Mathematisches Forschungsinstitut Oberwolfach

Report No. 33/2011

DOI: 10.4171/OWR/2011/33

Differentialgeometrie im Großen

Organised by
Olivier Biquard (Paris)
Xiuxiong Chen (Madison/Hefei)
Bernhard Leeb (München)
Gang Tian (Princeton/Beijing)

July 3rd – July 9th, 2011

Abstract. The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. Global Riemannian geometry with its connections to topology, geometric group theory and geometric analysis remained an important focus of the conference. Special emphasis was given to Einstein manifolds, geometric flows and to the geometry of singular spaces.

Mathematics Subject Classification (2000): 53Cxx, 51Fxx, 51Mxx, 51Kxx, 58Jxx, 32Qxx.

Introduction by the Organisers

The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. Traditionally, the conference series covers a wide scope of different aspects of global differential geometry and its connections with geometric analysis, topology and geometric group theory. The Riemannian aspect is emphasized, but the interactions with the developments in complex geometry and physics play also an important role. Within this spectrum each particular conference gives special attention to two or three topics of particular current relevance.

The scientific program of the last conference had consisted of only 17 talks which left ample time for informal discussions and worked out very well. Nevertheless, we returned this time to a program of 22 talks in order to be able to schedule more of the many interesting talk proposals, especially by young people who attended the conference for the first time.

This time, a main focus of the workshop were *Einstein manifolds* and related topics, represented by six talks concerned with regularity and rigidity results, the structure of 4-manifolds with Ricci curvature bounds, gravitational instantons and Kähler-Einstein metrics.

Another focus were *geometric flows*, with five talks in particular on the Ricci flow in dimensions three and four where its singularities and the long time asymptotics in certain equivariant situations were studied, but also on stability questions for the Ricci flow in higher dimension and noncompact situations, and on other geometric flows related to metrics with special holonomy.

A prominent theme was also the *geometry of singular spaces*, that is, metric spaces with upper or lower sectional curvature bounds (in the sense of Aleksandrov), with five talks discussing the smoothing problem for singular nonpositively curved structures, deformations of hyperbolic cone structures in dimension three, and Tits buildings from the perspective of comparison geometry.

Other talks presented results in complex geometry about extremal Kähler metrics and obstructions to Kähler-Einstein metrics, results on rigidity questions in conformal dynamics, on isoperimetric problems in Lorentzian geometry, and discussed regularity properties of metric spaces in connection with subriemannian (Carnot) geometry.

There were 53 participants from 12 countries, more specifically, 21 participants from Germany, 11 from the United States of America, 8 from France, 3 from Switzerland, 2 from China, 2 from Japan and respectively 1 from Belgium, England, Mexico, Poland, Russia and Spain. This time only 1 participant was a woman (whereas last time 6 women participated). 42% of the participants (22) were young researchers (less than 10 years after diploma or B.A.), both on doctoral and postdoctoral level.

The organizers would like to thank the institute staff for their great hospitality and support before and during the conference. The financial support for young participants, in particular from the Leibniz Association and from the National Science Foundation, is gratefully acknowledged.

Workshop: Differentialgeometrie im Großen

Table of Contents

Abstracts

On Hermitian, Einstein 4-Manifolds

Claude LeBrun

A Riemannian manifold (M, h) is said to be *Einstein* if it has constant Ricci curvature; this happens iff the Ricci tensor r of (M, h) satisfies

$$r = \lambda h$$

for some real number λ, called the Einstein constant [2]. On the other hand, if M has been made into a complex manifold by equipping it with an integrable almost-complex structure J, a Riemannian metric h on (M, J) is called *Hermitian* if, at each point of M, J is an orthogonal transformation with respect to h:

$$h(\cdot, \cdot) = h(J\cdot, J\cdot).$$

If (M, J, h) is Kähler-Einstein, then (M, h) is of course Einstein, and (M, J, h) is of course Hermitian. The theory of the complex Monge-Ampère equation moreover shows that Kähler-Einstein metrics exist on many compact complex manifolds. However, not every compact Hermitian, Einstein manifold is Kähler-Einstein; one of the best-known counter-examples is the Page metric [6] on $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$. More recently, in collaboration with Chen and Weber [3], the author showed that $\mathbb{CP}_2\#2\overline{\mathbb{CP}}_2$ carries an Einstein metric, referred to herein as the CLW metric, which is similarly Hermitian, but not Kähler. The purpose of this talk was to explain the following uniqueness result [5], which asserts that, in real dimension four, these are the only two exceptions to a general pattern:

Theorem 1. *Let (M^4, J) be a compact complex surface, and suppose that h is an Einstein metric on M which is Hermitian with respect to J. Then either*

- *(M, J, h) is Kähler-Einstein; or*
- *$M \approx \mathbb{CP}_2\#\overline{\mathbb{CP}}_2$, and h is a constant times the Page metric; or*
- *$M \approx \mathbb{CP}_2\#2\overline{\mathbb{CP}}_2$ and h is a constant times the CLW metric.*

One key step involves showing that any such metric h must be *conformal* to a Kähler metric g on (M, J); this is a specifically four-dimensional phenomenon, and fails in higher dimensions. Other 4-dimensional phenomena guarantee that this conformally related g is moreover an *extremal Kähler metric* [2] in the sense of Calabi, and this fact then plays a central role in the proof.

While the well-developed theory of Kähler-Einstein metrics [1, 7, 8] has given us a complete understanding of the existence and uniqueness problems for Kähler-Einstein metrics on compact complex surfaces, the existence statement is actually somewhat involved in the $\lambda > 0$ case. Fortunately, Theorem 1 tells us that the answer actually becomes more transparent if we relax the Kähler condition, and merely require that our Einstein metrics be Hermitian:

Theorem 2. *A compact complex surface (M^4, J) admits a Hermitian, Einstein metric if and only if its first Chern class "has a sign." More precisely, (M, J) admits a Hermitian, Einstein metric h with Einstein constant λ iff $c_1(M, J)$ can be expressed as λ times a Kähler class. For fixed $\lambda \neq 0$, this metric h is moreover* unique *modulo biholomorphisms of (M, J).*

It should be emphasized that when h is non-Kähler, the Kähler class mentioned in Theorem 2 is essentially *unrelated* to the Kähler class $[\omega]$ of the extremal Kähler metric g that is conformal to h. Indeed, for both the Page and CLW metrics, the line $\mathbb{R}[\omega] \subset H^2(M, \mathbb{R})$ spanned by the Kähler class of g has irrational slope, in the sense that it only intersects the integer lattice $H^2(M, \mathbb{Z})$ at the origin.

The proof of Theorem 1 depends on the study of the functional

$$\mathcal{C}(g) = \int_M s^2 \, d\mu_g$$

on the space of all Kähler metrics on (M, J); here $s = s_g$ denotes the scalar curvature of g. Any Hermitian, Einstein metric h must be conformal to a critical point g of this functional, and the heart of the argument involves showing that this g must in fact be an *absolute minimizer* of $\mathcal{C}$ on the space of all Kähler metrics (with the Kähler class allowed to vary). If this minimizer is not itself Kähler-Einstein, one then shows that its scalar curvature must be everywhere positive, and that, up to an overall constant, h must actually be given by $s^{-2}g$.

The detailed knowledge of $\mathcal{C}$ required to prove Theorem 1 can also be applied [4] to give a new proof of the existence of the CLW metric by bubbling off extremal Kähler metrics from $\mathbb{CP}_2\#3\overline{\mathbb{CP}}_2$.

Theorem 3. *There is an extremal Kähler metric g on $\mathbb{CP}_2\#2\overline{\mathbb{CP}}_2$ which minimizes the functional $\mathcal{C}$ among all Kähler metrics, and which is conformal to an Einstein metric. Moreover, there is a 1-parameter family $\{g_t \mid t \in [0, 1)\}$, of extremal Kähler metrics on $\mathbb{CP}_2\#3\overline{\mathbb{CP}}_2$ such that g_0 is Kähler-Einstein, and such that $g_{t_j} \to g$ in the Gromov-Hausdorff sense for a sequence $t_j \nearrow 1$.*

The Page metric on $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$ can also be reconstructed in exactly the same manner. But since the Page metric can actually be written down in closed form, this hardly seems worth the bother!

References

[1] T. Aubin, *Equations du type Monge-Ampère sur les variétés kählériennes compactes*, C. R. Acad. Sci. Paris, 283A (1976), pp. 119–121.

[2] A. L. Besse, *Einstein manifolds*, vol. 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987.

[3] X. X. Chen, C. LeBrun, and B. Weber, *On conformally Kähler, Einstein manifolds*, J. Amer. Math. Soc. **21** (2008), 1137–1168.

[4] C. LeBrun, *Einstein manifolds and extremal Kähler metrics.* e-print arXiv:1009.1270 [math.DG], 2010.

[5] C. LeBrun, *On Einstein, Hermitian 4-Manifolds*, e-print arXiv:1010.0238 [math.DG], 2010.

[6] D. Page, *A compact rotating gravitational instanton*, Phys. Lett., **79B** (1979), 235–238.

[7] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Inv. Math., 101 (1990), pp. 101–172.
[8] S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. USA, 74 (1977), pp. 1789–1799.

Rigidity and stability of Einstein metrics for quadratic curvature functionals

Jeff A. Viaclovsky
(joint work with Matthew J. Gursky)

This talk is concerned with the functional

$$\mathcal{F}_t[g] = \int |Ric|^2 \, dV + t \int R^2 \, dV, \tag{1}$$

in dimensions $n \geq 3$, where Ric is the Ricci tensor, R is the scalar curvature, and $t \in \mathbb{R}$ is a real parameter. In dimensions other than four, the functional $\mathcal{F}_t$ is not scale-invariant. Therefore, we will consider the volume-normalized functional

$$\tilde{\mathcal{F}}_t[g] = Vol(g)^{\frac{4}{n}-1} \mathcal{F}_t[g]. \tag{2}$$

Any Einstein metric is critical for $\tilde{\mathcal{F}}_t$.

On an Einstein manifold, the Lichnerowicz Laplacian is given by

$$\Delta_L h_{ij} = \Delta h_{ij} + 2R_{ipjq}h^{pq} - \frac{2}{n} R h_{ij},$$

and $\mathrm{spec}_{TT}(-\Delta_L)$ will denote the set of eigenvalues of $(-\Delta_L)$ restricted to transverse-traceless (TT) tensors. The term *infinitesimal rigidity* for a critical point of a Riemannian functional will refer to the non-existence of non-trivial TT solutions of the linearized equations. For an Einstein metric, infinitesimal rigidity with respect to the total scalar curvature functional is the condition

$$\frac{2}{n} R \notin \mathrm{spec}_{TT}(-\Delta_L).$$

The term *rigidity* will will refer to a metric being an isolated critical point of a functional in the moduli space of Riemannian metrics.

We first state a rigidity result for the functional $\tilde{\mathcal{F}}_0 = Vol^{\frac{4}{n}-1} \int |Ric|^2 dV$:

Theorem 1. Let (M, g) be an n-dimensional Einstein manifold, $n \geq 3$. If

$$\left\{ \frac{2}{n} R, \frac{4}{n} R \right\} \notin \mathrm{spec}_{TT}(-\Delta_L), \tag{3}$$

then g is rigid for $\tilde{\mathcal{F}}_0$. The same result holds when $R < 0$, assuming $n = 3$ or $n = 4$.

Next, we have a theorem regarding local minimization of $\tilde{\mathcal{F}}_t$, up to diffeomorphism and scaling:

Theorem 2. Let (M,g) be an n-dimensional Einstein manifold, $n \geq 3$, and let $t > -1/n$. If $R > 0$ and

$$\text{spec}_{TT}(-\Delta_L) \cap \Big[\frac{2}{n}R, \Big(\frac{4}{n} + 2t\Big)R\Big] = \varnothing, \tag{4}$$

then g is a strict local minimizer for $\tilde{\mathcal{F}}_t$. If $R < 0$ and $n = 3$ or $n = 4$, then the same result holds provided that endpoints in (4) are reversed.

By the Gauss-Bonnet theorem in dimension four, for $t = -1/3$, $\tilde{\mathcal{F}}_t$ is equivalent to the functional $\mathcal{W} \equiv \int |W|^2$, and is therefore conformally invariant. A critical metric satisfies the Euler-Lagrange equations

$$B_{ij} \equiv -4\Big(\nabla^k\nabla^l W_{ikjl} + \frac{1}{2}R^{kl}W_{ikjl}\Big) = 0. \tag{5}$$

The tensor B_{ij} is known as the Bach tensor. Einstein metrics are in particular Bach-flat, and our main result regarding their rigidity and stability is the following:

Theorem 3. Let (M^4, g) be an Einstein manifold. Assume that

$$\Big\{\frac{1}{3}R, \frac{1}{2}R\Big\} \notin \text{spec}_{TT}(-\Delta_L). \tag{6}$$

Then g is Bach-rigid.

If $R > 0$, and g moreover satisfies

$$\text{spec}_{TT}(-\Delta_L) \cap \Big[\frac{1}{3}R, \frac{1}{2}R\Big] = \varnothing, \tag{7}$$

then g is strict local minimizer for $\mathcal{W}$. The same result holds if $R < 0$, provided that the endpoints of the interval in (7) are reversed.

The simplest concrete example for the above theorems is given by the round sphere (S^n, g_S):

Theorem 4. On (S^n, g_S), or any constant curvature quotient thereof, if $n \geq 4$, g_S is a strict local minimizer for $\tilde{\mathcal{F}}_t$ provided that

$$\frac{4-3n}{2n(n-1)} < t < \frac{2}{n(n-1)}. \tag{8}$$

If $n = 3$, the same conclusion holds provided that

$$-\frac{3}{8} < t < \frac{1}{3}. \tag{9}$$

For any $n \geq 3$, g_S is a strict local minimizer for $\tilde{\mathcal{R}}$.

In the case of $n = 3$, there is an interesting variation of the round metric given by scaling the fibers of the Hopf fibration $S^1 \to S^3 \to S^2 = \mathbb{CP}^1$, known as *Berger spheres*. One can employ this variation to show that the upper endpoint in (9) is sharp: if $n = 3$, g_S is *not* a local minimizer for $t = 1/3$, that is, there is a path g_s with $g_0 = g_S$, with $\mathcal{F}(g_s) < \mathcal{F}(g_0)$ for $s < 0$.

Other examples for the above theorems are compact hyperbolic manifolds, complex projective space $(\mathbb{CP}^m, g_{FS})$, and products of round spheres $(S^m \times S^m, g_1 + g_2)$ with $m \geq 2$, and Ricci-flat metrics, see [1] for details and references.

The classical Bishop's inequality implies that if (M, g) is a closed manifold with $Ric(g) \geq Ric(S^n) = (n-1)g$, then the volume satisfies $Vol(g) \leq Vol(S^n)$, and equality holds only if (M, g) is isometric to the sphere. An interesting consequence of local minimization for $t = 0$ for Einstein metrics is that, locally, a "reverse Bishop's inequality" holds:

Theorem 5. Let (M, g) be an Einstein manifold with positive scalar curvature, normalized so that $Ric(g) = (n-1)g$. Assume g is a strict local minimizer for $\tilde{\mathcal{F}}_0$. Then there exists a $C^{2,\alpha}$-neighborhood U of g such that if $\tilde{g} \in U$ with $Ric(\tilde{g}) \leq (n-1)\tilde{g}$, then $Vol(\tilde{g}) \geq Vol(g)$ with equality if and only if $\tilde{g} = \phi^* g$ for some diffeomorphism $\phi : M \to M$.

It follows from some eigenvalue computations that if (M, g) is a sphere, space form, or complex projective space, then for metrics $\tilde{g}$ near g

$$Ric(\tilde{g}) \leq Ric(g) \Longrightarrow Vol(M, \tilde{g}) \geq Vol(M, g).$$

Theorem 5 has a counterpart for negative Einstein manifolds, which is related to the work of Besson-Courtois-Gallot. Again, we refer the reader to [1] for more details and references.

References

[1] Matthew J. Gursky and Jeff A. Viaclovsky, *Rigidity and stability of Einstein metrics for quadratic curvature functionals*, arXiv:1105.4648, 53 pages, 2011.

Quantitative Stratification and regularity for Einstein manifolds, harmonic maps and minimal surfaces

Aaron Naber
(joint work with Jeff Cheeger)

The focus of this talk is in constructing new regularity results for Einstein manifolds, harmonic maps between Riemannian manifolds, and minimal hypersurfaces. The key point is in taking ineffective, e.g. tangent cone or tangent map, behavior and deriving from this effective estimates on the original space. Although we will focus on these three contexts, it is worth emphasizing that these techniques are quite general and work on a large class of nonlinear pde's.

For Einstein manifolds the results include *apriori* L^p estimates on the curvature $|Rm|$ and the much stronger curvature scale

$$r_{|Rm|}(x) = max\{r > 0 : sup_{B_r(x)}|Rm| \leq r^{-2}\}\,. \tag{1}$$

If we assume additionally that the curvature lies in some L^q we are able to prove that $r^{-1}_{|Rm|}$ lies in weak L^{2q}. More precisely our main Theorem in this context is the following.

Theorem 1. *If M^n is an Einstein manifold $|Rc| \leq n-1$ and $Vol(B_2(x)) \geq v > 0$. Then the following hold.*

(1) *If M is real then for every $0 < p < 1$ we have*

$$\fint_{B_1(x)} |Rm|^p \leq \fint_{B_1(x)} (r^{-1}_{|Rm|})^{2p} \leq C(n, v, p).$$

(2) *If M is Kähler then for every $0 < p < 2$ we have*

$$\fint_{B_1(x)} |Rm|^p \leq \fint_{B_1(x)} (r^{-1}_{|Rm|})^{2p} \leq C(n, v, p).$$

(3) *If M is real and $\fint_{B_2(x)} |Rm|^q \leq \Lambda$, then $\fint_{B_1(x)} (r^{-1}_{|Rm|})^{2s} \leq C$ for every $s < q$.*

(4) *If M is Kähler and $\fint_{B_2(x)} |Rm|^q \leq \Lambda$, then $r^{-1}_{|Rm|}$ lives in weak L^{2q}.*

The fourth result above was also recently proved by Chen and Donaldson when $q \equiv 2$ using very different techniques.

