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Operator Theory and Harmonic Analysis

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Abstract. The major topics discussed in this workshop were the Feichtinger conjecture and related questions of harmonic analysis, the corona problem for the ball $\mathbb{B}^n$, the weighted approximation problem, and questions related to the model spaces, to multipliers, (hyper-)cyclicity, differentiability, Bezout and Fermat equations, traces and Toeplitz operators in different function spaces. A list of open problems raised at this workshop is also included.

Mathematics Subject Classification (2010): 26B35, 30D15, 30E10, 30H80, 31C25, 46E22, 46L30, 47A16, 47B35, 94A12.

Introduction by the Organisers

The "Paving Conjecture" of Kadison–Singer claims that given any linear contraction whose diagonal matrix elements are zero with respect to a given basis, there is a partition of the basis in a universal number of pieces such that the compression of the operator to the span of each piece of the partition (or the minor matrix generated by the piece, if one prefers) has norm bounded above by one half. Recently, it became clear that this conjecture is equivalent to several others including a conjecture generalizing the Bourgain–Tzafriri Restricted Invertibility Theorem and the Feichtinger conjecture, which claims that every frame in a Hilbert space is a finite union of Riesz basis sequences. In many reproducing kernels Hilbert spaces like the Hardy space and its model subspaces, the de Branges spaces, the Smirnov spaces, the Bergman space, the Fock space, and their weighted analogs, questions on frames and Riesz basis sequences of reproducing kernels are reformulated as deep problems concerning uniqueness, interpolation and sampling properties of analytic functions. See, for example, the work by Seip–Wallstén on the Fock space and the 2007 paper by Borichev–Dhuez–Kellay on the weighted Fock spaces and the references therein. Frequently, such problems are related to harmonic analysis applications, from wavelet theory, time-frequency analysis (Gabor frames), operator theory (spectral theory of Toeplitz operators), and systems theory (via, for example, spectral theory of Hankel operators).

Therefore, the Feichtinger conjecture was one of the main topics of the workshop. In the first talk of the workshop, P. Casazza described numerous equivalent reformulations of the Kadison–Singer problem and proposed a method to construct a counter-example using the Laurent operators. V. Vasyunin described his results on trace H^∞-algebras giving a negative answer to a stronger form of the Bourgain–Tzafriri restricted invertibility conjecture. N. Lev presented his results on the Riesz bases of exponentials with restrictions on the exponents on a finite union of intervals.

Yu. Lyubarskii described the asymptotics of the sampling constants (the condition number) for lattice families of reproducing kernels in the Fock space when the area of the fundamental domain of the lattice approaches the critical one. Yu. Belov discussed the systems biorthogonal to exact systems of reproducing kernels in Hilbert spaces of analytic functions. In particular, he answered an old question by Nikolski and constructed a model subspace and an exact system of reproducing kernels there with a non-complete biorthogonal system.

The model spaces K_Θ play an important role in complex analysis, harmonic analysis and Mathematical Physics. The term was coined by N. Nikolski a long time ago and is suggested by the Szőkefalvi-Nagy–Foiaş model theory of contractions on a Hilbert space. In the simplest (scalar) case, K_Θ is the orthogonal complement in the Hardy space H^2 of the Beurling shift invariant subspaces ΘH^2, where Θ is an inner function. The central result of Sz. Nagy–Foiaş theory is a theorem describing all contractions of a Hilbert space as compressions to a K_Θ of a unilateral shift operator. This theorem yields a functional model of a general, abstract linear operator, whence the term of a model space. Model spaces play an important role in approximation theory, too. The first results of these are due to Douglas, Shapiro, Shields and Tumarkin. Also, they are closely related to the phenomenon of pseudo-analytic continuation and to de Branges spaces of entire functions. We note that a de Branges space is isometric to a model subspace K_B where B is a meromorphic Blaschke product. An interesting open question is whether the model space K_Θ possesses an unconditional basis of reproducing kernels. This is related to a hard problem of J. B. Garnett and P. W. Jones on whether each inner function can be uniformly approximated by interpolating Blaschke products.

Motivated by recent work of Sarason, A. Baranov presented a variety of results concerning the truncated Toeplitz operators $P_\Theta M_\phi$, where P_Θ is the projector onto the model space K_Θ.

Another major topic of the meeting was the weighted approximation. J. Brennan discussed the relations between uniform rational approximation and L^p-polynomial approximation. H. Hedenmalm proved a uniqueness theorem for the Fourier transforms of measures with support on a hyperbola, related to the Klein–Gordon equation. A. Poltoratski discussed the type and the gap problems in weighted L^p spaces and their relations to the kernels of the Toeplitz operators.

One more topic of interest during the workshop was the corona problem in $\mathbb{B}^n$. B. Wick discussed BMO estimates for this problem using the Koszul complex technique, whereas T. Trent presented his results on the operator version of the corona problem for some multiplier spaces on $\mathbb{B}^n$.

R. Rochberg discussed in his talk geometrical (shape) structures associated with reproducing kernel Hilbert spaces.

N. Arcozzi presented an analog of the Fefferman theorem for the Dirichlet space.

K. Dyakonov presented his results on (local) *abc* theorems for analytic functions.

R.Zarouf discussed analogs of the Kreiss resolvent condition for matrices with restrictions on the spectrum.

J.-F. Olsen proved an F. and M. Riesz theorem for the Hardy space $H^1(\mathbb{T}^\infty)$.

E. Saksman established the optimal estimate for the growth of the frequently hypercylcic (with respect to the differentiation operator) entire functions. Namely, he proved that for every $c > 0$ there exists an entire frequently hypercyclic function f such that $|f(z)| \leq c|z|^{-1/4}e^{|z|}$, $|z| > 1$.

E. Abakumov discussed his results on translation cyclic vectors and generating sets in weighted $\ell^p(\mathbb{Z})$ and $L^p(\mathbb{R})$ spaces.

A. Aleksandrov presented his results on the perturbation (Hölder) smoothness of the functional calculus for the normal operators with respect to the (operator) norm and to the Schatten–von Neumann norm.

A. Nicolau obtained an analog of N. Makarov's result on the differentiability if the Zygmund class for the case $\mathbb{R}^d$, $d > 1$. In particular, he proved that every function in the small Zygmund class is differentiable at a set of points of Hausdorff dimension at least 1.

On Wednesday morning a problem session chaired by E. Saksman had been organized. Most of the problems discussed during that session are included at the end of this report. Further open questions were pointed out in many of the talks.

This workshop was organized by Alexander Borichev (Marseille), Raymond Mortini (Metz), Nicolai Nikolski (Bordeaux) and Kristian Seip (Trondheim). Unfortunately, Raymond Mortini, Nicolai Nikolski, and Kristian Seip were unable to participate. All the participants were grateful for the hospitality and the stimulating atmosphere of the Forschungsinstitut Oberwolfach.

Workshop: Operator Theory and Harmonic Analysis

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Abstracts

On completeness of translates in weighted spaces

Evgeny Abakumov
(joint work with Aharon Atzmon, Sophie Grivaux)

We discuss two types of questions about the completeness of translates in certain weighted spaces of sequences (functions).

The first problem concerns the cyclicity of the shift operator on weighted ℓ^p spaces of sequences on $\mathbb{Z}$.

Let $\omega : \mathbb{Z} \longrightarrow (0, +\infty)$ be a positive weight on $\mathbb{Z}$ such that

$$\sup_{n\in\mathbb{Z}} \frac{\omega(n+1)}{\omega(n)} < +\infty.$$

For $p \geq 1$, define

$$\ell^p_\omega(\mathbb{Z}) = \Big\{(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \ ; \ ||(a_n)||_{\ell^p_\omega(\mathbb{Z})} = \Big(\sum_{n\in\mathbb{Z}} |a_n|^p \omega(n)^p\Big)^{\frac{1}{p}} < +\infty\Big\}.$$

Clearly, the shift operator $S : (a_n)_{n\in\mathbb{Z}} \longmapsto (a_{n-1})_{n\in\mathbb{Z}}$ is bounded on all spaces $\ell^p_\omega(\mathbb{Z})$, $p \geq 1$.

Problem 1. *Given $p \geq 1$, characterize the weights ω for which there exists a sequence $x \in \ell^p_\omega(\mathbb{Z})$ such that the set of the right translates $(S^n x,\ n \geq 0)$ is complete in $\ell^p_\omega(\mathbb{Z})$.*

The above cyclicity problem goes back to works of Shields [9] and Nikolski [8], and is still open. See also [6, 7] for an early discussion of the question and some partial results. Problem 1 is also mentioned in [5], and later on in [10], where a complete characterization of the supercyclic weighted shifts is given.

First, we formulate an abstract result; namely, we give some conditions which guarantee that a bicyclic operator on a Banach space is cyclic.

Let X be a complex separable Banach space, and let T be a bounded linear operator on X. Recall that T is said to be cyclic if there exists a cyclic vector $x_0 \in X$, that is, such that the linear span of the vectors $T^n x_0$, $n \geq 0$, is dense in X. For an invertible operator T on X, $x_0 \in X$ is said to be a bicyclic vector for T if the vectors $T^n x_0$, $n \in \mathbb{Z}$, span a dense subspace of X.

Theorem 2. *Let X be a complex separable Banach space, and let T be a bounded invertible bicyclic operator on X. Suppose that*

(1) *there exists a nonnegative integer k such that $||T^n|| = O(n^k)$, $n \to \infty$;*
(2) *$\log ||T^{-n}||/\sqrt{n}$ tends to zero as $n \to \infty$.*

If $\sigma_p(T^*)$ does not include the unit circle, then T is cyclic.

An analogous result holds for injective (not necessarily invertible) operators.

We conjecture that condition (2) is optimal, so it cannot be replaced by the condition $\log ||T^{-n}|| = O(\sqrt{n})$. The fact that it cannot be replaced by the condition $\log ||T^{-n}|| = O(n/\log n)$ follows from Volberg's theorem [11].

To prove Theorem 2, we use a Baire Category argument for the shift operator acting on certain regular Banach algebras of sequences; given two non-empty open subsets U and V of X, it is possible to show that there exists a polynomial p such that $p(T)U \cap V$ is non-empty.

The following statement can be easily derived from Theorem 2.

Theorem 3. *Let X be a Banach space with separable dual, and let $U : X \to X$ be a surjective isometry. If U is bicyclic, then U is cyclic.*

If X is a Hilbert space, this result follows from [3].

Theorem 3 applies in particular to the shift operator S acting on the unweighted spaces $\ell^p(\mathbb{Z})$, $p > 1$:

Corollary 4. *If $p > 1$, then S is cyclic on $\ell^p(\mathbb{Z})$.*

Corollary 4 was proved independently by Olevskii in 1998 by a different method (unpublished manuscript).

Now we give a partial answer to Problem 1.

Theorem 5. *Let ω be a positive weight such that the sequence $(\omega(n+1)/\omega(n))_{n\in\mathbb{Z}}$ is bounded from above. Suppose that there exist a nonnegative integer k and a submultiplicative sequence $(\rho(n))_{n\geq 0}$ of positive numbers with*

$$\lim_{n\to+\infty} \frac{\log^+ \rho(n)}{\sqrt{n}} = 0$$

such that

(1) $\omega(n) = O(n^k),\ n \to \infty;$
(2) $\omega(-n) = O(\rho(n)),\ n \to \infty.$

If $p > 1$, then S is cyclic on $\ell^p_\omega(\mathbb{Z})$ if and only if

$$\sum_{n\in\mathbb{Z}} \frac{1}{\omega(n)^{p'}} = +\infty,$$

where $1/p + 1/p' = 1$; and S is cyclic on $\ell^1_\omega(\mathbb{Z})$ if and only if $\inf_{n\in\mathbb{Z}} \omega(n) = 0$.

Theorem 5 is a direct corollary from the above mentioned version of Theorem 2 for injective operators.

The second type of problems is related to the completeness of translates in function spaces on the real line.

Let X be a Banach space of functions on $\mathbb{R}$ which is translation-invariant; that is, for every function $f \in X$ and for every $t \in \mathbb{R}$ we have $S_t f \in X$, where the translates $S_t f$ are defined by $S_t f(x) = f(x - t)$.

Definition 6. *A subset Λ of $\mathbb{R}$ is called generating for X if there exists a function $f \in X$ whose Λ-translates $S_\lambda f$, $\lambda \in \Lambda$, span a dense subspace of X.*

Denote by $R(\Lambda)$ the completeness radius of Λ, that is, the supremum of the non-negative numbers r such that $\mathcal{E}_\Lambda$ is complete in $L^2([-r, r])$.

We mention here two theorems.

Theorem 7. *If $w : \mathbb{R} \to [1, +\infty)$ is a submultiplicative weight which satisfies*

$$\int_{\mathbb{R}} \frac{\log w(t)}{1+t^2} < +\infty,$$

then Λ is generating for $L^1_w(\mathbb{R})$ if and only if $R(\Lambda) = +\infty$.

Theorem 7 was proved earlier by Bruna, Olevskii and Ulanovskii [4] for $L^1(\mathbb{R})$ and by Blank [2] for the weighted case. We present a different proof based on a Baire Category argument and on the regularity of the Banach algebra $L^1_w(\mathbb{R})$.

The following result applies to a large class of Banach spaces of functions on the real line.

Theorem 8. *Let X be a separable translation-invariant Banach space of locally integrable functions on $\mathbb{R}$. Suppose that X embeds continuously into the Fréchet space $L^1_{loc}(\mathbb{R})$, and that the space $\mathcal{D}$ of all $\mathcal{C}^\infty$-functions with compact support is densely contained in X. For $t \in \mathbb{R}$, let S_t denote the operator of translation by t on X: $S_t f = f(\cdot - t)$, $\quad f \in X$. Suppose that*

$$\int_{\mathbb{R}} \frac{\log ||S_t||}{1+t^2} < +\infty.$$

If $\Lambda \subset \mathbb{R}$ is such that $R(\Lambda) = +\infty$, then Λ is a generating set for X.

See [1] for the proofs of the above theorems and further results.

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Perturbations of normal operators

Alexei Aleksandrov
(joint work with Vladimir Peller, Denis Potapov, Fedor Sukochev)

1. Main results. Let ω denote a modulus of continuity. Put

$$\omega_*(\delta) \stackrel{\text{def}}{=} \delta \int_\delta^\infty \frac{\omega(t)}{t^2}\,dt, \quad \delta > 0.$$

We denote by $\Lambda_\omega(\mathbb{C})$ the space of functions on $\mathbb{C}$ such that

$$\|f\|_{\Lambda_\omega(\mathbb{C})} \stackrel{\text{def}}{=} \sup_{\zeta \neq \xi} \frac{|f(\zeta) - f(\xi)|}{\omega(|\zeta - \xi|)} < \infty.$$

Put $\Lambda_\alpha(\mathbb{C}) \stackrel{\text{def}}{=} \Lambda_\omega(\mathbb{C})$ with $\omega(\delta) = \delta^\alpha$, where $0 < \alpha < 1$. We denote by $\mathrm{Lip}(\mathbb{C})$ the space $\Lambda_\omega(\mathbb{C})$ with $\omega(\delta) = \delta$.

Theorem 1. *There exists a positive number c such that for every modulus of continuity ω and every $f \in \Lambda_\omega(\mathbb{C})$,*

$$\|f(M) - f(N)\| \le c\,\|f\|_{\Lambda_\omega(\mathbb{C})}\,\omega_*\big(\|M - N\|\big)$$

for arbitrary normal operators M and N.

Corollary 1. *There exists a positive number c such that for every $\alpha \in (0,1)$ and every $f \in \Lambda_\alpha(\mathbb{C})$,*

$$\|f(M) - f(N)\| \le c\,(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha(\mathbb{C})}\,\|M - N\|^\alpha.$$

for arbitrary normal operators M and N.

Corollary 2. *There exists a positive number c such that for every $f \in \mathrm{Lip}(\mathbb{C})$,*

$$\|f(M) - f(N)\| \le c\,\|f\|_{\mathrm{Lip}(\mathbb{C})}\,\|M - N\|\left(1 + \log\frac{\|M\| + \|N\|}{\|M - N\|}\right).$$

for arbitrary normal operators M and N.

Let $\boldsymbol{S}_p$ denote the Schatten–von Neumann class.

Theorem 2. *Let $0 < \alpha < 1$ and $1 < p < \infty$. Then there exists a positive number c such that for every $f \in \Lambda_\alpha(\mathbb{C})$ and for arbitrary normal operators M and N with $M - N \in \boldsymbol{S}_p$, the operator $f(M) - f(N)$ belongs to $\boldsymbol{S}_{p/\alpha}$ and the following inequality holds:*

$$\big\|f(M) - f(N)\big\|_{\boldsymbol{S}_{p/\alpha}} \le c\,\|f\|_{\Lambda_\alpha(\mathbb{C})}\|M - N\|^\alpha_{\boldsymbol{S}_p}.$$

All the results extend to the case of commutators $f(M)R - Rf(M)$ and quasicommutators $f(M)R - Rf(N)$.

For example, Theorem 1 can be reformulated for quasicommutators as follows.

Theorem 1′. *There exists a positive number c such that for every modulus of continuity ω and every $f \in \Lambda_\omega(\mathbb{C})$,*

$$\|f(M)R - Rf(N)\| \le c\|f\|_{\Lambda_\omega(\mathbb{C})}\ \omega_*(\max(\|MR - RN\|, \|M^*R - RN^*\|)).$$

for bounded operators R with $\|R\| = 1$ and arbitrary normal operators M and N.

2. The key inequality. Denote by $C_{\rm b}(\mathbb{C})$ the set of bounded continuous (complex) functions on $\mathbb{C}$.

Theorem 3. *Let $f \in C_{\rm b}(\mathbb{C})$. Suppose that the Fourier transform of f is supported on the disc $\{|\zeta| \le \sigma\}$. Then*

$$\|f(M) - f(N)\| \le \mathrm{const}\,\sigma\,\|M - N\|$$

for arbitrary normal operators M and N with bounded difference.

We define the Haagerup tensor product $C_{\rm b}(\mathbb{C})\hat{\otimes}_{\rm h}C_{\rm b}(\mathbb{C})$ as the set of all functions Φ on $\mathbb{C}\times\mathbb{C}$ that admit a representation

$$\Phi(\zeta,\xi) = \sum_{n\in\mathbb{Z}} \varphi_n(\zeta)\psi_n(\xi), \quad \zeta,\xi \in \mathbb{C} \tag{1}$$

such that $\varphi_n \in C_{\rm b}(\mathbb{C})$, $\psi_n \in C_{\rm b}(\mathbb{C})$ and

$$\left(\sup_{\zeta\in\mathbb{C}}\sum_{n\in\mathbb{Z}}|\varphi_n(\zeta)|^2\right)^{1/2}\left(\sup_{\xi\in\mathbb{C}}\sum_{n\in\mathbb{Z}}|\psi_n(\xi)|^2\right)^{1/2} < \infty. \tag{2}$$

For $\Phi \in C_{\rm b}(\mathbb{C})\hat{\otimes}_{\rm h}C_{\rm b}(\mathbb{C})$, its norm in $C_{\rm b}(\mathbb{C})\hat{\otimes}_{\rm h}C_{\rm b}(\mathbb{C})$ is, by definition, the infimum of the left-hand side of (2) over all representations (1).

