Mathematisches Forschungsinstitut Oberwolfach

Report No. 20/2010

DOI: 10.4171/OWR/2010/20

Non-positive Curvature and Geometric Structures in Group Theory

Organised by
Martin Bridson, Oxford
Linus Kramer, Münster
Bertrand Remy, Lyon
Karen Vogtmann, Cornell

April 25th – May 1st, 2010

Abstract. The focus of this meeting was the use of geometric methods to study infinite discrete groups. Key topics included isometric actions of such groups on spaces of nonpositive curvature, such as CAT(0) cube complexes, buildings, and hyperbolic or symmetric spaces. These actions lead to a rich and fruitful interplay between geometry and group theoretic questions.

Mathematics Subject Classification (2000): 20Fxx, 57Mxx.

Introduction by the Organisers

The meeting focused on several areas of current excitement in geometric group theory, unified by the important role that non-positive curvature plays in each. The geometric approach to group theory dominates the modern study of finitely generated groups. A central idea in this approach is that one illuminates the nature of groups by studying their actions on spaces with appropriate geometric structure. The quality of information one gleans from the action depends on the richness of the geometric structure and the quality of the action (discrete and cocompact by isometries being the most desirable). A powerful illustration of this is provided by the study of isometric actions on spaces of non-positive curvature. The curvature hypothesis alone tells one a great deal about the algebraic structure of the group, but the theory becomes much richer when one imposes further hypotheses on the space. Prime illustrations of this are the the theory of buildings (J. Tits) and, most classically, the actions of discrete subgroups of semi-simple Lie groups on Riemannian symmetric spaces (É. Cartan).

The topics covered during this workshop can each be seen as a natural extension of an aspect of this last beautiful subject: rigidity, fixed point theorems, questions of linearity and residual finiteness, analysis at infinity, cohomological issues, *etc.* The diverse techniques involved in the topics that emerge under these headings typically lie far from these classical origins, and the spaces that arise are typically highly singular — buildings, CAT(0) cube complexes, asymptotic cones, the curve complex and other spaces related to Teichmüller space, Outer Space, *etc.* But the classical situation still provides a stimulating analogy.

This diversity within a common framework was widely reflected in the speakers of the workshop. We concentrated on specific topics that have seen recent exciting progress. These include: the study of new classes of buildings, of CAT(0) cube complexes, lattices in the isometry groups of the latter spaces and related embedding results; recent insights into the nature of mapping class groups of surfaces and automorphism groups of free groups; recent definitive results on the nature of the full isometry groups of CAT(0) spaces that admit parabolics; and the introduction of powerful new tools of an analytic nature. More details can be seen in the individual abstract below. We had 55 participants from a wide range of countries, and 23 lectures. In addition, there were two special sessions in the evening, with lectures by Arthur Bartels on the recent proof of the Farrell-Jones Conjecture for hyperbolic and CAT(0) groups and by Mark Sapir on conjugacy growth in groups.

The staff in Oberwolfach was—as always—extremely supportive and helpful. We are very grateful for the additional funding for five young PhD students and recent postdocs through Oberwolfach-Leibniz-Fellowships. In addition, there was one young student funded through the DMV Student's Conference. We think that this provided a great opportunity for these students.

We feel that the meeting was exciting and highly successful. The quality of all lectures was outstanding, and outside of lectures there was a constant buzz of intense mathematical conversations. We are confident that this conference will lead to both new and exciting mathematical results and to new collaborations.

Workshop: Non-positive Curvature and Geometric Structures in Group Theory

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Abstracts

Divergence and quasimorphisms of right-angled Artin groups

Ruth Charney
(joint work with Jason Behrstock)

Let X be a geodesic metric space and let ρ be a linear function. The *divergence function* of a geodesic α with respect to ρ is the function $\mathrm{div}_\rho\,\alpha\,(r)$ defined as the length of the shortest path from $\alpha(-r)$ to $\alpha(r)$ that stays outside the ball of radius $\rho(r)$ about $\alpha(0)$. The divergence of the space X is defined by $\mathrm{div}_\rho\,X\,(r) = \sup\{\mathrm{div}_\rho\,\alpha\,(r)\}$ where the supremum is taken over all geodesics α.

In this project we consider the divergence of geodesics in right-angled Artin groups. Given a finite, simplicial graph Γ, the *right-angled Artin group* A_Γ is the finitely presented group with generators corresponding to vertices of Γ and relators $[x, y] = 1$ whenever the vertices x and y are adjacent in Γ. Right-angled Artin groups form a rich family of groups interpolating between $\mathbb{Z}^n$, the group corresponding to the complete graph on n vertices, and the free group F_n, corresponding to the graph with n vertices and no edges.

If Γ is disconnected, then A_Γ is a free product of two infinite groups and the divergence in A_Γ is infinite. If Γ_1 and Γ_2 are two graphs, their *join* is the graph J obtained by connecting every vertex of Γ_1 to every vertex of Γ_2 by an edge. The Artin group A_J associated to a join is the direct product of two infinite groups, $A_{\Gamma_1} \times A_{\Gamma_2}$, hence it has linear divergence. It is not surprising then, that for any graph Γ, subgraphs that decompose as joins are central to our understanding of divergence of geodesics in A_Γ.

We define a notion of *join length* of an element $g \in A_\Gamma$, as follows.

$$\ell_J(g) = \min\{k \mid g = a_1 \dots a_k \text{ where } a_i \text{ lies in } A_J \text{ for some join } J \subset \Gamma\}.$$

For a bi-infinite geodesic word α in A_Γ, we say α has *finite join length* if the join lengths of its finite segments are bounded by a constant. We prove

Theorem 1. *Let Γ be a connected graph and let α be a bi-infinite geodesic in A_Γ. Then α has linear divergence if and only if it has finite join length.*

The proof uses the action of A_Γ on a CAT(0) cube complex, X_Γ, whose 1-skeleton is the Cayley graph of A_Γ. We show that the join length of a geodesic edge path g determines the behavior of the walls in X_Γ crossed by g. Two walls H_1, H_2 in X_Γ are said to be *strongly separated* if no wall of X_Γ intersects both. Define the *separation length* of g to be

$$\ell_S(g) = \max\{k \mid g \text{ crosses a sequence of } k \text{ strongly separated walls}\}.$$

We give a group theoretic characterization of when two walls are strongly separated from which it follows that the join length and separation length of a bi-infinite geodesic grow at the same rate.

We show that two strongly separated walls are connected by a unique minimal geodesic (called the bridge) and the distance between points on these walls diverges linearly with the distance from the bridge. It follows that the divergence function of a bi-infinite geodesic α is bounded below by a linear function of r times the separation length, or equivalently the join length, of the segments $[\alpha(-r), \alpha(r)]$.

Moreover, if Γ is connected, any two join subgroups can be connected by a sequence (of uniformly bounded length) of join subgroups in which any two consecutive subgroups have infinite intersection. Using the fact that divergence in each join subgroup is linear, it follows that the divergence function of α is bounded above by a linear function times its join length. We conclude

Corollary 2. *Let Γ be a connected graph. A_Γ has linear divergence if and only if Γ is a join; otherwise its divergence is quadratic.*

Divergence gives information about the large scale geometry of a group. By the work of Drutu, Mozes and Sapir [DS, DMS], divergence of geodesics in a group is closely related to cut points in asymptotic cones. As an application of our results, we obtain a complete characterization, for any asymptotic cone of A_Γ, of when two points can be separated by a cut-point.

A second application of divergence is to produce quasimorphisms of A_Γ. A function $\phi\colon G \to \mathbb{R}$ is a *homogeneous quasimorphism* if $\phi(g^n) = n\phi(g)$ for all $n > 0$, and there exists a constant $D \geq 0$ such that

$$|\phi(gh) - \phi(g) - \phi(h)| \leq D$$

for every $g, h \in G$. The vector space of homogeneous quasimorphisms, modulo the subspace of true homomorphisms, is denoted $\widetilde{QH}(G)$ and is related to the bounded cohomology of G.

Bestvina and Fujiwara [BF2] have shown that for (nice) group actions on a CAT(0) space, rank-one isometries (i.e. hyperbolic isometries whose axes do not bound a half-flat) give rise to non-trivial quasimorphisms. Since geodesics with super-linear divergence cannot bound a half-flat, we obtain the following.

Theorem 3. *Let Γ be an arbitrary graph.*

(1) If $G \subseteq A_\Gamma$ is any subgroup which is not contained in a conjugate of a join subgroup, then G contains a rank-one isometry of X_Γ.
(2) If G is any non-abelian subgroup of A_Γ, then $\widetilde{QH}(G)$ is infinite dimensional.

We remark that these results can also be derived from the work of of Bestvina-Fujiwara [BF1] or Caprace and Fujiwara [CF] by embedding the right-angled Artin group into a mapping class group or a right-angled Coxeter group.

Finally, from the work of Burger and Monod on nonexistence of quasimorphisms of higher rank lattices [BM1, BM2, Mon1], we conclude

Corollary 4. *If Λ is an irreducible lattice in a connected semisimple Lie group with finite center, no compact factors, and rank at least 2, then every homomorphism $\rho\colon \Lambda \to A_\Gamma$ is trivial.*

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Finiteness properties of S-arithmetic groups over global function fields

Ralf Gramlich

(joint work with Kai-Uwe Bux, Stefan Witzel)

The purpose of this note is to sketch how, for an S-arithmetic group Γ, Harder's reduction theory allows one to define a Γ-invariant Γ-cocompact Morse function on the appropriate product X of affine buildings with uniquely determined gradients whose sublevel sets locally look like the intersection of the complements of finitely many horoballs, and to indicate some applications.

Let K be a global function field over $\mathbb{F}_q$, let G be a reductive K-isotropic algebraic K-group considered as a subgroup of some matrix group $GL_m(K)$, let S be a finite non-empty set of places of K, let $X = \prod_{s\in S} X_s$ be the product of the affine buildings of the groups $G(K_s)$, and let Γ be the S-arithmetic lattice in $\prod_{s\in S} G(K_s)$.

Definition 1. Let P be a maximal K-parabolic subgroup of G and let $x = (x_s)_{s\in S} \in X$ be a tuple of special vertices $x_s \in X_s$. In analogy to [Har69, Section 1.3] define

$$\pi(P,x) := \int_{R_u(P(\mathbb{A}_K))\cap G(\mathbb{A}_K)_x} \omega,$$

where ω denotes a non-trivial volume form on $R_u(P)$ defined over K. This Haar integral does not depend on the choice of the volume form ω; cf. [Wei82, Section 2.4].

Theorem 2 ([Har69, Satz 1.3.2], [Har77, 1.4.1])**.** *Let P be a maximal K-parabolic and let $\chi\colon P \to G_{\mathbf{m}}$ be the character "sum of roots" (*[Har69, p. 39, Section 1.3]*).*

For each tuple of special vertices $x \in X$ and each $g \in \prod_{s \in S} P(K_s)$ one has

$$\log_q(\pi(P, gx)) = \log_q(\pi(P, x)) + \log_q(|\chi(g)|_S^{-1}) = \log_q(\pi(P, x)) + \sum_{s \in S} s(\chi(g)).$$

In particular, there exists a Busemann function $p(P, \cdot)\colon X \to \mathbb{R}$ whose restriction to the set of tuples of special vertices of X equals $\log_q(\pi(P, \cdot))$.

Let B be a minimal K-parabolic subgroup of G, let $R(B)$ be the radical of B, let $R_u(B)$ be the unipotent radical of B, let $T = R(B)/R_u(B)$, let $S \subseteq T$ be the maximal K-split subtorus of T, let $\pi = \{\alpha_1, ..., \alpha_r\} \subset X(S)$ be the system of simple roots, and let $X(B) = \mathrm{Hom}_K(B, G_{\mathbf{m}})$ be the module of K-rational characters of B so that $X(B) \otimes \mathbb{Q} = X(S) \otimes \mathbb{Q}$, let $P_i \supseteq B$ be the maximal parabolic of type $\pi - \{\alpha_i\}$, let $\chi_{P_i}\colon P_i \to G_{\mathbf{m}}$ be the sum of roots of P_i, and let $\chi_i := \chi_{P_i|B}$.

The χ_i form a basis of $X(B) \otimes \mathbb{Q}$ and, if $(\cdot, \cdot)$ is a positive definite bilinear form on $X(B) \otimes \mathbb{Q}$ which is invariant under the action of the Weyl group, then $(\chi_i, \alpha_j) = 0$, if $i \neq j$, and $(\chi_i, \alpha_i) > 0$ for all i.

Definition 3. Let $\alpha_1, ..., \alpha_r$ and $\chi_1, ..., \chi_r$ be as above and let $c_{i,j} \in \mathbb{Q}$ such that $\alpha_i = \sum_{j=1}^r c_{i,j}\chi_j$. In analogy to [Har69, p. 47] define

$$\begin{aligned} p_i(B, x) &:= p(P_i, x), \\ n_i(B, x) &:= \sum_{j=1}^r c_{i,j} p_j(B, x). \end{aligned}$$

Let $(c_i)_{1 \leq i \leq r}$ be a family of positive real numbers such that each $\overline{p}_i(B, \cdot) := c_i p_i(B, \cdot)$ is a Busemann function with respect to a unit speed geodesic.

Theorem 4 ([Har69, Satz 2.3.2], [Har77, 1.4.2])**.** *There exists $C_1 \in \mathbb{R}$ such that for each $x \in X$ there exists a minimal K-parabolic B with $n_i(B, x) \geq C_1$ for all $1 \leq i \leq r$.*

Theorem 5 ([Har68, Satz 2.2.13], [Har69, Satz 2.1.2], [Har69, Satz 2.3.3])**.** *There exist constants $C_2 > \gamma > C_1$ such that for $x \in X$, a minimal K-parabolic B with $n_i(B, x) \geq C_1$ for all $1 \leq i \leq r$, the family $(P_i)_{1 \leq i \leq r}$ of maximal K-parabolic subgroups of G containing B, and $j \in \{1, ..., r\}$ with $n_j(B, x) \geq C_2$, each minimal K-parabolic subgroup B' of G with $n_i(B', x) \geq C_1$ for all $1 \leq i \leq r$*

(1) satisfies $n_j(B', x) \geq \gamma$, and
(2) is contained in P_j.

Definition 6. A pair (B, x) consisting of a minimal K-parabolic subgroup B of G and an element $x \in X$ such that $n_i(B, x) \geq C_1$ for all $i \in I$ is called **reduced**.

Let B be a minimal K-parabolic subgroup of G and let $P_j \supseteq B$ be a maximal parabolic. An element $x \in X$ is called **close to the boundary of X with respect to** P_j, if (B, x) is a reduced pair and $n_j(B, x) \geq C_2$.

An element $x \in X$ is called **close to the boundary of** X, if there exists a maximal K-parabolic P such that x is close to the boundary of X with respect to P.

Let $x \in X$ be close to the boundary of X. Define

$$P_x := \bigcap \{P \subset G \mid x \text{ is close to the boundary of } X \text{ with respect to } P\}.$$

By Theorem 5 the group P_x is a K-parabolic subgroup of G. It is called the **isolated parabolic subgroup of G corresponding to x.**

Proposition 7. *Let $x \in X$ be close to the boundary of X, let P_x be the corresponding isolated parabolic subgroup of G, and let γ be a geodesic ray in X with $\gamma(0) = x$ and whose end point lies in the simplex of the building at infinity corresponding to P_x. Then each $y \in \gamma([0,\infty))$ is close to the boundary of X, one has $P_y = P_x$, and for each minimal K-parabolic B the pair (B, y) is reduced if and only if (B, x) is reduced.*

Proposition 8 ([Har69, Satz 2.2.2], [Har77, 1.4.3], [Beh98, Section 2.4])**.** *For $C \in \mathbb{R}$ the filtrations*

$$\begin{aligned} X^n(C) &= \{x \in X \mid (B,x) \text{ reduced implies } n_i(B,x) \le C \text{ for all } 1 \le i \le r\}, \\ X^{\overline{p}}(C) &= \{x \in X \mid (B,x) \text{ reduced implies } \overline{p}_i(B,x) \le C \text{ for all } 1 \le i \le r\} \end{aligned}$$

are Γ-cocompact and Γ-invariant.

Observation 9 ([Beh98, Proposition 1])**.** *There exists a constant C_3 such that $\overline{p}_i(B,x) \ge C_3$ implies $n_i(B,x) \ge C_2$ for each reduced pair (B,x) and all $1 \le i \le r$. In particular, $X^n(C_2) \subseteq X^{\overline{p}}(C_3)$.*

Proposition 10. *Let $x \in X \backslash X^{\overline{p}}(C_3)$ and let $\overline{x} \in X^{\overline{p}}(C_3)$ an element at which the function $X^{\overline{p}}(C_3) \to \mathbb{R}\colon z \mapsto d(x,z)$ assumes a local minimum. Then $P_{\overline{x}} = P_x$.*

Moreover, the function $X^{\overline{p}}(C_3) \to \mathbb{R}\colon z \mapsto d(x,z)$ assumes a global minimum at $\overline{x}$. Furthermore, there exists a unique unit speed geodesic ray $\gamma_x\colon [0,\infty) \to X$ with $\gamma(0) = x$ along which the function $X \backslash X^{\overline{p}}(C_3) \to \mathbb{R}\colon x \mapsto d(x, X^{\overline{p}}(C_3))$ assumes its steepest ascent; its end point lies in the simplex at infinity corresponding to P_x.

Definition 11. Define $h\colon X \to \mathbb{R}\colon x \mapsto d(x, X^{\overline{p}}(C_3))$. For $x \in X \backslash X^{\overline{p}}(C_3)$ the unit speed geodesic ray γ_x from Proposition 10 is called the **flow line of h in x.** The **gradient** $\nabla_x h$ is defined as the direction of γ_x at x.

In case the global rank of G is 1, the sublevel sets of the Morse function h locally look like the complement of one horoball. Investigation of connectedness properties of horospheres in affine buildings yields the following result.