For minimizing harmonic maps $f : M \to N$ between Riemannian manifolds we prove similar results. In particular if we define the regularity scale

$$r_f(x) = max\{r > 0 : sup_{B_r(x)} |\nabla f| \leq r^{-1}\}\,, \tag{2}$$

then we prove the estimates

$$\begin{aligned} \fint_{B_1} |\nabla f|^p \leq \fint_{B_1} (r_f^{-1})^p \leq C\,, \\ \fint_{B_1} |\nabla^2 f|^{\frac{p}{2}} \leq C\,, \end{aligned} \tag{3}$$

for all $0 < p < 3$. These are the first L^p estimates on the gradient for $p > 2$, and the first estimates of any sort on the hessian of f. These estimates are sharp, in that there exists minimizing harmonic maps for which $|\nabla f|$ does not live in L^3.

Finally we prove analagous results for minimizing hypersurfaces. Namely we prove L^p estimates for $p < 7$ for the second fundamental form and its regularity scale.

The proof of the results rely on a new quantitative dimension reduction, that in the process strengthens hausdorff estimates on singular sets to minkowski estimates.

References

[1] J. Cheeger and A. Naber, *Lower Bounds on Ricci Curvature and Quantitative Behavior of Singular Sets*, preprint

[2] J. Cheeger and A. Naber, *Quantitative Stratification and the Regularity of Harmonic Maps and Minimal Currents*, preprint

Regenerating hyperbolic cone 3-manifolds

Joan Porti

We start with an application of the main result. Fix n positive real numbers

$$0 < \beta_1, \ldots, \beta_n \leq \pi/2,$$

satisfying $\sum(\pi - \beta_i) > \pi$. By Andreev theorem, for any choice of $0 < \alpha < \pi/2$, satisfying $2\alpha + \beta_i > \pi$, there exists a unique hyperbolic polyhedron with the combinatorial type of a prism with an n-edged polygonal base, with dihedral angles at the "vertical" edges $\beta_1, \cdots, \beta_n$, and angle α at all "horizontal" edges. See Figure 1.

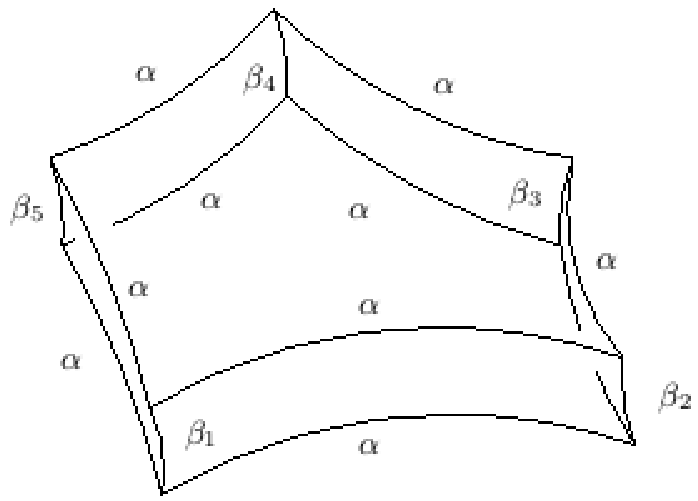

Figure 1. A hyperbolic prism as in Corollary 1.

Now assume that $\alpha \nearrow \pi/2$ and keep $\beta_1, \ldots, \beta_n$ fixed.

Corollary 1. *When $\alpha \nearrow \pi/2$, the prism converges to the n-edged polygon with angles $\beta_1, \ldots, \beta_n$ of minimal perimeter.*

Let $\mathcal{O}^3$ be a closed and orientable 3-orbifold, which is Seifert fibered over a Coxeter two orbifold P^2:

$$S^1 \to \mathcal{O}^3 \to P^2.$$

The branching locus of $\mathcal{O}^3$ is a link or a trivalent graph $\Sigma_{\mathcal{O}^3}$. Its edges and circles are grouped in two, horizontal (if they are transverse to the fibers) or vertical (if they are fibers):

$$\Sigma_{\mathcal{O}^3} = \Sigma^{hor}_{\mathcal{O}^3} \cup \Sigma^{vert}_{\mathcal{O}^3}.$$

Points in $\Sigma^{hor}_{\mathcal{O}^3}$ project to the mirror and dihedral points of P^2. Assume that the orbifold P^2 is a *hyperbolic* Coxeter group, generated by reflections on a hyperbolic polygon whose angles are π over an integer. Thus P^2 is a polygon with mirror points at the edges, and dihedral points at the vertices. We may assume also that P^2 has possibly a single cone point in its interior. For instance, S^3 with branching locus a Montesinos link, other than a two-bridge link, is an example of such fibration.

We view the Seifert fibration as a transversely hyperbolic foliation, hence with a developing map

$$D_0 : \tilde{\mathcal{O}}^3 \to \mathbf{H}^2$$

that factors through the universal covering of P^2.

According to [2] there is a unique point in the Teichmüller space that minimizes the perimeter of P^2 (this also follows from Kerckhoff's proof of Nielsen conjecture [1]). Let

$$P^2_{min}$$

denote the orbifold equipped with this hyperbolic structure.

The main result of this paper is the following:

Theorem 2. *Assume that P^2 has at most one cone point in its interior. There exists a family of hyperbolic cone manifold structures $C(\alpha)$ on $|\mathcal{O}^3|$, with singular locus $\Sigma_{\mathcal{O}^3}$ and cone angle $\alpha \in (\pi - \varepsilon, \pi)$ on $\Sigma^{hor}_{\mathcal{O}^3}$ and constant angles (the orbifold ones) on $\Sigma^{vert}_{\mathcal{O}^3}$, so that*

$$\lim_{\alpha \to \pi^-} C(\alpha) = P^2_{min}$$

for the Gromov-Hausdorff convergence. Moreover the developing maps converge to the developing map of the transversely hyperbolic foliation.

This result is generalized in two ways: by allowing vertical angles that are not integer divisors of 2π (as in Corollary 1) and by changing the speed of the horizontal angles. More details can be found in [3].

References

[1] Steven P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

[2] Joan Porti. Hyperbolic polygons of minimal perimeter with given angles. arXiv:1010.1380, 2010. To apperar in Geom. Dedicata.

[3] Joan Porti. Regenerating hyperbolic cone 3-manifolds from dimension 2. arXiv:1003.2494, 2010.

The deformation space of hyperbolic cone-3-manifolds

Grégoire Montcouquiol
(joint work with Hartmut Weiß)

The goal of this talk is to explain the local deformation theory for (closed, orientable) hyperbolic cone-3-manifolds with cone angles smaller that 2π, as presented in the preprint [4]. We recall that a hyperbolic cone-3-manifold X is a stratified metric space, composed of a regular part M and a singular locus Σ, such that:

- M is an incomplete hyperbolic 3-manifold and a dense open subset of X, whose metric completion is X;

- each point p on Σ has a neighborhood in X in which the metric can be expressed as

$$g = dr^2 + \sinh(r)^2 g_{\bar{L}}, \quad r \in [0, \varepsilon)$$

in spherical coordinates centered at p, where $(\bar{L}, g_{\bar{L}})$ is a spherical cone-surface called the link of p.

If the link of p is equal to $\mathbb{S}^2_\alpha$, the spherical suspension over a circle of length α, then in cylindrical coordinates around p the metric can be also expressed as

$$g = d\rho^2 + \sinh(\rho)^2 d\theta^2 + \cosh(\rho)^2 dz^2, \quad \rho \in [0, \varepsilon),\ \theta \in \mathbb{R}/\alpha\mathbb{Z},\ z \in [-c, c].$$

The set $\{\rho = 0\}$ is called a singular edge, of cone angle α. If the link of p has more than two cone points, then p is called a singular vertex, and lies at the endpoints of several edges.

In two independent papers by H. Weiß and myself [3, 5], the latter based on a collaboration with R. Mazzeo [2], it was proven that the deformations of X preserving the diffeomorphism type of the pair (X, Σ) are locally parametrized by the vector of the edges' cone angles. For general deformations, it is possible to split some vertices of valence greater than 4 into several lower valence vertices, creating new singular edges in the process; locally, these are the only possible modifications of (X, Σ) such that the diffeomorphism type of $M = X \setminus \Sigma$ is still preserved. Now to classify such splitting deformations topologically, one has to describe how the new singular edges sit within X : this is achieved by prescribing the meridians of these new edges. In particular, we will be interested in pants decompositions of the vertices' links.

More precisely, let ∂M be the boundary of a tubular neighborhood of the singular locus, and let g be its genus. For each singular edge e_i of X, we can choose a meridian $\mu_i \subset \partial M$; we denote by $\vec{\mu} = \{\mu_1, \ldots, \mu_N\}$ the meridian set of X. This set cuts ∂M into a disjoint union of punctured spheres S_j, such that S_j is naturally homeomorphic to the regular part L_j of the link of the j-th vertex v_j. Let $\vec{\nu}$ be a pants decomposition of $\coprod_{j=1}^k L_j$; so $\mathcal{C} = \vec{\mu} \cup \vec{\nu}$ gives a pants decomposition of ∂M. A deformation X' of X is then $\vec{\nu}$-compatible if the meridians of the new edges are up to homotopy contained in $\mathcal{C}$.

Our main result is that under some assumptions, the deformations of X are locally parametrized by the cone angles of the original edges and the lengths of the new ones:

Theorem 1. *Let X be a closed orientable hyperbolic cone-3-manifold with cone angles less than 2π and meridian set $\vec{\mu}$. Let $\vec{\nu}$ be a pants decomposition of $\coprod_{j=1}^k L_j$ such that $\mathcal{C} = \vec{\mu} \cup \vec{\nu}$ gives an admissible pants decomposition of ∂M. If all the curves in $\vec{\nu}$ are splittable, then the map*

$$(\alpha, \ell) : C_{-1}(X, \vec{\nu}) \to (0, 2\pi)^N \times \mathbb{R}_+^{3g-3-N}$$

sending a $\vec{\nu}$-compatible cone-manifold structure to the vector composed of its original edges' cone angles and new edges' lengths, is a local homeomorphism at the given structure.

Note that we recover the former parametrization result by setting $\ell = 0$. The first assumption (that $\mathcal{C}$ is admissible) is rather technical: it means that the induced holonomy representation on each pants of the decomposition is irreducible. The second one (that the curves are splittable), while not strictly necessary, is more essential: it ensures that the splittings can be realized geometrically. Indeed, the discrete set of pants decompositions is infinite, and one would expect that most choices of $\vec{\nu}$ do not lead to actual deformations. However, we will give examples in this talk of cone-manifolds admitting infinitely many different splitting deformations.

The main difficulty in the proof of Theorem 1 is to obtain an adapted local chart on $\mathrm{Def}(M)$, or equivalently (using standard results of the deformation theory of hyperbolic structures) on the character variety $X(\pi_1 M, \mathrm{SL}(2,\mathbb{C})) = \mathrm{Hom}(\pi_1 M, \mathrm{SL}(2,\mathbb{C}))/\,\mathrm{SL}(2,\mathbb{C})$. The strategy consists of constructing first a larger, adequate coordinate system on the simpler space $X(\pi_1 \partial M, \mathrm{SL}(2,\mathbb{C}))$, which is provided by action-angle variables: as explained by Goldman in [1], the character variety of $\pi_1 \partial M$ has a canonical complex-symplectic structure, and the traces of the curves in $\mathcal{C}$ yield a holomorphic completely integrable system. Using the infinitesimal rigidity, we can then show that part of these coordinates lifts via the natural restriction map $X(\pi_1 M, \mathrm{SL}(2,\mathbb{C})) \to X(\pi_1 \partial M, \mathrm{SL}(2,\mathbb{C}))$ to a local chart on $\mathrm{Def}(M)$ near X.

To construct actual deformations of the cone-manifold X, we follow the same strategy of beginning with a simpler problem, namely deforming a neighborhood of the singular locus. In particular, we explain how to construct model splitting deformations of a vertex along a given curve, and why the splitting condition (which states that the curve is transverse to a 1-form naturally defined along it) is needed for this construction to work. We also give examples, showing that the cone angles of the new edges can be arbitrarily large. Theorem 1 then follows from combining the deformation obtained near Σ with the corresponding hyperbolic deformation of M.

An interesting consequence of this result is that the cone-manifold deformation space of X is stratified, because of the choice of $\vec{\nu}$; we actually get a geometric realization of a sub-complex of the curve complex of $\coprod_j L_j$. Moreover, this stratification may not be locally finite, when there exists an infinite sequence of non-homotopic splittable curves. But since the character variety is locally compact, this implies that sequences of strata must accumulate. We give an example of this phenomenon, hinting that for the splitting curves we recover convergence in the measured lamination sense.

References

[1] W. Goldman, The complex-symplectic geometry of $SL(2,\mathbb{C})$-characters over surfaces. In *Algebraic groups and arithmetic*, pages 375–407. Tata Inst. Fund. Res., Mumbai, 2004.

[2] R. Mazzeo and G. Montcouquiol, Infinitesimal rigidity of cone-manifolds and the Stoker problem for hyperbolic and Euclidean polyhedra. To appear in *J. Differential Geom.*

[3] G. Montcouquiol, Deformations of hyperbolic convex polyhedra and 3-cone-manifolds. Preprint, available on arXiv:0903.4743, 2009.

[4] G. Montcouquiol and H. Weiss, Complex twist flows on surface group representations and the local shape of the deformation space of hyperbolic cone-3-manifolds. Preprint, available on arXiv:0904.4568, 2009.

[5] H. Weiss, The deformation theory of hyperbolic cone-3-manifolds with cone-angles less than 2π. Preprint, available on arXiv:0904.4568, 2009.

A variational approach to $\mathbf{G_2}$-geometry

Frederik Witt

(joint work with Hartmut Weiß)

In this talk we present a variational approach to G_2-geometry developed in [11] and [12].

G_2-geometry. Consider the standard orthonormal basis of octonions $e_0, \ldots, e_7 \in \mathbb{O}$. The product induces a commutator on the imaginary octonions $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ with skew-symmetric structure constants c_{ijk}, i.e. $[e_i, e_j] = \sum_k c_{ijk} e_k$. We therefore get a 3-form $\Omega = \sum_{i<j<k} c_{ijk} e^{ijk}$ (with $e^{ijk} = e^i \wedge e^j \wedge e^k$ being shorthand for the wedge product of the dual basis). We define Λ^3_+, the space of *positive* forms, to be the orbit of Ω under the action of the orientation-preserving automorphisms $GL(7)_+$. The stabiliser of Ω is conjugate to the simple compact Lie group G_2 and therefore, $\Lambda^3_+ \cong GL(7)_+/G_2$ is an *open* orbit and in fact an open *cone*. The group G_2 is one of the few possible holonomy groups in Berger's list of an irreducible, nonsymmetric Riemannian manifold, and the only one to occur in odd dimensions.

Over a smooth, oriented 7-manifold we can then consider the fibre bundle $\Lambda^3_+ M := P \times_{GL(7)_+} \Lambda^3_+$ associated with the principle bundle $P \to M$ of oriented frames. A global section $\Omega \in C^\infty(\Lambda^3_+ M)$ gives rise to a reduction of P to a principal G_2-bundle. As G_2 sits inside $SO(7)$ we then have an induced metric g_Ω, and via the orientation, a Hodge-$\star$ operator. The obstruction to the existence of Ω is the second Stiefel-Whitney class w_2 of M. Once $w_2 = 0$, any Riemannian metric is induced by such a positive 3-form (though not in a unique way). In particular, any metric with special holonomy is of this form. In fact, the holonomy of g_Ω is contained in G_2 (implying Ricci-flatness) if and only if $d\Omega = 0$, $d\star_{g_\Omega}\Omega = 0$ [5]. The latter condition can be regarded as a non-linear harmonic equation on $C^\infty(\Lambda^3_+ M)$. We refer to solutions as *torsion-free* forms (see for instance [2] for an explanation of this terminology). This characterisation of holonomy G_2-metrics was used for the construction of examples ([1], [3]) and in particular compact ones [9]. Yet a theorem à la Yau which under specific assumptions would ensure a priori existence is still missing.

The Dirichlet functional. Next assume that M is compact. We define the *Dirichlet functional* by

$$\mathcal{D} : C^\infty(\Lambda^3_+ M) \to \mathbb{R}, \quad \Omega \mapsto \tfrac{1}{2}\int_M (|d\Omega|^2_{g_\Omega} + |d\star_\Omega \Omega|^2_{g_\Omega})\, vol_{g_\Omega}\,.$$

This functional is invariant under $\mathit{Diff}(M)_+$, the orientation preserving diffeomorphisms, and positive homogeneous, i.e. $\mathcal{D}(\lambda\Omega) = \lambda^{5/3}\mathcal{D}(\Omega)$ for $\lambda > 0$. Since $\Lambda^3_+ M$ is open, $\mathcal{D}$ can be differentiated. One can show that the absolute minimisers (the torsion-free forms) are the only critical points. Furthermore, critical points subject to the constraint $\int_M vol_{g_\Omega} \equiv 1$ are given by $\operatorname{grad}\mathcal{D}(\Omega) = \lambda_0\Omega$ for a non-negative real constant λ_0. Examples are provided by so-called *weak holonomy* [7] or *nearly parallel* G_2-manifolds [6]. These are characterised by the equation $d\Omega = c\star_{g_\Omega}\Omega$ for $c \neq 0$ and induce an Einstein metric.

To detect critical points it is natural to consider the negative gradient flow.

Theorem 1. *Given a positive 3–form Ω_0, there exists $\varepsilon > 0$ and a smooth family of positive 3–forms $\Omega(t)$ for $t \in [0,\varepsilon]$ such that*

$$\frac{\partial}{\partial t}\Omega = -\operatorname{grad}\mathcal{D}(\Omega), \quad \Omega(0) = \Omega_0.$$

Furthermore, for any two solutions $\Omega(t)$ and $\Omega'(t)$ we have $\Omega(t) = \Omega'(t)$ whenever defined.