The proof of Theorem 3 is based on the following statement.

Lemma. *Let f satisfy the assumptions of Theorem* 3*. Then there exist functions $g, h \in C_{\rm b}(\mathbb{C})\hat{\otimes}_{\rm h}C_{\rm b}(\mathbb{C})$ such that*

$$f(\zeta) - f(\xi) = (\zeta - \xi)g(\zeta,\xi) + (\overline{\zeta} - \overline{\xi})h(\zeta,\xi)$$

and

$$\|g\|_{C_{\rm b}(\mathbb{C})\hat{\otimes}_{\rm h}C_{\rm b}(\mathbb{C})} + \|h\|_{C_{\rm b}(\mathbb{C})\hat{\otimes}_{\rm h}C_{\rm b}(\mathbb{C})} \le \mathrm{const}\,\sigma\|f\|_{L^\infty(\mathbb{C})}.$$

This lemma also implies the quasicommutator version of Theorem 3.

Note that the case of self-adjoint operators M and N was considered in [3], [4], [5] and [6].

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Function Spaces Related to the Dirichlet Space

Nicola Arcozzi
(joint work with Richard Rochberg, Eric T. Sawyer, Brett D. Wick)

We report on recent work related with the holomorphic Dirichlet space and we contextualize it within the general theory.

1. An old and prestigious story: the Hardy space.

Consider the Hardy space H^2 in the unit disc $\mathbb{D}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \implies \|f\|^2_{H^2} = \sum_{n=0}^{\infty} |a_n|^2.$$

The *multiplier space* of H^2, containing all g's holomorphic in $\mathbb{D}$ such that the operator $f \mapsto gf$ is bounded on H^2, is $\mathcal{M}(H^2) = H^\infty$, the space of the bounded holomorphic functions. We also have that

$$H^2 \cdot H^2 := \left\{h = fg : \ f, g \in H^2\right\} = H^1 \hookleftarrow H^2$$

is the product space of H^2, by inner/outer factorization and Cauchy–Schwarz inequality. It is interesting, then, to find the dual space of H^1. C. Fefferman [7] proved that, under the H^2 pairing (with some care), $(H^2 \cdot H^2)^* = (H^1)^* = BMO \cap H(\mathbb{D})$ is the space of the analytic functions with bounded mean oscillation. The definition of BMO, born out of a problem in elasticity theory [9], in our context is as follows. A complex valued function b on the torus $\mathbb{T}$ has *bounded mean oscillation* if there is a positive constant C such that

$$\frac{1}{|I|}\int_I \left| f(e^{i\theta}) - \frac{1}{|I|}\int_I f(e^{i\psi})d\psi \right|^2 d\theta \le C$$

for all subarcs I of $\mathbb{T}$. The BMO norm of f is the best C we can put in the inequality (assume $\int_{\mathbb{T}} f = 0$ to make it truly a norm). A different characterization of the BMO norm for analytic function was used in establishing this and other results. Let μ be a positive measure on the unit disc. The *Carleson measure norm* of b is

$$[\mu]_{CM(H^2)} := \sup_{f\neq 0} \frac{\int_{\mathbb{D}} |f|^2 d\mu}{\|f\|^2_{H^2}} \approx \sup_I \frac{\mu(S(I))}{|I|}.$$

In the term on the far right, $S(I) = \{z \in \mathbb{D} : \ z/|z| \in I, \ 1-|z| < |I|2\pi\}$ is the Carleson box based on the arc I, and the equivalence $\approx$ is Carleson's characterization of μ's in $CM(H^2)$ [4]. Let b an analytic function on $\mathbb{D}$ and let $d\mu_b = (1-|z|^2)|b'|^2 dA$ (dA is area measure on $\mathbb{D}$). Then, $\|b\|_{BMO} \approx [\mu_b]^{1/2}_{CM(H^2)}$ (the measure μ_b is an important object in H^2: $\|f\|_{H^2}^2 \approx \int_{\mathbb{D}} d\mu_f$). We have then a sequence of Banach spaces naturally arising in the Hilbertian theory of H^2:

$$H^\infty = \mathcal{M}(H^2) \hookrightarrow BMOA = (H^2 \cdot H^2)^* \hookrightarrow H^2 \hookrightarrow H^1 = H^2 \cdot H^2.$$

The story we are telling has a chapter concerning bilinear forms. Given b analytic in $\mathbb{D}$, let $T_b^{H^2} : H^2 \times H^2 \to \mathbb{C}$ be the bilinear Hankel form $T_b^{H^2}(f,g) =< fg, b >_{H^2}$. Nehari [12] proved that

$$\|T_b^{H^2}\|_{H^2 \times H^2} := \sup \frac{|T_b^{H^2}(f,g)|}{\|f\|_{H^2}\|g\|_{H^2}} = \|b\|_{(H^2 \cdot H^2)^*} \approx \|b\|_{BMO} \approx [\mu_b]_{CM(H^2)}^{1/2}$$

(the two $\approx$'s are Fefferman's fundamental contribution to the theory).

2. A developing story: the Dirichlet space.

Consider the Dirichlet space $\mathcal{D}$, containing the functions f holomorphic in $\mathbb{D}$ for which the seminorm

$$\|f\|_{\mathcal{D}} = \left(\int_{\mathbb{D}} |f'(z)|^2\right)^{1/2}$$

is finite. We assume throughout that $f(0) = 0$, so to make $\|f\|_{\mathcal{D}}$ into a norm. The multiplier space $\mathcal{M}(\mathcal{D})$ of $\mathcal{D}$ contains the functions g such that $f \mapsto gf$ is bounded on $\mathcal{D}$, and it is easily seen that it consists of those bounded functions g for which the measure $d\mu = d\mu_g = |g'|^2 dA$ satisfies

$$[\mu]_{CM(\mathcal{D})} := \sup_{f \neq 0} \frac{\int_{\mathbb{D}} |f|^2 d\mu}{\|f\|_{\mathcal{D}}^2} < +\infty.$$

Measures (not necessarily arising from a function g) with this imbedding property are called *Carleson measures for $\mathcal{D}$,* and they were characterized by Stegenga [13] in terms of a capacitary condition. Let $E = \cup_j I_j$ be the disjoint union of closed subarcs of the unit circle and let $S(E) = \cup_j S(I_j)$ be the union of the corresponding Carleson boxes. The Carleson measure norm in $\mathcal{D}$ of a positive measure μ is

$$[\mu]_{CM(\mathcal{D})} \approx \mu(\mathbb{D}) + \sup_E \frac{\mu(S(E))}{\mathrm{Cap}(E)},$$

where $\mathrm{Cap}(E)$ is the logarithmic capacity (the one for which $\mathrm{Cap}(I) \approx \log^{-1}(|I|^{-1})$ for small arcs I). In turn, $\|g\|_{\mathcal{M}(\mathcal{D})} \approx \|g\|_{H^\infty} + [\mu_g]_{CM(\mathcal{D})}$. (It is useful considering Carleson measures for $\mathcal{D}$ supported on the boundary of $\mathbb{D}$ for studying boundary values of Dirichlet functions, but we do not need them here). Following the lead of the Hardy theory, we might think that the right substitute of $BMOA$ in Dirichlet theory might be the space χ,

$$\|b\|_\chi := [\mu_b]_{CM(\mathcal{D})} = [|b'|^2 dA]_{CM(\mathcal{D})}.$$

Lacking inner/outer factorization, the analog of H^1 might be the *weak product space $\mathcal{D} \odot \mathcal{D}$,*

$$\|h\|_{\mathcal{D}\odot\mathcal{D}} = \inf\{\sum_j \|a_j\|_{\mathcal{D}}\|b_j\|_{\mathcal{D}} : \sum_j a_j b_j = h\}.$$

(For weak products in general, see [5]). Note that $H^2 \odot H^2 = H^2 \cdot H^2 = H^1$. Since $1 \in \mathcal{D}$, we have the chain of inclusions

$$H^\infty \cap \chi = \mathcal{M}(\mathcal{D}) \hookrightarrow \chi \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{D} \odot \mathcal{D}.$$

Theorem 1 ([1])**.** *$(\mathcal{D} \odot \mathcal{D})^* = \chi$ under $\mathcal{D}$ pairing.*

Theorem 1 might be seen as an analog of Fefferman's Theorem in Dirichlet theory. In proving it, we found it easier passing to an equivalent formulation in terms of Hankel type forms. Given b, holomorphic in $\mathbb{D}$, define $T_b^{\mathcal{D}}(f,g) = < fg, b >_{\mathcal{D}}$. Functional analytic considerations show that

$$\|T_b^{\mathcal{D}}\|_{\mathcal{D}\times\mathcal{D}} := \sup \frac{|T_b^{\mathcal{D}}(f,g)|}{\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}} = \|b\|_{(\mathcal{D}\odot\mathcal{D})^*}.$$

What one has to prove is then

Theorem 2. *$\|T_b^{\mathcal{D}}\|_{\mathcal{D}\times\mathcal{D}} \approx \|b\|_{\chi}$.*

This is done in [1], and it might be seen as a Nehari-type theorem. It is easily seen that $\|T_b^{\mathcal{D}}\|_{\mathcal{D}\times\mathcal{D}} \lesssim \|b\|_{\chi}$. In the other direction, we use Stegenga's capacitary characterization of Carleson measures of $\mathcal{D}$, discrete approximation of the extremal function for the capacity of a given set and estimates of holomorphic versions of these discrete functions. A discussion of the context surrounding these theorems is in [2].

Results of similar flavor have been obtained for a few other functions spaces. Ferguson and Lacey [8] considered the Hardy space on the polydisc, while Mazya and Verbitsky [11] have, as a consequence of a more general theory, analogous results for some Sobolev spaces.

3. Related questions.

We end the abstract with some open questions.

- Is there a better, more geometric characterization of the functions belonging to χ and $\mathcal{D} \odot \mathcal{D}$?
- Are there versions of the John–Nirenberg inequality [10] for functions belonging to the space χ?
- Are there analogous results for other holomorphic function spaces? The techniques used in [1] can not be easily transfered outside the Dirichlet case. It would be especially interesting to have results for the weighted Dirichlet spaces which are intermediate between Hardy and Dirichlet,

  $$\|f\|^2_{\mathcal{D}_a} = \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^a dA(z),\ 0 < a < 1,$$

  as well as results for the analytic Besov spaces [14].
- We single out the above question in the special case of the Drury–Arveson space [6, 3], in view of its importance as the analog of the Hardy space in multivariable operator theory.

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Truncated Toeplitz operators: existence of bounded symbols

Anton Baranov

Truncated Toeplitz operators are compressions of usual Toeplitz operators to star-invariant (model) subspaces of the Hardy space H^2 in the disc. Let Θ be an inner function and let $K_\Theta = H^2 \ominus \Theta H^2$ be a model subspace (in what follows we will also work with model subspaces of H^p defined by $K^p_\Theta = H^p \cap \overline{z}\overline{H^p}$). For a function $\phi \in L^2(\mathbb{T})$, define the truncated Toeplitz operator A_ϕ by the formula

$$A_\phi f = P_\Theta(\phi f)$$

for functions $f \in K_\Theta \cap H^\infty$. Here $P_\Theta f = f - \Theta P_+(\overline{\Theta} f)$ is the projector onto K_Θ. In contrast to the classical Toeplitz operators, a truncated Toeplitz operator may be sometimes extended to a bounded operator on the whole space K_Θ even for an unbounded symbol ϕ.

Some important special cases of truncated Toeplitz operators (in what follows, TTO) were extensively studied. Let us mention the following:

(*i*) If $\phi(z) = z$, then $A_z f = P_\Theta(zf)$ is the model operator in Sz.-Nagy–Foiaş theory. If $\phi \in H^\infty$, then $A_\phi = \phi(A_z)$.

(*ii*) If $\Theta(z) = z^n$, then $K_\Theta = \mathcal{P}_{n-1}$ is the set of polynomial of degree at most n, and truncated Toeplitz operators correspond to finite Toeplitz matrices $\{c_{m-k}\}_{m,k=0}^{n-1}$.

(iii) Wiener–Hopf convolution operators on an interval $(0, a)$ are unitarily equivalent (via the Fourier transform) to truncated Toeplitz operators on the space K_Θ in the half-plane with $\Theta(z) = \exp(iaz)$.

However, a systematic study of general truncated Toeplitz operators was started recently by D. Sarason [7]. This paper laid the basis of the theory and inspired much of the subsequent activity in the field (see, e.g., [2, 4, 5]).

The symbol of a truncated Toeplitz operator is not unique. Sarason obtained the description of symbols which generate zero TTO: $A_\phi = 0$ *if and only if* $\phi \in \Theta H^2 + \overline{\Theta H^2}$. Also, in [7] several characterizations of TTO by operator identities are obtained.

On the other hand, some basic questions about TTO remained open. One of such questions was the existence of a bounded symbol for a bounded TTO.

Question 1 (Sarason, 2007). *Let A be a bounded TTO. Whether there exists $\psi \in L^\infty(\mathbb{T})$ such that $A = A_\psi$? In other words, is any bounded TTO a restriction of a bounded Toeplitz operator in H^2?*

It follows from the results of R. Rochberg [6] that if $\Theta(z) = \exp\Big(-\frac{\zeta_0+z}{\zeta_0-z}\Big)$, $\zeta_0 \in \mathbb{T}$ (or $\Theta(z) = \exp(iaz)$, an inner function in the upper half-plane), then any bounded TTO has a bounded symbol.

However, the answer in general is negative. Moreover, there exists a rank one TTO in K_Θ which has no bounded symbol. The first example of a truncated Toeplitz operator without a bounded symbol was constructed in [2].

For $\lambda \in \mathbb{D}$, let $k_\lambda(z) = \frac{1-\overline{\Theta(\lambda)}\Theta(z)}{1-\overline{\lambda}z}$, and let $\tilde{k}_\lambda(z) = \frac{\Theta(z)-\Theta(\lambda)}{z-\lambda}$. Recall that k_λ is the reproducing kernel for the space K_Θ. Sarason [7] has shown that A is a rank one TTO if and only if (up to a constant factor) $A = k_\lambda \otimes \tilde{k}_\lambda$ or $A = k_\lambda \otimes k_\lambda$, $\lambda \in \mathbb{D}$, or $A = k_\zeta \otimes k_\zeta$,where $\zeta \in \mathbb{T}$ is a Carathéodory point for Θ. Here $g \otimes h$ is the rank one operator, $(g \otimes h)f = (f, h)g$.

Let $\zeta \in \mathbb{T}$. By the results of Ahern–Clark (for $p = 2$) and W.S. Cohn ($1 < p < \infty$) k_ζ belongs to H^p if and only if the zeros z_n of Θ and the associated singular measure ν satisfy

$$\sum_n \frac{1-|z_n|^2}{|z_n-\zeta|^p} + \int \frac{d\nu(\tau)}{|\tau-\zeta|^p} < \infty. \tag{1}$$

If ζ satisfies (1) for $p = 2$, then ζ is said to be a Carathéodory point for Θ; in this case $|\Theta(\zeta)| = 1$ (in the nontangential sense) and there is the nontangential limit $\lim_{z\to\zeta} \frac{\Theta(z)-\Theta(\zeta)}{z-\zeta}$, the so-called nontangential derivative of Θ.

Theorem 1 ([2]). *Let Θ be an inner function such that $k_\zeta \in H^2$, but $k_\zeta \notin H^p$ for some $p \in (2, \infty)$. Then the operator $k_\zeta \otimes k_\zeta$ has no bounded symbol (no symbol in L^p).*

A wider class of examples is given by the following

Theorem 2 ([2]). *If for some $p \in (2, \infty)$ we have*

$$\sup_{\lambda\in\mathbb{D}} \|k_\lambda\|_p / \|k_\lambda\|_2^2 = \infty, \tag{2}$$

then there exists a bounded TTO without a bounded symbol.

Though the answer to Question 1 is negative, new questions arise: *For which inner functions Θ any bounded TTO on K_Θ has a bounded symbol? How to identify TTO with bounded symbols among all bounded TTO?*

A natural candidate for the first question is the class of one-component inner functions. For $\varepsilon \in (0,1)$ put $\Omega_\varepsilon = \{z : |\Theta(z)| < \varepsilon\}$; then Θ is said to be *one-component* if Ω_ε is connected for some $\varepsilon \in (0,1)$. By a theorem due to A.B. Aleksandrov [1], an inner function Θ is one-component if and only if it satisfies $\sup_{\lambda\in\mathbb{D}} \|k_\lambda\|_\infty/\|k_\lambda\|_2^2 < \infty$ (compare with (2)).

Existence of a bounded symbol turns out to be closely connected with another class of problems about truncated Toeplitz operators. Given Θ and $p \in [1,\infty)$, denote by $\mathcal{C}_p(\Theta)$ the class of complex Borel measures in the closed disk $\overline{\mathbb{D}}$ such that there is the embedding $K^p_\Theta \subset L^p(|\mu|)$. The problem of the description of the class $\mathcal{C}_p(\Theta)$ was posed by W.S. Cohn in 1982; in spite of a number of partial results, the problem is still open.

Let $\mu \in \mathcal{C}_2(\Theta)$. Define the bounded operator A_μ on K_Θ by

$$(A_\mu f, g) = \int f\overline{g}d\mu, \qquad f, g \in K_\Theta.$$

Sarason [7] has shown that A_μ is a bounded TTO and asked the following question:

Question 2. *Is any bounded TTO of the form A_μ for some $\mu \in \mathcal{C}_2(\Theta)$?*

An important advance in our understanding of truncated Toeplitz operators was achieved in [3]. In particular, we give the positive answer to Question 2 and describe the class of TTO with a bounded symbol in terms of Carleson measure for $K^1_{\Theta^2}$.

Theorem 3 ([3]). (*i*) *Any bounded TTO is of the form A_μ for some $\mu \in \mathcal{C}_2(\Theta)$.*
(*ii*) *A bounded TTO A has a bounded symbol iff $A = A_\mu$ for some $\mu \in \mathcal{C}_1(\Theta^2)$.*

Our second main theorem describes those inner functions for which any bounded TTO has a bounded symbol. The description is either in terms of the classes $\mathcal{C}_p(\Theta)$ or in terms of a certain weak factorization of functions from $K^1_{\Theta^2}$.

