Sébastien Sereni)

A total coloring is a combination of a vertex coloring and an edge coloring of a graph: every vertex and every edge is assigned a color and any two adjacent/incident objects must receive distinct colors. One of the main open problems in the area of graph colorings is the Total Coloring Conjecture of Behzad and Vizing from the 1960's asserting that every graph has a total coloring with at most $\Delta + 2$ colors where Δ is its maximum degree.

Fractional colorings are linear relaxation of ordinary colorings. In the setting of fractional total colorings, the Total Coloring Conjecture was proven by Kilakos and Reed [1]. In the talk, we will present a proof of the following recent conjecture of Reed:

For every real $\varepsilon > 0$ and integer Δ, there exists g such that every graph with maximum degree Δ and girth at least Δ has total fractional chromatic number at most $\Delta + 1 + \varepsilon$.

For $\Delta = 3$ and $\Delta \in \{4, 6, 8, 10, \ldots, \}$, we prove the conjecture in a stronger form: there exists an integer g_Δ such that every graph with maximum degree Δ and girth at least g_Δ has total fractional chromatic number equal to $\Delta + 1$.

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Vertex cuts and tree decompositions

Bernhard Krön
(joint work with Martin J. Dunwoody)

Structure trees of graphs are trees which correspond to automorphism invariant tree-decompositions. Given a connected graph, in many cases it is possible to construct a structure tree that provides information about the ends of the graph or its connectivity, see [1]. For example, Stallings' theorem on the structure of groups with more than one end can be proved by analyzing the action of the group on a structure tree as in [2].

Theorem 1 (Stallings' Structure Theorem [5, 6])**.** *A finitely generated group has more than one end if and only if it splits over a finite subgroup.*

Tutte used a structure tree to investigate finite 2-connected graphs, that are not 3-connected in [4]. Most of these structure tree theories have been based on edge cuts, which are components of the graph obtained by removing finitely many edges. A new axiomatic theory is described using vertex cuts in [3]. These are certain components of the graph obtained by removing finitely many vertices. This generalizes Tutte's tree decomposition of 2-connected graphs to k-connected graphs for any k, in finite and infinite graphs. The theory can be applied to non-locally finite graphs with more than one vertex end, i.e. graphs with rays that can be separated by removing a finite number of vertices. This gives a decomposition for a group acting on such a graph, generalizing Stallings' theorem from finitely generated groups to arbitrary groups in the following way.

Theorem 2. *A group has a Cayley graph with more than one end if and only if it splits over a finite subgroup.*

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Extremal graphs and graph limits

László Lovász
(joint work with Balázs Szegedy)

The theory of graph limits has many points where it touches extremal graph theory. One could mention that Szemerédi's Regularity Lemma is one of the basic tools for graph limits, but also that the strongest form of the Regularity Lemma is a compactness statement for the space of limit objects [6]. One could also point at the recent proof by Razborov [10] of a long-standing conjecture of Lovász and Simonovits on the minimum number of triangles in a graph with given edge density.

We have to describe the elements of graph limit theory. For two simple graphs F and G, let $t(F,G)$ denote the density of F is G, defined as the probability that a random map $V(F) \to V(G)$ preserves edges. A sequence $G_1, G_2, \dots$ of simple graphs is called *convergent*, if $|V(G_n)| \to \infty$ and $t(F, G_n)$ has a limit for every fixed F as $n \to \infty$ (see [1, 2].

It was proved in [5] that every convergent sequence has a limit object in the form of a symmetric measurable function $W: [0,1]^2 \to [0,1]$. This represents the limit in the sense that

$$t(F,G_n) \to t(F,W) := \int_{[0,1]^V} \prod_{ij\in E} W(x_i,x_j) \prod_{i\in V} dx_i\,.$$

These limit functions are called *graphons*. Formalizing theorems and arguments in terms of graphons often makes the arguments much more clean and transparent.

In this talk we describe some new results which aim at answering general questions in extremal graph theory using the framework of graphons.

1. *What is the possible structure of extremal graphs?* We investigate graphons that are *finite forcible* in the sense that they are determined by a finite number of prescribed subgraph densities; this corresponds to a unique asymptotic structure in finite graphs forced by finitely many subgraph densities. We consider extremal problems in graph theory that can be formulated as minimizing or maximizing the density of some type of subgraphs subject to fixing other densities. One conjecture says that every such extremal problem has an extremal family whose limit is finitely forcible.

Constant graphons (which are limits of random graphs) can be forced by the densities of edges and 4-cycles, by a result of Graham, Chung and Wilson [3]. This can be generalized to stepfunctions (limits of generalized random graphs) [4]. It was conjectured that these are the only finitely forcible graphons, but recently further nontrivial and quite interesting families have been discovered. A complete charaterization is an exciting but difficult open problem.

2. *Is there a general method to prove extremal graph results?* Densities of different subgraphs in a fixed graphon can be characterized through the positive semidefiniteness of certain "connection matrices", and many inequalities between subgraph densities follow from this semidefiniteness. Such a proof can be translated to more direct arguments using the Cauchy–Schwarz inequality.

We don't know if every linear inequality between subgraph densities that holds for all simple graphs can be proved this way; we expect the answer is negative. But we can show that every such inequality can be relaxed arbitrarily little to get another inequality that is already provable this way.

3. *Which graphs have a low-dimensional structure?* Many intresting families of graphs have a low-dimensional structure (threshold graphs, Borsuk graphs). For each graphon, one can define a topology on $[0, 1]$ (different from the usual), which refects many interesting graph-theoretic properties. For example, if we exclude a fixed bipartite graph as the bipartite subgraph between two subsets of nodes, then we get a family of graphs whose limit objects are all compact and finite dimensional.

It can be proved [6, 9] that this dimension is closely related to the number of partition classes in the Regularity Lemma. A consequence of the result mentioned above is that if we exclude a fixed bipartite graph as above, then the number of partition classes for a (weak) regularity partition with error ϵ is bounded by a polynomial in $1/\epsilon$.

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Connectivity keeping subgraphs in k-connected graphs

Wolfgang Mader

All graphs (digraphs) considered here are finite without multiple edges (of the same direction) and without loops. A " graph " is always undirected. For a graph (digraph) G, " k-connected " means (srongly) k-vertex-connected and the connectivity number of G is denoted by $\kappa(G)$

Almost 40 years ago, G.Chartrand, A.Kaugars, and D.R.Lick [2] proved the following well known result.