We refer to this flow as the *Dirichlet flow.* Note that the $\mathit{Diff}(M)_+$-invariance gives rise to a non-trivial kernel of the principal symbol of the linearised operator $D_\Omega\operatorname{grad}\mathcal{D}$. However, one can show that the symbol is positive-definite with kernel tangent to the $\mathit{Diff}(M)_+$-orbits. Along the lines of deTurck's trick for Ricci flow [4], we define a geometrically perturbed operator $P_{\bar\Omega}$ of $-\operatorname{grad}\mathcal{D}$ depending on a fixed positive form $\bar\Omega$. This new operator is strongly elliptic so that standard parabolic theory applies.

The moduli space. Assume that $\bar\Omega$ is a positive 3-form which is torsion-free, and that M satisfies in addition $H^1(M,\mathbb{R}) = 0$. Then $P_{\bar\Omega}(\Omega) = 0$ if and only if Ω is torsion-free and Ω is perpendicular to the tangent space of the orbit of $\bar\Omega$ under $\mathit{Diff}(M)_0$, the diffeomorphisms isotopic to the identity. Put differently, $P^{-1}_{\bar\Omega}(0)$ is a *slice* for the $\mathit{Diff}(M)_0$-action on $\mathcal{X} = \{\Omega \in C^\infty(\Lambda^3_+ M) \,|\, d\Omega = d\star_{g_\Omega}\Omega = 0\}$. Hence, $\mathcal{M}_{\mathrm{G}_2} = \mathcal{X}/\mathit{Diff}(M)_0$, a G_2-analogon of Teichmüller space, is a smooth manifold. A Hodge theoretic argument then shows that $T_{\bar\Omega}P^{-1}_{\bar\Omega}(0)$ is isomorphic with $H^3(M,\mathbb{R})$. Hence $\dim\mathcal{M}_{\mathrm{G}_2} = b_3$, a fact, together with smoothness, previously established by Joyce [9].

Stability. Define a "G_2-soliton" to be a positive 3-form Ω such that there exists a real constant λ_0 and a vector field X with $-\operatorname{grad}\mathcal{D}(\Omega) = \lambda_0\Omega + \mathcal{L}_X\Omega$. One can show that such a soliton can only exist for $\lambda_0 \leq 0$ and $X = 0$, which is precisely the condition for the constrained critical points above. That is, unless Ω is torsion-free, it is a shrinker and dies in finite time. However, in vicinity of a torsion-free form, we can show:

Theorem 2. *Let $\bar{\Omega} \in \Omega^3_+(M)$ be a torsion–free* G_2*–form. For initial conditions sufficiently C^∞–close to $\bar{\Omega}$ the Dirichlet flow exists for all times and converges modulo diffeomorphisms to a torsion–free* G_2*-form.*

The key properties of the flow we use are "linear stability", i.e. $D^2_{\bar{\Omega}}\mathcal{D} \geq 0$ and the smoothness of the moduli space. Unlike for similar stability theorems for Ricci flow (cf. [10]) these properties hold automatically and need not to be imposed. A main ingredient for longtime existence is uniform existence of the Dirichlet flow on $[0, 1]$ for starting points close to $\bar{\Omega}$. This is done by an implicit function theorem argument in the vein of Huisken and Polden [8]. Convergence modulo diffeomorphisms comes from the analysis of the perturbed flow $\partial_t \Omega = P_{\bar{\Omega}} \Omega_t$ and parabolic regularity.

References

[1] R. Bryant, *Metrics with exceptional holonomy*, Ann. Math. **126** (1987), 525–576.

[2] R. Bryant, *Some remarks on* G_2*-structures*, in: Gökova Geometry/Topology Conference (GGT), pp. 75–109, Gökova (2006).

[3] R. Bryant, S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** no. 3 (1989), 829–850.

[4] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. **28** no. 1 (1983), 157–162.

[5] M. Fernández, A. Gray, *Riemannian manifolds with structure group* G_2, Ann. Mat. Pura Appl. **132** (1982), 19–45.

[6] T. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, *On nearly parallel* G_2*-structures*, J. Geom. Phys. **23** (1997), 259–286.

[7] A. Gray, *Weak holonomy groups*, Math. Z. **123** (1971), 290–300.

[8] G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces*, in: S. Hildebrandt, M. Struwe (ed.), Calculus of Variations and Geometric Evolution Problems LNM 1713, pp. 45–84, Springer, Berlin, 1999.

[9] D. Joyce, *Compact Riemannian 7-manifolds with holonomy* G_2*. I, II.* J. Differential Geom. **43** no. 2 (1996), 291–328, 329–375.

[10] N. Šešum, *Linear and dynamical stability of Ricci–flat metrics*, Duke Math. Jour. **133** no. 1 (2006), 1–26.

[11] H. Weiß and F. Witt, *A heat flow for special metrics*, Preprint 2010, arXiv:0912.0421.

[12] H. Weiß and F. Witt, *Energy functionals and soliton equations for* G_2*–forms*, in preparation.

Gap phenomena in the Ricci flow

Bing Wang

We develop some estimates under the Ricci flow and use these estimates to study the blowup rates of curvatures at singularities. As applications, we obtain some gap theorems: $\sup_X |Ric|$ and $\sqrt{\sup_X |Rm|} \cdot \sqrt{\sup_X |R|}$ must blowup at least at the rate of type-I. Our estimates also imply some gap theorems for shrinking Ricci solitons. The main estimates are listed as follows.

Theorem 1 (Riemannian curvature ratio estimate)**.** *There exists a constant $\varepsilon_0 = \varepsilon_0(m)$ with the following properties.*

Suppose $K \geq 0$, $\left\{(X, g(t)), -\frac{1}{8} \leq t \leq K\right\}$ is a Ricci flow solution on a complete manifold X^m, $Q(0) = 1$, and $Q(t) \leq 2$ for every $t \in [-\frac{1}{8}, 0]$. Suppose T is the first time such that $Q(T) = 2$, then there exists a point $x \in X$ and a nonzero vector $V \in T_x X$ such that

$$\left|\log \frac{\langle V, V\rangle_{g(0)}}{\langle V, V\rangle_{g(T)}}\right| > \varepsilon_0. \tag{1}$$

In particular, we have

$$\int_0^T P(t)dt > \varepsilon_0. \tag{2}$$

Consequently, we have

$$Q(K) < 2^{\frac{\int_0^K P(t)dt}{\varepsilon_0} + 1}. \tag{3}$$

Theorem 2 (Ricci curvature estimate). *Suppose $\left\{(X, g(t)), -\frac{1}{8} \leq t \leq 0\right\}$ is a Ricci flow solution satisfying the following properties.*

- *X is a complete manifold of dimension m.*
- *$|Rm|_{g(t)}(x) \leq 2$ whenever $x \in B_{g(0)}(x_0, 1)$, $t \in \left[-\frac{1}{8}, 0\right]$.*

Then there exists a large constant $A = A(m)$ such that

$$\sup_{B_{g(0)}\left(x_0, \frac{1}{2}\right) \times \left[-\frac{1}{16}, 0\right]} |Ric| \leq A \, \|R\|^{\frac{1}{2}}_{L^\infty\left(B_{g(0)}(x_0,1) \times \left[-\frac{1}{8}, 0\right]\right)}. \tag{4}$$

The estimates and applications can be found in [1] and [2].

References

[1] Bing Wang, *On the condtitions to extend Ricci flow(II)*, Int Math Res Notices (2011), doi:10.1093/imrn/rnr141.

[2] Xiuxiong Chen, Bing Wang *On the conditions to extend Ricci flow(III)*, arXiv: 1107.5110.

Ricci flow on 3-manifolds with symmetry

John Lott

(joint work with Natasa Sesum)

This report concerns the long-time behavior of a three-dimensional Ricci flow, under the assumption that the initial metric has continuous symmetries.

We first consider a global $U(1) \times U(1)$ symmetry. This would be relevant to the case when the manifold M is the total space of a principal $U(1) \times U(1)$ bundle over S^1. More generally, we consider a manifold that fibers over S^1, with T^2-fiber, equipped with an initial metric having a local $U(1) \times U(1)$ symmetry that is globally twisted by an element of $SL(2, \mathbf{Z}) \subset Aut(U(1) \times U(1))$.

The conclusion is that the flow approaches a particular locally homogeneous flow.

Theorem 1 : Let N be an orientable three-manifold that fibers over S^1 with T^2-fibers. Choosing an orientation for S^1, let $H \in SL(2, \mathbf{Z}) = \pi_0(Diff^+(T^2))$ be the holonomy of the torus bundle. We can consider N to be the total space of a twisted principal $U(1) \times U(1)$ bundle, where the twisting is determined by H.

Let $h(\cdot)$ be a Ricci flow solution on N. Suppose that $h(0)$ is invariant under the local $U(1) \times U(1)$ actions. Then the Ricci flow exists for all $t \in [0, \infty)$. There is a constant $C < \infty$ so that for all $p \in N$ and $t \in [0, \infty)$, one has $\left|Riem^N\right|(p,t) \leq \frac{C}{t}$.

(i) If H is elliptic, i.e. has finite order, then $\lim_{t\to\infty} h(t)$ exists and is a flat metric on N. The convergence is exponentially fast.

(ii) Suppose that H is hyperbolic, i.e. has two distinct real eigenvalues. We write $h(t)$ in the form

$$h(t)\colon =\colon g_{yy}(y,t)\colon dy^2\colon +\colon (dx)^T G(y,t)dx,$$

where $\{x^1, x^2\}$ are local coordinates on T^2 and $y \in [0,1)$ is a local coordinate on S^1. Then up to an overall change of parametrizations for S^1 and T^2, we have

$$\lim_{t\to\infty} \frac{g_{yy}(y,t)}{t}\colon =\colon \frac{1}{2}\colon Tr(X^2),$$
$$\lim_{t\to\infty} G(y,t)\colon =\colon e^{yX},$$

where X is the real symmetric matrix such that $e^X = H^T H$. The convergence is power-decay fast in t.

The second result is about an initial metric on $N = S^1 \times M$ with an $O(2)$-symmetry, i.e. a warped product metric. The conclusion is that the flow approaches a product flow.

Theorem 2 : Let $h(\cdot)$ be a Ricci flow solution on a closed connected orientable three-dimensional manifold N. Suppose that $N = S^1 \times M$ and $h(t)$ is a warped product metric

$$h(t) = g(t) + e^{2u(t)} d\theta^2$$

over the two-dimensional orientable base M.

(i) If $\chi(M) > 0$ then there is a finite singularity time $T < \infty$. As $t \to T^-$, the lengths of the circle fibers remain uniformly bounded above and below. For any $p \in N$, the pointed smooth limit $\lim_{t\to T^-}\left(N, p, \frac{1}{T-t}h(t)\right)$ exists and is the isometric product of $\mathbf{R}$ with a sphere S^2 of constant curvature $\frac{1}{2}$.

(ii) If $\chi(M) \leq 0$ then the Ricci flow exists for all $t \in [0, \infty)$. Also, there is a constant $C < \infty$ so that for all $p \in N$ and $t \in [0, \infty)$, one has $\left|Riem^N\right|(p,t) \leq \frac{C}{t}$.

(iii) If $\chi(M) = 0$ then $\lim_{t\to\infty} h(t)$ exists and is a flat metric on T^3. The convergence is exponentially fast.

(iv) If $\chi(M) < 0$, put $\widehat{g}(t) = \frac{g(t)}{t}$. For any $i_0 > 0$, define the i_0-thick part of $(M, \widehat{g}(t))$ by

$$X_{i_0}(t) = \{m \in M \,:\, injrad_{\widehat{g}(t)}(m) \geq i_0\}.$$

Then

$$\lim_{t\to\infty} \max_{x\in X_{i_0}(t)} |R_{\widehat{g}(t)}(x) + 1| = 0$$

and

$$\lim_{t\to\infty} \max_{x\in X_{i_0}(t)} |\widehat{\nabla} u|_{\widehat{g}(t)}(x) = 0.$$

For all sufficiently small i_0, if t is sufficiently large then $X_{i_0}(t)$ is nonempty.

References

[1] J. Lott and N. Sesum, *Ricci flow on three-dimensional manifolds with symmetry*, http://arxiv.org/abs/1102.4384

Rigidity results for some gravitational instantons

Vincent Minerbe

A Riemannian four-manifold (M^4, g) is hyperkähler if its holonomy is in $Sp(1)$, i.e. if it carries three covariant constant complex structures I, J and K satisfying the quaternionic relations ($IJ = JI = K$). Since $Sp(1) = SU(2)$, this is the same as requiring that parallel transport preserves a single complex structure I as well as a I-holomorphic volume $(2,0)$-form. The Ricci curvature of a Kähler manifold can be seen as the curvature of the canonical bundle : a hyperkähler manifold is therefore always Ricci-flat and, if M happens to be simply-connected, hyperkähler is indeed the same as Kähler and Ricci flat.

A gravitational instanton is a complete non-compact hyperkähler four-manifold whose curvature decays faster than quadratically at infinity: $|Riem| = O(r^{-2-\varepsilon})$, where r denotes the distance to some point and ε is a positive number. This assumption is a bit stronger than square integrable curvature: in view of a theorem of J. Cheeger and G. Tian [3], a Ricci-flat four-manifold with curvature in L^2 always satisfies $|Riem| = O(r^{-2})$. Gravitational instantons are nice examples of manifolds with special holonomy and appear naturally in theoretical physics, especially in Euclidean quantum gauge theory and string theory. Besides, they are relevant in the study of the moduli space of Einstein metrics on compact four-manifolds, since they are possible blow-up limits of sequences of such metrics.

Known examples of gravitational instantons arise with special asymptotic geometries. For instance, P. B. Kronheimer [5] used a symplectic quotient method to build examples on minimal resolutions of $\mathbb{C}^2/\Gamma$, where Γ is any finite subgroup of $SU(2)$. These examples are 'ALE' in that they ressemble the flat $\mathbb{C}^2/\Gamma$ at infinity.

Another kind of asymptotics, called 'ALF', is modelled on a circle fibration over a Euclidean base of dimension 3, with fibers of constant non-zero length. The base of this fibration at infinity is either $\mathbb{R}^3$ or $\mathbb{R}^3/\pm$: the former case is called 'ALF of cyclic type' while the latter is 'ALF of dihedral type'. For instance, the standard flat metric on $\mathbb{R}^3 \times \mathbb{S}^1$ and the explicit multi-Taub-Nut metrics are ALF of cyclic type, whereas the Atiyah-Hitchin metric on the moduli space of centered two-monopoles is ALF of dihedral type.

More gravitational instantons arise by solving the Monge-Ampère equation, either by the Tian-Yau continuity method [9, 10, 4] or by a PDE gluing technique [1]. In particular, this makes it possible to desingularize flat hyperkähler orbifolds, producing gravitational instantons with 'ALG' or 'ALH' asympotics: 'ALG' (resp. 'ALH') means the asymptotic geometry is modelled on a $\mathbb{T}^2$ (resp. $\mathbb{T}^3$)-fibration over a Euclidean base of dimension 2 (resp. 1).

It is somehow expected that all gravitational instantons have ALE, ALF, ALG or ALH asymptotics. From basic comparison geometry, the volume growth of balls of radius R is at least linear in R and at most quartic. Since the late eighties, it is known that gravitational instantons with exactly quartic volume growth are ALE [2] and indeed belong to Kronheimer's family of examples [6]. In [7], it is proved that ALF gravitational instantons can also be characterized by their volume growth: *gravitational instantons with cubic volume growth are ALF*. Moreover, there is no gravitational instanton with a volume growth less than quartic and more than cubic.

Finally, it is also natural to ask for a complete classification of ALF gravitational instantons. In [8], it is proved that *ALF gravitational instantons of cyclic type are* $\mathbb{R}^3 \times \mathbb{S}^1$ *and multi-Taub-Nut manifolds*. The proof consists in proving that such gravitational instantons are bound to possess a $\mathbb{S}^1$-symmetry and then exploit this to describe the hyperkähler structure.

References

[1] O. Biquard, V. Minerbe, *A Kummer construction for gravitational instantons*, to appear in Comm. Math. Phys..

[2] S. Bando, A. Kasue, H. Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. 97 (1989), no. 2, 313–349.

[3] J. Cheeger, G. Tian, *Curvature and injectivity radius estimates for Einstein 4-manifolds*, J. Amer. Math. Soc. 19 (2006), 487–525.

[4] H. J. Hein, *Complete Calabi-Yau metrics from $P^2\#9\bar{P}^2$*, arXiv:1003.2646.

[5] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. 29 (1989), no. 3, 665–683.

[6] P. B. Kronheimer, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. 29 (1989), no. 3, 685–697.

[7] V. Minerbe, *On the asymptotic geometry of gravitational instantons* , Ann. Sci. c. Norm. Supr. (4) 43 (2010), no. 6, 883-924.

[8] V. Minerbe, *Rigidity for Multi-Taub-NUT metrics.* J. Reine Angew. Math. 656 (2011), 47–58.

[9] G. Tian, S. T. Yau, *Complete Kähler manifolds with zero Ricci curvature I*, J. Am. Math. Soc. 3 (1990), no.3, 579-609.

[10] G. Tian, S. T. Yau, *Complete Kähler manifolds with zero Ricci curvature II*, Invent. Math. 106 (1991), no.1, 27-60.

Hyperbolic monopoles and Pluricomplex geometry

Lorenz Schwachhöfer

This talk is based on an ongoing research project with Roger Bielawski. Most of the results obtained so far have been published at arXiv:1104.2270: *Roger Bielawski and Lorenz Schwachhöfer*, Pluricomplex Geometry and Hyperbolic Monopoles.