Theorem 4 ([3]). *The following are equivalent:*

(*i*) *any bounded truncated Toeplitz operator on K_Θ admits a bounded symbol;*

(*ii*) $\mathcal{C}_1(\Theta^2) = \mathcal{C}_2(\Theta^2)$;

(*iii*) *for any $f \in H^1 \cap \bar{z}\Theta^2 H^1_-$ there exist $x_k, y_k \in K_\Theta$ with $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty$ such that $f = \sum_k x_k y_k$ (one can have only 4 summands).*

It was shown by Aleksandrov (see [1]) that for a one-component inner function the class $\mathcal{C}_p(\Theta)$ does not depend on the exponent p. Thus, we have

Corollary. *If Θ is one-component, then $\mathcal{C}_2(\Theta) = \mathcal{C}_1(\Theta^2)$, and so any bounded TTO has a bounded symbol.*

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System of reproducing kernels and their biorthogonal: completeness or non-completeness?

Yurii Belov

(joint work with Anton Baranov)

Let $\mathcal{H}$ be a separable Hilbert space. A sequence of vectors $\{v_n\}$ is said to be *complete* if $\overline{Span\{v_n\}} = \mathcal{H}$. If, moreover, the system $\{v_n\}$ fails to be complete when we remove any vector, then we say that the system is *exact*. For every exact system of vectors $\{v_n\}$ there exists a unique biorthogonal system $\{w_m\}$ such that $\langle v_n, w_m \rangle = \delta_{mn}$.

Suppose that $\mathcal{H}$ is a space of entire functions with reproducing kernels. Namely, for each $w \in \mathbb{C}$ there is an element $k_w \in \mathcal{H}$ such that $\langle f, k_w \rangle = f(w)$ for all $f \in \mathcal{H}$. We are looking for an answer to the following question:

Question 1. *Let $\{k_\lambda\}$ be an exact system of reproducing kernels in $\mathcal{H}$. Is it true that the biorthogonal system is also complete in $\mathcal{H}$?*

Of course, for an arbitrary sequence of vectors, its biorthogonal system may be non-complete. If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis, then system $\{e_n + e_1\}_{n=2}^{\infty}$ is complete, but biorthogonal system $\{e_n\}_{n=2}^{\infty}$ is non-complete. On the other hand, it is well known that if we restrict ourselves to a *system of reproducing kernels*, then the answer may be positive. R.M. Young [4] proved the completeness of such systems for the Paley–Wiener spaces; E. Fricain [3] extended this result to a class of de Branges spaces of entire functions (see discussion below).

Our aim is to exhibit some classes of spaces for which we know the answer (positive or negative). In particular, we answer the question posed by N.K. Nikolski and construct an example of a model (shift-coinvariant) subspace of the Hardy space H^2 with a non-complete biorthogonal system.

To make general problem more realistic we need some additional structure on $\mathcal{H}$, namely the existence of a *Riesz basis.* Recall that a system of vectors $\{v_n\}$ is said to be a Riesz basis if $\{v_n\}$ is an image of an orthonormal basis under a bounded and invertible operator in $\mathcal{H}$. We consider the class $\mathfrak{R}$ of spaces of entire functions satisfying three axioms:

(A1) $\mathcal{H}$ has a reproducing kernel k_λ at every point $\lambda \in \mathbb{C}$;

(A2) If function f is in $\mathcal{H}$ and $f(w) = 0$, then function $\frac{f(z)}{z-w}$ is also in $\mathcal{H}$;

(A3) There exists a sequence of distinct points $T = \{t_n\} \subset \mathbb{C}$ such that the sequence of normalized reproducing kernels $\{k_{t_n}/\|k_{t_n}\|_{\mathcal{H}}\}$ forms a Riesz basis for $\mathcal{H}$.

First example of such spaces is the Paley–Wiener space PW_π which is the space of entire functions of exponential type at most π that are in $L^2(\mathbb{R})$. In this case the sequence $\{\frac{\sin(\pi(z-n))}{\pi(z-n)}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of reproducing kernels and $(A3)$ is satisfied. Axioms $(A1), (A2)$ follow immediately.

More interesting examples are de Branges spaces. We say that an entire function E belongs to the Hermite–Biehler class if it has no real zeros and $|E(z)| > |E(\overline{z})|$ for any z in the upper half-plane $\mathbb{C}^+$. The de Branges space $\mathcal{H}(E)$ consists of all entire functions f such that $f(z)/E(z)$ and $\overline{f(\overline{z})}/E(z)$ belong to the Hardy space H^2 in $\mathbb{C}^+$. The norm in $\mathcal{H}(E)$ is given by

$$\|f\|^2_{\mathcal{H}(E)} = \int_{\mathbb{R}} \frac{|f(x)|^2}{|E(x)|^2} dx.$$

As in the Paley–Wiener space, in the de Branges spaces there exist orthonormal bases of reproducing kernels (see [2]). That means that de Branges spaces form a subclass of our class $\mathfrak{R}$.

We will use an explicit parametrization of the class $\mathfrak{R}$ from [1]. We can associate with the space $\mathcal{H} \in \mathfrak{R}$ a space of *meromorphic functions with prescribed poles.* Namely, given a sequence of distinct complex numbers $T = \{t_n\}$ and a weight sequence $b = \{b_n\}$ that satisfy $\sum_n \frac{b_n}{1+|t_n|^2} < \infty$, we introduce the space $\mathcal{H}(T, b)$ consisting of all functions of the form

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n b_n^{1/2}}{z - t_n}, \tag{1}$$

for which

$$\|f\|^2_{\mathcal{H}(T,b)} = \sum_{n=1}^{\infty} |a_n|^2 < +\infty.$$

The map $f \mapsto Ff$ is a unitary map from $\mathcal{H}(T, b)$ to $\mathcal{H}$ which maps reproducing kernels to reproducing kernels. So, for our approach, we can consider the pairs (T, b) as a parametrization of all spaces from $\mathfrak{R}$.

Now we are ready to state our main result.

Theorem 2. *If $\mathcal{H} \in \mathfrak{R}$ and $\sum_n b_n < +\infty$, then there exists an exact system of reproducing kernels such that its biorthogonal system is not complete.*

A converse result says that if b_n have no more than a power decay, then the biorthogonal systems are complete.

Theorem 3. *If* $\sum_n b_n = +\infty$ *and there exists* N *such that* $\inf_m(b_m(1+|t_m|)^N) > 0$*, then a system biorthogonal to an exact system of reproducing kernels is always complete in* $\mathcal{H}$*.*

The restriction on the decay of b_n in Theorem 3 is essential.

Example 4. *There exists a space* $\mathcal{H} \in \mathfrak{R}$ *such that for any Riesz basis of reproducing kernels we have* $\sum_n b_n = +\infty$*, but there exists an exact system of reproducing kernels such that its biorthogonal is not complete.*

Now we turn to the question of the "size" of the orthogonal complement of a biorthogonal system in the case when the system is not complete. Here we emphasize the following informal principle: *The size of the orthogonal complement of biorthogonal system depends on smallness of the sequence* $\{b_n\}$*. The orthogonal complement becomes bigger if* b_n *tend to zero faster.*

Nevertheless, if $\{b_n\}$ are extremely small, then the biorthogonal system has finite codimension.

Theorem 5. *Suppose* $t_n = n$*,* $n \in \mathbb{Z}$*, and* b_n *are so small that there is no nontrivial sequence* $\{c_n\}$*,* $|c_n| \le b_n^{1/2}$*, such that* $\sum_{n=-\infty}^{\infty} c_n n^k = 0$ *for any* $k \in \mathbb{N}_0$*. If* $\mathcal{H}$ *is the corresponding Hilbert space of the class* $\mathfrak{R}$*, then the system biorthogonal to an exact system of reproducing kernels always has finite codimension.*

We illustrate this by the following example.

Example 6. *If* $t_n = n$*,* $b_n = \exp(-|n|)$*,* $n \in \mathbb{Z}$*, then any biorthogonal system has finite codimension.*

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Analytic Capacity and Certain Problems in Approximation Theory

James E. Brennan

My purpose here is to discuss certain connections between uniform rational approximation, and approximation in the mean by either polynomials or rational functions on compact nowhere dense subsets of the complex plane $\mathbb{C}$. If X is compact $C(X)$ will denote the space of all continuous functions on X, and $R(X)$ will stand for the subspace of $C(X)$ consisting of all functions that can be uniformly approximated on X by rational functions whose poles lie outside of X. For each

$p, 1 \leq p < \infty$, let $H^p(X, dA)$ be the closed subspace of $L^p(X, dA)$ that is spanned by the polynomials, and let $R^p(X, dA)$ be the corresponding subspace spanned by the rational functions. Here dA denotes two-dimensional Lebesgue measure.

By definition a point $x \in X$ is a *peak point* for $R(X)$ if there exists a function $f \in R(X)$ such that $f(x) = 1$, but $|f(y)| < 1$ whenever $y \neq x$. It is a theorem of Errett Bishop (cf. [5]) that $R(X) = C(X)$ if and only if dA almost every point of X is a peak point for $R(X)$. If, on the other hand, $x_0 \in X$ is not a peak point for $R(X)$ it can be shown (cf. [3]) that

$$|P(x_0)| \leq C_p \, \|P\|_{L^p(X,dA)}$$

for every polynomial P, and some constant C_p depending only on p. In this context x_0 is said to yield a *bounded point evaluation* for $H^p(X, dA)$. Thus, if $R(X) \neq C(X)$ then $H^p(X, dA) \neq L^p(X, dA)$ for any p, $1 \leq p < \infty$. The proof makes essential use of Tolsa's theorem [13] on the semiadditivity of analytic capacity, and settles a question from 1973 which initially arose in connection with the invariant subspace problem for subnormal operators on a Hilbert space.

It can happen, however, that $R(X) \neq C(X)$, but nevertheless $R^p(X, dA) = L^p(X, dA)$ for all $p < \infty$. This was initially established by Sinanjan [12] in 1966. His argument, however, depends on earlier work of Mergeljan and is computationally rather difficult. A proof that is conceptually much clearer can be found in [3], and depends only on the fundamentally different behavior of q-capacity and analytic capacity under a contraction. In order to prove that $R^p(X, dA) = L^p(X, dA)$ it is sufficient to verify that if $k \in L^q(X, dA)$, $q = p/(p-1)$, and $\int fk \, dA = 0$ for all rational functions f, then the Cauchy integral

$$\hat{k}(\zeta) = \int \frac{k(z)}{z - \zeta} dA_z$$

vanishes almost everywhere. Since $\hat{k}$ is continuous if $q > 2$ it follows that $R^p(X, dA) = L^p(X, dA)$ for $1 \leq p < 2$ whenever X is compact and has empty interior. We may assume, therefore, that $1 < q \leq 2$. In this case $\hat{k}$ belongs to the Sobolev space W_1^q, and as such enjoys a certain residual continuity which is best described in terms of an associated capacity C_q (cf. [2,3]). More precisely, the Cauchy integral $\hat{k}$ is *q-finely continuous* at almost every point $x_0 \in X$ in the sense that there exists a set E that is *thin* or *sparse* in a potential theoretic sense at x_0 and

$$\lim_{x \to x_0, \, x \in \mathbb{C} \setminus E} \hat{k}(x) = \hat{k}(x_0).$$

In our case it is sufficient to know that E is *thick* at x_0 if

$$\limsup_{r \to 0} \frac{C_q(E \cap B_r)}{r^{2-q}} > 0, \tag{1}$$

where $B_r = B_r(x_0)$ is the disk with center at x_0 and radius r (cf. [2], p.221). Hence, if (1) is satisfied at almost every point of X when $E = \{z : \hat{k}(z) = 0\}$, then $R^p(X, dA) = L^p(X, dA)$.

To obtain a compact set X for which $R^p(X, dA) = L^p(X, dA)$ but $R(X) \neq C(X)$ we first iterate the construction of the *corner quarters Cantor set* in such a way that the resulting set E is dense in the unit square Q, the analytic capacity $\gamma(E) = 0$, and the orthogonal projection of $E \cap B_r$ onto the line $2y = x$ covers an interval of length $(3/\sqrt{5})r$ for any disk B_r contained in Q. The construction depends on ideas in the papers of Garnett [7], Ivanov [10], and Vitushkin [14]. The details can be found in [3]. Since q- capacity decreases modulo a multiplicative constant under a contraction (cf. [1], p.140), $C_q(E \cap B_r) \geq Kr^{2-q}$ for some constant K depending only on q and all disks B_r lying in Q. Because $\gamma(E) = 0$ we can choose a compact set X_0 lying inside Q whose area $|X_0| > 0$ and $X_0 \cap E = \varnothing$. Covering E by countably many sufficiently small open squares $\Omega_j, j = 1, 2, 3, ...$ having disjoint closures, none of which meets X_0, we arrive at a compact set $X = Q \setminus \cup_j \Omega_j$ and $X_0 \subseteq X$. If the Ω_j's are chosen sufficiently small we can arrange that

$$\lim_{r \to 0} \frac{\gamma(B_r(x) \backslash X)}{r} = 0 \tag{2}$$

at almost every point $x \in X_0$. By a theorem of Vitushkin [15] (cf. also [5], p.207) it follows from (2) that $R(X) \neq C(X)$, and from (1) that $R^p(X, dA) = L^p(X, dA)$ for $1 \leq p < \infty$.

If x_0 is not a peak point for $R(X)$ we have seen that $H^1(X, dA)$ has a bounded point evaluation at x_0, and so there exists a function $h \in L^\infty(X, dA)$ so that $P(x_0) = \int Ph\, dA$ for all polynomials P. In the case of $R(X)$ it can be shown that there exists an absolutely continuous measure $h\, dA$ with the property that

$$f(x_0) = \int fh\, dA \tag{3}$$

for all $f \in R(X)$. This can be deduced from Davie's theorem [4] on bounded pointwise approximation, and was apparently first noticed by Brian Cole (cf. also [6]). By our remarks in the preceding paragraph the most that can be said, in general, is that $h \in L^1(X, dA)$. To establish (3) it is evidently sufficient to verify that the evaluation functional $L(f) = f(x_0)$ is weak-$*$ continuous on $R(X)$. And, for this it follows from an extension of the Krein–Smulian theorem first employed by Hoffman and Rossi [9] that it is enough to show that if $\{f_n\}$ is a sequence of rational functions in $R(X)$ which converges pointwise and boundedly to 0 almost everywhere dA on X, then $L(f_n) = f_n(x_0) \to 0$. This proves that the kernel of L is weak-$*$ closed, from which it follows that L is weak-$*$ continuous.

Fix $r > 0$, let g be a smooth function supported in $B_r(x_0)$ such that $0 \leq g \leq 1$ with $g = 1$ in a neighborhood of x_0, and $\|\partial g/\partial \bar{z}\|_\infty \leq 4/r$. Set $f_n = 0$ in a region containing its singularities in such a way that the modified function, still denoted f_n, is analytic in a neighborhood of X and $\|f_n\|_\infty \leq 2\,\|f_n\|_X$. Next, consider the function F_n obtained from the Vitushkin localization operator. In particular, F_n is analytic everywhere where f_n is analytic, is analytic outside $B_r(x_0)$, is explicitly expressed as

$$F_n(\zeta) = g(\zeta)f_n(\zeta) + \frac{1}{\pi} \int \frac{f_n(z)}{z - \zeta} \frac{\partial g}{\partial \bar{z}}\, dA_z, \tag{4}$$

$\|F_n\| \le 16\,\|f_n\|$, and $F_n - f_n$ is analytic in a neighborhood of x_0 (cf. [5,15]). Since x_0 is not a peak point for $R(X)$, it follows from Mel'nikov's *peak point criterion* [11] (cf. also [8]) that

$$\sum_{n=1}^{\infty} 2^n\, \gamma(A_n(x_0)\backslash X) < \infty, \tag{5}$$

where $A_n(x_0) = \{z : 2^{-n-1} < |z - x_0| \le 2^{-n}\}$. On the other hand, $|F_n(x_0)|$ is dominated by a constant times the portion of (5) coming from the disk $B_r(x_0)$, and so if r is sufficiently small $|F_n(x_0)| < \epsilon$ for all n. Since $F_n(x_0) - f_n(x_0) \to 0$ as $n \to \infty$, $L(f_n) = f_n(x_0) \to 0$.

By a theorem of Øksendal [16] the resulting representing measure for $R(X)$ is carried by the set of non-peak points, and it can be taken to be a positive measure (cf. [5], p.33).

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The Kadison–Singer Problem in Harmonic Analysis

Peter G. Casazza

It is now known [2] that the famous, intractable 1959 Kadison–Singer Problem in C^*-Algebras is equivalent to famous unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and engineering. We will look at the Kadison–Singer Problem in Harmonic Analysis and the most viable approach to producing a counter-example.

1. The Kadison–Singer Problem

A **state** of a Von Neumann Algebra $\mathbb{R}$ is a linear functional f on $\mathbb{R}$ for which $f(I) = 1$ and $f(T) \geq 0$, whenever $T \geq 0$ (i.e. whenever T is a positive operator). The set of states of $\mathbb{R}$ is a convex subset of the dual space which is compact in the w^*-topology. By the Krein–Milman Theorem, this convex set is the closed onvex hull of its extreme points. The extremal elements in the space of states are called the **pure states**. For over 50 years the Kadison–Singer Problem [3] (see also [2]) has defied the best efforts of some of the most talented mathematicians of our time.

Kadison–Singer Problem. (KS) *Does every pure state on the (abelian) von Neumann Algebra $\mathbb{D}$ of bounded diagonal operators on ℓ_2 have a unique extension to a (pure) state on $B(\ell_2)$, the von Neumann Algebra of all bounded linear operators on the Hilbert space ℓ_2?*

2. The Paving Conjecture

In 1979, Anderson [1] showed that the Kadison–Singer Problem is equivalent to the *Paving Conjecture*.

Paving Conjecture. (PC) *For $\epsilon > 0$, there is a natural number r so that for every natural number n and every linear operator T on l_2^n whose matrix has zero diagonal, we can find a partition (i.e. a paving) $\{A_j\}_{j=1}^r$ of $\{1, \ldots, n\}$, such that*

$$\|Q_{A_j} T Q_{A_j}\| \leq \epsilon \|T\| \quad \text{for all } j = 1, 2, \ldots, r,$$

where Q_{A_j} is the natural projection onto the A_j coordinates of a vector.

The important point here is that r depends only on $\epsilon > 0$ and not on n or T. Operators satisfying the Paving Conjecture are called **pavable operators**. A projection P on $\mathcal{H}_n$ is (ϵ, r)**-pavable** if there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, n\}$ satisfying

$$\|Q_{A_j} P Q_{A_j}\| \leq \epsilon, \quad \text{for all } j = 1, 2, \ldots, r.$$

Anderson [1] showed that the Paving Conjecture is equivalent to the Kadison–Singer Problem.

3. The Feichtinger Conjecture

Recall that a family of vectors $\{f_i\}_{i\in I}$ is a **Riesz basic sequence** in a Hilbert space $\mathcal{H}$ if there are constants $A, B > 0$ so that for all families of scalars $\{a_i\}_{i\in I}$ we have:

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2.$$

If $A = 1-\epsilon$ and $B = 1+\epsilon$ we call $\{f_i\}_{i\in I}$ an ϵ-Riesz basic sequence. If $\|f_i\| = 1$ for all $i \in I$, we call the family a **unit norm** family. In [2] it was shown that the following conjecture is equivalent to the Kadison–Singer Problem.