Theorem 12 ([BW, Theorem 1.2])**.** *If G is connected, noncommutative, absolutely almost simple of K-rank 1, then the finiteness length of Γ is $\left(\sum_{s\in S} \mathrm{rk}_{K_s}(G)\right) - 1$.*

If G is an absolutely almost simple $\mathbb{F}_q$-group of rank $n \ge 1$, then $G(\mathbb{F}_q[t,t^{-1}])$ admits a root group datum. Hence the theory of twinnings of Bruhat–Tits buildings, which is a special case of reduction theory, applies and allows one to prove the following result.

Theorem 13 ([BGW, Theorem A])**.** *If G is an absolutely almost simple $\mathbb{F}_q$-group of rank $n \ge 1$, then the finiteness length of $G(\mathbb{F}_q[t])$ equals $n - 1$.*

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A geometric construction of lattices in two-dimensional affine buildings

Jan Essert

In recent years, lattices in the automorphism groups of non-positively curved polyhedral complexes have been studied by various authors. A very interesting survey on current problems and open questions is [FHT09].

We are interested in lattices in the automorphism groups of two-dimensional locally finite affine buildings. It is well known that there are countably many so-called *classical* two-dimensional affine buildings associated to isotropic algebraic groups over local fields, but also uncountably many *exotic* two-dimensional buildings with potentially small automorphism groups.

Classically, one obtains lattices in the automorphism group of a classical building X by considering arithmetic lattices in the associated algebraic group G and using a result by Tits in [Tit74], stating that G is always cocompact in $\mathrm{Aut}(X)$.

There are different, geometric constructions of lattices in both classical and exotic two-dimensional buildings by Köhler-Meixner-Wester [KMW84, KMW85], Ronan [Ron84], Kantor [Kan86] and Cartwright-Steger-Mantero-Zappa [CMSZ93a, CMSZ93b]. We propose a new construction which produces lattices acting simply transitively on a fixed type of panel of the building.

The lattices are obtained as fundamental groups of complexes of groups which are constructed using so-called *Singer polygons*, generalised polygons with a point-regular automorphism group. Using this construction, we obtain panel-regular lattices in buildings of type $\tilde{A}_2$ and $\tilde{C}_2$ with explicit presentations. These concise presentations also allow a simple description of the associated buildings as well as the calculation of group homology. In particular, we obtain the following results.

Classification If a lattice acts regularly on one type of panel in an $\tilde{A}_2$-building, it acts regularly on all types of panels. Using this fact, we obtain for each such lattice some combinatorial data involving finite projective planes. Conversely, for each such piece of data, one can construct an $\tilde{A}_2$-building with a panel-regular lattice.

The question whether two different pieces of data lead to isomorphic buildings or commensurable lattices remains open. Likewise, it is unclear whether the corresponding buildings are classical or exotic.

Lattices of type $\tilde{A}_2$: In buildings of type $\tilde{A}_2$, we obtain a very explicit construction of lattices using the notion of classical difference sets. In this case, the vertex stabilisers are cyclic groups. Using a list of these difference sets, it is easy to construct short presentations. The smallest example is

$$\Gamma = \langle x_1, x_2, x_3 \mid x_1^7 = x_2^7 = x_3^7 = x_1x_2x_3 = x_1^3x_2^3x_3^3 = 1\rangle.$$

It can be shown that this lattice is actually contained in $\mathrm{SL}_3(\mathbb{F}_2((t)))$, but this is unknown for all larger examples.

Lattices of type $\tilde{C}_2$: In buildings of type $\tilde{C}_2$, one can produce different series of examples acting regularly on one or two types of panels of the building. The vertex stabilisers are Heisenberg groups or elementary abelian 2-groups. The most curious lattice among these is

$$\Lambda = \big(\mathbb{Z}/q * \mathbb{Z}/q\big)/\big\langle [x,y] \colon \text{for certain pairs } (x,y) \in \mathbb{Z}/q \times \mathbb{Z}/q\big\rangle,$$

where $q = 2^k$ for $k > 1$. The associated building is necessarily exotic.

All results outlined in this abstract can be found in [Ess09] and [Ess10].

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McCool groups and stabilizers on the boundary of outer space

Gilbert Levitt
(joint work with Vincent Guirardel)

Let $\mathcal{C}$ be a finite set of conjugacy classes in a free group F_n. Let $\text{Out}_{\mathcal{C}}(F_n)$ be the pointwise stabilizer of $\mathcal{C}$ in $\text{Out}(F_n)$. If for instance $\mathcal{C}$ is the class of $[a,b][c,d]$ in $F(a,b,c,d)$, then $\text{Out}_{\mathcal{C}}(F_n)$ is a mapping class group. We call $\text{Out}_{\mathcal{C}}(F_n)$ a McCool group because of:

Theorem 1 (McCool [1])**.** $\text{Out}_{\mathcal{C}}(F_n)$ *is finitely presented.*

McCool's proof used peak reduction. Using JSJ theory and outer space, we prove:

Theorem 2. $\text{Out}_{\mathcal{C}}(F_n)$ *is VFL: some finite index subgroup has a finite* $K(\pi,1)$*.*

Here is a sketch of the proof. It also applies if $\mathcal{C}$ is infinite, or if F_n is replaced by a torsion-free hyperbolic group.

One considers splittings of F_n (equivalently, graphs of groups decompositions, or actions on trees) which are relative to $\mathcal{C}$: every element of $\mathcal{C}$ must be contained in a vertex group (i.e. fix a point in the tree). There are two cases.

If F_n is freely indecomposable rel $\mathcal{C}$, one considers its cyclic JSJ decomposition rel $\mathcal{C}$. There is an $\text{Out}_{\mathcal{C}}(F_n)$-invariant JSJ tree T and one understands $\text{Out}_{\mathcal{C}}(F_n)$ through its action on T.

If F_n is not freely indecomposable rel $\mathcal{C}$, there is no $\text{Out}_{\mathcal{C}}(F_n)$-invariant tree and one has to consider outer space rel $\mathcal{C}$. This is the set of projective classes of actions of F_n on simplicial trees, with edge stabilizers trivial and vertex stabilizers freely indecomposable rel the elements of $\mathcal{C}$ which they contain. $\text{Out}_{\mathcal{C}}(F_n)$ acts "cocompactly" on this contractible space, and stabilizers are controlled by the previous case.

McCool groups come up when studying the action of $\text{Out}(F_n)$ on the boundary of (ordinary) outer space. A point on this boundary is a projective class $[T]$ of actions of F_n on $\mathbb{R}$-trees, and we distinguish between $\text{Out}_{[T]}(F_n)$ (the stabilizer of the projective tree) and $\text{Out}_T(F_n)$ (the stabilizer of the $\mathbb{R}$-tree). The quotient $\text{Out}_{[T]}(F_n)/\text{Out}_T(F_n)$ embeds into the multiplicative reals, and a result by M. Lustig implies that the image is trivial or cyclic. It it thus enough to study $\text{Out}_T(F_n)$.

Theorem 3. $\text{Out}_T(F_n)$ *has a finite index subgroup* $\text{Out}^0_T(F_n)$ *fitting in an exact sequence*

$$1 \to F_{n_1} \times \cdots \times F_{n_p} \to \text{Out}^0_T(F_n) \to M_1 \times \cdots \times M_q \to 1$$

where F_{n_i} *is free and* M_i *is a McCool group.*

In particular, $\text{Out}_{[T]}(F_n)$ *and* $\text{Out}_T(F_n)$ *are VFL.*

If for example T is the Bass-Serre tree of a cyclic amalgam $F *_C F'$ where the amalgam identifies $a \in F$ with $b \in F'$, the exact sequence is

$$1 \to \mathbb{Z} \to \text{Out}^0_T(F_n) \to \text{Out}_a(F) \times \text{Out}_b(F') \to 1$$

with the kernel generated by the Dehn twist (acting on F as conjugation by a and on F' as the identity).

If $[T]$ is fixed by an irreducible automorphism of F_n, then $\mathrm{Out}_{[T]}(F_n)$ is virtually cyclic (Bestvina-Feighn-Handel).

To prove Theorem 3, one considers the preimage $\mathrm{Aut}_T(F_n)$ of $\mathrm{Out}_T(F_n)$ in $\mathrm{Aut}(F_n)$. The action of F_n on T extends to an isometric action of $\mathrm{Aut}_T(F_n)$, and we view an element H of $\mathrm{Aut}_T(F_n)$ as an isometry of T.

If T is simplicial, the first step is to restrict to the finite index subgroup $\mathrm{PG}(T) \subset \mathrm{Aut}_T(F_n)$ consisting of elements acting trivially on the quotient graph T/F_n. The letters P and G stand for "Piecewise Group" because each element $H \in \mathrm{PG}(T)$ *piecewise* agrees with an element of the *group* F_n: given an edge e of T, there exists $g \in F_n$ such that H and g agree on e.

In general, we define a subgroup $\mathrm{PG}(T) \subset \mathrm{Aut}_T(F_n)$ as follows: $H \in \mathrm{Aut}_T(F_n)$ belongs to $\mathrm{PG}(T)$ if and only if every arc in T may be subdivided into finitely many subarcs, and on each subarc H agrees with some element of F_n. Letting $\overline{\mathrm{PG}}(T)$ be the image of $\overline{\mathrm{PG}}(T)$ in $\mathrm{Out}_T(F_n)$, we show that $\overline{\mathrm{PG}}(T)$ has finite index and admits a description as in Theorem 3.

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The Farrell-Jones Conjecture for hyperbolic groups and CAT(0)-groups

Wolfgang Lück

(joint work with Arthur Bartels)

Let G be a discrete group and let R be an associative ring with unit. We explain and state the following conjectures and discuss their relevance.

Kaplanski Conjecture. If G is torsionfree and R is an integral domain, then 0 and 1 are the only idempotents in RG.

Conjecture. Suppose that G is torsionfree. Then $K_n(\mathbb{Z}G)$ for $n \leq -1$, $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish.

Novikov Conjecture. Higher signatures are homotopy invariants.

Borel Conjecture. An aspherical closed manifold is topologically rigid.

Serre's Conjecture. A group of type FP is of type FF.

Conjecture. If G is a finitely presented Poincaré duality group of dimension $n \geq 5$, then it is the fundamental group of an aspherical homology ANR-manifold.

Conjecture If G is a hyperbolic group with S^n as boundary, then there is a closed aspherical manifold M whose fundamental group is G.

Farrell-Jones Conjecture. Let G be torsionfree and let R be regular. Then the assembly maps for algebraic K- and L-theory

$$\begin{array}{rcl} H_n\big(BG;\mathbb{K}_R\big) & \to & K_n(RG); \\ H_n\big(BG;\mathbb{L}_R^{\langle -\infty\rangle}\big) & \to & L_n^{\langle -\infty\rangle}(RG), \end{array}$$

are bijective for all $n \in \mathbb{Z}$.

There is a more complicate version of the Farrell-Jones Conjectures which makes sense for all groups and rings and allows twistings of the group ring. We explain that it implies all the other conjectures mentioned above provided that in the Kaplanski Conjecture R is a field of characteristic zero, in the Novikov Conjecture and the Borel Conjecture the dimension is greater or equal to five and in the conjecture about boundaries of hyperbolic groups the dimension of the sphere is greater or equal to five.. We present the following result:

Theorem [Bartels-Lück]. Let $\mathcal{FJ}$ be the class of groups for which the Farrell-Jones Conjecture is true in its general form. Then:

(1) Hyperbolic groups belong to $\mathcal{FJ}$;
(2) CAT(0) groups belong to $\mathcal{FJ}$;
(3) Cocompact lattices in almost connected Lie groups belong to $\mathcal{FJ}$;
(4) Fundamental groups of (not necessarily compact) 3-manifolds possibly with boundary) belong to $\mathcal{FJ}$;
(5) If G_0 and G_1 belong to $\mathcal{FJ}$, then also $G_0 * G_1$ and $G_0 \times G_1$;
(6) If G belongs to $\mathcal{FJ}$, then any subgroup of G belongs to $\mathcal{FJ}$;
(7) Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). If each G_i belongs to $\mathcal{FJ}$, then also the direct limit of $\{G_i \mid i \in I\}$.
(8) Let $1 \to H \to G \xrightarrow{p} Q \to 1$ be an extension of groups. If Q and for all virtually cyclic subgroups $V \subseteq Q$ the preimage $p^{-1}(V)$ belongs to $\mathcal{FJ}$, then G belongs to $\mathcal{FJ}$;

Since certain prominent constructions of groups yield colimits of hyperbolic groups, the class $\mathcal{FJ}$ contains many interesting groups, e.g. limit groups, Tarski monsters, groups with expanders and so on. Some of these groups were regarded as possible counterexamples to the conjectures above but are now ruled out by the theorem above.

There are also prominent constructions of closed aspherical manifolds with exotic properties, e.g. whose universal covering is not homeomorphic to Euclidean space, whose fundamental group is not residually finite or which admit no triangulation. All these constructions yield fundamental groups which are CAT(0) and hence yield topologically rigid manifolds.

However, the Farrell-Jones Conjecture is open for instance for solvable groups, $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, mapping class groups or automorphism groups of finitely generated free groups.

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Asymptotic dimension of mapping class groups is finite

Mladen Bestvina

(joint work with Ken Bromberg and Koji Fujiwara)

The *asymptotic dimension* $\operatorname{asdim}(X)$ of a metric space X is said to be $\leq n$ if for every $R > 0$ there is a covering of X by sets U_i such that every metric R-ball in X intersects at most $n+1$ of the U_i's, and $\sup \operatorname{diam} U_i < \infty$. This definition is due to Gromov and it is invariant under quasi-isometries (or even coarse isometries). In particular, asymptotic dimension of a finitely generated group is well-defined. It is not hard to see that $\operatorname{asdim}(\mathbb{R}^2) \leq 2$ by considering the usual "brick decomposition" of $\mathbb{R}^2$ (with large bricks), and more generally, $\operatorname{asdim}(\mathbb{R}^n) \leq n$. This inequality is also easily seen using the product formula $\operatorname{asdim}(X \times Y) \leq \operatorname{asdim}(X) + \operatorname{asdim}(Y)$.

A generalization of the product formula is Bell-Dranishnikov's Hurewicz theorem: Suppose $f : X \to Y$ is a Lipschitz map between geodesic metric spaces such that for every $M > 0$ the family $\{f^{-1}(B(y, M))\}$ of preimages of metric balls of radius M has asymptotic dimension $\leq n$ *uniformly* (this means that coverings as in the definition can be found with a diameter bound independent of the center y). Then $\operatorname{asdim}(X) \leq \operatorname{asdim}(Y) + n$.

For example, if $1 \to A \to B \to C \to 1$ is a short exact sequence of finitely generated groups then $\mathrm{asdim}(B) \leq \mathrm{asdim}(A) + \mathrm{asdim}(C)$. Likewise, asymptotic dimension of the hyperbolic plane is ≤ 2 by considering the projection to a line whose fibers are horocycles tangent to a fixed point at infinity (e.g. the projection to the y-coordinate in the upper half-space model). More generally, if $G = KAN$ is a semi-simple Lie group with symmetric space $X = AN$, then the projection $X = AN \to A$ shows that $\mathrm{asdim}(X) \leq \mathrm{asdim}(A) + \mathrm{asdim}(N) = \dim(A) + \dim(N) = \dim(X)$. In this example equality holds by considering a factorization of the identity through the nerve (up to bounded error) and its effect on top dimensional locally finite homology.

A theorem of Yu says that groups of finite asymptotic dimension (and finite classifying space) satisfy the Novikov conjecture.

Gromov proved that δ-hyperbolic groups have finite asymptotic dimension. Here is a proof. Let X be the Cayley graph of the group and suppose that $R \gg \delta$ is an integer. For every vertex v at distance $5kR$ from 1, $k = 1, 2, 3, \cdots$, consider the set

$$U_v = \{x \in \Gamma \mid d(1, x) \in [5(k+1)R, 5(k+2)R] \text{ and } v \text{ lies on some geodesic } [1, x]\}$$

An easy thin triangle argument shows that if v, w are two vertices at distance $5kR$ such that both U_v and U_w intersect the same R-ball, then $d(v, w) \leq 2\delta$. This gives a bound on the number of U_v's that can intersect the same R-ball, and this bound is independent of R; thus $\mathrm{asdim}(\Gamma) < \infty$.

Bell-Fujiwara modified this argument to show that curve complexes have finite asymptotic dimension. They are hyperbolic by the celebrated work of Masur-Minsky, but not locally finite, resulting in an infinite bound. The trick is to use *tight* geodesics in place of arbitrary geodesics. Finiteness properties of tight geodesics proved by Bowditch imply that asymptotic dimension is finite.

We now outline a proof of the main theorem:

Theorem. Asymptotic dimension of mapping class groups is finite.

Let Σ be a surface of finite type and $\mathrm{MCG}(\Sigma)$ its mapping class group.

Step 1. Produce an action of $\mathrm{MCG}(\Sigma)$ on a finite product $X_1 \times X_2 \times \cdots \times X_k$ of metric spaces. An orbit map $\mathrm{MCG}(\Sigma) \to X_1 \times X_2 \times \cdots \times X_k$ is a quasi-isometric embedding. This reduces us to showing $\mathrm{asdim}(X_i) < \infty$ for all i.

Step 2. Show each X_i is hyperbolic and has a Lipschitz map $X_i \to T_i$ satisfying the Hurewicz theorem with fibers curve complexes of subsurfaces of Σ. This reduces us to showing $\mathrm{asdim}(T_i) < \infty$.

Step 3. Show that each T_i is quasi-isometric to a tree (i.e. it is a *quasi-tree*) and hence $\mathrm{asdim}(T_i) = 1$.