Theorem CKL (G.Chartrand, A.Kaugars, and D.R.Lick [2]). *Every k-connected graph G of minimum degree $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor$ has a vertex x with $\kappa(G-x) \geq k$.*

A short proof of this result has been given in [6]. The lower bound for $\delta(G)$ in Theorem CKL is best possible. If for the k-connected graph G even $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor - 1 + m$ holds for a positive integer m, then Theorem CKL obviously implies the existence of m vertices $x_1, \ldots, x_m$ such that $\kappa(G - \{x_1, \ldots, x_m\}) \geq k$. S.Fujita and K.Kawarabayashi asked in [4], if it is possible to choose these vertices in such a way that they span a connected subgraph in G, and they stated the following conjecture.

Conjecture FK (S.Fujita and K.Kawarabayashi[4]). *For all positive integers k, m, there is a (least) non-negative integer $f_k(m)$ such that every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor - 1 + f_k(m)$ contains a connected subgraph W of exact order m such that $\kappa(G - V(W)) \geq k$ holds.*

In their paper [4], S.Fujita and K.Kawarabayashi studied the case $m = 2$ and proved that $f_k(2)$ exists and $2 \leq f_k(2) \leq 3$ holds for all k. They constructed also examples in [4] showing $f_k(m) \geq m$ for all m and k. Our main result in [8] says that $f_k(m)$ always exists and $f_k(m) = m$ holds for all positive integers k, m.

Theorem 1 (W.Mader [8]) . *Every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ for positive integers k, m contains a graph P of order m such that $\kappa(G - V(P)) \geq k$ holds.*

In the proof of this result, it will be importent that we can let start our path from a prescribed vertex. So we proved first the following result (which is not stated in the following form in [8], but follows from Theorem 2 in [8] in a similar way as Theorem 1).

Theorem 2 (W.Mader [8]). *Let G be a $(k+1)$-connected graph with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ and choose $p \in V(G)$. Then there is a path P of order m starting from p such that $\kappa(G - V(P)) \geq k$ holds.*

Of course, now the question arises, if we can find instead of a path any other prescribed tree of order m, the deletion of which does not distroy the k-connectivity. In analogy to Conjecture FK, I conjectured in [8] that for every positive integer k and every tree T, there is a (least) non-negative integer $g_k(T)$ such that every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor - 1 + g_k(T)$ has a subgraph S isomorphic to T such that $\kappa(G - V(S)) \geq k$ holds. The next result shows that this conjecture is right.

Theorem 3 (W.Mader [9]). *For all positive integers k and all trees T, $g_k(T)$ exists and $g_k(T) \leq 2(k-1+|T|)^2 + |T| - \lfloor \frac{3k}{2} \rfloor$ holds.*

I think this upper bound for $g_k(T)$ far away from being best possible.

Conjecture 1 (W.Mader [8]). *For all positive integers k and all trees T, $g_k(T) = |T|$ holds.*

Attacking a conjecture of S.C.Locke (cf.[1] and [5]), A.A.Diwan and N.P.Tholiya ([3]) proved Conjecture 1 for $k = 1$ very recently.

There are results which correspond to Theorem CKL for digraphs. In a digraph, a "path " and a "circuit " are always continuously directed. A digraph is called *antisymmetric* , if it has no circuits of length two. The minimum outdegree and the minimum indegree of a digraph D are denoted by $\delta^+(D)$ and $\delta^-(D)$, respectively. The next two results are direct analogues to Theorem CKL

Theorem 4 (W.Mader [7]) *Every k-connected antisymmetric digraph D with $\delta^+(D) \geq \lfloor \frac{3k+1}{2} \rfloor$ and $\delta^-(D) \geq \lfloor \frac{3k+1}{2} \rfloor$ has a vertex x with $\kappa(D-x) \geq k$.*

Theorem 5 (W.Mader [7]). *Every k-connected digraph D with $\delta^+(D) \geq 2k$ and $\delta^-(D) \geq 2k$ has a vertex x with $\kappa(D-x) \geq k$.*

Both the values $\lfloor \frac{3k+1}{2} \rfloor$ and $2k$ are best possible, and a lower bound, no matter how large, for $\delta^+(D)$, say, is not enough for $k \geq 2$. Considering the results for graphs, Theorems 4 and 5 suggest the following conjectures.

Conjecture 2 . *Every k-connected antisymmetric digraph D with $\delta^+(D) \geq \lfloor \frac{3k+1}{2} \rfloor + m$ and $\delta^-(D) \geq \lfloor \frac{3k+1}{2} \rfloor + m$ for a non-negative integer m has a path P of length $2m$ with $\kappa(D - V(P)) \geq k$.*

Conjecture 3 *Every k-connected digraph D with $\delta^+(D) \geq 2k+m$ and $\delta^-(D) \geq 2k+m$ for a non-negative integer m has a path P of length m with $\kappa(D-V(P)) \geq k$.*

I think it very difficult to prove these conjectures, but perhaps it is easier to find counterexamples. It seems even hard to decide, if for sufficiently large $min\{\delta^+(D), \delta^-(D)\}$ (independent of the vertex number of D) such a path exists.

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Treewidth reduction for constrained separation and bipartization problems

Dániel Marx
(joint work with Barry O'Sullivan, Igor Razgon)

Our main combinatorial observation says roughly the following: for any two vertices s, t of a graph and integer k, all the inclusionwise minimal $s-t$ vertex cutsets are contained in a part of the graph that has treewidth bounded by a function of k. For the precise statement, we need the following notion. If C is a subset of vertices of G, then the *torso* of G with respect to G is a graph on the vertices C such that $a, b \in C$ are adjacent if and only if there is an $a-b$ path in G with no internal vertices in C (including the possibility that a and b are adjacent in G).

Lemma 1. *For every $s, t \in G$ and integer $k \geq 0$, there is a set C of vertices that contains every inclusionwise minimal $s-t$ vertex cutset of size at most k, and the treewidth of the torso of G with respect to C is at most $w(k)$ for some function w depending only on k. Furthermore, such a set can be found in time $f(k) \cdot |E(G)|$.*

Therefore, if we are looking for an s-t vertex cutset that has some additional property (say, it induces an independent set, induces a graph belonging to a hereditary class $\mathcal{G}$, contains certain number of colored vertices, etc.), then we can restrict our attention to this bounded-treewidth part of the graph. There are known standard techniques (dynamic programming) and powerful general results (Courcelle's theorem) for finding cutsets in bounded treewidth graphs in linear time (for every fixed bound w on treewidth). Putting these two components together we obtain a very robust method, which gives us linear-time algorithms for various generalizations of cut problems such as multicut and multiway cut. This generalizes earlier fixed-parameter tractability results for cut problems [2, 1] in a very strong way.