Conjecture 1. (R_ϵ-Conjecture) *For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of ϵ-Riesz basic sequences.*

Definition 2. *A family of vectors $\{f_i\}_{i\in I}$ is a* **frame** *for a Hilbert space $\mathcal{H}$ if there are constants $A, B > 0$ so that for all $f \in \mathcal{H}$ we have*

$$A\|f\|^2 \le \sum_{i\in I}|\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

*If we only have the right hand side inequality, we call this a B-***Bessel sequence.**

It follows that $\{f_i\}_{i\in I}$ is B-Bessel if and only if the operator $T : \ell_2(I) \to \mathcal{H}$ given by $Te_i = f_i$ satisfies $\|T\|^2 = B$, where $\{e_i\}_{i\in I}$ is the natural orthonormal basis of $\ell_2(I)$.

Conjecture 3. *(The Feichtinger Conjecture) Can every unit norm frame (or Bessel sequence) be partitioned into a finite number of Riesz basic sequences?*

In [2] it was shown that the Feichtinger Conjecture is equivalent to the Kadison–Singer Problem.

4. The Kadison–Singer Problem in Harmonic Analysis

Definition 4. *If $\phi \in L^\infty[0,1]$, the Laurent operator T_ϕ is defined by:*

$$T_\phi f = \phi \cdot f \quad \forall f \in L^2[0,1].$$

Much work was done in 1980's to solve PC for Laurent Operators by Bourgain/Tzafriri, and Berman/Halpern/Kaftal/Weiss (see the references in [2]). But the problem remains open today.

Definition 5. *If $A \subseteq \mathbb{Z}$, let*

$$S(A) = span\ \{e^{2\pi i n t}\}_{n\in A} \subseteq L^2[0,1].$$

Theorem 6. *(Berman, Halpern, Kaftal and Weiss) For every $\epsilon > 0$ and for every $[a,b] \le [0,1]$ there exists a partition of $\mathbb{Z}$ into arithmetic progressions $(A_j)_{j=1}^r$ so that for all j and $f \in S(A_j)$*

$$(1-\epsilon)(b-a)\|f\|^2 \le \|P_E f\|^2 \le (1+\epsilon)(b-a)\|f\|^2,$$

$P_E f = \chi_E \cdot f.$

Conjecture 7. *(The Harmonic Analysis Conjecture) For every measurable* $E \subseteq [0,1]$ *and for every* $\epsilon > 0$ *there exists a partition* $(A_j)_{j=1}^r$ *of* $\mathbb{Z}$ *such that for every* j *and* $f \in S(A_j)$

$$(1-\epsilon)|E|\|f\|^2 \leq \|P_E f\|^2 \leq (1+\epsilon)|E|\|f\|^2.$$

If we replace $1 \pm \epsilon$ *by universal* $0 < A < 1 < B < \infty$*, we call this* **weak H.A.**.

Theorem 8. *The following are equivalent:*

(1) *H.A. Conjecture*
(2) *Every* T_ϕ *is pavable*
(3) *There is a universal constant* K *such that for every measurable subset* $E \subseteq [0,1]$ *there exists a partition* $(A_j)_{j=1}^r$ *of* $\mathbb{Z}$ *so that for all* $f \in$ *span* $(e^{2\pi irt})_{n\in A_j}$

$$\|f \cdot \chi_E\|^2 \leq K|E|\|f\|^2.$$

Moreover: We may assume $|E| = 1/2$.

(B) Weak HA is equivalent to FC for Laurent operators.

Conjecture 9. *(The Feichtinger Conjecture for Laurent operators) Given* $E \subset [0,1]$ *measurable with* $|E| = 1/2$*, there is a partition* $\{A_j\}_{j=1}^r$ *of* $\mathbb{Z}$ *so that for all* $j = 1, 2, \ldots, r$*, the family*

$$\left\{e^{2\pi int}\chi_E\right\}_{n\in A_j},$$

is a Riesz basic sequence.

Theorem 10. *(Halpern, Kaftal, Weiss) If* $E \subset [0,1]$ *is measurable and* χ_E *is Riemann integrable, then there is a partition* $\{A_j\}_{j=1}^r$ *of* $\mathbb{Z}$ *into arithmetic progressions so that each*

$$\left\{e^{2\pi int}\chi_E\right\}_{i\in A_j},$$

is a Riesz basic sequence.

To produce a counter-example to FC, we need to work with a set $E \subset [0,1]$ satisfying:

1. $|E| = 1/2$.

2. Neither E nor E^c contains an interval.

At this time, for sets E as above, we do not know if a single one of them satisfies FC.

Definition 11. *A subset* $B \subset \mathbb{Z}$ *is called a* **syndetic set** *if there exists an* $M \in \mathbb{Z}$ *so that*

$$B \cap [nM, (n+1)M] \neq \phi, \quad \text{for all } n \in \mathbb{Z}.$$

Lawton [4] proved an important result concerning the Feichtinger Conjecture in harmonic analysis and syndetic sets.

Theorem 12. *(Lawton) If $E \subset [0,1]$ is measurable and*

$$\left\{e^{2\pi int}\chi_E\right\}_{n\in\mathbb{Z}}, \quad \textit{satisfies FC},$$

then there is a partition of $\mathbb{Z}$ into syndetic sets $\{B_j\}_{j=1}^M$ so that

$$\left\{e^{2\pi int}\chi_E\right\}_{n\in B_j} \quad \textit{is a Riesz sequence for every } j = 1, 2, \ldots, M.$$

We believe that the Lawton theorem is the best direction for a counter-example to the Feichtinger Conjecture for Laurent operators. In particular, as we have a classification of the measurable sets $E \subset [0,1]$ for which $\{e^{2\pi int}\chi_E\}_{n\in\mathbb{Z}}$ can be written as a finite union of Riesz sequences made up of arithmetic progressions in $\mathbb{Z}$, we believe there should be a similar classification of the measurable sets E for which our family can be written as a finite union of Riesz sequences made up of syndetic sets – and this will not contain all the measurable sets E. All the other sets will give counter-examples to FC.

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Local *ABC* theorems for holomorphic functions

Konstantin M. Dyakonov

Given a polynomial p (in one complex variable), write $\deg p$ for the degree of p and $\widetilde{N}(p) = \widetilde{N}_{\mathbb{C}}(p)$ for the number of its distinct zeros in $\mathbb{C}$. The so-called *abc* theorem, often referred to as Mason's theorem (but essentially due to Stothers [7]), reads as follows.

Theorem 1. *Suppose a, b and c are polynomials, not all constants, having no common zeros and satisfying $a + b = c$. Then*

$$\max\{\deg a, \deg b, \deg c\} \le \widetilde{N}(abc) - 1. \tag{1}$$

Various approaches to and consequences of Theorem 1 are discussed in [3, 4, 5, 6]. One impressive – and immediate – application is a simple proof of Fermat's Last Theorem for polynomials, saying that there are no nontrivial polynomial solutions to the equation $P^n + Q^n = R^n$ when $n \ge 3$. Besides, it was Theorem 1 that led (via the classical analogy between polynomials and integers) to the famous *abc conjecture* in number theory; see [3, 5].

We now obtain some *abc* type estimates that apply to a much more general situation. Namely, we consider a bounded simply connected domain $\Omega \subset \mathbb{C}$ with

$\partial\Omega$ a rectifiable Jordan curve, and we replace the polynomial equation $a + b = c$ by

$$f_0 + \cdots + f_n = f_{n+1}, \tag{2}$$

where the f_j's are analytic functions on an open neighborhood of $\Omega \cup \partial\Omega$.

With each f_j we associate the (finite) Blaschke product B_j built from the function's zeros in Ω. This means that B_j is given by

$$z \mapsto \prod_{k=1}^{s} \left(\frac{\phi(z) - \phi(a_k)}{1 - \overline{\phi(a_k)}\phi(z)} \right)^{m_k}, \qquad z \in \Omega, \tag{3}$$

where $a_k = a_k^{(j)}$ $(1 \le k \le s = s_j)$ are the distinct zeros of f_j in Ω, $m_k = m_k^{(j)}$ are their respective multiplicities, and ϕ is a conformal map from Ω onto the unit disk. Further, let $\mathbf{B}$ denote the *least common multiple* of the Blaschke products $B_0, \dots, B_{n+1}$ (defined in the natural way), and put

$$\mathcal{B} := \operatorname{rad}(B_0 B_1 \dots B_{n+1}).$$

Here, we use the notation $\operatorname{rad}(B)$ for the *radical* of a Blaschke product B; this is, by definition, the Blaschke product that arises when the zeros of B are all converted into simple ones. In other words, given a Blaschke product of the form (3), its radical is obtained by replacing each m_k with 1.

Finally, we write $W = W(f_0, \dots, f_n)$ for the *Wronskian* of the (analytic) functions $f_0, \dots, f_n$, so that

$$W := \begin{vmatrix} f_0 & f_1 & \dots & f_n \\ f_0' & f_1' & \dots & f_n' \\ \dots & \dots & \dots & \dots \\ f_0^{(n)} & f_1^{(n)} & \dots & f_n^{(n)} \end{vmatrix}. \tag{4}$$

We then introduce the quantities

$$\kappa = \kappa(W) := \|W'\|_{L^1(\partial\Omega)} \|1/W\|_{L^\infty(\partial\Omega)}, \quad \lambda = \lambda(W) := \|W'\|_{L^2(\Omega)} \|1/W\|_{L^\infty(\partial\Omega)}$$

and

$$\mu = \mu(W) := \|W\|_{L^\infty(\partial\Omega)} \|1/W\|_{L^\infty(\partial\Omega)}.$$

The underlying measures on Ω and $\partial\Omega$ are dA/π and $ds/(2\pi)$, respectively, where dA stands for area and ds for arc length.

Theorem 2. *Suppose f_j $(j = 0, 1, \dots, n+1)$ are analytic functions on $\Omega \cup \partial\Omega$, related by* (2) *and such that the Wronskian* (4) *vanishes nowhere on $\partial\Omega$. Then*

$$N_\Omega(\mathbf{B}) \le \kappa + n\mu N_\Omega(\mathcal{B}) \tag{5}$$

and

$$N_\Omega(\mathbf{B}) \le \lambda^2 + n\mu^2 N_\Omega(\mathcal{B}), \tag{6}$$

where $N_\Omega(\cdot)$ denotes the number of the function's zeros in Ω, counting multiplicities.

Both estimates are sharp, for any Ω, and each of them can be used to derive (a generalization of) the original *abc* theorem for polynomials. In fact, (1) follows upon applying either (5) or (6) to the three polynomials, letting $\Omega = \{z : |z| < R\}$ and then passing to the limit as $R \to \infty$. See [1, 2] for details and further developments.

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Heisenberg uniqueness pairs and the Klein–Gordon equation

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(joint work with Alfonso Montes-Rodríguez)

This reports on the work [2]. A Heisenberg uniqueness pair (HUP) is a pair (Γ, Λ), where Γ is a curve in the plane and Λ is a set in the plane, with the following property: any finite Borel measure μ in the plane supported on Γ, which is absolutely continuous with respect to arc length, and whose Fourier transform $\widehat{\mu}$ vanishes on Λ, must automatically be the zero measure. This is analogous to the concept of mutually annihilating pairs [1]. We prove that when Γ is the hyperbola $x_1 x_2 = 1$, and Λ is the lattice-cross

$$\Lambda = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where α, β are positive reals, then (Γ, Λ) is an HUP if and only if $\alpha\beta \le 1$; in this situation, the Fourier transform $\widehat{\mu}$ of the measure solves the one-dimensional Klein–Gordon equation. Phrased differently, we show that

$$e^{\pi i \alpha n t}, \; e^{\pi i \beta n / t}, \qquad n \in \mathbb{Z},$$

span a weak-star dense subspace in $L^\infty(\mathbb{R})$ if and only if $\alpha\beta \le 1$. In order to prove this theorem, some elements of linear fractional theory and ergodic theory are needed, such as the Birkhoff Ergodic Theorem. An idea parallel to the one exploited by Makarov and Poltoratski [3] (in the context of model subspaces) is also needed. As a consequence, we solve a problem on the density of algebras generated by two inner functions raised by Matheson and Stessin [4].

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Sampling of band-limited signals and quasicrystals

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(joint work with Gady Kozma)

A band-limited signal is an entire function F of exponential type, square-integrable on the real axis. By the classical Paley–Wiener theorem, F is the Fourier transform of an L^2-function supported by a bounded (measurable) set $S \subset \mathbb{R}$, which is called the spectrum of F. We denote by PW_S the Paley–Wiener space of all functions $F \in L^2(\mathbb{R})$ which are Fourier transforms of functions from $L^2(S)$,

$$F(t) = \int_S f(x)\, e^{-2\pi i t x}\, dx, \quad f \in L^2(S).$$

A discrete set $\Lambda \subset \mathbb{R}$ is called a *set of sampling* for PW_S if every signal with spectrum in S can be reconstructed in a stable way from its 'samples' $\{F(\lambda), \lambda \in \Lambda\}$, that is, there are positive constants A, B such that the inequalities

$$A\|F\|_{L^2(\mathbb{R})} \leq \Big(\sum_{\lambda\in\Lambda} |F(\lambda)|^2 \Big)^{1/2} \leq B\|F\|_{L^2(\mathbb{R})}$$

hold for every $F \in PW_S$. Equivalently, this means that the exponential system $E(\Lambda) = \{\exp 2\pi i\lambda t,\ \lambda \in \Lambda\}$ is a *frame* in the space $L^2(S)$.

A necessary condition for the sampling property of Λ was given by Landau [7] who proved that if Λ is a set of sampling for PW_S then $D^-(\Lambda) \geq \operatorname{mes} S$. Here we denote by $D^-(\Lambda)$ the lower uniform density of Λ.

Olevskii and Ulanovskii [11, 12] discovered that there exist "universal" sampling sets Λ of given uniform density $D(\Lambda) = d$, which provide a stable reconstruction of any signal whose spectrum is a compact set of Lebesgue measure $< d$. An interesting example of universal sampling sets, based on so-called "simple quasicrystals", was presented by Matei and Meyer in [9, 10]. Let α be an irrational real number, and consider the sequence of points $\{n\alpha\}$, $n \in \mathbb{Z}$, on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Given an interval $I = [a, b) \subset \mathbb{T}$ we define the set

$$\Lambda(\alpha, I) := \{n \in \mathbb{Z} : a \leq n\alpha < b\}.$$

It is well-known that the points $\{n\alpha\}$ are equidistributed (and moreover, they are well-distributed) on the circle $\mathbb{T}$. This implies that the set $\Lambda(\alpha, I)$ has a uniform density $D(\Lambda(\alpha, I)) = |I|$, where $|I|$ denotes the length of the interval I. It was proved in [9, 10] that the exponential system $E(\Lambda(\alpha, I))$ is a frame in $L^2(S)$ for every compact set $S \subset \mathbb{T}$ of measure $< |I|$.

In our joint paper with Gady Kozma [5] we study the *Riesz basis* problem for the exponential system $E(\Lambda(\alpha, I))$ in L^2 on a finite union of intervals. Equivalently, does $\Lambda(\alpha, I)$ provide a *stable and non-redundant sampling* of signals with a 'multi-band' spectrum? Our first result shows that the question admits a positive answer provided that a certain diophantine condition, relating α and the length of the interval I, holds:

Theorem 1. *Let $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$. Then the exponential system $E(\Lambda(\alpha, I))$ is a Riesz basis in $L^2(S)$ for every set $S \subset \mathbb{T}$, $\operatorname{mes} S = |I|$, which is the union of finitely many disjoint intervals whose lengths belong to $\mathbb{Z} + \alpha\mathbb{Z}$.*

Remark that the condition $\operatorname{mes} S = |I|$ in Theorem 1 is necessary for the Riesz basis property in $L^2(S)$, as follows from Landau's inequalities [7]. Theorem 1 extends results from the papers [1, 8] on the existence of exponential Riesz bases in L^2 on multiband sets (that is, finite unions of intervals).

Our second result complements the picture by clarifying the role of the diophantine assumption $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$ in Theorem 1. It turns out that this condition is not only sufficient, but also necessary, for the stable and non-redundant sampling on multiband spectra:

Theorem 2. *Suppose that $|I| \notin \mathbb{Z} + \alpha\mathbb{Z}$. Then $E(\Lambda(\alpha, I))$ is not a Riesz basis in $L^2(S)$, for any set $S \subset \mathbb{T}$ which is the union of finitely many intervals.*

Our approach to the problem above is based on its connection to the theory of equidistribution and discrepancy for the irrational rotation of the circle. It is well-known that if α is an irrational real number then the sequence $\{n\alpha\}$ is uniformly distributed modulo 1. Given a finite union of intervals S on the circle $\mathbb{T}$, consider the *discrepancy* function defined by

$$D(n, S) = \nu(n, S) - n \operatorname{mes} S,$$

where $\nu(n, S)$ denotes the number of integers $0 \leq k \leq n-1$ such that $k\alpha \in S$. The uniform distribution of $\{n\alpha\}$ means that $D(n, S) = o(n)$ as $n \to \infty$.

Better estimates for the discrepancy can be obtained based on diophantine properties of the number α, for example $D(n, S) = O(\log n)$ if α is a quadratic irrational number.

It was discovered by Hecke that if S is a single interval whose length belongs to the group $\mathbb{Z} + \alpha\mathbb{Z}$ then the discrepancy is actually bounded, $D(n, S) = O(1)$. Erdös and Szüsz conjectured [2] that also the converse to Hecke's result should be true. This conjecture was confirmed in 1966 by Kesten [4].

In collaboration with G. Kozma [5] we study the *bounded mean oscillations* of the discrepancy. Let BMO denote the space of sequences with bounded mean oscillations (analogous to the classical John–Nirenberg BMO space of functions on $\mathbb{R}$). This space contains all bounded sequences, but, as is well-known, it also contained unbounded ones. In [5] we extend Kesten's theorem to this space:

Theorem 3. *Let α be an irrational number, and $S \subset \mathbb{T}$ be a finite union of intervals. Let $\nu(n, S)$ denote the number of integers $0 \leq k \leq n-1$ such that $k\alpha \in S$. If the sequence $\{\nu(n, S) - n \operatorname{mes} S\}$, $n = 1, 2, 3, \dots$, belongs to* BMO*, then* $\operatorname{mes} S \in \mathbb{Z} + \alpha\mathbb{Z}$.

The link between the Riesz basis problems for $E(\Lambda(\alpha, I))$ and the theory of discrepancy for irrational rotations is an idea due to Meyer, which we refer to as the 'duality principle'. Meyer's duality principle allows us to reduce the problem about exponential Riesz bases in $L^2(S)$ to a similar problem in $L^2(I)$, where I is a single interval. It is then possible to invoke known results about exponential Riesz bases in $L^2(I)$.

