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Calculus of Variations

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Abstract. Since its invention by Newton, the calculus of variations has formed one of the central techniques for studying problems in geometry, physics, and partial differential equations. This trend continues even today. On the one hand, slow but steady progress is made on long-standing questions concerning minimal surfaces, curvature flows, and related geometric objects. Basic questions also remain in such areas as mathematical physics and general relativity. On the other hand, new types of question emerge, driven by applications from economics and engineering to materials science, whose solution will depend on developing ideas and techniques in this classical branch of analysis. The July 2010 Oberwolfach workshop on the Calculus of Variations showcased a blend of continued progress in traditional areas with surprising developments which emerged from the exploration of new lines of research.

Mathematics Subject Classification (2000): 49-xx, 35Jxx, 53Cxx, 58Exx.

Introduction by the Organisers

This workshop attracted 49 participants, including 13 recent PhDs and 3 women. Its main themes could be divided into four large groups (i) geometry (ii) partial differential equations; (iii) physics and materials; (iv) optimal transportation and its applications. Doctoral students and postdoctoral fellows accounted for nearly a third of the 21 presentations which took place 19-23 July 2010.

The first general area encompassed the role of calculus of variations in differential geometry, including minimal surface theory and general relativity. Some of the most exciting developments here concern rigidity questions in Riemannian geometry described by Simon Brendle and Andre Neves. Brendle's lecture was devoted to the construction of a counterexample to a conjecture of Min-Oo. For a manifold which is asymptotically Euclidean and has non-negative scalar curvature, the positive mass theorem asserts that the ADM mass be non-negative, vanishing only in the case of Euclidean space. Min-Oo established a similar result in the asymptotically hyperbolic setting — namely, that no compact perturbation exists whose only effect is to increase the positive scalar curvature locally. He conjectured the same would be true in the positive curvature setting of the hemisphere, a conjecture disproved by Brendle's example with Marques and Neves. Neves, on the other hand, devoted his talk to positive results, including sharp bounds on the area of a minimizing surface in a compact oriented 3-manifold whose scalar curvature exceeds that of the sphere; the case of equality is attained by products of spheres.

Curvature-driven flows were addressed in talks by Peter Topping, John Head and Brian White. Peter Topping explained conditions for there to be a unique Ricci flow which instantaneously completes an incomplete surface. The talks of Head and White concerned flows of embedded submanifolds by mean curvature. Here a program by Huisken and Sinestari has succeeded in classifying singularities of the flow; as in the Ricci flow case, such singularities can be bypassed using surgery. John Head described doctoral work showing that in the limit, the flow obtained by postponing the surgeries for as long as possible coincides with the one arising from the viscosity solution of the level-set formulation of mean-curvature flow. On the other hand, Brian White explained how he and Tom Ilmanen have exploited the classification of singularities for mean-curvature flow to resolve a classical problem in geometric measure theory. Under a mild topological assumption, they were able to show that the density of an area-minimizing hypersurface exceeds the square root of two at each singularity. Among dimension independent bounds, this result is sharp. Emanuel Spadaro described his doctoral work with Camillo DeLellis, which focused on simplifying Almgren's thousand page proof of regularity results for minimal varieties of codimension two and higher. Finally, Mu-Tao Wang described his definition with S.T. Yau of the quasi-local mass (or total energy) bounded by a closed spacelike surface in general relativity. Their approach, which involves extremizing over isometric embeddings of the geometry into a Lorentzian spacetime, is reminiscent of Gromov's definition of the Hausdorff distance between two abstract metric spaces.

Turning to variational problems in physics and materials science, we may mention the review of Felix Otto, devoted to establishing ansatz-free upper bounds on nonlinear rates of coarsening in dynamical settings, and on branching formations in static patterns. Here interpolation inequalities between function spaces sensitive to competing energies in the physical system play crucially. Robert Seiringer described classical and quantum mechanical models for electrons moving through a dielectric medium, and explained how screening effects must be taken into account when analyzing the ground state energy of the system, to preclude the possibility of binding. Nicola Fusco discussed the variational problems governing the equilibrium configurations of an epitaxially strained crystalline film.

Exciting developments were also reported in the theory of partial differential equations which arise as Euler-Lagrange equations for variational problems. One of the highlights was Neil Trudinger's description of affine maximal hypersurfaces. This geometric problem dates back to Chern and Calabi, and involves proving regularity for 4th order analogs of the elliptic Monge-Ampère equation. For two-dimensional surfaces, the problem was solved some fifteen years ago by Trudinger and Wang. In recent work, they have succeeded in extending their result to all dimensions. Giuseppe Mingione described new techniques for showing boundary regularity of solutions to minimization problems below the level at which the Euler-Lagrange equation becomes effective; the key technical problem is that coefficients in this equation depend on the solution, hence need not be smooth, a priori. Alessio Figalli described regularity results with Luis Caffarelli for an obstacle problem from mathematical finance involving the fractional Laplacian, while Daniel Faraco described the lack of uniqueness for solutions of the incompressible porous medium equation, following similar results in fluid mechanics dating back to Shnirelman. Yet another highlight was new PhD Charles Smart's lecture on optimal Lipschitz extensions. Here he described the differentiability proved with Lawrence C Evans for the viscosity solution of the infinity-Laplace equation, and his simplification with Armstrong of Jensen's argument for its uniqueness. He concluded by describing preliminary results concerning the vector-valued analog of this problem, which is to construct a mapping whose Lipschitz constant is the minimum possible (relative to its boundary conditions) on every subdomain of a given domain.

Turning to questions in optimal transportation, Brendan Pass described doctoral research on the multiple marginal problem of optimally correlating $m \geq 3$ distributions in several dimensions with respect to a given cost function. He described existence, uniqueness, and rectifiability results, some of which were new even for two marginals. Most striking among these is difference between the dimension of the maximizer and the minimizer when $m \geq 3$, and the fact that the solution is a spacelike manifold with respect to 2^{m-1} pseudo-metrics. Young-Heon Kim described progress with Figalli and McCann concerning regularity of optimal maps on a Riemannian manifold in the two marginal case. They have overcome the subtleties associated with the cut-locus on products of round spheres, which provide a reasonably robust model for the singularities displayed by more general superdifferentiable costs. Alexander Plakhov described problems of minimizing aerodynamic resistance, in which optimal transportation plays a role. His delicate constructions establish the surprising ability to lower the resistance to zero in certain (unstable) directions. To do so requires recapturing lost momentum through multiple scattering. Finally, Guillaume Carlier described an economic model for optimal transportation with congestion.

Apart from the lectures classified above, there were several which defy categorization, such as Paul Lee's results on bracket conditions which guarantee the continuity of sub-Riemannian actions in a control-theoretic context, and Bob Jerrard's talk on Lorentzian analogues of variational questions modeling the limiting geometry of singularities arising from phase-field models in a singular limit. In addition to formal lectures, many lively discussions between new and seasoned researchers took place throughout the week, affirming the vitality of this flourishing subject. We hope the collection of extended abstracts supplied by the speakers below helps to convey a sense of the excitement and possibilities shared by the participants and researchers working at the scientific frontier in the calculus of variations.

Workshop: Calculus of Variations

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Abstracts

Branching, coarsening, and interpolationestimates

Felix Otto

In many mesoscopic model of phase transitions, the order parameter has a finite number of preferred values and thus displays domains, i. e. regions where it is nearly constant, and which are of a characteristic width w, separated by comparatively sharp transition layers. We now give three examples, of different physical background, where in equilibrium configurations the average width w of these domains decreases substantially towards an edge of the sample — a decrease that is mediated by **branching** of the domains:

1) Type-I superconductors under an external magnetic field. Here the order parameter is a scalar, namely the density of superconducting electrons, and branching has been analyzed by Landau.
2) Strongly uniaxial ferromagnets, where the order parameter is the magnetization, a vector field. This phenomenon has been first analyzed by Hubert and Privorotskii.
3) Shape memory alloys, where Martensitic twins branch towards (not a sample surface but) an interface with the Austenite phase. Here the order parameter is the lattice distortion (w. r. t. the high-symmetry Austenite phase) and thus a tensor. This phenomenon has first been (rigorously) analyzed by Kohn & Müller.

The challenge for mathematical analysis in these variational models consists in proving that the minimizers display branching. This is translated into a question on the scaling of the minimal energy. The task is then to establish lower bounds that scale with the system volume (and have optimal scaling in the nondimensional parameters). This is sometimes called "Ansatz-free lower bounds" in contrast to the upper bounds that come from a specific, physically motivated Ansatz.

We now turn to time-dependent mesoscopic models for phase transitions, like the Cahn-Hilliard equation that describes the demixing of an initially spatially homogeneous two-component mixture after a sudden reduction of temperature ("quench"). After an initial stage, the scalar order parameter (i. e. the relative concentration) forms domains. The average width $w(t)$ of these domains increases over time — a phenomenon called **coarsening**. The analogy is obvious: In branching, the domain width w increases with the distance z to the sample edge; in coarsening, w increases with the time t. Matched asymptotic analysis allows to relate the late stages of coarsening to a curvature-driven geometric motion of the (ideally sharp) interface. This geometric motion typically has a scale invariance that heuristically predicts the exponent α in $w \sim t^{\alpha}$. These heuristics lead to the numerically and experimentally confirmed coarsening laws in several examples:

1) $w \sim t^{1/3}$ for a shallow quench (where the geometric motion is Mullins-Sekerka);

2) $w \sim t^{1/4}$ for a deep quench (surface diffusion);
3) $w \sim t$ if the coarsening is mediated by an underlying flow that is limited by viscous dissipation (Siggia's growth);
4) and $w \sim t^{1/2}$ for grain growth (flow by mean curvature).

Kohn and Otto developed an approach that yields upper bounds on w that are optimal in t-scaling (and hold for all system sizes). It applies to examples 1)-3), but not to 4). The approach relies on the gradient flow structure of these evolutions (which separates the driving energetics from the limiting dissipation mechanism). It converts an estimate on how fast the energy decreases as a function of distance to the well-mixed state into an estimate on how fast it decreases as a function of time. This requires a good understanding of the distance on configuration space; as in classical differential geometry it is the distance "in the large" induced by the infinitesimal distance (i. e. the metric tensor) defined through the dissipation mechanism. In example 1), the geometry comes from an ambient Euclidean one. In 2) and 3) the induced distance can be estimated from below by a Wasserstein metric (with linear cost function in 2) and logarithmic cost function in 3)). In 4), the induced distance is trivial — leading to a failure of this approach.

Both the Ansatz-free lower bounds in branching and the estimates on the slope of the energy landscape lead to **interpolation estimates**. Examples 1) and 3) in branching, and example 1) in coarsening can be tackled by the *same* Gagliardo-Nirenberg estimate

$$\|u\|_{L_{4/3}} \lesssim \|u\|_{\dot{H}^1_1}^{1/2} \|u\|_{\dot{H}^{-1}_2}^{1/2}. \tag{1}$$

Because of the L_1-type norm on the r. h. s. , (1) was only recently established by Cohen & Dahmen & Daubechies & DeVore by wavelet methods. Note that (1) is scale invariant in any dimension — and, crucially for our application, it scales with the system volume (we think of periodic boundary conditions) as can be seen from rewriting it as $\int |u|^{4/3} dx \lesssim \left(\int |\nabla u| dx\right)^{2/3} \left(\int |j|^2 dx\right)^{1/3}$ for flux j with $\nabla \cdot j = u$. The relevance of (1) is clearest in case of example 1) in coarsening, where the late stages are described by the Mullins-Sekerka interfacial motion. The latter is the gradient flow of the perimeter (this brings in $\int |\nabla u|$) with respect to the H^{-1}-inner product (this brings in $\|u\|_{\dot{H}^{-1}_2}$ as induced distance), restricted to characteristic functions $u \in \{-1, 1\}$ (this is leveraged by $\|u\|_{L_{4/3}}$).

The interpolation estimates needed for superconductors (i. e. the branching example 2)) are more subtle because a) the superconducting and the normal phases are not symmetric, since only the normal phase carries the magnetic flux imposed by the external field (Meißner's effect) b) next to the strength Φ of the external field, the model contains an additional dimensionless parameter $\nu \ll 1$. It turns out that there are *two* scaling regimes for $\Phi \ll 1$. For $\nu^{6/7} \ll \Phi$
1, the relevant interpolation estimate reads: For all $u(x) \geq 0$

$$\|(u-2)_+\|_{w-L_{4/3}} \lesssim \|u\|_{\dot{H}^1_1}^{1/2} W(u,1)^{1/2}, \tag{2}$$

where $W(u,1)$ denotes the Wasserstein distance (with quadratic cost functional) between u and the uniform density. Compared with (1), the H^{-1}-norm has been replaced by the Wasserstein distance, which reflects the fact that the flux j is only supported in the normal phase and acts as a velocity. The exponent 4/3 is a coincidence of dimension $d=2$ (the physical dimension for the cross-section). It is not clear whether the weak space $w-L_{4/3}$ can be sharpened to $L_{4/3}$.

In the regime $\Phi \ll \nu^{6/7}$, the relevant interpolation estimate (again for dimension $d=2$) is the following:

$$\|u\|_{w-L_{9/7}} \lesssim \|u\|_{\dot{H}_1^1}^{4/9} \left(\sup_{\nu} \inf_{v(x)\geq 0} \left(\nu^{2/3} W^2(u,v) + \nu^{-1/3} \|v\|^2_{\dot{H}_2^{-1/2}} \right)^{3/5} \right)^{5/9}.$$

In comparison with (2), the Wasserstein distance $W(u,1)$ has been replaced by the 1-homogeneous expression $\sup_{\nu} \inf_{v(x)\geq 0} \left(\nu^{2/3} W^2(u,v) + \nu^{-1/3} \|v\|^2_{\dot{H}_2^{-1/2}} \right)^{3/5}$ which interpolates between the Wasserstein distance and the $H^{-1/2}$-norm. This reflects the fact that in this regime, the magnetic flux $v(x)$ is not uniform at the sample edge, so that the field energy outside of the sample has to be accounted for by $\nu^{-1}\|v\|^2_{\dot{H}_2^{-1/2}}$. This interpolation estimate highlights the nature of the problem: A given magnetic flux quantum has to be transported by an optimal strategy that consists of flux tubes in the sample which branch towards the sample edge and thus spread the flux somewhat, and the flux spreading freely outside of the sample.

This is an account of joint work with R. V. Kohn, R. Choksi, S. Conti, B. Niethammer, S. Serfaty, Y. Brenier, T. Viehmann, and C. Seis [1].

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Counterexample to Min-Oo's conjecture

Simon Brendle

Consider a compact Riemannian manifold M of dimension n whose boundary ∂M is totally geodesic and is isometric to the standard sphere S^{n-1}. A natural conjecture of Min-Oo asserts that if the scalar curvature of M is at least $n(n-1)$, then M is isometric to the hemisphere S^n_+ equipped with its standard metric. This conjecture is inspired by the positive mass theorem in general relativity, and has been verified in many special cases (see e.g. [1], [3], [5], [6]). I will present joint work with F.C. Marques and A. Neves which shows that Min-Oo's conjecture fails in dimension $n \geq 3$. The details will appear in [4].

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Absence of Binding in Pekar's Polaron Model

Robert Seiringer

The binding of polarons, or its absence, is an old and subtle topic. In this talk we consider the Pekar model for polarons. For $\psi \in L^2(\mathbb{R}^{3N})$ with $\|\psi\|_2 = 1$, it is given by

$$\mathcal{E}_N^{\alpha,U}(\psi) = \sum_{i=1}^N \int_{\mathbb{R}^{3N}} |\nabla_i \psi|^2 + \sum_{i<j} \int_{\mathbb{R}^{3N}} \frac{|\psi|^2}{|x_i - x_j|} - \frac{\alpha}{2} \int_{\mathbb{R}^6} \frac{\rho_\psi(x)\rho_\psi(y)}{|x-y|}$$

where

$$\rho_\psi(x) = \sum_{i=1}^N \int_{\mathbb{R}^{3(N-1)}} |\psi(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_N)|^2$$

denotes the particle density, $U > 0$ is the electronic Coulomb repulsion and $\alpha > 0$ is the polaron coupling constant.