Reed, Vetta, and Smith [3] proved that for every fixed k, there is a quadratic-time algorithm for deciding if a graph G can be made bipartite by the deletion of a set S of at most k vertices. We can generalize this problem by requiring some additional properties on the set S, such as it induces an independent set. The algorithm in [3] reduces the problem to a series of minimum cut computations. By plugging in our results for constrained vertex cutset problems, we can immediately obtain quadratic-time algorithms for the constrained bipartization problems where S has to induce an independent set or a graph belonging to a hereditary class $\mathcal{G}$.

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Two surprising results about crossing numbers

Bojan Mohar

(joint work with Sergio Cabello, Zdeněk Dvořák)

The theory of crossing numbers provides many beautiful results and offers applications in diverse areas of mathematics and theoretical computer science. However, some of the very basic questions about crossing numbers remain unsolved.

Two recent results by the speaker, one coauthored with Zdeněk Dvořák, the other with Sergio Cabello, brought some new surprises. These results were presented and discussed in the talk.

Towards a rough structure theorem for crossing-critical graphs. A graph is *k-crossing-critical* (or simply *k-critical*) if its crossing number is at least k, but every proper subgraph has crossing number smaller than k. Using the Excluded Grid Theorem of Robertson and Seymour, it is not hard to argue that k-crossing-critical graphs have bounded tree-width [4]. However, all known constructions of crossing-critical graphs suggested that their structure is "path-like". Salazar and Thomas conjectured (cf. [4]) that they have bounded path-width. This problem was solved by Hliněný [6], who proved that the path-width of k-critical graphs is bounded above by $2^{f(k)}$, where $f(k) = (432 \log_2 k + 1488)k^3 + 1$. In the late 1990's, additional conjectures were proposed.

Conjecture 1 (Richter [12])**.** *For every positive integer k, there exists an integer $D(k)$ such that every k-crossing-critical graph has maximum degree less than $D(k)$.*

The second conjecture was proposed as an open problem in the 1990's by Carsten Thomassen and formulated as a conjecture by Richter and Salazar.

Conjecture 2 (Richter and Salazar [12, 13])**.** *For every positive integer k, there exists an integer $B(k)$ such that every k-crossing-critical graph has bandwidth at most $B(k)$.*

Conjecture 2 would be a strengthening of Hliněný's theorem about bounded path-width and would also imply Conjecture 1. Hliněný and Salazar [9] recently made a step towards Conjecture 1 by proving that k-crossing-critical graphs cannot contain a subdivision of $K_{2,N}$ with $N = 30k^2 + 200k$.

Somewhat surprisingly, both of the above conjectures are false.

Theorem 3 (Dvořák & Mohar [3]). *For every $k \geq 171$ and every d, there exists a k-crossing-critical graph H containing a vertex of degree at least d.*

Furthermore, Dvořák and Mohar (work in progress) provided further evidence that k-critical graphs essentially adhere to the bounded bandwidth structure. Their goal is to show a rough structure theorem for k-critical graphs. The following should be a rough characterization of k-crossing-critical graphs we are aiming for:

(a) Their path-width is bounded (in terms of k).
(b) They have $O(k)$ vertices of large degree, and the neighborhood of each of these vertices adheres to the "projective" structure exhibited by the examples in [3].
(c) The rest of the graph has bounded bandwidth.

Crossing number of near-planar graphs. A graph is *near-planar* if it contains an edge e such that $G - e$ is planar. Near-planarity is a very weak relaxation of planarity, and hence it is natural to study the crossing number of near-planar graphs. Graphs embeddable in the torus and apex graphs are a superfamily of near-planar graphs.

The crossing number of near-planar graphs has been studied in [1, 5, 8, 10, 14]. For instance the early result of Riskin [14] shows that the crossing number of a 3-connected cubic near-planar graph can be computed in polynomial time. This was extended to non-3-connected near-planar graphs of maximum degree 3 in [1]. The following result came as a big surprise.

Theorem 4 (Cabello & Mohar [2]). *Computing the crossing number of near-planar graphs is NP-hard.*

This result is not only surprising but also fundamental. It provides evidence that computing crossing numbers is an extremely challenging task, even for the simplest families of non-planar graphs.

The reduction is based on considering the following optimization problem: draw two planar graphs inside a disk with some of its vertices at prescribed positions of the boundary, so as to minimize the number of crossings in the drawing.

Our approach can be used to prove hardness of some other geometric problems. As an interesting consequence we obtain a new, geometric proof of NP-completeness of the crossing number problem, even when restricted to cubic graphs. Hardness of the crossing number problem for cubic graphs was originally established by Hliněný [7], who asked if one can prove this result by a reduction from an NP-complete geometric problem instead of the Linear Arrangement problem used in his proof. In particular, we provide a positive answer to this task. The proofs also provide NP-hardness results for the rectilinear crossing number of near-planar graphs and for crossing number problems for graphs with rotations [11].

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A weighted version of Mader's Theorem on disjoint S-paths

Gyula Pap

The disjoint S-paths problem is a common generalization of maximum cardinality non-bipartite matching, and disjoint s-t-paths, that is defined as follows. We are given an undirected graph $G = (V, E)$, a subset $S \subseteq V$ of nodes (called "terminals"). A path in G is called an S-path if both of its ends are in S. The disjoint S-paths problem is, given G and S, to determine tha maximum number of node-disjoint S-paths. Mader's Theorem [1] provides a good characterization, and a polynomial time algorithm follows from Lovász theory of matroid matching. A natural question to ask is whether there is a weighted generalization of this result, that generalizes Mader's Theorem in the way maximum weighted matching generalizes maximum cardinality matching.

The weighted version of this problem is that, besides $G = (V, E)$ and $S \subseteq V$, we are given a weight function $w : \binom{S}{2} \to \mathbb{N}$ to define the non-negative weight of pairs of terminals.

Weighted Disjoint S-Paths Problem Given G, S, w, determine the following maximum

$$\nu(G,S,w) := \max_{\mathcal{P}} \sum_{P\in\mathcal{P}} w(s(P), t(P)), \tag{1}$$

which is taken over sets $\mathcal{P}$ of node-disjoint S-paths.

The main result in the talk is that this problem can be solved in time $poly(|V|)$, assuming that w is representable by tree-distances. The definition of this notion – first related to S-flows, the LP relaxation of the above Weighted Problem, by Hirai [2] – goes as follows. Assume that there is a tree $T = (L, F)$ (outside of graph G) such that for every node $s \in S$ there is a subtree $R(s)$ of T such that $w(s, s') = dist_T(R(s), R(s'))$ for all $s, s' \in S$. Then w is said to be representable by tree-distances.