The last step is the most interesting and leads one to wonder which groups admit interesting actions on quasi-trees. An axiomatic construction is as follows:

Let $\mathbb{Y}$ be a set and assume that for every $Y \in \mathbb{Y}$ we have a function $d_Y : (\mathbb{Y} - \{Y\})^2 \to [0, \infty)$ such that

- $d_Y(A, B) = d_Y(B, A)$,
- $d_Y(A, C) \leq d_Y(A, B) + d_Y(B, C)$,
- there is $\xi > 0$ such that for any $A, B, C \in \mathbb{Y}$ at most one of
$$d_A(B, C), d_B(A, C), d_C(A, B)$$
is $> \xi$, and
- there is K_0 such that for any A, B the set
$$\{C \in \mathbb{Y} \mid d_C(A, B) > K_0\}$$
is finite.

The simplest example comes from considering a discrete group of isometries of hyperbolic space containing a loxodromic element g with axis A. Take $\mathbb{Y}$ to be the set of translates of A and for $B, C, D \in \mathbb{Y}$ put $d_B(C, D) = \text{diam}(\pi_B(C) \cup \pi_B(D))$, where π_B is the nearest point projection to B. Similar examples can be obtained from hyperbolic groups, or $CAT(0)$ groups with rank 1 elements, or $Out(F_n)$ (thanks to the work of Yael Algom-Kfir). Of interest to us is the setting of subsurface projections, where $\mathbb{Y}$ is a family of essential subsurfaces of Σ such that $A, B \in \mathbb{Y}$ implies $\partial A \cap \partial B \neq \emptyset$. The distance $d_A(B, C)$ is given by restricting ∂B and ∂C to A and measuring the distance in the curve complex of A. The axioms for this case are part of the work of Masur-Minsky and Behrstock.

One can then consider the *projection complex* $S_K(\mathbb{Y})$. Fix a large $K > 0$. The vertices of $S_K(\mathbb{Y})$ are the elements of $\mathbb{Y}$, and two vertices A, B are joined by an edge if $d_C(A, B) < K$ for all C. (Technically, we first perturb d by $\leq 2\xi$ before defining $S_K(\mathbb{Y})$; this is ignored here.) We then argue that $S_K(\mathbb{Y})$ is a quasi-tree.

To finish the argument, we divide the collection of all essential subsurfaces of Σ into finitely many classes $\mathbb{Y}_1, \cdots, \mathbb{Y}_k$ so that each $\mathbb{Y}_i$ satisfies the above "transversality" property. Moreover, this can be done so that each $\mathbb{Y}_i$ is invariant under a certain fixed finite index subgroup $G \subset \text{MCG}(\Sigma)$. We obtain quasi-trees $T_1, \cdots, T_k$, and replacing each vertex $A \in T_i$ by the corresponding curve complex $C(A)$ gives rise to X_i. The fact that orbit maps are QI-embeddings follows from a distance formula due to Masur-Minsky.

A Survey on Measure Equivalence

Damien Gaboriau

There are several excellent surveys with different approaches to measure equivalence to which the reader is referred for further information and references, for instance [Gab05, Sha05, Pop07, Fur09]. This note essentially follows [Gab10], where (many) more references can be found.

Two groups Γ_1 and Γ_2 are **virtually isomorphic** if there exist $F_i \triangleleft \Lambda_i < \Gamma_i$ such that $\Lambda_1/F_1 \simeq \Lambda_2/F_2$, where F_i are finite groups, and Λ_i has finite index in Γ_i. This condition is equivalent with: Γ, Λ admit commuting actions on a set Ω such that each of the actions $\Gamma \curvearrowright \Omega$ and $\Lambda \curvearrowright \Omega$ has finite quotient set and finite stabilizers.

A finite set admits two natural generalizations, a topological one (compact set) leading to **geometric group theory** and a measure theoretic one (finite measure set) leading to **measured group theory**.

Definition 1 ([Gro93]). Two countable groups Γ_1 and Γ_2 are **measure equivalent (ME)** if there exist commuting actions of Γ_1 and Γ_2, that are (each) measure preserving, free, and with a finite measure fundamental domain, on some standard (infinite) measure space (Ω, m).

The ratio $[\Gamma_1 : \Gamma_2]_\Omega := m(\Omega/\Gamma_2)/m(\Omega/\Gamma_1)$ of the measures of the fundamental domains is called the **index** of the **coupling** Ω. The typical examples, besides virtually isomorphic groups, are lattices in a common (locally compact second countable) group G with its Haar measure, acting by left and right multiplication. See [Fur99] for the basis.

The topological analogue was shown to be equivalent with **quasi-isometry (QI)** between finitely generated groups [Gro93], thus raising **measured group theory** (i.e. the study of groups up to ME) to parallel **geometric group theory**. Measure equivalence and orbit equivalence are intimately connected by considering the relation between the quotient actions $\Gamma_1 \curvearrowright \Omega/\Gamma_2$ and $\Gamma_2 \curvearrowright \Omega/\Gamma_1$. In fact two groups are ME iff they admit SOE free actions.

Definition 2 (Stable Orbit Equivalence). Two p.m.p. actions $\Gamma_i \curvearrowright (X_i, \mu_i)$ are **stably orbit equivalent (SOE)** if there are Borel subsets $Y_i \subset X_i$, $i = 1, 2$ which meet almost every orbit of Γ_i and a measure-scaling isomorphism $f : Y_1 \to Y_2$ s.t.

$$f(\Gamma_1.x \cap Y_1) = \Gamma_2.f(x) \cap Y_2 \quad \text{a.e.}$$

The **index** or **compression constant** of this SOE f is $[\Gamma_1 : \Gamma_2]_f = \frac{\mu(Y_2)}{\mu(Y_1)}$.

The state of the art ranges from quite well understood ME-classes to mysterious and very rich examples. For instance, the finite groups obviously form a single ME-class. The infinite amenable groups form a single ME-class (Ornstein-Weiss). The ME-class of a lattice in a center-free simple Lie group G with real rank ≥ 2 (like $\mathrm{SL}(n, \mathbb{R})$, $n \geq 3$) consists of those groups that are virtually isomorphic with a lattice in G (Furman). If Γ is a non-exceptional mapping class group, its ME-class consists only in its virtual isomorphism class (Kida). Kida extended this kind of result to some amalgamated free products.

On the opposite, the ME-class $ME(\mathbf{F}_2)$ of the (mutually virtually isomorphic) free groups $\mathbf{F}_r$ ($2 \leq r < \infty$) remains far from being understood. It contains the free products $*_{i=1}^{r} A_i$ of infinite amenable groups, surface groups $\pi_1(\Sigma_g)$ ($g \geq 2$). It also contains certain branched surface groups (Gaboriau) and more generally, elementarily free groups (Bridson-Tweedale-Wilton).

Question 1. *Are all limit groups ME with a free group?*

The latter follows from the fact that the commutator $[a, b]$ in the free group $\mathbf{F}_2 = \langle a, b \rangle$ appears to be a free factor **in a measurable sense** [Gab05]. This is a particular feature of measure equivalence, that the study of subgroups is extended to a much flexible family of sub-objects. For instance, Gaboriau-Lyons proved a measurable version of von Neumann's problem [GL09]: every non-amenable group Γ contains a free group $\mathbf{F}_2$ **in a measurable sense**, i.e. there are p.m.p. free actions $\Gamma \curvearrowright (X, \mu)$ and $\mathbf{F}_2 \curvearrowright (X, \mu)$ such that for almost every $x \in X$, $\mathbf{F}_2.x \subset \Gamma.x$.

There is a considerable list of ME-invariants (see [Gab05, Gab10] and the references therein). For instance Kazhdan property (T), Haagerup property, the sign of the Euler characteristic (when defined)... Recently **exactness** and belonging to the class $\mathcal{S}$ of Ozawa were proved to be ME-invariants (Brown-Ozawa, Sako). There are also numerical invariants which are preserved under ME modulo multiplication by the index: $\mathrm{Cost}(\Gamma) - 1$, the ℓ^2-Betti numbers $(\beta_n^{(2)}(\Gamma))_{n\in\mathbb{N}}$.

ME is stable under some basic constructions:
(a) if $\Gamma_i \overset{\mathrm{ME}}{\sim} \Lambda_i$ for $i = 1, \cdots, n$ then $\Gamma_1 \times \cdots \times \Gamma_n \overset{\mathrm{ME}}{\sim} \Lambda_1 \times \cdots \times \Lambda_n$
(b) if $\Gamma_i \overset{\mathrm{ME}}{\sim} \Lambda_i$ with index 1, then $\Gamma_1 * \cdots * \Gamma_n \overset{\mathrm{ME}}{\sim} \Lambda_1 * \cdots * \Lambda_n$ (with index 1).

In order to study when the converses hold (Monod-Shalom, Ioana-Peterson-Popa, Chifan-Houdayer, Alvarez-Gaboriau...), one has of course to impose some irreducibility conditions on the building blocks, and these conditions have to be strong enough to resist the measurable treatment. These requirements are achieved
(a) (for direct products) if the Γ_i, Λ_i belong to the class $\mathcal{C}_{\mathrm{reg}}$ of Monod-Shalom (for instance if they are non-amenable non-trivial free products): the non-triviality of the bounded cohomology $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^2(\Gamma))$ is an ME-invariant preventing Γ to decompose (non-trivially) as a direct product;
(b) (for free products) if the Γ_i, Λ_i are $\mathcal{MFI}$ (for instance if they have $\beta_1^{(2)} = 0$ and are non-amenable) [AG10]: they are not ME with a (non-trivial) free product.
We prove for instance: If $\Gamma_1 * \cdots * \Gamma_n \overset{\mathrm{ME}}{\sim} \Lambda_1 * \cdots * \Lambda_p$, where both the Γ_i's and the Λ_j's belong to distinct ME-classes and are $\mathcal{MFI}$, then $n = p$ and up to a permutation of the indices $\Gamma_i \overset{\mathrm{ME}}{\sim} \Lambda_i$ [AG10].

Ioana-Peterson-Popa and Chifan-Houdayer considered such a situation, where the groups have Kazhdan property (T), or are direct products, under extra ergodicity hypothesis. The delicate point of removing ergodicity assumptions in [AG10] was achieved by using a measurable version of Kurosh's theorem due to Alvarez [Alv09].

Similar "deconstruction" results were obtained by Sako for building blocks made of direct products of non-amenable exact groups by taking free products with amalgamation over amenable subgroups or by taking wreath product with amenable base.

Refinements of the notion of ME were considered by Sauer, Shalom and their collaborators.

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Boundary amenability of groups acting on buildings

Jean Lécureux

One of the main utility of buildings is to study and understand the structure of reductive groups over local fields and their lattices. However, there are also some other groups which act on buildings, such as for example Kac-Moody groups. We prove that all these groups have a topologically amenable action on a compact space. This compact space is described geometrically as a combinatorial boundary of a building.

1. Buildings

First, let me recall briefly the definition and the basic vocabulary about buildings. A *Coxeter group* is a group defined by a presentation of the form $\langle s \in S \mid (st)^{m_{st}} = 1\rangle$, where m_{st} are natural integers (or possibly infinite) such that $m_{ss} = 1$: the generators have order two.

For example, regular tilings of the euclidean plane or the hyperbolic plane give rise to Coxeter groups, generated by the reflections with respect to the lines (or geodesics) of the tiling. More generally, to every Coxeter group W is associated a metric simplicial complex, called the *Davis complex*, which is CAT(0), and on which W acts.

Let us introduce a few words of vocabulary. A *reflection* in W is a conjugate of an element of S. A *wall* is the set of fixed points of a reflection in Σ. A *chamber* is the closure of a connected component of Σ deprived of all its walls. A *panel* is an intersection of two adjacent chambers.

A building is a tessellation of such Davis complexes. More precisely:

Definition. Let (W, S) be a Coxeter group, and Σ its Davis complex.

A *building* of type (W, S) is a gluing of chambers along their panels, covered by subcomplexes called *apartments*, such that:

- Every apartment is isomorphic to Σ;
- Every two chambers are contained in some apartment;

- For every apartments A and A', there is an isomorphism from A to A' which fixes $A \cap A'$.

2. Combinatorial boundary of buildings

This section is a joint work with Pierre-Emmanuel Caprace [1].

Recall that there is a notion of *projection* in buildings: the projection of a chamber C on a panel σ is the unique chamber which is adjacent to σ and at minimal distance from C.

Let X a building. Denote $\mathrm{ch}(X)$ the set of chambers of X and, for any panel σ, $\mathrm{lk}(\sigma)$ the link of σ (*i.e.* the set of chambers containing σ). From this notion of projection, we get an injection

$$i_{\mathrm{ch}} : \mathrm{ch}(X) \to \prod \mathrm{lk}(\sigma),$$

the product being taken over all panels σ in X. Each of the links $\mathrm{lk}(\sigma)$ is endowed with the discrete topology, and the product is endowed with the product topology.

Definition. The *combinatorial compactification* $\mathscr{C}_{\mathrm{ch}}(X)$ of a building X is the closure of $i_{\mathrm{ch}}(\mathrm{ch}(X))$.

One of the interests of this compactification is that it parametrizes amenable subgroups of the automorphism group:

Theorem ([1])**.** *Let X be a locally finite building, and G a group acting properly on X. Then G is amenable if and only if it virtually fixes a point in $\mathscr{C}_{\mathrm{ch}}(X)$.*

Remark. It is also possible to define a compactification of the set of spherical residues instead of the set of chambers. In this case, the theorem above remains true even without the hypothesis that X is locally finite.

3. Boundary amenability

The notion of a topological amenable action is defined as follows:

Definition. Let G be a locally compact group acting on a locally compact space S. The action of G on S is *topologically amenable* if there exists a sequence of continuous maps $\mu_n : S \to \mathrm{Prob}(G)$ such that

$$\lim_{n\to+\infty} \|\mu_n(gs) - g\mu_n(s)\| = 0,$$

uniformly on every compact subset of $G \times S$.

This notion has many applications. In particular, the class of discrete groups that admit a topologically amenable action on a discrete space is a very interesting one (see for example [5]). It implies for example that the group can be coarsely embedded into a Hilbert space [3], which in turn implies that it satisfies the Novikov conjecture [6].

For groups acting on buildings, we prove the following theorem:

Theorem ([4])**.** *Let G be a group acting properly on a locally finite building X. Then the action of G on $\mathscr{C}_{\mathrm{ch}}(X)$ is topologically amenable.*

Let us give two important elements of the proof. The first one is the notion of *generalized sectors* in a building. Such a sector should be seen as kind of combinatorial convex hull of a chamber and a point at infinity. More precisely, the sector $Q(x, \xi)$ is defined as the pointwise limit of the convex hulls between x and C_n, where (C_n) is a sequence converging to ξ.

These sectors are very useful because of two points: a sector is contained in an apartment, and two sectors converging to the same point always intersect. These two features allow us, by constructing our measure $\mu_n(\xi)$ with support in a sector converging to ξ, to reduce the problem to the construction of $\mu_n(\xi)$ in an apartment.

In restriction to an apartment, we use a second idea, which was already used in a similar context in [2]: a Coxeter complex – and in fact its combinatorial compactification – can be embedded equivariantly into a finite product of trees.

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Trees of manifolds as Gromov boundaries of hyperbolic groups

Jacek Świątkowski

We present a rich class of explicitly defined metric compacta X that can be realized, up to homeomorphism, as Gromov boundaries $\partial\Gamma$ of word hyperbolic groups Γ. If one restricts to compacta that are connected and have no local cut points (which corresponds to JSJ-indecomposability of groups), the only so far known examples seem to be: spheres S^n with $n \geq 2$, Sierpinski compacta $M_{n,n+1}$ in all dimensions $n \geq 1$, Menger universal compacta $M_{n,2n+1}$ in dimensions $n = 1, 2, 3$ and Pontryagin surfaces Π_p for prime p – certain 2-dimensional compacta.

Trees of manifolds are compacta obtained as inverse limits of appropriate sequences of iterated connected sums of copies of manifolds M from a family $\mathcal{M}^n$ (infinite or finite), where each M is closed, connected, oriented and n-dimensional. By a theorem of Jakobsche [Jak91], for any family $\mathcal{M}^n$ an inverse limit as above is unique up to homeomorphism; therefore we denote it $X(\mathcal{M}^n)$.

Theorem. *Given a finite family $\mathcal{M}^n$, suppose Y is a piecewise hyperbolic CAT(-1) complete oriented pseudomanifold with 0-dimensional set Σ of topological singularities, such that*

(1) $\forall p \in \Sigma\ \mathrm{Lk}(p, Y) \overset{PL}{\cong} M \in \mathcal{M}^n$, and

(2) $\forall M \in \mathcal{M}^n$ the subset $\Sigma_M = \{p \in \Sigma \mid \mathrm{Lk}(p, Y) = M\}$ is a net in Y.

Then ∂Y is homeomorphic to $X(\mathcal{M}^n)$.

The theorem, combined with the procedure of *oriented strict hyperbolization* of Charney-Davis-Gromov-Januszkiewicz, yields:

Corollary. *Let $\mathcal{M}^n = \{M_1, \dots, M_k\}$ be a finite family of manifolds as before, and suppose that the disjoint union $\coprod_{i=1}^k k_i \cdot M_i$ bounds a compact oriented connected $(n+1)$-manifold W^{n+1}, where the k_i are positive integers. Then $X(\mathcal{M}^n)$ realizes as $\partial\Gamma$, the Gromov boundary of a hyperbolic group Γ.*

Sketch of proof. Cap all components of ∂W^{n+1} disjointly with cones, getting X, strictly hyperbolize it to get X_h. Then the universal cover $\widetilde{X}_h$ is as in the above theorem, and hence $\Gamma = \pi_1(X_h)$ does the job. □

Question. *Is $X(\mathbb{C}P^2)$ homeomorphic to the Gromov boundary $\partial\Gamma$ of any hyperbolic group Γ?*

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Isomorphism problem for complex hyperbolic lattices and beyond

Vincent Guirardel
(joint work with François Dahmani)

We give a solution of the isomorphism problem for some classes of relative hyperbolic groups with residually finite parabolic subgroups. This gives a solution to the isomorphism problem for fundamental groups of manifolds with pinched negatively curved manifolds with finite volume.

