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Motives and Homotopy Theory of Schemes

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Abstract. The 2010 program on Motives and Homotopy Theory of Schemes consisted of a lively and varied series of 19 one-hour lectures on the latest developments in the field, presenting a wide range of aspects of this multi-faceted subject. Besides the lectures, we were happy to welcome a wide range of nationalities and age groups to the conference.

Mathematics Subject Classification (2000): 14F, 14G, 19E.

Introduction by the Organisers

The confluence of algebraic geometry and homological algebra known as the theory of motives has experienced an amazing resurgence of activity in the last twenty years. More recently, the growth of *motivic homotopy theory* has expanded the area to allow for a systematic treatment of a wide variety of "motivic" phenomena, embedding K-theory, motivic cohomology, quadratic forms into a single larger field. At the same time, the theory allows for the transfer of constructions and techniques from classical homotopy theory to problems in algebraic geometry.

Here in more detail are the topics which were discussed.

Motives, varieties and algebra. We had three talks on applications of motives to the study of varieties over non-algebraically closed fields. Using a version of the Rost motive, Semenov described a surprising restriction on the Rost invariant for homogeneous spaces for E_6, Gille extended the property of Rost nilpotence to geometrically rational surfaces over fields of characteristic zero, Zainoulline gave a uniform bound for the torsion part of codimension 2 algebraic cycles on certain projective homogeneous varieties. In addition, Krashen explained how the patching techniques of Harbater and Hartmann were applied (in a joint work with these two) to gives a new local-global principles for galois cohomology.

Categories of motives. Déglise described his work with Cisinski constructing a category of motives (with $\mathbb{Q}$-coefficients) over a general base satisfying the Grothendieck six operations formalism. Barbieri-Viale showed how Nori's construction of a category of motives gives a finer construction of a category of n-motives, i.e., motives of varieties of dimension $\leq n$, with $n = 0$ being the category of Artin motives, $n = 1$ Deligne's category of 1-motives. Park described his construction (with Krishna) of a triangulated category of motives over $k[t]/t^{n+1}$, based on modifications of the Bloch-Esnault additive Chow groups. Wildeshaus showed how he applied the technique of weight structures on a triangulated category, developed by Bondarko, to study motives of Shimura varieties.

Tannaka groups and fundamental groups. Esnault described her proof (with Mehta) of Gieseker's conjecture, that the vanishing of the étale fundamental group of a smooth projective variety X over an algebraically closed field of positive characteristic implies that there are no non-trivial $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules on X. Terasoma described his construction (with K. Kimura) of a mixed cycle-theoretic and representation-theoretic differential graded algebra, whose co-modules may be viewed as "mixed elliptic motives". Furusho described his work giving simplfied relations defining the Grothendieck-Teichmüller group, and showing that all elements of the Grothendieck-Teichmüller group satisfy the "double-shuffle relations".

Arithmetic. Geisser discussed Parshin's conjecture, that the rational higher K-theory of a smooth and proper variety over a finite field is torsion, and related this conjecture to finite generation properties of motivic cohomology and motivic Borel-Moore homology, as well as the statement that rational motivic homology and cohomology are dual vector spaces. Flach reported on progress (including joint work with Morin) in Lichtenbaum's program of describing the vanishing order and leading term of zeta functions of arithmetic schemes in terms of Weil-étale cohomology. In particular, Flach and Morin have defined a Weil-étale topos for a regular proper scheme over $\operatorname{Spec} \mathbb{Z}$ which gives the correct answer for the zeta value at 0. Holmstrom reported on his work (with J. Scholbach) on lifting the Deligne regulator to a map in the motivic stable homotopy category, and using this to define Arakelov motivic cohomology via a cone construction.

Motivic homotopy theory. Ostvar discussed his computations (with Ormsby) giving information on the coefficient rings for MGL, kgl and the motivic sphere spectrum, using versions of the Adams spectral sequence and the Adams-Novikov spectral sequence. Pelaez presented his recent work on the functoriality of the slice filtration, which as an application gives a good definition of an integral category of motives over a base-scheme S for S a scheme over a field of characteristic zero. Yagunov showed us his computation of the first non-trivial differential in the motivic cohomology to K-theory spectral sequence, after localization at a given prime. His main result is that this differential is expressible in terms of the motivic Steenrod operations. Asok reported on a joint work with Morel and Haesemeyer, in which they compute the maps in the motivic stable homotopy category from $\mathrm{Spec}\, k$ to a smooth proper scheme X as the group of oriented 0-cycles on X (as defined by Barge-Morel and extended by Fasel). Hornbostel gave us a description of a motivic version of a result of Lurie in the stable homotopy category, namely, that the suspension spectrum of $\mathbb{CP}^\infty$ classifies "preorientations of the derived multiplicative group". This motivic version gives an as application an intrinsic description of algebraic K-theory, namely, that it represents orientations of the derived motivic multiplicative group.

Workshop: Motives and Homotopy Theory of Schemes

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Abstracts

Beilinson motives and the six functors formalism

Frédéric Déglise
(joint work with Denis-Charles Cisinski)

Notations

We denote by $\mathscr{S}$ the category of excellent noetherian scheme of finite dimension. Without precision, schemes are considered to be objects of this category.

Monoidal categories (resp. functors) are always assumed to be symmetric.

1. Introduction

Let $\mathscr{T}ri^{\otimes}$ be the 2-category of triangulated monoidal categories, with weakly monoidal triangulated natural transformations as 2-morphisms.

Definition 1. A triangulated category satisfying the six functor formalism consists of the following data:

(1) For any scheme S, we consider a triangulated closed monoidal category $\mathcal{T}(S)$, with unit object $\mathbb{I}_S$.

(2) For any morphism $f : T \to S$, a pair of adjoint functors

$$f^* : \mathcal{T}(T) \to \mathcal{T}(S) : f_*$$

such that f^* is monoidal and $S \mapsto \mathcal{T}(S), f \mapsto f^*$ is a contravariant 2-functor from $\mathscr{S}$ to $\mathscr{T}ri^{\otimes}$.

(3) For any separated morphism of finite type $f : T \to S$, a pair of adjoint functors

$$f_! : \mathcal{T}(T) \to \mathcal{T}(S) : f^!$$

such that $S \mapsto \mathcal{T}(S), f \mapsto f_!$ is a 2-functor from the category of schemes with morphisms separated of finite type to $\mathscr{T}ri^{\otimes}$.

These data are assumed to satisty the following properties:

(4) For any separated morphism of finite type, there exists a natural transformation $f_! \to f_*$ compatible with composition which is an isomorphism when f is proper.

Let S be a scheme and $p : \mathbb{P}^1_S \to S$ (resp. $s : S \to \mathbb{P}^1_S$) be the canonical projection (resp. infinite section) of the projective line over S. Define the Tate twist as:

$$\mathbb{I}_S(1) = s^* p^! (\mathbb{I}_S)[-2].$$

For any integer $n \geq 0$, we let $\mathbb{I}_S(n)$ be the n-th tensor power of $\mathbb{I}_S(1)$ and for any object M of $\mathcal{T}(S)$, we put $M(n) = M \otimes \mathbb{I}_S(n)$.

(5) For any smooth quasi-projective morphism f of constant relative dimension n, there exists a natural isomorphism $f^! \to f^*(n)[2n]$ compatible with composition.

(6) For any cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

in which f is separated of finite type, there exists natural isomorphisms:

$$g^* f_! \longrightarrow f'_! g'^*,$$
$$g'_* f'^! \longrightarrow f^! g_*.$$

(7) For any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exist natural isomorphisms

$$(f_! K) \otimes_X L \longrightarrow f_!(K \otimes_X f^* L)\,,$$
$$\underline{\mathrm{Hom}}_X(f_!(L), K) \longrightarrow f_*\, \underline{\mathrm{Hom}}_Y(L, f^!(K))\,,$$
$$f^!\, \underline{\mathrm{Hom}}_X(L, M) \longrightarrow \underline{\mathrm{Hom}}_Y(f^*(L), f^!(M))\,.$$

The first example of such a formalism was given in [SGA4]. More recently, the six funtors formalism has been constructed by J. Ayoub in [Ayo07] for the stable homotopy category of schemes $SH(S)$ defined by F. Morel and V. Voevodsky.[1]

In the next section, we propose a definition of a rational triangulated category which satisfies the six functors formalism and which we propose as a category of triangulated mixed motives. The justification for this claim is that our category extends the definition of Voevodsky known over (perfect) fields. We refer the interested reader to [CD09] for more details on our construction.

2. Beilinson motives

1. Recall that for any scheme S, there exists a ring spectrum $\mathbf{K}_S$ in $SH(S)$ such that:

- For any morphism of schemes $f : T \to S$,

(1) $$f^*(\mathbf{K}_S) = \mathbf{K}_T.$$

- When S is regular, for any integer n,

(2) $$\mathrm{Hom}(\Sigma^\infty X_+[n], \mathbf{K}_S) = K_n(S)$$

where the right hand side denotes Quillen algebraic K-theory.

Let us denote by $SH(S, \mathbb{Q})$ the rationalisation of the stable homotopy category.[2] We denote by $\mathbf{K}_S^{\mathbb{Q}}$ the object defined by the above spectrum in $SH(S, \mathbb{Q})$. The idea of the following definition comes from topology:

Definition 2. Consider the notations above.

[1] In the stable homotopy category though, one should be aware that in property (5), one has to replace the twist by a tensor product with a *Thom space*.

[2] The category with same objects but the Hom groups are tensored with $\mathbb{Q}$.

(1) We say an object $\mathbf{E}$ of $SH(S, \mathbb{Q})$ is $\mathbf{K}$-acyclic if $\mathbf{E} \otimes \mathbf{K}_S^{\mathbb{Q}} = 0$.
(2) We say a morphism $f : \mathbf{E} \to \mathbf{F}$ in $SH(S, \mathbb{Q})$ is a $\mathbf{K}$-equivalence if a cone of f is $\mathbf{K}$-acyclic.
(3) We say an object M of $SH(S, \mathbb{Q})$ is a *Beilinson motive* if for all $\mathbf{K}$-acyclic spectrum $\mathbf{E}$, $\mathrm{Hom}(\mathbf{E}, M) = 0$.

We let $DM_{\mathrm{B}}(S)$ be the full subcategory of $SH(S, \mathbb{Q})$ made by the Beilinson motives.

According to the theory of Bousfield localization, the category $DM_{\mathrm{B}}(S)$ can be described as the localization of the category $SH(S, \mathbb{Q})$ with respect to $\mathbf{K}$-equivalences. Moreover, we get an adjunction of triangulated categories:

$$L_{\mathrm{B}} : SH(S, \mathbb{Q}) \leftrightarrows DM_{\mathrm{B}}(S) : \mathcal{O}_{\mathrm{B}}$$

where $\mathcal{O}_{\mathrm{B}}$ is the natural forgetful functors. As the $\mathbf{K}$-equivalences are stable by base change (using (1)) and tensor product, we get using the main result of [Ayo07] the following theorem:

Theorem 3 ([CD09, §13.2]). *The triangulated category DM_{B} satisfies the six functors formalism.*

Note moreover that L_{B} is monoidal and commutes with operations such as f^* and $f_!$.

2. Let S be any regular scheme. We will consider on $K_n(S) \otimes \mathbb{Q}$ the γ-filtration together with its graded pieces which give a canonical decomposition:

$$K_n(S) \otimes \mathbb{Q} = \bigoplus_{i \in \mathbb{N}} Gr_\gamma^i \big(K_n(S) \otimes \mathbb{Q}\big). \tag{3}$$

We will use the following theorem of J. Riou:

Theorem 4 ([Rio06]). *Let S be a scheme. There exists a canonical decomposition in $SH(S, \mathbb{Q})$ of the form:*

$$\mathbf{K}_S = \bigoplus_{i \in \mathbb{Z}} K_S^{(i)} \tag{4}$$

stable by base change and such that, whenever S is regular, for any integer $n \in \mathbb{Z}$, the induced decomposition on the cohomology represented by $\mathbf{K}_S$ coincide with (3) *through the identification* (2).

According to Riou, we define the Beilinson spectrum over any scheme S as $\mathbf{H}_{\mathrm{B},S} = \mathbf{K}_S^{(0)}$. Note that Bott periodicity for K-theory implies that (4) can be rewritten as:

$$\mathbf{K}_S = \bigoplus_{i \in \mathbb{Z}} \mathbf{H}_{\mathrm{B},S}(i)[2i] \tag{5}$$

where $\mathbf{H}_{\mathrm{B},S}(i)$ is the i-th Tate twist in $SH(S, \mathbb{Q})$.

The following result is a key point of our construction:

Proposition 1 ([CD09, 13.1.5, 13.1.6]). *The spectrum $\mathbf{H}_{\text{Б},S}$ admits a ring structure in $SH(S,\mathbb{Q})$ such that its multiplication map*

$$\mu : \mathbf{H}_{\text{Б},S} \wedge \mathbf{H}_{\text{Б},S} \to \mathbf{H}_{\text{Б},S}$$

is an isomorphism.