To formulate the mi-max formula, we need the linear extension of a tree T, denoted by $\overline{T}$, which comes from replacing every edge by a line segment of length one. In this abstract we omit the rigorous definition, and simply say that this means that for every edge of T we have a unit-length segment, and certain segments are glued together at one of their ends. Distances between points of the extended tree are defined naturally.

Now the dual of the min-max formula is a quituple of U_i, B_i, y_i, x, q, where $x : V \to \mathbb{R}_+$ are node-weights, $q : V \to \overline{T}$ node-potentials valued in the extended tree, a family $U_1, \cdots, U_k$ of subsets of V (for some non-negative integer k), subsets $B_i \subseteq U_i$ for all $i = 1, \cdots, k$, weights $y_i \geq 0$ for $i = 1, \cdots, k$. Then U_i, B_i, y_i, x, q is called a feasible dual solution, if for all terminals $s \in S$, the following inequality holds:

$$2dist(q(s), R(s)) \leq x(s), \tag{2}$$

and moreover, for all edges $uv \in E$, the following inequality holds:

$$2dist(q(u), q(v)) \leq x(u) + x(v) + \sum_{\substack{i:u\in B_i,\\ v\in U_i}} y_i + \sum_{\substack{j:v\in B_j,\\ u\in U_j}} y_j - \sum_{\substack{i:v\notin U_i,\\ u\in U_i - B_i}} y_i - \sum_{\substack{j:u\notin U_j,\\ v\in U_j - B_j}} y_j. \tag{3}$$

A feasible dual solution defines an upper bound, and the main result is that this upper bound provides a tight min-max equation to determine the maximum weight of a collection of node-disjoint S-paths.

Theorem 1 (Min-max)**.** *For G, S, w as above,*

$$\nu(G,S,w) = \min \sum_{v\in V} x(v) + \sum_{i=1}^{k} y_i \left\lfloor \frac{|B_i|}{2} \right\rfloor . \tag{4}$$

where the minimum is taken over $x(v), y_i, B_i$ as above, with $k \leq 2|V|^2$.

The straightforeward reason that this actually is a good characterization is that if one specifies the sets U_i, B_i, and specifies which unit segments the potentials $x(v)$ are taken from, then the residual problem becomes an LP. The less straigthforeward reason is that one can prove a little more than this, by showing that all node-potentials are either end-points, or half-way points of the extended tree, and all the values $x(v)$ and y_i are integral. Thus the dual optimum has a polynomial description.

In a sense, this result is tight. We say that a family of weight-functions w is closed under taking representations, if whenever $w : \binom{S}{2} \to \mathbb{N}$ is in the class, then $w' : \binom{S'}{2} \to \mathbb{N}$ is also in the class, if there is a function $\pi : S' \to S$ such that $w'(a,b) = w(\pi(a), \pi(b))$. This basically means that we can expand a point in S by a set with pairwise weights zero. The following result means that the main result is, in a sense the strongest we can hope for.

Theorem 2. *The class of weights representable by tree-distances is the largest class that is closed under representations in which the weighted disjoint S-paths problem is solvable in poly$(|V|)$.*

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Szemerédi's Regularity Lemma for matrices and sparse graphs

Alexander Scott

Let X and Y be disjoint sets of vertices in a graph G. We say that the pair (X, Y) is *ϵ-regular* if, for every $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$ we have

$$|d(X', Y') - d(X, Y)| < \epsilon,$$

where $d(X, Y)$ is the density between X and Y. Note that ϵ plays two roles here, bounding both the size of the subsets X' and Y' and the difference in density.

We shall consider partitions $V_0 \cup \cdots \cup V_k$ of $V(G)$ with a specified vertex class V_0, which we shall refer to as the *exceptional set.* A partition $V(G) = V_0 \cup \cdots \cup V_k$ with exceptional set V_0 is *balanced* if $|V_i| = |V_j|$ for all $i, j \geq 1$. We say that a partition $V(G) = V_0 \cup \cdots \cup V_k$ with exceptional set V_0 is *ϵ-regular* if it is balanced, $|V_0| < \epsilon|G|$ and all but at most ϵk^2 pairs (V_i, V_j) with $i > j \geq 1$ are ϵ-regular (we will suppress explicit mention of the exceptional set).

Szemerédi's Regularity Lemma [4] then says the following.

Theorem 1 (Szemerédi's Regularity Lemma)**.** *For every $\epsilon > 0$ and every integer $m \geq 1$ there is an integer M such that every graph G with $|G| \geq M$ has an ϵ-regular partition $\mathcal{P}$ with $|\mathcal{P}| \in [m, M]$.*

Szemerédi's Regularity Lemma is a graph-theoretic tool of great importance. However, for sparse graphs the property of ϵ-regularity is not so useful: if the graph does not contain a large set of vertices that induces a reasonably dense subgraph then *every* balanced partition (into not too many classes) is ϵ-regular, and so a regular partition may tell us nothing about the structure of our graph.

It is therefore natural to look for a version of Szemerédi's Regularity Lemma that provides structural information about sparse graphs.

We say that a pair (X, Y) is (ϵ, p)-*regular* if, for every $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$ we have

$$|d(X', Y') - d(X, Y)| < \epsilon p.$$

We say that (X, Y) is (ϵ)-*regular* if it is (ϵ, d)-regular, where d is the density of G. A partition $V_0 \cup \cdots \cup V_k$ with exceptional set V_0 is (ϵ)-*regular* if all but at most ϵk^2 pairs (V_i, V_j) with $i > j \geq 1$ are (ϵ, d)-regular.

A Regularity Lemma for sparse graphs was proved by Kohayakawa and Rödl (see [2, 3, 1]); however, it only applies to graphs that do not have large dense parts. More precisely, we say that a graph with density d is (η, D)-*upper-uniform* if, for all disjoint $X, Y \subset V$ with $\min\{|X|, |Y|\} \geq \eta|G|$, we have $e(X, Y) \leq Dd|X||Y|$.

Theorem 2. [3] *For every $\epsilon, D > 0$ and every integer $m \geq 1$ there are $\eta > 0$ and an integer M such that every (η, D)-upper uniform graph G has an (ϵ)-regular partition $\mathcal{P}$ with $|\mathcal{P}| \in [m, M]$.*

The upper uniformity condition is an annoying technical condition here; as noted by Gerke and Steger in their survey paper [1], it is not known whether the restriction is required. Our aim in this talk is to prove the following theorem.

Theorem 3 (SRL for sparse graphs)**.** *For every $\epsilon > 0$ and every positive integer m there is a positive integer M such that every graph G with at least M vertices has an (ϵ)-regular partition $\mathcal{P}$ with $|\mathcal{P}| \in [m, M]$.*

Thus the upper regularity condition is not necessary. More generally (after making suitable definitions), we get a Regularity Lemma for arbitrary matrices.