In this research project, we are concerned with a new type of differential geometry, which we call *pluricomplex geometry*. It is a generalization of hypercomplex geometry: we still have a 2-sphere of complex structures, but they no longer behave like unit imaginary quaternions. We still require, however, that the 2-sphere of complex structures determines a decomposition of the complexified tangent space as $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

It turns out that the geometry of such structures is very rich and can be profitably studied from various points of view. For instance, the pluricomplex geometry of a manifold M is the same as a special type of hypercomplex geometry on the bundle

$$T_{\triangle}M = \{(v, \overline{v}); v \in T^{\mathbb{C}}M\}.$$

In fact, an integrable pluricomplex structure (i.e. a 2-sphere of integrable complex structures satisfying the above condition) on a manifold M can be viewed as an integrable hypercomplex structure on an complex thickening $M^{\mathbb{C}}$ of M, commuting with the tautological complex structure of $M^{\mathbb{C}}$ (thus, the pluricomplex geometry of M can be viewed as a biquaternionic geometry of $M^{\mathbb{C}}$). It follows, remarkably enough, that any integrable pluricomplex structure has an associated canonical torsion-free connection (generally without special holonomy).

Throughout this work we are motivated by a particular example: the moduli space $M_{k,m}$ of (framed) $SU(2)$-monopoles of charge k on the hyperbolic 3-space with curvature $-1/m^2$. It is well known that the moduli space of Euclidean monopoles $M_k = M_{k,\infty}$ has a natural hyper-Kähler metric, which is of great physical significance. The moduli space of hyperbolic monopoles is a deformation of M_k. Furthermore, it can be constructed via twistor methods (at least if $2m \in \mathbb{Z}$), which leads one to expect a natural geometry. Hitchin constructed a natural self-dual Einstein metric on the moduli space of centred hyperbolic monopoles of charge $k = 2$. The general case, however, has resisted a solution.

A significant progress has been made by O. Nash, who found a new twistorial construction of $M_{k,m}$ and described the complexification of the natural geometry of $M_{k,m}$, whatever it might be. However, it was evident from this work that the real geometry of $M_{k,m}$ is subtly different form the real geometry of a (complexified) hyper-Kähler metric.

Our aim is to identify this real geometry. As we show it is a strongly integrable pluricomplex geometry. Thus, in particular, there is a natural torsion-free connection on $M_{k,m}$.

References

[1] Roger Bielawski and Lorenz Schwachhöfer, *Pluricomplex Geometry and Hyperbolic Monopoles*, arXiv:1104.2270

Entropy rigidity and coarse geometry

Mario Bonk

According to *Sullivan's dictionary* [10] there is a close correspondence between the dynamics of Kleinian groups and the dynamics of rational maps under iteration.

To discuss one explicit example, let Γ be a *Fuchsian group*, i.e., a uniform lattice in the isometry group of the (real) hyperbolic plane $\mathbb{H}^2$. Suppose $\Gamma \curvearrowright \mathbb{H}^3$ is a *quasi-Fuchsian action* of Γ on hyperbolic 3-space $\mathbb{H}^3$, i.e., an action that is isometric, properly discontinuous, and convex cocompact. Then we have $\dim_H \Lambda(\Gamma) \geq 1$ for the Hausdorff dimension of the limit set $\Lambda(\Gamma) \subseteq \partial_\infty \mathbb{H}^3 \approx \mathbb{S}^2$ with equality if and only if the action stabilizes an isometric copy of $\mathbb{H}^2$ in $\mathbb{H}^3$. Here and in the following we denote by $\partial_\infty X$ the *boundary at infinity* of a space X whenever this is meaningful.

This theorem is due to R. Bowen [6]. Note that in the above situation the limit set $\Lambda(\Gamma)$ is a topological circle (actually a quasi-circle). In view of this, the following statement in complex dynamics attributed to D. Sullivan can be seen as an analog of Bowen's theorem: Let f be a hyperbolic rational map (i.e., a rational map that is expanding on its Julia set in a suitable way), and suppose that its Julia set $\mathcal{J} \subseteq \widehat{\mathbb{C}}$ is a topological circle. Then $\dim_H \mathcal{J} \geq 1$ with equality if and only if $\mathcal{J}$ is a geometric circle.

The purpose of my talk was to discuss generalizations of these type of rigidity theorems in various contexts. The starting point was Hamenstädt's Entropy Rigidity Theorem [8] that can be formulated as follows: Let (M, g_0) be a compact n-manifold, $n \geq 3$, with negatively curved locally symmetric Riemannian metric g_0 normalized so that $K_u(g_0) = -1$, where $K_u(g_0)$ is the *least upper curvature bound*, i.e., the minimal upper bound for the sectional curvatures of (M, g_0). If g is another Riemannian metric on M with $K_u(g) \leq -1$, then we have the inequality $h_{\text{top}}(g) \geq h_{\text{top}}(g_0)$ for the topological entropies of the geodesic flows on the Riemannian manifolds (M, g) and (M, g_0) with equality if and only if (M, g) and (M, g_0) are isometric.

A stronger version of this theorem more in the spirit of metric geometry was established by M. Bourdon [5]. In our context it is convenient to state it in the following way: Suppose Γ is the fundamental group of a locally symmetric Riemannian manifold (M, g_0) as in Hamenstädt's theorem. Let $\Gamma \curvearrowright X$ be an action of Γ on a (proper and geodesic) CAT(-1)-space X so that the action is isometric, properly discontinuous, and quasi-convex cocompact. If $\Lambda(\Gamma)$ is the limit set of the action and S is the Riemannian universal cover of (M, g_0) (S is a negatively curved rank-one symmetric space), then we have the inequality $\dim_H \Lambda(\Gamma) \geq \dim_H \partial_\infty S$ with equality if and only if Γ stabilizes an isometric copy of S in X.

Here $\Lambda(\Gamma)$ and $\partial_\infty S$ are equipped with natural *visual metrics*. The space S is a hyperbolic space $\mathbb{H}^n_{\mathbb{F}}$ modeled on $\mathbb{F}$, where $\mathbb{F}$ is the algebra $\mathbb{R}$ (the real numbers), $\mathbb{C}$ (the complex numbers), $\mathbb{H}$ (the quaternions), or $\mathbb{O}$ (the octonians, in which case necessarily $n = 2$) of real dimension $k = 1, 2, 4, 8$, respectively. If $S = \mathbb{H}^n_{\mathbb{F}}$, then we have $\dim_H \partial_\infty S = nk + k - 2$.

Note that if the setup is as in Hamenstädt's theorem, then the fundamental group Γ of the manifold M induces an action $\Gamma \curvearrowright X$ on the Riemannian universal cover X of (M, g). Moreover, X is a CAT(-1)-space and we have $\dim_H \Lambda(\Gamma) = h_{\mathrm{top}}(g)$.

It would be very interesting to remove the reference to the symmetric spaces in Bourdon's theorem and replace it by an assumption on Γ more in the spirit of metric geometry. This is partially achieved in the following result by B. Kleiner and myself [2, 3].

Theorem 1. *Let X be a* CAT(-1)*-space, and $\Gamma \curvearrowright X$ be a group action that is isometric, properly discontinuous, and quasi-convex cocompact. If $\Lambda(\Gamma) \subseteq \partial_\infty X$ is the limit set of the action, then $\dim_H \Lambda(\Gamma) \geq \dim_{\mathrm{top}} \Lambda(\Gamma) =: n$ with equality if and only if Γ stabilizes an isometric copy of (real) hyperbolic space $\mathbb{H}^{n+1}$.*

Here $\dim_{\mathrm{top}} Z$ denotes the topological dimension of a space Z. This theorem recovers the real hyperbolic version of Bourdon's theorem.

Recently, my student Qian Yin [11] proved a rigidity theorem in complex dynamics that can be seen as an analog of Hamenstädt's Entropy Rigidity Theorem and provides another entry in Sullivan's dictionary. To formulate Yin's result we consider branched covering maps $f\colon S^2 \to S^2$ on a 2-sphere S^2 (for more background on the following discussion see [4]). The *critical points* of f are those points in S^2 where the map is not locally injective. The map f is called a *Thurston map* if the forward orbit of each critical point under iteration of f is finite. This is equivalent with the requirement that the *postcritical set*

$$\mathrm{post}(f) := \bigcup_{n\in\mathbb{N}} \{f^n(c) : c \text{ is a critical point of } f\}$$

of f is a finite set. Here f^n denotes the n-th iterate of f.

Let $f\colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subseteq S^2$ a Jordan curve with $\mathrm{post}(f) \subseteq \mathcal{C}$. By pulling $\mathcal{C}$ back under the iterate f^n we get a natural cell decomposition $\mathcal{D}^n$ of S^2 whose 1-skeleton is the set $f^{-n}(\mathcal{C})$. The 2-dimensional cells in this cell decompositions are the closures of the complementary components of $f^{-n}(\mathcal{C})$ and are called *tiles of level n* or simply *n-tiles*. The cell decompositions $\mathcal{D}^n$ encode much information on the dynamics of the map f and are the basis for studying Thurston maps from a combinatorial point of view.

We say that the Thurston map f is *expanding* if the diameter of n-tiles (for a fixed base metric on S^2) approaches 0 uniformly as $n \to \infty$. If the Thurston map is a rational map on the Riemann sphere, then it is expanding if and only if it has no periodic critical point, or equivalently, if and only if its Julia set is the whole Riemann sphere.

The points in $\text{post}(f)$ decompose $\mathcal{C}$ into arcs. We say that a connected set $K \subseteq S^2$ *joins opposite sides of* $\mathcal{C}$ if K meets two of these arcs that are non-adjacent for $\#\,\text{post}(f) \geq 4$ or if it meets all three arcs for $\#\,\text{post}(f) = 3$. We denote by $D_n(f, \mathcal{C})$ the minimal number of n-tiles needed to form a connected set K joining opposite sides of $\mathcal{C}$. If f is expanding, then $D_n(f, \mathcal{C}) \to \infty$ as $n \to \infty$. Actually, in this case one can show that $D_n(f, \mathcal{C})$ grows exponentially fast and that the exponential growth rate is independent of $\mathcal{C}$.

In the other direction, we always have $D_n(f, \mathcal{C}) \lesssim \deg(f)^{n/2}$, where $\deg(f)$ is the topological degree of f. For a certain class of maps, the so-called Lattès maps (see [9]), we have a maximal growth rate in the sense that $D_n(f, \mathcal{C}) \asymp \deg(f)^{n/2}$ as $n \to \infty$. Q. Yin showed that this property essentially characterizes Lattès maps among expanding Thurston maps [11].

Theorem 2. *Let $f \colon S^2 \to S^2$ be an expanding Thurston map. Then f is topologically conjugate to a Lattès map if and only if the following conditions are true:*

(i) *f has no periodic critical points.*

(ii) *There exists $c > 0$, and a Jordan curve $\mathcal{C} \subseteq S^2$ with* $\text{post}(f) \subseteq \mathcal{C}$ *such that*

$$D_n(f, \mathcal{C}) \geq c \deg(f)^{n/2} \tag{1}$$

for all $n \in \mathbb{N}_0$.

The connection of this theorem to the Entropy Rigidity Theorem is not immediate, but will become clearer after several remarks. First, one can show that $h_{\text{top}}(f) = \log(\deg(f))$ for the topological entropy of f. Moreover, for a given expanding Thurston $f \colon S^2 \to S^2$ and a Jordan $\mathcal{C} \subseteq S^2$ with $\text{post}(f) \subseteq \mathcal{C}$ one can define a *tile graph* $\mathcal{G}(f, \mathcal{C})$ as follows: its set of vertices V is given by the set of tiles on all levels; here one includes $X^{-1} := S^2$ as a tile of level -1 and considers tiles as different if their levels are different even if the underlying sets are the same (which may happen in some exceptional cases). Moreover, one connects two distinct vertices in the graph as represented by an n-tile X^n and a k-tile Y^k by an (unoriented) edge iff $|n - k| \leq 1$ and $X^n \cap Y^k \neq \emptyset$. In this graph the n-tiles form the sphere of radius $n + 1$ centered at the basepoint X^{-1}. One can show that $\mathcal{G}(f, \mathcal{C})$ is Gromov hyperbolic, and that up to *rough isometry* (see [4] for the definition) this graph is independent of $\mathcal{C}$.

Hence one can associate an *asymptotic upper curvature* $K_u(\mathcal{G})$ to $\mathcal{G} = \mathcal{G}(f, \mathcal{C})$ (this notion was introduced in [4]). This quantity only depends on f, but not on $\mathcal{C}$. Moreover, $K_u(\mathcal{G})$ is related to the growth rate of $D_n(f, \mathcal{C})$. One can show that the limit

$$\Lambda_0(f) := \lim_{n \to \infty} D_n(f, \mathcal{C})^{1/n}$$

exists and that

$$K_u(\mathcal{G}) = -\log^2(\Lambda_0(f)).$$

Then the statement $D_n(f, \mathcal{C}) \lesssim \deg(f)^{n/2}$ translates into $K_u(\mathcal{G}) \geq -\frac{1}{4} \log^2(\deg(f))$ which is true for all expanding Thurston maps. One would like to say that here equality occurs for an expanding Thurston map without periodic critical points if and only if f is topologically conjugate to a Lattès map. The "only if" part of this statement is not quite true and one has to make the slightly stronger assumption (1) for the desired conclusion.

The Entropy Rigidity Theorem can be formulated in a similar vein: Let (M, g_0) be as in this theorem, but now g is a negatively curved metric on M such that $h_{\text{top}}(g_0) = h_{\text{top}}(g)$. So we normalize the topological entropy of the geodesic flow instead of imposing a normalizing condition on the least upper curvature bound $K_u(g)$ of the metric g. Then $K_u(g) \geq -1$ with equality if and only if (M, g) and (M, g_0) are isometric.

It is very intriguing that the crucial condition (1) in Yin's theorem only involves combinatorial data of the dynamical system. It is tempting to search for analogs of this in settings such as Theorem 1 where the assumptions on metric geometry are relaxed to "coarser" information. To formulate a possible candidate for such a statement, suppose X is a proper geodesic Gromov hyperbolic space that is an $\mathrm{AC}_u(-1)$-space (see [1]; this is slightly stronger than requiring that $K_u(X) \leq -1$). Suppose that $\Gamma \curvearrowright X$ is an action that is isometric, properly discontinuous, and cocompact, and let $N(R) := \#\{\gamma \in \Gamma : \mathrm{dist}(\gamma(p), p) \leq R\}$, be the *orbit growth function*, where p is a fixed basepoint in X. Suppose that $\partial_\infty X$ is a topological n-sphere S^n and consider the *critical exponent*

$$e(\Gamma) := \limsup_{R \to \infty} \frac{\log N(R)}{R}$$

of the group action (see [7] for more background). *Is it true that in this situation $e(\Gamma) \geq n$ with equality if and only if X is rough isometric to $\mathbb{H}^{n+1}$?*

It seems that by using the results in [2] one can deduce that X is quasi-isometric to $\mathbb{H}^{n+1}$ in case of equality $e(\Gamma) = n$. The desired conclusion that X is rough isometric to $\mathbb{H}^{n+1}$ is much stronger (and may be overly ambitious).

References

[1] M. Bonk and Th. Foertsch, *Asymptotic upper curvature bounds in coarse geometry*, Math. Z. **253** (2006), 753–785.

[2] M. Bonk and B. Kleiner, *Rigidity for quasi-Möbius group actions*, J. Differential Geom. **61** (2002), 81–106.

[3] M. Bonk and B. Kleiner, *Rigidity for quasi-Fuchsian actions on negatively curved spaces*, Int. Math. Res. Not. 2004, no. **61**, 3309–3316.

[4] M. Bonk and D. Meyer, *Expanding Thurston maps*, Preprint, arXiv:1009.3647, 2010.

[5] M. Bourdon, *Sur le birapport au bord des* CAT(−1)*-espaces*, Inst. Hautes Études Sci. Publ. Math. **83** (1996), 95–104.

[6] R. Bowen, *Hausdorff dimension of quasicircles*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 11–25.

[7] M. Coornaert, *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. **159** (1993), 241–270.

[8] U. Hamenstädt, *Entropy-rigidity of locally symmetric spaces of negative curvature*, Ann. of Math. (2) **131** (1990), 35–51.

[9] J. Milnor, *On Lattès maps*, Dynamics on the Riemann sphere, pp. 9–43, Eur. Math. Soc., Zürich, 2006.

[10] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. (2) **122** (1985), 401–418.
[11] Q. Yin, *Lattès maps and combinatorial expansion*, PhD thesis, University of Michigan, 2011.

Geodesic metric spaces with unique blow-up almost everywhere: properties and examples

Enrico Le Donne

In this report we deal with metric spaces that at almost every point admit a tangent metric space. These spaces are in some sense generalizations of Riemannian manifolds. We will see that, at least at the level of the tangents, there is some resemblance of a differentiable structure and of (sub)Riemannian geometry. I will present some results and give examples.

Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be metric spaces. Fix $x_0 \in X$ and $y_0 \in Y$. If there exists $\lambda_j \to \infty$ such that, in the Gromov-Hausdorff convergence,

$$(X, \lambda_j d_X, x_0) \to (Y, d_Y, y_0), \qquad \text{as } j \to \infty,$$

then (Y, y_0) is called *a tangent* (or a *weak tangent*, or a *blow-up*) of X at x_0.

Some remarks are due. Fixed $x_0 \in X$, there might be more than one tangent. Moreover, in general there might not exist any tangent. However, if the distance is doubling, then, by the work of Gromov [Gro81], then tangents exists. Namely, for any sequence $\lambda_j \to \infty$, there exists a subsequence $\lambda_{j_k} \to \infty$ such that $(X, \lambda_{j_k} d_X, x_0)$ converges as $k \to \infty$. A tangent is well defined up to pointed isometry. Thus we define the set of all tangents of X at x_0 as

$$\operatorname{Tan}(X, x_0) := \{\text{tangents of } X \text{ at } x_0\}/\text{pointed isometric equivalence}.$$

We consider two questions: how big is $\operatorname{Tan}(X, x_0)$? what happens when the tangent is unique? The rough answer that we will give are the following. Under some 'standard' assumptions, if $(Y, y_0) \in \operatorname{Tan}(X, x_0)$, then $(Y, y) \in \operatorname{Tan}(X, x)$, for all $y \in Y$. Moreover, in the case of unique tangents, such tangents are very special, however, not much can be said about the initial space X.