In order to prove Theorem 2 we combine the duality principle with a theorem due to Pavlov [14] (see also [3]) which describes completely the exponential Riesz bases in $L^2(I)$. Pavlov's theorem allows us to conclude that the discrepancy must be in BMO, and we can then apply Theorem 3.

For Theorem 1 we use the fact that if S is the union of finitely many disjoint intervals whose lengths belong to $\mathbb{Z} + \alpha\mathbb{Z}$, then the discrepancy function is bounded. In fact this remains true under the following, somewhat more general, condition: the indicator function $\mathbf{1}_S$ can be expressed as a finite linear combination of indicator functions of intervals whose lengths belong to $\mathbb{Z} + \alpha\mathbb{Z}$. It is interesting to remark that the latter condition is not only sufficient, but also necessary, for the boundedness of the discrepancy function (see [13]).

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Sampling near the critical density

Yurii Lyubarskii
(joint work with Alexander Borichev and Karlheinz Gröchenig)

We study the stability problem for the expansions of functions on the real line with respect to a discrete set of phase-space shifts of a Gaussian, precisely

$$f(x) = \sum_{k,l\in\mathbb{Z}} c_{kl} e^{2\pi i l a x} e^{-\pi(x-bk)^2} . \tag{1}$$

Expansions of such form (with $a = 1$, $b = 1$) were introduced by D. Gabor in his classical article [3]. Now expansions of type (1), so-called Gabor expansions, appear in signal processing, quantum mechanics, time-frequency analysis, the theory of pseudodifferential operators, and other applications.

During the last decades an extensive theory of expansions (1) as well as more general Gabor expansions has been developed (see, for instance [4, 2] and the references therein). However, not much is known about numerical stability property of such expansions.

In modern language, the existence of Gabor expansions is derived from frame theory. To fix terminology and notation, take some $g \in L^2(\mathbb{R})$, it will be called a window function, and let $\Lambda = M\mathbb{Z}^2 \subset \mathbb{R}^2$ be a lattice in $\mathbb{R}^2$, where M is a 2×2 invertible real-valued matrix. Given a point $\lambda = (x, \xi)$ in phase-space $\mathbb{R}^2$, the corresponding time-frequency shift is

$$\pi_\lambda f(t) = e^{2\pi i \xi t} f(t - x), \qquad t \in \mathbb{R} .$$

The set of functions $\mathcal{G}(g, \Lambda) = \{\pi_\lambda g : \lambda \in \Lambda\}$ is called the *Gabor system* generated by g and Λ. We say that such a system is a *Gabor frame* or *Weyl–Heisenberg frame*, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R})$,

$$A\|f\|^2_{L^2(\mathbb{R})} \le \sum_{\lambda\in\Lambda} |\langle f, \pi_\lambda g\rangle_{L^2(\mathbb{R})}|^2 \le B\|f\|^2_{L^2(\mathbb{R})} . \tag{2}$$

The (best possible) constants $A = A(\Lambda, g)$ and $B = B(\Lambda, g)$ in (2) are called the *lower and upper frame bounds* for the frame $\mathcal{G}(g, \Lambda)$.

The ratio $B(\Lambda)/A(\Lambda)$ plays the role of the condition number for the frame $\mathcal{G}(g, \Lambda)$. We investigate behavior of this ration as the density approaches the critical value one. We deal with Gabor frames for the Gaussian window $\mathcal{G}(g_0, \Lambda)$ for the square lattice $\Lambda(a) = a\mathbb{Z} \times a\mathbb{Z}$, $g_0(t) = e^{-t^2}$ and study the behavior of its frame constants $A(a) = A(\Lambda(a))$ and $B(a) = B(\Lambda(a))$ near the critical density $d(\Lambda) = 1$.

The main result of our paper [1] reads as follows

Theorem *There exist constants* $0 < c < C < \infty$ *such that for each* $a \in (1/2, 1)$ *the frame bounds* $A(a)$, $B(a)$ *for the frame* $\mathcal{G}(g_0, \Lambda(a))$ *satisfy*

$$c(1 - a^2) \leq A(a) \leq C(1 - a^2)$$

and

$$c < B(a) < C.$$

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Differentiability of functions in the Zygmund class

Artur Nicolau

The Zygmund class $\Lambda_*(\mathbb{R}^d)$ is the class of bounded continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ for which

$$\|f\|_* = \sup\{\frac{|f(x+h) + f(x-h) - 2f(x)|}{\|h\|} : x, h \in \mathbb{R}^d\} < \infty.$$

The small Zygmund class $\lambda_*(\mathbb{R}^d)$ is the subclass formed by those functions $f \in \Lambda_*(\mathbb{R}^d)$ which satisfy

$$\lim_{\|h\|\to 0} \sup_{x\in\mathbb{R}^d} \frac{|f(x+h) + f(x-h) - 2f(x)|}{\|h\|} = 0.$$

These spaces were introduced by Zygmund in the forties when he observed that the conjugate function of a Lipschitz function in the unit circle does not need to be Lipschitz but it is in the Zygmund class [8]. For $0 < \alpha \leq 1$, let $\Lambda_\alpha(\mathbb{R}^d)$ be the Holder class of bounded functions $f\colon \mathbb{R}^d \to \mathbb{R}$ for which there exists a constant $C = C(f)$ such that $|f(x+h) - f(x)| \leq C\|h\|^\alpha$, for any $x, h \in \mathbb{R}^d$. It is well known that $\Lambda_1(\mathbb{R}^d) \subset \Lambda_*(\mathbb{R}^d) \subset \Lambda_\alpha(\mathbb{R}^d)$ for any $0 < \alpha < 1$ and actually the Zygmund class $\Lambda_*(\mathbb{R}^d)$ is the natural substitute of $\Lambda_1(\mathbb{R}^d)$ in many different contexts. For instance, the Hilbert transform of a compactly supported function in $\Lambda_1(\mathbb{R}^d)$ may not be in $\Lambda_1(\mathbb{R}^d)$, while standard Calderón–Zygmund operators map compactly supported functions in $\Lambda_*(\mathbb{R}^d)$ (respectively in $\Lambda_\alpha(\mathbb{R}^d)$ for some fixed $0 < \alpha < 1$) into $\Lambda_*(\mathbb{R}^d)$ (respectively into $\Lambda_\alpha(\mathbb{R}^d)$). See [3]. The Zygmund class can also be described in terms of harmonic extensions, Bessel potentials or best polynomial approximation and again it is the natural substitute of the Lipschitz class $\Lambda_1(\mathbb{R}^d)$ in these contexts, see [8].

A classical result of Rademacher says that any function in $\Lambda_1(\mathbb{R}^d)$ is differentiable at almost every point. However functions in the Zygmund class as the Hardy–Weierstrass function f_b given by

$$f_b(x) = \sum_{n=0}^{\infty} b^{-n}\cos(2\pi b^n x), \qquad x \in \mathbb{R}, b > 1, \tag{1}$$

may not be differentiable at any point. More generally, let g be an almost periodic function of class $\mathcal{C}^2$ in the real line. Then for any $b > 1$, the function

$$f(x) = \sum_{n=0}^{\infty} b^{-n} g(b^n x), \qquad x \in \mathbb{R},$$

is in the Zygmund class $\Lambda_*(\mathbb{R})$ and under mild assumptions on the function g, Heurteaux has proved that f is nowhere differentiable [5].

Similarly there exist functions in the small Zygmund class which are differentiable at almost no point. However it was already observed by Zygmund in [8] that any function in $\lambda_*(\mathbb{R})$ is differentiable at a dense set of points of the real line. Similarly a function in the Zygmund class $\Lambda_*(\mathbb{R})$ has bounded divided differences at a dense set of points. In the eighties, N. Makarov proved that Zygmund functions on the real line have bounded divided differences at sets of Hausdorff dimension one, see [7].

Theorem 1. (Makarov)
(a) Let $f \in \Lambda_*(\mathbb{R})$. Then the set

$$\left\{x \in \mathbb{R} : \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} < \infty \right\}$$

has Hausdorff dimension 1.
(b) Let $f \in \lambda_*(\mathbb{R})$. Then the function f is differentiable at a set of points of Hausdorff dimension 1.

The main purpose of the talk is to discuss the situation in higher dimensions and present the results in our joint work with J. J. Donaire and J. G. Llorente [1]. Given a function $f \in \Lambda_*(\mathbb{R}^d)$ and a unit vector $e \in \mathbb{R}^d$, let $E(f,e)$ be the set of points where the divided differences of f in the direction of e are bounded, that is,

$$E(f,e) = \left\{x \in \mathbb{R}^d : \limsup_{\mathbb{R} \ni t \to 0} \frac{|f(x+te) - f(x)|}{|t|} < \infty \right\}.$$

There exist functions $f \in \Lambda_*(\mathbb{R}^d)$ such that, for any unit vector $e \in \mathbb{R}^d$, the set $E(f,e)$ has Lebesgue measure zero. However the one dimensional result of Makarov gives that for any function $f \in \Lambda_*(\mathbb{R}^d)$ and any fixed unit vector $e \in \mathbb{R}^d$, the set $E(f,e)$ has Hausdorff dimension d. Similarly for a function $f \in \lambda_*(\mathbb{R}^d)$ and a unit vector $e \in \mathbb{R}^d$, the set

$$\left\{x \in \mathbb{R}^d : \lim_{\mathbb{R} \ni t \to 0} \frac{f(x+te) - f(x)}{t} \text{ exists} \right\}$$

may have Lebesgue measure zero but it has Hausdorff dimension d. For a fixed direction e, the divided differences in this direction, $(f(x+te)-f(x))/t$, $x \in \mathbb{R}^d$, satisfy a certain mean value property with respect to Lebesgue measure in $\mathbb{R}^d$. This is the main point in the proof of Makarov's result as well as in the arguments leading to the fact that $\dim E(f,e)=d$. In this talk we want to study the size of the set $E(f)$ of points where the divided differences in any direction are simultaneously bounded, that is,

$$E(f)=\left\{x \in \mathbb{R}^d : \limsup_{\|h\|\to 0} \frac{|f(x+h)-f(x)|}{\|h\|} < \infty\right\}.$$

Let $\{e_i : i=1,\ldots,d\}$ be the canonical basis of $\mathbb{R}^d$. If $f \in \Lambda_*(\mathbb{R}^d)$ it turns out that

$$E(f)=\bigcap_{i=1}^{d} E(f,e_i).$$

So, the main difficulty in the higher dimensional situation is to obtain a simultaneous control of the divided differences in different directions e_i, $i=1,\ldots,d$. The main result is the following.

Theorem 2.
(a) Let f be a function in $\Lambda_*(\mathbb{R}^d)$. Then the set $E(f)$ has Hausdorff dimension bigger or equal to 1.
(b) Let f be a function in $\lambda_*(\mathbb{R}^d)$. Then f is differentiable at a set of points of Hausdorff dimension bigger or equal to 1.

The result is local in the sense that given $f \in \Lambda_*(\mathbb{R}^d)$ and a cube $Q \subset \mathbb{R}^d$ the set $E(f)\cap Q$ has Hausdorff dimension bigger or equal 1. Similarly given $f \in \lambda_*(\mathbb{R}^d)$ and a cube $Q \subset \mathbb{R}^d$, the function f is differentiable at a set of points in the cube Q which has Hausdorff dimension bigger or equal to 1.

The proof of this result consists in constructing a Cantor type set on which the function f has bounded divided differences. The construction of the Cantor type set uses a stopping time argument based on a certain one dimensional mean value property that the divided differences of f satisfy. Roughly speaking, the divided differences distribute their values in a certain uniform way when measured with respect to length. This is the main new idea in the proof and it allows us to obtain a simultaneous control of the divided differences in the coordinate directions. Moreover the result is sharp in the following sense.

Theorem 3. *There exists a function f in the small Zygmund class $\lambda_*(\mathbb{R}^d)$ such that the set $E(f)$ has Hausdorff dimension* 1.

The one dimensional case may suggest that a natural candidate for the function f in Theorem 3 is a lacunary series. However this is not the case. Actually it turns out that natural lacunary series f in $\Lambda_*(\mathbb{R}^d)$ satisfy $\dim E(f)= d$ ([2]). Instead, the function f will be constructed as $f=\sum g_k$, where $\{g_k\}$ will be a sequence of smooth functions defined recursively with $\sum \|g_k\|_\infty < \infty$. The main idea is to construct them in such a way that $\nabla g_{k+1}(x)$ is *almost orthogonal* to $\nabla \sum_{j=1}^{k} g_j(x)$

and $\sum \|\nabla g_k(x)\|^2 = \infty$ for *most* points $x \in \mathbb{R}^d$. Since one cannot hope to achieve both requirements at all points $x \in \mathbb{R}^d$, an exceptional set A appears. It turns out that the function f is in the small Zygmund class and it is not differentiable at any point in $\mathbb{R}^d \setminus A$. The construction provides the convenient *one dimensional* estimates of the size of the set A.

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A remark on Hardy spaces in infinite variables

Jan-Fredrik Olsen

(joint work with Alexandru Aleman, Anders Olofsson)

This talk is about a work in progress. The motivation is to contribute to the theory of Hardy spaces of ordinary Dirichlet series, i.e. functions of the type $\sum_{n\in\mathbb{N}} a_n n^{-s}$. The most natural of these spaces are the Dirichlet–Hardy spaces

$$\mathcal{H}^2 = \left\{ \sum_{n\in\mathbb{N}} a_n n^{-s} : \sum_{n\in\mathbb{N}} |a_n|^2 < \infty \right\},$$

and

$$\mathcal{H}^\infty = \left\{ \sum_{n\in\mathbb{N}} a_n n^{-s} : \sup_{\Re s>0} \left| \sum_{n\in\mathbb{N}} a_n n^{-s} \right| < \infty \right\}.$$

They were introduced by Hedenmalm, Lindqvist and Seip in [6]. In the natural inner product, the space $\mathcal{H}^2$ has translates of the Riemann zeta function

$$\zeta(s) = \sum_{n\in\mathbb{N}} n^{-s}$$

as its reproducing kernel. It is a result of Hedenmalm, Lindqvist and Seip that $\mathcal{H}^\infty$ is the multiplier algebra of $\mathcal{H}^2$. This is perhaps surprising as the functions in $\mathcal{H}^2$ are in general only defined on the half-plane $\mathbb{C}_{1/2} = \{\Re s > 1/2\}$, while functions in $\mathcal{H}^\infty$ are always analytic on the strictly larger half-plane $\mathbb{C}_0 = \{\Re s > 0\}$.

The theory of the spaces $\mathcal{H}^2$ has had many recent contributions, some of which also to weighted counter-parts. These include analogs to Bergman, Dirichlet and Drury–Arveson spaces of Dirichlet series. See e.g. [4, 7, 1, 8, 10, 11, 9].

We focus on the scale $\mathcal{H}^p$ of Dirichlet–Hardy spaces. One way to realise these spaces is to observe that for Dirichlet polynomials one has the identity

$$\|D\|_{\mathcal{H}^2}^2 = \lim_{T\to\infty} \int_{-T}^{T} |D(\mathrm{i}t)|^2 \mathrm{d}t. \tag{1}$$

(This is a special case of a theorem of F. Carlson [3].) By changing the exponent $p = 2$, it is possible to verify that one has a norm.

Another, equivalent, but perhaps more illuminating way of obtaining these spaces, is to use the observation due to H. Bohr [2] that Dirichlet series can be identified with power series in a countably infinitely many variables. The trick is to identify each monomial z_i by the function $s \longmapsto p_i^{-s}$. In this way $z_1^{\nu_1} \cdots z_k^{\nu_k}$ corresponds to n^{-s}, where $n = p_1^{\nu_1} \cdots p_k^{\nu_k}$, and one obtains the Bohr lift

$$\sum_{n\in\mathbb{N}} a_n n^{-s} \longleftrightarrow \sum_{n\in\mathbb{N}} a_n z_1^{\nu_1} \cdots z_k^{\nu_k}.$$

As this correspondence respects multiplication, the Euler product for the Riemann zeta function yields

$$\zeta(s+\bar{w}) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s-\bar{w}}} \longleftrightarrow \prod_{p \text{ prime}} \frac{1}{1-z_i p_i^{-\bar{w}}}.$$

In other words, the Riemann zeta function is nothing but a product of Szegő kernels.

As the distinguished boundary $\mathbb{T}^\infty$ of the infinite polydisk $\mathbb{D}^\infty$ is a compact abelian group, it has a Haar measure $\mathrm{d}\mu$. With this, one may define the spaces $L^p(\mathbb{T}^\infty)$ in the natural way. Identifying $\chi = (z_1, z_2, \ldots) \in \mathbb{T}^\infty$ with functionals on the positive rational numbers $\mathbb{Q}_+$ by setting $\chi(p_i) = z_i$, and extending multiplicatively, we see that each $f \in L^p(\mathbb{T}^\infty)$ has a Fourier series

$$f \sim \sum_{r\in\mathbb{Q}_+} a_r \chi(r).$$

We say that f is analytic and belongs to $f \in H^p(\mathbb{T}^\infty)$ if $a_r = 0$ whenever $r \notin \mathbb{N}$. In view of the Bohr correspondence, this allows us to define the space $\mathcal{H}^p$ as the isometric image of a subspace of $L^p(\mathbb{T}^\infty)$. It is a consequence of the Birkhoff–Khinchin ergodic theorem and a theorem of Kroenecker that for $p \in [1, \infty)$, the corresponding relation (1) still holds (see [6, 1] and the references therein).

As a preliminary result in our current research effort, we establish an analogue for $H^1(\mathbb{T}^\infty)$ of the theorem of F. and M. Riesz which says that if a function vanishes on a set of measure greater than zero, then this function has to be the zero function. Given $f \in H^1(\mathbb{T}^\infty)$, our approach is to consider the functions $f_\chi(s) = \sum a_n \chi(n) n^{-s}$ for $\chi \in \mathbb{T}^\infty$. As was originally shown by Helson [5], and later proved in this setting in [6, 1], these functions are analytic on $\mathbb{C}_0$ and in the Hardy space $H^1(\mathrm{d}t/(1+t^2))$ for almost every $\chi \in \mathbb{T}^\infty$. By considering such slice functions, it is possible to define a Hardy–Littlewood maximal operator M, bounded from $H^1(\mathbb{T}^\infty)$ into $L^1(\mathbb{T}^\infty)$. Also, it follows that we can extend the Fatou theorem obtained by Saksman and Seip for the space $H^\infty(\mathbb{T}^\infty)$ to $H^1(\mathbb{T}^\infty)$. With these tools in hand, we use the subharmonicity of the slices $f_\chi(s)$ inside of $\mathbb{C}_0$ to prove that $\log|f| \in L^1(\mathbb{T}^\infty)$, whence the desired result follows.