We give a proof of two things. First, the transition from many-body collapse to the existence of a thermodynamic limit for N polarons occurs precisely at $U = \alpha$. I.e., for $U > \alpha$,

$$E_N(\alpha, U) = \inf_\psi \mathcal{E}_N^{\alpha,U}(\psi) \geq -CN$$

for some N-independent constant C. For $U < \alpha$, it is easy to see that such a bound fails, and $E_N \sim -N^3$ instead.

Second, if U is large enough, there is no multi-polaron binding of any kind. More precisely, there exists a constant $\widetilde{C} > 0$ such that $U \geq \widetilde{C}\alpha$ implies that

$$E_N(\alpha, U) = NE_1(\alpha)$$

for all $N \geq 2$. Considering the known fact that there is binding for some $U > \alpha$, these conclusions are not obvious and their proof has been an open problem for some time.

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Boundary regularity in variational problems

Giuseppe R. Mingione

0.1. **Problems.** We will describe a few results obtained together with Jan Kristensen (Oxford) in [7]. The matter concerns the boundary regularity of solutions $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ to Dirichlet variational problems of the type

$$\min_{v\in W^{1,p}(\Omega)} \int_\Omega F(x,v,Dv)\,dx\,, \qquad v \equiv u_0 \quad \text{on} \quad \partial\Omega \tag{1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, $N > 1$ and $n \geq 3$ and the boundary datum u_0 are assumed to be suitably smooth: Ω is a $C^{1,\alpha}$ domain and $u_0 \in C^{1,\alpha}(\Omega,\mathbb{R}^N)$ for some $\alpha \in (0,1]$. The restriction $n \geq 3$ comes from the fact that in the two dimensional case $n = 2$ specific techniques apply, allowing to prove everywhere boundary regularity of minima under reasonable assumptions. For this we refer to [1] and related bibliography. The interior regularity available for minima prescribes that solutions to 1 are *partially regular* i.e. they are of class $C^{1,\alpha/2}$ outside a negligible closed subset and singularities appear [10, 11]. It is then natural to try to extend the almost everywhere regularity up to the boundary. The only results available prior to [7] were due to Jost & Meier [5], who proved the everywhere boundary Hölder continuity of minima in the special case $F(x,v,Dv) = a(x,v)|Dv|^2$, a result later generalized in [2] by considering the degenerate version $F(x,v,Dv) = a(x,v)|Dv|^p$. Such results strongly rely on the special structure of the functionals considered i.e. the dependence on Dv via $|Dv|$, which is known to allow for everywhere regularity since the work of Uhlenbeck [12]. In the general case 1 *even the existence of one regular boundary point was an open problem*, while on the other hand singularities are known to appear at the boundary no matter the smoothness of u_0 and $\partial\Omega$ [4]. In [7] we give the first boundary regularity results valid for classes of general functionals as in 1, proving that almost every boundary point, with respect to the natural surface measure, is regular. Here a boundary point $x_0 \in \partial\Omega$ is called regular iff there exists a ball $B(x_0,R)$ such that Du is Hölder continuous in the closure $\overline{\Omega \cap B(x_0,R)}$. One of the main difficulties is that under the assumptions we shall consider the functional in 1 does not possess the Euler-Lagrange system, therefore the available boundary theory regularity theory for solutions to general elliptic systems [3] does's help. Moreover, the available interior singular sets estimates for minima of variational integrals [6] are not sufficient when carried up to the boundary.

0.2. **Results.** The main idea of the proof is to observe that the low regularity of $(x, y) \mapsto F(x, y, \cdot)$ is a potential source of singularities [8]. Therefore we consider integrands $F(x, y, \cdot)$ enjoining different regularity conditions with respect to the coefficients (x, u); in particular, higher regularity will be imposed to $y \mapsto F(\cdot, y, \cdot)$ to avoid that the dependence $x \mapsto F(x, u(x), \cdot)$ looks too rough when $u(x)$ is considered as a coefficient. The precise assumptions are therefore

(2)
$$\begin{cases} \nu|z|^p \leq F(x,y,z) \leq L(1+|z|^2)^{\frac{p}{2}} \\ \nu(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle F_{zz}(x,y,z)\lambda,\lambda\rangle \leq L(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \\ |F(x_1,y_1,z)-F(x_2,y_2,z)| \leq L\left[\omega_\alpha(|x_1-x_2|)+\omega_\beta(|y_1-y_2|)\right](1+|z|^2)^{\frac{p}{2}} \\ |F_z(x_1,y_1,z)-F_z(x_2,y_2,z)| \leq L\omega_\alpha(|x_1-x_2|+|y_1-y_2|)(1+|z|^2)^{\frac{p-1}{2}}, \end{cases}$$

to be satisfied for all x, x_1, $x_2 \in \Omega$, y, y_1, $y_2 \in \mathbb{R}^N$ and z, $\lambda \in \mathbb{R}^{nN}$, where $p \geq 2$, $0 < \nu \leq L$, where $\omega_\alpha(s) := \min\{s^\alpha, 1\}$ and $\omega_\beta(s) := \min\{s^\beta, 1\}$. $\omega_\alpha(s) := \min\{s^\alpha, 1\}$ and $\omega_\beta(s) := \min\{s^\beta, 1\}$. The first two lines in 2 describe the ellipticity and growth properties of the functional in question, while the following two serve to specify the regularity assumed with respect to coefficients (x, y). We will start with a low dimensional case, namely when $p \leq n+2$, that is, when a remote effect of certain Caccioppoli estimates combined with Sobolev embedding theorem, it is possible to provide an additional integral control on the oscillations of u. A sample result in this case is the following:

Theorem 1. *Under the assumptions 2 with* $n \leq p + 2$, $\alpha > 1/2$ *and* $\beta > \max\{1 - 2/n, 2/3\}$, *let* $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ *be a solution to the Dirichlet problem 1. Then* $\mathcal{H}^{n-1}$*-almost every boundary point is regular for* u.

When considering the case $n > p + 2$ we compensate the lack of additional regularity with some structure properties. The model case has splitting structure:

$$\int_\Omega c(x)f(Dv) + h(x, v, Dv)\,dx \tag{3}$$

with $0 \leq h(x, v, Dv) \leq L(1 + |Dv|^\gamma)$ with $\gamma < p$. When considering the general case we then add an additional reduced growth condition of the type

$$|F_z(x, y_1, z) - F_z(x, y_2, z)| \leq L\omega_\beta(|y_1 - y_2|)(1 + |z|^2)^{\frac{\gamma-1}{2}}, \quad y_1, y_2 \in \mathbb{R}^N \tag{4}$$

Theorem 2. *Under the assumptions 2 and 4 with* $\gamma < p$, *take the parameter* $2/3 \leq s \leq p/(p-1)$ *and assume that* $\alpha > 1/2$, $\beta > s$ *and* $\gamma \leq ps + 2ps/(n-2)$. *Let* $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$ *be a solution to the Dirichlet problem 1. Then* $\mathcal{H}^{n-1}$*-almost every boundary point is regular for* u.

The parameter s "tunes" the structure properties of the functional: the larger we allow γ to be i.e. the more growth we assume on Dv, the more regularity we assume on the respective coefficient. Moreover, when taking model examples as in 4, we are able to relax the Hölder continuity of $c(x)$ in fractional differentiability, allowing for rough coefficients. We refer anyway to [7] for more results.

0.3. **Techniques.** The techniques employed in [7] make use of several regularity methods and ingredients to achieve certain up-to-the-boundary fractional differentiability properties of Du. This implies in turn singular sets estimates [6]. The general strategy might be resumed as follows:

- Step 1: Morrey regularity up to the boundary on reduced subsets of small Hausdorff codimension
- Step 2: Sharp form of certain Caccioppoli type inequalities to include also rough coefficients
- Step 3: variational nonlinear Calderón-Zygmund theory to establish suitable up-to-the-boundary higher integrability of Du of reduced subsets of small codimension
- Step 4: Du belongs to a suitable fractional Sobolev space; this goes via
- Step 4.1: Nonlinear atomic decomposition of Besov spaces replacing atoms based on harmonic functions by atoms made based on solutions to nonlinear systems, and using related low regularity properties of the "kernel"
- Step 4.2: Use of higher integrability of the gradient to improve the decomposition
- Step 4.3: Iteration of Step 4.2 to reach maximal exponents and bounds

For further results about singular sets and boundary singularities we refer to the recent survey papers [8, 9].

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Optimal transport with congestion, weak flows and degenerate elliptic equations

Guilaume Carlier

This talk is based on a recent joint work with Lorenzo Brasco and Filippo Santambrogio [2]. Following [3], we consider a continuous model of congested transport, where the unknown is some probability measure Q on a set of curves in a given domain domain Ω (a city, say) which captures the overall transport pattern in the city. We are also given two probabilty measures μ_0 and μ_1 that capture respectively the distribution of residents and services in the city, so that Q should satisfy the obvious mass conservation conditions

$$e_{0\#}Q = \mu_0, \; e_{1\#}Q = \mu_1 \tag{1}$$

where e_t denotes the evaluation map at time $t \in [0,1]$. The measure Q (concentrated on the set C of absolutely curves from $[0,1]$ to $\overline{\Omega}$) induces an intensity of traffic $i_Q \in \mathcal{M}(\overline{\Omega})$, defined by

$$\int \varphi di_Q := \int_{C([0,1],\overline{\Omega})} \Big(\int_0^1 \varphi(\gamma(t))|\dot\gamma(t)|dt \Big) dQ(\gamma)$$

for all $\varphi \in C(\overline{\Omega}, \mathbb{R})$. The congestion effect is then captured through a metric:

$$\xi_Q(x) := g(i_Q(x)), \text{ for } i_Q \ll \mathcal{L}^d \;(+\infty \text{ otherwise}).$$

where g is a given increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$. Denoting by $\mathcal{Q}(\mu_0, \mu_1)$ the set of probability measures on C that satify the mass conservation constraint (1), roughly speaking, an equilibrium is a $Q \in \mathcal{Q}(\mu_0, \mu_1)$ that is supported on geodesics for the conformal Riemannian metric ξ_Q. This generalizes a well-known concept of equilibrium due to Wardrop in the discrete network setting [5]. We then relate equilibria to the variational problem

$$\inf_{Q \in \mathcal{Q}(\mu_0,\mu_1)} \int_\Omega H(i_Q(x))dx \tag{2}$$

where $H' = g$, $H(0) = 0$. Under mild assumptions this proves existence of equilibria and gives a variational characterization.

We then look for a more explicit construction of equilibria by a flow formulation. For $Q \in \mathcal{Q}(\mu_0, \mu_1)$, let us define the vector-measure σ_Q by : $\forall X \in C(\overline{\Omega}, \mathbb{R}^d)$:

$$\int_{\overline{\Omega}} X(x) d\sigma_Q(x) = \int_{C([0,1],\overline{\Omega})} \left(\int_0^1 X(\gamma(t)) \cdot \dot\gamma(t) dt \right) dQ(\gamma)$$

which is a kind of vectorial traffic intensity. It is easy to check that :

$$\operatorname{div}(\sigma_Q) = \mu_0 - \mu_1, \; \sigma_Q \cdot n = 0, \text{ and } |\sigma_Q| \le i_Q.$$

Since H is increasing, it proves that the value of the scalar problem (2) is larger than that of the minimal flow problem (setting : $\mathcal{H}(\sigma) = H(|\sigma|)$):

$$\inf_{\operatorname{div}(\sigma)=\mu_0-\mu_1} \int_\Omega \mathcal{H}(\sigma(x))dx. \tag{3}$$

Conversely, if σ is a minimizer of (3) and $Q \in \mathcal{Q}(\mu_0, \mu_1)$ is such that $i_Q = |\sigma|$ then Q solves the scalar problem (2) (i.e. is an equilibrium). If σ is Lipschitz (and μ_0 and μ_1 have Lipschitz densities bounded away from zero) such a measure Q is easy to construct by Moser's deformation argument:

$$Q := \delta_{X(t,.)} \otimes \mu_0$$

where X denotes the flow of the vector field $(t, x) \mapsto \sigma(x)/((1-t)\mu_0(x) + t\mu_1(x))$. Without regularity, it is still possible to relate the two problems thanks to the superposition principle of Ambrosio and Crippa [1].

The solution of (3) is $\sigma = \nabla \mathcal{H}^*(\nabla u)$ where $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$ and u solves the PDE:

$$\begin{cases} \operatorname{div}\nabla \mathcal{H}^*(\nabla u) &= \mu_0 - \mu_1, & \text{in } \Omega, \\ \nabla \mathcal{H}^*(\nabla u) \cdot \nu &= 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Let us recall that $H' = g$ where g is the congestion function, it is therefore natural to have $g(0) > 0$: the metric is positive even if there is no traffic, so that the radial function $\mathcal{H}$ is not differentiable at 0 which implies $\nabla \mathcal{H}^* = 0$ on a ball which makes (4) very degenerate. A reasonable model of congestion is $g(t) = 1 + t^{p-1}$ for $t \geq 0$, with $p > 1$, in which case

$$\sigma = \Big(|\nabla u| - 1\Big)_+^{q-1} \frac{\nabla u}{|\nabla u|}, \; q = \frac{p}{p-1}$$

where u solves the very degenerate PDE:

$$\operatorname{div}\Big(\Big(|\nabla u| - 1\Big)_+^{q-1} \frac{\nabla u}{|\nabla u|}\Big) = \mu_0 - \mu_1, \;\; \text{in } \Omega, \tag{5}$$

with Neumann boundary condition

$$\Big(|\nabla u| - 1\Big)_+^{q-1} \frac{\nabla u}{|\nabla u|} \cdot \nu = 0, \; \text{on } \partial\Omega.$$

For $q \geq 2$, we prove that u is globally Lipschitz and σ has some Sobolev regularity which enables us to define a flow à la DiPerna-Lions. Finally, we discuss a dual formulation that consists in finding a metric minimizing a functional that depends on the corresponding geodesic distance, we also present some numerical simulations based on this formulation.

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A variational problem from quasilocal mass

Mu-Tao Wang

This talk is based on joint work with S.-T. Yau [1], [2], [3] and P. Cheng and S.-T. Yau [4]. Quasilocal mass is a notion in general relativity that is associated with a closed spacelike 2-surface Σ in spacetime. We recently discovered a new prescription for quasilocal mass that satisfy essential requirements for a valid definition. The new definition is closely tied to the rigidity problem of isometric embeddings of surfaces and a variational problem naturally arises from minimizing the quasilocal energy.

For a closed spacelike 2-surface Σ in spacetime M, we consider a "quasilocal observer" (X, T_0) where $X : \Sigma \to \mathbb{R}^{3,1}$ is an isometric embedding of (Σ, σ) with the induced metric from M and $T_0 \in \mathbb{R}^{3,1}$ is a constant future timelike vector. To each (X, T_0), we attached a quasilocal energy $E(X, T_0)$ which corresponds to the energy seen by the quasilocal observer (X, T_0). The *quasilocal mass* of Σ is then obtained by minimizing $E(X, T_0)$ among all "admissible" observers:

$$\min_{(X,T_0)} E(X, T_0).$$

$E(X, T_0)$ is defined to be difference between the physical surface Hamiltonian in M and the reference surface Hamiltonian in $\mathbb{R}^{3,1}$. On the physical side, we assume the mean curvature vector H of Σ in M is spacelike. Thus, we can find a future timelike normal vector field that is orthogonal to H. These directions together define a connection one-form α_H of the normal bundle of Σ. The physical data only depends on $(\sigma, |H|, \alpha_H)$.