Theorem 4 (SRL for matrices)**.** *For every $\epsilon > 0$ and every positive integer L there is a positive integer M such that, for all $m, n \geq M$, every real m by n matrix A has an (ϵ)-regular block partition $(\mathcal{P}, \mathcal{Q})$ with $|\mathcal{P}|, |\mathcal{Q}| \in [L, M]$.*

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Unfriendly partitions for rayless graphs

Philipp Sprüssel
(joint work with Henning Bruhn, Reinhard Diestel, Agelos Georgakopoulos)

An *unfriendly partition* of a graph is a partition of its vertex set for which every vertex has at least as many *opponents*, neighbours in the other class of the partition, as it has *friends*, neighbours in its own class. It is easy to see that every finite graph has an unfriendly partition—each partition maximizing the number of cross-edges is unfriendly.

For infinite graphs, finding an unfriendly partition is much harder. It is known that there are (uncountable) graphs with no unfriendly partition [4], but the countable case is still open. The Unfriendly Partition Conjecture is one of the best-known open problems in infinite graph theory:

Conjecture 1 (Unfriendly Partition Conjecture)**.** *Every countable graph has an unfriendly partition.*

A few cases of the Unfriendly Partition Conjecture are easy: For *locally finite* graphs, graphs in which every vertex has finite degree, one obtains an unfriendly partition using the finite theorem and compactness. On the other end of the spectrum, for countable graphs with all vertex degrees infinite one can construct an unfriendly partition in an ω-sequence of steps, in each step giving some vertex one new opponent.

Aharoni, Milner, and Prikry [1] proved the first—and for a long time the only—substantial result towards the solution of the Unfriendly Partition Conjecture: They show that every graph (countable or not) with only finitely many vertices of infinite degree has an unfriendly partition. In [2], we have been able to show that the class of rayless graphs[1] also admits unfriendly partitions:

Theorem 2. *Every rayless graph has an unfriendly partition.*

For the proof of this theorem we use a tool developed by Schmidt [3], which assigns to every rayless graph an ordinal number, its *rank*. This is defined as follows: A graph G has rank 0 if it is finite, and it has rank $\alpha > 0$ if there is a finite vertex set S such that every component of $G - S$ has rank smaller than α. It is not hard to see that the graphs that have a rank are precisely the rayless ones.

The rank function makes the class of rayless graphs accesible to induction proofs, and indeed we use it to prove Theorem 2 by transfinite induction. In fact, we prove the stronger statement that every rayless graph G is *pre-partitionable*, that is, for every subset S of its vertices (possibly empty) and every partition of S, this partition can be extended to a partition of all of $V(G)$ which is unfriendly on $V(G) \setminus S$.

Theorem 3. *Every rayless graph is pre-partitionable.*

[1]A graph is *rayless* if it does not contain a (one-way) infinite path.

In the proof of Theorem 3, it turns out that we do not really need the graph to be rayless—all we need are the following properties of the rank function:

- Every graph of rank 0 is pre-partitionable.
- The disjoint union of finitely many graphs of rank at most α has rank at most α.
- Adding at finitely many vertices to a graph (and connecting them arbitrarily) does not increase the rank.

For rank $\alpha > 0$, the latter two properties are simple consequences of the recursive definition of the rank function, for rank 0 they are an obvious property of the class of finite graphs. Inspired by this observation, call a class of graphs *finitely closed* if it is closed under taking finite disjoint unions and adding finitely many vertices. Every finitely closed class of graphs can be used to define a rank function: Let the graphs of rank 0 be the graphs in our finitely closed class and define rank $\alpha > 0$ as above.

Theorem 4. *If $\mathcal{U}$ is a finitely closed class of pre-partitionable graphs, then every graph with a rank (respect to $\mathcal{U}$) is pre-partitionable (and hence has an unfriendly partition).*

Examples for finitely closed classes of pre-partitionable graphs are the class of graphs with only finitely many vertices of infinite degree or the class of countable graphs with only finitely many vertices of finite degree.

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Limits of trees

Gábor Tardos
(joint work with Gábor Elek)

Consider the following sampling procedure for finite trees: uniformly select k edges of the tree and contract all other edges. The sampling procedure defines a distribution on k-edge trees and naturally (analogously to the work of Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi on limits of dense graphs) defines convergence. We call a sequence of trees with sizes tending to infinity *convergent* if the distributions on k-samples are convergent for any k. As limit objects we get compact real trees (T, d) equipped with a probability Borel measure μ on T satisfying that for any points $x, y \in T$ for their distance $d(x, y)$ we have $d(x, y) \leq \mu([x, y])$ where $[x, y]$ denotes the unique path in T connecting x and y. We call such a triple (T, d, μ) a *limit tree*. We identify a (finite, graph theoretic) tree T with n edges with the limit tree (T', d, μ) where (T', d) is obtained from T by replacing the edges with intervals of length $1/n$ connecting the corresponding vertices and μ is the uniform measure.

A point x in a real tree (T, d) is called an *endpoint*, a *regular point*, or a *branching point*, if $T \setminus \{x\}$ has one, two, respectively more connected components. Note that the set of branching points in a compact real tree is countable. For a real tree $\mathcal{T} = (T, d, \mu)$ we define the probability space $\mathcal{T}^* = (T \times \{0,1\}, \mu^*)$ such that μ^* is concentrated on $T \times \{0\} \cup R \times \{1\}$, where R is the set of regular points in (T, d) and such that μ^* satisfies $\mu^*(H \times \{0,1\}) = \mu(H)$ for any Borel $H \subseteq T$ and $\mu^*([x, y] \times \{1\}) = d(x, y)$. We define *separation* among points in $T \times \{0,1\}$ as follows: (x, b) separates (x', b') from (x'', b'') if $b = 1$ and x separates y from z in the real tree (T, d).

We define k-sampling of a limit tree $\mathcal{T} = (T, d, \mu)$ using the probability space $\mathcal{T}^*$. We start by selecting k independent uniform samples from that space corresponding to the edges of the k-sample and build a tree from these edges such that the edge separation in the sample tree is the same as the separation among the random points in $\mathcal{T}^*$. The sample tree is well defined if whenever $(x, 1)$ is among our sample points we do not have another sample point of the form $(x, 0)$ or $(x, 1)$. This is satisfied with probability one. The k-sampling of (finite) trees and the corresponding limit trees results in identical distributions if we allow repetitions in the case of finite trees (and are close to each other otherwise).