3. Recall that the category $SH(S,\mathbb{Q})$ is the homotopy category of a monoidal model category $Sp(S,\mathbb{Q})$. One deduces from the previous theorem that $\mathbf{H}_{\text{Б},S}$ there exists a (commutative) monoid $\bar{\mathbf{H}}_{\text{Б},S}$ in $Sp(S,\mathbb{Q})$ which coincides in $SH(S,\mathbb{Q})$ with $\mathbf{H}_{\text{Б},S}$.[3] This allows to define the triangulated category $\mathbf{H}_{\text{Б},S} - \mathrm{mod}$ of $\mathbf{H}_{\text{Б},S}$-modules.[4] By construction, we get a canonical adjunction:

$$L_{\mathbf{H}_\text{Б}} : SH(S,\mathbb{Q}) \leftrightarrows \mathbf{H}_{\text{Б},S} - \mathrm{mod} : \mathcal{O}_{\mathbf{H}_\text{Б}}.$$

such that $L_{\mathbf{H}_\text{Б}}(\mathbf{E}) = \mathbf{E} \wedge \mathbf{H}_{\text{Б},S}$. As a corollary of the previous result, we get the following theorem:

Theorem 5 ([CD09, 13.2.9]). *Consider the notations above. There exists a canonical functor $\varphi : DM_\text{Б}(S) \to \mathbf{H}_{\text{Б},S} - \mathrm{mod}$ which fits into the commutative diagram:*

$$\begin{array}{ccccc} SH(S,\mathbb{Q}) & & \xrightarrow{\quad L_{\mathbf{H}_\text{Б}} \quad} & & \mathbf{H}_{\text{Б},S} - \mathrm{mod}. \\ & {\scriptstyle L_\text{Б}}\searrow & & \nearrow{\scriptstyle \varphi} & \\ & & DM_\text{Б}(S) & & \end{array}$$

Moreover, φ is an equivalence of triangulated monoidal categories.

Corollary 1. *For any regualr scheme S and any couple of integers $(n,p) \in \mathbb{Z}^2$, one has:*

$$\mathrm{Hom}_{DM_\text{Б}(S)}(\mathbb{I}_S, \mathbb{I}_S(p)[n]) = K^{(p)}_{2p-n}(S).$$

For a non necessarily regular scheme S, we will define *Beilinson motivic cohomology* of S as the left hand side in the above identification.

Example 1. Let X be a smooth S-scheme. Define the (homological) motive of X/S as $M(X) = L_\text{Б}(\Sigma^\infty X_+)$.

If in addition, X/S is projective of constant dimension d, then one shows $M(X)$ is strongly dualisable with strong dual $M(X)(-d)[-2d]$.

Assuming that S is regular, one can define the category $\mathcal{M}^{rat}(S)$ of Chow motives as usual. Applying the previous corollary, one gets a fully faithful functor:

$$\mathcal{M}^{rat}(S)^{op} \to DM_\text{Б}(S), h(X) \mapsto M(X).$$

Corollary 2. *Let S be any scheme, $\mathbf{E}$ be an object of $SH(S,\mathbb{Q})$ and $u : \mathbf{S}^0 \to \mathbf{H}_{\text{Б},S}$ be the unit of ring spectrum $\mathbf{H}_{\text{Б},S}$. Then the following conditions are equivalent:*

(i) $\mathbf{E}$ *is a Beilinson motive.*
(ii) $\mathbf{E}$ *admits a structure of an $\mathbf{H}_{\text{Б},S}$-module in $SH(S,\mathbb{Q})$.*
(iii) *The morphism $u \wedge Id_\mathbf{E} : \mathbf{E} \to \mathbf{H}_{\text{Б},S} \wedge \mathbf{E}$ is an isomorphism.*

[3]One says also that $\mathbf{H}_{\text{Б},S}$ is a *strict* ring spectrum.

[4]One constructs according to Schwede and Shipley a model category on the category of modules over $\bar{\mathbf{H}}_{\text{Б},S}$; $\mathbf{H}_{\text{Б},S} - \mathrm{mod}$ is its homotopy category.

Moreover, when these conditions are satisfied, the structure of an $\mathbf{H}_{\text{Б},S}$-module on $\mathbf{E}$ is unique.[5]

3. Proper descent and Voevodsky motives

4. Consider again a scheme S.

Let us recall that Voevodsky has introduced the h-topology on the category $\mathscr{S}_S^{ft}$ of finite type S-schemes: its coverings are made of the universal topological epimorphism $f : W \to X$.[6] We let $\mathrm{Sh}_h(S, \mathbb{Q})$ be the category of sheaves of $\mathbb{Q}$-vector spaces on $\mathscr{S}_S^{ft}$ for the h-topology.

Voevodsky then defines the category of (rational) h-motives $\underline{DM}_h^{eff}(S, \mathbb{Q})$ as the $\mathbb{A}^1$-localization of the derived category of the abelian category $\mathrm{Sh}_h(S, \mathbb{Q})$. Any S-scheme X of finite type defines an object of $\mathrm{Sh}_h(S, \mathbb{Q})$ denoted by $\underline{\mathbb{Q}}^h(X)$. We then define the Tate twist $\underline{\mathbb{Q}}_S^h(1)$ in $\underline{DM}_h^{eff}(S, \mathbb{Q})$ as the cokernel of the split monomorphism $\underline{\mathbb{Q}}^h(S) \to \underline{\mathbb{Q}}^h(\mathbb{P}^1_S)$ defined by the inclusion of the infinite S-point.

In fact, one can show that $\underline{DM}_h(S, \mathbb{Q})$ is the homotopy category of a suitable Quillen model category on the category of complexes on $\mathrm{Sh}_h(S, \mathbb{Q})$. Moreover, this model category is monoidal: it defines a (derived) closed monoidal structure on $\underline{DM}_h(S, \mathbb{Q})$. Moreover, we can define the so called $\mathbb{P}^1$-*stabilisation* of this category: this is the universal homotopy category $\underline{DM}_h(S, \mathbb{Q})$ of a monoidal model category given with a left derived monoidal functor

$$\Sigma^\infty : \underline{DM}_h^{eff}(S, \mathbb{Q}) \longrightarrow \underline{DM}_h(S, \mathbb{Q})$$

such that $\Sigma^\infty \underline{\mathbb{Q}}_S^h(1)$ is $\otimes$-invertible.

One can recognize in this construction the steps needed to define the stable homotopy category $SH(S)$: in the former, one simply starts from complexes of $\mathbb{Q}$-sheaves for the h-topology on $\mathscr{S}_S^{ft}$ instead of simplicial sheaves of sets for the Nisnevich topology on smooth S-schemes. The analogy between the tow constructions allow to define a canonical triangulated monoidal functor:

$$a_h : SH(S) \to \underline{DM}_h(S, \mathbb{Q})$$

which factors through the rational stable homotopy category. One of the main theorem of [CD09] is the following:

Theorem 6. *There exists a unique functor $\psi : DM_{\text{Б}}(S) \to \underline{DM}_h(S, \mathbb{Q})$ which makes the following diagram (essentially) commutative:*

$$\begin{array}{ccc} SH(S, \mathbb{Q}) & \xrightarrow{\quad a_h \quad} & \underline{DM}_h(S, \mathbb{Q}). \\ & {\scriptstyle L_{\text{Б}}}\searrow \qquad \nearrow{\scriptstyle \psi} & \\ & DM_{\text{Б}}(S) & \end{array}$$

Moreover, ψ is fully faithful and monoidal.

[5] And can be lifted in the monoidal category of symmetric spectra.

[6] That is the topology of X is the final topology relative to f, and this property remains true after any base change. The basic examples of such coverings: faithfully flat morphisms, proper surjective morphisms.

In fact, ψ sends the Beilinson motive $M_S(X)$ of a smooth S-scheme X to the object $\mathbb{Q}_S^h(X)$ and the essential image of ψ is made by the localizing subcategory of the triangulated category $\underline{DM}_h(S)$ generated by the objects $\mathbb{Q}_S^h(X)(i)$ for a smooth S-scheme X and an integer $i \in \mathbb{Z}$.

5. Consider a spectrum $\mathbf{E}$ over a scheme S. Given a scheme X/S of finite type, with structural morphism f, we define the cohomology of X with coefficients in $\mathbf{E}$ as:

$$\mathbf{E}^{n,p}(X) = \mathrm{Hom}_{SH(X,\mathbb{Q})}\left(\Sigma^\infty X_+, f^*(\mathbf{E})(p)[n]\right), (n,p) \in \mathbb{Z}^2.$$

This definition can be extended to simplicial objects of $\mathscr{S}_S^{ft}$ and defines in fact a contravariant functor.
One says that $\mathbf{E}$ satisfies h-descent if for any smooth S-scheme X and any h-cover $\pi : \mathcal{V}_\bullet \to X$ the induced morphism:

$$\pi^* : \mathbf{E}^{n,p}(X) \to \mathbf{E}^{n,p}(\mathcal{V}_\bullet)$$

is an isomorphism. One can reformulate the previous theorem by the equivalence of the following conditions for a rational spectrum $\mathbf{E}$:

(i) $\mathbf{E}$ is a Beilinson motive.
(iv) $\mathbf{E}$ satisfies h-descent.

Note in particular that Beilinson motivic cohomology satisfies h-descent – thus proper and faithfully flat descent.

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Chow motives and the Rost invariant

Nikita Semenov

The following results are my joint investigations with Skip Garibaldi and Viktor Petrov.

Let G be a linear algebraic group of inner type over a field k, X a projective homogeneous G-variety defined over k, and p a prime number. Denote by $\mathrm{Ch}^*(X)$ the Chow ring of X with $\mathbb{F}_p$-coefficients and set $\overline{X} = X \times_k k_s$, where k_s stands for a separable closure of k.

Let D be a subset of the vertices of the Dynkin diagram of G containing the Tits index of $G_{k(X)}$, and $b \in \mathrm{Ch}^l(\overline{X})$ a cycle defined over k. We give an inductive definition. Assume that b does not lie in any *shell* strictly contained in D, and let K be a generic field extension of k with respect to the property that the Tits index of G_K is D. We say that b lies in the shell corresponding to D, if there is a cycle $a \in \mathrm{Ch}_l(\overline{X})$ defined over K such that $\deg(ab) = 1$.

Generalizing results of Karpenko [Ka09] and Vishik [Vi98] one can show the following theorem.

Theorem 1. *In the above notation let $b \in \mathrm{Ch}^l(\overline{X})$ be the generic point of an indecomposable direct summand M of the Chow motive of X with $\mathbb{F}_p$-coefficients. Let $\alpha \in \mathrm{Ch}^t(\overline{X})$ be a cycle defined over k. Assume that $b' = b \cdot \alpha$ lies in the same shell as b.*

Then the Tate twist $M(t)$ is isomorphic to an indecomposable direct summand of the motive of X with generic point b'.

Using this theorem one can provide new motivic decompositions of projective homogeneous varieties, new restrictions on the J-invariant of linear algebraic groups, and give the following example, which is due to Garibaldi.

Let G be a simply connected group of inner type E_6 of rank two and with a non-trivial Tits algebra. Let

$$r_G\colon H^1(k, G) \to H^3(k, \mathbb{Z}/6)$$

be the Rost invariant of G. Then $2r_G$ has trivial kernel.

In particular, if $z \in H^1(k, G)$ and the modulo 3 part of $r_G(z)$ equals zero, then $r_G(z) = 0$. This property does not depend on the field k; cf. [Se95, Example 9.5].

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On the Algebraic Fundamental Group

Hélène Esnault

Let X be a smooth projective variety defined over an algebraically closed field, let $x \to X(k)$ be a rational point. If $k = \mathbb{C}$, the Riemann-Hilbert correspondence is an equivalence of categories between local systems of $\mathbb{C}$-vector spaces and $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. As the categories are $\mathbb{C}$-linear, rigid, abelian, this yields an isomomorphism of $\mathbb{C}$-proalgebraic groups $\mathrm{Aut}^{\otimes}(-, \omega_x)$, where ω_x is the neutralization which assigns to a local system $\mathcal{V}$, resp. a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module E, its fiber $\mathcal{V}_x$, resp. E_x at x. This is an isomorphism of type Betti $\leftrightarrow$ de Rham. The topological fundamental group $\pi_1^{\mathrm{top}}(X, x)$ is finitely generated. By definition, the Betti version is $\left(\pi_1^{\mathrm{top}}(X, x)\right)^{\mathrm{alg}} = \varprojlim H$, where H is the Zariski closure of the monodromy group of a complex linear representation $\pi_1^{\mathrm{top}}(X, x) \to GL(n, \mathbb{C})$, thus is residually finite. So one concludes that if $\pi_1^{\mathrm{et}}(X, x) = \{1\}$, there are no nontrivial $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules.

If k has characteristic $p > 0$, the Katz' equivalence ([3, Theorem 1.3]) is an equivalence between $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules and stratified bundles. As the categories are k-linear, rigid, abelian, this yields an isomomorphism of k-proalgebraic groups $\mathrm{Aut}^{\otimes}(-, \omega_x)$, where ω_x is the neutralization which assigns to a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module E, resp. a stratified bundle $\big((E, E_1, \ldots, E_n, \ldots), (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)\big)$ its fiber E_x at x. This is an analog to de Rham $\leftrightarrow$ Betti over the complex numbers. The study of the categories, the use of Langer's moduli and of Hrushovsky's theorem allow to give a positive answer to Gieseker conjecture in [3, p.8]

Theorem 1. *([2, Theorem 1,1]) If $\pi_1^{\mathrm{et}}(X \otimes_k \bar{k}, x) = \{1\}$, then there are non nontrivial $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules.*

More generally we show that if $X_S \to S$ is a good model of X/k, where S is a smooth affine variety over $\mathbb{F}_p$, then stable *torsion* bundles E on X_s, where $s \to S$ is a closed point, that is bundles which satisfy $(F^N_{X_s})^*E \cong E$, where F_{X_s} is the Frobenius of $X_s = X_S \otimes_X s$, are dense in the Verschiebung divisible sublocus of Langer's moduli M_S ([2, Theorem 3.14]).