On the reference side in $\mathbb{R}^{3,1}$, first we take $\tau = -\langle X, T_0\rangle$ be the quasilocal observer's time function. The projection of $X(\Sigma)$ onto the orthogonal complement of T_0 in $\mathbb{R}^{3,1}$ is an embedded surface $\hat{\Sigma}$ in $\mathbb{R}^3$. We restrict ourself to convex quasilocal observers, i.e. (X, T_0) such that $\hat{\Sigma}$ is a convex surface in $\mathbb{R}^3$. The reference surface Hamiltonian is then $-\frac{1}{8\pi}\int_{\hat{\Sigma}} \hat{H}$, or a constant multiple of the total mean curvature of $\hat{\Sigma}$. Finally, we define

$$E(X, T_0) = \frac{1}{8\pi}\int_{\hat{\Sigma}} \hat{H} - \frac{1}{8\pi}\int_{\Sigma}[|H|\sqrt{1+|\nabla\tau|^2}\cosh\theta + \theta\Delta\tau - \alpha_H(\nabla\tau)],$$

where $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$, and ∇ and Δ are the gradient and Laplace operator with respect to σ.

In fact, $\int_{\hat{\Sigma}} \hat{H}$ can be written exactly as the second integral by replacing H by H_0, the mean curvature vector of $X(\Sigma)$ in $\mathbb{R}^{3,1}$.

Given physical data $(\sigma, |H|, div_\Sigma \alpha_H)$ for Σ in M, the Euler Lagrange equation for quasilocal mass seeks for isometric embeddings (X, T_0) that satisfy

(1)

$$-\Delta\theta + div_\Sigma\left(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H|\right) - (\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} = div_\Sigma\alpha_H$$

and

$$\langle dX, dX\rangle = \sigma,$$

with $\tau = -\langle X, T_0\rangle$. The equation should be read in the following way. Take a function τ on Σ, consider $\hat{\sigma} = \sigma + (d\tau)^2$ on Σ. Isometrically embed $(\Sigma, \hat{\sigma})$ into $\mathbb{R}^3$. Pick up $\hat{h}_{ab}$ and $\hat{H}$ from this isometric embedding and we look for τ that satisfies (1), a fourth-order elliptic equation.

We proved that $m(\Sigma)$ has positivity and rigidity properties and approaches the correct limits for large spheres. In the talk, I discuss how to solve the Euler-Lagrangian equation (1) and show that the solution is locally energy-minimizing at spatial infinity of an asymptotically flat spacetime.

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The multi-marginal optimal transportation problem

Brendan Pass

Optimal transportation is an active and exciting area of research; for background and an extensive list of references, see the books by Villani [3, 4]. However, most of the progress made in this field to date has been restricted to problems with two marginals; problems with three or more marginals have thus far received relatively little attention. This abstract briefly summarizes recent progress made by the author on these multi-marginal problems; a more detailed exposition can be found in [1] and [2].

The multi-marginal transportation problem asks how to couple several distributions of mass with maximal efficiency, as measured by a prescribed surplus function. More precisely, for $i = 1, 2, ..., m$, let M_i be a compact smooth manifold of dimension n_i, endowed with a Borel probability measure μ_i and let $s : M_1 \times M_2 \times ... \times M_m \to \mathbb{R}$ be a C^2 smooth function, which we will call the surplus function. The optimal transportation problem then has two formulations. In the *Monge* formulation, the goal is to maximize

$$\textbf{(M)} \qquad S(G_2, G_3, ..., G_m) := \int_{M_1} s(x_1, G_2(x_1), G_3(x_1), ..., G_m(x_1)) d\mu_1$$

among all $(m-1)$-tuples of measurable maps $(G_2, G_3, ..., G_m)$, where $G_i : M_1 \to M_i$ pushes μ_1 forward to μ_i for all $i = 2, 3, ..., m$.

In the *Kantorovich*, or relaxed, formulation of the problem one maximizes

$$\textbf{(K)} \qquad S(\mu) := \int_{M_1 \times M_2 \times ... \times M_m} s(x_1, x_2, x_3, ..., x_m) d\mu$$

among all positive Borel measures μ on $M_1 \times M_2 \times ... \times M_m$ such that the canonical projection

$$\pi_i : M_1 \times M_2 \times ... \times M_m \to M_i$$

pushes μ forward to μ_i for all i. Heuristically, in the Monge formulation, mass at almost every point $x_1 \in M_1$ must be coupled with mass at unique points $x_i \in M_i$ for $i = 2, 3, ..., m$, whereas in the Kantorovich formulation a coupling may *split* a piece of mass at x_1 among two or more destination points in M_i for $i = 2, 3, ..., m$.

It is straightforward to show that a solution μ to the Kantorovich problem exists. When $m = 2$ and $n_1 = n_2$, it is possible, under weak conditions on s and μ_1, to show that the solution is concentrated on the graph of a function over x_1. This function then solves the Monge problem and in this case it is not hard to show that the solutions to both the Monge and Kantorovich problems are unique. Gangbo and Święch, Heinich, and Carlier have extended these results to the multi-marginal setting for certain special surplus functions; a complete list of references may be found in [1].

In [1], I develop a geometric framework to study the multi-marginal optimal transportation problem. I define a convex family G of semi-Riemannian metrics on $M_1 \times M_2 \times ... \times M_m$, derived from the mixed, second order partial derivatives of s. For any semi-metric g in this family, I then prove that near a point $\vec{x} := (x_1, x_2, ..., x_m) \in M_1 \times M_2 \times ... \times M_m$, the support of the optimal measure μ is contained in a Lipschitz submanifold of $M_1 \times M_2 \times ... \times M_m$, whose dimension is the number of non-timelike directions in the signature of g at $\vec{x}$. This generalizes a similar result in the two marginal setting, due to McCann, Warren and myself, which asserts that when $m = 2$ and $n_1 = n_2 := n$, μ is supported on an n-dimensional Lipschitz submanifold, under a weak local condition on s. In that case, the family G contains only one semi-metric, whose signature is (n, n); this is exactly the semi-metric used by Kim and McCann to study the regularity of solutions to the Monge problem. When $m \geq 3$, the dimension of the support of μ depends on an entire family of semi-metrics whose signatures may vary; generically, each g will have between n_{max} and $N - n_{max}$ non-timelike directions, where $n_{max} := \max_i n_i$ and $N := \sum_{i=1}^m n_i$.

In certain cases, all the elements of G have many non-timelike directions, and we demonstrate by example that in such cases the support of the solution may actually concentrate on a high dimensional submanifold. In particular, it will not be concentrated on the graph of a function over the first marginal; moreover, I show that in some of these cases the solution is non-unique. This stands in stark contrast to the two marginal case; in addition, it suggests that stronger conditions must be assumed in order to prove the existence and uniqueness of solutions to the Monge problem as well as uniqueness of solutions to the Kantorovich problem.

These questions are resolved in [2]. The conditions I impose on s are much stronger than conditions required to prove analogous results for two marginal problems, as one would expect given the preceding discussion. Nonetheless, they apply to the surplus functions considered by Gangbo and Święch and Heinich as well as several other interesting examples, which are outlined in [2].

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Regularity for the parabolic obstacle problem with fractional Laplacian

Alessio Figalli

In recent years, there has been an increasing interest in studying constrained variational problems with a fractional diffusion. One of the motivations comes from mathematical finance: jump-diffusion processes where incorporated by Merton [4] into the theory of option evaluation to introduce discontinuous paths in the dynamics of the stock's prices, in contrast with the classical lognormal diffusion model of Black and Scholes [1]. These models allow to take into account large price changes, and they have become increasingly popular for modeling market fluctuations, both for risk management and option pricing purposes.

Let us recall that an American option gives its holder the right to buy a stock at a given price prior (but not later) than a given time $T > 0$. If $v(\tau, x)$ represents the rational price of an American option with a payoff ψ at time $T > 0$, then v will solve (in the viscosity sense) the following obstacle problem:

$$\begin{cases} \min\{\mathcal{L}v, v - \psi\} = 0, \\ v(T) = \psi. \end{cases}$$

Here $\mathcal{L}v$ is a (backward) parabolic integro-differential operator of the form

$$\mathcal{L}v = -v_\tau - rv + \sum_{i=1}^n (r-d_i)x_i v_{x_i} - \frac{1}{2}\sum_{i,j=1}^n x_i x_j \sigma_{ij} v_{x_i x_j}$$
$$- \int \Big[v\big(\tau, x_1 e^{y_1}, \dots, x_n e^{y_n}\big) - v(\tau,x) - \sum_{i=1}^n (e^{y_i}-1)x_i v_{x_i}(\tau,x)\Big]\,\mu(dy),$$

where $r>0$, $d_i \in \mathbb{R}$, $\sigma = (\sigma_{ij})$ is a non-negative definite matrix, and μ is a jump measure. When the matrix σ is uniformly elliptic, after the change of variable $x_i \mapsto \log(x_i)$ the equation becomes uniformly parabolic (backward in time) and the diffusion part dominates. In particular, if no jump part is present (i.e., $\mu \equiv 0$), then the regularity theory is pretty well-understood.

In [2] we assume that there is no diffusion (i.e., $\sigma \equiv 0$), so all the regularity should come from the jump part. We also assume that the jump part behaves, at least at the leading order, as a fractional power of the Laplacian, so that the equation takes the form

$$\mathcal{L}v = -v_\tau - rv - b\cdot\nabla u + (-\Delta)^s v + \mathcal{K}v, \qquad s \in (0,1),$$

where $b = (d_1 - r, \dots, d_n - r)$, and $\mathcal{K}v$ is a non-local operator of lower order with respect to $(-\Delta)^s v$. As explained in [2, Section 5], when $s > 1/2$ the regularity theory for the above equation is essentially the same as the one for the model equation

$$\text{(1)} \qquad \begin{cases} \min\{-v_\tau + (-\Delta)^s v, v-\psi\} = 0 & \text{on } [0,T]\times\mathbb{R}^n, \\ v(T) = \psi & \text{on } \mathbb{R}^n. \end{cases}$$

In [2] we decide to focus on (1), since this allows to avoid technicalities which may obscure the main ideas behind the regularity theory that we develop.

Our main result is the following [2, Theorem 2.1]:
Given $s \in (0,1)$ and $\psi \in C^2(\mathbb{R}^n)$, with

$$\|\nabla\psi\|_{L^\infty(\mathbb{R}^n)} + \|D^2\psi\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s\psi\|_{C_x^{1-s}(\mathbb{R}^n)} < +\infty,$$

let u be the unique continuous viscosity solution of (1). Then u is globally Lipschitz in space-time on $[0,T]\times\mathbb{R}^n$, and satisfies

$$\begin{cases} u_\tau \in \log\mathrm{Lip}_t C_x^{1-s}([0,T)\times\mathbb{R}^n), \ (-\Delta)^s u \in \log\mathrm{Lip}_t C_x^{1-s}([0,T)\times\mathbb{R}^n) & \text{if } s \le 1/3; \\ u_\tau \in C_{t,x}^{\frac{1-s}{2s}-0^+,1-s}([0,T)\times\mathbb{R}^n), \ (-\Delta)^s u \in C_{t,x}^{\frac{1-s}{2s},1-s}([0,T)\times\mathbb{R}^n) & \text{if } s > 1/3. \end{cases}$$

Let us make some comments. First of all we recall that, for the stationary version of the obstacle problem, solutions belong to $C_x^{1+s}(\mathbb{R}^n)$ (or equivalently, $(-\Delta)^s u \in C_x^{1-s}(\mathbb{R}^n)$), and such a regularity result is optimal [5, 3]. Hence, at least concerning the spatial regularity, our result is optimal, too. Actually, this may look a bit surprising. Indeed, as shown in [2, Remark 3.7], for any $\beta \in (0,1)$ one can find a traveling wave solution to the equation $\min\{-v_\tau + (-\Delta)^{1/2}v, v-\psi\} = 0$ which is $C^{1+\beta}$ both in space and time, but not $C^{1+\gamma}$ for any $\gamma > \beta$. Hence, in order to prove that solutions to (1) are C^{1+s} in space, one has to exploit the crucial fact that v coincides with the obstacle at time T.

Once the C_x^{1-s}-regularity of $(-\Delta)^s u$ is established, the fact that $s = 1/3$ plays a special role is not surprising: indeed, the operator $-\partial_\tau + (-\Delta)^s$ is invariant under the scaling $(\tau, x) \mapsto (\lambda^{2s}\tau, \lambda x)$. Hence, a spatial regularity C^{1-s} naturally corresponds to a time regularity $C^{\frac{1-s}{2s}}$, provided $\frac{1-s}{2s} < 1$, that is $s > 1/3$.

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Rigidity results for manifolds with positive scalar curvature metrics

Andre Neves

A classical question in Differential Geometry is the search for theorems which, under some curvature condition, imply certain inequality and the equality being attained implies rigidity of the original metric. To that effect, Min-Oo conjectured that metrics with $R(g) \geq n(n-1)$ which agree with the round metric on a neighborhood of the equator in the northern hemisphere must be round. Very recently, in a joint work with Brendle and Marques [3], we disproved this conjecture and so there is a clear interest in knowing which type of rigidity statements hold. I will talk about two theorems in these direction.

Theorem 1 (with Bray, Brendle, and Eichmair [1])**.** *Let (M, g) be a three manifold with scalar curvature $R(g) \geq 6$. Then*

$$\inf\{\text{area}(\Sigma) \,|\, \Sigma \textit{ is embedded projective plane}\} \leq 2\pi$$

and equality implies $(M, g) = (\mathbb{RP}^3, g_{std})$.

The idea to prove the inequality is to use a Hersch-type trick. The idea to prove the rigidity is to use Ricci flow.

Theorem 2 (with Bray and Brendle [2])**.** *Let (M, g) be a three manifold with scalar curvature $R(g) \geq 6$. Then*

$$\inf\{\text{area}(\Sigma) \,|\, \Sigma \textit{ embedded sphere non-trivial in homology}\} \leq 4\pi/3$$

and equality implies $(M, g) = (S^2 \times S^1, g_{std} + d\theta^2)$.

The inequality follows at once from combining Gauss equation with the second variation formula. The rigidity statement comes from using a c.m.c. foliation argument.

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On the affine Plateau problem

Neil S. Trudinger

The affine Plateau problem is the affine invariant analogue of the classical Plateau problem for minimal surfaces. Roughly put, it involves finding a locally convex hypersurface which maximizes affine area among a class of of locally convex hypersurfaces with prescribed boundary and Gauss map image. In this talk we report on a recent preprint [4], with Xu-jia Wang, where we establish, under appropriate conditions, the smooth solvability in all dimensions, thereby extending our earlier results in [2] for the two dimensional case.

The affine area of a smooth (C^2) locally convex hypersurface $\mathcal{M}$ in $\mathbb{R}^{n+1}$ is given by

$$A(\mathcal{M}) = \int_{\mathcal{M}} K^{1/(n+2)},$$

where $K, (\geq 0)$, denotes the Gauss curvature of $\mathcal{M}$. The affine area functional A is invariant under uni-modular affine transformations in $\mathbb{R}^{n+1}$. When $\mathcal{M} = \mathcal{M}_u$ is the graph of a locally convex function $u \in C^2(\Omega)$, over a domain $\Omega \in \mathbb{R}^n$, we have

$$A(\mathcal{M}) = A[u] = \int_{\Omega} (\det D^2 u)^{1/(n+2)}.$$

Let $\mathcal{M}_0$ be a bounded, connected hypersurface in $\mathbb{R}^{n+1}$ with C^2 smooth boundary Γ and assume that $\mathcal{M}_0 \cup \Gamma$ is C^2 smooth and locally uniformly convex up to the boundary. Denote by $S[\mathcal{M}_0]$ the set of locally convex C^2 smooth hypersurfaces with boundary Γ, which can be smoothly deformed from $\mathcal{M}_0$ in the family of locally convex hypersurfaces whose Gauss map images lie in that of $\mathcal{M}_0$.