The isomorphism problem for a class of groups $\mathcal{C}$ asks for an algorithm that takes as input two presentations of groups G, G' in $\mathcal{C}$, and which decides whether G is isomorphic to G'. This is known to be unsolvable for the class of all finitely presented groups since the 50's [Ady55, Rab58]. In fact, the isomorphism problem is unsolvable for some very natural classes of groups, including the class of free-by-free groups (Miller [Mil71]), the class of [free abelian]-by-free groups (Zimmermann [Zim85]) or the class of solvable groups of derived length 3 (Baumslag-Gildenhuys-Strebel [BGS85]).

On the positive side, the isomorphism problem is known to be decidable for the class of nilpotent groups and virtually polycyclic groups (Grunewald-Segal [GS80], Segal [Seg90]), and, following Sela, for the class of hyperbolic groups ([Sel95, DG08, DG10]), and toral relatively hyperbolic groups [DG08]. As a corollary, Dahmani and Groves give a solution to the isomorphism problem for fundamental groups of hyperbolic manifolds with finite volume [DG08].

In pinched variable negative curvature, the parabolic subgroups are virtually nilpotent instead of virtually abelian. Our initial motivation is to generalize this solution for fundamental groups of such manifolds, and more generally, to a class of relative hyperbolic groups with virtually nilpotent parabolic subgroups. However, one cannot rely on the same approach. Indeed, the solutions to the isomorphism problem for classes of hyperbolic and relative hyperbolic groups mentioned above fundamentally rely on a solution of the equations problem in these groups. But this problem is known to be unsolvable in the class of nilpotent groups [Rom79].

Instead, our strategy is to use Dehn filling theorems by Groves-Manning and Osin [GM08, Osi07] to produce sequences of canonical hyperbolic quotients of the given groups, and then to use our solutions of the isomorphism problem for hyperbolic groups with torsion to compare these Dehn fillings. The success of this approach might be surprising since there exists non-isomorphic nilpotent groups having the same finite quotients.

Theorem 1. *The isomorphism problem is solvable for the class of relative hyperbolic groups with virtually polycyclic parabolic groups, and which do not split over virtually polycyclic groups relative to their non virtually cyclic parabolic subgroups.*

In particular, the isomorphism problem is solvable for the class of fundamental groups of manifolds with pinched negative curvature and finite volume. In fact, we prove the following more general statement.

Theorem 2. *Let $\mathcal{C}$ be a recursively enumerable class of finitely presented groups that are residually finite and universally parabolic. Then there is an algorithm as follows. It takes as input two presentations of groups G, G' such that*

(1) G is hyperbolic relative to some family of groups $P_1, \dots P_n$ in $\mathcal{C}$, G non-elementary

(2) G does not split over an elementary subgroup, relative to $P_1, \dots, P_n$

and similarly for G'.

Then it says whether G is isomorphic to G'.

In this statement, P is *universally parabolic* if whenever P is contained in a relative hyperbolic group, P is contained in parabolic subgroup.

The theorem applies in particular to the class $\mathcal{C}$ of semi-direct products $F_r \ltimes \mathbb{Z}^n$ with $r, n \geq 2$. Since the isomorphism problem in $\mathcal{C}$ is unsolvable, the following corollary might be surprising.

Corollary 3. *The isomorphism problem is solvable for the class of non-elementary relative hyperbolic groups with parabolic groups in $\mathcal{C}$ that do not split over an elementary subgroup, relative to its parabolic subgroups.*

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Almost automorphisms of trees and lattices

Shahar Mozes

(joint work with Uri Bader, Pierre-Emmanuel Caprace, and Tsachik Gelander)

Recall that by a theorem of A. Borel every simple Lie group contains a lattice. There are locally compact groups which do not admit lattices. The easiest way to show the existence of such a group is to observe that a locally compact group which admits a lattice must be unimodular. One also has examples of certain nilpotent groups which do not admit lattices. In the talk we showed an example of a simple group not admitting a lattice.

Let $d \geq 2$ be a fixed integer, T be a (non-rooted) $(d+1)$-regular tree and G the group of **almost automorphisms** (also sometimes called **spheromorphisms**) of T. An element in G is defined by a triple (A, B, φ) where $A, B \subset T$ are finite subtrees with $|\partial A| = |\partial B|$ and $\varphi : T \setminus A \to T \setminus B$ is an isomorphism between the complements, and two such triples define the same element in G if and only if up to enlarging A, B they coincide.

The group G was first introduced by Neretin [3]; it is known to be abstractly simple [2]. For each vertex $v \in T$, the stabilizer $\mathrm{Aut}(T)_v$ is a compact open subgroup of $\mathrm{Aut}(T)$ and it is not difficult to see that G commensurates $\mathrm{Aut}(T)_v$. (In fact, the group G can be identified with the group of all **abstract commensurators** of $\mathrm{Aut}(T)_v$ or, equivalently, with the group of **germs of automorphisms** of $\mathrm{Aut}(T)$, see [1, Th. E].)

We endow G with the group topology defined by declaring that the conjugates of $\mathrm{Aut}(T)$ in G form a sub-basis of identity neighbourhoods. Since G commensurates the compact open subgroups of $\mathrm{Aut}(T)$, it follows that the embedding $\mathrm{Aut}(T) \hookrightarrow G$ continuous. In this way, the group G becomes a totally disconnected locally compact group containing $\mathrm{Aut}(T)$ as an open subgroup. In particular elements of G close to the identity can be realised as true automorphisms of T. As a locally compact group, the group G is compactly generated; in fact it contains a dense copy of the Higman–Thompson group $\Gamma_{d,2}$, which is finitely generated.

Theorem 1. *G does not contain any lattice.*

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Cannon Coxeter Groups

Marc Bourdon

(joint work with Bruce Kleiner)

Every word hyperbolic group Γ has a canonical action on its boundary at infinity $\partial\Gamma$; with respect to any visual metric on $\partial\Gamma$, this action is by uniformly quasi-Moebius homeomorphisms. This structure has a central role in the proofs of Mostow's rigidity theorem and numerous other results in the same vein, which are based on the analytic theory of quasiconformal homeomorphisms of the boundary.

Using a combinatorial analogue of the classical modulus on the 2-sphere, we obtain a proof of Cannon's conjecture in the special case of Coxeter groups (see [2] Theorem 1.3):

Theorem 1. *Let Γ be a word hyperbolic Coxeter group whose boundary is homeomorphic to the 2-sphere. Then there is a properly discontinuous, cocompact, and isometric action of Γ on the 3-dim real hyperbolic space $\mathbb{H}^3$.*

This result was essentially known (Mike Davis has communicated to us a different proof). Our view is that the principal value of the proof is that it illustrates the feasibility of the asymptotic approach (using the ideal boundary and modulus estimates), and it may suggest ideas for attacking the general case. It also gives a new proof of the Andreev's theorem on realizability of polyhedra in $\mathbb{H}^3$, in the case when the prescribed dihedral angles are submultiples of π.

We now give an indication of the ideas that go into the proof. Let Z be a compact metric space. For every $k \in \mathbb{N}$, let G_k be the incidence graph of a ball cover $\{B(x_i, 2^{-k})\}_{i\in I}$, where $\{x_i\}_{i\in I} \subset Z$ is a maximal 2^{-k}-separated subset. Given $p \geq 1$ and a curve family $\mathcal{F}$ in Z, we denote by $\mathrm{Mod}_p(\mathcal{F}, G_k)$ the G_k-combinatorial p-modulus of $\mathcal{F}$.

By [4], if Γ is a hyperbolic group and $\partial\Gamma$ is quasi-Moebius homeomorphic to the Euclidean 2-sphere, then Γ admits a properly discontinuous, cocompact, isometric action on $\mathbb{H}^3$. Also, as a consequence of the uniformization theorem established in [1], we obtain:

Corollary 2. *Suppose Z is an approximately self-similar metric space homeomorphic to the 2-sphere. Assume that for $d_0 > 0$ small enough, there exists a constant $C = C(d_0) \geq 1$ such that for every $k \in \mathbb{N}$ one has*

$$\mathrm{Mod}_2(\mathcal{F}_0, G_k) \leq C. \tag{1}$$

Then Z is quasi-Moebius homeomorphic to the Euclidean 2-sphere.

Thus, we are reduced to verifying the hypotheses of the above corollary when Z is the boundary of a Coxeter group Γ. We note that an alternate reduction to the same assertion can be deduced using [3].

One of the main results of the paper [2] is the existence of a finite number of "elementary curves families", whose moduli govern the modulus of every (thick enough) curve family in $\partial\Gamma$. Each elementary curve family is associated to a conjugacy class of an infinite parabolic subgroup.

In consequence, to obtain the bound (1), it is enough to establish that every connected parabolic limit set ∂P enjoys the following property: there exists a non constant continuous curve $\eta \subset \partial P$, such that letting $\mathcal{U}_\epsilon(\eta)$ be the ϵ-neighborhood of η in the C^0 topology, the modulus $\mathrm{Mod}_2(\mathcal{U}_\epsilon(\eta), G_k)$ is bounded independently of k, for $\epsilon > 0$ small enough.

To do so, two cases are distinguished: either ∂P is a circular limit set *i.e.* it is homeomorphic to the circle, or it is not.

In the second case one can find two crossing curves $\eta_1, \eta_2 \subset \partial P$. Since $\partial\Gamma$ is a planar set, one gets that $\min_{i=1,2} \mathrm{Mod}_2(\mathcal{U}_\epsilon(\eta_i), G_k)$ is bounded independently of k, for ϵ small enough.

Let $r > 0$, and denote by $\mathcal{F}_1$ the subfamily of $\mathcal{F}_0$ consisting of the curves $\gamma \in \mathcal{F}_0$ which do not belong to the r-neighborhood $N_r(\partial P)$ of any circular limit set ∂P. At this stage one knows that for r small enough, $\mathrm{Mod}_2(\mathcal{F}_1, G_k)$ is bounded independently of k. To bound the modulus of $\mathcal{F}_0 \setminus \mathcal{F}_1$, we proceed as follows. Consider a curve $\gamma \in \mathcal{F}_0$ contained in $N_r(\partial P)$, where ∂P is a circular parabolic limit set. The idea is to break γ into pieces $\gamma_1, \ldots, \gamma_i$, such that for each $j \in \{1, \ldots, i\}$, the maximal distance $\max\{d(x, \partial P) \; ; \; x \in \gamma_j\}$ is comparable to $\mathrm{diam}(\gamma_j)$. Then for each j, applying a suitable group element $g \in \Gamma$, we can arrange that both $g\gamma_j$

and $g\partial P$ have roughly unit diameter. Since $g\,\gamma_j$ lies close to $g\,\partial P$, but not too close, it cannot lie very close to a circular limit set; it follows that $g\,\gamma_j$ belongs to a curve family with controlled modulus. We then apply g^{-1} to the corresponding admissible function, and renormalize it suitably; by summing the collection of functions which arise in this fashion from all such configurations, we arrive at an admissible function for all such curves γ. The fact that the conformal dimension of S^1 is < 2 allows us to bound the 2-mass of this admissible function, and this yields the desired bound (1).

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Bruhat-Tits Buildings

Richard M. Weiss

Let (W, S) be a Coxeter system and let $\Sigma = \Sigma(W, S)$ be the corresponding Coxeter complex. The conjugates of elements of S in W are called *reflections.* Each reflection gives rise to a partition of Σ into two halves. Each of these halves is called a *root* of Σ. For each subset J of S, the pair $(\langle J\rangle, J)$ is also a Coxeter system. A Coxeter system is *irreducible* if it is not a direct product of Coxeter systems. If $|W| < \infty$, then (W, S) is called *spherical.* In this case, Σ has a canonical embedding in a sphere of dimension $|S| - 1$. The irreducible spherical Coxeter systems of rank $\ell \geq 3$ are $\mathsf{A}_\ell, \mathsf{B}_\ell = \mathsf{C}_\ell, \mathsf{D}_\ell, \ldots, \mathsf{H}_\ell$ (with $\ell = 3$ and 4).

A *building of type* (W, S) is a geometrical structure two of whose principal features are its apartments and its residues. The *apartments* are substructures isomorphic to $\Sigma(W, S)$ and the *residues* (more precisely, the J*-residues*, where $J \subset S$) are certain distinguished subbuildings of type $(\langle J\rangle, J)$. A building of type (W, S) is called *irreducible* if (W, S) is irreducible. The cardinality $|S|$ is called the *rank* of the building, and thus the rank of a J-residue is $|J|$. (We are implicitly assuming that all buildings in this essay are *thick* as defined in [5, 1.6].)

A *generalized polygon* is a bipartite graph whose diameter is half the length of a shortest circuit. A generalized polygon is also called a generalized n-gon, where n denotes its diameter. A spherical building of rank 2 (i.e. a building of spherical type (W, S) with $|S| = 2$) is the same thing as a generalized n-gon. It is irreducible if and only if $n \geq 3$. Its apartments are the circuits of length $2n$.

A root in a building is, by definition, a root of one of its apartments (and, in fact, a root of many of its apartments). Thus a root in a generalized n-gon Δ is simply a path of length n. To each root $\alpha = (x_0, x_1, \ldots, x_n)$ in a generalized n-gon Δ we associate the corresponding *root group* U_α. This is the pointwise stabilizer in $\mathrm{Aut}(\Delta)$ of all the vertices at distance at most 1 from some vertex in the set $\{x_1, x_2, \ldots, x_{n-1}\}$. See [5, 11.1] for the definition of a root group in an irreducible spherical building of arbitrary rank.

A spherical building of rank at least 2 is called *Moufang* if it is irreducible and if for each root α, the root group U_α acts transitively on the set of apartments containing α. This notion was introduced by Tits, who named it in honor of Ruth Moufang. Its importance stems from the following facts:
(a) The generalized polygon associated with an absolutely simple algebraic group of k-rank 2 is always Moufang (see [4, 43.2.6]).
(b) An irreducible spherical building of rank greater than 2 is always Moufang as are all its irreducible residues of rank at least 2 (see [5, 11.6 and 11.8]).
(c) An irreducible spherical building of rank greater than 2 is uniquely determined by its irreducible residues of rank 2 (see [5, 10.2]).

Moufang polygons were classified in [4]. There are nine families, one each of triangles, hexagons and octagons and six families of quadrangles. (In particular, $n = 3$, 4, 6 and 8. The fact that there are no Moufang pentagons implies by (b) that there are no irreducible spherical buildings of type H_3 or H_4. It was this discovery that led Tits to the idea that Moufang polygons could be classified.)

The polygons in a given family are classified by a corresponding family of algebraic structures. For example, Moufang triangles are classified by fields, skew fields and octonion division algebras. What we mean by this is that for each field, skew field or octonion division algebra K, there is a corresponding Moufang triangle $\mathcal{T}(K)$ that arises by applying a certain recipe to K, that every Moufang triangle arises in this way and that non-isomorphic K's yield non-isomorphic Moufang triangles (i.e. K is an invariant of the triangle).

The first family of Moufang quadrangles are those classified by anisotropic quadratic spaces (K, L, q)—here K is a commutative field, L is a vector space over K and q is a quadratic form on L such that $q(u) = 0$ if and only if $u = 0$. (Actually, it is only the similarity class of (K, L, q) that is an invariant of the corresponding quadrangle.) The second family of Moufang quadrangles are those classified by "involutory sets," by which we mean pairs (K, σ), where K is a field or skew field and σ is an involution of K, that is to say, an anti-automorphism of order 2.

We will not say anything about the remaining Moufang quadrangles (or the Moufang hexagons and octagons) except to observe that among them are the Moufang quadrangles of type E_6, E_7 and E_8. These are the spherical buildings associated to certain rank 2 forms of absolutely simple algebraic groups of type E_6, E_7 and E_8 (as opposed to the split forms of these algebraic groups which give rise to buildings of type E_6, E_7 and E_8).

Irreducible spherical buildings of rank greater than 2 can be described in terms of their irreducible rank 2 residues. For example, buildings of type E_8 are classified by fields, and for a given field K, the irreducible rank 2 residues of the building $\mathsf{E}_8(K)$ are all isomorphic to the Moufang triangle $\mathcal{T}(K)$. To give another example, there is for each anisotropic quadratic space (K, L, q) a unique building of type B_ℓ whose irreducible rank 2 residues are isomorphic either to $\mathcal{T}(K)$ or to the Moufang quadrangle that arises from (K, L, q)—but there are several other families of buildings of type B_ℓ as well.

In every case, the algebraic structure that classifies the corresponding family of Moufang irreducible spherical buildings of rank $\ell \geq 2$ involves a field or skew field or octonion division algebra K, and this K is an invariant (with some minor caveats) of the corresponding building Δ. We call K the *defining field* of Δ; see [6, 30.29]. (If Δ is the spherical building associated with an absolutely simple algebraic group of k-rank 2, then k is either the center of K or, in those cases involving an involution σ of K, the set of fixed points of σ in the center of K.)

The classification of irreducible spherical buildings of rank greater than 2 was first carried out in [2]. A "second generation" proof based on (c) above and the classification of Moufang polygons can be found in Chapter 40 of [4]. The results of the classification are summarized in Appendix B in [6].

The *affine* Coxeter systems are those whose Coxeter complex has a natural embedding in a Euclidean space. The irreducible affine Coxeter systems are precisely the Coxeter systems $\tilde{\mathsf{A}}_\ell$, $\tilde{\mathsf{B}}_\ell$, $\tilde{\mathsf{C}}_\ell$, etc., associated with the extended Dynkin diagrams (where ℓ is the dimension of the Euclidean space), and an irreducible *affine building* X is one whose type is one of these Coxeter systems. A building X of type $\tilde{\mathsf{A}}_\ell$, $\tilde{\mathsf{B}}_\ell$, $\tilde{\mathsf{C}}_\ell$, etc., has, in addition to its apartments and residues, a third principal feature, namely its *building at infinity* X^∞. This is a spherical building of type A_ℓ, B_ℓ, C_ℓ, etc. It is usual to call X a *Bruhat-Tits building* if, in addition, X^∞ is Moufang.