Definition and examples. Let $(X_j, x_j), (Y, y)$ be pointed geodesic metric spaces. We write $(X_j, x_j) \to (Y, y)$ in the Gromov-Hausdorff convergence if, for all $R > 0$, we have $d_{GH}(B(x_j, R), B(y, R)) \to 0$. Here

$$d_{GH}(A, B) := \inf\{d_H^Z(A', B') : Z \text{ metric space}, A', B' \subseteq Z, A \overset{isom}{=} A', B \overset{isom}{=} B'\},$$

and $d_H^Z(\cdot, \cdot)$ is the Hausdorff distance in the space Z.

Example 1. When $\mathbb{R}^n$ is endowed with the Euclidean distance (or more generally a norm), we have $\operatorname{Tan}(\mathbb{R}^n, p) = \{(\mathbb{R}^n, 0)\}, \forall p \in \mathbb{R}^n$.

Example 2. Let (M, d) be a Riemannian manifold (or more generally a Finsler manifold), we have $\operatorname{Tan}(M, d, p) = \{(\mathbb{R}^n, \|\cdot\|, 0)\}, \forall p \in \mathbb{R}^n$.

Definition 3 (Carnot group). Let $\mathfrak{g}$ be a stratified Lie algebra, i.e., $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$, with $[V_j, V_1] = V_{j+1}$, for $1 \leq j \leq s$, where $V_{s+1} = \{0\}$. Let $\mathcal{G}$ be the simply-connected Lie group whose Lie algebra is $\mathfrak{g}$. Fix $\|\cdot\|$ on V_1. Define, for any $x, y \in \mathcal{G}$,

$$d_{CC}(x, y) := \inf \left\{ \int_0^1 \|\dot{\gamma}(t)\| dt \mid \gamma \in C^\infty([0,1]; \mathcal{G}), \gamma(0) = x, \gamma(1) = y, \dot{\gamma} \in V_1 \right\}.$$

The pair $(\mathcal{G}, d_{CC})$ is called *Carnot group.*

In particular, any Carnot group $\mathcal{G}$ is a metric space homeomorphic to the Lie group $\mathcal{G}$. Moreover, by the work of Pansu and Gromov [Pan83], the Carnot groups are the blow-downs of left-invariant Riemannnian/Finsler distances on $\mathcal{G}$. Namely, if $\|\cdot\|$ is a norm on $\mathrm{Lie}(\mathcal{G})$ extending the one on V_1 and $d_{\|\cdot\|}$ is the corresponding Finsler distance,

$$(\mathcal{G}, \lambda d_{\|\cdot\|}, 1) \overset{\lambda \to 0}{\longrightarrow} (\mathcal{G}, d_{CC}, 1).$$

Example 4. If $(\mathcal{G}, d_{CC})$ is a Carnot group, then $\mathrm{Tan}(\mathcal{G}, d_{CC}, 1) = \{(\mathcal{G}, d_{CC}, 1)\}$. Indeed, for all $\lambda > 0$, there is a group homomorphism $\delta_\lambda : \mathcal{G} \to \mathcal{G}$ such that $(\delta_\lambda)_*|_{V_1}$ is the multiplication by λ. Consequently, $(\delta_\lambda)_* d_{CC} = \lambda d_{CC}$. QED

Results. Our main theorem is the following.

Theorem 5 ([LD11]). Let (X, d) be a geodesic metric space. Let μ be a doubling measure. Assume that, for μ-almost every $x \in X$, the set $\mathrm{Tan}(X, x)$ contains only one element. Then, for μ-almost every $x \in X$, the element in $\mathrm{Tan}(X, x)$ is a Carnot group.

Example 6 (SubRiemannian manifolds). Let M be a Riemannian manifold (or more generally Finsler). Let $\Delta \subseteq TM$ be a smooth sub-bundle. Let $\mathcal{X}^1(\Delta)$ be the vector fields tangent to Δ. By induction, define $\mathcal{X}^{k+1}(\Delta) := \mathcal{X}^k(\Delta) + [\mathcal{X}^1(\Delta), \mathcal{X}^k(\Delta)]$. Assume that there exists $s \in \mathbb{N}$ such that $\mathcal{X}^s(\Delta) = TM$ and that, for all k, the function $p \mapsto \dim \mathcal{X}^k(\Delta)(p)$ is constant. Define, for any $x, y \in M$,

$$d_{CC}(x, y) := \inf\{\mathrm{Length}(\gamma) \mid \gamma \in C^\infty([0,1]; M), \gamma(0) = x, \gamma(1) = y, \dot{\gamma} \in \Delta\}.$$

Then (M, d_{CC}) is called an *(equiregular) subFinsler manifold.* In such a case, by a theorem of Mitchell, see [Mit85, MM95],

$$\mathrm{Tan}(M, d_{CC}, p) = \{(\mathcal{G}, d_{CC}, 1)\}, \qquad \forall p \in M,$$

with $(\mathcal{G}, d_{CC})$ a Carnot group, which might depend on p.

Theorem 5 is proved using the following general property.

Theorem 7 ([LD11]). Let (X, μ, d) be a doubling-measured metric space. Then, for μ-almost every $x \in X$, if $(Y, y) \in \mathrm{Tan}(X, x)$, then $(Y, y') \in \mathrm{Tan}(X, x)$, for all $y' \in Y$.

If $\#\,\mathrm{Tan}(X, x_0) = 1$, then $(Y, y_0) = (Y, y)$, for all $y \in Y$. In other words, the isometry group $\mathrm{Isom}(Y)$ acts on Y transitively. Thus we use the following.

Theorem 8 (Gleason-Montgomery-Zippin, [MZ74]). Let Y be a metric space that is complete, proper, connected, and locally connected. Assume that the isometry group $\mathrm{Isom}(Y)$ of Y acts transitively on Y. Then $\mathrm{Isom}(Y)$ is a Lie group with finitely many connected components.

Regarding the conclusion of the proof of Theorem 5, since moreover Y is geodesic, being X so, then Y is a subFinsler manifold, by [Ber88]. From Mitchell's Theorem and the fact that $\{Y\} = \mathrm{Tan}(Y, y)$, Y is a Carnot group. QED

Comments and more examples. There are other settings in which the tangents are (almost everywhere) unique. The snow flake metrics $(\mathbb{R}, \|\cdot\|^\alpha)$ with $\alpha \in (0,1)$ are such examples. Some examples on which the tangents are Euclidean spaces are the Reifenberg vanishing flat metric spaces, which have been considered in [CC97, DT99]. Alexandrov spaces have Euclidean tangents almost everywhere, [BGP92].

However, even in the subRiemannian setting, the tangents are not local model for the space. Indeed, there are subRiemannian manifolds with a different tangent at each point, [Var81]. In fact, there exists a nilpotent Lie group equipped with left invariant sub-Riemannian metric that is not locally biLipschitz equivalent to its tangent, see [LDOW11]. Such last fact can be seen as the local counterpart of a result by Shalom, which states that there exist two finitely generated nilpotent groups Γ and Λ that have the same blow-down space, but they are not quasi-isometric equivalent, see [Sha04].

Another pathological example from [HH00] is the following. For any $n > 1$, there exists a geodesic space X supporting a doubling measure μ such that at μ-almost all point of X the tangent is $\mathbb{R}^n$, but X has no manifold points.

References

[Ber88] Valeriĭ N. Berestovskiĭ, *Homogeneous manifolds with an intrinsic metric. I*, Sibirsk. Mat. Zh. **29** (1988), no. 6, 17–29.

[BGP92] Yuriĭ Burago, Mikhail Gromov, and Grigoriĭ Perel'man, *A. D. Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk **47** (1992), no. 2(284), 3–51, 222.

[CC97] Jeff Cheeger and Tobias H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480.

[DT99] Guy David and Tatiana Toro, *Reifenberg flat metric spaces, snowballs, and embeddings*, Math. Ann. **315** (1999), no. 4, 641–710.

[Gro81] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73.

[HH00] Bruce Hanson and Juha Heinonen, *An n-dimensional space that admits a Poincaré inequality but has no manifold points*, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3379–3390.

[LD11] Enrico Le Donne, *Metric spaces with unique tangents*, accepted for publication in the Annales Academiae Scientiarum Fennicae Mathematica (2011).

[LDOW11] Enrico Le Donne, Alessandro Ottazzi, and Ben Warhurst, *Ultrarigid tangents of sub-riemannian nilpotent groups*, Preprint, submitted (2011).

[Mit85] John Mitchell, *On Carnot-Carathéodory metrics*, J. Differential Geom. **21** (1985), no. 1, 35–45.

[MM95] Gregori A. Margulis and George D. Mostow, *The differential of a quasi-conformal mapping of a Carnot-Carathéodory space*, Geom. Funct. Anal. **5** (1995), no. 2, 402–433.

[MZ74] Deane Montgomery and Leo Zippin, *Topological transformation groups*, Robert E. Krieger Publishing Co., Huntington, N.Y., 1974, Reprint of the 1955 original.
[Pan83] Pierre Pansu, *Croissance des boules et des géodésiques fermées dans les nilvariétés*, Ergodic Theory Dynam. Systems **3** (1983), no. 3, 415–445.
[Sha04] Yehuda Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. **192** (2004), no. 2, 119–185.
[Var81] A. N. Varčenko, *Obstructions to local equivalence of distributions*, Mat. Zametki **29** (1981), no. 6, 939–947, 957.

Moduli space of test configurations

Toshiki Mabuchi

1. Introduction

This is a work partly joint with Yasufumi Nitta. Let (M, L) be an n-dimensional connected projective algebraic manifold, so that L is a very ample line bundle on M. Let T be a maximal algebraic torus in the group $\mathrm{Aut}(M)$ of all holomorphic automorphisms of M. The problem we have in mind is the following extremal Kähler version of Donaldson-Tian-Yau's Conjecture:

Conjecture A (Donaldson-Tian-Yau-Székelyhidi): *If (M, L) is K-stable relative to T, then there exists an extremal Kähler metric in the polarization class $c_1(L)$.*

Here (M, L) is called *K-stable relative to T* if $F_1(\mathcal{M}, \mathcal{L}) < 0$ for all non-trivial test configurations $(\mathcal{M}, \mathcal{L})$ for (M, L) in Donaldson's sense associated to one-parameter groups orthogonal to T. Note that, in the case $\mathrm{Aut}(M)$ is discrete, Conjecture A is nothing but the original Donaldson-Tian-Yau's Conjecture. In this note, for simplicity, we assume that $\mathrm{Aut}(M)$ is discrete (and hence T is trivial), though such assumtion of discreteness is not necessary.

2. Test configurations with fixed components

For a test configuration $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ for (M, L) of exponent 1 in Donaldson's sense, let $\tilde{\pi} : \tilde{\mathcal{M}} \to \mathbb{C}$ be the associated $\mathbb{C}^*$-equivariant projective morphism. Consider the normalization $\nu : \mathcal{M} \to \tilde{\mathcal{M}}$ of the complex variety $\tilde{\mathcal{M}}$. Let D_i, $i = 1, 2, \dots, r$, be the irreducible components of the scheme-theoretic fiber $\mathcal{M}_0$ of $\mathcal{M}$ over the origin 0. For an effective $\mathbb{R}$-divisor

$$D := \sum_{i=1}^{r} \alpha_i D_i$$

on $\mathcal{M}$, we consider the formal line bundle

$$\mathcal{L} := \nu^* \tilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{M}}(D)$$

over $\mathcal{M}$. Then the pair $(\mathcal{M}, \mathcal{L})$ together with flat $\mathbb{C}^*$-equivariant projective morphism $\pi = \tilde{\pi} \circ \nu : \mathcal{M} \to \mathbb{C}$ is called a *test configuration for (M, L) with fixed components* (cf. [2]). Here $\nu^* \tilde{\mathcal{L}}$ and $\mathcal{O}_{\mathcal{M}}(D)$ are called the *moving part* and the *fixed part* of $\mathcal{L}$, respectively. By abuse of terminology, $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ is also called the moving part of $(\mathcal{M}, \mathcal{L})$, and D is called the fixed components for $(\mathcal{M}, \mathcal{L})$.

A test configuration $(\mathcal{M}, \mathcal{L})$ for (M, L) with fixed components is called $\mathbb{R}$-*nef*, if there exist a proper $\mathbb{C}^*$-equivariant desingularization $\mu : \hat{\mathcal{M}} \to \mathcal{M}$, isomorphic over $\mathcal{M} \setminus \mathcal{M}_0$, and an $\mathbb{R}$-divisor $\hat{D}$ on $\hat{\mathcal{M}}$ with support in $\hat{\mathcal{M}}_0$ such that $\mu_* \hat{D} = D$ as cycles on $\mathcal{M}$ and that

$$c_1(\hat{\mathcal{L}})[C] \geq 0$$

for $\hat{\mathcal{L}} := \mu^* \nu^* \tilde{\mathcal{L}} \otimes \mathcal{O}_{\hat{\mathcal{M}}}(\hat{D})$ and all irreducible closed curves C on $\hat{\mathcal{M}}_0$. Here $c_1(\hat{\mathcal{L}})[C]$ in the left-hand side means $c_1(\mu^* \nu^* \tilde{\mathcal{L}})[C] + c_1(\hat{D})[C]$. In addition given an $\mathbb{R}$-nef test configuration $(\mathcal{M}, \mathcal{L})$ with fixed components, if D is a $\mathbb{Q}$-divisor on $\mathcal{M}$, then $(\mathcal{M}, \mathcal{L})$ is called $\mathbb{Q}$-*nef*.

For each positive integer ℓ, let $\mathcal{T}_\ell(M, L)$ denote the set of all test configurations for (M, L) of exponent ℓ in Donaldson's sense. We then consider the set $\mathcal{B}(M, L)$ of all test configurations for (M, L) with fixed components. We now define $\mathcal{B}^{\mathbb{Q}}_{nef}(M, L)$ and $\mathcal{B}^{\mathbb{R}}_{nef}(M, L)$ by

$$\mathcal{B}^{\mathbb{Q}}_{nef}(M, L) := \{ (\mathcal{M}, \mathcal{L}) \in \mathcal{B}(M, L) \,;\, (\mathcal{M}, \mathcal{L}) \text{ is } \mathbb{Q}\text{-nef} \},$$
$$\mathcal{B}^{\mathbb{R}}_{nef}(M, L) := \{ (\mathcal{M}, \mathcal{L}) \in \mathcal{B}(M, L) \,;\, (\mathcal{M}, \mathcal{L}) \text{ is } \mathbb{R}\text{-nef} \}.$$

Then for each $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ in $\mathcal{T}_\ell(M, L)$, its ℓ-th root $(\mathcal{M}', \mathcal{L}') \in \mathcal{B}^{\mathbb{Q}}_{nef}(M, L)$ exists in the following sense (cf. [3]):

Theorem B: *For each $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}}) \in \mathcal{T}_\ell(M, L)$, let $\nu : \mathcal{M} \to \tilde{\mathcal{M}}$ be the normalization of $\tilde{\mathcal{M}}$. Then for some $(\mathcal{M}', \mathcal{L}')$ in $\mathcal{B}^{\mathbb{Q}}_{nef}(M, L)$, there exists a $\mathbb{C}^*$-equivariant birational map $\iota : \mathcal{M} \to \mathcal{M}'$, injective in codimension 1, such that $(\iota^* \mathcal{L}')^\ell = \nu^* \tilde{\mathcal{L}}$ up to codimension ≥ 2 subvarieties of $\mathcal{M}$.*

Here a $\mathbb{C}^*$-equivariant birational map $\iota : \mathcal{M} \to \mathcal{M}'$ is called *injective in codimension* 1, if there exist $\mathbb{C}^*$-invariant subvarieties Z and Z' of $\mathcal{M}$ and $\mathcal{M}'$, respectively, such that $\operatorname{codim}_{\mathcal{M}} Z \geq 2$, and that ι restricts to a $\mathbb{C}^*$-equivariant isomorphism of $\mathcal{M} \setminus Z$ onto $\mathcal{M}' \setminus Z'$. We observe that, not only for elements in $\mathcal{T}_\ell(M, L)$, but also for $(\mathcal{M}, \mathcal{L}) \in \mathcal{B}^{\mathbb{R}}_{nef}(M, L)$, we can define the Donaldson-Futaki invariant $F_1(\mathcal{M}, \mathcal{L}) \in \mathbb{R}$. Then in Theorem B above,

$$F_1(\tilde{\mathcal{M}}, \tilde{\mathcal{L}}) \;=\; F_1(\mathcal{M}', \mathcal{L}').$$

It is possible to show that all fixed components for $(\mathcal{M}', \mathcal{L}')$ appearing in Theorem B, with ℓ running through the set of all positive integers, form a bounded family.

3. Test configurations possibly with irrational weights

Use the same notation as in the previous sections. Let $(\mathcal{M}, \mathcal{L}) \in \mathcal{B}^{\mathbb{R}}_{nef}(M, L)$, and consider the vector bundle E obtained as the direct image sheaf

$$E \;:=\; \tilde{\pi}_* \tilde{\mathcal{L}} \;\; (= \pi_* \mathcal{L})$$

over $\mathbb{C}$. Then an affirmative solution of equivariant Serre's Conjecture allows us to obtain a $\mathbb{C}^*$-equivariant isomorphism

$$E \;\cong\; E_0 \times \mathbb{C},$$

where E_0 denotes the fiber of E over the origin. Note that the fiber E_1 of E over $1 \in \mathbb{C}$ is naturally identified with $H^0(M, L)$. Fix a Hermitian metric h_1 for L such that $\omega_1 := c_1(L; h_1)$ is Kähler on M. Then by (ω_1, h_1), we have a natural Hermitian metric ρ_1 for the vector space $H^0(M, L) = E_1$. Then by an observation of Donaldson, the $\mathbb{C}^*$-equivariant isomorphism $E \cong E_0 \times \mathbb{C}$ can be chosen in such a way that the Hermitian metric ρ_1 corresponds to a hermitian metric ρ_0 on E_0 fixed by the action of $S^1 \subset \mathbb{C}^*$. Then the moving part $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ of $(\mathcal{M}, \mathcal{L})$ is recovered from the associated 1-parameter group

$$\psi : \mathbb{C}^* \to \mathrm{SL}(E_0)$$

induced by the representation of $\mathbb{C}^*$ on E_0. To obtain a well-defined ψ, we choose an unramified cover of $\mathbb{C}^*$ if necessary. Note that we may assume that the test configuration $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ is nontrivial. Then $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ is recovered from

$$\tilde{A} := \frac{A}{\|A\|},$$

where $A := \psi_*(1)$ and $\|A\| := \sqrt{\mathrm{tr}({}^t A \bar{A})}$. By fixing an orthonormal basis for the vector space (E_0, ρ_0), we can write $A \in \sqrt{-1}\,\mathfrak{su}(E_0, \rho_0)$ in the form

$$A \;= U\,\Lambda\,U^{-1},$$

where $\Lambda = \Lambda(\psi)$ is a diagonal matrix with the i-th diagonal elements $\lambda_i(\psi) \in \mathbb{Q}$. In a more general situation, by allowing the real number $\lambda_i(\psi)$ to be irrational, we have *test configurations* $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ *with irrational weights*. Such generalized test configurations are parametrized by the compact set

$$\{\, \tilde{A} \in \sqrt{-1}\,\mathfrak{su}(E_0, \rho_0)\,;\; \|\tilde{A}\| = 1 \,\}.$$

This gives a compact completion of the set of all $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$, where $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}})$ are the moving parts of all $(\mathcal{M}, \mathcal{L}) \in \mathcal{B}^{\mathbb{Q}}_{nef}(M, L)$.