Theorem 1. *There exists a locally uniformly convex C^∞ smooth hypersurface $\mathcal{M}$ maximizing A over $S[\mathcal{M}_0]$ if and only if the Gauss map image of M_0 does not cover a hemisphere.*

For the graph case, we let Ω be a bounded C^2 domain in $\mathbb{R}^n$ and φ a uniformly convex function in $C^2(\bar{\Omega})$ and denote by $S[\varphi, \Omega]$ the set of convex functions u in $C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$, satisfying $u = \varphi$ on $\partial\Omega$ and whose normal images $N_u(\Omega)$ lie in $N_\varphi(\bar{\Omega})$.

Theorem 2. *There exists a unique locally uniformly convex function $u \in C^\infty(\Omega)$ maximizing A over the set $S[\varphi, \Omega]$.*

The proof of Theorem 1 depends on a reduction to the graph case of Theorem 2. The proof of Theorem 2 is accomplished by first solving a relaxed variational problem and then showing that strictly convex maximizers are smooth. These stages are already accomplished in [2] with techniques that carry over automatically for more general functionals, namely Monge-Ampère integrals of the form,

$$\mathcal{F}[u] = \mathcal{F}[u,\Omega] = \int_\Omega \{F(\det D^2u) - fu\},$$

where $F \in C^3(0,\infty), F' > 0, F(\infty) = \infty, f \in L^\infty(\Omega)$ and $u \in C^2(\Omega)$ is locally convex. The Euler-Lagrange equation for $\mathcal{F}$ is the fourth order nonlinear PDE,

$$L[u] := \mathcal{U}^{ij} D_{ij} F'(\det D^2u) = f,$$

which is well defined in a classical sense if $u \in C^4(\Omega)$ is locally uniformly convex, that is $D^2u > 0$, and elliptic if $F'' \neq 0$. Here the coefficient matrix $[\mathcal{U}^{ij}]$ is the cofactor matrix of the Hessian D^2u. Furthermore if we also assume that $F(t)$ is a concave function of $t^{1/n}$, then solutions of the PDE will be local maximizers of $\mathcal{F}$. Conversely, smooth, locally uniformly convex local maximizers will be solutions. When $F(t) = t^{1/(n+2)}$, we obtain the prescribed affine mean curvature equation, which becomes the affine maximal surface equation when $f = 0$. When $F(t) = \log t$, we obtain Abreu's equation, which arises in complex geometry [1].

The variational problem in Theorem 2 is relaxed as follows. First we extend the functional $\mathcal{F}$ to general convex functions u by extending D^2u to vanish on the null set where u is not twice differentiable; (this is feasible if $F(t) = o(t)$ for large t). Next we let φ be a convex function defined in a neighbourhood of $\overline{\Omega}$ and extend the set S by defining $\overline{S}[\varphi,\Omega]$ to be the set of convex functions u in $C^{0,1}(\Omega)$, satisfying $u = \varphi$ on $\partial\Omega$ and $N_u(\Omega) \subset N_\varphi(\overline{\Omega})$. It follows that $\mathcal{F}(u)$ is well defined in $[-\infty,\infty)$, for $u \in \overline{S}$, with $\mathcal{F}(u) > -\infty$ if $F(0) > -\infty$.

Theorem 3. *Suppose $|\partial\Omega| = 0$ and $\mathcal{F}[u] > -\infty$ for some $u \in \bar{S}[\varphi,\Omega]$. Then under the above conditions on F, there exists a unique maximizer of $\mathcal{F}$ over the set $\bar{S}[\varphi,\Omega]$*

For regularity, we have the following result from [2].

Theorem 4. *For $F(t) = t^\theta, 0 < \theta \leq 1/n$, a strictly convex maximizer of $\bar{S}[\varphi,\Omega]$ lies in $W^{4,p}$ for all $p < \infty$ and is a locally uniformly convex strong solution of the Euler-Lagrange equation.*

Higher regularity follows from linear theory. In particular if $f \in C^\infty(\Omega)$, then strictly convex maximizers are also in $C^\infty(\Omega)$. More general functions F for which $F(0) > -\infty$ are permitted in Theorem 4 for arbitrary dimension $n \geq 2$. However the corresponding result for $F(t) = \log t$ was obtained recently by Zhou [5] for two dimensions, along with a corresponding extension of Theorem 2.

The essential issue in proving Theorem 2 is thus that of strict convexity of maximizers. This is much more complicated than the analogous results for generalized solutions of Monge-Ampère equations [3] and the new arguments in [4] lead as well to a simplified proof of the two dimensional case in [2]. A critical new idea is a secondary penalization in the regularity proof which enables the dual functional to be employed in the strict convexity arguments.

Finally we remark that the behaviour of the gradient of solutions, (or the Gauss mapping in the hypersurface case), at the boundary still remains an open problem. We could partly formulate this problem by asking what conditions on φ and Ω in Theorem 2 ensure that $Du = D\varphi$ on $\partial\Omega$ in an appropriate sense.

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Sharp lower density bounds for area-minimizing cones

Brian White
(joint work with Tom Ilmanen)

In this lecture, I will describe some sharp lower bounds on densities of area-minimizing hypercones or, equivalently, on volumes of certain closed minimal hypersurfaces in round spheres. The results are joint work with Tom Ilmanen.

I begin by indicating why such density bounds are of interest. Recall that if M is an m-dimensional minimal variety in a Riemannian manifold and if x is an interior point of M, then the density of M at x is

$$\Theta(M,x) := \lim_{r\to 0} \frac{\text{area}\,(M\cap \mathbf{B}(x,r))}{\omega_m r^m}, \tag{1}$$

where ω_m is the m-dimensional volume of the unit ball in $\mathbb{R}^m$. The limit exists by the monotonicity formula. The density is 1 at any multiplicity 1 regular point, and it is strictly greater than 1 at any singular point (by Allard's regularity theorem). If M is a cone with vertex x, then the ratio in (1) is independent of r; in that case, we write $\Theta(M) = \Theta(M,x)$.

In this lecture, I will focus on the case of area-minimizing hypersurfaces (either integral currents or flat chains mod 2). Consider the following question:

> **Q1**. What is the infimum of $\Theta(M,x)$ among all pairs (M,x) where M is an area minimizing hypersurface (in some Riemannian manifold) and x is an interior singular point of M?

Here "interior point of M" means "point in the support of M but not in the support of ∂M".

Note that if x is an interior singular point of M and if C is a tangent cone to M at x, then C is an area minimizing hypercone in Euclidean space with a singularity at its vertex, and $\Theta(C) = \Theta(M, x)$. Furthermore, standard dimension reducing arguments show that either C has an isolated singularity at its vertex, or else there is another area minimizing hypercone C' of lower dimension such that C' has an isolated singularity at vertex and such that $\Theta(C') \le \Theta(C)$. Thus the question Q1 is equivalent to:

> **Q2**. What is the infimum of $\Theta(C)$ among all area-minimizing hypercones C such that C has an isolated singularity at the origin?

Ilmanen and I were able to give a sharp answer to question Q2 provided one restricts the cones C to those that are topologically nontrivial. In particular, we proved:

Theorem 1. *Suppose that $C \subset \mathbb{R}^n$ is an area-minimizing hypercone with an isolated singularity at the origin. Suppose also that C is topologically nontrivial in the following sense: at least one of the two components of $\mathbb{R}^n \setminus C$ is non-contractible. Then the density of C at the origin is greater than $\sqrt{2}$.*

If one wants a constant independent of the dimension of the dimension n, then $\sqrt{2}$ is the best possible because the Simons' cone

$$C_{m,m} := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m} : |x| = |y|\}$$

is both topologically nontrivial and area-minimizing for $m \ge 4$, and by a straightforward calculation its density $\Theta(C_{m,m})$ tends to $\sqrt{2}$ as m tends to ∞.

In the lecture, I will describe the proof of Theorem 1.

Let C be a cone as in Theorem 1. Since one of the components of $\mathbb{R}^n \setminus C$ is non-contractible, one of its homotopy groups, say the k^{th} homotopy group, is nontrivial. One can get a better lower bound for $\Theta(C)$ if one allows a constant that depends on k. In particular, Ilmanen and I proved that

$$\Theta(C, \mathbf{O}) > d_k = \left(\frac{k}{2\pi e}\right)^{k/2} \sigma_k$$

where d_k is the Gaussian density of a shrinking k-dimensional sphere and σ_k is the area of the unit k-dimensional sphere. (Gaussian density, which was discovered by Huisken, plays the role in mean curvature flow that density does is minimal surface theory. See [6].)

As before, this result is sharp in the sense that for any $\epsilon > 0$, there is an n and a cone $C \subset \mathbb{R}^n$ such that C satisfies the hypotheses of the theorem and such that

$$\Theta(C) < d_k + \epsilon.$$

0.1. **Open Problems.** There are many interesting open problems about lower bounds for density. For example:

(1) (conjectured by Bruce Solomon.) For $m \geq 1$, prove that the Simon's cone $C_{m,m}$ is the $(2m+1)$-dimensional minimal (or area-minimizing) hypercone of least possible density. For $m = 1$, one has to exclude cones with soap-film-like triple junctions, since the density of 3 half planes meeting along a common edge is $3/2$, which is less than $\Theta(C_{1,1})$. However, in higher dimensions this exclusion is not necessary since $\Theta(C_{m,m}) < 3/2$ for $m > 1$.
(2) (Conjectured by Bruce Solomon.) Prove that the cone $C_{m,m+1}$ is the $(2m+2)$-dimensional minimal (or area-minimizing) hypercone of least possible density.
(3) Prove that $C_{m,n}$ has realizes the least possible density among all $(m+n+1)$-dimensional minimal (or area-minimizing) hypercones C such that at least one of the components of the complement has nontrivial m^{th} homotopy group.
(4) Prove lower density bounds for minimal or for area-minimizing cones of codimension > 1.

Here $C_{m,k}$ denotes the Simon's type cone $\{(x,y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{k+1} : m|x| = k|y|\}$, which is the cone over $\mathbf{S}^m \times \mathbf{S}^k$ (with appropriate radii.)

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Optimal Control and Weak KAM Theory

Paul Lee

In this abstract, we outline the recent results in [1] concerning continuity of some optimal control costs and its application to the weak KAM theory. For simplicity, many of the results in this abstract are stated with assumptions stronger than those in [1].

Let M be a compact smooth manifold without boundary and let $H : T^*M \to \mathbb{R}$ be a smooth function on the cotangent bundle T^*M of the manifold M, called a Hamiltonian. Let us consider the following Hamilton-Jacobi equation

$$H(x, df_x) = h. \tag{1}$$

Suppose that we have a family of classical solutions $f(x, P)$ to (1)

$$H(x, \partial_x f(x, P)) = h(P)$$

which satisfies the condition $\det(\partial_x \partial_P f) \neq 0$. Under these assumptions, we can define a change of variables $(x, p) \mapsto (X, P)$ by

$$p = \partial_x f(x, P), \quad X = \partial_P f(x, P).$$

This change of variables transforms the Hamiltonian system

$$\dot{x} = \partial_p H, \quad \dot{p} = -\partial_x H \tag{2}$$

to a much simpler system

$$\dot{X} = \partial_P h, \quad \dot{P} = 0.$$

In particular, the entries of $P = (P_1, ..., P_n)$ are constants of motion and the Hamiltonian system (2) can be integrated according to Liouville-Arnold Theorem (see [2]). In other words, (2) is completely integrable.

All the above assumes that we have a family of classical solutions to the Hamilton-Jacobi equation (1). One natural question is the following: "what can we say about (1) if the corresponding Hamiltonian system is not completely integrable?" One answer is given by the following weak KAM theorem:

Theorem 1. *(Weak KAM Theorem [4, 3]) Assume that the Hamiltonian H satisfies the following assumptions:*

- *The matrix $(\partial_{p_i} \partial_{p_j} H)$ is positive definite,*
- *$H(x, p) > C|p| + K$.*

Then there exists a unique constant h such that the Hamilton-Jacobi equation (1) has a viscosity solution.

In [1], we consider Hamiltonians arising from some optimal control problems. More precisely, let $X_0, X_1, ..., X_k$ be vector fields defined on the manifold M. Consider the following affine control system:

$$\dot{x}(t) = F(x(t), u(t)), \tag{3}$$

where $F(x, u) := X_0(x) + \sum_{i=1}^k u_i X_i(x)$ and $u : [0, T] \to \mathbb{R}^k$ is any L^2 integrable function.

Let $L : M \times \mathbb{R}^k \to \mathbb{R}$ be a smooth function called Lagrangian. The optimal control problem is the following minimization problem:

$$c_T(x, y) := \inf \int_0^T L(x(t), u(t))dt, \tag{4}$$

where the infimum above is taken over the set of all pairs $(x(\cdot), u(\cdot))$ satisfying (3).

Finally, we define the Hamiltonian H corresponding to the above optimal control problem by

$$H(x,p) := \sup_{u\in\mathbb{R}^k} \left(p(F(x,u)) - L(x,u)\right). \tag{5}$$

The following is a version of the weak KAM theorem corresponding to the Hamiltonian defined in (5).

Theorem 2. [1] *Assume that the function $(t,x,y) \mapsto c_t(x,y)$ is continuous. Then there exists a unique constant h such that the Hamilton-Jacobi equation (1) with Hamiltonian given by (5) has a viscosity solution.*

Thanks to Theorem 2, it remains to consider when the cost function c_T is continuous. Before stating the result, we need the following definition. The family of vector fields $\{X_1, ..., X_k\}$ is said to be m-generating if the vector fields X_i and their iterated Lie brackets up to $m-1$ order spanned each tangent space TM. More precisely, the following holds for each point x in the manifold M

$$T_xM = \text{span}\{[X_{i_1}, [X_{i_2}, ..., [X_{i_{l-1}}, X_{i_l}]]](x) | 1 \le i_j \le k, 1 \le l \le m-1\}.$$

Theorem 3. [1] *Assume that the Lagrangian L satisfies the following conditions*

$$C_1|u|^2 + K_1 \le L(x,u) \le C_2|u|^2 + K_2, \quad \frac{\partial L}{\partial x} \le C_3|u|^2$$

for some constants $C_1, C_2, C_3 > 0$ and the Hessian of L in the u variable is positive definite. If the family of vector fields $\{X_1, ..., X_k\}$ is 3-generating, then the cost function $(t,x,y) \mapsto c_t(x,y)$ defined in (4) is continuous.

In [1], we also showed that Theorem 3 is sharp. More precisely, we considered the following control system on $\mathbb{R}^2$:

$$(\dot{x}_1, \dot{x}_2) = (0, x_1^2) + u_1(1,0) + u_2(0, x_1^3) = (u_1, x_1^2 + u_2 x_1^3). \tag{6}$$

Note that the family of vector fields $\{(1,0), (0, x_1^3)\}$ is 4-generating but not 3-generating.

Theorem 4. [1] *The cost function c_1 defined by (4), the control system (6), and the Lagrangian $L(x,u) = \frac{1}{2}\left(u_1^2 + u_2^2\right)$ is not continuous at $((0,0),(0,0))$.*

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Billiards, optimization of resistance and invisibility

Alexander Plakhov

We consider problems of minimal and maximal resistance in billiards and related problems in optics.

1. We start with discussing the classical problem of minimal resistance, first stated by Newton for convex and axially symmetric bodies, and its generalizations studied in 1990's and 2000's. Actually, Newton never mentioned the convexity assumption; without it, the solution turns out to be different and the minimum of resistance becomes smaller $2 \div 4$ times [7]. On the other hand, the problem for convex but generally non-symmetric bodies was stated in 1993 by Buttazzo and Kawolh [3] and gave rise to several interesting works. The solution in this extended class of bodies exists and does not coincide with Newton's optimal body, but until now not much is known about this solution.