Let A be an apartment of a Bruhat-Tits building X. Every root of A belongs to a unique "parallel class" consisting of a discrete chain of roots $\cdots \subset \alpha_{i-1} \subset \alpha_i \subset \alpha_{i+1} \subset \cdots$ of A. Corresponding to A is an apartment A^∞ of $\Delta := X^\infty$. Now let α be a root of A^∞ and let U_α be the corresponding root group of Δ. Tits showed that each element g of U_α^* is induced by a unique element $\tilde{g}$ of $\mathrm{Aut}(X)$. Furthermore, there is a parallel class of roots $\{\alpha_i \mid i \in \mathbb{Z}\}$ of A and a map φ_α from U_α^* to $\mathbb{Z}$ such that for each $g \in U_\alpha^*$, the fixed point set of $\tilde{g}$ in A is $\alpha_{\varphi_\alpha(i)}$. Writing U_α additively, even though it might not be abelian, and setting $\varphi_\alpha(0) = \infty$, we observe that $\varphi_\alpha(-g) = \varphi_\alpha(g)$ and $\varphi_\alpha(g+h) \geq \min\{\varphi_\alpha(g), \varphi_\alpha(h)\}$ for all $g, h \in U_\alpha$. It follows that the formula

$$d_\alpha(g, h) = 2^{-\varphi_\alpha(g-h)}$$

defines a metric d_α on U_α. The group U_α, it turns out, must be complete with respect to this metric. Moreover, the root α can be chosen so that U_α (or a canonical subgroup of U_α in certain cases) is isomorphic to the additive group of the defining field K of Δ and for this choice of α, the map φ_α (or its restriction to the canonical subgroup) is given by a discrete valuation on K with respect to which K is complete.

Suppose conversely that Δ is an arbitrary Moufang spherical building of rank at least 2 whose defining field K is complete with respect to a discrete valuation. There is a recipe (depending on the family to which the building belongs) that starts with the valuation on K and produces a set of functions $\varphi_\alpha\colon U_\alpha^* \to \mathbb{Z}$, one for each root α in an apartment of Δ. The classification of Bruhat-Tits buildings consists of a uniqueness assertion and an existence assertion. The uniqueness assertion says that if $\Delta = X^\infty$ for some Bruhat-Tits building X, then X is uniquely determined by Δ. The existence assertion is more difficult to state. It says that if the maps φ_α satisfy certain compatibility conditions which include the requirement that the map d_α defined in terms of φ_α as in the formula above is a metric on U_α, then there exists a Bruhat-Tits building such that $X^\infty = \Delta$.

In [1], the compatibility conditions on the maps φ_α were shown to hold in every case except when Δ is a Moufang quadrangle of type E_6, E_7 or E_8. The compatibility condition was recently shown (in [7]) to hold in the case that Δ is a Moufang quadrangle of type E_6 or E_7. The case E_8 remains an open problem.

We thus have the following conclusion: Bruhat-Tits buildings of dimension $\ell \geq 2$ (with the possible—but unlikely—exception of those whose building at infinity is a Moufang quadrangle of type E_8) are classified by irreducible Moufang spherical buildings of rank ℓ whose defining field K is complete with respect to a discrete valuation such that each of its root groups U_α is complete with respect to the metric d_α referred to above.

(If Δ is the spherical building associated to an absolutely simple algebraic group of k-rank at least 2—but not a quadrangle of type E_8—and k is complete with respect to a discrete valuation, then K is also complete with respect to a discrete valuation and all its roots groups are automatically complete. Hence X always exists in this case. See [6, 27.7 and 30.32].)

The classification of Bruhat-Tits buildings—and much more—was first carried out in [1] and [3]. See [6] for a reworking of the proof.

If a Bruhat-Tits building X is locally finite, then in addition to being complete with respect to a discrete valuation, the defining field K of $\Delta = X^\infty$ must have finite residue field $\bar{K}$. This puts strong constraints on the algebraic structures that can occur. For example, there are no such octonion division algebras, and anisotropic quadratic spaces over such a field exist only in dimension at most 4. In Chapter 28 of [6] a precise description of all the possibilities is given and locally finite Bruhat-Tits building are split up into 35 families according to the dimensions of the various algebraic structures involved and the structure of the residues.

The irreducible residues of rank at least 2 of a Bruhat-Tits building are always Moufang. For example, the irreducible rank 2 residues of the building $\tilde{E}_8(K)$ are all isomorphic to the Moufang triangle $\mathcal{T}(\bar{K})$. (Here $\tilde{E}_8(K)$ denotes the unique Bruhat-Tits building whose building at infinity is $E_8(K)$.) For any given prime power q, there are infinitely many complete fields K such that $\bar{K} \cong \mathbb{F}_q$. This observation yields examples of distinct Bruhat-Tits buildings which agree on balls of any given radius. In particular, Bruhat-Tits buildings, in contrast to spherical buildings, are not uniquely determined by their rank 2 residues.

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The Current State of Knowledge of Cohomological Dimension and other Homological Finiteness conditions for Soluble Groups, Elementary Amenable Groups and Groups in General, in a nutshell

Peter H. Kropholler

How much easier it would be to make a short crisp title if we had short crisp answers. Much is known yet some relatively innocent looking questions remain tough. In this talk I simply take a little historical survey without the details and begin with some landmarks:

- [Hall, 1959] *Every finitely generated abelian-by-nilpotent group is residually finite.* Philip Hall was one of the first to see deep connections between soluble group theory and other branches of mathematics. In this case he was motivated by Hilbert's Basissatz and Nullstellensatz both of which are embedded in some form in this theorem.
- [Jateogaonkar, Roseblade, 1973] *Every finitely generated abelian-by-polycyclic group is residually finite.*
- [Bieri–Strebel, 1982] *The characterization of finitely presented metabelian groups.*
- [Kropholler, 1984] *Every finitely generated soluble group either*
  - *is minimax; or*
  - *has a section (S/T for some $T \lhd S \leq G$) isomorphic to a wreath product $C_p \wr C_\infty$ where p is a prime.*

  Notation: C_∞ is the abstract notation for the additive group $\mathbb{Z}$, and C_p is what you think it is. The base of the wreath product is an elementary abelian p-group of infinite rank, the infinite cyclic group permutes a basis of this p-group regularly. A soluble group is called *minimax* if and only if there is a series of finite length $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ in which the factors G_{i+1}/G_i are either cyclic or quasicyclic. A *quasicyclic group* is one of the groups C_{p^∞} for some prime p. Each quasicyclic group is the colimit (directed union) of finite cyclic groups of some fixedprime power order and these groups are seen concretely as the groups of p-roots of unity in the field $\mathbb{R}$ of complex numbers. Minimax groups are not fashionable but they are here to stay.

- [Gildenhuys, Bieri, Kropholler, Strebel, Nucinkis, Martinez-Perez, Mislin] *For any soluble group, or more generally, any elementary amenable group G, the following are equivalent:*
  - *G is of type* FP_∞*;*
  - *G admits a cocompact model for the classifying space $\underline{E}G$ of proper actions;*
  - *the Hirsch length and virtual cohomological dimension of G are finite and equal;*
  - *G is constructible in the sense that it can be built from the trivial group by a series of ascending HNN-extensions and finite extensions.*

  Moreover, groups satisfying these conditions are virtually soluble minimax.

In general, Bieri, Strebel and Gildenhuys use the term constructible to allow arbitrary HNN-extensions and amalgamated free products from pieces already constructed, as well as finite extensions. For soluble groups the only amalgamated free product that is relevant is one in which both indices are two: it is the same as extending by an infinite dihedral group D_∞ and so is accounted for by a stationary HNN-extension and a finite (index 2) extension.

It is time for examples: two examples will be more than enough!

- The lamplighter group $C_2 \wr C_\infty$ and its cousins, the traffic light groups $C_p \wr C_\infty$. We have already seen how these groups fit neatly into a dichotomy of finitely generated soluble groups.
- Fox's group $F = \langle x, y;\ y^{-1}xy = x^2\rangle$ and its siblings the ascending HNN-extensions $\langle x, y;\ y^{-1}xy = x^n\rangle$ where n is a non-zero integer.

Fox's group is a lovely example. It is an ascending HNN extension and also a soluble minimax group. It is a member of the family of Baumslag–Solitar groups, although not generic in that family by a very long way. It is not the fundamental group of any 3-manifold but it has infinite cyclic derived factor group, trivial Schur multiplier and is finitely presented which means that it arises as the fundamental group of a higher knot complement. According to Kervaire's theorem, since it additionally has a presentation of deficiency 1 as you can see, it is the fundamental group of a complement of S^2 in S^4. So why do I call it Fox's group? Inside Fox's group is the series

$$F \rhd \langle x^F\rangle \rhd \langle x\rangle \rhd 1.$$

in which the factors are C_∞, C_{2^∞}, C_∞. It has Hirsch length 2.

Now, let's look at the natural conjectures on dimension of soluble groups. For measuring dimension we will fix a commutative ring k and write

$$\mathrm{cd}_k(G) := \sup\{n;\ H^n(G, M) \neq 0 \text{ for some } kG\text{-module } M\}$$

$$\mathrm{hd}_k(G) := \sup\{n;\ H_n(G, M) \neq 0 \text{ for some } kG\text{-module } M\}$$

for the cohomological and homological dimensions of G over k. Note that if $k_1 \to k_2$ is a ring homomorphism then $\mathrm{cd}_{k_1} \geq \mathrm{cd}_{k_2}$ and $\mathrm{hd}_{k_1} \geq \mathrm{hd}_{k_2}$, and since $\mathbb{Z}$ is an initial object in the category of rings, the dimensions of $\mathbb{Z}$ always bound above the dimensions over other rings.

Conjecture 1. *Let G be an elementary amenable group. Then G has finite cohomological dimension over k if and only if the following three conditions are satisfied:*

- *G has finite Hirsch length;*
- *G has cardinality less than $\aleph_\omega$; and*
- *finte subgroups of G have orders invertible in k.*

Theorem 2. *Let G be a group. Then*

- $\mathrm{cd}_k(G) = 0$ *if and only if G is finite with order invertible in k;*
- $\mathrm{hd}_k(G) = 0$ *if and only if G is locally finite with orders of finite subgroups invertible in k;*
- $\mathrm{cd}_k(G) \leq 1$ *if and only if G admits an action on a tree with finite vertex and edge stabilizers whose orders are invertible in k.*
- $\mathrm{cd}_k(G) \leq n$ *if G admits a model for an n-dimensional model of $\underline{E}G$ and the finite subgroups of G have order invertible in k. The converse is true for $n \neq 2$. There is a counterexample when $n = 2$.*
- $\mathrm{hd}_k(G) \leq n$ *if G is a filtered colimit of groups with* $\mathrm{cd}_k \leq n$.

Conjecture 3. *For any group G,* $\mathrm{hd}_k(G) \leq 1$ *if and only if G is a filtered colimit of groups with* $\mathrm{cd}_k \leq 1$.

Lemma 4. *If G is an elementary amenable group then* $\mathrm{hd}_k(G) \leq 1$ *if and only if G is a filtered colimit of groups with* $\mathrm{cd}_k \leq 1$.

Conjecture 5. *Let G be an elementary amenable group. Then* $\mathrm{cd}_k(G) = 2$ *if and only if G admits an action on a 2-dimensional contractible complex with stabilizers finite of orders invertible in k and this happens if and only if exactly one of the following three conditions is satisfied.*

(1) There is a non-zero integer n such that G has a subgroup of finite index isomorphic to $\langle x, y;\ y^{-1}xy = x^n \rangle$.
(2) G has a countably infinite locally finite normal subgroup T such that G/T is either infinite cyclic or infinite dihedral.
(3) G is locally finite and has cardinality $\aleph_1$.

Quasi-isometry invariance of splittings and the lamplighter group

Panos Papasoglu

Stallings showed that a finitely generated group has more than one end if and only if it splits over a finite group. This gives a geometric characterization of finitely generated groups which split over a finite group, where geometric here means in the sense of quasi-isometries.

One has a similar geometric characterization for splittings over 2-ended groups which applies to finitely presented groups. It was shown in [2] that a one ended finitely presented group, which is not virtually a surface group, splits over a 2-ended group if and only if its Cayley graph is coarsely separated by a quasi-line.It is natural to ask whether the characterization given in [2] applies in fact to all one ended finitely generated groups (as it is the case for Stallings' theorem).

We show that there is a one-ended finitely generated group (the lamplighter group) which is coarsely separated by a quasi-line but does not split over a 2-ended group. It turns out that the same group can be used to answer a question of Kleiner ([3], problem 4.5): we show the Cayley graph of the lamplighter group is coarsely separated by quasi-circles. Note that if the Cayley graph of a finitely presented groups is coarsely separated by quasi-circles then the group is virtually a surface group (see [1]).

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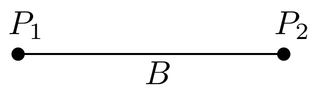
Cocompact lattices in complete Kac–Moody groups

Anne Thomas

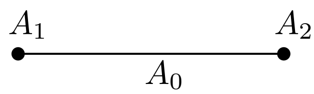
(joint work with Inna (Korchagina) Capdeboscq)

A classical theorem of Siegel [4] states that the minimum covolume among lattices in $G = SL_2(\mathbb{R})$ is $\frac{\pi}{21}$, and determines the lattice which realises this minimum. More recently, Lubotzky [5] and Lubotzky–Weigel [6] constructed the lattice of minimal covolume in $G = SL_2(K)$, where K is a nonarchimedean local field, such as $\mathbb{Q}_p$ or the field $\mathbb{F}_q((t^{-1}))$ of formal Laurent series.

The group $G = SL_2(\mathbb{F}_q((t^{-1})))$ is the first example of a complete Kac–Moody group of rank 2 over a finite field. Such Kac–Moody groups are locally compact, totally disconnected topological groups, which in general are non-linear. The Kac–Moody groups G that we consider have Bruhat–Tits building a regular tree X (see [2]), and the action of G on X induces an edge of groups

$\mathbb{G} =$ 

where P_1 and P_2 are the standard parahoric subgroups of G, and $B = P_1 \cap P_2$ is the standard Iwahori subgroup. Now let m, n be integers ≥ 2. An (m, n)*–amalgam* is a free product with amalgamation $A_1 *_{A_0} A_2$, where the group A_0 has index m in A_1 and index n in A_2. The amalgam is *faithful* if A_0, A_1 and A_2 have no common normal subgroup. In Bass–Serre theory, an (m, n)–amalgam is the fundamental group Γ of an edge of groups

$\mathbb{A} =$ 

with universal cover the (m, n)–biregular tree, and this amalgam is faithful if and only if $\Gamma = \pi_1(\mathbb{A}) \cong A_1 *_{A_0} A_2$ acts faithfully on the universal cover.

The question of classifying amalgams is, in general, difficult. A deep theorem of Goldschmidt [3] established that there are only 15 faithful $(3, 3)$–amalgams of finite groups, and classified such amalgams. Goldschmidt and Sims conjectured that when both m and n are prime, there are only finitely many faithful (m, n)–amalgams of finite groups. On the other hand, Bass–Kulkarni [1] showed that if either m or n is composite, there are infinitely many faithful (m, n)–amalgams of finite groups. This result implies that in the automorphism group of an (m, n)–biregular tree, there is no positive lower bound on the set of covolumes of lattices.

Now let Γ be a cocompact lattice in the complete Kac–Moody group G which acts transitively on the edges of the Bruhat–Tits tree X. Then Γ is the fundamental group of an edge of groups $\mathbb{A}$ as above, with A_0, A_1 and A_2 finite groups. Hence to classify the edge-transitive cocompact lattices in G, we classify the amalgams $A_1 *_{A_0} A_2$ which embed in G. We note that, since the action of G on X is not in general faithful, an amalgam Γ may embed as a cocompact edge-transitive lattice in G even though it is not faithful.

Our first main result is Theorem 1 below, which classifies the edge-transitive lattices in G. There are some exceptional edge-transitive lattices for small values of p and q, which we do not discuss here for reasons of space. The group G in our results is a *topological* Kac–Moody group, meaning that it is the completion of a minimal Kac–Moody group Λ with respect to some topology. We use the completion in the 'building topology' (see [2]).

Our notation is as follows. We write C_n for the cyclic group of order n and S_n for the symmetric group on n letters. Since for a finite field $\mathbb{F}_q$ and the root system A_1 there exist at most two corresponding finite groups of Lie type (one isomorphic to $SL_2(\mathbb{F}_q)$, and the other to $PSL_2(\mathbb{F}_q)$), to avoid complications we use Lie-theoretic notation, and write $A_1(q)$ which stands for both of these groups. We denote by T a fixed maximal split torus of G with $T \leq P_1 \cap P_2$. The centre $Z(G)$ of G is then contained in T, and T is isomorphic to a quotient of $\mathbb{F}_q^* \times \mathbb{F}_q^*$ (the particular quotient depending upon G). Finally, since each parabolic/parahoric subgroup P_i, $i = 1, 2$, admits a Levi decomposition, we denote by L_i a Levi complement of P_i, $i = 1, 2$. Since $L_i = M_iT$, where $M_i \cong A_1(q)$ is normalized by T, let H_i be a non-split torus of M_i such that $N_T(H_i)$ is as big as possible.

We say that two edge-transitive cocompact lattices $\Gamma = A_1 *_{A_0} A_2$ and $\Gamma' = A_1' *_{A_0'} A_2'$ in G are *isomorphic* if $A_i \cong A_i'$ for $i = 0, 1, 2$ and the obvious diagram commutes; our classification of edge-transitive lattices is up to isomorphism. In particular, this means that we assume $A_i \leq P_i$ for $i = 1, 2$.