Based on this and on the Mordell-Weil theorem, Raynaud conjectures that if k is a field of finite type over $\mathbb{F}_p$, and X is smooth projective over k, then stratified bundles with underlying stable E_i are torsion over $\bar{k}$. In fact, Raynaud's conjecture can be thought of as an equal characteristic $p > 0$ version of Grothendieck's p-curvature conjecture, in the same spirit as André's formulation [1, Section II] of an equal characteristic 0 version of it.

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Motivic invariants of the rational numbers

Paul Arne Østvær

(joint work with Kyle Ormsby)

This is a report on joint work in progress with Kyle Ormsby. Our main object is to access the algebraic cobordism groups and the motivic stable stems of the field of rational numbers. Let $S^{m+n\alpha}$ be the smash product of the mth simplicial circle with the nth Tate circle. We shall index motivic homology theories accordingly by writing $MGL_{m+n\alpha}$ and $\pi_{m+n\alpha}\mathbf{1}$ for the algebraic cobordism groups and the motivic stable stems, respectively. For $n \geq 0$ and all perfect fields there exists an isomorphism $MGL_{-n\alpha} \cong K^M_n$ due to Morel [2]. Here K^M_* denotes the Milnor K-theory (indexed by $\mathbb{Z}$). The rational part of the algebraic cobordism groups $MGL_{m+n\alpha}$ is known over number fields by work of Naumann-Spitzweck-Østvær [3]. The motivic stable stem $\pi_{m+n\alpha}\mathbf{1}$ is known for $m \leq 0$ by work of Morel [2] (it vanishes if $m < 0$ and identifies with the Milnor-Witt ring if $m = 0$). Our approach for the rationals combines the computational machinery of algebraic topology and local-to-global principles deduced from the thesis work of Ormsby [4].

From now on all motivic spectra will implicitly be completed at the prime 2. The Adams spectral sequence for the motivic Brown-Peterson spectrum MBP takes the form

$$Ext^*_{\mathcal{A}_*}(M\mathbb{Z}/2_*, M\mathbb{Z}/2_*MBP) \Rightarrow MBP_*.$$

In this spectral sequence, $M\mathbb{Z}/2$ denotes the mod 2 motivic Eilenberg-MacLane spectrum and $\mathcal{A}_* = M\mathbb{Z}/2_*M\mathbb{Z}/2$ is the dual mod 2 motivic Steenrod algebra. Likewise, we note that the Adams spectral sequence for the motivic connective K-theory spectrum kgl takes the form

$$Ext^*_{\mathcal{A}_*}(M\mathbb{Z}/2_*, M\mathbb{Z}/2_*kgl) \Rightarrow kgl_*.$$

Both of these spectral sequences are trigraded and strongly convergent over local and global number fields by work of Hu-Kriz-Ormsby [1]. The Adams spectral sequence for kgl computes algebraic K-groups and provides valuable insight into the somewhat more complicated computations for MGL. To begin with we show there are isomorphisms of $\mathcal{A}_*$-comodules

$$M\mathbb{Z}/2_*kgl \cong \mathcal{A}_*\Box_{\mathcal{E}(1)}M\mathbb{Z}/2_*$$

and

$$M\mathbb{Z}/2_*MBP \cong \mathcal{A}_*\Box_{\mathcal{E}(\infty)}M\mathbb{Z}/2_*$$

for the quotient Hopf algebroids

$$\mathcal{E}(1) = \mathcal{A}_*//(\xi_1, \xi_2, \ldots) + (\tau_2, \tau_3, \ldots)$$

and

$$\mathcal{E}(\infty) = \mathcal{A}_*//(\xi_1, \xi_2, \ldots).$$

Via the change-of-rings isomorphism for Ext-groups these isomorphisms greatly facilitate computations. For the local fields $\mathbb{R}$ and $\mathbb{Q}_p$ where p is a prime number and the rational numbers $\mathbb{Q}$ the Adams spectral sequence for kgl yields complete computations of the 2-completed connective K-groups.

Similar local-to-global type of computations can be carried out for MBP, and consequently for MGL at the prime 2. We employ the structure of MBP_* for the purpose of computing the Adams-Novikov spectral sequence

$$Ext^*_{MBP_*MBP}(MBP_*, MBP_*) \Rightarrow \pi_*\mathbf{1}$$

converging to the motivic stable stems. In topology the so-called α-family gives rise to infinitely many non-trivial elements in the stable stems. In the motivic setup we recover these and a whole host of new elements. The details of our work will appear in [5].

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Nori n-motives

Luca Barbieri-Viale

In this talk I introduced *n-motives* by showing that Nori's construction of mixed motives can be applied to varieties of dimension $\leq n$. I then have linked Nori 0-motives to Artin motives and Nori 1-motives to Deligne 1-motives (with torsion). It will be nice to compare Nori 2-motives with Ayoub 2-motives in [2].

Nori's construction of the category of effective homological (resp. cohomological) mixed motives EHM (resp. ECM) is drafted in [7], [8], [6] and generalized in [1]. For details on EHM and ECM just mainly refer to [7] (which is published and consistent with Nori's notation in [8]). Recall that there exists a functor $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \to D_b(\mathsf{EHM})$ from Voevodsky triangulated category of effective geometrical motives [9] to the bounded derived category of EHM.

The main mentioned tasks of the talk that are already contained in [4] can be summarized as follows. Fix a field k and an embedding $k \subseteq \mathbb{C}$. Let ${}^D(Sch_k)_{\leq n}$ be the full subdiagram of Nori's diagram ${}^D Sch_k$ whose objects are triples (X, Y, i) where $X \in Sch_k$ of dimension $\leq n$, $Y \subseteq X$ is closed and i is an integer. Then just apply Nori's construction in [6] to $H_* : {}^D(Sch_k)_{\leq n} \to R\text{-}\mathsf{mod}$ the representation given by $(X, Y, i) \rightsquigarrow H_i(X, Y; R)$ the singular homology R- module of the pair. Define $\mathsf{EHM}_n := \mathcal{C}(H_*)$ for R=$\mathbb{Z}$. For all non negative integers n these EHM_n are abelian categories along with canonical faithful exact functors $\mathsf{EHM}_n \to \mathsf{EHM}_{n+1}$ such that the 2-colimit yields

$$\operatorname*{Colim}_{n\geq 0} \mathsf{EHM}_n = \mathsf{EHM}$$

There is a tensor pairing

$$\otimes_{n,n'} : \mathsf{EHM}_n \times \mathsf{EHM}_{n'} \longrightarrow \mathsf{EHM}_{n+n'}$$

for all $n, n' \geq 0$. Similarly, we have ECM_n and the same assertions hold. Actually, we have a duality antiequivalence

$$(\ )^* : \mathsf{EHM}_n \xrightarrow{\simeq} \mathsf{ECM}_n$$

of abelian categories. We also get a functor

$$M_{\leq n} : (Sch_k)_{\leq n} \to D_b(\mathsf{EHM}_n)$$

where $(Sch_k)_{\leq n}$ is the category of k-schemes of dimension$\leq n$.

Further, the functor LAlb constructed in [5] yields a map of diagrams ^{D}LAlb from Nori's diagram ${}^D(Sch_k)$ to 1-motives with cotorsion ${}_t\mathcal{M}_1$. Dually, RPic yields a map of diagrams ^{D}RPic from the opposite diagram ${}^D(Sch_k)^{op}$ to 1-motives with torsion ${}^t\mathcal{M}_1$. Applying the Betti realisation $T_{\mathbb{Z}}({}^D\mathrm{RPic})$ we get a representation which is nothing but the singular cohomology representation H^* when restricted to the subdiagrams ${}^D(Sch_k)_{\leq n}$ for $n = 0, 1$. Thus, by universality, we get canonical (exact, faithful) functors

$$C_n : \mathsf{ECM}_n \to {}^t\mathcal{M}_n$$

for $n = 0, 1$. These functors are compatible with realizations. For $n = 0$ the functor C_0 is clearly an equivalence with quasi-inverse the functor

$$T_0 : {}^t\mathcal{M}_0 \to \mathsf{ECM}_0$$

where $T_0(\mathcal{F}) = \mathcal{F}(\ell) = \mathcal{F}(\bar{k})$ as the discrete sheaf $\mathcal{F}$ yields a $\mathrm{Gal}(\bar{k}/k)$-module $\mathcal{F}(\ell)$ for a suitable finite Galois extension ℓ of k (see also [8]). For $n = 1$ we need the thickness of ECM_1 in ECM to get the (exact, faithful) functor

$$T_1 : {}^t\mathcal{M}_1 \to \mathsf{ECM}_1$$

lifting $T_{\mathbb{Z}}$ (= the Betti or Hodge realisation of 1-motives). Over $k = \mathbb{C}$ the functor T_1 is then providing a quasi-inverse to C_1 so that

$$\mathsf{ECM}_1 = \mathsf{MHS}_1$$

is equivalent to mixed Hodge structures of level ≤ 1.

In general, the canonical exact functors $\mathsf{EHM}_n \to \mathsf{EHM}$ induce triangulated functors $D_b(\mathsf{EHM}_n) \to D_b(\mathsf{EHM})$. I don't know if the latter functor is fully faithful. If the case $M_{\leq n}$ extends to a canonical functor

$$R_n : d_{\leq n}\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to D_b(\mathsf{EHM}_n)$$

from Voevodsky triangulated category of geometrical n-motives $d_{\leq n}\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (= the thick subcategory of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ generated by motives of smooth varieties of dimension $\leq n$, see [9, §3.4]). See also [3] for an approach to $d_{\leq n}\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ via n-motivic sheaves.

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Chow motives and Rost nilpotence

Stefan Gille

In my talk I reported on Rost nilpotence and my recent work [5], where I proved this property for geometrically rational surfaces over fields of characteristic zero.

Let k be a field with algebraic closure $\bar{k}$ and R a commutative ring (with 1). Let $\mathfrak{Chow}(k,R)$ be the category of effective Chow motives with coefficients in R. Let further $\mathrm{CH}_i(X)$ be the Chow group of dimension i-cycles modulo rational equivalence of the k-scheme X and $\mathrm{A}_0(X)$ the torsion part of $\mathrm{CH}_0(X)$.

In Voevodsky's [8] proof of the Milnor conjecture the following exact triangle (in Voevodsky's derived category of effective motives over k) plays a crucial role:

$$M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \longrightarrow M_{\underline{a}} \longrightarrow M(\mathcal{X}_{\underline{a}})\,.$$

Here $M(\mathcal{X}_{\underline{a}})$ is the simplizial motive of the splitting quadric $Q_{\underline{a}}$ of the symbol $\underline{a} = \{a_1, \ldots, a_n\} \in \mathrm{K}_n^M(k)/2$, and $M_{\underline{a}}$ is a direct summand of $Q_{\underline{a}}$.

This exact triangle is a corollary of Rost's [7] decomposition theorem for the Chow motive of the projective quadric $Q_{\underline{a}}$, where the (so called) Rost motive $M_{\underline{a}}$ shows up. Rost's construction of the direct summand $M_{\underline{a}}$ uses the following property of quadrics (discovered by himself): If ρ is an endomorphism of a quadric Q in $\mathfrak{Chow}(k,R)$, such that ρ_E is an idempotent for some field extension E/k then there exists an idempotent $\tilde{\rho}$ of Q in $\mathfrak{Chow}(k,R)$, such that $\rho_E = \tilde{\rho}_E$. This property is a consequence of Rost nilpotence:

On says that Rost nilpotence is true for X in $\mathfrak{Chow}(k,R)$ if the kernel of the restriction map

$$\mathrm{res}_{E/k} : \mathrm{End}_{\mathfrak{Chow}(k,R)}(X) \longrightarrow \mathrm{End}_{\mathfrak{Chow}(E,R)}(E \times_k X)\,, \quad \alpha \longmapsto \alpha_E$$

consists of nilpotent elements for all field extensions E/k.

This has been proven for projective quadrics by Rost [7] and more generally for projective homogeneous varieties in [2]. Note that if X is a projective homogeneous variety with $X(k) \neq \emptyset$ then $\mathrm{A}_0(X) = 0$.

Remarks.

1.) Rost nilpotence plays not only a role in the proof of the Milnor conjecture, but is also essential for the recent progress in the algebraic theory of quadratic forms, see the Bourbaki report of Kahn [6].

2.) Rost nilpotence is closely related to torsion questions in Chow theory. For instance, it is easy to show that Rost nilpotence is true for any motive in $\mathfrak{Chow}(k,R)$ if R is a field of characteristic 0.

To state the main result recall that a k-surface S is called geometrically rational if $S_{\bar{k}}$ is rational. By a result of Coombes [4] there is then always a Galois extension L/k, such that S_L is rational. Any extension field L of k with S_L rational will be called a splitting field of S. Note that there exists geometrically rational surfaces S with $S(k) \neq \emptyset$ but $\mathrm{A}_0(S) \neq 0$.

Theorem. *Let k be a field of characteristic 0 and S a geometrically rational surface. Then Rost nilpotence is true for S in $\mathfrak{Chow}(k,R)$ if $R=\mathbb{Z}$ or $R=\mathbb{Z}/\mathbb{Z}m$ for some integer $m\geq 2$.*

By means of elementary algebra using the fact that $\mathrm{End}_{\mathfrak{Chow}(\bar{k},\mathbb{Z}/\mathbb{Z}m)}(S_{\bar{k}})$ is finite for all integers $m\geq 2$ one reduces[1] to $R=\mathbb{Z}$ which is proven in [5]. In this case the idea of proof is as follows.