We state the question: do there exist bodies of zero resistance? Surprisingly, the answer is positive; such a body is shown in Fig. 1.

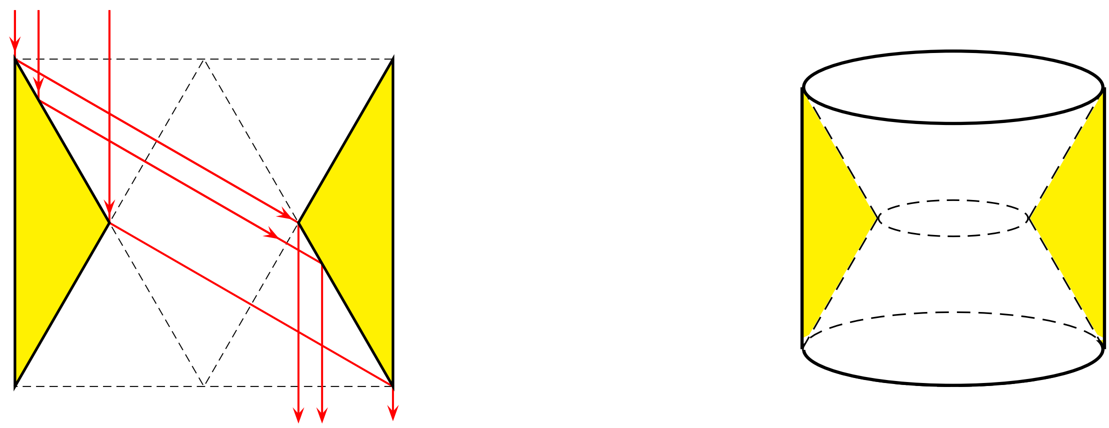

Figure 1. A body of zero resistance and its vertical central cross section.

A spacecraft of this shape could travel infinitely long time in the cloud without slowing down its motion, provided that the particles-spacecraft collisions are elastic.

By doubling this body, we get a body (with mirror surface) invisible in one direction; see Fig. 2.

When looking at it from a sufficiently large distance in vertical direction, we will not see it. Each particle (photon) makes exactly 4 reflections from its surface. For a more detailed study of invisible and zero resistance bodies, see [1].

An elegant construction proposed by allows one to get an invisible body with only 3 reflections. We also explain a nice construction of a body invisible in two mutually perpendicular directions proposed by Vera Roshchina, and prove that invisibility/zero resistance in *all* directions is impossible. Thus, the following theorem is proved.

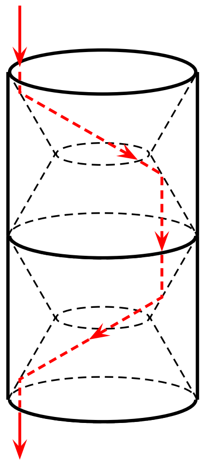

Figure 2. A body invisible in one direction obtained by gluing together two bodies of zero resistance.

Theorem 1. *(a) For any two mutually perpendicular directions* v_1 *and* v_2*, there exists a body that is invisible/has zero resistance in these directions.*

(b) There do not exist bodies invisible/having zero resistance in all directions $v \in S^2$*.*

It is unknown if there exist bodies invisible in 3 or more directions. We believe that there do not exist bodies invisible/having zero resistance in a positive measure set of directions, but cannot prove it.

2. In the second part of the talk we discuss retroreflectors: bodies that have maximal resistance in all directions. Each particle (or ray of light) incident on such a body will change its direction to the opposite. A well-known example of retroreflector based on light refraction is the Eaton lens: a transparent ball with radially symmetric refraction index going to infinity at the center of the ball [4]. Here we concentrate on billiard retroreflectors. Actually, it is unknown if they exist or not; however it is possible to construct an asymptotical retroreflector: a family of bodies B_ε, $\varepsilon > 0$ whose reflection properties converge, in a sense, to the property of retroreflection as $\varepsilon \to 0$. We provide a collection of 3 asymptotical retroreflectors in 2 dimensions. Their reflecting properties are determined by hollows on their boundary. The corresponding asymptotically retroreflecting hollows are (i) Mushroom, (ii) Tube and (iii) Notched Angle. The Mushroom and the body with mushroom-shaped hollows are shown in figures 3 and 4. The asymptotic retroreflectivity of these bodies has been proved, respectively, in [8, 2, 6].

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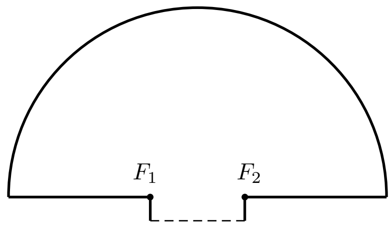

Figure 3. Mushroom is a union of a semi-ellipse with foci F_1 and F_2 and the rectangle whose upper side coincides with F_1F_2. The focal distance equals ε, the large semiaxis of the ellipse equals 1, and the height of the rectangle equals ε^3.

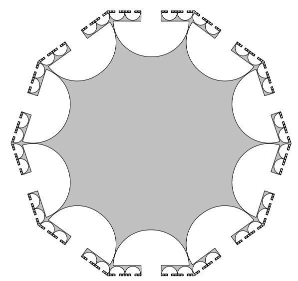

Figure 4. A retroreflector with mushroom-shaped hollows.

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Instantaneously complete Ricci flows

Peter Topping

The Ricci flow [3] takes a Riemannian metric g on a manifold $\mathcal{M}$ and deforms it under the nonlinear PDE

$$\frac{\partial g}{\partial t} = -2Ric(g).$$

In the talk, we described the theory of Hamilton and Chow which tells us exactly what happens to a compact Riemannian surface under this flow, including existence, uniqueness, maximal existence time and asymptotics. We then asked what happens in the noncompact case.

In that direction, we first surveyed a number of results of others which tell us what happens asymptotically when we start with various special surfaces. We then discussed the notion of instantaneously complete Ricci flow [4] and proposed it as the right class of solutions when we start with a surface which may be incomplete or of unbounded curvature. The following generalises a number of results in the literature.

Theorem 1 (Main theorem, special case, joint with Giesen [2].). *Let* $(\mathcal{M}^2, g_0)$ *be a smooth Riemannian surface which need not be complete, and could have unbounded curvature. Depending on the conformal type, we define* $T \in (0, \infty]$ *by*

$$T := \begin{cases} \frac{1}{8\pi} Vol_{g_0}\mathcal{M} & \text{if } (\mathcal{M}, g_0) \cong \mathcal{S}^2, \\ \frac{1}{4\pi} Vol_{g_0}\mathcal{M} & \text{if } (\mathcal{M}, g_0) \cong \mathbb{C} \text{ or} (\mathcal{M}, g_0) \cong \mathbb{R}P^2, \\ \infty & \text{otherwise.} \end{cases}$$

Then there exists a smooth Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ *such that*

(1) $g(0) = g_0$;
(2) $g(t)$ *is instantaneously complete (i.e. complete for* $t \in (0, T)$*);*
(3) $g(t)$ *is maximally stretched (i.e. any other Ricci flow* $\tilde{g}(t)$ *with* $\tilde{g}(0) \leq g_0$ *satisfies* $\tilde{g}(t) \leq g(t)$*),*

and this flow is unique in the sense that if $\big(\tilde{g}(t)\big)_{t\in[0,\tilde{T})}$ *is any other Ricci flow on* $\mathcal{M}$ *satisfying 1,2 and 3, then* $\tilde{T} \leq T$ *and* $\tilde{g}(t) = g(t)$ *for all* $t \in [0, \tilde{T})$.

If $T < \infty$, *then we have*

$$Vol_{g(t)}\mathcal{M} = \left\{ \begin{array}{ll} 8\pi(T-t) & \text{if} (\mathcal{M}, g_0) \cong \mathcal{S}^2, \\ 4\pi(T-t) & \text{otherwise,} \end{array} \right\} \longrightarrow \quad 0 \quad \text{as } t \nearrow T,$$

and in particular, T *is the maximal existence time. Alternatively, if* $\mathcal{M}$ *supports a complete hyperbolic metric* H *conformally equivalent to* g_0 *(in which case* $T = \infty$*) then we have convergence of the rescaled solution*

$$\frac{1}{2t} g(t) \longrightarrow H \quad \text{smoothly locally as } t \to \infty.$$

If additionally there exists a constant $M > 0$ *such that* $g_0 \leq MH$ *then the convergence is global: for all* $t > 0$ *we have* $\left\| \frac{1}{2t} g(t) - H \right\|_{C^0(\mathcal{M},H)} \leq \frac{C}{t}$.

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Regularity of optimal transport maps on multiple products of spheres

Young-Heon Kim
(joint work with Alessio Figalli and Robert J. McCann)

This abstract reports regularity of optimal transportation maps on Riemannian manifolds, in the case of multiple products of round spheres. In the following presentation, our assumptions are not necessarily the most optimal/general ones (see [4] for details).

Let M be a compact n-dimensional Riemannian manifold equipped with two probability (bounded measurable) densities ρ, $\bar\rho > 0$, and with transportation cost $c(x,\bar x) = \mathrm{dist}^2(x,\bar x)/2$ for $(x,\bar x) \in M\times M$. By the results of many people including Monge, Kantorovich, Brenier and McCann, there exists unique (Borel measurable) optimal map $T : M \to M$ that minimizes the total transportation cost

$$\int_M c(x, T(x))\rho(x)dvol_M$$

among all maps transporting the density ρ to $\bar\rho$, i.e. $\int_M f(T(x))\rho(x)dvol_M = \int_M f(\bar x)\bar\rho(\bar x)dvol_M$ for all continuous $f : M \to \mathbb{R}$. Moreover, this optimal map has a very nice characterization as $T(x) = \exp_x \nabla u(x)$ *a.e.*, where the function u, called the c-potential, is given in a pair $(u, \bar u)$ as

$$(1) \qquad u(x) = \sup_{\bar x\in M} -c(x,\bar x) - \bar u(\bar x), \qquad \bar u(\bar x) = \sup_{x\in M} -c(x,\bar x) - u(x),$$

and u is Lipschitz continuous and semi-convex.

For the Euclidean case, such potential u satisfies classical Monge-Ampère equation $\det(D^2u(x)+I) = \frac{\rho(x)}{\bar\rho(\nabla u(x)+x)}$ in a weak sense, relating optimal transportation problem to fully nonlinear partial differential equations. Moreover, there is a connection to geometric variational problems as shown in [10]. Namely, if T is smooth, then its graph gives a maximal space-like Lagrangian submanifold (thus with zero mean curvature) in the product space $M \times M \setminus$ cut locus, equipped with a symplectic form ω and a pseudo-Riemannian metric $h_c^{\rho,\bar\rho}$, which are given at $(x,\bar x)$ as $\omega = -\partial_{x^i}\partial_{\bar x^j} c\, dx^i \wedge d\bar x^j$ and

$$h_c^{\rho,\bar\rho} = -\left(\frac{\rho(x)\bar\rho(\bar x)}{|\det(D_xD_{\bar x}c(x,\bar x))|}\right)^{\frac{1}{n}} \partial_{x^i}\partial_{\bar x^j} c\,(dx^i\otimes d\bar x^j + d\bar x^j \otimes dx^i)/2.$$

Though $h_c^{\rho,\bar\rho}$ has n positive and n negative eigenvalues, it induces a Riemannian metric on the graph of T for smooth T, making the graph of T space-like and thus defining its volume. Also, the volume-maximality is obtained through calibration with the n-form $\rho(x)dvol(x) + \bar\rho(\bar x)dvol(\bar x)$. (In fact, this pseudo-Riemannian formulation holds for more general costs and domains. See [8, 10].)

As the above suggests, it is natural in both PDE and geometry, to consider regularity of the optimal map T. More precisely, we ask whether $T \in C^\infty/C^0$ for $\log\rho, \log\bar\rho \in C^\infty/L^\infty$. In [4], we answer this question affirmatively when $M = S^{n_1}_{r_1} \times \cdots \times S^{n_k}_{r_k}$ is the multiple product of round spheres of arbitrary dimension and size. To the author's knowledge, this is the first regularity result of optimal maps transporting densities supported on non-flat manifolds that allow zero sectional curvature. For flat manifolds (including domains in $\mathbb{R}^n$) the result is known due to the work of Delanoë, Caffarelli, Urbas, and Cordero-Erausquin. For positively curved manifolds such regularity is recently known for the sphere and its small perturbations and quotients, due to Loeper, Kim, McCann, Delanoë, Ge, Figalli, Rifford and Villani [13, 8, 9, 14, 1, 5, 16, 6], among others. A key notion for regularity on non-flat manifolds is the MTW condition formulated by Ma, Trudinger and Wang [15], which is shown to be a necessary condition for regularity by Loeper [12]. As found in [8], this condition can be understood as follows: Let R_c denote the Riemann curvature tensor of the pseudo-metric $h_c := -\partial_{x^i}\partial_{\bar{x}^j} c\,(dx^i \otimes d\bar{x}^j + d\bar{x}^j \otimes dx^i)/2$ on the product space $M \times M \setminus$ cut locus. The MTW condition (denoted by $MTW^{\perp} \geq 0$) requires on c that

$$MTW(p, \bar{p}) := R_c\left((p \oplus 0) \wedge (0 \oplus \bar{p}), (p \oplus 0) \wedge (0 \oplus \bar{p})\right) \geq 0$$

for all $p \in T_x M$, $\bar{p} \in T_{\bar{x}} M$ with $h_c(p \oplus 0, 0 \oplus \bar{p}) = 0$. Use $MTW^{\perp} > 0$ to denote the same condition but with strict inequality. It is known by Loeper [12] that along the diagonal $\{x = \bar{x}\}$, MTW coincides with the sectional curvature of the original metric on M, thus to have regularity of optimal maps the manifolds need to be non-negatively curved; but, the converse does not hold [7].

Before our work [4], all the previous regularity results given on non-flat manifolds use the strict condition $MTW^{\perp} > 0$ where there are strong $C^2/C^{1,\alpha}$ estimates for c-potentials developed by Ma, Trudinger and Wang / Loeper [15, 12], respectively. Such strong estimates are not available for $M = S^{n_1}_{r_1} \times \cdots \times S^{n_k}_{r_k}$ where MTW tensor degenerates (due to flat directions), and we use instead a recently developed local analysis in [3] where the condition $MTW \geq 0$ (without $h_c(p \oplus 0, 0 \oplus \bar{p}) = 0$), which holds for products of spheres [9], enables one to transform c-potentials to convex functions in certain coordinate charts. In [3], assuming $MTW \geq 0$ for c, continuity and injectivity of optimal maps are shown in domains in $\mathbb{R}^n$, which applied to the result of Liu, Trudinger and Wang [11] shows also higher regularity. However, to apply the local analysis to global domains such as manifolds, where dist has singularity, one has to show that the optimal map has to *stay away from the singularity* of the cost function. This is unavoidable also for the cases of $MTW^{\perp} > 0$, especially to get higher regularity results [2, 13, 1, 14, 9]; however, in these cases the stay away from singularity property can be proved relatively easily, using the simple structure of the cut locus plus the strong condition $MTW^{\perp} > 0$. The main result in [4] shows the stay away from singularity property for the multiple products of spheres. More precisely,

Theorem 1 (Stay-away from cut-locus). *Let $M = S^{n_1}_{r_1} \times \cdots \times S^{n_k}_{r_k}$ be the product of round spheres, and let $c = \mathrm{dist}^2/2$. Assume a c-potential function u satisfies for some $\lambda > 0$,*

$$\lambda \,\mathrm{vol}(\Omega) \leq \mathrm{vol}(\partial^c u(\Omega)) \leq \frac{1}{\lambda}\,\mathrm{vol}(\Omega) \quad \textit{for any Borel set } \Omega \subset M,$$

where the c-subdifferential $\partial^c u$ is defined as a set-valued map $\partial^c u(x) = \{\bar{x} \in M \mid u(x) + \bar{u}(\bar{x}) = -c(x, \bar{x})\}$. Here, $\bar{u}$ is the dual c-potential of u as in (1). Then, *there exists a constant* $C(\lambda) > 0$ *such that*

$$\operatorname{dist}\big(\partial^c u(x), Cut(x)\big) \geq C(\lambda), \qquad \forall\, x \in M,$$

where $Cut(x)$ *denotes the cut-locus of* x.