Theorem 1. *Let G be a topological Kac–Moody group of rank 2 defined over a finite field $\mathbb{F}_q$ of order $q = p^a$ where p is prime, with symmetric generalised Cartan matrix $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$. Assume that $q \geq 60$. Then G has edge-transitive cocompact lattices Γ of each of the following isomorphism types, and every edge-transitive cocompact lattice Γ in G is isomorphic to one of the following amalgams.*

(1) If $p = 2$ *then* $\Gamma = A_1 *_{A_0} A_2$ *where for* $i = 1, 2$*:*
(a) $A_i = A_0 H_i$ *with* $H_i \cong C_{q+1}$*; and*
(b) $A_0 \leq Z(G)$.
(2) If p *is odd and* $L_i/Z(L_i) \cong PSL_2(q)$*, then one of the following holds:*
(a) $q \equiv 1 \pmod 4$ *and* G *does not contain any edge-transitive cocompact lattices.*
(b) $q \equiv 3 \pmod 4$ *and* $\Gamma = A_1 *_{A_0} A_2$ *where for* $i = 1, 2$*:* $A_i = A_0 N_{M_i}(H_i)$ *where* $A_0 \leq Z(G)$.
(3) If p *is odd and* $L_i/Z(L_i) \cong PGL_2(q)$*, then*
(a) When $q \equiv 1 \pmod 4$*, let* $Q_i \in \mathcal{S}yl_2(Z(L_i))$ *and* Q_i^0 *be a unique subgroup of* Q_i *index* 2*. Then:*
(i) If $Q_i^0 \neq 1$ *is not contained in* $Z(G)$*, then there are no edge-transitive lattices.*
(ii) If $Q_i^0 \leq Z(G)$*, then one of the following holds:*
(A) $\Gamma = A_1 *_{A_0} A_2$ *where for* $i = 1, 2$*,* $A_1 \cong A_2$ *and* $A_i = H_i O_2(C_i)\langle t_i\rangle Z_0$ *where* $C_i := C_{L_i}(H_i)$*,* $t_i \in N_T(H_i) - C_T(H_i)$ *is of order* 2*,* $Z_0 \leq Z(G)$ *and* $A_0 = Q_i\langle t_i\rangle Z_0$.
(B) If $Q_i \not\leq Z(G)$*, there are no other edge-transitive lattices, but if* $Q_i \leq Z(G)$*,* $\Gamma = A_1 *_{A_0} A_2$ *where for* $i = 1, 2$*,* $A_1 \cong A_2$ *with* $A_i = H_i O_2(C_i) Z_0$ *where* $C_i = C_{L_i}(H_i)$*,* $Z_0 \leq Z(G)$ *and* $A_0 = Q_i Z_0$.
(b) When $q \equiv 3 \pmod 4$*:*
(i) $\Gamma = A_1 *_{A_0} A_2$ *with* $A_1 \cong A_2$ *such that* $A_i = C_i' T_0 Z_0$ *with* $C_i' \leq C_{L_i}(H_i)$ *and* $|C_i' : H_i| = 2$*. Moreover,* $T_0 \in \mathcal{S}yl_2(T)$*,* $Z_0 \leq Z(G)$ *and* $A_0 = T_0 Z_0$.
(ii) If $Z(M_i) \not\leq Z(G)$*, then there are no other edge-transitive lattices; but*
(iii) If $Z(M_i) \leq Z(G)$*, then also* $\Gamma = A_1 *_{A_0} A_2$ *with* $A_1 \cong A_2$ *where either* $A_i = C_i' A_0$ *with* C_i' *is as described above,* $C_i' \cap A_0 = Z(M_i)$ *and* $A_0 \leq Z(G)$*, or* $2(b)$ *holds.*

Our second main result, on covolumes, is Theorem 2 below. The Haar measure μ on G may be normalised so that the covolume $\mu(\Gamma\backslash G)$ of an edge-transitive cocompact lattice $\Gamma = A_1 *_{A_0} A_2$ is equal to $|A_1|^{-1} + |A_2|^{-1}$. Using this normalisation, we obtain the following.

Theorem 2. *Let* G *be as in Theorem 1 above and* $q \geq 540$*. Then*

$$\min\{\mu(\Gamma\backslash G) \mid \Gamma \text{ a cocompact lattice in } G\} = \frac{2}{(q+1)|Z(G)|\delta_0}$$

where $\delta_0 \in \{1, 2, 4\}$ *(depending upon the particular group* G*). Moreover, the cocompact lattice of minimal covolume in* G *is edge-transitive whenever* G *admits such a lattice, and otherwise acts on the Bruhat–Tits tree for* G *with two orbits of edges and two orbits of vertices.*

A key tool for the proofs of Theorems 1 and 2 above is Proposition 3 below, which we use together with classical results of finite group theory.

Proposition 3. *Let G be as in Theorem 1 above. If Γ is a cocompact lattice in G, then Γ does not contain p–elements.*

Proposition 3 is proved geometrically, by considering the action of root groups on the tree for G, and the stabilisers of ends of this tree.

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Rays in locally symmetric spaces and singular vectors

Cornelia Drutu
(joint work with Irina Tita)

This talk investigates the relationship between families of linear forms with exceptional behaviour, in particular singular, and geodesic rays in locally symmetric spaces.

This talk presents one of the numerous topics situated at the interface of the geometry of non-positively curved spaces, that of spherical buildings, and the geometry of numbers. In Geometry of Numbers an important object of study is the non-increasing function defined, for an arbitrary matrix L with ℓ rows and m columns, as $\psi_L : \mathbb{N} \to \mathbb{R}_+$,

$$\psi_L(k) = \inf_{\|\mathbf{q}\|_{max} \leq k, \mathbf{q} \in \mathbb{Z}^m, \mathbf{p} \in \mathbb{Z}^\ell} \|L\mathbf{q} - \mathbf{p}\|_{max} .$$

By Dirichlet's theorem for every L, $\psi_L(k) \leq k^{-\frac{m}{\ell}}$.

One can naturally associate to a matrix L a locally geodesic ray in the locally symmetric space $X_n = SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n)$, where $n = \ell + m$, and to the behaviour of ψ_L the behaviour of this ray in the cuspidal end of X_n. The first to note this was Dani in [Dan85].

The generic matrices satisfy two important properties:

(1) There exists a constant $C = C(\ell, m)$ such that for almost every $L \in M_{\ell,m}(\mathbb{R})$
$$\limsup_{k\to\infty} k^{\frac{m}{\ell}} \psi_L(k) > C .$$
This corresponds to the fact that an arbitrary ray in X_n returns in a large enough compact infinitely often.

(2) Groshev's theorem implies that for almost every $L \in M_{\ell,m}(\mathbb{R})$,

- for every $\epsilon > 0$, $\liminf_{k\to\infty} k^{\frac{m}{\ell}} (\ln k)^{\frac{1}{\ell}+\epsilon} \psi_L(k) = \infty$.
- $\liminf_{k\to\infty} k^{\frac{m}{\ell}} (\ln k)^{\frac{1}{\ell}} \psi_L(k) < 1$.

These statements correspond to what is known as the logarithm law, established by Sullivan and Kleinbock-Margulis.

The set of exceptional matrices L, that is matrices with non-generic behaviour, contains many interesting subclasses. Two natural questions to ask are:

(1) what are all the possible exceptional behaviours ?

(2) how large are the corresponding 'exceptional sets' ?

More is known about the exceptional behaviour with respect to the generic statement (2). Given a matrix L one can define the supremum α_L of all the exponents $\beta > 0$ such that

$$\liminf_{k\to\infty} k^{\beta} \psi_L(k) < 1 .$$

Groshev's theorem implies that for almost every L the exponent α_L is equal to $\frac{m}{\ell}$. For every real number $\alpha > \frac{m}{\ell}$, the set of matrices $L \in M_{\ell,m}(\mathbb{R})$ such that $\alpha_L = \alpha$, even though of Lebesgue measure zero, has Hausdorff dimension $(m-1)\ell + \frac{m+\ell}{1+\alpha}$ ([Dod92], [DV97]). The calculation of the Hausdorff dimension of the set vectors on a rational quadric with $\alpha_L = \alpha > 1$ can be found in [Dru05].

These sets of exceptional matrices and vectors correspond to rays in the symmetric space X_n (respectively in a locally symmetric space determined by the rational quadric) that infinitely often rise at the time t at some height βt (height measured with respect to the appropriate Busemann function), where β is related by a precise equation to the parameter α.

As for the 'exceptional matrices' with respect to the generic statement (1), much less is known. Following the terminology in [Cas57], Chapter V, Section 7, a matrix $L \in M_{\ell,m}(\mathbb{R})$ is called *singular* if

$$\lim_{k\to\infty} k^{\frac{m}{\ell}} \psi_L(k) = 0 . \tag{1}$$

Such a matrix corresponds to a geodesic ray that eventually leaves every compact in the locally symmetric space X_n .

For $\ell = m = 1$ the only singular matrices (i.e. numbers) are the rational numbers. It is proved in [Che07] that for every $n \geq 2$ the set of n-tuples $(x_1, x_2, ..., x_n)$ such that

$$\lim_{k\to\infty} k \min_{i\in\{1,2,...n\}} \psi_{x_i}(k) = 0 , \tag{2}$$

has Hausdorff dimension $n - \frac{1}{2}$. Note that the set of n-tuples contained in rational hyperplanes is of Hausdorff dimension $n-1$, thus most of the n-tuples with the above property are independent over $\mathbb{Q}$.

When $\max(\ell, m) > 1$ singular matrices do exist. As shown in [Cas57, Chapter V, Section 7, Theorem XIV], for instance in the case $m = 1, \ell = 2$, for every function $\omega\colon \mathbb{N} \to \mathbb{R}_+$ there exists a linear form $L : \mathbb{R}^2 \to \mathbb{R}$ with rationally independent coefficients such that

$$\psi_L(k) \le \omega(k) \text{ for every } k \in \mathbb{N}\,. \tag{3}$$

The purpose of this talk is to explain how the following two results in Geometry of Numbers can be obtained from results in [Wei04], and arguments of non-positive curvature in various symmetric spaces and of geometry of spherical buildings in their boundaries.

Theorem 1. *Let $\omega\colon (0,\infty) \to (0,\infty)$ be an arbitrary continuous function.*

(a) ***(systems of linear forms)*** *For every $\ell \ge 1$ and every $m \ge 2$ there exists an irrational matrix L in $M_{\ell,m}(\mathbb{R})$ and a T_0 such that for every $T \ge T_0$ the following system has a solution $\overline{q} \in \mathbb{Z}^m$*

$$\left\{ \begin{array}{ccc} \|\overline{q}\| & \le & T \\ \|L\overline{q}\|_{\mathbb{Z}^\ell} & \le & \omega(T)\,. \end{array} \right. \tag{4}$$

(b) ***(vectors)*** *Assume that ω is continuous non-increasing and $T \mapsto T\omega(T)$ is increasing and unbounded. Then for every $\ell \ge 2$ there exists an irrations column vector $\mathbf{x}$ in $M_{\ell,1}(\mathbb{R})$ and a T_0 such that for every $T \ge T_0$ the following system has a solution $q \in \mathbb{Z}$*

$$\left\{ \begin{array}{ccc} |q| & \le & T \\ \|q\mathbf{x}\|_{\mathbb{Z}^\ell} & \le & \omega(T)\,. \end{array} \right. \tag{5}$$

Assume now that the function $T \mapsto T\omega(T)$ is bounded. Then any vector $\mathbf{x}$ such that the system (5) has a solution $q \in \mathbb{Z}$ for every T greater than some T_0 is contained in an affine line over $\mathbb{Q}$.

Assume that $\lim_{T\to\infty} T\omega(T) = 0$. Then any vector $\mathbf{x}$ such that the system (5) has a solution $q \in \mathbb{Z}$ for every T greater than some T_0 is a rational vector.

Theorem 2. *Consider a rational quadric $\mathfrak{Q}_\mathfrak{q}$ (i.e. a hypersurface defined by a rational quadratic equation) containing an affine line over $\mathbb{Q}$.*

(a) *Let $\omega\colon (0,\infty) \to (0,\infty)$ be an arbitrary continuous non-increasing function such that the function $T \mapsto T\omega(T)$ is increasing and unbounded. Then there exists an irrational vector $\mathbf{x} \in \mathfrak{Q}_\mathfrak{q}$ and a T_0 such that for every $T \ge T_0$ the following system has a solution $\frac{1}{q}\mathbf{p} \in \mathfrak{Q}_\mathfrak{q}$*

$$\left\{ \begin{array}{ccc} |q| & \le & T \\ |q|\|\mathbf{x} - \frac{1}{q}\mathbf{p}\| & \le & \omega(T)\,. \end{array} \right. \tag{6}$$

(b) *Assume now that $\omega\colon (0,\infty) \to (0,\infty)$ is a function such that $T \mapsto T\omega(T)$ is bounded. Then any vector $\mathbf{x}$ such that the system (6) has a solution $\frac{1}{q}\mathbf{p} \in \mathfrak{Q}_\mathfrak{q}$ for every T greater than some T_0 is contained in an affine line over $\mathbb{Q}$ entirely contained in $\mathfrak{Q}_\mathfrak{q}$. More precisely, there exist two rational vectors $\mathbf{r}$ and $\mathbf{r}'$ in $\mathbb{Q}^\ell$ such that for some $t \in \mathbb{R}$*

$$\mathbf{x} = \mathbf{r} + t\mathbf{r}'\,,$$

moreover $\mathbf{r} + \mathbb{R}\mathbf{r}' \subset \mathfrak{Q}_{\mathfrak{q}}$. *If moreover* $\lim_{T\to\infty} T\omega(T) = 0$ *then* $\mathbf{x}$ *is a rational vector.*

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Contractibility of the Kakimizu complex

Piotr Przytycki

(joint work with Jennifer Schultens)

We study a generalisation $MS(E)$ of the following simplicial complex $MS(L)$ defined by Kakimizu [4]. Let $E = E(L)$ be the exterior of a tubular neighbourhood of a knot L in $\mathbb{S}^3$. A *spanning surface* is a surface properly embedded in E, which is contained in some Seifert surface for L. Let $\mathcal{MS}(L)$ be the set of isotopy classes of spanning surfaces which have minimal genus. The vertex set of $MS(L)$ is defined to be $\mathcal{MS}(L)$. Vertices $\sigma, \sigma' \in \mathcal{MS}(L)$ span an edge if they have representative spanning surfaces which are disjoint. Simplices are spanned on all complete subgraphs of the 1–skeleton. In other words, $MS(L)$ is the *flag* complex spanned on its 1–skeleton.

The general setting in which we define $MS(E(L))$, or more generally $MS(E, \gamma, \alpha)$, is the following. Let E be a compact connected orientable, irreducible and ∂–irreducible 3–manifold. In particular, for any non-splittable link L in $\mathbb{S}^3$, the complement $E(L)$ of a regular neighbourhood of L satisfies these conditions. Let γ be a union of oriented disjoint simple closed curves on ∂E, which does not separate any component of ∂E. For $E = E(L)$ an example of γ is the set of longitudes of all link components (or its subset). We fix a class α in the homology group $H_2(E, \partial E, \mathbb{Z})$ satisfying $\partial\alpha = [\gamma]$. For $E = E(L)$ and γ the set of longitudes, there is only one choice for α. It is the homology class dual to the element of $H^1(E, \mathbb{Z})$ mapping all oriented meridian classes onto a fixed generator of $\mathbb{Z}$. A *spanning surface* is an oriented surface properly embedded in E in the homology class α whose boundary is homotopic with γ.

We now define the simplicial complex $MS(E, \gamma, \alpha)$, which we abbreviate to $MS(E)$, if $E = E(L)$ and γ is the set of all longitudes. The vertex set of $MS(E, \gamma, \alpha)$ is defined to be $\mathcal{MS}(E, \gamma, \alpha)$, the set of isotopy classes of spanning surfaces which have minimal genus. However, we span an edge on $\sigma, \sigma' \in \mathcal{MS}(E, \gamma, \alpha)$ only if they have representatives $S \in \sigma, S' \in \sigma'$ such that the (connected) lift of $E \setminus S'$ to the infinite cyclic cover associated with α intersects exactly two lifts of $E \setminus S$. This is not always true for disjoint S, S' (because they are allowed to be disconnected).

For every link L it is a basic question to determine the complex $MS(E(L))$ which encodes the structure of the set of all minimal genus spanning surfaces. This has been done for all prime knots of at most 10 crossings by Kakimizu [5, Theorem A]. Moreover, questions about common properties of all $MS(E(L))$ (or rather $MS(L)$) have been asked. Here is a brief summary (for a broader account, see [6]).

Scharlemann–Thompson proved [9, Proposition 5] that $MS(E(L))$ is connected, in the case where L is a knot. Later Kakimizu [4, Theorem A] provided another proof for links. Schultens [10, Theorem 6] proved that, in the case where L is a knot, $MS(E(L))$ is simply-connected (see also [8] for atoroidal genus 1 knots). For atoroidal knots bounds on the diameter of $MS(E(L))$ have been obtained ([6, 8]). Kakimizu conjectured (see [7, Conjecture 0.2]) that $MS(L)$ is contractible. This was verified for special aborescent links by Sakuma [7, Theorem 3.3 and Proposition 3.11], and announced for special prime alternating links by Hirasawa–Sakuma [3]. In the present article, we confirm this conjecture, under no hypothesis, for the complex $MS(E, \gamma, \alpha)$.

Theorem 1. *$MS(E, \gamma, \alpha)$ is contractible.*

Using the same method we are also able to establish the following. Note that for $E = E(L)$ all mapping classes of E fix α and the homotopy class of γ.

Theorem 2. *Let G be a finite subgroup of the mapping class group of E fixing α and the homotopy class of γ. We consider its natural action on $MS(E, \gamma, \alpha)$. Then there is a simplex in $MS(E, \gamma, \alpha)$ fixed by all elements of G.*

Sakuma argued [7, Proposition 4.9(1)] (see also [10, Theorem 5] for knots) that the set of vertices of any simplex of $MS(E, \gamma, \alpha)$ can be realised as a union of pairwise disjoint spanning surfaces. Hence in the language of spanning surfaces Theorem 2 amounts to the following.