Let $\alpha\in\mathrm{End}_{\mathfrak{Chow}(k,R)}(S)$. By a special case of a lemma of Rost [7] the correspondence α is nilpotent if $\alpha_{k(S)\,*}(\mathrm{A}_0(S_{k(S)}))=0$.

Assume that $\alpha_E=0$ for some field extension E/k. Then also $\alpha_L=0$ for any splitting field L/k of S. To show that $\alpha_{k(S)\,*}(\mathrm{A}_0(S_{k(S)}))=0$ for this endomorphism α one uses the Bloch [1] map $\Phi_{S_{k(S)}}:\mathrm{A}_0(S_{k(S)})\longrightarrow \mathrm{H}^1(G,\mathrm{H}_1(\mathrm{C}_\bullet(S_{L(S)})))$, where L/k is a Galois splitting field of S with group $G=\mathrm{Gal}(L/k)$ and $\mathrm{C}_\bullet$ is the cycle complex

$$\mathrm{K}_2^M(k(S)) \xrightarrow{d_2^S} \bigoplus_{x\in S_{(1)}} k(x)^\times \xrightarrow{d_1^S} \bigoplus_{x\in S_{(0)}} \mathbb{Z}\,,$$

where $S_{(i)}$ denotes the set of $x\in S$ with $\dim\overline{\{x\}}=i$.

Since $S(k(S))\neq\emptyset$ the homomorphism $\Phi_{S_{k(S)}}$ is injective by a theorem of Colliot-Thélène [3] and so Rost nilpotence is true for geometrically rational surfaces over fields[2] of characteristic 0 by the following fact.

Lemma. ([5, Thm. 4.8]) *Let k be a perfect field, and S a geometrically rational k-surface. Let further L/k be a (finite) Galois splitting field of S with group $G=\mathrm{Gal}(L/k)$. Then the following diagram*

$$\begin{array}{ccc} \mathrm{A}_0(S) & \xrightarrow{\Phi_S} & \mathrm{H}^1(G,\mathrm{H}_1(\mathrm{C}_\bullet(S_L))) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \mathrm{H}^1(G,\alpha_{L\,*})} \\ \mathrm{A}_0(S) & \xrightarrow{\Phi_S} & \mathrm{H}^1(G,\mathrm{H}_1(\mathrm{C}_\bullet(S_L)))\,. \end{array}$$

commutes for any $\alpha\in\mathrm{End}_k(S)=\mathrm{CH}_2(S\times_k S)$.

It is likely that this lemma is true for any field (and so also Rost nilpotence for geometrically rational surfaces over fields of positive characteristic), but the proof in [5] uses resolution of singularities in dimension 2.

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Relative DGA, associated DG category and mixed elliptic motif

Tomohide Terasoma

(joint work with Kenichiro-Kimura)

Bloch and Kriz defined a category of mixed Tate motives over a field k as the category of comodules over a Hopf algebra $H^0(B(A,\epsilon))$. Here A is the Bloch higher cycle DGA and $B(A,\epsilon)$ is the bar complex of A. Moreover, they construct a comodule P_n over $H^0(B(A,\epsilon))$ associated to polylogarithm. The motif corresponding to the comodule P_n is called the polylog motif. In this lecture, we consider elliptic analog of this construction, which is called mixed elliptic motives.

We want to construct the category of mixed elliptic motives as that of comodules over a Hopf algebra. Let E be an elliptic curve over k without complex mutiplication. The category of pure elliptic motives of E is a smallest subcategory of motif containg $h^1(E)$ and $\mathbf{Q}(1)$, and closed under taking direct sums, tensor products, direct summands, and duals. Roughly spleaking, mixed elliptic motif is the smallest full subcategory of mixed motif containg the category of pure elliptic motives and closed under extensions. Since the category of pure elliptic motives is equivalent to that of representations of GL_2, usual bar construction is insufficient to make a correct Hopf algebra. We use a method of relative bar construction. Originally, the relative bar construction is defined by R.Hain to construct a relative completion of the fundamental group $\pi_1(X)$ of a differentiable manifold X reltaive to a monodromy representation $\rho : \pi_1(X) \to S$, where S is a reductive group. We give a slightly different formalism of reltaive bar construction using the notion of reltaive DGA, which is suitable for making a DG category. Using this framework, it is enough to construct a relative DGA A_{EM} over $\Gamma(GL_2, \mathcal{O})$ associated to the elliptic curve E.

The relative DGA A_{EM} is defined as follows. Let V be a two dimensional vector space and $Sym^m(V)$ be the symmtric tensor representation of $GL(V)$. The relative DGA A_{EM} is defined by

$$\begin{aligned} A_{EM} = \oplus_{m,m',p,p'} & Sym^m(V)(p) \\ & \otimes \underline{Hom}^\bullet(Sym^m(h^1(E))(p), Sym^{m'}(h^1(E))(p')) \\ & \otimes (Sym^{m'}(V)(p'))^* \end{aligned}$$

where

$$\underline{Hom}^\bullet(Sym^m(h^1(E))(p), Sym^{m'}(h^1(E))(p'))$$
$$=Sym^m Sym^{m'} Z_-^{m+p'-p}(E^m \times E^{m'}, m-m'+2(p'-p)-i),$$

$Z_-^\bullet(*,\bullet)$ is the $(-)$-part of Bloch cycle complex, and Sym^m is the symmetrizing projector. To introduce a product structure and an antipodal, we use a quasi-isomorphism

$$B_{GL_2}(A_{EM},\epsilon) \to B^v_{GL_2}(A_{EM},\epsilon)$$

where $B^v_{GL_2}(A_{EM},\epsilon)$ is called the "virtual bar complex".

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Weil-étale cohomology of regular arithmetic schemes

Matthias Flach

This talk was concerned with Zeta functions $\zeta(X,s)$ of arithmetic schemes X and the description of their vanishing order and leading Taylor coefficient at any integer $s=n$ in terms of Weil-étale cohomology groups, developing an idea of Lichtenbaum. We first discussed the simplest case $s=0$ and the rather complete results one has in this case for X (separated and of finite type) over a finite field by work of Lichtenbaum and Geisser. We then discussed Lichtenbaum's work, as completed by the speaker, for $s=0$ and X the spectrum of the ring of integers of a number field. We then reported on joint work with Baptiste Morin giving a definition of a Weil-étale topos for an arbitrary regular scheme X, proper over Spec($\mathbb{Z}$) which yields Weil-étale cohomology groups with $\mathbb{R}$-coefficients having the expected relation to the Zeta-function at $s=0$, provided one knows the meromorphic continuation and functional equation of the Hasse-Weil L-functions attached to $X \otimes \mathbb{Q}$. This is the case, for example, if X is a regular model of $E \times \cdots \times E$ where E is an elliptic curve over a totally real field F by recent work of Harris, Taylor, Shin et al. Finally, we reported on work in progress which reformulates the Tamagawa number conjecture of Bloch and Kato for the Dedekind Zeta function of a number field at any integer n in terms of Weil-étale cohomology. This kind of reformulation should be possible for any X and any integer n but we are lacking a good definition of the Weil-etale topos in characteristic zero. In the case where X is the spectrum of the integer ring in a number field we get around this problem by defining Weil-étale cohomology as the hypercohomology of certain explicitly constructed complexes of sheaves $R\mathbb{Z}(n)$ in the étale topos.

Preorientations of the derived motivic multiplicative group

Jens Hornbostel

Recently, Jacob Lurie [Lu1] gave a description of the spectrum tmf (= "topological modular forms") as the solution of a moduli problem in derived algebraic geometry. The latter here is constructed with commutative ring spectra as the affine derived schemes, and the moduli problem is to classify derived oriented elliptic curves with all terms defined appropriately. Lurie's point of view is that the best language to state and prove the theorem is the one of infinity categories rather than the one of model categories, and we have no reason to doubt he is right.

The above description of tmf (corresponding to height 2 and the second chromatic layer) has an analog in height 1 which is much easier to state and to prove, and is also due to Lurie [Lu1, section 3]. Namely, real topological K-theory KO classifies oriented derived multiplicative groups. The key step for proving this is to show that the suspension spectrum of $\mathbf{CP}^\infty$ classifies preorientations of the derived multiplicative group. Here the derived multiplicative group is by definition $\mathbf{G}_m := \Sigma^\infty \mathbf{Z}_+$, the name being justified by classical algebraic geometry over a base field k, where the multiplicative group is $Spec(k[\mathbf{Z}])$. As usual, the object $Rmap_{AbMon(Sp^\Sigma)}(\Sigma^\infty \mathbf{Z}_+, -))$ it represents via the derived version of the Yoneda embedding will still be called the multiplicative group. We are able provide a proof of this result in the language of model categories and symmetric spectra Sp^Σ, and present some of its ingredients in our talk. The result reads as follows in general, the special case $N = \mathbf{CP}^\infty$ being the one discussed above:

Theorem 1. *(Lurie) For any abelian monoid A in symmetric spectra Sp^Σ (based on simplicial sets) and any abelian group N in simplicial sets, we have a natural isomorphism of abelian groups*

$$\begin{aligned} &Hom_{Ho(AbMon(Sp^\Sigma))}(\Sigma^\infty N_+, A) \\ \simeq\ &Hom_{Ho(AbMon(\S Sets))}(N, Rmap_{AbMon(Sp^\Sigma)}(\Sigma^\infty \mathbf{Z}_+, A)) \\ =\ &Hom_{Ho(AbMon(\S Sets))}(N, \mathbf{G}_m(A)). \end{aligned}$$

Here $Ho(-)$ denotes the homotopy category, $Rmap$ means the derived mapping space and the weak equivalences between abelian monoids are always the underlying ones, forgetting the abelian monoid structure. We explain the model structures involved in this theorem, which are due to Hovey-Shipley-Smith, Harper and others (see in particular [HSS], [Sh], [Ha]). Among the ingredients of the proof we then discuss are a model category refinement of the recognition principle, a theorem of Schwede-Shipley [SS] comparing chain complexes with $H\mathbf{Z}$-modules and a new non-positive model structure for E-modules in Sp^Σ where E is the Barratt-Eccles operad. Using a theorem of Snaith [Sn], Lurie's definition of an orientation and the above theorem then imply his above theorem about KO.

We then discuss the motivic generalization of this theorem, that is to motivic symmetric spectra $Sp^{\Sigma,T}(\mathcal{M})$ on the site $\mathcal{M} = (Sm/k)_{Nis}$ with k an arbitrary base field. For this, we must establish various motivic model structures on categories built from motivic symmetric spectra with respect to both circles S^1 and

$\mathbf{P}^1$ and suitable model structures, the first results here being due to Hovey and Jardine. Once we have established all necessary model structures and some of their properties, the main theorem then can be stated as follows.

Theorem 2. *Let $\mathcal{M} = (Sm/k)_{Nis}$ and $T = S^1$ or $T = \mathbf{P}^1$. Then for any abelian monoid A in motivic symmetric T-spectra $Sp^{\Sigma,T}(\mathcal{M})$ and any abelian group N in the category $\Delta^{op}PrShv(\mathcal{M})$ of simplicial presheaves on $\mathcal{M}$, we have a natural isomorphism of abelian groups*

$$Hom_{Ho(AbMon(Sp^{\Sigma,T}(\mathcal{M})))}(\Sigma_T^\infty N_+, A)$$
$$\simeq Hom_{Ho(AbMon(\Delta^{op}PrShv(\mathcal{M})))}(N, Rmap_{AbMon(Sp^{\Sigma,T}(\mathcal{M}))}(\Sigma_T^\infty \mathbf{Z}_+, A))$$

Appliying this theorem to $T = \mathbf{P}^1$ pointed at ∞ and to $N = \mathbf{P}^\infty$ which is not a variety but still a simplicial presheaf, and using the recently established motivic version of Snaith's theorem [GS], [SO], it will imply that *algebraic K-theory represents motivic orientations of the derived motivic multiplicative group*, provided one works with the correct motivic generalizations of the concept of derived algebraic groups and of orientations. We mention some non-trivial motivic ingredients of the proof, notably Morel's [Mo] stable $\mathbf{A}^1$-connectivity theorem which is the reason why we work over a field rather than a more general base scheme.

One of the many motivations is that the generalizations of the language of derived algebraic geometry from classical to motivic spectra should ultimately lead to a definition of a motivic version of tmf, generalizing the above Theorem 1 of Lurie about height 2 to the motivic set-up as well.

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The motivic Galois group, the Grothendieck-Teichmüller group and the double shuffle group

Hidekazu Furusho

Let $DM(\mathbf{Q})_{\mathbf{Q}}$ be the triangulated category of *mixed motives* over $\mathbf{Q}$ constructed by Hanamura, Levine and Voevodsky. *Tate motives* $\mathbf{Q}(n)$ $(n \in \mathbf{Z})$ are Tate objects of the category. Let $DMT(\mathbf{Q})_{\mathbf{Q}}$ be the triangulated sub-category of $DM(\mathbf{Q})_{\mathbf{Q}}$ generated by Tate motives $\mathbf{Q}(n)$ $(n \in \mathbf{Z})$. By the work of Levine a neutral tannakian $\mathbf{Q}$-category $MT(\mathbf{Q}) = MT(\mathbf{Q})_{\mathbf{Q}}$ of *mixed Tate motives over* $\mathbf{Q}$ can be extracted by taking a heart with respect to a t-structure of $DMT(\mathbf{Q})_{\mathbf{Q}}$. Deligne and Goncharov [1] defined the full subcategory $MT(\mathbf{Z}) = MT(\mathbf{Z})_{\mathbf{Q}}$ of *unramified mixed Tate motives*, whose objects are mixed Tate motives M (an object of $MT(\mathbf{Q})$) such that for each subquotient E of M which is an extension of $\mathbf{Q}(n)$ by $\mathbf{Q}(n+1)$ for $n \in \mathbf{Z}$, the extension class of E in $Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(n), \mathbf{Q}(n+1)) = Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Q}^\times \otimes \mathbf{Q}$ is equal to $\mathbf{Z}^\times \otimes \mathbf{Q} = \{0\}$. The category $MT(\mathbf{Z})$ forms a neutral tannakian $\mathbf{Q}$-category with the fiber functor $\omega_{\mathrm{can}} : MT(\mathbf{Z}) \to Vect_{\mathbf{Q}}$ ($Vect_{\mathbf{Q}}$: the category of $\mathbf{Q}$-vector spaces) sending each motive M to $\oplus_n Hom(\mathbf{Q}(n), Gr^W_{-2n}M)$.