We define the *separation edit distance* between the two limit trees T_1 and T_2 as the infimum of the following probability for all couplings C between T_1^* and T_2^*. Here a coupling is a distribution on pairs (u, v) where u is distributed according to T_1^* and v is distributed according to T_2^* and the probability we want to minimize is that of u_1 separating u_2 from u_3 without v_1 separating v_2 from v_3 or the opposite happening, where (u_1, v_1), (u_2, v_2) and (u_3, v_3) are independent random samples from C.

For a finite tree T one can define a directed three uniform hypergraph whose vertices are the edges of T and the hyperedges describe separation. For finite trees T_1 and T_2 of equal size the separation edit distance between the corresponding limit trees is close to the normalized edit distance between the corresponding hypergraphs with the best overlay of their vertex set.

Theorem 1. The Gromov-Prohorov distance and the separation edit distance define the same topology on limit trees. A sequence of finite trees of growing size is convergent if and only if the sequence of the corresponding limit trees is convergent in this topology.

Theorem 2. For any limit tree $\mathcal{T}$ the sequence of its k-samples form a convergent tree sequence converging to $\mathcal{T}$ with probability one.

Another sampling procedure we consider for a finite tree T is as follows: take

k uniform random vertices of T and label them with distinct labels, then take the smallest subtree of T containing all labeled vertices, finally replace all maximal paths whose internal vertices are unlabeled and have degree two with edges. We prove that this yields a strictly finer convergence structure and we prove similar results on this structure in terms of global distance and limit objects.

Four-critical graphs on surfaces

Robin Thomas
(joint work with Zdeněk Dvořák and Daniel Král')

A graph is 4-critical if every proper subgraph of G is 3-colorable, but G itself is not.

THEOREM. There exists an absolute constant K such that if G is a 4-critical graph in a surface of Euler genus g drawn with no homotopically non-trivial cycles of length at most four and t is the number of triangles in G, then G has at most $K(g+t)$ faces of size at least five, each of size at most $K(g+t)$.

This has several consequences. When applied to graphs of girth at least five it implies a theorem of Thomassen stating that there are only finitely many 4-critical graphs of girth at least five on any given surface. We have a linear bound on the size of such graphs.

A second corollary is a proof of correctness of our linear-time algorithm to decide whether a triangle-free graph drawn in a fixed surface is 3-colorable.

A third consequence is a solution of a problem of Havel from 1969: there exists an absolute constant d such that if G is a planar graph and every two triangles in G are at distance at least d, then G is 3-colorable. For this application it is crucial that the bound in our theorem is linear in t.

A fourth consequence is a theorem of Kawarabayashi and Thomassen: there exists an absolute constant c such that if G is a triangle-free graph drawn in a surface of Euler genus g, then there exists a set $X \subseteq V(G)$ of size at most cg such that $G \backslash X$ is 3-colorable.

Fifth, it follows that for every orientable surface Σ there exists an integer ρ such that every triangle-free graph that can be drawn in Σ with representativity at least ρ is 3-colorable. This generalizes a theorem of Hutchinson, who proved the same for graphs with all faces even.

Finally, there is a version of the last result for non-orientable surfaces if one assumes that G has no subgraph that quadrangulates Σ.

Coloring dense graphs via VC-dimension

Stéphan Thomassé
(joint work with Tomasz Łuczak)

The *chromatic number* of a graph $G = (V, E)$, i.e. the minimum number of parts of a partition of its vertex set into edgeless subsets (*stable sets*), is one of the most studied parameters in graph theory. However, this parameter cannot be directly interpreted as a measure of the complexity of G since very simple graphs, like *cliques* K_n of size n inducing all possible edges, have chromatic number n.

The picture becomes completely different when a graph G has large chromatic number for non obvious reasons like the containment of a large clique. For instance, in the case of triangle-free graphs, achieving high chromatic number is not a straightforward exercise. Indeed this simple question is the starting point of several areas like random and topological graphs.

Specifically some constructions of triangle-free graphs with high chromatic number were provided first by Zykov [15], and then by Mycielski [11]. Erdős [4] proposed a construction based on random graphs with arbitrarily large girth. Geometric constructions based on the Borsuk-Ulam theorem provide examples with arbitrarily large odd girth. However, all graphs constructed in the above way are sparse, i.e. have small minimum degree with respect to the number of vertices.

In their seminal paper, Erdős and Simonovits [5] asked for a bound on the chromatic number of triangle-free graphs with minimum degree larger than $n/3$. This question was first solved by Thomassen in [12] where he provided a bound on the chromatic number when the degree is larger than $(1/3 + \varepsilon)n$, for every $\epsilon > 0$. This result was sharpened by Łuczak [10] who showed that, up to homomorphism, there are only finitely many maximal triangle-free graphs with minimum degree larger than $(1/3 + \varepsilon)n$. The finiteness relies on the partition provided by the regularity lemma. Finally, Brandt and Thomassé [3] proved that all such graphs have chromatic number at most four, using a complete characterization of the family.

Our goal is to show how Vapnik-Červonenkis theory can be used for the Erdős-Simonovits type problems. The three main results of this paper are the following.

- We give a new short proof of the existence of a bound on the chromatic number of triangle-free graphs with minimum degree larger than $n/3$. This direct consequence of classical VC-dimension even allows to break the $1/3$ barrier. For instance, it provides a bound for minimum degree $n/3$ minus a constant. This was completely out of reach using the previous methods of [3], [10], and [12].
- Introducing a new parameter, the *paired VC-dimension*, we characterize the graphs H such that the class of H-free graphs (with respect to homomorphism) has chromatic threshold 0 (i.e. have bounded chromatic number as soon as the minimum degree is larger than cn, where $c > 0$). From this characterization follows that the chromatic threshold of H-free

graphs is either 0 or at least 1/3. For instance 0 is achieved by the pentagon, 1/3 by the triangle, but no value in between can be realized.
- Thomassen [13] recently proved that the class of pentagon-free graphs (with respect to subgraph) has chromatic threshold 0. We give a new proof of this result based on paired VC-dimension. Using our method we construct a wide class of non-bipartite graphs (which includes graph without a copy of the Petersen graph) for which the threshold is also 0.

Our central theorem gives an upper bound on the chromatic number of a graph in terms of its minimum degree and paired VC-dimension. We believe that this result can be both generalized and sharpened (the implicit bounds for the chromatic number which follow from our argument are rather poor). Another key observation is the "duality" between VC-dimension, which provides upper bounds on the chromatic number of dense graphs, and the Borsuk-Ulam theorem, which gives constructions achieving lower bounds.