Corollary 3. *Let G be a finite subgroup of the mapping class group of E fixing α and the homotopy class of γ. Then there is a union of pairwise disjoint spanning surfaces of minimal genus which is G–invariant up to isotopy.*

In the case where E is atoroidal and ∂E is a union of tori, its interior admits, by the work of Thurston and the theorem of Prasad, a unique complete hyperbolic structure. Then the mapping class group of E coincides with the isometry group of its interior, hence it is finite. Moreover, after deforming the metric in a way discussed in [6, Chapter 10] we can assume that each element of $\mathcal{MS}(E,\gamma,\alpha)$ has a unique representative of minimal area. In this case Corollary 3 gives the following.

Corollary 4. *If E is atoroidal and ∂E is a union of tori, then there is a union of pairwise disjoint spanning surfaces of minimal genus which is invariant under any isometry fixing α (the homotopy class of γ is then fixed automatically). In particular, if $E = E(L)$, then this union is invariant under any isometry.*

A related result concerning periodic knots was proved in Edmonds [1].

Finally, Theorem 1 turns out to be a special case (G trivial) of the following.

Theorem 5. *Let G be any subgroup of the mapping class group of E fixing α and the homotopy class of γ. Then its fixed-point set $\mathrm{Fix}_G(MS(E,\gamma,\alpha))$ is either empty or contractible.*

Outline of the idea. We now outline the main idea of the article. The central object is the *projection map* π, which assigns to a pair of vertices $\sigma, \rho \in \mathcal{MS}(E,\gamma,\alpha)$ at distance $d > 0$ a vertex $\pi_\sigma(\rho)$ adjacent to ρ at distance $d-1$ from σ. Kakimizu [4] used the projection to prove that $MS(E(L))$ is connected, but in fact he did not need to verify that it is well-defined — he worked only with representatives of vertices. We verify that π is well-defined using a result of Oertel on *cut-and-paste operations* on surfaces with *simplified intersection.*

We explain how to prove contractibility of $MS(E,\gamma,\alpha)$. Assume for simplicity that $MS(E,\gamma,\alpha)$ is finite (which is the case for E atoroidal by the work of Haken [2]). We fix some $\sigma \in \mathcal{MS}(E,\gamma,\alpha)$. Then we prove that among vertices farthest from σ there exists a vertex ρ which is *strongly dominated* by $\pi_\sigma(\rho)$. This means that all the vertices adjacent to ρ are also adjacent to or equal $\pi_\sigma(\rho)$. Hence there is a homotopy retraction of $MS(E,\gamma,\alpha)$ onto the subcomplex spanned by all the vertices except ρ. Proceeding in this way we retract the whole complex onto σ.

Remaining questions. Finally, we indicate that questions about the structure of the set of all incompressible spanning surfaces remain open. Kakimizu [4] considers the complex $IS(L)$ whose vertices are isotopy classes of spanning surfaces which are incompressible and ∂–incompressible but not necessarily of minimal genus. The edges of $IS(L)$ are defined like edges of $MS(L)$, in particular we have an embedding of $MS(L)$ into $IS(L)$. Kakimizu asks if $IS(L)$ is contractible as well. He proves that $IS(L)$ is connected, using a composition of the projection π with an additional operation, which we do not know how to make well-defined on the set of isotopy classes of surfaces. This is why we do not know if we can extend Theorem 5 or even Theorem 1 to the complex $IS(L)$ (or rather to $IS(E,\gamma,\alpha)$, appropriately defined). Note however that, since $MS(E,\gamma,\alpha)$ would be a subcomplex of $IS(E,\gamma,\alpha)$, Theorem 2 would trivially carry over to $IS(E,\gamma,\alpha)$.

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Rank rigidity for CAT(0) cube complexes

Pierre-Emmanuel Caprace
(joint work with Michah Sageev)

Let X be a CAT(0) space. A **rank one isometry** of X is a hyperbolic isometry $g \in \mathrm{Is}(X)$ none of whose axes bounds a flat half-plane. If X is Gromov hyperbolic, then every hyperbolic isometry of X is a rank one isometry. However, there are examples of CAT(0) spaces which admit rank one isometries but also contain (not necessarily isolated) Euclidean flats of arbitrary large dimensions. However, in all cases, the existence of a rank one isometry implies that the space X is subjected to some kind of *hyperbolic behaviour*, at least when X is locally compact. The following conjecture suggests that rank one isometries play a major role in the structure of general locally compact CAT(0) spaces.

Conjecture (Ballmann–Buyalo [4])**.** *Let X be a locally compact geodesically complete* CAT(0) *space and Γ be a discrete group acting properly and cocompactly by isometries on X. Then one of the following (mutually exclusive) assertions holds.*

(a) X splits as a product $X \cong X_1 \times X_2$ with non-compact CAT(0) *factors.*
(b) Γ contains a rank one isometry.
(c) X is an irreducible symmetric space of non-compact type and rank ≥ 2.
(d) X is an irreducible Euclidean building of dimension ≥ 2.

The main evidence for this conjecture is provided by the case of complete simply connected Riemannian manifolds of non-positive curvature, where the conjecture holds (see [1] and references therein). Other known cases include piecewise Euclidean CAT(0) cell complexes in dimension ≤ 3 (see [2] and [3]). The main result of this note is the following.

Theorem A. *Let X be a finite-dimensional* CAT(0) *cube complex and $\Gamma \leq \mathrm{Aut}(X)$. If $(X \cup \partial X)^\Gamma = \varnothing$, then there is a Γ-invariant convex sub-complex Y satisfying one of the following (mutually exclusive) assertions.*

(a) Y splits as a product $Y \cong Y_1 \times Y_2$ with non-compact CAT(0) *cubical factors.*
(b) Γ contains an element whose restriction to Y is a rank one isometry.

If in addition X is infinite and locally compact and if Γ acts properly and cocompactly, then the same conclusion holds even without the assumption that $(X \cup \partial X)^\Gamma = \varnothing$.

A first application is a purely geometric proof of the Tits alternative for groups acting on finite-dimensional CAT(0) cube complexes, a result which was first established by M. Sageev and D. Wise [8]. Amongst other applications, we mention the following result related to the *Flat Closing Conjecture*.

Theorem B. *Let X be a locally compact* CAT(0) *cube complex and $\Gamma \leq \mathrm{Aut}(X)$ be a discrete group acting properly and cocompactly. If X splits as a product of n non-compact* CAT(0) *cubical factors, then Γ contains a free Abelian subgroup of rank n.*

In order to present another application, we recall that a **quasi-morphism** of a group G is a map $f\colon G \to \mathbf{R}$ such that $\sup_{g,h\in G} |f(gh)-f(g)-f(h)| < \infty$. The set $\mathrm{QH}(G)$ of all quasi-morphisms is a real vector space containing the space $\ell^\infty(G)$ of all bounded functions as well as the space $\mathrm{Hom}(G, \mathbf{R})$ of usual homomorphisms. We set $\widetilde{\mathrm{QH}}(G) = \mathrm{QH}(G)/\big(\ell^\infty(G) \oplus \mathrm{Hom}(G, \mathbf{R})\big)$. The vector space $\widetilde{\mathrm{QH}}(G)$ can be identified with the kernel of the canonical map $H^2_{\mathrm{b}}(G, \mathbf{R}) \to H^2(G, \mathbf{R})$ from the bounded cohomology of G with trivial coefficients to the usual Hochschild cohomology in degree two. For a non-elementary Gromov hyperbolic group G as well as numerous other groups admitting a 'rank one behaviour', the space $\widetilde{\mathrm{QH}}(G)$ is infinite-dimensional (see [5]). On the other hand, for an irreducible lattice Γ in an adjoint semi-simple Lie group of higher rank, the space $\widetilde{\mathrm{QH}}(\Gamma)$ vanishes (see [6]).

Theorem C. *Let X be a locally compact* CAT(0) *cube complex such that* $\mathrm{Aut}(X)$ *acts cocompactly and let $\Gamma \leq \mathrm{Aut}(X)$ be a lattice. Then the following assertions are equivalent.*

(i) $\widetilde{\mathrm{QH}}(\Gamma)$ *is finite-dimensional.*
(ii) $\widetilde{\mathrm{QH}}(\Gamma) = 0$.
(iii) X admits an $\mathrm{Aut}(X)$*-invariant convex subcomplex Y which admits a canonical splitting $Y \cong T_1 \times \cdots \times T_n$, where each T_i is a quasi-tree and the Γ-action on T_i is quasi-distance-transitive.*

Here, a **quasi-tree** is a space which is quasi-isometric to a simplicial tree. The Γ-action on T_i is called **quasi-distance-transitive** if there is a constant $\delta > 0$ such that for all pairs (x, y) and $(x', y') \in T_i \times T_i$, there is some $\gamma \in \Gamma$ such that $\mathrm{dist}(x, \gamma.x') < \delta$ and $\mathrm{dist}(y, \gamma.y') < \delta$.

The implication (iii) ⇒ (ii) is due to Burger–Monod [6], while the implication (i) ⇒ (iii) requires to combine Rank Rigidity with the constructions of quasi-morphisms elaborated by Bestvina–Fujiwara [5]. For more details as well as other applications, we refer to [7].

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Cohomology computations for relatives of Coxeter groups

Michael Davis
(joint work with Boris Okun)

We compute the group cohomology $H^*(G;\mathbb{Z}G)$ or the reduced L^2-cohomology, $\mathcal{H}^*(G;L^2G)$ when G is either

- a graph product of infinite groups,
- an Artin group, or
- a Bestvina-Brady group.

The proofs will appear in [8].

Suppose (W,S) is a Coxeter system. A subset $T\subset S$ is *spherical* if it generates a finite subgroup of W. Let $\mathcal{S}$ denote the poset of spherical subsets of S. The *nerve* of (W,S) is the simplicial complex L with vertex set S and with one simplex for each nonempty $T\in\mathcal{S}$. Let $K:=\mathrm{Flag}(\mathcal{S})$ be the simplicial complex of all flags in $\mathcal{S}$ (the *geometric realization of* $\mathcal{S}$). Note that K is isomorphic to the cone on the barycentric subdivision of L. For each $s\in S$, put $K_s:=\mathrm{Flag}(\mathcal{S}_{\geq\{s\}})$ and for each $T\leq S$, put

$$K_T:=\bigcap_{s\in T}K_s \quad\text{and}\quad K^T:=\bigcup_{s\in T}K_s.$$

(K is the *Davis chamber* and K_s is a *mirror* of K.) Also, for each $T\in\mathcal{S}$, $\partial K_T:=\mathrm{Flag}(\mathcal{S}_{>T})$ (which is isomorphic to the barycentric subdivision of the link of T in L).

Previous results. The following theorem was proved in [2] (also see [3]).

Theorem A. ([2, 5]).

$$H^*(W;\mathbb{Z}W)=\bigoplus_{T\in\mathcal{S}}H^*(K,K^{S-T})\otimes A_T,$$

where A_T is a certain nontrivial free abelian group.

Essentially the same result holds for the compactly supported cohomology of any locally finite building of type (W, S) (except that the free abelian group A_T is larger), cf. [4]. In particular, since any graph product of finite groups acts properly and cocompactly on a right-angled building, Theorem A also holds for graph products of finite groups.

Theorem B. (Davis-Leary [7]). *Suppose A is the Artin group associated to (W, S) and that X is its Salvetti complex. Then*

$$\mathcal{H}^*(X; L^2W) = H^*(K, \partial K) \otimes L^2(A).$$

When the $K(\pi, 1)$-Conjecture holds for A, this formula computes $\mathcal{H}^*(A; L^2W)$. (Recall that the $K(\pi, 1)$-Conjecture asserts that $X = BA$.)

The von Neumann dimension of $\mathcal{H}^k(X; L^2A)$ is called the k^{th} *L^2-Betti number*, and denoted $L^2b^k(X; A)$. So, Theorem B gives: $L^2b^k(X; A) = b^k(K, \partial K)$, where $b^k(K, \partial K)$ denotes the ordinary Betti number of $(K, \partial K)$.

Theorem C. (Jensen-Meier [9]). *Suppose A is the right-angled Artin group (the RAAG) associated to the right-angled Coxeter system (a RACS) (W, S). Then*

$$H^n(A; \mathbb{Z}A) = \bigoplus_{T \in \mathcal{S}} H^{n-|T|}(K_T, \partial K_T) \otimes B,$$

where B is a certain free abelian group.

Computations. Suppose that Γ is a graph with vertex set S and that (W, S) is the associated RACS. Let $(G_s)_{s \in S}$ be a family of groups and $G = \prod_\Gamma G_s$, the corresponding graph product. For each spherical subset T, define G_T to be direct product $\prod_{s \in T} G_s$ (it is a subgroup of G). The proofs of following computations use a spectral sequence and in the end, only an associated graded module to a cohomology group is computed which we denote $\operatorname{Gr} H^*(\ )$.

Theorem 1. *Suppose each G_s is infinite. Then*

$$\operatorname{Gr} H^n(G; \mathbb{Z}G) = \bigoplus_{T \in \mathcal{S}} \bigoplus_{p+q=n} H^p(K, \partial K_T; H^q(G_T; \mathbb{Z}G)).$$

Similarly, for L^2-cohomology, we have,

$$L^2b^n(G) = \sum_{T \in \mathcal{S}} \sum_{p+q=n} b^p(K_T, \partial K_T) L^2b^q(G_T).$$

We note that $H^q(G_T; \mathbb{Z}G)$ and $L^2b^q(G_T)$ can be calculated from their values for the G_s by using the Künneth Formula. We also note that if each $G_s = \mathbb{Z}$, then G is a RAAG. Moreover, $G_T = \mathbb{Z}^T$, so $H^q(\mathbb{Z}^T; \mathbb{Z}\mathbb{Z}^T)$ is nonzero only in degree $q = |T|$ (where it is $= \mathbb{Z}$) and we recover Theorem C. Since all L^2-Betti numbers of $\mathbb{Z}^T$ vanish for $T \neq \emptyset$ we also recover Theorem B in the right-angled case.

It is known that any Artin groups A_T of spherical type is a $|T|$-dimensional duality groups, i.e., $H^*(A_T; \mathbb{Z}A_T)$ is concentrated in degree $|T|$ and is free abelian.

Theorem 2. *Suppose A is the Artin group associated to (W,S) and that X is its Salvetti complex. Then*

$$\operatorname{Gr} H^n(A;\mathbb{Z}A) = \bigoplus_{T\in\mathcal{S}} H^{n-|T|}(K_T,\partial K_T)\otimes(\mathbb{Z}A\otimes_{A_T} H^{|T|}(A_T;\mathbb{Z}A_T)).$$

Suppose A is the RAAG associated to the RACS (W,S) with nerve L. Let BB_L denote the kernel of the map $A\to\mathbb{Z}$ which sends each standard generator to 1. If L is acyclic, then BB_L is called a *Bestvina-Brady group* (in which case Bestvina and Brady proved it was type FP).

Theorem 3. *Suppose BB_L is a Bestvina-Brady group. Then its cohomology with group ring coefficients is the same as that of the corresponding Artin group, shifted in degree by* 1*,*

$$\operatorname{Gr} H^n(BB_L;\mathbb{Z}BB_L) = \bigoplus_{T\in\mathcal{S}_{>\emptyset}} H^{n-|T|+1}(K_T,\partial K_T)\otimes\mathbb{Z}(BB_L/(BB_L\cap A_T)).$$

Similarly, for L^2-cohomology, we have,

$$L^2b^n(BB_L) = \sum_{s\in S} b^n(K_s,\partial K_s).$$

A spectral sequence. For all of these computations the proofs involve the following lemma concerning a Mayer-Vietoris type spectral sequence. Suppose a CW-complex X is a union of a collection of subcomplexes $\{X_a\}_{a\in\mathcal{P}}$ indexed by a poset $\mathcal{P}$ giving it the structure of a "poset of spaces" as in [8]. There is a spectral sequence converging to $H^*(X)$ with E_1-page:

$$E_1^{p,q} = C^p(\operatorname{Flag}(\mathcal{P});\mathcal{H}^q),$$

where $\mathcal{H}^q$ denotes the constant coefficient system $\sigma\mapsto X_{\min\sigma}$, where $\min\sigma$ denotes the minimum element of the flag, σ. For each $a\in\mathcal{P}$, put $X_{<a} := \bigcup_{b<a} X_b$. Consider the following hypothesis:

(*) For each $a\in\mathcal{P}$, the map induced by the inclusion, $H^*(X_a)\to H^*(X_{<a})$ is the 0-homomorphism.

Lemma. *Suppose (*) holds. Then the spectral sequences decomposes into a direct sum at E_2 and $E_2 = E_\infty$:*

$$E_\infty^{p,q} = E_2^{p,q} = \bigoplus_{a\in\mathcal{P}} H^p(\operatorname{Flag}(\mathcal{P}_{\geq a}),\operatorname{Flag}(\mathcal{P}_{>a});H^q(X_a)).$$

In each case in which we are interested, $\mathcal{P}=\mathcal{S}$; moreover, the appropriate space associated to the group in question is covered by subcomplexes indexed by $\mathcal{S}$ and condition (*) is easily verified.

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Isoperimetric and finiteness properties of arithmetic groups

Kevin Wortman
(joint work with Mladen Bestvina and Alex Eskin)

A *coarse manifold* Σ in a metric space X is a function from the vertices of a triangulated manifold M into X. In a slight abuse of language, we refer to the image of a coarse manifold as a coarse manifold, thus a coarse manifold in X will be regarded as a subset of X.

Given a coarse manifold Σ, we define $\partial\Sigma$ as the restriction of the function defining Σ to ∂M.

We say Σ has *scale* $r > 0$ if every pair of adjacent vertices in M map to within distance r of each other in X. We define the *volume* of Σ to be the number of vertices in M.

If M is an n-manifold, we call Σ a coarse n-manifold. If Σ' is a coarse manifold as well whose domain is the triangulated manifold M', we say that Σ and Σ' have the *same topological type* if M and M' are homeomorphic.

Let K be a global field (number field or function field), and let S be a set of finitely many inequivalent valuations of K including one from each class of archimedean valuations. The ring $\mathcal{O}_S \subseteq K$ will be the corresponding ring of S-integers.

For any $v \in S$, we let K_v be the completion of K with respect to v so that K_v is a locally compact field.