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Completeness of systems of exponentials in L^2-spaces

Alexei Poltoratski

Problems discussed in this talk belong to the area often called the Uncertainty Principle in Harmonic Analysis. This name first appeared as the title of the book [4] by Havin and Jöricke, which covers a large collection of results that could be described by the statement "it's impossible for a non-zero function and its Fourier transform to be simultaneously very small." For example, if a function is supported on a small interval, then the set of zeros of its Fourier transform is sparse. Another example: a small amount of information about the potential of a Schrödinger operator requires a large amount of information about the spectral measure to determine the operator uniquely. Various completeness problems for systems of exponential functions or special functions in L^2-spaces also belong to the same group.

Results I plan to discuss concern families of exponential functions in L^p-spaces. Such questions belong to the very foundations of analysis. The natural problem of approximation of a general wave by combinations of elementary harmonics gave Harmonic Analysis its present name.

Generalizations concerning other special functions, such as Airy or Bessel functions, and other function spaces, can also be treated with similar methods, see examples provided in [5].

Let μ be a finite positive Borel measure on $\mathbb{R}$. Let us consider the family E_Λ of exponential functions $\exp(i\lambda t)$ on $\mathbb{R}$ whose frequencies λ belong to a certain set $\Lambda \subset \mathbb{C}$:

$$E_\Lambda = \{\exp(i\lambda t) |\ \lambda \in \Lambda\}. \tag{1}$$

The classical completeness problem is to find conditions on μ and Λ that ensure completeness of the system E_Λ in $L^p(\mu)$, i.e. density of finite linear combinations of functions from E_Λ in $L^p(\mu)$.

For $1 < p$ the question can be restated via duality: Does there exist $f \in L^p(\mu), f \perp E_\Lambda$? Special cases $p = 1$ and $p = \infty$ have also been studied.

Historically, the following three versions of this problem received most attention from the analytic community.

The Beurling–Malliavin (BM) Problem: $\Lambda = \{\lambda_n\}$ is a sequence, μ is Lebesgue measure on an interval $[0, a]$, $p = 2$: When is $E_{\{\lambda_n\}}$ complete in $L^2(0, a)$?

The Type Problem: $\Lambda = [0, a]$ is an interval, μ is any finite positive measure on $\mathbb{R}$, $p = 2$: When is $E_{[0,a]}$ complete in $L^2(\mu)$?

The Gap Problem: $\Lambda = [0, a]$ is an interval, μ is any finite positive measure on $\mathbb{R}$, $p = 1$: Does there exist $f \in L^1(\mu),\ f \perp E_{[0,a]}$?

The first problem was solved by Beurling and Malliavin in a series of papers in the early sixties, see [1, 2]. The so-called BM-theory created in these papers is considered to be one of the deepest parts of Harmonic Analysis. Over the past 50 years, numerous attempts to generalize BM-theory were undertaken. Some of such results and further open problems are contained in [5, 6].

A solution to the Gap problem was recently suggested in [8]. The solution brings up a number of related questions that will be discussed in the talk.

Perhaps the most popular of the three, the Type Problem remains open in the general case to this day. Partial results, obtained recently in [3, 7], and further questions will be discussed in the talk.

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Metrics From Reproducing Kernel Hilbert Spaces

Richard Rochberg
(joint work with Nicola Arcozzi, Eric T. Sawyer, Brett D. Wick)

Given H, a reproducing kernel Hilbert space, RKHS, there is an associated set, X, such that the elements of H are realized as functions on X. There are ways to use these functions to define a metric on X. Here we discuss the properties of one of these metrics. Further discussion of this metric and of various related metrics is in [3]. Both this note and that paper are exploratory and these topics await, and in our view, invite, systematic study.

1. Introducing a Metric

Suppose H is a reproducing kernel Hilbert space on a set X. It has reproducing kernels $\{k_x(\cdot)\}_{x\in X}$ or $K(y,x)=k_x(y)$. We denote the normalized kernels by $\hat{k}_x = k_x/\|k_x\|$. We define a metric δ on X by $\delta(x,y)=\delta_H(x,y)=\sqrt{1-\left|\left\langle \hat{k}_x,\hat{k}_y\right\rangle\right|^2}$. The metric measures how close the unit vectors $\hat{k}_x$ and $\hat{k}_y$ are to being parallel. If θ is the angle between the two then $\delta(x,y)=\sqrt{1-\cos^2\theta}=|\sin\theta|$ and this observation can be used as the starting point for demonstrating that δ satisfies the triangle inequality [1, Pg. 128]. We will give an alternative approach to the triangle inequality in a moment. For the Hardy space, H^2, δ is the pseudohyperbolic metric ρ on the disk:

$$\delta_{H^2}(z,w)=\sqrt{1-\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}}=\sqrt{\left|\frac{z-w}{1-\bar{z}w}\right|^2}=\rho(z,w),\qquad z,w\in\mathbb{D}.$$

2. The Metric Occurs in Several Places

A number of quantities related to the function theory and operator theory on H can be expressed in terms of δ. Here are several.

Proposition 1 (Coburn [6])**.** *For $x,y\in X$ let P_x and P_y be the self adjoint projections onto the span of k_x and k_y respectively. With this notation*

$$\delta(x,y)=\|P_x-P_y\|=\frac{1}{2}\|P_x-P_y\|_{\text{Trace}}.$$

Proof Discussion. P_x-P_y, is a rank two self adjoint operator with trace zero and so it has two eigenvalues, $\pm\lambda$ for some $\lambda\geq 0$. All the quantities of interest can be expressed in terms on λ. □

Corollary 2. *δ satisfies the triangle inequality.*

Straightforward computation with the skew adjoint rank two operator $[P_a, P_b]$ leads to the following:

Proposition 3. $\|[P_a, P_b]\|^2 = \delta(a,b)^2\left(1 - \delta(a,b)^2\right)$.

Suppose A is a bounded linear map of H to itself. The Berezin transform of $A, \hat{A}$, is a scalar function on X which is a valuable tool for studying A, see for instance [2]. It is traditionally defined by the formula $\hat{A}(x) = \left\langle A\hat{k}_x, \hat{k}_x \right\rangle$ however it can also be described by $\hat{A}(x) = \mathrm{Trace}\left(P_x A\right)$. With this in hand we obtain the sharp modulus of continuity estimates for Berezin transforms.

Proposition 4. *If A is a bounded linear operator on H, $x, y \in X$ then*

$$\left|\hat{A}(x) - \hat{A}(y)\right| \leq 2\|A\|\,\delta(x,y).$$

This estimate is sharp. Given H, x, and y one can select A so that equality holds.

If m is a multiplier for H and M is the corresponding multiplication operator then $\hat{M} = m$. Hence we have the following:

Corollary 5.

$$|m(x) - m(y)| \leq 2\|M\|\,\delta(x,y).$$

3. Invariant and Coinvariant Subspaces

Suppose we are given several RKHSs on a set X and linear maps between them. We would like to use the associated metrics on X to study the relation between the function spaces and to study the linear maps between them. The goal is broad and vague. Here we just report on a particular case where some structure appears in the answer. We suppose we are given H and are given J, a closed subspace of H which is invariant under multiplication by all bounded multipliers of H. Set $J^{\perp} = H \ominus J$. We are interested in the relationship between the metrics δ_H, δ_J, and $\delta_{J^{\perp}}$ associated with H, J, and $J^{\perp}$.

3.1. Triples of Points and the Shape Invariant.

Suppose H is a RKHS on X with kernel functions $\{k_x\}$ and associated distance function δ. Select distinct $x, y, z \in X$. We consider the invariant subspace J of functions which vanish at x. We want to compute $\delta_J(y,z)$ in terms of other data.

For any $\alpha, \beta \in X$ we define $\theta_{\alpha\beta}$ with $0 \leq \theta_{\alpha\beta} \leq \pi$ and $\phi_{\alpha\beta}$ with with $0 \leq \phi_{\alpha\beta} < \pi$ by

$$k_\alpha(\beta) = \langle k_\alpha, k_\beta \rangle = \|k_\alpha\|\,\|k_\beta\|\left(\cos\theta_{a\beta}\right)e^{i\phi_{\alpha\beta}}.$$

and we set

$$\Upsilon = \cos\theta_{xy}\cos\theta_{yz}\cos\theta_{zx}\cos\left(\phi_{xy} + \phi_{yz} + \phi_{zx}\right).$$

We write δ_{xy} for $\delta_H(x,y)$, etc. A straightforward but slightly lengthy computation gives the following very symmetric formula:

$$\frac{\delta_J(y,z)}{\delta_H(y,z)} = \frac{\sqrt{\delta_{xy}^2 + \delta_{xz}^2 + \delta_{zy}^2 - 2 + 2\Upsilon}}{\delta_{xy}\delta_{xz}\delta_{yz}}. \tag{1}$$

An intriguing aspect of this is the appearance of Υ. That quantity is a classical invariant of projective and hyperbolic geometry called the *shape invariant.* It is closely related to the geometry of the triangle in the projective space over H whose vertices are the spans of the kernel functions k_x, k_y and k_z. For more information about the shape invariant see [4], [5], or [9].

One reason for mentioning this is that Υ was the one new term that appeared in (1) and it is slightly complicated. The fact that this quantity has a life of its own in geometry suggests that perhaps the computations we are doing are somewhat natural and may lead to somewhere interesting.

3.2. **Spaces with Complete Nevanlinna Pick Kernels.** There are some classes of RKHS where there are close relations between K_J, the reproducing kernel for the invariant subspace J and K, the kernel function of H. We give one instance in this section and another in the next. In both cases there are consequences for the associated metric functions that follow directly from the definitions, algebraic manipulation and applications of the Cauchy Schwarz inequality, details and variations are in [3]. We will be informal about some of the technical details.

Suppose that H is a RKHS on X with a complete NP kernel $K(\cdot,\cdot)$. Suppose also, and this is for convenience, that we have a distinguished point $\omega \in X$ such that for all x in X, $K(\omega, x) = 1$. The following information about invariant subspaces of H is due to McCullough and Trent [12], further information is in [8].

Proposition 6. *Suppose J is a closed multiplier invariant subspace of H. There are multipliers $\{m_i\}$ so that the reproducing kernel for J is of the form*

$$K_J(x,y) = \left(\sum m_i(x)\overline{m_i(y)}\right) K(x,y). \tag{2}$$

Corollary 7. *If H has a complete NP kernel and J is any closed multiplier invariant subspace of H then for all $x, y \in X$*

$$\delta_J(x,y) \geq \delta_H(x,y) \geq \delta_{J^\perp}(x,y).$$

Remark 8. *The RKHS on the disk with kernel function $K(x,y) = (1-\bar{y}x)^{-\alpha}$, $0 < \alpha \leq 1$ satisfy the hypotheses of the Corollary.*

3.2.1. *Inequalities in the Other Direction; Bergman Type Spaces.* There is a class of RKHS which share many properties of the classical Bergman space, the so-called Bergman type spaces studied in [10] and [11]. Rather than give the full definition we mention that the class includes the classical Bergman space as well as the weighted Bergman spaces between the Hardy space and the classical Bergman space; that is, it includes the spaces in the Remark below. Suppose we have H, a Bergman type space and J, a closed subspace of H that is invariant under all the multiplier operators on H. Suppose further that J has index 1, that is $\dim J \ominus zJ = 1$. Let $\{k_z\}$ be the reproducing kernels for H and $\{j_z\}$ be those for J.

Proposition 9 (Corollary 0.8 of [11])**.** *There is a function $G \in H$ and a positive semidefinite sesquianalytic function $A(z,w)$ so that for $z, w \in \mathbb{D}$*

$$j_z(w) = \overline{G(z)}G(w)(1 - \bar{z}wA(z,w))k_z(w).$$

Corollary 10.

$$\delta_J(z,w) \leq \delta_H(z,w).$$

Remark 11. *The RKHS on the disk with kernel function* $K(x,y) = (1-\bar{y}x)^{-\alpha}$, $1 \leq \alpha \leq 2$ *satisfy the hypotheses of the Corollary.*

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On frequently hypercyclic entire functions

Eero Saksman

(joint work with David Drasin)

Let T be a linear operator on a separable topological vector space E. The operator T is *hypercyclic* if there exists $x \in E$ such that the set of iterates $\{T^n x : n \geq\}$ is dense in E. In this situation x is sometimes called a *universal* element.

Recently there has been considerable interest on a related, more stringent notion. The operator T (and likewise the element $x \in E$) is called *frequently hypercyclic* if $T^n x$ visits any given neighbourhood with a relatively constant rate. More precisely, given any open set $U \subset E$ one asks that the set $A = \{n \geq 1 \ : \ T^n x \in U\}$ has positive density, i.e.

$$\liminf_{n\to\infty} \frac{1}{n} \#(A \cap \{1,\ldots,n\}) > 0.$$

This notion was introduced by Bayart and Grivaux [2] and has been studied in many papers devoted to operators in Hilbert, Banach, or general topological vector spaces. [6] and [1] and the references therein contain more information.

In our talk (based on the note [5]) we considered the classical operator of differentiation $D : \mathcal{E} \to \mathcal{E}$, where $Df(z) := f'(z)$, and the space $\mathcal{E}$ consists of entire functions on the complex plane $\mathbf{C}$, equipped with the standard compact-open topology. The question that was studied is the following: *how slowly can a frequently hypercyclic entire function grow?* This question was posed and first results on it were given by Bonillla and Grosse-Erdmann in [3], [4]. Quite recently, Blasco, Bonilla and Grosse-Erdmann proved lower and upper bounds for the minimal growth of an frequently hypercyclic entire function: any such f satisfies

$$\liminf_{r\to\infty} \frac{M_f(r)}{e^r r^{-1/4}} > 0, \tag{1}$$

where one denotes $M_f(r) := \sup_\theta |f(re^{i\theta})|$. Moreover, they showed that for any given function $\phi : (0,\infty) \to [1,\infty)$ with $\lim_{r\to\infty} \phi(r) = \infty$ there exists a D-frequently hypercyclic f with

$$M_f(r) \leq e^r \phi(r) \qquad \text{for } r \geq 1.$$

They also consider estimates of the growth in terms of the average L^p-norms, that are stated in terms of $M_{f,p}(r) := \big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\big)^{1/p}$.

Our main result determines the optimal growth rate of entire D-frequently hypercyclic functions. It turns out that the sharp result actually corresponds to the lowest possible rate (1) allowed by [1]. In order to get the sharp result one needs to control possible cancellations in a very precise manner, whence the proof invokes the Rudin–Shapiro polynomials and applies a simple heat kernel estimate. The result is the following:

Theorem. (i) *For any $c > 0$ there is an entire frequently hypercyclic function such that*

$$M_f(r) \leq c\frac{e^r}{r^{1/4}} \qquad \text{for all } r \geq 1.$$

This estimate is optimal by (1).

(ii) *More generally, given $p \in [1,\infty]$ there is an entire D-frequently hypercyclic function such that*

$$M_{f,p}(r) \leq c\frac{e^r}{r^{a(p)}} \qquad \text{for all } r > 0,$$

where $a(p) = 1/4$ for $p \in [2,\infty]$ and $a(p) = 1/(2p)$ for $p \in [1,2]$. This estimate is also optimal.

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Corona Theorems and 1-positive Square

Tavan T. Trent

Let Ω be a domain in $\mathbb{C}^n$ and let $\mathcal{H}(\Omega)$ denote a reproducing kernel Hilbert space of analytic functions on Ω. Denote the multiplier algebra on $\mathcal{H}(\Omega)$ by $\mathcal{M}(\mathcal{H}(\Omega))$. Motivated by the classical Carleson corona theorem on $H^\infty(\mathbb{D})$ [4], we are interested in whether the *corona theorem* holds for $\mathcal{M}(\mathcal{H}(\Omega))$. More generally, we also consider related corona problems. Consider the following three possible theorems:

$\boxed{\text{Corona Thm for } \mathcal{M}(\mathcal{H}(\Omega))}$ Assume that $\{f_j\}_{j=1}^n \subseteq \mathcal{M}(\mathcal{H}(\Omega))$ and $\sum |f_j(z_j)|^2 \geq \epsilon^2 > 0$ on Ω. Then there exists

$$\{g_j\}_{j=1}^n \subseteq \mathcal{M}(\mathcal{H}(\Omega)) \text{ with } \sum f_j g_j = 1 \text{ in } \Omega.$$

(A) $\boxed{\mathcal{H}(\Omega)\text{-Corona Thm}}$ Assume that $\{f_j\}_{j=1}^n \subseteq \mathcal{M}(\mathcal{H}(\Omega))$ and $\sum |f_j(z_j)|^2 \geq \epsilon^2 > 0$ on Ω. Let $T_F = (T_{f_1}, \ldots, T_{f_n})$ acting on $B(\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}(\Omega), \mathcal{H}(\Omega))$ denote multiplication by the $f_j' s$.

$$\text{Then there exists a } \delta > 0, \text{ such that } T_F T_F^* \geq \delta^2 I_{\mathcal{H}(\Omega)}$$

(i.e. T_F is *onto*).

(B) $\boxed{\text{Operator Corona Thm for } \mathcal{M}(\mathcal{H}(\Omega))}$ If $T_F T_F^* \geq \delta^2 I_{\mathcal{H}(\Omega)}$ for some $\delta > 0$,

$$\text{then there exists } \{g_j\}_{j=1}^n \subseteq \mathcal{M}(\mathcal{H}(\Omega)) \text{ s.t. } T_F T_G = I_{\mathcal{H}(\Omega)}.$$

It is easy to see that the corona theorem for $\mathcal{M}(\mathcal{H}(\Omega))$ holds if and only if both theorems (A) and (B) are valid. In the following, we will concern ourself with the question of when theorem (B) holds for $\mathcal{M}(\mathcal{H}(\Omega))$.

We say that the reproducing kernel Hilbert space, $\mathcal{H}(\Omega)$ has 1−*positive square*, if its reproducing kernel, $k_w(z)$, can be written as :

$$\frac{1}{k_w(z)} = a_0(z)\,\overline{a_0(w)} - \sum_{n=1}^{\infty} a_n(z)\,\overline{a_n(w)},$$

where $\{a_n\}_n^\infty$ is contained in $\mathcal{M}(\mathcal{H}(\Omega))$. See Agler [1], Quiggen [9], and McCullough [7] for more about such kernels, which are often referred to in the literature as *complete Nevanlinna–Pick* kernels. Basic examples of $\mathcal{H}(\Omega)$ with 1-positive square include, $H^\infty(\mathbb{D})$, Dirichlet space on $\mathbb{D}$, and Drury–Arveson space on $\mathbb{B}^n$. Non-examples include Bergman spaces and Hardy spaces on the unit ball and polydisk in dimensions higher than 1.