4. Precompactness of the moduli space

By letting ℓ run through the set of all positive integers, we conclude from remarks at the end of Sections 2 and 3 that the set of all possible $(\mathcal{M}', \mathcal{L}')$ appearing in Theorem B has compact completion. This fact is called the *precompactness of the moduli space of test configurations*, and gives some affirmative application related to Conjecture A. For instance, one can show that K-stability implies asymptotic Chow stability under the assumption that $\mathrm{Aut}(M)$ is discrete. A relative version of the implication also holds even when $\mathrm{Aut}(M)$ is not discrete.

References

[1] C. Arezzo, A. Della Vedova, and G. La Nave, *Singularities and K-semistability*, Math.DG, arXiv:0906.2745.
[2] T. Mabuchi, *Test configurations with fixed components*, in preparation.
[3] T. Mabuchi and Y. Nitta, *Completion of the space of test configurations*, in preparation.
[4] G. Székelyhidi, *Extremal metrics and K-stability*, Bull. London Math. Soc. 39 (2007), 76–84.

Integral invariants in complex differential geometry

Akito Futaki

Let M be a compact complex manifold of $\dim_{\mathbb{C}} M = m$. Suppose we are given the following (A) and (B) seperately.

(A)

- A holomorphic principal G-bundle with a complex Lie group G;
- A subgroup H of $\mathrm{Aut}(M)$ acting on P_G from the left commuting with the right G-acton, with Lie algebra $\mathfrak{h}$;
- A type $(1,0)$-connection θ of P_G.

(B)

- A Kähler class $\Omega \in H^2_{DR}(M)$;
- A Kähler form $\omega \in \Omega$;
- For each $X \in \mathfrak{h}$ regarded as a holomorphic vector field on M, we are given a complex valued smooth function u_X such that

$$i(X)\omega = -\overline{\partial} u_X \tag{1}$$

with the normalization

$$\int_M u_X \omega^m = 0. \tag{2}$$

Let $I^k(G)$ be the set of all $\mathrm{Ad}(G)$-invariant polynomials of $\mathfrak{g}$ into $\mathbb{C}$ of degree k. For $\phi \in I^k(G)$ we define $\mathcal{F}_\phi : \mathfrak{g} \to \mathbb{C}$ by

$$\mathcal{F}_\phi(X) = (m-k+1)\int_M \phi(\Theta) \wedge u_X \omega^{m-k} + \int_M \phi(\theta(\tilde{X}) + \Theta) \wedge \omega^{m-k+1}, \tag{3}$$

where $\tilde{X}$ is a holomorphic vector field on P_G induced by $X \in \mathfrak{g}$ and Θ is the curvature form of θ.

It is shown in [7] that $\mathcal{F}_\phi$ is independent of the choice of θ in (A) and ω in (B). But a similar formula can be found in a paper of H. Cartan, and this result may be attributed to him.

There are three useful cases:

(a) Take P_G to be the frame bundle and $\phi = c_1$, then we obtain an *obstruction for a Kähler class to admit a Kähler form of harmonic k-th Chern form* for each $k = 1, \cdots, m$. When $k = 1$, this is an obstruction for the Kähler class to admit a

Kähler metric of constant scalar curvature. In particular if M is a Fano manifold and we take Ω to be the first Chern class $c_1(M)$, this is an obstruction to the existence of a Kähler-Einstein metric. See [5], [2], [1] and [6].

(b) Let (M, L) be a polarized manifold. Take P_G to be the frame bundle, ϕ to be the k-th Todd class $Todd^{(k)}$ and Ω to be $c_1(L)$. Then we obtain obstructions for asymptotic Chow semistability ([11], [7], [10]) . Note that $\mathcal{F}_{Todd^{(1)}} = \mathcal{F}_{\frac{1}{2}c_1}$ so that the first one coincides with the one in (a) up to a constant. There is known example of 7-dimensional toric Fano manifold with $\mathcal{F}_{Todd^{(1)}} = 0$ and $\mathcal{F}_{Todd^{(k)}} \neq 0$ for $k = 2, \cdots, m$. This example was suggested by Nill and Paffenholz [12], and the computation of $\mathcal{F}_{Todd^{(k)}}$ was done by [14]. This example is a Kähler-Einstein manifold by [15] but is asymptotically unstable. Note that Donaldson [3] showed that if a polarized manifold admits a constant scalar curvature metric and if the automorphism group is discrete then (M, L) is asymptotically Chow stable. Note also that Odaka [13] showed that Donaldson's result does not hold for orbifolds.

(c) If we take P_G to be the frame bundle and ϕ to be c_1^{m+1}. Then $\mathcal{F}_{c_1^{m+1}}$ has nothing to do with the Kähler form ω, and it turns out that it is an invariant depending only on the complex struture. In other words it can be defined for possibly non-Kähler manifolds. This was used in [8] to show that the invariant vanishes for all Vaisman manifolds.

Finally notice that the three families (a), (b) and (c) intersect with the obstruction to the existence of Kähler-Einstein metrics.

References

[1] S. Bando : An obstruction for Chern class forms to be harmonic, Kodai Math. J., 29(2006), 337-345.

[2] E. Calabi : Extremal Kähler metrics II, Differential geometry and complex analysis, (I. Chavel and H.M. Farkas eds.), 95-114, Springer-Verlag, Berline-Heidelberg-New York, (1985).

[3] S.K. Donaldson : Scalar curvature and projective embeddings, I, J. Differential Geometry, 59(2001), 479-522.

[4] S.K. Donaldson : Scalar curvature and stability of toric varieties, J. Differential Geometry, 62(2002), 289-349.

[5] A. Futaki : An obstruction to the existence of Einstein Kähler metrics, Invent. Math. **73**, 437-443 (1983).

[6] A. Futaki : Kähler-Einstein metrics and integral invariants, Lecture Notes in Math., vol.1314, Springer-Verlag, Berline-Heidelberg-New York,(1988)

[7] A. Futaki : Asymptotic Chow semi-stability and integral invariants, Intern. J. Math., **15**, 967-979, (2004).

[8] A. Futaki, K. Hattori and L. Ornea : An integral invariant from the view point of locally conformally Kähler geometry, arXiv:1105.4774.

[9] A. Futaki and S. Morita : Invariant polynomials of the automorphism group of a compact complex manifold, J. Differential Geometry, **21**, 135–142 (1985).

[10] A. Futaki, H. Ono and Y. Sano : Hilbert series and obstructions to asymptotic semistability, Advances in Math., 226 (2011), 254–284.

[11] T. Mabuchi : An obstruction to asymptotic semi-stability and approximate critical metrics, Osaka J. Math., 41(2004), 463-472. math.DG/0404210.

[12] B. Nill and A. Paffenholz : Examples of non-symmetric Kähler-Einstein toric Fano manifolds, preprint. arXiv:0905.2054.

[13] Y. Odaka, *The Calabi conjecture and K-stability,* arXiv:1010.3597, to appear in IMRN.
[14] H.Ono, Y.Sano and N.Yotsutani : An example of asymptotically Chow unstable manifolds with constant scalar curvature, to appear in Annales de L'Institut Fourier. arXiv:0906.3836.
[15] X.-J. Wang and X. Zhu : Kähler-Ricci solitons on toric manifolds with positive first Chern class, Adv. Math. 188 (2004), no. 1, 87–103.

A gauge theoretic approach to the anti-self-dual Einstein equations

Joel Fine
(joint work with Dmitri Panov)

In the 1977 Plebanski explained how to reformulate the anti-self-dual Einstein equations with non-zero scalar curvature as a PDE not for a metric, but rather for a *connection* on an SO(3)-bundle [1]. The aim of this talk is to lay the foundations for further study of this equation. Our approach is motivated by an analogy we describe with instantons and the Yang–Mills functional.

We investigate the solutions to the PDE inside the space $\mathcal{D}$ of "definite connections", an open set of connections satisfying a certain curvature inequality. Solutions of the PDE are zeros of a non-linear differential operator defined on $\mathcal{D}$. We show the operator is elliptic at all points of $\mathcal{D}$. We describe an energy functional on $\mathcal{D}$ whose topological minima are precisely the solutions to the PDE. The downward gradient of this functional defines a flow which we show is parabolic modulo gauge and hence exists for short time.

We explain how our techniques also apply to the zero scalar-curvature (hyperkähler) setting and to a conjecture of Donaldson [2]. We formulate a positively-curved version of Donaldson's conjecture and show that the negatively-curved analogue is false.

References

[1] J. F. Plebanski, *On the separation of Einsteinian substructures*, J. Math. Phys. **18** (1977), 2511–2520.
[2] S. K. Donaldson, *Two-forms on four-manifolds and elliptic equations*, Nankai Tracts Math. *Inspired by S. S. Chern* **11** (2006), 153–172.

On isoperimetric surfaces in initial data sets

Jan Metzger
(joint work with Michael Eichmair)

In this abstract we present the main results from [4].

We study initial data sets that are $\mathcal{C}^0$-asymptotic to Schwarzschild with mass $m > 0$. These are complete three dimensional manifolds (M, g) that are diffeomorphic to $(\mathbb{R}^3 \setminus B(0,1), g)$ outside some bounded open set, such that there is a constant $C > 0$ so that in the Euclidean coordinates (x_1, x_2, x_3) on $\mathbb{R}^3 \setminus B(0,1)$ we have

$$(1) \qquad r^2|(g - g_m)_{ij}| + r^2|\partial_k g_{ij}| + r^3|\partial_k \partial_l g_{ij}| \leq C \text{ for all } i, j, k, l \in \{1, 2, 3\}.$$

Here, $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$ denotes the Euclidean distance to the origin and g_m is the conformally flat spatial Schwarzschild metric with mass $m > 0$:

$$(g_m)_{ij} = \left(1 + \tfrac{m}{2r}\right)^4 \delta_{ij}.$$

We are interested in the *isoperimetric profile* of (M, g):

$$A_g(V) := \inf \left\{ \mathcal{H}^2_g(\partial\Omega) \mid \Omega \subset M \text{ is a smooth region and } \mathcal{L}^3_g(\Omega) = V \right\}.$$

Here, $\mathcal{H}^2_g$ and $\mathcal{L}^3_g$ denote respectively the two-dimensional Hausdorff measure and the Lebesgue measure with respect to g.

A smooth region $\Omega \subset M$ is called *isoperimetric* if $A_g(\mathcal{L}^3_g(\Omega)) = \mathcal{H}^2_g(\partial\Omega)$. The boundary $\Sigma = \partial\Omega$ of an isoperimetric region Ω is a compact constant mean curvature surface that is also stable for the area functional with respect to volume preserving deformations. Hence

$$\int_\Sigma (|h|^2 + \mathrm{Ric}(\nu,\nu)) f^2 \, \mathrm{d}\mathcal{H}^2_g \le \int_\Sigma |\nabla f|^2 \, \mathrm{d}\mathcal{H}^2_g \qquad \text{for all} \quad f \in \mathcal{C}^1(\Sigma) \quad \text{with} \quad \int_\Sigma f \, \mathrm{d}\mathcal{H}^2_g = 0,$$

where h denotes the second fundamental form of Σ and $\mathrm{Ric}(\nu,\nu)$ the Ricci curvature of g evaluated in direction normal to Σ.

In compact manifolds it is well known that there exist isoperimetric regions for all volumes smaller than half the volume of the manifold. The situation in the non-compact case is much more complicated. Explicit solutions to the isoperimetric problem are only known in very few, highly symmetric cases [11]. Bray showed in [1] that the centered spheres in the Schwarzschild manifold with $m > 0$ are the unique isoperimetric surfaces relative to the horizon at $r = \frac{m}{2}$. In [2] it was shown that Bray's technique also applies to a class of rotationally symmetric manifolds.

The analysis of stable constant mean curvature surfaces in initial data sets has been initiated by Christodoulou and Yau in [3]. Huisken and Yau [7] showed that the asymptotic end of a manifold that is asymptotic to Schwarzschild (in a stronger sense than our assumption (1) here) is foliated by stable constant mean curvature spheres. Huisken and Yau used this foliation to define a geometric center of mass for such manifolds. A crucial question in this context is whether this center of mass is uniquely defined. Huisken and Yau answered this question affirmatively by establishing a strong uniqueness result for the surfaces in the foliation used to define the center of mass: they showed that for each $q > 1/2$ there exists $H_0 > 0$ such that for each $H \in (0, H_0)$ there is exactly one stable constant mean curvature sphere with mean curvature H containing $B_{H^{-q}}(0)$. Related existence and uniqueness results for stable constant mean curvature surfaces have also been obtained by Ye [12]. Qing and Tian [10] showed that uniqueness holds outside a compact set that is independent of H. Further work has been done to generalize both existence [6] and uniqueness [8] of foliations by surfaces with constant mean curvature to more general asymptotics, as well as more general prescribed mean curvature surfaces [9]. In [5] we complement these uniqueness results in initial data sets with non-negative scalar curvature by showing that stable constant mean curvature surfaces with large area cannot intersect a given compact set on which the scalar curvature is strictly positive.

The surfaces considered by Huisken and Yau are locally isoperimetric since they pass the second derivative test for the associated variational problem. It is natural to ask whether these surfaces are also global minimizers, i.e. whether they are isoperimetric regions. This question was asked by Bray in [1, p. 44] in view of his result on the isoperimetric property of the centered spheres in Schwarzschild. Our main result in [4] confirms this conjecture and completely clarifies the isoperimetric structure of initial data sets for large volumes.

Theorem 1 (cf. [4, Theorem 1.1]). *Let (M, g) be an initial data set as described above that is $\mathcal{C}^0$-asymptotic to Schwarzschild with mass $m > 0$. Then there exists $V_0 > 0$ with the following property: For every $V \geq V_0$ there is an isoperimetric region Ω_V with volume V. Its boundary is connected and close to a centered coordinate sphere.*

An immediate consequence is that the isoperimetric profile $A_g(V)$ of (M, g) is asymptotic to that of the exact Schwarzschild metric.

Our proof is based on direct minimization of the area functional subject to a volume constraint. In non-compact manifolds this is delicate, since part of the volume of a minimizing sequence may drift to infinity so that the limit may have less volume than one started with. To show that the positivity of mass prevents this volume loss, we derive an explicit comparison theorem. This implies that in manifolds which are $\mathcal{C}^0$-asymptotic to Schwarzschild, centered regions are isoperimetrically superior to regions which have a fraction of their boundary area off center in the following sense:

Definition 1 ([4, cf. Definition 3.2]). Given $\tau > 1$ and $\eta \in (0, 1)$. A region Ω in (M, g) is called (τ, η)-off center if:

(1) $\mathcal{L}^3_g(\Omega)$ is so large that there exists a coordinate sphere $S_r = \partial B_r$ with $\mathcal{L}^3_g(B_r) = \mathcal{L}^3_g(\Omega)$, and if
(2) $\mathcal{H}^2_g(\partial\Omega \setminus B_{\tau r}) \geq \eta \mathcal{H}^2_g(S_r)$.

Our comparison theorem then tells us the following:

Theorem 2 ([4, cf. Theorem 3.4]). *Let (M, g) be an initial data set that is $\mathcal{C}^0$-asymptotic to Schwarzschild with mass $m > 0$. For each pair $(\tau, \eta) \in (1, \infty) \times (0, 1)$ and constant $\Theta > 0$ there exists a constant $V_0 > 0$ such that the following holds: given a bounded region Ω that is (τ, η)-off center with $\mathcal{H}^2_g(\partial\Omega)\mathcal{L}^3_g(\Omega)^{-1/3} \leq \Theta$ and such that $\mathcal{H}^2_g(\partial\Omega \cap B_\sigma) \leq \Theta\sigma^2$ holds for all $\sigma \geq 1$ one has*

$$\mathcal{H}^2_g(S_r) + \frac{\eta m \pi}{300}\left(1 - \frac{1}{\tau}\right)^2 r \leq \mathcal{H}^2_g(\partial\Omega)$$

where $S_r \subset M$ is the coordinate sphere that encloses the same g-volume as Ω.

This comparison theorem not only allows us to argue that a minimizing sequence does not lose volume in the limit, but it also yields position estimates for the resulting isoperimetric region. In fact, by the comparison theorem, any sequence of isoperimetric regions Ω_{V_i} with volume $V_i \to \infty$ scaled down to volume $\frac{4\pi}{3}$ converges to a centered round ball with radius 1.

We conclude that the Huisken-Yau geometric center of mass of manifolds that are $\mathcal{C}^2$-asymptotic to Schwarzschild with mass $m > 0$ is indeed also the isoperimetric center of mass.

References

[1] H. L. Bray, *The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature (thesis)*, arXiv:0902.3241 (1998).