Definition 1. The *motivic Galois group* of unramified mixed Tate motives $MT(\mathbf{Z})$ is defined to be the pro-algebraic group $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) := \underline{Aut}^{\otimes}(MT(\mathbf{Z}) : \omega_{\mathrm{can}})$.

The action of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ on $\omega_{\mathrm{can}}(\mathbf{Q}(1)) = \mathbf{Q}$ defines a surjection $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \to \mathbf{G}_m$ and its kernel $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1$ is the unipotent radical of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$. In [1] §4 they constructed the *motivic fundamental group* $\pi_1^{\mathcal{M}}(X : \overrightarrow{01})$ with $X = \mathbf{P}^1 \backslash \{0, 1, \infty\}$, which is an ind-object of $MT(\mathbf{Z})$. This is an affine group $MT(\mathbf{Z})$-scheme. It induces the morphism $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \to \underline{Aut}\underline{F_2}$ where $\underline{F_2} = \omega_{\mathrm{can}}(\pi_1^{\mathcal{M}}(X : \overrightarrow{01}))$ is the free pro-unipotent algebraic group of rank 2. Denote its restriction into the unipotent part by

$$\Psi : \mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})_1 \to \underline{Aut}\underline{F_2}. \tag{1}$$

This map is expected to be injective.

Let us fix notations: Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure and $U\mathfrak{F}_2 = k\langle\langle X_0, X_1\rangle\rangle$ a non-commutative formal power series ring with two variables X_0 and X_1. Its element $\varphi = \varphi(X_0, X_1)$ is called *group-like* if it satisfies $\Delta(\varphi) = \varphi \otimes \varphi$ with $\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0$ and $\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1$ and its constant term is equal to 1. For a monic monomial W, $c_W(\varphi)$ means the coefficient of W in φ. For any k-algebra homomorphism $\iota : U\mathfrak{F}_2 \to S$ the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(X_0), \iota(X_1))$.

Definition 2 ([2])**.** The *Grothendieck-Teichmüller group* GRT_1 is defined to be the pro-unipotent algebraic variety whose set of k-valued points consists of group-like series $\varphi \in U\mathfrak{F}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$ satisfying *Drinfel'd's two hexagon equations* in $U\mathfrak{F}_2$:

$$\varphi(t_{13}, t_{12})\varphi(t_{13}, t_{23})^{-1}\varphi(t_{12}, t_{23}) = 1, \tag{2}$$

$$\varphi(t_{23}, t_{13})^{-1}\varphi(t_{12}, t_{13})\varphi(t_{12}, t_{23})^{-1} = 1 \tag{3}$$

and *his pentagon equation* in $U\mathfrak{a}_4$:

(4) $\varphi(t_{12}, t_{23}+t_{24})\varphi(t_{13}+t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\varphi(t_{12}+t_{13}, t_{24}+t_{34})\varphi(t_{12}, t_{23}).$

Here $U\mathfrak{a}_4$ means the universal enveloping algebra of the *completed pure braid Lie algebra* $\mathfrak{a}_4$ over k *with 4 strings*, generated by t_{ij} ($1 \leqslant i,j \leqslant 4$) with defining relations $t_{ii}=0$, $t_{ij}=t_{ji}$, $[t_{ij}, t_{ik}+t_{jk}]=0$ (i,j,k: all distinct) and $[t_{ij}, t_{kl}]=0$ (i,j,k,l: all distinct).

By the multiplication below, GRT_1 really forms a group

$$\varphi_2 \circ \varphi_1 := \varphi_1(\varphi_2 X_0 \varphi_2^{-1}, X_1)\cdot\varphi_2 = \varphi_2\cdot\varphi_1(X_0, \varphi_2^{-1}X_1\varphi_2). \tag{5}$$

The group was introduced by Drinfel'd [2] in his study of quasitriangular quasi-Hopf quantized universal enveloping algebras, certain types of quantum groups. Let $\underline{F_2}$ be the free pro-unipotent algebraic group with two generators e^{X_0} and e^{X_1} and $\underline{Aut}\underline{F_2}$ be the pro-algebraic group which represents $k \mapsto Aut\underline{F_2}(k)$. By the map sending $X_0 \mapsto X_0$ and $X_1 \mapsto \varphi X_1 \varphi^{-1}$, the group GRT_1 is regarded as a subgroup of $\underline{Aut}\underline{F_2}$. By geometric interpretations of the equations (2)$\sim$ (4), it is shown that $\mathrm{Im}\Psi$ is contained in GRT_1. Actually it is expected that they are isomorphic. Our first result here is on defining equations of GRT_1.

Theorem 3 ([4])**.** *Let $\varphi=\varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$. Suppose that φ satisfies the pentagon equation* (4)*. Then it also satisfies two hexagon equations* (2) *and* (3)*.*

This theorem claims that the pentagon equation (4) is essentially a single defining equation of the Grothendieck-Teichmüller group.

Again let us fix notations: Let $\pi_Y : k\langle\langle X_0, X_1\rangle\rangle \to k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ be the k-linear map between non-commutative formal power series rings that sends all the words ending in X_0 to zero and the word $X_0^{n_m-1}X_1\cdots X_0^{n_1-1}X_1$ ($n_1,\dots,n_m \in \mathbf{N}$) to $(-1)^m Y_{n_m}\cdots Y_{n_1}$. Define the coproduct Δ_* on $k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ by $\Delta_* Y_n = \sum_{i=0}^n Y_i \otimes Y_{n-i}$ with $Y_0 := 1$. For $\varphi = \sum_{W:\text{word}} c_W(\varphi)W \in k\langle\langle X_0, X_1\rangle\rangle$, put $\varphi_* = \exp\left(\sum_{n=1}^\infty \frac{(-1)^n}{n} c_{X_0^{n-1}X_1}(\varphi) Y_1^n\right)\cdot\pi_Y(\varphi)$. For a group-like series $\varphi \in U\mathfrak{F}_2$ the *generalised double shuffle relation* means the equality

$$\Delta_*(\varphi_*) = \varphi_* \widehat{\otimes} \varphi_*. \tag{6}$$

Definition 4 ([5])**.** The *double shuffle group* DMR_0 is the pro-unipotent algebraic variety whose set of k-valued points consists of the group-like series $\varphi \in U\mathfrak{F}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$ which satisfy (6).

The generalized double shuffle relation (6) arises from the generalized (regularised) double shuffle relations among multiple zeta values, which are expected to be the strongest relation among them. In [5] it is proved that DMR_0 is closed by the multiplication (5) as GRT_1. By the same way to the GRT_1-case, the group DMR_0 is regarded as a subgroup of $\underline{Aut}\underline{F_2}$. It is also shown that $\mathrm{Im}\Psi$ is contained in DMR_0. Actually it is expected that they are isomorphic. And DMR_0 is also expected to be isomorphic to GRT_1. Our second result here is a relationship between them.

Theorem 5 ([3]). $GRT_1 \subset DMR_0$.

We note that this realizes the project of Deligne-Terasoma where they indicated a different approach. Their arguments concerned multiplicative convolutions whereas our methods are based on a bar construction calculus (cf. [3]).

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On rational K-theory in characteristic p

Thomas Geisser

Parshin's conjecture states that for X smooth and proper over a finite field, the group $K_i(X)$ is torsion for $i > 0$. This conjecture is motivated by the idea that higher algebraic K-groups are related to extensions in a conjectural category of mixed motives, whereas over a finite field, such a category would be semi-simple.

In [2], we showed that if Tate's conjecture holds and rational and numerial equivalence agree up to torsion, then Parshin's conjecture holds.

In this talk, we gave an overview over the articles [3], [6] and [5], which deal with consequences and approaches to Parshin's conjecture for rational motivic theories for schemes over a finite field.

1. Motivic theories

Recall from [1] that we have four motivic theories: Motivic cohomology, motivic cohomology with compact support, motivic homology and Borel-Moore motivic homology. All four theories are homotopy invariant and satisfy a projective bundle formula. The theories are related by the following diagram

$$\begin{array}{ccc} H^i_c(X,\mathbb{Q}(n)) & \xrightarrow{proper} & H^i(X,\mathbb{Q}(n)) \\ {\scriptstyle smooth}\downarrow & & {\scriptstyle smooth}\downarrow \\ H_j(X,\mathbb{Q}(m)) & \xrightarrow{proper} & H^c_j(X,\mathbb{Q}(m)) \end{array}$$

The horizontal maps are isomorphisms for proper X, and the vertical maps are isomorphisms if X is smooth of pure dimension d, and $m+n=d$ and $j+i=2d$. The groups diagonally opposite should be in some form of duality; we will see that with rational coefficients, this is equivalent to deep conjectures.

Since $K_i(X)_{\mathbb{Q}} = \oplus_n H^{2n-i}(X,\mathbb{Q}(n))$, Parshin's conjecture is equivalent to the following conjecture for all n and m, respectively.

Conjecture P^n: *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, and all $i \neq 2n$, the group $H^i(X, \mathbb{Z}(n))$ is torsion.*

Conjecture P^n is known for $n = 0, 1$ and is trivial for $n < 0$.

Conjecture P_m: *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, and all $i \neq 2m$, the group $H_i^c(X, \mathbb{Z}(m))$ is torsion.*

This conjecture is not known for any m. By the projective bundle formula one gets $P^n \Rightarrow P^{n-1}$ and $P_m \Rightarrow P_{m-1}$.

2. Applications to Suslin homology

Suslin homology $H_i^S(X, A)$ of X with coefficients in the abelian group A is the homology of $C_*^X \otimes A$. Here C_*^X is the complex which in degree $-i$ is the free abelian group generated by closed irreducible subschemes of $X \times \Delta^i$ which are finite and surjective over Δ^i, and differentials given by alternating maps of pull-backs along face maps.

Proposition 2. *Under resolution of singularities, the following statements are equivalent:*

1. *Conjecture P_0*
2. *The groups $H_i^S(X, \mathbb{Q})$ are finite dimensional and vanish unless $0 \leq i \leq \dim X$. If X is smooth, then they vanish unless $i = 0$.*
3. *The groups $H_i^S(X, \mathbb{Z})$ are finitely generated for all X of finite type over a finite field.*
4. *For all X smooth over a finite field, there are short exact sequences*

$$0 \to H_{i+1}^S(\bar{X}, \mathbb{Z})_G \to H_i^S(X, \mathbb{Z}) \to H_i^S(\bar{X}, \mathbb{Z})^G \to 0.$$

   Here G is the free abelian group generated by the Frobenius endomorphism, and $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$.

The proof of finite generation uses recent work of Jannsen, Kerz and Saito on the so called Kato-conjecture. In [4], we proved the analog of the above for smooth and proper X and higher Chow groups, and later realized that using Suslin homology, the properness becomes unnecessary.

3. Niveau spectral sequences

Parshin's conjecture can be analyzed with niveau spectral sequences for Borel-Moore homology and motivic cohomology with compact support. Due to space constraints we only consider the second case [6], the other case is in [3].

In order to avoid derived inverse limits, we work with the dual of motivic cohomology with compact support

$$H_c^i(X, \mathbb{Q}(n))^* := \mathrm{Hom}(H_c^i(X, \mathbb{Z}(n)), \mathbb{Q}).$$

For a point $x \in X$ we define $H^i_c(k(x), \mathbb{Q}(n))^* := \mathrm{colim}_{U \cap \overline{\{x\}} \neq \emptyset} H^i_c(U \cap \overline{\{x\}}, \mathbb{Q}(n))^*$. Then the usual yoga with exact couples gives a homological spectral sequence

$$E^1_{s,t} = \bigoplus_{x \in X_{(s)}} H^{s+t}_c(k(x), \mathbb{Q}(n))^* \Rightarrow H^{s+t}_c(X, \mathbb{Q}(n))^*. \tag{1}$$

Using this spectral sequence, one sees that Conjecture P^n holds if and only if $H^i_c(X, \mathbb{Q}(n))$ vanishes for $i < 2n$ and all schemes X over $\mathbb{F}_q$, if and only if $H^i_c(k, \mathbb{Q}(n))^*$ vanishes for $i < 2n$ and all finitely generated fields $k/\mathbb{F}_q$.