Whenever $\mathcal{H}(\Omega)$ has 1-positive square, then theorem (B), the operator corona theorem for $\mathcal{H}(\Omega)$, always holds. This follows from the commutant lifting theorem of Ball, Trent, and Vinnikov [3]. In this case, the corona theorem for $\mathcal{M}(\mathcal{H}(\Omega))$ follows, if the appropriate operators, T_F, are shown to be surjective. This strategy for proving the corona theorems first appeared in Trent [10]. The most outstanding example of this approach to date is the remarkable work of Costea, Sawyer, and Wick [5], which includes the corona theorem for multipliers on Drury–Arveson spaces.

If T_F and T_H are analytic Toeplitz operators acting on $B(\overset{\infty}{\underset{n=1}{\oplus}} H^2(\mathbb{D}))$, an old result of Leech (see [8]) says that if

$$T_F T_F^* \geq T_H T_H^*, \qquad (*)$$

then there exists an analytic Toeplitz operator, $T_G \in B(\overset{\infty}{\underset{n=1}{\oplus}} H^2(\mathbb{D}))$ with $T_F T_G = T_H$ and $\|T_G\| \leq 1$. Of course, from Douglas's lemma [6], condition (*) already gives us *some* operator $C \in B(\overset{\infty}{\underset{n=1}{\oplus}} H^2(\mathbb{D}))$ with $T_F C = T_H$ and $\|C\| \leq 1$, but not necessarily an *analytic Toeplitz* one.

We say that an algebra, $\mathcal{A}$, of $B(\mathcal{H}(\Omega))$ has the *Douglas property* if whenever

$A_{ij}, B_{ij} \in \mathcal{A}$ with $[A_{ij}], [B_{ij}] \in B(\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}(\Omega))$ with $[A_{ij}][A_{ij}]^* \geq [B_{ij}][B_{ij}]^*$,

then there exists $C_{ij} \in \mathcal{A}$ with $[C_{ij}] \in B(\overset{\infty}{\underset{n=1}{\oplus}} \mathcal{H}(\Omega))$ satisfying

$$\begin{aligned} &(1) \qquad [A_{ij}][C_{ij}] \;=\; [B_{ij}] \\ \text{and} \quad &(2) \qquad \|[C_{ij}]\| \;\leq\; 1. \end{aligned}$$

The commutant lifting theorem of [3] implies that if $\mathcal{H}(\Omega)$ has 1-positive square then $\mathcal{M}(\mathcal{H}(\Omega))$ has the *Douglas property*. So theorem (B), the operator corona theorem for $\mathcal{M}(\mathcal{H}(\Omega))$, follows.

Recently, McCullough and I have discovered a partial converse. We say that $\mathcal{H}(\Omega)$ is *nice* if it satisfies the properties of Agler–McCarthy [2] and if the reproducing kernel of $\mathcal{H}(\Omega)$ can be written as:

$$\frac{1}{k_w(z)} = \sum_{n=1}^{N} a_n(z)\,\overline{a_n(w)} - \sum_{n=1}^{M} b_n(z)\,\overline{b_n(w)},$$

where N and M are finite and $\{a_n\}_1^N$ and $\{b_n\}_1^M$ are contained in $\mathcal{M}(\mathcal{H}(\Omega))$. Examples of such *nice* spaces include the Bergman and Hardy spaces on the unit ball and polydisk in all dimensions. Then

Theorem 1. (McCullough–Trent) *Assume that $\mathcal{H}(\Omega)$ is nice. $\mathcal{H}(\Omega)$ has 1-positive square $\Longleftrightarrow$ $\mathcal{M}(\mathcal{H}(\Omega))$ has the Douglas property.*

The above theorem says that the multiplier algebras on the Bergman spaces $B^2(\mathbb{B}^n)$ and $B^2(\mathbb{D}^n)$ for any $n \geq 1$ and for all Hardy spaces, $H^2(\mathbb{B}^n)$ and $H^2(\mathbb{D}^n)$ with $n > 1$ do not have the *Douglas property*. Furthermore, in the above cases it can be shown that a Leech-type theorem also fails.

For example, fix any $n > 1$. Then there exist analytic Toeplitz operators in $B(\overset{\infty}{\underset{n=1}{\oplus}} H^2(\mathbb{D}^n))$ such that

$$T_F T_F^* \geq T_H T_H^*. \qquad (1)$$

But if $C \in B(\overset{\infty}{\underset{n=1}{\oplus}} H^2((D))$ satisfies $T_F C = T_H$, then C is not an analytic Toeplitz operator. Thus there is no solution to $T_F X = T_H$, with X an analytic Toeplitz operator.

We conclude with two problems.

(a) If $\mathcal{H}(\Omega)$ has 1-positive square, does the corona theorem hold for $\mathcal{M}(\mathcal{H}(\Omega))$?

(b) Can the example of the failure of Leech's theorem for $H^2((D^n))$ $(n > 1)$ be chosen so that $T_H = I$ in (1)?

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Trace H^∞-algebras with a given critical constant

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(joint work with Nikolai Nikolski)

We deal with a numerical control of inverses (condition numbers) for functions $T = f(A)$ of large matrices in terms of the lower spectral parameter

$$\delta = \delta(T) = \min |\lambda_j(T)|.$$

Precisely, our problem is the following. Given a sequence $\sigma = \{\lambda_j\}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ of the complex plain, we consider all normalized matrices A, $\|A\| \le 1$ (or Hilbert space operators) such that $\sigma(A) \subset \sigma$ (counting multiplicities) and look for a numerical function $c(\delta) = c(\delta, \sigma)$ bounding the inverses

$$\| T^{-1}\| \le c(\delta)$$

for all $T = f(A)$ having $\delta \le |\lambda_j(T)| \le \|T\| \le 1$, where $\lambda_j(T)$ mean eigenvalues of $T = f(A)$. The best possible upper bound $c(\delta)$ is called $c_1(\delta) = c_1(\delta, \sigma)$,

$$c_1(\delta,\sigma) = \sup\left\{\|T^{-1}\|\colon T = f(A), \delta \le |\lambda_j(T)| \le \|T\| \le 1,\ \sigma(A) \subset \sigma,\ \|A\| \le 1\right\}.$$

Here f can be a polynomial (if A is a finite matrix) or an H^∞ function (if A is a Hilbert space contraction). Recall that

$$H^\infty = \left\{f\colon f \text{ holomorphic on } \mathbb{D} \text{ and } \|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty\right\}.$$

Since $\delta \mapsto c_1(\delta, \sigma)$, $0 < \delta < 1$, is a decreasing function, we can define a *critical constant* (or, an *invertibility threshold*) $\delta_1 = \delta_1(\sigma)$, $0 \le \delta_1 \le 1$, by the following properties

$$0 < \delta < \delta_1 \implies c_1(\delta) = \infty\,,$$
$$\delta_1 < \delta \le 1 \implies c_1(\delta) < \infty\,.$$

The number δ_1 can be considered as a threshold of bounded invertibility or as a threshold for an operator algebra to be *inverse closed*: operators T from our collection with a "*scattered*" spectral data (i.e., $\inf_j |\lambda_j(T)| < \delta_1$, $\|T\| = 1$) are, in general, not invertible, whereas those with "*flat*" spectral data $\delta_1 < \delta \le |\lambda_j(T)| \le \|T\| \le 1$ are invertible.

We show that for H^∞ trace algebras this critical constant δ_1 can take any value in the interval (0,1). Till now there was described only the case $\delta_1 = 0$. We present an effective construction, namely, for an arbitrary $\alpha > 0$ and $\rho > 0$ we take

$$\beta = \frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\,,$$

and put

$$B = \prod_{n=0}^{\infty} \frac{e^{\pi i \beta^n \rho z} + e^{-\pi\alpha}}{1 + e^{\pi(i\beta^n\rho z - \alpha)}}\,.$$

Theorem.
$$\delta_1(H^\infty/BH^\infty) = \frac{1}{\sqrt{1+2\alpha^2}}\,.$$

For a more sophisticated Blaschke product B it is possible to prove the existence of a noninvertible element in the algebra $\delta_1(H^\infty/BH^\infty)$. As an application of this result a counterexample to a stronger form of the Bourgain–Tzafriri restricted invertibility conjecture for bounded operators is exhibited. The classical *Restricted Invertibility Conjecture* asserts that for every bounded operator T on a Hilbert space H and every orthogonal basis $\{e_j\}_{j\in\mathbb{N}}$ satisfying $\inf_j \frac{\|Te_j\|}{\|e_j\|} > 0$, there exists a finite partition $\bigcup_{s=1}^r I_s = \mathbb{N}$ such that all restrictions $T|H_{I_s}$ are left invertible. This conjecture is neither proved nor disproved till now. Using the Blaschke product with a given constant $\delta_1(H^\infty/BH^\infty)$, it is possible to construct a counterexample to a stronger conjecture, where an "orthogonal (or unconditional) basis" is replaced by a "summation basis".

BMO Estimates for the $H^\infty(\mathbb{B}_n)$ Corona Problem

Brett D. Wick
(joint work with Şerban Costea, Eric T. Sawyer)

In 1962 Lennart Carleson demonstrated in [4] the absence of a corona in the maximal ideal space of $H^\infty(\mathbb{D})$ by showing that if $\{g_j\}_{j=1}^N$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

$$1 \geq \sum_{j=1}^N |g_j(z)| \geq \delta > 0, \qquad z\in\mathbb{D}, \tag{1}$$

then there are functions $\{f_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^N f_j(z)\,g_j(z) = 1,\ z\in\mathbb{D} \quad\text{and}\quad \sum_{j=1}^N \|f_j\|_\infty \leq C. \tag{2}$$

Later, Hörmander noted a connection between the Corona problem and the Koszul complex, and in the late 1970's Tom Wolff gave a simplified proof using the theory of the $\overline{\partial}$ equation and Green's theorem. This proof has since served as a model for proving corona type theorems for other Banach algebras. While there is a large literature on such corona theorems in one complex dimension (see e.g. [8]), progress in higher dimensions has been limited. Indeed, apart from the simple cases in which the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$, no corona theorem has been proved in higher dimensions until the recent work of the authors [6] on the Drury–Arveson Hardy space multipliers. Instead, partial results have been obtained, which we will discuss more below.

We of course have the analogous question in several complex variables when we consider $H^{\infty}(\mathbb{B}_n)$. The Corona problem for the Banach algebra $H^{\infty}(\mathbb{B}_n)$ is to show that if $g_1, \ldots, g_N \in H^{\infty}(\mathbb{B}_n)$ satisfy

$$1 \geq \sum_{j=1}^{N} |g_j(z)| \geq \delta \quad \forall z \in \mathbb{B}_n,$$

then the ideal generated by $\{g_j\}_{j=1}^{N}$ is all of $H^{\infty}(\mathbb{B}_n)$, equivalently $\sum_{j=1}^{N} f_j(z) g_j(z) = 1$ for all $z \in \mathbb{B}_n$ for some $f_1, \ldots, f_N \in H^{\infty}(\mathbb{B}_n)$. This famous problem has remained open for $n > 1$ since Lennart Carleson proved the $n = 1$ dimensional case in 1962, but there are some partial results.

Most notably, there is the classical result of Varopoulos where $BMOA(\mathbb{B}_n)$ estimates were obtained for solutions f to the Bézout equation $f_1 g_1 + f_2 g_2 = 1$ [9]. The restriction to just $N = 2$ generators provides some algebraic simplifications to the problem. Note also that the more general equation

$$f_1 g_1 + f_2 g_2 = h, \qquad h \in H^{\infty}(\mathbb{B}_n),$$

can then be solved for $f \in H^{\infty}(\mathbb{B}_n) \cdot BMOA(\mathbb{B}_n)$.

Over two decades later, the case $2 \leq N \leq \infty$ was studied by Andersson and Carlsson [2] in 2000 who obtained $H^{\infty}(\mathbb{B}_n) \cdot BMOA(\mathbb{B}_n)$ solutions f to the infinite Bézout equation $\sum_{i=1}^{\infty} f_i g_i = 1$, and hence also to the more general equation

$$\sum_{i=1}^{\infty} f_i g_i = h, \qquad h \in H^{\infty}(\mathbb{B}_n). \tag{3}$$

To see that $H^{\infty}(\mathbb{B}_n) \cdot BMOA(\mathbb{B}_n)$ is strictly larger than $BMOA(\mathbb{B}_n)$, recall that the multiplier algebra of $BMOA(\mathbb{B}_n)$ is a *proper* subspace of $H^{\infty}(\mathbb{B}_n)$ satisfying a vanishing Carleson condition (see e.g. Theorem 6.2 in [2]).

Our proof uses the methods of [6]. Key to these new estimates are the almost invariant holomorphic derivatives from Arcozzi, Rochberg and Sawyer [3]. Consequently our proof can be used to handle any number of generators N with no additional difficulty and always yields $BMOA(\mathbb{B}_n)$ solutions f to (3). This leads to the main result of [7] in which we obtain $BMOA(\mathbb{B}_n)$ solutions to the $H^{\infty}(\mathbb{B}_n)$ Corona Problem (3) with infinitely many generators.

Theorem 1. *There is a constant $C_{n,\delta}$ such that given $g = (g_i)_{i=1}^{\infty} \in H^{\infty}(\mathbb{B}_n; \ell^2)$ satisfying*

$$1 \geq \sum_{j=1}^{\infty} |g_j(z)|^2 \geq \delta^2 > 0, \qquad z \in \mathbb{B}_n, \tag{4}$$

there is for each $h \in H^{\infty}(\mathbb{B}_n)$ a vector-valued function $f \in BMOA\left(\mathbb{B}_n; \ell^2\right)$ satisfying

$$\begin{aligned} \|f\|_{BMOA(\mathbb{B}_n;\ell^2)} &\leq C_{n,\delta} \|h\|_{H^{\infty}(\mathbb{B}_n)}, \\ \sum_{j=1}^{\infty} f_j(z) g_j(z) &= h(z), \qquad z \in \mathbb{B}_n. \end{aligned} \tag{5}$$

Our method of proof uses the notation and techniques from [6]. First, we show the well known fact that the space $BMOA(\mathbb{B}_n;\ell^2)$ can be identified as the space of Carleson measures for $H^2(\mathbb{B}_n)$, denoted $\mathcal{CM}\left(\mathbb{B}_n;\ell^2\right)$. Namely, we show

Lemma 2. *For $g \in H^2\left(\mathbb{B}_n;\ell^2\right)$ we have*

$$c\left\|g\right\|_{BMOA(\mathbb{B}_n;\ell^2)} \leq \left\|\left(1-|z|^2\right)^{\frac{n}{2}+1} g'\left(z\right)\right\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)} \leq C\left\|g\right\|_{BMOA(\mathbb{B}_n;\ell^2)}. \tag{6}$$

We next use the Koszul complex to reduce the problem to estimates for certain $\overline{\partial}$ problems. Solutions to the $\overline{\partial}$ problem are given by Theorem I.1 on p. 127 of [5], giving the following formula for $(0,q)$-forms:

Theorem 3 (Charpentier, [5])**.** *For $q \geq 0$ and all forms $f\left(\xi\right) \in C^1\left(\overline{\mathbb{B}_n}\right)$ of degree $(0,q+1)$, we have for $z \in \mathbb{B}_n$:*

$$f\left(z\right) = C_q \int_{\mathbb{B}_n} \overline{\partial} f\left(\xi\right) \wedge \mathcal{C}_n^{0,q+1}\left(\xi,z\right) + c_q \overline{\partial}_z \left\{\int_{\mathbb{B}_n} f\left(\xi\right) \wedge \mathcal{C}_n^{0,q}\left(\xi,z\right)\right\}.$$

Here one can compute explicitly that for $0 \leq q \leq n-1$

$$\mathcal{C}_n^{0,q}\left(w,z\right) = \sum_{\nu \in P_n^q} \left(-1\right)^q \Phi_n^q\left(w,z\right) sgn\left(\nu\right)\left(\overline{w_{i_\nu}} - \overline{z_{i_\nu}}\right) \bigwedge_{j \in J_\nu} d\overline{w_j} \bigwedge_{l \in L_\nu} d\overline{z_l} \bigwedge \omega_n\left(w\right).$$

Here, J_ν is a multi-index of length $n-q-1$, L_ν is an index of length q and P_ν is the set of permutations on $\{1,\ldots,n\}$. We also have set

$$\Phi_n^q\left(w,z\right) \equiv \frac{\left(1-w\overline{z}\right)^{n-1-q}\left(1-|w|^2\right)^q}{\triangle\left(w,z\right)^n}, \qquad 0 \leq q \leq n-1, \text{ and} \tag{7}$$

$$\triangle(w,z) \equiv |1-w\overline{z}|^2 - \left(1-|w|^2\right)\left(1-|z|^2\right).$$

Using the Charpentier solution operators $\mathcal{C}_n^{0,q}$ on $(0,q+1)$-forms we have for $\Omega_0^1 h = \frac{\overline{g_j} h}{|g|^2}$ and Γ_0^2 an alternating two-tensor of functions that

$$f \quad = \quad \Omega_0^1 h - \Gamma_0^2(g,\cdot) \equiv \mathcal{F}^0 + \mathcal{F}^1 + \cdots + \mathcal{F}^n \tag{8}$$

is analytic on $\mathbb{B}_n$ and that $f \cdot g = h$. Here, the $\mathcal{F}_j$ are an iteration of certain Charpentier solution operators arising from an application of the Koszul complex. One then must show that the solutions arising from the Koszul complex and via the explicit representation of the solution operators belong to $BMOA(\mathbb{B}_n;\ell^2)$. To accomplish this, one shows that positive operators preserve the class $\mathcal{CM}\left(\mathbb{B}_n\right)$.

Lemma 4. *Let $a,b,c \in \mathbb{R}$. Then the operator*

$$T_{a,b,c} h\left(z\right) = \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^a \left(1-|w|^2\right)^b \left(\sqrt{\triangle\left(w,z\right)}\right)^c}{|1-w\overline{z}|^{n+1+a+b+c}} h\left(w\right) dV\left(w\right) \tag{9}$$

is bounded on $\mathcal{CM}\left(\mathbb{B}_n\right)$ if $c > -2n$ and $-2a < -n < 2\left(b+1\right)$.

One then concludes the proof by observing that for appropriate choices of the parameters a, b, c it is possible to control the terms arising from the Koszul complex. In particular, one shows that for $T_l = T_{a_l,b_l,c_l}$ the following estimates hold

$$\|\mathcal{F}_j\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)} \lesssim \left\|T_1 T_2 \cdots T_j \Omega_0^1 h\right\|_{\mathcal{CM}(\mathbb{B}_n;\ell^2)} \lesssim C_{\delta,n} \|h\|_{H^\infty(\mathbb{B}_n)} .$$

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A resolvent estimate for operators with finite spectrum

Rachid Zarouf

Introduction

Let $T : (\mathbb{C}^n, |\cdot|) \mapsto (\mathbb{C}^n, |\cdot|)$ be an operator acting on a finite dimensional Banach space. We suppose that T satisfies the following *power boundedness condition*:

(PBC) $$P(T) = \sup_{k\geq 0} \left\|T^k\right\|_{E\to E} < \infty .$$

We denote by $\sigma(T) = \{\lambda_1, ..., \lambda_n\}$ the spectrum of T, $r(T) = \max_i |\lambda_i|$ its spectral radius (which satisfies $r(T) \leq 1$ since $P(T) < \infty$), and $R(z, T) = (zId - T)^{-1}$ the resolvent of T at point z, for $z \in \mathbb{C} \setminus \sigma(T)$, Id being the identity operator. Our problem here is to study the quantity $\|R(z, T)\|$.