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The probability of a hereditary graph property

Andrew Thomason
(joint work with Edward Marchant)

A graph property $\mathcal{P}$ is *hereditary* if, whenever a graph G has $\mathcal{P}$ and F is an induced subgraph of G, then H also has $\mathcal{P}$. The property is *monotone* if it is closed further under the taking of any subgraph (not just induced): hence a monotone property is also hereditary. For example, the properties of "being triangle-free" and of "being 4-colourable" are both monotone, whereas the property of having no induced 5-cycle is hereditary but not monotone.

Given a class $\mathcal{H}$ of graphs, the property Forb($\mathcal{H}$) comprises those graphs containing no induced subgraph isomorphic to a member of $\mathcal{H}$. Note that Forb($\mathcal{H}$) is hereditary for every $\mathcal{H}$ and that, conversely, every hereditary property $\mathcal{P}$ is of the form Forb($\mathcal{H}$) for some class $\mathcal{H}$ (just let $\mathcal{H}$ consist of those graphs not having $\mathcal{P}$). If $\mathcal{H}$ contains just a single graph then we call the property Forb($\mathcal{H}$) *principal.* So for example the property of having no induced 5-cycle is principal.

Following a long tradition of the study of the number of graphs of order n having a given monotone property, Prömel and Steger [6] showed that the number of labelled graphs on vertex set $\{1, \ldots, n\}$ having the principal property $\mathcal{P}$ is $2^{(1-1/t+o(1))\binom{n}{2}}$, where $t = t(\mathcal{P})$ is defined as follows. Given non-negative integers a and b , let $\mathcal{P}(a, b)$ be the class of all graphs whose vertices can be partitioned into a complete graphs and b independent sets. Then $t(\mathcal{P})$ is the maximum value of $a + b$ such that $\mathcal{P}(a, b) \subseteq \mathcal{P}$. The same result was proved for general hereditary properties by Alexseev [1] and by Bollobás and Thomason [2].

The theorem of Prömel and Steger is equivalent to stating that if a graph of order n is generated by choosing its edges independently at random with probability 1/2 then the probability of the graph having property $\mathcal{P}$ is $2^{(-1/t+o(1))\binom{n}{2}}$. Bollobás and Thomason [3] investigated the probability of having $\mathcal{P}$ if the edges are chosen with constant probability p. They noted that this probability is of the form $2^{-c_p(\mathcal{P})(1+o(1))\binom{n}{2}}$ for some constant $c_P(\mathcal{P})$ depending on p and $\mathcal{P}$, but that if $p \neq 1/2$ then the properties $\mathcal{P}(a, b) \subseteq \mathcal{P}$ do not determine $c_p(\mathcal{P})$.

To overcome the inadequacy of the properties $\mathcal{P}(a, b)$ they introduced a more general class of properties. A *type* τ is a complete graph, each of whose vertices is coloured either red or blue, and each of whose edges is coloured either red, blue or green. The *elementary property* $\mathcal{P}(\tau)$ consists of all graphs G for which $V(G)$ has a partition witnessing τ, which is to say, a partition $\{V_v : v \in V(\tau)\}$ such that $G[V_v]$ is complete or empty according as v is red or blue, and such that the bipartite graph $G[V_u, V_v]$ is complete or empty according as uv is red or blue. In the case that uv is green, there is no restriction on $G[V_u, V_v]$. It follows that the property $\mathcal{P}(\tau)$ is hereditary. If τ has a red and b blue vertices, and all its edges are green, then $\mathcal{P}(\tau) = \mathcal{P}(a, b)$.

If $\mathcal{P}(\tau) \subseteq \mathcal{P}$ then evidently $c_p(\mathcal{P}(\tau)) \geq c_p(\mathcal{P})$. In [3] it is proved that the properties $\mathcal{P}(\tau)$ determine $c_p(\mathcal{P})$ in the following weak sense: for every $\epsilon > 0$ there is a type τ with $\mathcal{P}(\tau) \subseteq \mathcal{P}$ and $c_p(\mathcal{P}) \leq c_p(\mathcal{P}(\tau)) \leq c_p(\mathcal{P}) + \epsilon$. However, an example was put forward to show that there might be no τ with $\mathcal{P}(\tau) \subseteq \mathcal{P}$ and $c_p(\mathcal{P}) = c_p(\mathcal{P}(\tau))$.

It turns out, though, that this example is incorrect (as was pointed out by Uri Stav). In fact, it is now possible to say much more about the probability of a hereditary property $\mathcal{P}$, such as the following statements.

(1) There always exists some τ with $\mathcal{P}(\tau) \subseteq \mathcal{P}$ and $c_p(\mathcal{P}) = c_p(\mathcal{P}(\tau))$.
(2) For some properties $\mathcal{P}$ there are infinitely many such τ that are "minimal" (i.e. not containing some τ' with $c_p(\mathcal{P}) = c_p(\mathcal{P}(\tau'))$.
(3) There are "non-compact" properties $\mathcal{P} = \mathrm{Forb}(\mathcal{H})$, such that if $\mathcal{H}'$ is any finite subset of $\mathcal{H}$ and if $\mathcal{P}' = \mathrm{Forb}(\mathcal{H}')$, then $c_p(\mathcal{P}) < c_p(\mathcal{P}')$.
(4) The structure of the relevant types τ is of a restricted kind and hence it is sometimes possible to actually evaluate $c_p(\mathcal{P})$.

These facts about $c_p(\mathcal{P})$ emerged somewhat unexpectedly from a study by Marchant and Thomason [4] of extremal functions for weighted 2-coloured graphs. A 2-coloured graph H is a graph whose edges are coloured red, blue or green. It is "contained" in another 2-coloured graph G if it is a subgraph in the obvious way, except that green edges of G can contain edges of H of any colour (red, blue or green). We call G *complete* if its underlying graph is complete: thus G is the same as a type except that its vertices are not coloured. In particular a complete green graph of order n contains every 2-coloured graph of order at most n.

If we weight the edges of G with weight p for a red edge, $q = 1 - p$ for a blue edge and 1 for a green edge then we write $w_p(G) = pe(G_r) + qe(G_b)$ for the total edge weight. The extremal problem for a class $\mathcal{H}$ of 2-coloured graphs is then to determine the function

$$\kappa_p(\mathcal{H}) = \lim_{n\to\infty} \max \left\{ \frac{w_p(G)}{\binom{n}{2}} : |G| = n,\ G \text{ complete},\ H \not\subset G \text{ for all } H \in \mathcal{H} \right\}.$$

The reasons for defining $\kappa_p(\mathcal{H})$ in this way, and the motivation for studying it, are set out in [4].