The spectral sequence (1) is concentrated below and on the line $t = n$, and on the line $t = n$, the terms $E^1_{s,n}$ vanish for $s < n$. We define $\tilde{H}^j_c(X, \mathbb{Q}(n))^* = E^2_{j-n,n}$ to be the homology of the line $E^1_{*,n}$

$$\bigoplus_{x \in X_{(n)}} H^{2n}_c(k(x), \mathbb{Q}(n))^* \leftarrow \cdots \leftarrow \bigoplus_{x \in X_{(d)}} H^{n+d}_c(k(x), \mathbb{Q}(n))^* \tag{2}$$

where the term indexed by $X_{(i)}$ is in degree $n + i$. We obtain canonical maps

$$\tilde{H}^i_c(X, \mathbb{Q}(n))^* \xrightarrow{\alpha^*} H^i_c(X, \mathbb{Q}(n))^* \tag{3}$$

The map α^* is an isomorphism for all X if and only if the groups $H^i_c(k, \mathbb{Q}(n))^*$ vanish for $i \neq n + \mathrm{trdeg}\, k$, if and only if Conjecture P^n holds, and for smooth and projective X we have

$$\tilde{H}^i_c(X, \mathbb{Q}(n))^* \cong \begin{cases} CH^n(X)^* & i = 2n; \\ 0 & \text{else.} \end{cases}$$

To unify the homological and cohomological approaches, we consider Beilinson's

Conjecture $D(n)$**:** *For all smooth and proper schemes X over the finite field $\mathbb{F}_q$, the intersection pairing gives a functorial isomorphism*

$$CH^n(X)_{\mathbb{Q}} \cong \mathrm{Hom}(CH_n(X), \mathbb{Q}).$$

A more general statement is the following:

Conjecture $BP(n)$**:** *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, the cup product pairing*

$$H^i(X, \mathbb{Q}(n)) \times H^{2d-i}(X, \mathbb{Q}(d-n)) \to \mathbb{Q}$$

is perfect.

The latter is a combination of Parshin's and Beilinson's conjecture, and relates the groups on the opposite sides of the diagram in the beginning:

Proposition 3. *For fixed n, the following statements are equivalent:*

1. *Conjecture $BP(n)$.*
2. *Conjectures $D(n)$, P^n and P_n.*

(3) *There are perfect pairings of finite dimensional vector spaces*

$$H_i^c(X, \mathbb{Q}(n)) \times H_c^i(X, \mathbb{Q}(n)) \to \mathbb{Q}$$

for all X, respectively all smooth projective X.

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Weight and boundary

Jörg Wildeshaus

The purpose of this talk was to give an introduction to the notion of *weight structure* on a triangulated category, and to illustrate its usefulness for motives. An extended version of the present notes can be found in the first section of [W3].

The following definition is due to Bondarko.

Definition 1. Let $\mathcal{C}$ be a triangulated category. A *weight structure on* $\mathcal{C}$ is a pair $w = (\mathcal{C}_{w \le 0}, \mathcal{C}_{w \ge 0})$ of full sub-categories of $\mathcal{C}$, such that, putting

$$\mathcal{C}_{w \le n} := \mathcal{C}_{w \le 0}[n] \quad , \quad \mathcal{C}_{w \ge n} := \mathcal{C}_{w \ge 0}[n] \quad \forall\, n \in \mathbb{Z} \; ,$$

the following conditions are satisfied.

(1) The categories $\mathcal{C}_{w \le 0}$ and $\mathcal{C}_{w \ge 0}$ are Karoubi-closed: for any object M of $\mathcal{C}_{w \le 0}$ or $\mathcal{C}_{w \ge 0}$, any direct summand of M formed in $\mathcal{C}$ is an object of $\mathcal{C}_{w \le 0}$ or $\mathcal{C}_{w \ge 0}$, respectively.

(2) (Semi-invariance with respect to shifts.) We have the inclusions

$$\mathcal{C}_{w \le 0} \subset \mathcal{C}_{w \le 1} \quad , \quad \mathcal{C}_{w \ge 0} \supset \mathcal{C}_{w \ge 1}$$

of full sub-categories of $\mathcal{C}$.

(3) (Orthogonality.) For any pair of objects $M \in \mathcal{C}_{w \le 0}$ and $N \in \mathcal{C}_{w \ge 1}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(M, N) = 0 \; .$$

(4) (Weight filtration.) For any object $M \in \mathcal{C}$, there exists an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, such that $A \in \mathcal{C}_{w \le 0}$ and $B \in \mathcal{C}_{w \ge 1}$.

Our convention concerning the sign of the weight is actually opposite to the one from [Bo2, Def. 1.1.1], i.e., we exchanged the roles of $\mathcal{C}_{w\leq 0}$ and $\mathcal{C}_{w\geq 0}$. Note that in condition (4), "the" weight filtration is not assumed to be unique. Weight structures are relevant to motives thanks to the following result.

Theorem 2 (Bondarko). *Let F be a commutative flat $\mathbb{Z}$-algebra, and assume k to admit resolution of singularities. Then there is a canonical weight structure on the category $DM_{gm}^{eff}(k)_F$, the F-linear version of Voevodsky's category of effective* geometrical motives. *It is uniquely characterized by the requirement that its heart equal (the opposite of) $CHM^{eff}(k)_F$, the F-linear version of Grothendieck's category of effective* Chow motives.

A concise presentation of the ingredients of the proof is given in [W2, Thm. 1.13]. The following result is formally implied by Theorem 1, and the fondamental properties of the category $DM_{gm}^{eff}(k)_F$, notably *localization* [V, Prop. 4.1.5]. For details of the proof, we refer to [W2, Cor. 1.14] ([Bo1, Thm. 6.2.1 (1) and (2)] it k is of characteristic zero).

Corollary 3. *Let X/k be a variety. Assume k to admit resolution of singularities.*
(a) The motive with compact support *$M_{gm}^c(X)$ lies in $DM_{gm}^{eff}(k)_{F,w\geq 0}$.*
(b) If X is smooth, then the motive *$M(X)$ lies in $DM_{gm}^{eff}(k)_{F,w\leq 0}$.*

Fix a smooth variety X over k. Recall that by construction [W1, Def. 2.1, Prop. 2.2], the *boundary motive* $\partial M_{gm}(X)$ of X lies in an exact triangle

$$\partial M_{gm}(X) \longrightarrow M(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1]$$

in $DM_{gm}^{eff}(k)_F$. The following was the main new result presented in the talk. Its proof uses only the formal properties of weight structures, and the information on weights from Corollary 3.

Corollary 4. *There is a natural bijection between*
(1) the isomorphism classes of weight filtrations of $\partial M_{gm}(X)$,
(2) the isomorphism classes of effective Chow motives $M_0 \in CHM^{eff}(k)_F$, together with a factorization

$$M(X) \xrightarrow{\pi_0} M_0 \xrightarrow{i_0} M_{gm}^c(X)$$

of the canonical morphism $M(X) \to M_{gm}^c(X)$, such that both i_0 and $\pi_0[1]$ can be completed to give weight filtrations of $M_{gm}^c(X)$ and of $M(X)[1]$, respectively.

Note that the motive $M(\widetilde{X})$ of any smooth compactification $\widetilde{X}$ of X yields data of type (2), hence an isomorphism class of weight filtrations of $\partial M_{gm}(X)$. As we pointed out, there should be weight filtrations of $\partial M_{gm}(X)$ other than those obtained this way, for example those associated to the (hypothetical) *intersection motive* of a singular compactification of X. Indeed, the bijection of Corollary 2 should potentially serve to construct such motives.

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On the functoriality of the slice filtration

Pablo Pelaez

Let $g : X \to Y$ be a map of finite type between Noetherian separated schemes of finite Krull dimension, and $\mathcal{SH}_X$ the Morel-Voevodsky stable homotopy category of T-spectra [1], where T is the pointed simplicial presheaf represented by $S^1 \wedge \mathbb{G}_m$.

Given an integer $q \in \mathbb{Z}$, we consider the following family of T-spectra

$$C^q_{eff} = \{F_n(S^r \wedge \mathbb{G}^s_m \wedge U_+) \mid n, r, s \geq 0; s - n \geq q; U \in Sm_X\}$$

where Sm_X is the smooth Nisnevich site over X and F_n is the left adjoint to the n-evaluation functor between the category of T-spectra and the unstable category of pointed simplicial presheaves on Sm_X

$$ev_n : Spt(\mathcal{M}_X) \to \mathcal{M}_X$$

Voevodsky [5] defines the slice filtration as the following family of triangulated subcategories of $\mathcal{SH}_X$

$$\cdots \subseteq \Sigma^{q+1}_T \mathcal{SH}^{eff}_X \subseteq \Sigma^q_T \mathcal{SH}^{eff}_X \subseteq \Sigma^{q-1}_T \mathcal{SH}^{eff}_X \subseteq \cdots$$

where $\Sigma^q_T \mathcal{SH}^{eff}_X$ is the smallest full triangulated subcategory of $\mathcal{SH}_X$ which contains C^q_{eff} and is closed under arbitrary coproducts.

It follows from the work of Neeman [2], [3] that the inclusion

$$i_q : \Sigma^q_T \mathcal{SH}^{eff}_X \to \mathcal{SH}_X$$

has a right adjoint $r_q : \mathcal{SH}_X \to \Sigma^q_T \mathcal{SH}^{eff}_X$, and the following functors

$$f_q : \mathcal{SH}_X \to \mathcal{SH}_X$$

$$s_q : \mathcal{SH}_X \to \mathcal{SH}_X$$

are exact, where f_q is defined as the composition $i_q \circ r_q$, s_q fits in the following distinguished triangle for every $E \in \mathcal{SH}_X$

$$f_{q+1}E \xrightarrow{\rho_q^E} f_qE \xrightarrow{\pi_q^E} s_qE \longrightarrow \Sigma_T^{1,0} f_{q+1}E$$

and $\mathrm{Hom}_{\mathcal{SH}_X}(F, s_qE) = 0$ for every $F \in \Sigma_T^{q+1}\mathcal{SH}_X^{eff}$. We will refer to f_qE as the $(q-1)$-connective cover of E, and to s_qE as the q-slice of E.

Our main goal is to discuss the behavior of the pullback functor

$$\mathbf{L}g^* : \mathcal{SH}_Y \to \mathcal{SH}_X$$

with respect to the slice filtration in $\mathcal{SH}_X$ and $\mathcal{SH}_Y$. It is easy to see that

$$\mathbf{L}g^*(\Sigma_T^q \mathcal{SH}_Y^{eff}) \subseteq \Sigma_T^q \mathcal{SH}_X^{eff}$$

and thus for any integer $q \in \mathbb{Z}$, we have a pair of natural transformations

$$\alpha_q : \mathbf{L}g^* \circ f_q \to f_q \circ \mathbf{L}g^*$$

$$\beta_q : \mathbf{L}g^* \circ s_q \to s_q \circ \mathbf{L}g^*$$

such that for every $E \in \mathcal{SH}_Y$ the following diagram

$$\begin{array}{ccccccc}
\mathbf{L}g^*(f_{q+1}E) & \xrightarrow{\mathbf{L}g^*(\rho_q^E)} & \mathbf{L}g^*(f_qE) & \xrightarrow{\mathbf{L}g^*(\pi_q^E)} & \mathbf{L}g^*(s_qE) & \longrightarrow & \mathbf{L}g^*(\Sigma_{T_Y}^{1,0} f_{q+1}E) \\
\downarrow{\scriptstyle \alpha_{q+1}(E)} & & \downarrow{\scriptstyle \alpha_q(E)} & & \downarrow{\scriptstyle \beta_q(E)} & & \downarrow{\scriptstyle \Sigma_{T_X}^{1,0}(\alpha_{q+1}(E))} \\
f_{q+1}(\mathbf{L}g^*E) & \xrightarrow[\rho_q^{\mathbf{L}g^*E}]{} & f_q(\mathbf{L}g^*E) & \xrightarrow[\pi_q^{\mathbf{L}g^*E}]{} & s_q(\mathbf{L}g^*E) & \longrightarrow & \Sigma_{T_X}^{1,0} f_{q+1}(\mathbf{L}g^*E)
\end{array}$$

is commutative and its rows are distinguished triangles in $\mathcal{SH}_X$.

Lemma 1. *Let $E \in \mathcal{SH}_Y$ and $q \in \mathbb{Z}$. If the following condition holds:*

$$f_{q+1}(\mathbf{L}g^*(s_qE)) = 0 \tag{1}$$

then the natural maps:

$$\alpha_{q+1}(f_qE) : \mathbf{L}g^*(f_{q+1}f_qE) \longrightarrow f_{q+1}(\mathbf{L}g^*(f_qE))$$
$$\alpha_q(f_qE) : \mathbf{L}g^*(f_qf_qE) \longrightarrow f_q(\mathbf{L}g^*(f_qE))$$
$$\beta_q(f_qE) : \mathbf{L}g^*(s_qf_qE) \longrightarrow s_q(\mathbf{L}g^*(f_qE))$$

are all isomorphisms in $\mathcal{SH}_X$.

Proof. For the details we refer the reader to [4]. □

Proposition 4. *If the condition (1) in lemma 1 holds for every spectrum in $\mathcal{SH}_Y$ and for every integer $\ell \in \mathbb{Z}$, then $\beta_q : \mathbf{L}g^* \circ s_q \to s_q \circ \mathbf{L}g^*$ is a natural isomorphism for every $q \in \mathbb{Z}$.*

Proof. For the details we refer the reader to [4]. □

In the rest of this note we assume that $Y = \mathrm{Spec}(k)$ where k is a field with resolution of singularities. Our main result is the following

Theorem 1. *Let E be a spectrum in $\mathcal{SH}_k$ and $q \in \mathbb{Z}$ an arbitrary integer. Then*

$$f_{q+1}(\mathbf{L}g^*(s_qE)) = 0$$

and as a consequence we have that $\beta_q : \mathbf{L}g^* \circ s_q \to s_q \circ \mathbf{L}g^*$ is a natural isomorphism for every $q \in \mathbb{Z}$.