The multiple product of spheres is a model case for more general manifolds on which the cost c satisfies the necessary conditions (c.f. [6]) for regularity of optimal transport maps. The method we develop in [4] demonstrates one approach to handling complex singularities of the cost.

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Optimal Lipschitz extensions

Charles K. Smart

1. Introduction

Suppose $U \subseteq \mathbb{R}^n$ is a bounded and open set and $g : \partial U \to \mathbb{R}^m$ is Lipschitz. A classical theorem of Kirszbraun states that g has a Lipschitz extension $u : \bar{U} \to \mathbb{R}^m$ that satisfies

$$\mathrm{Lip}(u, \bar{U}) = \mathrm{Lip}(g, \partial U).$$

In general, there are infinitely many such extensions. Our goal is to identify and study *optimal* Lipschitz extensions.

2. The scalar case

2.1. **Existence and uniqueness.** When $m = 1$, the correction notion of optimal Lipschitz extension was identified by Aronsson [2, 3]. A locally Lipschitz function $u \in C(U)$ is *absolutely minimizing Lipschitz* if it satisfies

$$\mathrm{Lip}(u, V) = \mathrm{Lip}(u, \partial V) \quad \text{for all } V \subset\subset U. \tag{1}$$

Jensen [7] proved the existence and uniqueness of AML extensions. He also showed that $u \in C(U)$ is AML if and only if it is a viscosity solution of

$$-\Delta_\infty u = -u_{x_i} u_{x_j} u_{x_i x_j} = 0 \quad \text{in } U.$$

Several new proofs of uniqueness have appeared since Jensen's original proof. A fundamentally new proof using random-turn games was discovered by Peres, Schram, Sheffield, and Wilson [8]. Armstrong and S. [1] extracted the "analytic heart" of [8] to give short, easy proof.

2.2. **Regularity.** Lipschitz estimates for AML functions are immediate from the definition. Aronsson showed that the function

$$u(x, y) := x^{4/3} - y^{4/3},$$

is AML on all of $\mathbb{R}^2$. This shows that the best possible regularity for AML functions is $C^{1,1/3}$.

Savin [9] proved that AML functions on $\mathbb{R}^2$ are C^1. Evans and Savin [5] proved that if $U \subseteq \mathbb{R}^2$, $u \in C(U)$ is AML, and $V \subset\subset U$, then

$$\|u\|_{C^{1,\alpha}(V)} \leq C\|v\|_{L^\infty(U)},$$

where $\alpha, C > 0$ depend only on U and V.

We prove the following result [6].

Theorem 1 (Evans-S.). *If $U \subseteq \mathbb{R}^n$ and $u \in C(U)$ is AML, then u is everywhere differentiable.*

The main innovation is the following estimate, which we prove using Bernstein's trick.

Lemma 2. *If $u^\varepsilon \in C^\infty(\mathbb{R}^n)$ satisfies*

$$-\Delta_\infty u^\varepsilon - \varepsilon\Delta u^\varepsilon = 0 \quad \textit{and} \quad |u^\varepsilon - x_n| \le \lambda < 1 \quad \textit{in } B(0,4),$$

then

$$|Du^\varepsilon| \le C \quad \textit{and} \quad |Du^\varepsilon|^2 - u^\varepsilon_{x_n} \le C\lambda^{1/4} \quad \textit{in } B(0,1),$$

where $C > 0$ depends only on n.

Everywhere differentiability follows easily from this estimate and the "blow-up plane" result of Crandall and Evans [4].

3. The vector case

Comparatively little is known in the case $m > 1$. We do know that the condition (1) does not characterize a unique extension. Suppose $n = m = 2$ and $u, v : \bar{B}(0,1) \to \bar{B}(0,1)$ are given by

$$u(z) := z^2 \quad \text{and} \quad v(z) := z^2/|z|.$$

One can check that both u and v satisfy (1). Moreover, one can check that

$$\mathrm{L}\, u < \mathrm{L}\, v \quad \text{in } B(0,1),$$

where

$$\mathrm{L}\, w(x) := \inf_{r>0} \mathrm{Lip}(w, B(x,r)),$$

is the *local Lipschitz constant.* Thus any "reasonable" notion of optimal should prefer u to v.

Sheffield and S. [10] propose a new notion of optimal extension. Suppose $U \subseteq \mathbb{R}^n$ and $u, v : \bar{U} \to \mathbb{R}^m$ are Lipschitz. If $u = v$ on ∂U and

$$\sup\{\mathrm{L}\, v : \mathrm{L}\, u < \mathrm{L}\, v\} > \sup\{\mathrm{L}\, u : \mathrm{L}\, v > \mathrm{L}\, u\},$$

then we say that u is *tighter* than v. We say that u is *tight* if there is no v tighter than u.

We discuss two results that partially characterize the smooth tight functions.

Theorem 3 (Sheffield-S.)**.** *Suppose $U \subseteq \mathbb{R}^n$, $u \in C^3(U, \mathbb{R}^m) \cap C(\bar{U})$, and there exists a unit vector field $a \in C^2(U, \mathbb{R}^n)$ such that $a(x)$ spans the principal eigenspace of $Du(x)^t Du(x)$ for every $x \in U$. Then u is tight if and only if*

$$-(u_a)_a = 0 \quad \textit{in } U,$$

where $w_a := \sum_i w_{x_i} a^i$.

Theorem 4 (Sheffield-S.)**.** *Suppose $U \subseteq \mathbb{R}^2$ and $u \in C^\infty(\bar{U})$ is analytic in a neighborhood of $\bar{U}$. Then u is tight if and only if either $u'' \equiv 0$ or*

$$\Re\frac{u'u'''}{(u'')^2} \le 2 \quad \textit{wherever it is defined.}$$

We also discuss two possible approaches to proving the existence of tight extensions.

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Higher integrability and approximation of minimal currents

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(joint work with Camillo De Lellis)

One of the main results in geometric measure theory is the interior partial regularity for area-minimizing integral currents, arising as generalized solutions of the classical least area problem given a fixed boundary. It is well known that the regularity strongly depends on the dimension of the ambient space. Indeed, if for hypersurfaces the first singularities appear in dimension 8, in higher codimension there are already examples of singular two dimensional currents. It was shown by Federer, following previous ideas of Wirtinger, that any complex variety is in fact a locally minimizing current, so that a branched complex curve as, for instance,

$$\mathcal{V} = \{(z, w) : z^2 = w^3\} \subseteq \mathbb{C}^2 \simeq \mathbb{R}^4, \tag{1}$$

is an example of a singular minimal current.

The most general results known for the case of higher codimension has been proven by Almgren [1] and Chang [3], and can be summarized in the following two theorems (note that both the results are optimal thanks to the examples provided by Federer).

Theorem 1 (Almgren). *Any* m*-dimensional area-minimizing current* T *in* $\mathbb{R}^{m+n}$ *(or, more generally, in any* $(m+n)$*-dimensional Riemannian manifold) is an* *analytic embedded manifold in its interior except possibly for a closed set of singular points Σ of Hausdorff dimension at most $m-2$.*

Theorem 2 (Chang). *For 2-dimensional area-minimizing currents, the set of interior singular points Σ consists of isolated points.*

It is clear already from the example in (1) that one of the main (and a posteriori the only) obstructions to the regularity in higher codimension is the presence of branching points, which prevent the application of the classical regularity approach through the approximation via harmonic functions. In particular, it is evident how in any neighborhood of the origin the minimal current $\mathcal{V}$ cannot be described as the graph of a function. For this reason the results in [1] (and those in [3] which build on it) need the developments of some new concepts and ideas which have been only partially exploited up to now (Almgren's big regularity paper has been written in the early '80s but has been published only recently in a volume of nearly one thousand pages). The principal contributions of Almgren's work can be summarized in the following three points which correspond roughly to the subdivision in chapters of [1]:

(1) the theory of multiple valued functions minimizing the Dirichlet energy, called Dir-minimizing Q-valued functions;
(2) the approximation of minimal currents through the graphs of multiple valued functions;
(3) the construction of the center manifold.

In a previous work in collaboration with C. De Lellis [4] we shortened and developed some of the features of the theory of Q-valued functions, also suggesting a new, intrinsic "metric" approach to the theory. Here I present some new contributions obtained in collaboration with C. De Lellis [5] to the understanding and the investigation of the second main step in Almgren's result, namely the approximation of minimal currents. We show how this approximation is closely related to an analytical a priori estimate which can be phrased in terms of higher integrability of the excess density (the terminology will be explained below), which in turn depends on a higher integrability property for the gradient of Dir-minimizing Q-valued functions. This work adds a new step in the program of making Almgren's partial regularity result manageable.

Almgren's approximation theorem

In order to illustrate the results, we introduce the following notation. We consider integer rectifiable m-dimensional currents T in some open cylinders:

$$\mathcal{C}_r(y) = B_r(y) \times \mathbb{R}^n \subset \mathbb{R}^m \times \mathbb{R}^n,$$

and denote by $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ the orthogonal projection. We will always assume that the current T is without boundary and, roughly speaking, is a Q covering of the horizontal plane:

$$\pi_\# T = Q\,[\![B_r(y)]\!] \qquad \text{and} \qquad \partial T = 0, \tag{2}$$

where Q is a fixed positive integer. For a current as in (2), we define the basic regularity quantity for the sequel, the *cylindrical excess*:

$$\mathrm{Ex}(T, \mathcal{C}_r(y)) := \frac{\|T\|(\mathcal{C}_r(y))}{\omega_m r^m} - Q, \tag{3}$$

where ω_m is the measure of the m-dimensional unit ball. Finally, we recall from [4] the notation $\mathcal{A}_Q(\mathbb{R}^n)$ for the space of unordered Q-tuples of points in $\mathbb{R}^n$ (the reader unfamiliar with this theory can think to the case $Q = 1$, where $\mathcal{A}_Q(\mathbb{R}^n)$ simply reduces to $\mathbb{R}^n$, and interpret the rest of the notation in the usual way – as we will comment below, Almgren's approximation theorem has some new significant feature also in this special case).

The following is Almgren's approximation theorem and is proved in the third chapter of the big regularity paper [1].

Theorem 3 (Almgren)**.** *There exist constants $C, \delta, \varepsilon_0 > 0$ with the following property. Assume T is an area-minimizing, integer rectifiable m-dimensional current T in $\mathcal{C}_4$ satisfying* (2)*. If $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$, then there exist a Q-valued function $f \in \mathrm{Lip}(B_1, \mathcal{A}_Q(\mathbb{R}^n))$ and a closed set $K \subset B_1$ such that*

$$\mathrm{Lip}(f) \le CE^{\delta}, \tag{4a}$$

$$\mathrm{graph}(f|_K) = T\llcorner(K \times \mathbb{R}^n) \quad \textit{and} \quad |B_1 \setminus K| \le CE^{1+\delta}, \tag{4b}$$

$$\left|\mathbf{M}\big(T\llcorner\mathcal{C}_1\big) - Q\,\omega_m - \int_{B_1} \frac{|Df|^2}{2}\right| \le C\,E^{1+\delta}. \tag{4c}$$

Theorem 3 has been proved by De Giorgi in the case $n = Q = 1$. In its generality, the main aspects of this result are two: the use of multiple valued functions (necessary when $n > 1$, as for the case of branched complex varieties outlined above) and the gain of a small power E^{δ} in the three estimates (4). Regarding this last point, we recall that, for general codimension, the usual Lipschitz approximation theorems cover the case $Q = 1$ and stationary currents, and give an estimate with $\delta = 0$.

The proof of Theorem 3 can be deduce as a consequence of a general approximation scheme for integer rectifiable currents and a key estimate proved by Almgren. To state them, we introduce the following further notation:

$$\mathfrak{e}_T(A) := \mathbf{M}(T\llcorner A \times \mathbb{R}^n) - Q\,|A| \quad \text{for every Borel } A \subseteq B_{4s}(x),$$

and M_T for the maximal function of the excess,

$$M_T(x) := \sup_{s>0} \mathrm{Ex}(T, \mathcal{C}_s(x)).$$

Proposition 4. *Let T be an integer rectifiable m-dimensional current in $\mathcal{C}_{4s}(x)$ satisfying* (2)*. Set $E = \mathrm{Ex}(T, \mathcal{C}_{4s}(x))$ and $K := \big\{M_T < E^{2\alpha}\big\} \cap B_{3s}(x)$, where $\alpha \in (0, \frac{1}{2m})$. Then, there exists $f \in \mathrm{Lip}(B_{3s}(x), \mathcal{A}_Q(\mathbb{R}^n))$ such that:*

$$\mathrm{graph}(f|_K) = T\llcorner(K \times \mathbb{R}^n), \quad \mathrm{Lip}(f) \le CE^{\alpha},$$
$$|B_{3s}(x) \setminus K| \le C\,E^{-2\alpha}\,\mathfrak{e}_T\big(\{M_T > \eta/2^m\} \cap B_{4s}(x)\big).$$

Proposition 5 (Almgren's strong estimate)**.** *There are constants $\sigma, C > 0$ with the following property. Let T be an area-minimizing, integer rectifiable m-dimensional current T in $\mathcal{C}_4$ satisfying* (2)*. If $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$, then*

$$\mathfrak{e}_T(A) \leq C\, E\left(E^\sigma + |A|^\sigma\right) \quad \textit{for every Borel } \; A \subset B_{4/3}.$$

Proposition 4 is proved using the metric theory of normal currents developed by Ambrosio and Kirchheim [2] and of Q-valued functions proposed in [4], extending to the this context the "gradient truncation" method with the maximal function. In particular, we exploit a modification of the key BV estimate for the slice of normal currents due to Jerrard and Soner.

The proof of Proposition 5, which in [1] is obtained as a consequence of several complicated covering algorithms, can be instead deduce in a simpler way from a new estimate which can be phrased in terms of higher integrability of the excess density.

Higher integrability

Given a current T as in (2), we consider the following quantity which we call *excess density*,

$$\mathfrak{d}_T(x) := \limsup_{s \to 0} \mathrm{Ex}(\mathcal{C}_s(x)).$$

The new a priori estimate concerns the higher integrability of $\mathfrak{d}_T$ under the usual hypothesis on the smallness of the excess. Note that, in principle, the excess density $\mathfrak{d}_T$ is a L^1 function. Our analysis shows that there exists $p > 1$ such that, in the regions where $\mathfrak{d}_T$ is small, its L^p norm is controlled by its L^1 norm, that is the excess.

Theorem 6. *There exist constants $p > 1$ and $C, \varepsilon_0 > 0$ with the following property. Let T be an area-minimizing, integer rectifiable m-dimensional current T in $\mathcal{C}_4$ satisfying* (2)*. If $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$, then*

$$\int_{\{\mathfrak{d}_T \leq 1\} \cap B_2} \mathfrak{d}_T^p \leq C\, E^p. \tag{5}$$

Theorem 6 is the main contribution of paper which allows us to give a shorter and conceptually clearer proof of Theorem 3. Moreover, we think that Theorem 6 may have an independent interest, which could be useful in other situations. Indeed, although in the case $Q = 1$ we know a posteriori that T is a $C^{1,\alpha}$ submanifold in $\mathcal{C}_2$, however, for $Q \geq 2$ this conclusion does not hold and Theorem 6 gives an a priori regularity information. Furthermore, we notice that (5) cannot be improved (except for optimizing the constants p, C and ε_0). For example, for $Q = 2$ and $p = 2$, the conclusion of Theorem 6 is false no matter how ε_0 and C are chosen.