Having a brief look at published papers on this subject one can notice that $\|R(z, T)\|$ is

1. sometimes associated with the quantity $(|z| - 1)$, (see for instance [2-9]) and
2. sometimes with the quantity $\operatorname{dist}(z, \sigma(T))$, (see for instance [1, 7]).

As regards point 1, we are lead to the so-called *Kreiss resolvent condition* (KRC). In the same spirit, point 2 leads to a strong version of the classical KRC, see for instance [7, Section 5].

1. Kreiss resolvent conditions

1.1. Known results: the classical KRC. The classical KRC is

$$\rho(T) = \sup_{|z|>1} (|z|-1) \|R(z,\,T)\| < \infty. \tag{KRC}$$

There exists a link between the conditions (KRC) and (PBC): they are equivalent. Indeed,

$$\rho(T) \underbrace{\leq}_{(1)} P(T) \underbrace{\leq}_{(2)} en\rho(T),$$

but we have to be careful: (1) is true for every power bounded operator (not necessarily acting on a finite dimensional Banach space) and is very easy to check (by a power series expansion of $R(z,\,T)$), whereas (2) is much more difficult to verify and has been proved only for the Hilbert norm $|\cdot| = |\cdot|_2$. In fact, the statement

$$\text{(KRC)} \Longrightarrow \text{(PBC)},$$

is known as *Kreiss Matrix Theorem* [2]. According to Tadmor, it has been shown originally by Kreiss (1962) with the inequality $P(T) \leq \textit{const}\,(\rho(T))^{n^n}$. It is useful in proofs of stability theorems for finite difference approximations to partial differential equations. Until 1991, the inequality of Kreiss has been improved successively by Morton, Strang, Miller, Laptev, Tadmor, Leveque and Trefethen [4] (with the inequality $P(T) \leq 2en\rho(T)$), and finally Spijker [8] with the inequality (2) above, in which the constant en is sharp.

1.2. A possible extension of the classical KRC. In this paragraph, we focus on the above inequality (1) and assume that $\alpha \in (0,\,1)$. We notice that in this case, $(|z|-1)^\alpha \gg |z|-1$ as $|z| \to 1^+$. As a consequence, we ask the following **question**: is it possible to find a constant $C_\alpha > 0$ such that $\|R(z,\,T)\| \leq C_\alpha \frac{P(T)}{(|z|-1)^\alpha}$, for all $|z| > 1$ and for all T?

The **answer** is “No” if $r(T) = 1$ and “Yes” if $r(T) < 1$ but with a constant $C_\alpha = C_\alpha(n,\,r(T))$ which depends on the size n of T and on its spectral radius $r(T)$.

More precisely, we define

$$\rho_\alpha(T) = \sup_{|z|>1} (|z|-1)^\alpha \|R(z,\,T)\|,$$

and show that $\rho_\alpha(T) < \infty$ if and only if $r(T) < 1$. Moreover, we prove in this case that we can choose

$$C_\alpha(n,\,r(T)) = K_\alpha \frac{n}{(1-r(T))^{1-\alpha}},$$

where K_α is a constant depending only on α. As regards the asymptotic sharpness as $n \to \infty$ and $r(T) \to 1^-$ of the above constant C_α, we show that there exists a contraction A_r on the Hilbert space $(\mathbb{C}^n, |\cdot|_2)$ of spectrum $\{r\}$ such that

$$\liminf_{r\to 1^-}(1-r)^{1-\alpha-\beta}\rho_\alpha(A_r) \geq \cot\left(\frac{\pi}{4n}\right) \geq P(A_r)\cot\left(\frac{\pi}{4n}\right),$$

for all $\beta \in (0, 1-\alpha)$. In this inequality, β is a "parasite" parameter which one can probably avoid.

Finally we mention the fact that the inequality (2) of paragraph 1.1 still holds:

$$P(T) \leq en\rho_\alpha(T),$$

for all $\alpha \in (0, 1]$.

2. A strong version of the classical KRC

What happens if we replace "$(|z|-1)$" by "$\text{dist}(z, \sigma(T))$" in Paragraph 1.1? We define the quantity

$$\rho^{strong}(T) = \sup_{|z|\geq 1} \text{dist}\,(z, \sigma(T))\,\|R(z, T)\|,$$

which satisfies the inequality $\rho^{strong}(T) \geq \rho(T)$, since $r(T) \leq 1$.

In this section we sharpen a result by B. Simon and E.B. Davies [1] which is the following: if $|\cdot| = |\cdot|_2$ is the Hilbert norm on $\mathbb{C}^n$, then

$$\|R(z, T)\| \leq \left(\frac{3n}{\text{dist}(z, \sigma(T))}\right)^{3/2} P(T),$$

for all $|z| \geq 1$, $z \notin \sigma(T)$. They conjecture in [1] that the power $3/2$ is not sharp. In [10], we improve their result (earning a square root at the denominator of the above inequality) and prove that

$$\rho^{strong}(T) \leq \left(\frac{5\pi}{3} + 2\sqrt{2}\right) n^{3/2} P(T).$$

However, we still feel that the constant $n^{3/2}$ is not sharp (n being probably the sharp one).

Remark 1. *Our estimates of $\rho_\alpha(T)$ in Paragraph 1.2 and of $\rho^{strong}(T)$ in Section 2, hold for operators T acting on a Banach space $(E, |\cdot|)$ not necessarily of finite dimension and not necessarily of Hilbert type, but with a finite spectrum $\sigma(T)$.*

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List of open problems

Special Session

The following list contains some of the problems which were posed during the problem session of the workshop.

1. Nicola Arcozzi: A problem

Problem. *Find a geometric characterization of the multipliers and of the Carleson measures for the infinite dimensional Drury–Arveson space.*

Discussion. The *d-dimensional Drury–Arveson space* is the closure of the complex polynomials on the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ with respect to the norm

$$\Big\| \sum_{n\in\mathbb{N}^d} a_n z^n \Big\|^2_{DA_d} = \sum_{n\in\mathbb{N}^d} |a_n|^2 \frac{n!}{|n|!}.$$

Alternatively, DA_d is the Hilbert function space having reproducing kernel $K(z,w) = (1-\overline{z}\cdot w)^{-1}$. The space DA_d and its multiplier space $M(DA_d)$ were introduced by Drury [3] in connection with the multivariable, commutative version of von Neumann's inequality for contractions. The combinatorial, dimensionless nature of the coefficients and the applications to Nevanlinna–Pick theory [1] motivate the interest in the infinite dimensional version of DA_d. A function g is a *multiplier* of DA_d if $f \mapsto M_g f = gf$ has finite operator norm $|||M_g|||_d$ on DA_d. A measure μ on $\mathbb{B}_d$ is a *Carleson measure* for DA_d if the imbedding $DA_d \hookrightarrow L^2(\mu)$ has bounded norm $[\mu]^{1/2}_{CM(d)}$. Since DA_d can be viewed as a weighted Dirichlet space on $\mathbb{B}_d$, for fixed integer d one has that $|||M_g|||^2_d \approx \big[|R^{(m)}g(z)|^2(1-|z|^2)^{2m-d}dV\big]_{CM(d)}$ if $m > (d-1)/2$ is fixed. (Here, R is the complex radial derivative in $\mathbb{B}_d$). Unfortunately, this estimate depends on d, hence finding dimension independent Carleson measure and multiplier estimates are, at the current state of knowledge, two distinct problems.

Geometric characterizations of Carleson measures for DA_d were found in [2], then in [4] and [5]. All proofs make use of dyadic decompositions and the behavior of constants with respect to dimension is certainly not the right one. Functional analysis, however, tells us that $[\mu]_{CM(d)}$ is comparable (independently of d) with the best constant $C(\mu)$ in the bilinear estimate

$$\int_{\mathbb{B}_d} d\mu(z) \int_{\mathbb{B}_d} d\mu(w)\varphi(z)\varphi(w)\Re K(z,w) \leq C(\mu) \int_{\mathbb{B}_d} \varphi^2 d\mu,$$

restricted to measurable $\varphi \geq 0$ (see [2]).

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2. James E. Brennan: Approximation by polynomials on domains of crescent type

Let Ω be a bounded simply connected region of crescent type in the complex plane $\mathbb{C}$; that is, a region whose closure $\overline{\Omega}$ is a compact set having two complementary components, a bounded component G and an unbounded component Ω_∞ with $\partial G \cap \partial\Omega_\infty \neq \varnothing$. We do not require that $\partial G \cap \partial\Omega_\infty$ be a singleton.

For each $z \in \partial G$ let $\delta(z) = \text{dist}(z, \Omega_\infty)$ and let ω be harmonic measure on ∂G relative to some fixed point $x_0 \in G$. Denote by dA two-dimentional Lebesgue measure and for $1 \leq p < \infty$ let $H^p(\Omega, dA)$ be the closed subspace of $L^p(\Omega, dA)$ that is spanned by the complex analytic polynomials. Thus $H^p(\Omega, dA) \subseteq L^p_a(\Omega, dA)$, the apparently larger of the two spaces consisting of those functions in $L^p(\Omega, dA)$ which are analytic in Ω. It is known that there exists a universal constant τ, $0 < \tau < 1$, such that if

$$\int_{\partial G} \log \delta(z)\, d\omega(z) = -\infty, \tag{1}$$

then $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ whenever $1 \leq p < 3 + \tau$.

Problem 1. *Does the divergence of the integral in* (1) *imply that* $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ *for all* p, $1 \leq p < \infty$ *?*

The completeness problem for crescent domains has been studied extensively over the years by Keldysh, Djrbashjan, Shaginjan, Mergeljan, Havin and Maz'ja as well as by the author. If ∂G is sufficiently smooth the answer to the question raised here is yes. In general, $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ for $1 \leq p < 3 + \tau$ whenever

(1) is satisfied and

$$\int_G |\varphi'|^{3+\tau} dA < \infty, \tag{2}$$

where φ is a conformal map of the bounded complementary region G onto the open unit disk. Historically, this was the motivation behind the question:

Problem 2. *For which values of τ is the integral in (2) finite, independent of G?*

Evidently, in Problem 2 the correct upper bound is most likely $\tau < 1$; that is $3 + \tau < 4$. The best known exponent for which (2) is finite in all cases is approximately 3.75, and is due to Hedenmalm and Shimorin [2]. It is not clear that there is any finite upper bound for Problem 1. For an in-depth discussion of the background and history of both problems see [1].

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3. Konstantin M. Dyakonov: Two problems on star-invariant subspaces

Given an inner function θ on the unit disk $\mathbb{D}$, consider the *star-invariant subspace*

$$K^p_\theta := H^p \cap \theta\, \overline{H^p_0}, \qquad 1 \le p \le \infty,$$

of the Hardy space H^p.

Problem 1. Characterize the extreme points of the unit ball of K^∞_θ.

Problem 2. Prove or disprove *Fermat's Last Theorem* for K^p_θ: there are no solutions $f, g, h \in K^p_\theta$ to the equation $f^n + g^n = h^n$ with $n \ge 3$, unless the three functions lie in a one-dimensional subspace of K^p_θ.

In [1], Problem 1 was solved for the simplest inner function $\theta(z) = z^{N+1}$, in which case K^∞_θ reduces to the space of polynomials of degree at most N. On the other hand, a similar problem for K^1_θ was solved in [2] with an arbitrary θ (and even in greater generality).

In connection with Problem 2, we remark that Fermat's Last Theorem is known to be true for polynomials and (equivalently) for rational functions.

References

[1] K. M. Dyakonov, *Extreme points in spaces of polynomials*, Math. Res. Lett. **10** (2003) 717–728.

[2] K. M. Dyakonov, *Interpolating functions of minimal norm, star-invariant subspaces, and kernels of Toeplitz operators*, Proc. Amer. Math. Soc. **116** (1992) 1007–1013.

4. Håkan Hedenmalm: A space of functions associated with the Dirichlet space

This problem comes from [2]. It is related, e.g., to the well-known Brennan conjecture. Let $\mathcal{D}_0$ be the usual Dirichlet space with orthonormal basis $n^{-1/2}z^n$, for $n = 1, 2, 3, \dots$. Let $\mathcal{D}_0^{\odot 2}$ be the space of all functions F of the form

$$F(z) = \sum_{j=1}^{+\infty} t_j f_j(z) g_j(z),$$

where t_j is a bounded sequence of nonnegative reals, and f_j is an orthonormal basis for $\mathcal{D}_0$, and so is g_j. The norm of F is the infimum of $\sup_j t_j$, taken over all possible such representations of F. What can we say about the boundary behavior of functions in $\mathcal{D}_0^{\odot 2}$? In particular, for F of norm < 1 in $\mathcal{D}_0^{\odot 2}$, how fast is the growth of

$$\int_{|z|=r} \exp\left\{ \frac{|F(z)|^2}{\log \frac{1}{1-r^2}} \right\}$$

as $r \to 1^-$? See the survey paper [1] for other spaces associated with the Dirichlet space.

References

[1] N. Arcozzi, R. Rochberg, E. Sawyer, B. Wick, *The Dirichlet space: A Survey.* arXiv:1008.5342

[2] H. Hedenmalm, S. Shimorin, work in preparation.

5. Karim Kellay: A problem

Let f be a function in the Nevanlinna class of unit disc $\mathbb{D}$, we assume that its radial limit f_* is in $L^2(\partial\mathbb{D})$. Let $f = \sum_n a_n z^n$ and suppose that

$$\sum_{n\geq 1} \frac{|a_n|^2}{\omega(n)^2} < \infty \quad \text{et} \quad \sum_{n\geq 0} |\widehat{f}(-n)|^2 \omega(n)^2 < \infty,$$

where $\omega(n) \nearrow +\infty$ when $n \to \infty$. Is it true that $\widehat{f}(-n) = 0$ for all $n \geq 1$ implies $f \in H^2$?

In the case $(\omega(n))_n$ log-concave and $\sum_n \frac{\log \omega(n)}{n^{3/2}} = \infty$ the answer is yes by Nikolski [1]. For a large class of sequences $(\omega(n))_n$ log-concave this is also true, see [2].

References

[1] N. K. Nikolskii, *Selected Problems of Weighted Approximation and Analysis*, Proc. Steklov. Inst. Math. **120** (1974); Amer. Math. Soc., Providence, R.I., 1976.

[2] A. Bourhim, O. El-Fallah, K. Kellay, *Boundary behaviour of functions of Nevanlinna class*, Indiana Univ. Math. J. **53** (2004) 2, 347–395.

6. Artur Nicolau: Two open questions

1. For $k = 1, 2, \ldots$, let $I_k(x)$ be the dyadic interval of length 2^{-k} which contains the point $x \in \mathbb{R}$. Let f be a locally integrable function in the real line and let f_I denote the mean of the function on the interval I. For $n = 1, 2, \ldots$, consider the square function given by

$$\langle f\rangle^2_{n,\infty}(x) = \sum_{k\geq n} (f_{I_{k+1}(x)} - f_{I_k(x)})^2$$

Is it true that

$$\limsup_{n\to\infty} \frac{|f_{I_n(x)} - f(x)|}{\sqrt{\langle f\rangle^2_{n,\infty}(x)|\log|\log\langle f\rangle^2_{n,\infty}(x)||}} < \infty$$

at almost every point $x \in \mathbb{R}$?

2. Describe the smooth 1-periodic functions $\phi : \mathbb{R} \to \mathbb{R}$ for which for any $\lambda > 1$ one has that

$$\sup_N |\sum_{n=1}^N \phi(\lambda^n x)| = \infty,$$

for almost every $x \in \mathbb{R}$.

7. Carl Sundberg: Inverting functions in the Drury–Arveson space

Recall that the Drury–Arveson space H^2_d is the space of analytic functions in the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ such that, if $f(z) = \sum_{n\in\mathbb{N}^d} a_n z^n$, then

$$\|f\|^2_{H^2_d} = \sum_{n\in\mathbb{N}^d} |a_n|^2 \frac{n!}{|n|!} < \infty,$$

where, for $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ we set $|n| = n_1 + \cdots + n_d$ and $n! = n_1! \ldots n_d!$.

Question. *If $f \in H^2_d$ and $|f(z)| \geq 1$ for all $z \in \mathbb{B}_d$, is $\frac{1}{f} \in H^2_d$?*

For $d = 1$, this is trivial since $H^2_1 = H^2$ is the ordinary Hardy space. For $d = 2, 3$, it is also easy since

$$\|f\|^2_{H^2_2} \approx |f(0)|^2 + \int_{\mathbb{B}_d} |Rf(z)|^2 dV_3(z)$$

and

$$\|f\|^2_{H^2_3} \approx |f(0)|^2 + \int_{\partial\mathbb{B}_d} |Rf(z)|^2 d\sigma_3(z),$$

where $Rf(z) = \sum_{j=1}^d z_j \frac{\partial f}{\partial z_j}$, and dV_3, $d\sigma_3$ are the obvious measures on $\mathbb{B}_2$, $\partial\mathbb{B}_3$. A positive answer to our question now follows from $R\left(\frac{1}{f}\right) = -\frac{Rf}{f^2}$.

To illustrate the problem that arises when $d > 3$, let us consider, e.g., $d = 5$. It can be shown that, for $f \in H(\mathbb{B}_5)$,

$$(*) \qquad \|f\|^2_{H^2_5} \approx \int_{\partial\mathbb{B}_5} \|f_\lambda\|^2_{W^2_1} d\sigma_5(\lambda),$$

where for $\lambda \in \partial\mathbb{B}_5$, $f_\lambda \in H(\mathbb{D})$ is given by

$$f_\lambda(z) = f(\lambda z)$$

for $z \in \mathbb{D}(= \mathbb{B}_1)$, and for $f \in H(\mathbb{D})$

$$\|f\|^2_{W^2_2} = \|f\|^2_{H^2} + \|f'\|^2_{H^2} + \|f''\|^2_{H^2}.$$

By differentiating $1/f$ twice, we see our problem for $d = 5$ reduces to proving the inequality

$$\left\| \frac{(f')^2}{f^3} \right\|_{H^2} \leq C\|f\|_{W^2_2}$$

for $f \in W^2_2$. This is a nonlinear Sobolev inequality of a type which does not seem to have been considered before, at least not to my knowledge. It is, I believe, unknown even in the real-valued case.

For $d > 5$ a formula like (*) also holds, but more derivatives get involved, which of course makes the problem even worse.

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