Proof. For the details we refer the reader to [4]. □

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Mixed motives over $k[t]/(t^{m+1})$

Jinhyun Park

(joint work with Amalendu Krishna)

For a preadditive dg-category $\mathcal{C}$, Bondal and Kapranov [2] gave a formalism that associates a triangulated category $\mathrm{Tr}(\mathcal{C})$ to $\mathcal{C}$ as follows:

$$\mathcal{C} \rightsquigarrow \mathcal{C}^{\oplus} \rightsquigarrow \mathrm{PreTr}(\mathcal{C}) \rightsquigarrow \mathrm{Tr}(\mathcal{C}).$$

Here, $\mathcal{C}^{\oplus}$ is the category obtained from $\mathcal{C}$ by formally throwing in finite coproducts of objects of $\mathcal{C}$, while $\mathrm{PreTr}(\mathcal{C})$ is the dg-category of so-called *twisted complexes*, (A_i, p_{ij}) where $A_i \in \mathcal{C}^{\oplus}$ and $p_{ij} : A_i \to A_j$ is a morphism of degree $i - j + 1$, with certain morphisms that form complexes with the differential given by some operator D. The category $\mathrm{Tr}(\mathcal{C})$ is obtained by taking the cohomology H^0 with respect to the differential D.

The basic central result in the paper [5] is to prove that this formalism can be extended to what we call *partial dg-categories*, motivated by Hanamura's work on motives in [3]. Roughly speaking, a partial dg-category is a dg-category-like collection of objects with morphisms, while the main difference is that (1) each pair of objects A, B has the complex $\hom(A, B)$ together with a class $\mathcal{S}(A, B)$ of quasi-isomorphic subcomplexes of $\hom(A, B)$, called *the distinguished subcomplexes* of $\hom(A, B)$, subject to certain compatibility conditions, and (2) the compositions of morphisms are not necessarily defined, but defined on the level of some distinguished subcomplexes.

We prove that under the above assumptions, we can still execute the formalism of Bondal and Kapranov, and the obtained collection $\mathrm{PreTr}(\mathcal{C})$ is also a partial dg-category with respect to the differentials D such that, after applying the cohomology functor H^0, we obtain an honest triangulated category $\mathrm{Tr}(\mathcal{C})$.

As the first example, one can take $\mathcal{C}_H$ as follows: let k be a fixed field. The objects of $\mathcal{C}_H$ are the pairs (X, r) where X is a smooth projective variety over k, and $r \in \mathbb{Z}$. The morphisms between $(X, r), (Y, s)$ are given by $\hom((X, r), (Y, s)) := z^{\dim X - r + s}(X \times Y, -\bullet)$, the higher Chow complex of $X \times Y$ ([1]). The distinguished subcomplexes are given by $z_{\mathcal{W}}^{\dim X - r + s}(X \times Y, -\bullet)$, where $\mathcal{W}$ are some finite collections of algebraic sets subject to certain conditions. That this $\mathcal{C}_H$ is a partial dg-category follows from the moving lemma for higher Chow groups. Here, $\mathrm{Tr}(\mathcal{C}_H)^\sharp$, where $\sharp$ is the pseudo-abelian hull, then becomes the integral version of Hanamura's triangulated category $D\mathcal{M}^H(k)$ of mixed motives over k. Its rational version was considered in [3].

The second result of this paper is about an extension of the category of mixed motives of Hanamura. Let $\mathcal{C}$ be defined by the same objects as $\mathcal{C}_H$, but for the morphisms, we take $\hom_{\mathcal{C}}((X, r), (Y, s)) = z^{\dim X - r + s}(X \times Y, -\bullet) \oplus Tz^{\dim X - r + s}(X \times Y, -\bullet)$, both the higher Chow complex and the additive higher Chow complex ([4]) of $X \times Y$. Since we now have the moving lemma for additive higher Chow groups by [4], we deduce that $\mathcal{C}$ is a partial dg-category. Hence, we define the triangulated category of mixed motives $D\mathcal{M}(k; m)$ with the modulus m augmention to be $\mathrm{Tr}(\mathcal{C})^\sharp$. They capture some of what one may call *mixed motives over* $k[t]/(t^{m+1})$.

This category $D\mathcal{M}(k; m)$ can be regarded as a kind of "square-zero" extension of the category $D\mathcal{M}^H(k)$ of Hanamura.

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Torsion in Chow groups of codimension 2 cycles for homogeneous varieties.

Kirill Zainoulline

This is a report on the joint project with R.S. Garibaldi. Let $CH^i(X)$ denote the Chow group of codimension i algebraic cycles on a smooth projective variety X modulo the rational equivalence relation. In the present talk we provide a uniform bound for the torsion part of codimension 2 algebraic cycles on certain projective homogeneous varieties. Our main result is the following:

Theorem 1. *Let X be a variety of Borel subgroups of a strongly inner simple linear algebraic group G over a field k. Then the torsion part of the second quotient of the Grothendieck γ-filtration on X is a cyclic group of order equal to the Rost number N of G. In particular, the torsion part of $CH^2(X)$ is a cyclic group of order dividing N. The number N depends only on the type of G and is equal to*

Type:	N
$A_n,\ C_n$	1
$B_n,\ D_n,\ G_2$	2
$F_4,\ E_6$	6
E_7	*12*
E_8	*60*

We recall that G is *strongly inner* if the simply connected cover of G is isomorphic to G_s twisted by a cocycle in $H^1_{\text{ét}}(k, G_s)$, where G_s denotes the simply connected split group of the same Killing-Cartan type as G.

Note that the theorem places no restriction on the field k; it extends easily to the semisimple case and to the case when X is any generically split projective homogeneous G-variety.

All previous computations of torsion in CH^2 of projective homogeneous varieties have been dealing with quadrics [3], Severi-Brauer varieties and their products [2], groups of types B_n and D_n [1] and certain varieties of small dimensions [5]. Our theorem provides a lot of new examples, e.g. varieties of groups of exceptional types.

The proof of the theorem is constructive in the sense that we provide an explicit generator θ of that cyclic group. It turns out that for prime orders θ also generates the torsion part of the respective *generalized Rost motives* which appear in the proof of the Bloch-Kato conjecture, hence, providing a link between the combinatorics of the root system of G_s, cohomological invariants of G_s (via the Rost invariant) and Voevodsky's generalized Rost motives. Torsion parts of these motives were computed by Merkurjev-Suslin in [4] using completely different techniques. Note that θ also measures the difference between the third terms of topological and gamma filtrations on K_0. In particular, for varieties of Borel subgroups θ generates the respective quotient.

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Arakelov motivic cohomology and zeta values

Andreas Holmstrom
(joint work with Jakob Scholbach)

The aim of the talk was to present a construction of a new cohomology theory for arithmetic schemes, which we call Arakelov motivic cohomology. The motivation for constructing this cohomology theory comes from three sources.

Firstly, our main motivation is the recent insight obtained independently and in different forms by Soulé and Scholbach, that a new type of cohomology is needed as a crucial input in the study of special values of zeta functions and L-functions (see [Sou09] and [Sch10]). Some version of this cohomology theory should also play a role in the Weil-etale framework described by Flach in this volume.

Secondly, it is natural to ask if the relation between Chow groups and motivic cohomology has a counterpart in the arithmetic setting, i.e. if the arithmetic Chow groups defined by Gillet-Soulé [GS90] and Burgos [Bur97] have an extension/generalization analogous to motivic cohomology.

Thirdly, an idea which I believe goes back to Beilinson is that the Beilinson regulator from motivic cohomology to Deligne cohomology should be interpreted as a kind of boundary map in a localization sequence for the inclusion of an arithmetic scheme into its Arakelov compactification. In such a long exact sequence, motivic cohomology and Deligne cohomology would be two of three components, and Arakelov motivic cohomology would be the third.

Working with a number field as base scheme, the last two points have been realized to a large extent by the work of Burgos and Feliu [BF09] on higher arithmetic Chow groups. The main advantage of our new construction is that it also works for schemes and motives over arithmetic base schemes such as Spec $\mathbb{Z}$. This is crucial for all applications to special values. We expect to prove comparison theorems over a number field between Arakelov cohomology groups and the higher arithmetic Chow groups of Burgos and Feliu. A consequence of such a comparison should be the transfer of certain properties, including proper push-forwards, to higher arithmetic Chow groups. This is interesting because it is needed for the formulation of higher arithmetic Riemann-Roch theorems, something which we also hope to address in the future.

The main technical problem in our construction, as well as in constructions of higher arithmetic Chow groups, is to find a lift of the Beilinson regulator to some category in which one can consider its cone (or homotopy fiber). This is a difficult problem. Goncharov [Gon05] constructed such a lift to a map between certain complexes, but was not able to prove that the induced map on cohomology groups actually agrees with the Beilinson regulator (however, this was proved later by Burgos and Feliu). The thesis of Feliu [Fel07] contains a different solution to the problem, but the lift obtained is not an actual map but rather a zig-zag of maps between complexes, and this zig-zag can not be constructed over an arithmetic base scheme. The key idea in our construction is that one can work with motivic spectra instead of complexes. Thanks to the recent foundational work of Ayoub, Cisinski, Déglise and Riou ([Ayo07a], [Ayo07b], [CD07], [CD09], [Rio09]), the problem of lifting the regulator becomes much easier in this setting, and this is what we exploit to construct the Arakelov motivic cohomology groups.

We briefly summarize the main points of the construction. First we must construct a ring spectrum which represents (real) Deligne cohomology for smooth varieties over $\operatorname{Spec}\mathbb{Q}$. This can be done using a slight modification of the method used by Cisinski and Déglise for mixed Weil cohomologies [CD07], provided one uses a good choice of Deligne complexes. After proving that the resulting Deligne spectrum is orientable, one can apply a general theorem of Cisinski and Déglise [CD09, Cor. 13.2.15], which produces a map of ring spectra from Riou's Beilinson spectrum to the Deligne spectrum. (The Beilinson spectrum is constructed by Riou via Adams operations on the algebraic K-theory spectrum, and is weakly equivalent to Voevodsky's Eilenberg-MacLane spectrum with rational coefficients, see [Rio09] and [CD09, 13.1.2].) This map will induce the Beilinson regulator on the level of cohomology groups. The rough idea now is to precompose this regulator map with the canonical map from the integral coefficient Eilenberg-MacLane spectrum, and define the Arakelov motivic cohomology spectrum as the homotopy fiber of this composition. Writing $\eta : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ for the generic point of $\operatorname{Spec}\mathbb{Z}$ this produces a cofiber sequence

$$\widehat{\mathrm{H}} \to \mathrm{H}_{\mathcal{M}} \to \eta_* \mathrm{H}_{\mathcal{D}}$$

in the model category underlying $\mathbf{SH}(\operatorname{Spec}\mathbb{Z})$. Here $\widehat{\mathrm{H}}$ is the Arakelov motivic cohomology spectrum, $\mathrm{H}_{\mathcal{M}}$ is the Eilenberg-MacLane spectrum (with integral coefficients), and $\mathrm{H}_{\mathcal{D}}$ is the Deligne spectrum.

This definition of Arakelov motivic cohomology appears to be the right one for the special value conjectures formulated by Scholbach and by Soulé. However, for the Weil-etale topology framework of Flach, we will need a modified definition, probably using etale motivic cohomology, and Deligne cohomology with integral coefficients instead of real coefficients. This is something we hope to come back to in a future paper.

Many good functoriality properties of Arakelov motivic cohomology follow rather formally from Ayoub's six functors formalism. In joint work with Peter Arndt, we use the recent results of Hornbostel [Hor10] on model structures on algebras over operads to equip the Arakelov motivic cohomology groups with a product structure.

For more details on the construction and on the applications to zeta values, we refer to the preprint [HS10] and other forthcoming papers.

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Field Patching and Local-Global Principles for Galois Cohomology

Daniel Krashen
(joint work with D. Harbater and J. Hartmann)

In this joint work with D. Harbater and J. Hartmann we use the technique of field patching to obtain local-global principles for the Galois cohomology of fields of the form $F = K(X)$ where K is a complete discretely valued field and X/K is a curve. These principles take the form of injective maps

$$H^n(F, \mathbb{Z}/m(n-1)) \hookrightarrow \coprod_u H^n(F_u, \mathbb{Z}/m(n-1)) \oplus \coprod_p H^n(F_p, \mathbb{Z}/m(n-1))$$

where $n \geq 2$ and $\mathrm{char}(F) \nmid m$.

The fields F_p and F_u are as follows: Choose $\hat{X}$ a regular model of X where the closed fiber $\overline{X}$ has nodal singularities. The p's are certain closed points of $\overline{X}$ and F_p is the fraction field of $\hat{\mathcal{O}}_{\hat{X},p}$, the complete _ ring at p. The u's are the components of the complement of the p's in $\overline{X}$, and F_u is the fraction field of the completion of the ring $\mathcal{O}_{\hat{X},u}$ at its Jacobsen radical.

On Some Differentials in the Motivic Cohomology Spectral Sequence

Serge Yagunov

This talk gives an overview of work in progress mostly presented in author's preprint [Ya]. Let k be a field of characteristic 0. Let us fix some prime $p > 2$ and denote by $\mathbb{Z}_{(p)}$ the localization of the ring of integers $\mathbb{Z}$ outside the prime ideal (p). Consider the Motivic Cohomology Spectral Sequence (MCSS) (see [FS]) with coefficients in $\mathbb{Z}_{(p)}$:

$$E_2^{i,j} = H^{i-j}(X, \mathbb{Z}_{(p)}(-j)) \Rightarrow K_{-i-j}(X, \mathbb{Z}_{(p)}).$$

(Everywhere in this report H denotes the Motivic cohomology.) Differentials in this spectral sequence act as: $d_n\colon E_n^{i,j} \to E_n^{i+n,j-n+1}$.