Given a type τ we say that a 2-coloured graph H is *τ-colourable* if $V(H)$ has a partition $\{V_v : v \in V(\tau)\}$, such that any edges in $H[V_v]$ are the same colour as v, and such that if uv is red or blue then any edges in the bipartite graph $H[V_u, V_v]$ are the same colour. If the edge uv is green then there is no restriction on the colours of edges in $H[V_u, V_v]$. It is easy to verify that the maximum weight of a τ-colourable graph of order n is approximately $\lambda_p(\tau)\binom{n}{2}$, where $\lambda_p(\tau)$ is the maximum value of the quadratic form

$$\sum_v x_v^2 w(v) + 2\sum_{uv} x_u x_v w(uv)$$

over all $\mathbf{x} \in [0,1]^{V(\tau)}$ with $\sum_v x_v = 1$.

There is a close relationship between hereditary properties and 2-coloured graphs, with edges and non-edges in hereditary properties corresponding to red and blue edges of 2-coloured graphs. In this relationship, the property $\mathcal{P}(\tau)$ corresponds to the class of τ-colourable graphs. In this context, the statements 1–4 above correspond to similar statements for 2-coloured graphs proved in [4]. More precisely, statement 1 comes from Theorem 3.25 of [4], which states that for every class $\mathcal{H}$ of 2-coloured graphs and for $0 < p < 1$, there exists a τ such that no graph in $\mathcal{H}$ is τ-colourable and $\lambda_p(\tau) = \kappa_p(\mathcal{H})$. Statement 2 follows from Theorem 3.27. Statement 3, coming from Theorem 3.25 of [4], is that in a minimal type τ all edges must be green, except that if $p < 1/2$ then some edges joining two red vertices might be blue, or if $p > 1/2$ then some edges joining two blue vertices might be red.

Finally, an illustration of statement 4 is that if $\mathcal{P}$ is the property of containing no induced subgraph isomorphic to a six-cycle with a single diagonal, then

$$\begin{aligned} \text{For } p \leq 1/2,\ c_p(\mathcal{P}) &= -\left[\frac{\log p + \log q}{\log p + 3\log q}\right] \log_2 q\ . \\ \text{For } p \geq 1/2,\ c_p(\mathcal{P}) &= -\frac{1}{2}\log_2 p\ . \end{aligned}$$

The relationship between 2-coloured graphs and hereditary properties is described in [5]. The more interesting of the general theorems about hereditary graph properties in [4] are parallel to extremal theorems about multigraphs due to Brown, Erdős and Simonovits, and to counterexamples in the theory of multigraphs due to Rödl and Sidorenko.

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Is The Missing Axiom of Matroid Theory Lost Forever?

Geoff Whittle

Evidence is accumulating that for any given finite field $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids is well behaved. In particular it appears likely that the class is well-quasi-ordered under the minor order; that any minor-closed property can be recognised in polynomial time; and that Rota's Conjecture holds in that $\mathbb{F}$-representability can be characterised by a finite number of forbidden minors.

In the talk analogous problems for infinite fields are considered; the general theme being that none of the attractive properties for matroids representable over finite fields extend to the infinite case. If $\mathbb{F}$ is an infinite field, then the class of $\mathbb{F}$-representable matroids is not well-quasi-ordered, members of the class cannot be recognised in polynomial time, and there are an infinite number of excluded minors for matroids representable over $\mathbb{F}$. Even worse, it is shown in [1] that for every $\mathbb{F}$-representable matroid M there is an excluded minor for $\mathbb{F}$-representability that has M as a minor.

A question raised by Whitney was also discussed. In his seminal paper [2], the question of finding a satisfactory axiomatisation of real-representable matroids was raised. In other words, is it possible to add extra axioms to the matroid axioms to characterise the class of matroids representable over the reals? This question was addressed in a 1976 paper of Vámos [3] entitled "The missing axiom of matroid theory is lost forever" where he proves that, using a certain logic, it is impossible to add a finite number of axioms to the matroid axioms to characterise real representability. This led to a widespread belief within the matroid community that the problem was hopeless. However Vámos' logic is extremely weak and matroids themselves cannot be characterised with a finite number of axioms in this logic. The question as to whether real representability can be characterised using a finite number of axioms in a more natural logic for matroids is still entirely open.

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A shorter proof of the unique linkage theorem

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(joint work with Ken-ichi Kawarabayashi)

The theory of graph minors developed by Robertson and Seymour has had a wide range of consequences beyond the proof of Wagner's Conjecture that graphs are well quasi ordered under the minor operation. The main algorithmic result of this theory is a polynomial-time algorithm for testing the existence of a fixed minor [2]. An immediate corollary of well quasi ordering is that every minor closed class has a finite list of forbidden minors. Together, these two results yield a polynomial time algorithm to test membership in any fixed minor closed class. The development of the algorithm includes a polynomial time algorithm for the k disjoint paths problem for fixed values of k, an interesting problem in its own right. The existence of such a polynomial time algorithm has in turn been used to show the existence of efficient algorithms for several other graph problems, and also leads to the framework of parameterized complexity developed by Downey and Fellows [1], which is perhaps one of the most active areas in the study of algorithms.

The proof of the algorithmic result hinges upon the Unique Linkage Theorem. We recall that a *linkage* is a graph where every component is a path. The *order* of a linkage is the number of components. We say two linkages $\mathcal{P}$ and $\mathcal{P}'$ are *equivalent* if they have the same order and for every component P of $\mathcal{P}$, there exists a component P' of $\mathcal{P}'$ such that P and P' have the same endpoints.

Theorem 1 (**The Unique Linkage Theorem** [4]). *For all $k \geq 1$, there exists a value $w(k)$ such that the following holds. Let $\mathcal{P}$ be a linkage of order k in a graph G with $V(G) = V(\mathcal{P})$. If there does not exist a linkage $\mathcal{P}'$ in G which is equivalent to $\mathcal{P}$ and uses strictly fewer vertices than $\mathcal{P}$, then the tree-width of G is at most $w(k)$.*

The original proof of the Unique Linkage Theorem, given by Robertson and Seymour [4], requires the full power of the theory of graph minors. Specifically, the proof uses the Structure Theorem [3, Theorem 1.3], which lies at the center of the theory of graph minors. The theorem describes the general structure of all graphs excluding a fixed graph as a minor. At a high level, the theorem says that every such graph can be decomposed into a collection of graphs each of which can "almost" be embedded into a bounded-genus surface, combined together in a tree structure. Much of the Graph Minors series of articles is devoted to the proof of this structure theorem.

Robertson and Seymour, however, predicted that there exists a simpler proof of the Unique Linkage Theorem which avoids the use of the Structure Theorem. Our main contribution is to confirm that they are right – we provide such a short proof. In fact, our proof is less than 30 pages, and gives rise to an explicit bound for the tree-width $w(k)$, while the original algorithm does not.

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