Theorem 6, which in principle does not involve any approximation, is in fact closely linked to the problem of approximating area-minimizing currents. Indeed, its proof depends on two ingredients: the derivation of the harmonic approximation of minimizing currents, in the spirit of the original work of De Giorgi generalized to the case of multiple valued functions; and a higher integrability property of the gradient of Dir-minimizing Q-valued functions.

Proposition 7 (Harmonic approximation)**.** *For every $\eta > 0$, there exists $\varepsilon_1 > 0$ with the following property. Let T be a rectifiable, area-minimizing m-dimensional current in $\mathcal{C}_{4s}(x)$ satisfying (2). If $E = \mathrm{Ex}(T, \mathcal{C}_{4s}(x)) \leq \varepsilon_1$ and f is the approximation in Proposition 4, then*

$$\int_{B_{2s}(x)\setminus K} |Df|^2 \leq \eta\, E\, s^m,$$

and there exists a Dir*-minimizing $w \in W^{1,2}(B_{2s}(x), \mathcal{A}_Q(\mathbb{R}^n))$ such that*

$$\int_{B_{2s}(x)} \big(|Df| - |Dw|\big)^2 \leq \eta\, E\, s^m.$$

Proposition 8. *Let $\Omega' \subset\subset \Omega \subset\subset \mathbb{R}^m$ be open domains. Then, there exist $p > 2$ and $C > 0$ such that*

$$\|Du\|_{L^p(\Omega')} \leq C\, \|Du\|_{L^2(\Omega)} \quad \textit{for every } \mathrm{Dir}\textit{-minimizing } u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)).$$

Notice that curiously, though Almgren's monograph contains statements about the energy of Dir-minimizing functions in various regions, Proposition 8 is stated nowhere and there is no hint to higher integrability.

Proposition 7 and Proposition 8 together imply the following estimate, which leads to Theorem 6 via an elementary higher integrability paradigm.

Proposition 9. *For every $\kappa > 0$, there exists $\varepsilon_2 > 0$ with the following property. Let T be an area-minimizing, integer rectifiable m-dimensional current in $\mathcal{C}_{4s}(x)$ satisfying (2). If $E = \mathrm{Ex}(T, \mathcal{C}_{4s}(x)) \leq \varepsilon_2$, then*

$$\mathfrak{e}_T(A) \leq \kappa\, E s^m \quad \textit{for every Borel } A \subset B_s(x) \textit{ with } |A| \leq \varepsilon_2\, s^m.$$

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Equilibrium configurations of epitaxially strained crystalline films

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(joint work with I. Fonseca, G. Leoni, and M. Morini)

We present some recent results on the equilibrium configurations of a variational model for the epitaxial growth of a thin film on a thick substrate introduced by Bonnetier–Chambolle in [1]. In the model only two dimensional morphologies are considered corresponding to three-dimensional configurations. The reference configuration of the film is

$$\Omega_h = \{ z = (x,y) \in \mathbb{R}^2 : \, 0 < x < b, \, 0 < y < h(x) \} ,$$

where $h : [0,b] \to [0,\infty)$ and its graph Γ_h represents the free profile of the film. Denoting by $u : \Omega_h \to \mathbb{R}^2$ the planar displacement of the film with respect to the reference configuration, the strain is given by

$$E(u) = \frac{1}{2}(\nabla u + \nabla^T u)$$

and the energy associated to a smooth configuration (h,u) is

$$G(h,u) = \int_{\Omega_h} \Big[\mu |E(u)|^2 + \frac{\lambda}{2}(\mathrm{div} u)^2 \Big] dz + \sigma_f \mathcal{H}^1(\Gamma_h) + (\sigma_s - \sigma_f)\mathcal{H}^1(\Gamma_h \cap \{y=0\})$$

where μ and λ represent the *Lamé coefficients* of the film, σ_f is the surface tension on the profile, σ_s the surface tension on the 'exposed' part of the substrate, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Here we assume that $\sigma_s > \sigma_f$, corresponding to the so-called 'wetting regime'. One seeks to minimize G among all configurations (h,u) such that $h(0) = h(b)$, $u(x,0) = e_0(x,0)$, for $0 < x < b$, $e_0 > 0$, $u(b,y) = u(0,y) + e_0(b,0)$ for $0 < y < b$, satisfying the volume constraint $|\Omega_h| = d > 0$.

However, smooth minimizing sequences may converge to irregular configurations, where the profile h is just a lower semicontinuous function of bounded variation. In particular, the extended graph of h may contain vertical segments and cuts. Let us denote by X the class of all reachable configurations (h,u), i.e., the class of all configurations such that $h : \mathbb{R} \to [0,\infty)$ is a b-periodic lower semicontinuous function of finite total variation in $(0,b)$ and $u \in H^1_{\mathrm{loc}}(\Omega_h;\mathbb{R}^2)$ satisfies the Dirichlet boundary condition $u(x,0) = e_0(x,0)$ and the periodicity assumption $u(b,y) = u(0,y) + e_0(b,0)$. It has been proved in [1] (see also [2] for a variant of the model) that the relaxed energy associated to any pair $(h,u) \in X$ is given by

$$F(h,u) = \int_{\Omega_h} \Big[\mu |E(u)|^2 + \frac{\lambda}{2}(\mathrm{div} u)^2 \Big] dz + \sigma_f \mathcal{H}^1(\Gamma_h) + 2\sigma_f \mathcal{H}^1(\Sigma_h) ,$$

where

$$\Gamma_h = \{(x,y) : \, 0 \le x < b, h^-(x) \le y \le h^+(x)\},$$

$$\Sigma_h = \{(x,y) : \, 0 \le x < b, \, h(x) \le y < h^-(x)\}$$

Here, $h^-(x) = \min\{h(x-), h(x+)\}$, $h^+(x) = \max\{h(x-), h(x+)\}$, and $h(x\pm)$ denote the right and left limit at x. Notice that in the representation formula for F

the *vertical cracks* (contained in Σ_h) are counted twice since they arise as limit of regular profiles. With this formula at hand one has (see [1]) the following existence result.

Theorem 1. *The minimum problem*

$$\min\{F(g,v):\,(g,v)\in X,\,|\Omega_g|=d\} \tag{1}$$

has always a solution for any $d>0$.

We say that an admissible configuration $(h,u)\in X$ is a *local minimizer for F* if there exists $\delta>0$ such that

$$F(h,u)<F(g,v)$$

for all pairs $(g,v)\in X$, with $|\Omega_g|=|\Omega_h|$, such that $0<d_H(\Gamma_h\cup\Sigma_h,\Gamma_g\cup\Sigma_g)<\delta$. Here, for any two subsets A,B in $\mathbb{R}^2$, $d_H(A,B)=\inf\{\varepsilon>0: B\subset\mathcal{N}_\varepsilon(A)\text{ and }A\subset\mathcal{N}_\varepsilon(B)\}$, where $\mathcal{N}_\varepsilon(A)$ denotes the ε-neighborhood of A. The use of d_H in measuring how far g is from h is due to the presence of the vertical cracks which are not seen by other kinds of possible distances such as the L^1 or the L^∞ one. However, if h is continuous, requiring that $d_H(\Gamma_h\cup\Sigma_h,\Gamma_g\cup\Sigma_g)$ is small is equivalent to requiring that $\sup\{|h(x)-g(x)|:\,0\le x\le b\}$ is small.
In order to state the regularity result proved in [2] we need another definition. We say that $(x,h^-(x))$, $x\in[0,b)$, is an *inward cusp point* if $g^-(x)=g(x)$ and $g'(x+)=-g'(x-)=+\infty$. The set of all cusp points in $[0,b)$ will be denoted by $\Sigma_{h,c}$.

Theorem 2. *Let $(h,u)\in X$ be a local minimizer for F. Then*

(i) *cusp points and vertical cracks are at most finite in $[0,b)$, i.e.,*

$$\operatorname{card}\big(\{x\in[0,b):\,(x,y)\in\Sigma_h\cup\Sigma_{h,c}\ \textit{for some}\ y\ge 0\}\big)<+\infty\,;$$

(ii) *the curve Γ_h is of class C^1 away from $\Sigma_h\cup\Sigma_{h,c}$;*
(iii) *$\Gamma_h\cap\{h>0\}$ is of class $C^{1,\alpha}$ away from $\Sigma_h\cup\Sigma_{h,c}$ for all $\alpha\in(0,1/2)$;*
(iv) *let $A:=\{x\in\mathbb{R}:\,h(x)>0$ and h is continuous at $x\}$. Then A is an open set of full measure in $\{h>0\}$ and h is analytic in A.*

Notice that statement (ii) of Theorem 2 implies in particular the so-called *zero contact angle condition* (that is $h'=0$) at the interface between film and substrate. We remark also that the regularity results in [2] refer to a slightly different model than the one considered here and to a slightly stronger notion of local minimality. However they apply also to the model under discussion.
We now come to the qualitative properties of solutions. The results presented here will appear in the forthcoming paper [3]. A first issue that will be discussed in the paper is to find sufficient conditions, based on a suitable notion of second variation for F, for an admissible configuration to be a local minimizer. To this aim, given a pair $(h,u)\in X$, with $h\in C^2([0,b])$, we say that (h,u) is a *critical point* for F if

it satisfies the following set of Euler-Lagrange equations:

$$
(2)\qquad \begin{cases} \mu\Delta u + (\lambda+\mu)\nabla(\mathrm{div}u) = 0 & \text{in } \Omega_h,\\ N(u)[\nu] = 0 & \text{on } \Gamma_h \cap \{y>0\},\\ N(u)(0,y)[\nu] = -N(u)(b,y)[\nu] & \text{for } 0<y<h(0)=h(b),\\ k+\mu\big|E(u)|^2 + \frac{\lambda}{2}(\mathrm{div}u)^2 = \mathrm{const} & \text{on } \Gamma_h \cap \{y>0\}, \end{cases}
$$

where $N(u) = \mu\big(\nabla u + \nabla^T u\big) + \lambda I \mathrm{div} u$, ν is the exterior normal to Ω_h and k is the curvature of Γ_h (here and in the following we assume $\sigma_f = 1$). From the definition of F one has immediately that any sufficiently smooth local minimizer satisfies (2), hence is a critical point. Notice also that the *flat configuration* (h, u_0) *of volume* d, where

$$
h \equiv \frac{d}{b}, \qquad u_0(x,y) = e_0\Big(x, \frac{-\lambda}{2\mu+\lambda}y\Big),
$$

is always a critical point, i.e., satisfies (2). The first result proved in [3] deals with the local minimality of the flat configuration. In order to state it we need to introduce the *Grinfeld function* K defined (see [4]) for $y \geq 0$ as

$$
(3)\quad K(y) = \max_{n\in\mathbb{N}} \frac{1}{n} J(ny), \quad \text{where } J(y) := \frac{y + (3-4\nu_p)\sinh y \cosh y}{4(1-\nu_p)^2 + y^2 + (3-4\nu_p)\sinh^2 y},
$$

ν_p being the *Poisson modulus* of the elastic material, i.e., $\nu_p = \frac{\lambda}{2(\lambda+\mu)}$.

Theorem 3. *Let* $d_{\mathrm{loc}} : (0,+\infty) \to (0,+\infty]$ *be defined as* $d_{\mathrm{loc}}(b) := +\infty$*, if* $0 < b \leq \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$*, and as the solution to*

$$
(4)\qquad K\Big(\frac{2\pi d_{\mathrm{loc}}(b)}{b^2}\Big) = \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}\frac{1}{b},
$$

otherwise. Then the flat configuration $(d/b, u_0)$ *is a local minimizer for* F *if* $0 < d < d_{\mathrm{loc}}(b)$*.*
The threshold d_{loc} *is critical: indeed, for* $d > d_{\mathrm{loc}}(b)$ *there exists* $(g,v) \in X$*, with* $|\Omega_g| = d$*, and* $d_H(\Gamma_{d/b}, \Gamma_g \cup \Sigma_g)$ *arbitrarily small such that* $F(g,v) < F(d/b, u_0)$*.*

A crucial point in the proof of Theorem 3 is a local minimality criterion, based on the positive definiteness of a suitable notion of second variation of F. To define it, let us consider a critical point $(h,u) \in X$, with $h \in C^\infty([0,b])$, $h > 0$. Given a variation $\psi \in H^1(0,b)$, $\psi(0) = \psi(b)$, with $\int_0^b \psi\, dx = 0$, for $|t|$ small we set $h_t = h + t\psi$ and u_t the corresponding minimizer of the elastic energy in Ω_{h_t} under the usual Dirichlet and periodicity assumptions. Thus $(h_t, u_t) \in X$ and $|\Omega_h| = |\Omega_{h_t}|$. The second variation of F at (h,u) along the direction ψ is then defined as

$$
(5)\qquad \frac{d^2}{dt^2} F(h_t, u_t)_{|t=0}\,.
$$

We say that the second variation at (h,u) is *positive definite* if (5) is positive for all $\psi \neq 0$.

Theorem 4. *Let $(h, u) \in X$ be a critical point for F, with $h \in C^\infty([0, b])$ and $h > 0$, and assume that the second variation of F at (h, u) is positive definite. Then (h, u) is a local minimizer.*

To the best of our knowledge, this result is the first example of a local minimality criterion based on the second variation in the framework of free boundary problems and we believe that many of the ideas introduced in [3] can be used in a large number of similar variational problems.
To conclude, we state a result dealing with the global minimality properties of the flat configuration. This theorem, as well as other qualitative properties of non-flat minimizers, is also contained in the forthcoming paper [3].

Theorem 5. *The following two statements hold.*

(i) *For every $b > 0$, there exists $0 < d_{\mathrm{glob}}(b) \le d_{\mathrm{loc}}(b)$ (see Theorem 3) such that the flat configuration $(d/b, u_0)$ is a global minimizer if and only $0 < d \le d_{\mathrm{glob}}(b)$. Moreover, if $0 < d < d_{\mathrm{glob}}(b)$, then $(d/b, u_0)$ is the unique global minimizer.*

(ii) *There exists $0 < b_{\mathrm{crit}} \le \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$ such that $d_{\mathrm{glob}}(b) = +\infty$ if and only if $0 < b \le b_{\mathrm{crit}}$, i.e., the flat configuration $(d/b, u_0)$ is the unique global minimizer for all $d > 0$ if and only if $0 < b \le b_{\mathrm{crit}}$.*

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Lack of uniqueness for weak solutions of the incompressible porous media equation

Daniel Faraco

The aim of the talk is to investigate the question of existence and uniqueness for weak solutions to equations describing the motion of an active scalar transported by an incompressible flow. The method is based on the approach by DeLellis and Szkelyhidi [6] to construct wild solutions to the Euler system, understanding it as a differential inclusion. The main result concerns the incompressible porous media equation.

The incompressible 2-D porous media equation (IPM) is described by

$$\rho_t + \nabla \cdot (v\rho) = 0$$

where the scalar $\rho(x,t)$ is the density of the fluid. The incompressible velocity field

$$\nabla \cdot v = 0$$

is related with the density by the well-known Darcy's law [1]

$$\frac{\mu}{\kappa} v = -\nabla p - (0, g\rho)$$

where μ represents the viscosity of the fluid, κ is the permeability of the medium, p is the pressure of the fluid and g is acceleration due to gravity. Without lost of generality we will consider $\mu/\kappa = g = 1$.