As it was shown by Levine [Le], MCSS with rational coefficients collapses at its E_2-term. On the other hand, the case of integer coefficients is tangled, due to the interplay of its localizations at different prime numbers. Hence, the p-local case makes a good model to study the nature of the MCSS differentials.

Theorem.

$$d_n = \begin{cases} 0 \text{ for } p-1 \nmid n-1 \\ \alpha(\mathfrak{B} \circ P^1 \circ \mathrm{red}) \text{ for } n = p. \end{cases}$$

Here $\alpha \in (\mathbb{Z}/p)^\times$ is a constant, the operations red, $\mathfrak{B}$, *and P^1 are coefficient reduction, Bockstein homomorphism (see below), and first $\mathbb{Z}/p$-Steenrod power operation* [Vo2], *correspondingly.*

Remark 1. The condition char $k = 0$ is inherited from Voevodsky's computation of the Motivic Steenrod algebra. For uniformity we restrict ourselves to the case of odd primes. However, we expect similar statement to be fulfilled for the case $p = 2$.

The proof strategy:

• Using Adams operations, we show that $d_n = 0$ for $p-1 \nmid n-1$. At this stage we also obtain the equality $pd_p = 0$.

• To prove that the first potentially non-trivial differential d_p can be identified with a bi-stable cohomological operation of degree $(2p-1, p-1)$ on motivic cohomology.

• To show that operations $\alpha(\mathfrak{B} \circ P^1 \circ \mathrm{red})$ exhaust the set of all operations of finite order in the motivic Steenrod algebra with coefficients in $\mathbb{Z}_{(p)}$.

• To find an example of a smooth algebraic variety such that $d_p \neq 0$ that shows $\alpha \neq 0$.

Remark 2. Since $d_p \neq 0$, the second potentially non-trivial differential d_{2p-1} evidently can not be interpreted as a cohomological operation but rather as a secondary operation.

Let us now discuss the proof in some more details. In the first step of the proof we follow the strategy used by Victor Buchstaber to compute differentials in the

Atiyah–Hirzebruch spectral sequence [Bu]. As it was shown by Marc Levine [Le], for $k \neq p$ the Adams operations ψ_k on $K_*(X)$ have representation on the Motivic Cohomology spectral sequence. Moreover, their action on the E_2-term is given by the relation: $\psi_k(\gamma) = k^{-q}\gamma$ for $\gamma \in H^*(X, \mathbb{Z}(q))$. Therefore, all topological arguments proposed in [Bu] work in this case as well. Since Adams operations commute with differentials, we have:

$$d_n\psi_k = \psi_k d_n : H^*(X, \mathbb{Z}(q)) \to H^{*+2n-1}(X, \mathbb{Z}(q + n - 1)).$$

Therefore, $k^{1-n-q}(k^{n-1}-1)d_n = 0$. Since k is an arbitrary integer mutually prime to p (and therefore, invertible in $\mathbb{Z}_{(p)}$), the differential d_n annihilates as multiplied by the

$$\underset{\substack{k>1\\(p,k)=1}}{\text{g.c.d.}}\{k^{n-1} - 1\} \stackrel{\text{def}}{=} KM(n - 1).$$

These numbers are sometimes called Kervaire–Milnor [KM] constants, probably after Adams who calculated them in [Ad] and showed, in particular, that for $p > 2$

$$\nu_p(KM(n)) = \begin{cases} 1 + \nu_p(n) \text{ for } n \equiv 0 (\text{mod } p - 1) \\ 0 \text{ else,} \end{cases}$$

where ν_p denotes the greatest dividing p-exponent *i.e.*, for every positive integer l one has: $l = 2^{\nu_2(l)}3^{\nu_3(l)}5^{\nu_5(l)}\ldots$

Let us pass to the computation of the Motivic Steenrod algebra with $\mathbb{Z}_{(p)}$-coefficients. Denote the set of bistable cohomological operations of degree (i, j), sending motivic cohomology with coefficients in a group S to one with coefficients in some group T by $\mathcal{OP}^{i,j}(S, T)$. In particular, $\mathcal{A}^{i,j}(S) = \mathcal{OP}^{i,j}(S, S)$ is the Motivic Steenrod algebra. In the table below one can find groups of operations with different coefficients and their generators. The first row was taken from Voevodsky's computation of the Motivic Steenrod algebra with finite coefficients [Vo].

$i=$	2p-2	2p-1	2p
$\mathcal{A}^{i,p-1}(\mathbb{Z}/p)$	$\mathbb{Z}/p$ P^1	$\mathbb{Z}/p \oplus \mathbb{Z}/p$ $P^1 \circ \beta$,$\beta \circ P^1$	$\mathbb{Z}/p$ $\beta \circ P^1 \circ \beta$
$\mathcal{OP}^{i,p-1}(\mathbb{Z}/p, \mathbb{Z}_{(p)})$	0	$\mathbb{Z}/p$ $\mathfrak{B} \circ P^1$	$\mathbb{Z}/p$ $\mathfrak{B} \circ P^1 \circ \beta$
$\mathcal{OP}^{i,p-1}(\mathbb{Z}_{(p)}, \mathbb{Z}/p)$	$\mathbb{Z}/p$ $P^1 \circ \text{red}$	$\mathbb{Z}/p$ $\beta \circ P^1 \circ \text{red}$	0
$\mathcal{A}^{i,p-1}(\mathbb{Z}_{(p)})_{tors}$	0	$\mathbb{Z}/p$ $\mathfrak{B} \circ P^1 \circ \text{red}$	0

(All these groups in degrees $(i, p - 1)$ are shown to be 0 as i lies outside the considered interval.) Here red is the coefficient reduction operation from $\mathbb{Z}_{(p)}$ to $\mathbb{Z}/p$, β (resp. $\mathfrak{B}$) denotes Bockstein operation, corresponding to the coefficient short exact sequence

$$0 \to \mathbb{Z}/p \stackrel{\times p}{\to} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0 \text{ (resp. } 0 \to \mathbb{Z}_{(p)} \stackrel{\times p}{\to} \mathbb{Z}_{(p)} \to \mathbb{Z}/p \to 0).$$

We consecutively compute all groups of cohomological operations mentioned in the table. As a result we obtain an expression for the torsion part in $\mathcal{A}^{2p-1,p-1}(\mathbb{Z}_{(p)})$ and for its generator.

Finally, let me show an example required by the last part of the proof. (The author thanks Alexander Merkurjev for his help with the variety construction.) Let $\mathcal{D}$ be a central simple algebra over k of prime degree p. Denote by G the variety $SL_{1,\mathcal{D}}$. G is a twisted form of SL_p, therefore, its dimension is p^2-1.

Consider the MCSS, corresponding to the variety G with coefficients in $\mathbb{Z}_{(p)}$. We show that in this case the differential $d_p\colon E_2^{1,-2} \to E_2^{p+1,-p-1}$ is non-trivial.

The proof is based on the comparison of the MCSS with the Brown-Gersten-Quillen spectral sequence, starting from $\mathcal{K}$-cohomology and converging to the K-groups of G. We show that non-triviality of the differential in question follows from the statement $CH^{p+1}(G) \neq 0$ that is also demonstrated in [Ya].

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Rational points vs. 0-cycles of degree 1 in stable $\mathbb{A}^1$-homotopy

Aravind Asok

(joint work with Christian Häsemeyer, Fabien Morel)

Suppose k is a field, and X is a smooth variety over k. Let $\mathcal{H}(k)$ denote the $\mathbb{A}^1$-homotopy category of smooth schemes over k [MV99]; abusing notation, we write X for the isomorphism class of a smooth scheme in $\mathcal{H}(k)$. Let $\mathcal{SH}(k)$ denote the stable $\mathbb{A}^1$-homotopy category of smooth schemes over k, i.e., the category of $\mathbb{P}^1$-spectra over k [Mor05]. The suspension spectrum $\Sigma^\infty_{\mathbb{P}^1} \operatorname{Spec} k_+$, denoted $\mathbb{S}^0$ for notational convenience, is called the motivic sphere spectrum.

If U is another smooth variety, write $[U,X]_{\mathbb{A}^1}$ for the set $\hom_{\mathcal{H}(k)}(U,X)$ and write $[U,X]_{st}$ for the abelian group $\hom_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1} U_+, \Sigma^\infty_{\mathbb{P}^1} X_+)$. Define $\pi_0^{\mathbb{A}^1}(X)$ to be the Nisnevich sheaf on $\mathcal{S}m_k$ associated with the presheaf $U \mapsto [U,X]_{\mathbb{A}^1}$ and $\pi_0^s(X)$ to be the Nisnevich sheaf on $\mathcal{S}m_k$ associated with the presheaf $U \mapsto [U,X]_{st}$.

Each of these sheaves determines "by restriction" a functor on the category of finitely generated separable extensions L/k.

Stable homotopy theory and rational points. If $\pi_0^{\mathbb{A}^1}(X)(k)$ is non-empty, we say that X has a rational point up to unstable $\mathbb{A}^1$-homotopy. It is known that if X has a rational point up to unstable $\mathbb{A}^1$-homotopy, then X has a rational point [MV99]. Thus, existence of a rational point is an unstable $\mathbb{A}^1$-homotopy invariant.

Similarly, say that X has a rational point up to stable $\mathbb{A}^1$-homotopy if the canonical map $\pi_0^s(X) \to \pi_0^s(\mathbb{S}^0)$ is a split epimorphism; a choice of a splitting will be called a rational point up to stable $\mathbb{A}^1$-homotopy. Any rational point up to unstable $\mathbb{A}^1$-homotopy determines a rational point up to stable $\mathbb{A}^1$-homotopy by taking iterated $\mathbb{P}^1$-suspensions. If X is smooth and proper, there is a group homomorphism from $\pi_0^s(X)(k)$ to the group of 0-cycles of degree 1; *a priori* it is not clear that this map is either surjective or injective.

Theorem 1. *Assume k is a field having characteristic 0. If X is a smooth proper k-variety, then X has a 0-cycle of degree 1 if and only if X has a rational point up to stable $\mathbb{A}^1$-homotopy.*

Sheaves of connected components. We deduce the above result from a description of the sheaf $\pi_0^s(X)$ for any smooth proper variety. The description is motivated by foundational work of Morel describing the sheaf $\pi_0^s(\mathbb{S}^0)$ in terms of the Grothendieck-Witt ring [Mor04]. There is a "Hurewicz" functor from the stable $\mathbb{A}^1$-homotopy category to Voevodsky's derived category of motives. The analog of the stable π_0 computed in Voevodsky's derived category of motives is the 0-th Suslin homology sheaf. For a smooth proper variety X, the sections of this sheaf over fields coincide with the Chow group of 0-cycles on X_L (*cf.* [Dég08, §3.4]).

We use the theory of oriented Chow groups, or Chow-Witt groups, as invented by J. Barge and F. Morel [BM00], and developed in detail by J. Fasel [Fas08, Fas07]. For any n-dimensional smooth proper k-scheme X, one can define the oriented Chow group $\widetilde{CH}_0(X)$ by means of a certain "oriented Chow cohomology group" $\widetilde{CH}^n(X, \omega_X)$ (see [Fas08, Definition 10.2.17] for details). This latter group is defined by means of an explicit Gersten resolution, and has functorial pushforwards for proper morphisms.

Theorem 2. *If X is a smooth proper k-variety over a field k having characteristic 0, then there is an isomorphism (natural with respect to X) between the functor $L \mapsto \pi_0^s(X)(L)$ and the functor $L \mapsto \widetilde{CH}_0(X_L)$.*

Sketch of proof of Theorem 2. One first reduces to the case where X is projective, and deals with an associated "abelianized" problem using a version of $\mathbb{A}^1$-homology that has been stabilized with respect to $\mathbb{G}_m$. When X is projective, the idea of the proof is to use Spanier-Whitehead duality: the Spanier-Whitehead dual of a smooth scheme X is the Thom space of the negative tangent bundle (see, e.g., [Hu05, Theorem A.1] or [Rio05, Théorème 2.2]).

When X has trivial tangent bundle, one can prove the result by proving a $\mathbb{P}^1$-bundle formula for the oriented Chow group of 0-cycles—this involves some facts about contractions of the sheaf $\mathbf{K}_n^{MW}$ as discussed at the end of [Mor06, §2.3]. In the general case, one has to show that the twist arising from non-triviality of the negative tangent bundle only appears through the canonical bundle ω_X of X. Locally the tangent bundle is trivial, and a careful patching argument (using the fact that any element of GL_n is $\mathbb{A}^1$-homotopic to its determinant) can be used to finish the proof; this involves an "unstable" construction of the map inducing duality as given by Voevodsky in [Voe03]. □

Sketch of proof of Theorem 1. The "only if" direction is straightforward. For the "if" direction, it suffices to show that the "forgetful" morphism $\widetilde{CH}_0(X_L) \to CH_0(X_L)$—functorial in L and X—is always a surjection. For any field F, the canonical map $GW(F) \to \mathbb{Z}$ given by the rank homomorphism is always surjective. Each of these groups is computed by means of a Gersten resolution. One then just uses the fact that X has Nisnevich cohomological dimension n. □

Remark 3. In fact, we prove a more precise result. The sheaf $\pi_0^s(X)$ is a strictly $\mathbb{A}^1$-invariant sheaf of groups by [Mor05, Theorem 6.2.7] and therefore "unramified" in an appropriate sense; one can then describe the sections of the sheaf $\pi_0^s(X)$ over a smooth scheme U in terms of sections over $k(U)$ together with information coming from discrete valuations associated with codimension 1 points of U.

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