Let $\mathbf{G}$ be a noncommutative, absolutely almost simple K-isotropic K-group. Let G_S be the semisimple Lie group

$$G_S = \prod_{v \in S} \mathbf{G}(K_v)$$

endowed with a left-invariant metric.

Under the diagonal embedding, the arithmetic group $\mathbf{G}(\mathcal{O}_S)$ is a lattice in G_S. The lattice being nonuniform is equivalent to the assumption that $\mathbf{G}$ is K-isotropic. The metric on G_S restricts to a metric on $\mathbf{G}(\mathcal{O}_S)$.

Denote the euclidean, or geometric, rank of G_S by $k(\mathbf{G}, S)$, so that

$$k(\mathbf{G}, S) = \sum_{v \in S} \mathrm{rank}_{K_v} \mathbf{G}$$

The interest is in proving the following conjecture from [7]

Conjecture 1. *Given $\mathbf{G}(\mathcal{O}_S)$ as above and a scale factor r_1, there exists a linear polynomial f and a scale factor r_2 such that if $\Sigma \subseteq G_S$ is a coarse n-manifold of scale r_1, with $\partial\Sigma \subseteq \mathbf{G}(\mathcal{O}_S)$, and $n < k(\mathbf{G}, S)$, then there is a coarse n-manifold $\Sigma' \subseteq \mathbf{G}(\mathcal{O}_S)$ of scale r_2 and of the same topological type as Σ such that $\partial\Sigma' = \partial\Sigma$ and $vol(\Sigma') \leq f(vol(\Sigma))$.*

The theorem of Lubotzky-Mozes-Raghunathan that higher rank irreducible lattices in semisimple Lie groups quasi-isometrically embed in their ambient Lie groups is a special case of the above conjecture [10], as would be that any irreducible lattice in a semisimple Lie group of rank at least 3 has a quadratic Dehn function. More generally, a proof of the above conjecture would imply Euclidean isoperimetric inequalities for irreducible lattice in low dimensions, where "low" is determined by the rank of the ambient semisimple group. In particular, it would imply that $\mathbf{G}(\mathcal{O}_S)$ is of type $F_{k(\mathbf{G},S)-1}$.

Known cases from the above paragraph include the aforementioned Lubotzky-Mozes-Raghunathan, Druţu's theorem that $\mathbb{Q}$-rank one lattices have a Dehn function that is asymptotically bounded by $n \mapsto n^{2+\varepsilon}$ [9], Young's theorem that $\mathbf{SL}_n(\mathbb{Z})$ has quadratic Dehn function when $n \geq 5$ [13], Raghunathan's proof that $\mathbf{G}(\mathcal{O}_S)$ is of type F_∞ when $\mathcal{O}_S$ is a classical ring of algebraic integers [11], Borel-Serre which established that $\mathbf{G}(\mathcal{O}_S)$ is of type F_∞ when K is a number field [5], and various special cases of the conjecture that $\mathbf{G}(\mathcal{O}_S)$ is of type $F_{k(\mathbf{G},S)-1}$ when K is a function field that have been shown by Stuhler, Behr, Abels, Abramenko, Bux-Wortman, and Bux-Gramlich-Witzel [12], [4], [1],[2], [3], [8], [6].

Notice that $|S| \leq k(\mathbf{G}, S)$, as $|S|$ measures the number of irreducible factors of G_S.

In joint work with Mladen Bestvina and Alex Eskin, we have established the following weakened and special case of Conjecture 1

Theorem 2. *Given $\mathbf{G}(\mathcal{O}_S)$ as above and a scale factor r_1, there exists a polynomial f of unspecified degree and a scale factor r_2 such that if $\Sigma \subseteq G_S$ is a coarse n-manifold of scale r_1, with $\partial\Sigma \subseteq \mathbf{G}(\mathcal{O}_S)$, and $n < |S|$, then there is a coarse n-manifold $\Sigma' \subseteq \mathbf{G}(\mathcal{O}_S)$ of scale r_2 and of the same topological type as Σ such that $\partial\Sigma' = \partial\Sigma$ and $vol(\Sigma') \leq f(vol(\Sigma))$.*

If $n = 1$, then we may assume f is linear.

Theorem 2 implies polynomial isoperimetric inequalities in low dimensions and that $\mathbf{G}(\mathcal{O}_S)$ is of type $F_{|S|-1}$.

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Quasi-isometric classification of 3–manifold groups

Jason Behrstock
(joint work with Walter D. Neumann)

The word metric on a finitely generated group, although non-unique — as it depends on a choice of finite generating set — is canonical when considered up to *quasi-isometry* (i.e., maps of bounded multiplicative and additive distortion). A fundamental question in group theory, as discussed in Gromov [7], is to classify finitely generated groups up to quasi-isometry.

An important and rich family of groups are provided by fundamental groups of 3–manifolds, and the quasi-isometric geometry of these groups has received significant attention, e.g., [4, 5, 6, 8, 9, 10, 11, 13, 14]. Through work of Papasoglu–Whyte on quasi-isometries of free-product decompositions [12], the quasi-isometric classification of general 3–manifolds reduces to the classification of irreducible ones.

Perelman's Geometrization Theorem show that any irreducible 3–manifold (with zero Euler characteristic) can be decomposed along tori and Klein bottles (the JSJ-decomposition) into pieces which admit geometric structures; when the collection of tori and Klein bottles is non-empty, such a manifold is called *non-geometric.* The quasi-isometric classification in the case of geometric closed manifolds is an immediate consequence of the Milnor–Švarc Lemma. For geometric 3-manifolds with boundary the quasi-isometric and commensurability classifications agree: this is a deep theorem of R. Schwartz in the hyperbolic case [14], the Seifert fibered space case was first proven by Kapovich–Leeb.

Until recently, the main results in the non-geometric case were due to Kapovich–Leeb who showed that quasi-isometries preserve the decomposition into geometric pieces and that quasi-isometries preserve the presence of hyperbolic components [9, 10, 11]. This work of Kapovich–Leeb prompted them to ask about the quasi-isometric classification of fundamental groups of closed graph manifolds [11]. In joint work with W. Neumann, we resolved this question with the following result, which as a special case shows that any two closed non-geometric graph manifolds have bilipschitz homeomorphic universal covers and hence, in particular, have quasi-isometric fundamental groups.

Theorem 1 (QI classification of graph manifolds; [2]). *If M and M' are non-geometric graph manifolds (possibly with boundary) then the following are equivalent:*

1. *The universal covers, $\widetilde{M}$ and $\widetilde{M}'$, are bilipschitz homeomorphic.*
2. *$\pi_1(M)$ and $\pi_1(M')$ are quasi-isometric.*
3. *The Bass-Serre trees for M and M' are isomorphic as two-colored trees. (Where the vertex groups of $\pi_1(M)$, resp. $\pi_1(M')$, are colored corresponding to whether the associated Seifert fibered pieces does or does not contain boundary components of M, resp. M'.)*
4. *The minimal two-colored graphs in the bisimilarity classes of the decomposition graphs associated to M and M' are isomorphic. (Again, the vertices are colored corresponding to whether the associated Seifert fibered piece does or does not contain boundary components of M, resp. M'.)*

Bisimilarity — a notion which arises in computer science — is an algorithmically checkable equivalence relation on colored finite graphs. Each equivalence class has a unique, canonical element which we call *minimal*. One can list the minimal two-colored graphs of small size, using Theorem 1 this allows us to conclude that, for instance, there are exactly $2, 6, 26, 199, 2811, 69711, 2921251, 204535126, \dots$ quasi-isometry classes of fundamental groups of non-geometric graph manifolds composed of at most $1, 2, 3, 4, 5, 6, 7, 8, \dots$ Seifert fibered pieces [2], see also [15].

For non-geometric manifolds with hyperbolic pieces we have discovered a similar, albeit more intricate, classification in terms of bisimilarity of certain labelled graphs. For non-geometric manifolds with only hyperbolic pieces of which at least one is non-arithmetic (we call these *NAH–manifolds*), the classifying objects are also minimal labelled graphs, but now the edges are labelled as well as the vertices, and the labellings are more complicated: each vertex is labelled by the isomorphism type of an orientable, complete, hyperbolic orbifold and each edges is labelled by a linear isomorphism between certain 2-dimensional $\mathbb{Q}$–vector spaces. We call such a graph an *NAH–graph*. Bisimilarity for NAH–graphs is defined similarity as for two-colored graphs, namely these are open graph-homomorphisms which preserve labels in a controlled way. In the case of such manifolds, with Neumann, we proved:

Theorem 2 (QI classification of NAH-manifolds; [3]). *Each NAH–manifold has an associated minimal NAH–graph and two such manifolds have quasi-isometric fundamental groups (in fact: bilipschitz equivalent universal covers) if and only if their minimal NAH–graphs are isomorphic.*

In certain cases we have also related the commensurability and quasi-isometric classification of NAH-manifolds via the following:

Theorem 3 (Commensurability and quasi-isometry; [3]). *Assuming CCC_3, if two NAH-manifolds have quasi-isometric fundamental groups and their common minimal NAH-graph is a tree with manifold labels then they (and in particular, their fundamental groups) are commensurable.*

Here CCC_3 is the 3-dimensional version of the following conjecture about the structure of hyperbolic manifolds; in [3] we showed that this conjecture follows from the conjecture that all hyperbolic groups are residually finite (although we find the present conjecture much more plausible).

Cusp Covering Conjecture. *Let M be a finite-volume hyperbolic n-manifold. Then for each cusp C of M there exists a sublattice Λ_C of $\pi_1(C)$ such that, for any choice of a sublattice $\Lambda'_C \subset \Lambda_C$ for each C, there exists a finite cover M' of M whose cusps covering each cusp C of M are the covers determined by Λ'_C.*

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When does a Coxeter group surject onto a virtually free group

Tadeusz Januszkiewicz

Let (W, S) be a Coxeter group, with the presentation given by the Coxeter matrix m_{st}: $\langle S | s^2 = 1 = (st)^{m_{st}} \rangle$. Results of Gonciulea, Cooper-Long-Reid and (a corollary of) Vinberg-Margulis assert that if W is infinite, then it admits a subgroup of finite index W^+ which surjects onto $\mathbf{Z}$. Moreover if W is not a product of finite and affine groups, then it is *large*, i.e., it admits a subgroup of finite index surjecting onto a free group on two generators.

It is natural to ask what can one say about quotients of W rather than W^+. For example: can a quotient of W be virtually free. It turns out that there is a simple answer to this.

Proposition. (W, S) *surjects onto a virtually free group if and only if at least one* m_{st} *is* ∞*, that is if* W *is not 2-spherical.*

Moreover W *surjects onto a nonelementary virtually free group if and only if it is not a product of a 2-spherical group with (some number of) infinite dihedral groups.*

Sketch of the proof. One implication is fairly well known: a 2-spherical group has Property FA, cf. [Ser03], hence cannot surject onto a virtually free group. The other direction uses the Tits representation:

If $m_{st} = \infty$, then W is free product of $W_{S-s} *_{W_{S-(s,t)}} W_{S-t}$.

Let $T : W_S \to GL_S(R)$ be the Tits representation, cf. [Bou02]. The entries of the matrices representing elements of W are algebraic (in fact cyclotomic) integers. Hence if we pick an ideal p and reduce modulo it, the image of W_S will be a finite group. Moreover the reduction is functorial for maps induced by inclusions of subgroups of W_S.

We apply the reduction procedure to all groups in the amalgam $W_{S-s} *_{W_{S-(s,t)}} W_{S-t}$ (but not to W_S). The resulting amalgam of finite groups clearly receives a surjection from W_S and gives an action on the Bass-Serre tree.

One further sees that the tree is a line if and only if $W_{\{s,t\}}$ commutes with $W_{S-s,t}$. This and a simple induction gives the statement of the Proposition. □

The considerations above are a warmup to the following question:

Question. *Does a large Coxeter group surject onto a hyperbolic group?*

Largeness is clearly a necessary condition. I have a candidate for the hyperbolic quotient.

Let W_S be a large Coxeter group, let W_A be a minimal large parabolic subgroup (it is then either a cocompact or a cofinite volume hyperbolic group with (perhaps ideal) simplex fundamental domain). Let $B = S - A$.

One takes then a simplex of groups, which to a proper subset $T \subset A$ associates the reduction mod p of the image in Tits representation of W_S of the parabolic subgroup $W_{T\cup B}$. It is clearly a quotient of W_S, and my hope, supported by low dimensional examples, is that it is hyperbolic.

After my talk Mark Sapir pointed out that one does not need to answer the Question to obtain that a Coxeter group is large if and only if it is SQ universal (this was one of my original motivations to look at the Question). This follows from a theorem of Peter Neumann, see [Neu73].

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Reporter: Stefan Witzel

Participants

Dr. Aurelien Alvarez
Departement de Mathematiques et d'Informatique
Universite d'Orleans
B. P. 6759
F-45067 Orleans Cedex 2

Prof. Dr. Goulnara N. Arzhantseva
Universite de Geneve
Section de Mathematiques
2-4, rue du Lievre
Case Postale 64
CH-1211 Geneve 4

Dr. Arthur Bartels
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Laurent Bartholdi
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Dr. Jason Behrstock
Department of Mathematics
Herbert H. Lehman College
CUNY
Bedford Park, Blvd. West
Bronx , NY 10468
USA

Prof. Dr. Mladen Bestvina
Department of Mathematics
College of Sciences
University of Utah
155 South 1400 East, JWB 233
Salt Lake City , UT 84112-0090
USA

Prof. Dr. Marc Bourdon
Mathematiques
UMR 8524 CNRS
Universite de Lille 1
F-59655 Villeneuve d'Ascq.

Prof. Dr. Martin R. Bridson
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Kai-Uwe Bux
Fakultät für Mathematik
Abteilung II
Universität Bielefeld
Postfach 100131
33501 Bielefeld

Dr. Pierre-Emmanuel Caprace
Institut de Mathematique
Universite Catholique de Louvain
Chemin du Cyclotron, 2
B-1348 Louvain-la-Neuve

Prof. Dr. Ruth Charney
Department of Mathematics
Brandeis University
Waltham , MA 02454-9110
USA

Owen Cotton-Barratt
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Michael W. Davis
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus , OH 43210-1174
USA

Prof. Dr. Thomas C. Delzant
Institut de Recherche
Mathematique Avancee
ULP et CNRS
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Prof. Dr. Cornelia Drutu Badea
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Dr. Jan Dymara
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Jan Essert
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Mark E. Feighn
Department of Mathematics
Rutgers University
Newark , NJ 07102
USA

Myriam Finster
Institut für Algebra und Geometrie
Karlsruhe Institute of Technology
Kaiserstr. 89-93
76133 Karlsruhe

Prof. Dr. Damien Gaboriau
Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Dr. Ralf Gramlich
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Prof. Dr. Daniel Groves
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago , IL 60607-7045
USA

Dr. Vincent Guirardel
Institut de Mathematiques de Toulouse
Universite Paul Sabatier
118, route de Narbonne
F-31062 Toulouse Cedex 9

Dipl.-Math. Guntram Hainke
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. James Howie
Department of Mathematics
Heriot-Watt University
Riccarton
GB-Edinburgh EH 14 4AS

Prof. Dr. Tadeusz Januszkiewicz
Department of Mathematics
The Ohio State University
231 West 18th Avenue
Columbus , OH 43210-1174
USA

Dr. Fanny Kassel
Laboratoire de Mathematique
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Linus Kramer
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Peter H. Kropholler
Department of Mathematics
University of Glasgow
University Gardens
GB-Glasgow G12 8QW

Benno Kuckuck
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Ian J. Leary
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus , OH 43210-1174
USA

Jean Lecureux
Institut Camille Jordan
Universite Lyon 1
Bat. Braconnier
21, Avenue Claude Bernard
F-69622 Villeurbanne Cedex

Prof. Dr. Enrico Leuzinger
Universität Karlsruhe
Institut für Algebra und Geometrie
Kaiserstr. 89-93
76133 Karlsruhe

Prof. Dr. Gilbert Levitt
University of Caen
Department of Mathematics
LMNO
BP 5186
F-14032 Caen Cedex

Prof. Dr. Wolfgang Lück
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Martin Lustig
LATP(Mathematiques)
Faculte St. Jerome
Universite Aix-Marseille III
Avenue Escadrille Normandie-N.
F-13397 Marseille Cedex 20

Prof. Dr. Shahar Mozes
Institute of Mathematics
The Hebrew University
Givat-Ram
91904 Jerusalem
ISRAEL

Prof. Dr. Bernhard M. Mühlherr
Mathematisches Institut
Justus-Liebig-Universität Gießen
Arndtstr. 2
35392 Gießen

Prof. Dr. Graham Niblo
School of Mathematics
University of Southampton
Highfield Campus
GB-Southampton SO17 1BJ

Prof. Dr. Panos Papazoglu
Department of Mathematics
University of Athens
Panepistemiopolis
157 84 Athens
GREECE

Prof. Dr. Piotr Przytycki
Institute of Mathematics of the
Polish Academy of Sciences
P.O. Box 21
ul. Sniadeckich 8
00-956 Warszawa
POLAND

Prof. Dr. Bertrand Remy
Institut Camille Jordan
Universite Lyon 1
Bat. Braconnier
21, Avenue Claude Bernard
F-69622 Villeurbanne Cedex

Prof. Dr. Mark V. Sapir
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville TN 37240-0001
USA

Prof. Dr. Roman Sauer
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Lars Scheele
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Gabriela Schmithüsen
Karlsruher Institut für Technologie(KIT)
Institut für Algebra & Geometrie
76128 Karlsruhe

Dr. Petra N. Schwer
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Jacek Swiatkowski
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. Katrin Tent
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Anne Thomas
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Karen L. Vogtmann
Department of Mathematics
Cornell University
436 Malott Hall
Ithaca , NY 14853-4201
USA

Richard Wade
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Richard M. Weiss
Department of Mathematics
Tufts University
503 Boston Ave.
Medford , MA 02155
USA

Stefan Witzel
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Dr. Kevin Wortman
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City , UT 84112-0090
USA