[2] H. L. Bray and F. Morgan, *An isoperimetric comparison theorem for Schwarzschild space and other manifolds*, Proc. Amer. Math. Soc. **130** (2002).

[3] D. Christodoulou and S.-T. Yau, *Some remarks on quasi-local mass*, Mathematics and general relativity (Santa-Cruz, 1986), Contemp. Math., 71 (1988).

[4] M. Eichmair and J. Metzger, *Large isoperimetric surfaces in initial data sets*, arXiv:1102.2999 (2011).

[5] M. Eichmair and J. Metzger, *On large volume preserving stable CMC surfaces in initial data sets*, arXiv:1102.3001 (2011).

[6] L. H. Huang, *Foliations by stable spheres with constant mean curvature for isolated systems with general asymptotics*, Comm. Math. Phys. **300** (2010).

[7] G. Huisken and S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Invent. Math. **124** (1996).

[8] S. Ma, *Uniqueness of the foliation of constant mean curvature spheres in asymptotically flat 3-manifolds*, arXiv: 1012.3231 (2011).

[9] J. Metzger, *Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature*, J. Differential Geom. **77** (2007).

[10] J. Qing and G. Tian, *On the uniqueness of the foliation of spheres of constant mean curvature in asymptotically flat 3-manifolds*, J. Amer. Math. Soc. **20** (2007).

[11] A. Ros, *The isoperimetric problem*, Global theory of minimal surfaces, Clay Math. Proc., 2 (2005)

[12] R. Ye, *Foliation by constant mean curvature spheres on asymptotically flat manifolds*, Geometric analysis and the calculus of variations, Intl. Press, Cambridge, MA, 1996.

Positively curved polar manifolds and buildings

Gudlaugur Thorbergsson
(joint work with Fuquan Fang, Karsten Grove)

This talk is based on the paper [3].

An isometric action on a complete Riemannian manifold is called *polar* if there is an (immersed) submanifold, a so-called *section*, that meets all orbits orthogonally. The concept, that was introduced independently by Szenthe [9] and Palais-Terng [7], goes back to isotropy representations of symmetric spaces. Also, as a special case, the adjoint action of a compact Lie group on itself is polar with section a maximal torus. An exceptional but important special case is that of cohomogeneity one actions and manifolds, i.e., actions with 1-dimensional orbit space.

The *exceptional case* of positively curved cohomogeneity one manifolds was studied in [4]. Aside from the rank one symmetric spaces, this also includes infinite families of other manifolds, most of which are not homogeneous even up to homotopy. In contrast, our work here has the following result as a corollary:

Theorem 1. *A polar action on a simply connected, compact, positively curved manifold of cohomogeneity at least three is equivariantly diffeomorphic to a polar action on a compact rank one symmetric space.*

It is our conjecture that the same conclusion holds in cohomogeneity two as well. In fact, by Theorem 4 below, we know this for all so-called reducible actions, and the only irreducible ones of cohomogeneity two correspond to Coxeter geometries of type A_3 or C_3. It is also a corollary of our main result that all irreducible polar actions of type $\mathsf{A}_k, k \geq 3$, are equivarianly diffeomorphic to linear polar representations on spheres. As we will see, however, the C_3 geometry is significantly different from all others. However, we are able to confirm our conjecture in high dimensions

Theorem 2. *A polar action on a simply connected, compact, positively curved manifold of dimension* ≥ 72 *is equivariantly diffeomorphic to a polar action on a compact rank one symmetric space.*

All polar actions on the simply connected, compact rank one symmetric spaces, i.e., the spheres and projective spaces, $\mathbb{S}^n, \mathbb{CP}^n, \mathbb{HP}^n$ and $\mathrm{Ca}\mathbb{P}^2$ were classified in [2] and [8]. In all cases but $\mathrm{Ca}\mathbb{P}^2$ they are either linear polar actions on a sphere or they descend from such actions to a projective space.

By work of Dadok [2], Tits [10], and Burns-Spatzier [1], the (maximal) irreducible polar linear actions are in 1-1 correspondence with *topological spherical buildings*. On $\mathrm{Ca}\mathbb{P}^2$ any polar action has either cohomogeneity one or two, and in the second case all but one have a fixed point. It follows from our work, that the cohomogeneity two action without fixed points has associated to it a *chamber complex* whose universal cover is a geometry of type C_3 which *is not a building*.

Our point of departure is the following description of sections and their (effective) stabilizer groups referred to as *polar groups* in [5] and Weyl groups in [9] and [7]:

Theorem 3. *The polar group of a simply connected positively curved polar manifold of cohomogeneity at least two is a Coxeter group or a* $\mathbb{Z}_2$ *quotient thereof. Moreover, the section with this action is equivariantly diffeomorphic to a sphere or a real projective space with a linear reflection group action.*

This naturally divides the investigation into two parts, according to whether the action of the polar group is *reducible* (including the case of fixed points) or *irreducible*. By abuse of language, we will simply say that the action is reducible or not if the same is true for its polar group action on the section. It is somewhat surprising that one can determine the reducible actions without knowing the irreducible ones. In fact, using direct geometric arguments, we prove:

Theorem 4. *A simply connected positively curved manifold with a reducible polar action of cohomogeneity at least two is equivariantly diffeomorphic to a rank one symmetric space with a polar action.*

Whether the action is reducible or not, the fundamental domains for the action of the polar group on a section are isometric to the orbit space, and we refer to these as *chambers*. The group acts simply transitively on the set of chambers, and one naturally builds a *chamber complex* out of these chambers. The chambers are either simplices or joins of simplices with a sphere. Even when the chambers are simplices, this complex is frequently *not simplicial* when the action is reducible.

Our starting point in the irreducible case is the following:

Theorem 5. *The chamber complex associated with a simply connected positively curved irreducible polar manifold of cohomogeneity at least two is a connected simplicial complex. Moreover, this complex naturally supports a metric which is locally* CAT(1)*.*

Following Tits, a chamber complex associated to a Coxeter group has a universal cover, and when the rank is at least four (corresponding to cohomogeneity at least three), this cover is some general (spherical) building, not necessarily associated with a simple group over a classical field. In our case, we show that the universal cover supports a natural topology inherited from the "base" manifold, making it into a topological building in the sense of Burns and Spatzier:

Theorem 6. *The universal cover of the chamber complex associated with a simply connected positively curved irreducible polar manifold of cohomogeneity at least two inherits a topology from the base complex. If the universal cover is a building, it becomes a compact topological building with this topology.*

From the work of Burns-Spatzier [1] and Tits [10], a topological spherical building, is the building of the sphere at infinity of a noncompact symmetric space U/K, and the action of K at the sphere at infinity is the linear polar action whose chamber complex is the building. In our case, the deck transformation group π of the cover becomes a compact normal subgroup of $\hat{\mathsf{G}} \subset \mathsf{K}$ acting freely on the sphere with quotient our manifold with the action by $\mathsf{G} = \hat{\mathsf{G}}/\pi$. Moreover, the actions by $\hat{\mathsf{G}}$ and K on the sphere are orbit equivalent. This already proves our Theorem 1 up to equivariant homeomorphism, and the rest follows, e.g., from the Recognition Theorem in [5].

We like to mention that the strategy used in the case of irreducible action has been independently developed by Lytchak [6] to determine the polar singular foliations of cohomogeneity at least three in symmetric spaces.

References

[1] K. Burns, and R. Spatzier, On topological Tits buildings and their classification. *Inst. Hautes etudes Sci. Publ. Math.* **65** (1987), 534.

[2] J. Dadok, Polar coordinates induced by actions of compact Lie groups, *Trans. Amer. Math. Soc.* **288** (1985), 125–137.

[3] F. Fang, K. Grove and G. Thorbergsson, *Tits Geometry and Positive Curvature.* Preprint 2011.
[4] K. Grove, B. Wilking and W.Ziller, *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry*, J. Differential Geom. **78** (2008), 33–111.
[5] K. Grove and W.Ziller, *Polar Actions and Manifolds*, in preparation.
[6] A. Lytchak, *Equifocal submanifolds of symmetric spaces.* Preprint 2011.
[7] R. S. Palais and C.-L. Terng, *A general theory of canonical forms*, Trans. Amer. Math. Soc., **300**, (1987), 771–789.
[8] F. Podestà & G. Thorbergsson, Polar actions on rank-one symmetric spaces. *J. Differential Geom.* **53** (1999), 131–175.
[9] J. Szenthe, Orthogonally transversal submanifolds and the generalizations of the Weyl group. *Period. Math. Hungar.* **15** (1984), 281299.
[10] J. Tits, *Buildings of spherical type and finite BN-pairs.* Lecture Notes in Mathematics, **386**. Springer-Verlag, Berlin-New York, 1974.

Polygons in Euclidean buildings of rank 2

Carlos Ramos-Cuevas

Let $X = G/K$ be a symmetric space of noncompact type, i.e. with nonpositive sectional curvature and no Euclidean factors. As in Euclidean space the only isometry invariant of a segment is its length, we can also define a generalized notion of length of an oriented segment in X by considering its equivalence class modulo the action of the identity component $G = Isom_0(X)$ of its isometry group. We obtain in this way a vector in the Euclidean Weyl chamber $\Delta_{euc} := (X \times X)/G$ which we call the Δ-valued length of the segment (cf. [6]). If (S, W) is the spherical Coxeter complex associated to X, then the Euclidean Weyl chamber Δ_{euc} is isometric to the complete Euclidean cone over the Weyl chamber $\Delta := S/W$.

Let now X be a thick Euclidean building modeled in the Euclidean Coxeter complex (E, W_{aff}) with associated spherical Coxeter complex (S, W). The same notion of Δ-valued length can be defined in this case, see [7].

We are interested in the following geometric question: *Which are the possible Δ-valued lengths of oriented polygons in X?* In the special case of the symmetric space $X = SL(m, \mathbb{C})/SU(m)$ this question is closely related with the so-called *Eigenvalue Problem* which asks: *How are the eigenvalues of two Hermitian matrices related to the eigenvalues of their sum?* We refer to [6] for more information about the relations between these two questions and [5] to read about the history on the Eigenvalue Problem.

Let $\mathcal{P}_n(X) \subset \Delta_{euc}^n$ denote the set of all possible Δ-valued lengths of oriented n-gons in X. If X is a symmetric space, the structure of $\mathcal{P}_n(X)$ is known ([8], [1], [4], [9], [6], [2]), namely, it is a convex finite sided polyhedral cone. It is also shown in [6] and [7] that $\mathcal{P}_n(X)$ depends only on the associated spherical Coxeter complex (S, W) regardless of whether X is a symmetric space or a Euclidean building. It follows that for a Euclidean building with a spherical Coxeter complex which also occurs for symmetric spaces the set $\mathcal{P}_n(X)$ is also a convex finite sided polyhedral cone.

However there exist *exotic* Coxeter complexes in the sense that they are associated to Euclidean buildings but not to symmetric spaces. According to a result of Tits [11] they only exist in rank 2, actually, all dihedral groups occur as Weyl groups of Euclidean buildings. In our talk we discussed the description of the set $\mathcal{P}_n(X)$ in these remaining cases (compare [10, Theorem 6.14]). This result was independently proven by Berenstein and Kapovich in [3] by different methods.

Theorem 1. *Let X be a thick Euclidean building of rank 2. The set $\mathcal{P}_n(X) \subset \Delta^n_{euc}$ of possible Δ-valued lengths of oriented n-gons in X is a convex finite sided polyhedral cone. The inequalities defining the faces of $\mathcal{P}_n(X)$ can be given in terms of the combinatorics of the spherical Coxeter complex associated to X.*

References

[1] P. Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compos. Math. 129 no. 1 (2001), 67-86.

[2] P. Belkale, S. Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. 166 (2006), no. 1, 185-228.

[3] A. Berenstein, M. Kapovich, *Stability inequalities and universal Schubert calculus of rank 2*, Preprint 2010. arXiv:1008.1773v1.

[4] A. Berenstein, R. Sjamaar, *Projections of coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, J. Amer. Math. Soc. 13 (2000), 433-466.

[5] W. Fulton, *Eigenvalues, invariant factors, highest weights and schubert calculus*, Bull. AMS 37 (2000), no. 3, 209-249.

[6] M. Kapovich, B. Leeb, J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, J. Differ. Geom. 81 (2009), 297-354.

[7] M. Kapovich, B. Leeb, J. Millson, *Polygons in buildings and their refined side lengths*, Geom. Funct. Anal. 19, no. 4, 1081-1100 (2009).

[8] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Mathematica 4 (1998), 419-445.

[9] L. O'Shea, R. Sjamaar, *Moment maps and Riemannian symmetric pairs*, Math. Ann. 317 (2000), 415-457.

[10] C. Ramos-Cuevas, *The generalized triangle inequalities in thick Euclidean buildings of rank 2*, Preprint, 2011. arXiv:1009.1316v1.

[11] J. Tits, *Endliche Spiegelungsgruppen, die als Weylgruppen auftreten*, Invent. Math. 43, 283-295 (1977).

Stability of symmetric spaces of noncompact type under Ricci flow

Richard H. Bamler

In this talk, we present stability results for symmetric spaces of noncompact type under Ricci flow, i.e. we show that any small perturbation of the symmetric metric is flown back to the original metric under an appropriately rescaled Ricci flow. These results can be found in [3]. It is important for us which smallness assumptions we have to impose on the initial perturbation. We will find that as long as the symmetric space does not contain any hyperbolic or complex hyperbolic factor, we don't have to assume any decay on the perturbation. Furthermore, in the hyperbolic and complex hyperbolic case, we show stability under a very weak assumption on the initial perturbation generalizing results by Schulze, Schnürer and Simon ([11]) in the hyperbolic case. The proofs of those results make use of an improved L^1-decay estimate for the heat kernel in vector bundles over symmetric spaces as well as of elementary geometry of negatively curved spaces.

As a motivation for our results, we consider the following question:

Question 1. Is there an $\varepsilon_n > 0$ such that every compact Riemannian manifold (M^n, g) with $-1 - \varepsilon_n < \sec_g < -1 + \varepsilon_n$ is automatically hyperbolic?

The answer is yes for $n = 2, 3$ (by uniformization the solution of the Geometrization Conjecture) and no for $n \geq 4$ (see [7] and [5]). We can generalize this question to other locally symmetric spaces $(M_{\text{sym}}, \overline{g})$. Recall that these are spaces which locally admit an isometric reflection at every point. By the de Rham Decomposition Theorem, the universal cover $\widetilde{M}_{\text{sym}}$ can be expressed as a product $M_1 \times \ldots \times M_m$ of irreducible symmetric spaces which turn out to be Einstein. M_{sym} is said to be of *noncompact type* if the Einstein constants of the M_i are all negative (for more details see e.g. [8]). A reasonable generalization of Question 1 is

Question 2. For which symmetric space of noncompact type $(M_{\text{sym}}, \overline{g})$ is there an $\varepsilon = \varepsilon(M_{\text{sym}}) > 0$ such that:
If (M, g) is a compact Riemannian manifold which satisfies the property that the universal cover $\widetilde{B}_1(p)$ of every 1-ball $B_1(p) \subset M$ is ε-close to a 1-ball $B_1(p_0) \subset M_{\text{sym}}$ (in some C^m-sense), them M is actually diffeomorphic to a geometric quotient of M_{sym}.

Observe that Question 2 is still false for $M_{\text{sym}} = \mathbb{H}^n$ for $n \geq 4$. Positive results towards this question in the case in which ε is allowed to depend on an upper bound on the volume of M are due to Min-Oo ([10]). We are unable to anwer Question 2 at this point. However, the analysis of the following related question gives hope that there might be a positive answer for certain M_{sym}:

Question 3. Let $(M_{\text{sym}}, \overline{g})$ be a locally symmetric space of noncompact type which is Einstein with Einstein constant λ, i.e. $\overline{g}$ is a fixed point of the rescaled Ricci flow equation

$$\partial_t g_t = -2 \operatorname{Ric}_{g_t} + 2\lambda g_t. \tag{1}$$

Is $\overline{g}$ *stable* under (1), i.e. is there an $\varepsilon = \varepsilon(M_{\text{sym}}) > 0$ such that if

$$(1 - \varepsilon)\overline{g} < g_0 < (1 + \varepsilon)\overline{g},$$

then there is a solution $(g_t)_{t \in [0,\infty)}$ to (1) with initial metric g_0 and as $t \to \infty$ we have convergence $g_t \longrightarrow \overline{g}$ in the pointed smooth Cheeger-Gromov sense, i.e. there is a family of diffeomorphisms Ψ_t of M_{sym} such that $\Psi_t^* g_t \longrightarrow \overline{g}$ in the smooth sense on every compact subset of M_{sym}?

Observe that we do not impose any spatial decay assumptions on the perturbation $g_0 - \overline{g}$ here. Surprisingly, Question 3 can be answered positively for many of the “higher” symmetric spaces:

Theorem 1 (cf [3]). Let $(M_{\text{sym}}, \overline{g})$ be a locally symmetric space of noncompact type which is Einstein. Assume that the de Rham decomposition of $\widetilde{M}_{\text{sym}}$ contains no factors which are homothetic to hyperbolic space $\mathbb{H}^n, (n \geq 2)$ or complex hyperbolic space $\mathbb{CH}^{2n}, (n \geq 1)$. Then $(M_{\text{sym}}, \overline{g})$ is stable in the sense of Question 3.

On the other hand, by results of Graham-Lee ([6]) and Biquard ([4]), the spaces $\mathbb{H}^n, (n \geq 4)$ and $\mathbb{CH}^{2n}, (n \geq 2)$ admit deformations g which are Einstein, are not isometric to $\overline{g}$ and satisfy $(1-\varepsilon)\overline{g} \leq g \leq (1+\varepsilon)\overline{g}$. Hence for those spaces we cannot hope for a result which is as strong as that of Theorem 1. We remark here that by a result of the author, Question 3 can still be answered positively for certain quotients of $\mathbb{H}^n$:

Theorem 2 (cf [2]). Any complete hyperbolic manifold $(M^n, \overline{g})$ of finite volume and dimension $n \geq 3$ is stable in the sense of Question 3.