For initial data in the Sobolev class $H^s(\mathbb{T}^2)$ $(s > 2)$ there is local-existence and uniqueness of solutions in a classical sense and global existence is an open problem [4]. It is known the existence of weak solutions, where the motion takes place in the interface between fluids with different constant densities, modeling the contour dynamics Muskat problem [3]. The existence of weak solutions for general initial data is not known. In this context we emphasize that the solutions we construct satisfy

$$\limsup_{t \to 0^+} \|\rho\|_{H^s}(t) = +\infty$$

for any $s > 0$.

From Darcy's law and the incompressibility of the fluid we can write the velocity as a singular integral operators with respect to the density as follows

$$v(x,t) = PV \int_{\mathbb{R}^2} \Omega(x-y)\, \rho(y,t) dy \; - \frac{1}{2}\,(0, \rho(x))\,, \quad x \in \mathbb{R}^2,$$

where the kernel is of Calderon-Zygmund type.

The integral operator is defined in the Fourier side by

$$\widehat{v}(\xi) = (\frac{\xi_1 \xi_2}{|\xi|^2}, -\frac{(\xi_1)^2}{|\xi|^2})\widehat{\rho}(\xi).$$

This system is analogous to the 2-D surface Quasi-geostrophic equation (SQG) [4], in the sense that is an active scalar that evolves by a nonlocal incompressible velocity given by singular integral operators. It follows that, for Besov spaces, if the weak solution ρ is in $L^3([0,T] \times B_3^{s,\infty})$ with $s > \frac{1}{3}$ then the L^2 norm of ρ is conserved [20]. This result frames IPM in the theory of Onsager's conjecture for weak solutions of 3-D Euler equations [2],[15]. However there is an extra cancelation, for SQG, due to the symmetry of the velocity given by

$$\widehat{v}(\xi) = i(-\frac{\xi_2}{|\xi|}, \frac{\xi_1}{|\xi|})\widehat{\rho}(\xi)$$

that provides global existence for weak solution with initial data in $L^2(\mathbb{T}^2)$ [16]. Furthermore, one can find a substantial difference between both systems for weak solutions of constant ρ in complementary domains, denoted in the literature as patches [13]. For IPM the Muskat problem presents instabilities of Kelvin-Helmholtz's type [3] and there is no instabilities for SQG ([17],[8]).

We remark that in contrast with IPM, the question of uniqueness of weak solutions for SQG remains open. The method breaks because there is no obvious way to put SQG in the compensated compactness framework, that is as a linear local PDE and a pointwise constraint. However for IPM we have

Theorem 1. *For every $T > 0$ there exists infinitely many non trivial weak solutions $(\rho, v) \in L^\infty(\mathbb{T}^2 \times [0,T])$ to the 2D IPM system such that $\rho(x, 0) = 0$.*

As we said the method of the proof follows the lines of [6]. However there are some differences which might be of interest in related problems. We conclude the report with some remarks.

- It is needed to work with the genuine Λ hull.
- The natural variable $q = \rho v$ yields a set K with the unpleasant property $K \in \partial K^\lambda$. There is two ways to solve this. The first is to notice that for the issue of weak solutions it is enough that the original variables (ρ, v) attain the correct boundary values. The second (pointed out by Székelyhidi) is that for solutions with $|\rho| = 1$ there is a way to symmetrize the equation which bypass this difficulty.
- We do not need to compute K^Λ for a suitable K but instead we work with degenerate $T4$ configurations. In some sense, this translates the difficulty from Λ convexity to standard convexity.

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The Surgery and Level-Set Approaches to Mean Curvature Flow

John Head

Given a smooth hypersurface immersion $F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$, the evolution of $\mathcal{M}_0^n = F_0(\mathcal{M}^n)$ by mean curvature flow is the one-parameter family of smooth immersions $F : \mathcal{M}^n \times [0,T) \to \mathbb{R}^{n+1}$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = \overrightarrow{H}(p,t) = -H(p,t)\nu(p,t), \quad p \in \mathcal{M}^n, t \geq 0, \tag{1}$$

$$F(\cdot,0) = F_0, \tag{2}$$

where $\overrightarrow{H}(p,t)$, $H(p,t)$ and $\nu(p,t)$ denote the mean curvature vector, mean curvature and outer unit normal respectively at the point $F(p,t)$ on the surface $\mathcal{M}_t^n = F(\cdot,t)(\mathcal{M}^n)$.

It is a well-known theorem due to Huisken that in dimensions two and higher any convex initial data will contract smoothly to a point in finite time and in an asymptotically round fashion. In the more general two-convex setting (with $n \geq 3$), in which the sum of any two of the principal curvatures is non-negative, Huisken and Sinestrari [4] make precise the intuitive picture that unless the surface is uniformly convex, any high-curvature region must contain a neck - that is, a piece of the surface which can be represented (up to a homothety) as a graph over a cylinder with small C^k-norm for a suitable k. They furthermore define a surgery algorithm according to which the smooth flow is stopped shortly before the singular time, and each neck is excised and replaced with spherical caps.

Huisken and Sinestrari introduce a set of parameters $H_0 < H_1 < H_2 < H_3$ which determine when and where surgery is performed. In particular, when the curvature exceeds a certain value $H_0 = H_0(\mathcal{M}_0^n)$ the geometry of the surface is controlled by *a priori* curvature estimates. The smooth flow is then stopped when the curvature reaches a maximum value H_3, and surgery is performed away from the point of maximum curvature at a smaller scale $H_1 = \xi H_3$ ($\xi = \xi(\mathcal{M}_0^n) < 1$) such that the maximum of the curvature after surgery drops by a fixed factor to H_2.

The starting point for the work in [3] is the observation made by Huisken and Sinestrari that H_3 is not unique - it can in fact be chosen arbitrarily large. For fixed H_0 and ξ, we therefore consider an increasing sequence of surgery parameters $H_{3,i}$, corresponding to a whole sequence of mean curvature flows with surgeries along which the surgery times approach the singular time and necks removed during surgery become smaller and smaller. In this lecture we present results from [3] for mean curvature flow with surgeries which establish that

Theorem *The $L^p(\mathcal{M}^n)$-norm of H is bounded on any finite time interval for all $p < n-1$.*

This result has an interesting geometric interpretation and can be used to improve the bound from [4] on the required number of surgeries. The dependence of the estimates on the surgery parameters is made explicit.

There is a well-developed theory of weak solutions available in the literature (see for example [1, 2]) which provides an alternative approach to extending mean curvature flow beyond the singular time. While the solution of mean curvature flow with surgeries is smooth and only solves the initial value problem (1)-(2) up to small errors resulting from surgery, the appropriately defined weak solution is canonical and of course possesses weaker regularity properties. Both concepts can, however, be discussed within the level-set framework of weak solutions. We control the position of the surgery solution relative to the weak solution using techniques introduced by Brakke, and it then follows from the above theorem that

Corollary *The sequence of mean curvature flows with surgeries described above will converge in the limit as $H_{3,i} \longrightarrow \infty$ to the unique weak solution of the level-set flow.*

We discuss several types of convergence and show that the rate of convergence is again controlled explicitly in terms of the surgery parameters. The novelty of this result is that it can be used to establish regularity properties for the weak solution. We note that a version of the corollary above was recently obtained by Lauer in [5].

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Lorentzian analogues of some classical variational problems

Robert L. Jerrard

A classical line of research in the calculus of variations seeks to establish relationships between variational elliptic equations of the form

$$-\Delta u + \frac{1}{\varepsilon^2} f(u) = 0 \tag{1}$$

for suitable nonlinearities f, and variational geometric problems involving the mean curvature. Some (mostly) well-known examples include the following:

1. Suppose that u is a scalar function on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and f is the derivative of a symmetric double-well potential, eg $f(s) = 2(s^2-1)s = F'(s)$ for $F(s) = \frac{1}{2}(s^2-1)^2$.

Then (1) is related to the problem of minimal hypersurfaces in Ω. This relationship manifests itself in several ways. For example, under suitable conditions, solutions of (1) have an interface that is approximately a minimal surface, or exhibit energy concentration around a minimal surface. Moreover, given a minimal surface, one can find solutions of (1) that are "nearby" in a precise sense.

1'. More generally, suppose that u is as above, and that f is the derivative of a double-well potential with two wells of (in general) *unequal* depth, eg $f(s) = (s^2-1)(2s-\varepsilon\kappa)$. Then $f(s) = F'(s)$ for a function $F(s)$ with a global minimum at $s=-1$ and a local minimum at $s=1$, with $F(1) - F(-1) = O(\varepsilon\kappa)$.

Then (1) is related to hypersurfaces of constant mean curvature κ. The relationships between the PDE and the geometric problem parallel those described above in the case $\kappa = 0$

2. Suppose that $\Omega \subset \mathbb{R}^n$ for some $n \geq 3$, that u takes values in $\mathbb{R}^2$, and that $f = \nabla F$ for some potential $F : \mathbb{R}^2 \to [0,\infty)$ that vanishes exactly on the unit circle. The model example is $f(s) = 2(|s|^2-1)s = \nabla F(s)$, for $F(s) = \frac{1}{2}(|s|^2-1)^2$.

In this case (1) is related to the problem of *codimension 2* minimal surfaces in Ω. For example, solutions with suitably bounded energy exhibit energy concentration around codimension 2 minimal surfaces.

2'. Related problems include the situation where u and f are as above, and (1) is coupled to an equation for a magnetic field. In this case again the PDE is associated to the problem of codimension 2 minimal surfaces, in ways that closely parallel the case **2** described above in which the magnetic field is absent.

3. Generalizations of case **2'** include more general gauge theories, such as various forms of the elliptic Yang-Mills-Higgs system. These are expected to be related to minimal surface problems of various codimensions, depending on the model under consideration, although not many results in this direction are established.

Lorentzian analogues of the above problems are obtained by replacing the Laplace operator $-\Delta$ by the wave operator $\Box := \partial_t^2 - \Delta$, or more generally by the Laplace-Beltrami operator $\Box_g$ on a Lorentzian manifold (M, g). The basic example of the wave operator of course corresponds to the case when the manifold is just a Minkowski spacetime. Focusing for simplicity on the Minkowskian case, one might then ask whether there are ways in which suitable semilinear wave equations are related to geometric problems involving surfaces of vanishing or prescribed *Minkowskian* mean curvature. Parallel to the elliptic problems discussed above, we will focus on the case where $k = 1$ or 2 and $n > k$.

The slight existing literature on $0 < \varepsilon \ll 1$ asymptotics of scaled semilinear wave equations deals mainly with the quite different situation in which energy concentrates around points rather than submanifolds. This can occur for $n = k = 2$ with $f(u) = (|u|^2 - 1)u$, see [3, 6, 2], or for $n \geq 2$ when f is a nonlinearity of focussing type, see [7]. A recent result [1] on scattering of flat kinks in certain nonlinear wave equations addresses related issues.

We first state a result that provides a hyperbolic analogue of those discussed in **1** and **1'** above. Consider the nonlinear wave equation

$$-\Box u + \frac{1}{\varepsilon^2}(u^2 - 1)(2u - \varepsilon\kappa) = 0, \qquad u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}. \tag{2}$$

Theorem 1. *Assume that $1 \leq N \leq 4$, and let $\Gamma \subset (-T, T) \times \mathbb{R}^N$ be a smooth timelike hypersurface of constant Minkowskian mean curvature κ.*

Then given $T_0 < T$, there exists a neighborhood $\mathcal{N}$ of Γ in $(-T_0, T_0) \times \mathbb{R}^N$ in which there exists a smooth solution $d : \mathcal{N} \to \mathbb{R}$ of the problem

$$d = 0 \ \text{on } \Gamma, \qquad -d_t^2 + |\nabla d|^2 = 1 \ \text{near } \Gamma. \tag{3}$$

(In other words, d is the signed Minkowski *distance to Γ.) Moreover, there exists a solution u of (2) such that for any $T_0 < T$,*

$$\|u - \tanh(\frac{d}{\varepsilon})\|_{L^2(\mathcal{N})} \leq C\sqrt{\varepsilon}. \tag{4}$$

In addition,

$$\int_{\mathcal{N}} \frac{1}{2}\bar{d}^2 \left[(u_t^2 + |\nabla u|^2) + \frac{1}{\varepsilon^2}(u^2 - 1)^2\right] dt\ dx \leq C\varepsilon \tag{5}$$

$$\text{for } \bar{d}(t,x) := \begin{cases} d(t,x) & \text{if } (t,x) \in \mathcal{N} \\ 1 & \text{otherwise.} \end{cases}$$

Finally,

$$\|\kappa_1 \mathcal{T}_\varepsilon(u) - \mathcal{T}(\Gamma)\|_{W^{-1,1}((-T_0,T_0)\times\mathbb{R}^N)} \leq C\varepsilon. \tag{6}$$

for a normalization constant κ_1, where

$$\mathcal{T}^{\varepsilon}_{\alpha\beta}(u) := \eta_{\alpha\beta}\left(\frac{\varepsilon}{2}\eta^{\gamma\delta}u_{x^\gamma}u_{x^\delta} + \frac{1}{2\varepsilon}(u^2-1)^2\right) - \varepsilon u_{x^\alpha}u_{x^\beta}$$

with $(\eta^{\alpha\beta}) = (\eta_{\alpha\beta}) := diag(-1,1,\ldots,1)$*; and* $\mathcal{T}(\Gamma)$ *is the measure-valued tensor defined by*

$$\mathcal{T}_{\alpha\beta}(\Gamma)(A) := \int_A P_{\alpha\beta}(t,x)\ d\lambda_\Gamma.$$

Here λ_Γ *denotes the Minkowski area density of* Γ*, and where* $P(t,x) = (P_{\alpha\beta}(t,x))$ *is the tensor corresponding to Minkowski orthogonal projection onto* $T_{(t,x)}\Gamma$*, for* λ_Γ *a.e.* $(t,x) \in \Gamma$*.*

In all the above conclusions, $C = C(T_0, \Gamma)$ *is independent of* ε*.*

The theorem as stated combines results from [4] and [5]. The former paper studies the case $\kappa = 0$ under the restriction that the surface Γ be homeomorphic to $(-T,T)\times\mathbb{T}^N$. The latter paper considers general $\kappa \in \mathbb{R}$ (and in fact proves similar results when κ is a smooth function of (t,x)) and does not impose any restrictions on the topology of Γ. To minimize technicalities, however, [5] considers hypersurfaces Γ that are stationary at time $t = 0$, and also does not carry out in detail the rather minor adaptations to the case $\kappa \neq 0$ of the lengthy arguments in [4] in which conclusions such as (6) are extracted from more basic estimates.

The restriction $1 \le N \le 4$ in Theorem 1 is needed only to assure that equation (2) is well-posed in the energy space. Corresponding results are valid in arbitrary dimensions, if one is willing to modify the nonlinearity as necessary to guarantee global well-posedness, and correspondingly modify certain auxiliary quantities.

The case **2** is also addressed in [4], which proves results describing energy concentration around codimension 2 timelike submanifolds of zero Minkowskian mean curvature in solutions, for well-prepared initial data, of the equation

$$-\Box u + \frac{1}{\varepsilon^2}(|u|^2-1)u = 0, \qquad u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^2, \quad N = 3 \text{ or } 4.$$

These results also hold for equations in arbitrary dimensions $n \ge 3$ with qualititatively similar nonlinearities satisfying appropriate growth conditions. And in joint work with M. Czubak, we are currently developing analogous results for the Abelian Higgs model as well as certain nonabelian gauge theories. These results will provide Lorentzian analogues of the elliptic phenomena discussed in **2'** and **3** above, respectively.

In every case, the argument starts with a change of variables that amounts to rewriting the equation in variables that follow a given minimal surface or surface of prescribed curvature, as the case may be. One then needs to show stability of well-chosen initial data for the transformed dynamics. In every case, the fact that the change of variables is built around a submanifold with the "correct" geometry endows the transformed equation with good structural properties that render this stability analysis possible.

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