A stability result for the case in which M is compact, has previously been found by Ye ([12]). In the simply-connected case however, we have to impose stronger decay assumptions on the perturbation $g_0 - \overline{g}$ to guarantee stability:

Theorem 3 (cf [3]). Let $(M, \overline{g})$ be either $\mathbb{H}^n$ for $n \geq 3$ or $\mathbb{CH}^{2n}$ for $n \geq 2$, choose a basepoint $x_0 \in M$ and let $r = d(\cdot, x_0)$ denote the radial distance function.
There is an $\varepsilon_1 > 0$ and for every $q < \infty$ an $\varepsilon_2 = \varepsilon_2(q) > 0$ such that the following holds: If $g_0 = \overline{g} + h$ and $h = h_1 + h_2$ satisfies

$$|h_1| < \frac{\varepsilon_1}{r+1} \qquad \text{and} \qquad \sup_M |h_2| + \left(\int_M |h_2|^q dx \right)^{1/q} < \varepsilon_2,$$

then there is a solution $(g_t)_{t \in [0,\infty)}$ to (1) and we have convergence $g_t \longrightarrow \overline{g}$ in the pointed smooth Cheeger-Gromov sense.

Earlier results in this direction are due to Schulze, Schnürer and Simon ([11]) who showed that in the case $M = \mathbb{H}^n$, $n \geq 4$ there is stability for every perturbation h for which $\|h\|_{L^\infty(M)}$ is bounded by a small constant depending on $\|h\|_{L^2(M)}$. Moreover, Li and Yin ([9]) have established stability in the case $M = \mathbb{H}^n$, $n \geq 3$ when the Riemannian curvature approaches the hyperbolic curvature like $\varepsilon_1(\delta)e^{-\delta r}$.

The proofs of Theorems 1 and 3 rely on a geometric analysis of the heat kernel associated to the linearized Ricci deTurck flow equation. It turns out that the geometry of the vector bundle $\text{Sym}_2 T^*M$ in which the perturbation lives, improves the L^1-decay rate of the heat kernel associated to this linearization. In [3] we are able to give bounds on the L^1-decay rate of any heat kernel living in a homogeneous vector bundle over a symmetric space and those bounds are optimal in many cases. It then remains an algebraic problem to compute those bounds and we find that the obstruction against a good decay, originate from the existence of so called cusp deformations which correspond to the "trivial Einstein deformations" in [1] and can be seen as algebraic deformations of cusp cross-sections. Cusp deformations turn out to exist only for the spaces $\mathbb{H}^n, (n \geq 3)$ and $\mathbb{CH}^{2n}, (n \geq 2)$. Theorems 1 and 3 for type h_2 perturbations follow then easily from these heat kernel estimates. In order to allow type h_1 perturbations, we make use of certain visibility properties of the geometry of negatively curved spaces.

References

[1] R. Bamler, *Construction of Einstein metrics by generalized Dehn filling*, arXiv:0911.4730 (November 24, 2009), http://arxiv.org/abs/0911.4730

[2] R. Bamler, *Stability of hyperbolic manifolds with cusps under Ricci flow*, arXiv:1004.2058 (April 12, 2010), http://arxiv.org/abs/1004.2058

[3] R. Bamler, *Stability of symmetric spaces of noncompact type under Ricci flow*, arXiv:1011.4267 (November 18, 2010), http://arxiv.org/abs/1011.4267

[4] O. Biquard, *Asymptotically symmetric Einstein metrics*, vol. 13, SMF/AMS Texts and Monographs (Providence, RI: American Mathematical Society, 2006)

[5] F. T. Farrell, L. E. Jones, *Negatively curved manifolds with exotic smooth structures*, Journal of the American Mathematical Society 2, no. 4 (1989): 899–908.

[6] C. R. Graham, J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. 87 (1991), no. 2, 186-225

[7] M. Gromov, W. Thurston, *Pinching constants for hyperbolic manifolds*, Inventiones Mathematicae 89, no. 1 (1987): 1–12

[8] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, American Mathematical Society (2001)

[9] H. Li, H. Yin, *On stability of the hyperbolic space form under the normalized Ricci flow*, arXiv:0906.5529 (June 30, 2009), http://arxiv.org/abs/0906.5529

[10] M. Min-Oo, *Almost symmetric spaces*, Astérisque, no. 163-164 (1988): 7, 221–246, 283 (1989)

[11] O. C. Schnürer, F. Schulze, M. Simon, *Stability of hyperbolic space under Ricci flow*, arXiv:1003.2107 (March 10, 2010), http://arxiv.org/abs/1003.2107

[12] R. Ye, *Ricci flow, Einstein metrics and space forms*, Trans. Amer. Math. Soc. 338 (1993), no. 2, 871-896

Normalized Ricci flow on 4-manifolds

Fuquan Fang

A *Ricci flow* on a manifold (M, g) is the following evolution equation with initial metric g

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)), \tag{1}$$

If the manifold (M, g) has finite volume, the normalized Ricci flow reads as

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)) + \frac{2}{n} r(t) g(t) \tag{2}$$

where $Ric(g(t))$ is the Ricci tensor of the metric $g(t)$, and $r(t) = \frac{\int_M R(g(t)) dv_{g(t)}}{\int_M dv_{g(t)}}$ is the average scalar curvature of $g(t)$.

In this talk I will briefly review my joint works with Yuguang Zhang and Zhenlei Zhang in recent five years on Ricci flow on 4-manifolds. By exploring Perelman's monotonicity for $\mathcal{F}$ and $\mathcal{W}$ functionals, the following is proved in [FZZ1]:

Theorem 1. *If $g(t)$ is a normalized Ricci flow on a 4-manifold M with uniformly bounded sectional curvature for all $t \in [0, \infty)$, then the Euler characteristic $\chi(M) \geq 0$.*

Moreover, if M has non-positive Perelman's λ-invariant, or equivalently, M does not admit a metric with positive scalar curvature, then

Theorem 2. *If $g(t)$ is a normalized Ricci flow on a closed oriented 4-manifold M with uniformly bounded scalar curvature for all $t \in [0, \infty)$ and $\bar{\lambda}_M \leq 0$, then*

$$2\chi(M) - 3|\tau(M)| \geq \frac{1}{96\pi^2} \bar{\lambda}_M^2,$$

where $\tau(M)$ is the signature of M.

The above can be regarded as a generalized Hitchin-Thorpe inequality for Einstein 4-manifolds. By combining with Seiberg-Witten theory we have proved a strengthened version of the above inequality for 4-manifolds with non-trivial monopole class, which can be regarded as Miyaoka-Yau type inequality:

Theorem 3. *Let M be a closed oriented 4-manifold with nontrivial Seiberg-Witten invariant and $\overline{\lambda}_M \leq 0$. Let $g(t)$ be a solution to the normalized Ricci flow with uniformly bounded scalar curvature for all $t \in [0, \infty)$, then*

$$\chi(M) \geq 3\tau(M)$$

Moreover, when the equality in the above theorem holds we have the following decomposition theorem for the manifold, which serves as an analogy of Thurston's geometrization picture for a large class of 4-manifolds:

Theorem 4. *Let $(M, g(t))$ a closed oriented 4-manifold with nontrivial Seiberg-Witten invariant. Assume that $b_2^+ > 1$ and $\chi(M) = 3\tau(M)$. If $g(t)$ is a solution to the normalized Ricci flow with uniformly bounded sectional curvature for all $t \in [0, \infty)$, then M admits a thick-thin decomposition where the thick part admits a complex hyperbolic metric of finite volume, the thin part admits pure F-structure of positive rank.*

References

[1] J. Cheeger and M. Gromov, *Collapsing Riemannian Manifolds while keeping their curvature bounded I*, J.Diff.Geom. 23, (1986), 309-364.

[2] J. Cheeger and M. Gromov, *Collapsing Riemannian Manifolds while keeping their curvature bounded II*, J.Diff.Geom. 32, (1990), 269-298.

[3] F.Q. Fang, Y.G. Zhang and Z.L. Zhang, *Non-singular solutions to the normalized Ricci flow equation*, Math. Ann. 2006.

[4] F.Q. Fang, Y.G. Zhang and Z.L. Zhang, *Maximal solutions of normalized Ricci flow on 4-manifolds*, Comm. Math. Phys., 283 (2008), 1-24.

[5] F.Q. Fang, Y.G. Zhang and Z.L. Zhang, *Non-singular solutions to the Ricci flow on manifolds of finite volume*, J. Geom. Anal. 2010.

[6] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17 (1982) 255-306.

More on gravitational instantons

Hans-Joachim Hein

1. Introduction

The topic of this lecture is complete, non-compact, non-flat, Ricci-flat 4-manifolds. A classification of such spaces would be the basic input for a local structure theory of compact 4-manifolds with bounded Ricci curvature. It is then natural in view of Gauß-Bonnet to also assume $\mathrm{Rm} \in L^2$, which implies $|\mathrm{Rm}| < Cr^{-2}$, [2], and is likely equivalent to finite topology. By volume comparison, $cr < |B(x_0, r)| < Cr^4$. If we do allow infinite topology, there exist examples of volume growth r^β for any $\beta \in (3, 4)$, but the only known non-integer rate with finite topology is $\beta = \frac{4}{3}$.

We emphasize right away that there are only two known examples with reduced holonomy SO(4): the Schwarzschild metric on $\mathbb{R}^2 \times S^2$, and the Taub-Bolt metric on $\mathcal{O}_{\mathbb{C}P^1}(1)$. Both are asymptotic to S^1-bundles of constant fiber length over $\mathbb{R}^3$, i.e. of class ALF ("asymptotically locally flat"). As in the compact case, there are currently no tools whatsoever to construct or to classify such manifolds.

We thus concentrate on the remaining reduced holonomy $\mathrm{SU}(2) = \mathrm{Sp}(1)$, where methods of complex geometry become available; such manifolds are usually called "gravitational instantons." The case of maximal volume growth is well understood: Even without assuming $\mathrm{Rm} \in L^2$, [6], all such spaces are asymptotic to $\mathbb{C}^2/\Gamma$ for some $\Gamma < \mathrm{U}(2)$ at rate r^{-4}, and hence coincide with one of Kronheimer's ALE spaces, or with certain finite quotients of these [8].

Much less is known about gravitational instantons of *less* than maximal volume growth, even with $\mathrm{Rm} \in L^2$. Concerning examples and also classification, there is a folklore picture that might be summarized as follows.

(1) Take a space M from the following list: flat ($\mathbb{R}^4$, $\mathbb{R}^3 \times S^1$, $\mathbb{R}^2 \times T^2$, $\mathbb{R} \times T^3$), Taub-NUT (an ALF metric on $\mathbb{R}^4$), or Atiyah-Hitchin (ALF on $\mathcal{O}_{\mathbb{C}P^1}(-4)$).

(2) Find finite groups Γ acting on M by hyperkähler isometries with a finite but non-empty set of fixed points. Construct complete hyperkähler metrics on crepant resolutions of M/Γ that are geometrically asymptotic to M/Γ at infinity.

(3) Determine the full deformation space of the metrics obtained in (2).

(4) Find all finite isometric coverings or subcoverings of the metrics from (3).

For $M = \mathbb{R}^4$, this recovers the ALE story. For the other choices of M, (2) is now understood as well [1], and there has been definite progress towards (3) in the ALF case [5]. It is important to note that the method of [1] reconstructs all known ALF spaces up to deformation, but also produces several new geometries:

- ALG: asymptotic at rate $r^{-2-(1/\theta)}$ to a twisted product of a flat 2-cone with cone angle $\theta \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ and a flat 2-torus, and
- ALH: exponentially asymptotic to a flat half-cylinder $\mathbb{R}^+ \times (T^3/\Gamma)$.

Theorem 1 (see [4]). *There exists a 29-dimensional moduli space of gravitational instantons with the following properties.*

- *The main stratum is smooth and consists of* ALH *spaces, locally comprising all small Ricci-flat deformations of these spaces.*

• *There are lower dimensional strata consisting of* ALG *spaces with cone angles* $\theta \in \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}\}$ *and generic decay* r^{-2} *to the flat model.*
• *Finally, there are strata containing two new types of geometries:*

$$|B(x_0, r)| \sim r^{\frac{4}{3}}, \quad |\mathrm{Rm}| \sim r^{-2}, \quad \mathrm{inj} \sim r^{-\frac{1}{3}},$$

$$|B(x_0, r)| \sim r^2, \quad |\mathrm{Rm}| \sim r^{-2}(\log r)^{-1}, \quad \mathrm{inj} \sim (\log r)^{-\frac{1}{2}}.$$

These have an asymptotic T^2*-fibration as well, but the fibers degenerate.*

Conjecture 2. *This is a classification of gravitational instantons with* $\mathrm{Rm} \in L^2$ *that are not* ALE *or* ALF*. The new geometries should be the only ones with slow curvature decay, and should not occur as singularity dilations.*

The idea is to solve a version of the Calabi conjecture on an appropriate open complex manifold M, endowed with a complete Kähler metric ω which is already asymptotically Ricci-flat. This general approach originates from [7].

2. Analysis for the complex Monge-Ampère equation

We can give a fairly precise theorem that also answers a question in [7].

Theorem 3. *Let* (M^n, ω) *be a complete Kähler manifold with bounded curvature and* $\mathrm{Ric} \geq -Cr^{-2}$ *such that* $|B(x_0, r)| \sim r^\beta$ *for some fixed* $\beta > 0$*. Let* $f : M \to \mathbb{R}$ *be smooth with* $|f| \leq Cr^{-2-\varepsilon}$ *for some* $\varepsilon > 0$*. If* $\beta \leq 2$*, assume* $\int(e^f - 1)\omega^n = 0$*. Then there exists a smooth solution* u *to* $(\omega + i\partial\bar{\partial}u)^n = e^f\omega^n$ *with uniform* $C^{4,\alpha}$ *bounds globally on* M*. If* $\beta \leq 2$*, then in addition* $\int |\nabla u|^2\omega^n < \infty$*.*

Remark 4. (i) Local collapsing is fine as long as large balls have large volume.
(ii) Since we only know how to make L^∞ solutions, a cancellation condition on f is needed if $\beta \leq 2$ to compensate for unboundedness of the Green's function.
(iii) When constructing Ricci-flat metrics, f will be the Ricci potential of ω. If f decays like r^{-2} or slower, or does not satisfy the cancellation condition, it may be possible to first improve ω, e.g. by solving an auxiliary linear equation.
(iv) If $\beta \leq 2$, we are able to exploit $\int |\nabla u|^2\omega^n < \infty$ to get asymptotics for the solution: $u = O(r^{-\varepsilon})$ if the diameter growth is linear, and $u = O(\exp(-\varepsilon r^{1-\gamma}))$ if the diameter growth is $O(r^\gamma)$, $\gamma \in [0, 1)$. These are new, and sharp in order.

3. Finding candidates for M *and* ω

In order to construct a complete Ricci-flat metric, we need M to admit a (possibly multivalued) nowhere vanishing holomorphic $(n, 0)$-form Ω with $\int_M \Omega \wedge \bar{\Omega} = \infty$. This is a highly restrictive condition. Examples come about in the form $M = X \setminus D$ where X is a compact complex variety and D is an anticanonical divisor. If $n = 2$ and both X and D are smooth, this leaves $\mathbb{C}P^2$ blown up in any number of points along a smooth cubic, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^1 \times E$ for an elliptic curve E, and $\mathbb{F}_2$.

Tian-Yau [7] had considered the general n-dimensional case where D is smooth and either ample or a fiber of a morphism. We let X be $\mathbb{C}P^2$, blown up in the 9 base points of a pencil of cubics, and D any fiber of the resulting elliptic fibration. There is an overlap with [7] if D is smooth, but [7] did not prove any asymptotics. The main novelty though is that we are able to treat singular fibers as well.

Constructing an asymptotically Ricci-flat metric ω is key. To do so, we exploit the elliptic fibration structure by making a semi-flat ansatz [3]. In principle, this generalizes to various higher-dimensional situations, but the asymptotic geometry of the metrics seems very hard to actually compute then.

We strongly believe that there is no hope of treating singular divisors in general, and that the normal crossings case should not be any easier than the general case. As a basic example (admittedly not normal crossings), the complement of 3 lines through one point in $\mathbb{C}P^2$ does not even admit complete Riemannian metrics with $\mathrm{Ric} \geq 0$ because π_1 of this space is the free group on 2 generators.

4. Classification issues

Even the more exotic geometries in Theorem 1 fit in well with a belief that, except for Atiyah-Hitchin, the asymptotic geometry of gravitational instantons with $\mathrm{Rm} \in L^2$ should be modelled by a Gibbons-Hawking ansatz; they correspond to integer multiples of the Green's function on $\mathbb{R} \times T^2$ and $(\mathbb{R}^2 \times S^1)/\mathbb{Z}_2$, respectively. Thus, both geometries fit into an infinite sequence indexed by $b \in \mathbb{N}$, but only the cases $b \in \{1, ..., 9\}$ and $b \in \{1, ..., 4\}$ can be filled by elliptic surfaces. The singular fibers at infinity are then of Kodaira type I_b and I_b^*, respectively.

Now consider the Tian-Yau metrics on the complement of a smooth cubic in a del Pezzo surface of degree b. These do not seem to fit into our discussion because $|B(x_0, r)| \sim r^{4/3}$ but $|\mathrm{Rm}| = O(r^{-2/3})$ from [7], so that apparently $\mathrm{Rm} \notin L^2$, but a more careful computation shows $|\mathrm{Rm}| \sim r^{-2}$. I am currently working on a proof that they are in fact globally isometric to the I_b examples from Theorem 1. As a toy model, notice that hyperkähler rotation with respect to a flat metric turns the affine variety $\mathbb{C}^* \times \mathbb{C}^*$ into $\mathbb{C} \times$ elliptic curve.

Conjecture 5. *If $X \setminus D$ carries a gravitational instanton metric, then $X \setminus D$ is biholomorphic to one of the known examples up to hyperkähler rotation.*

Example 6. Generic hyperkähler rotations of ALH spaces can be holomorphically compactified as $\mathbb{C}P^2$ blown up in 9 general points. The divisor D is then a smooth elliptic curve with $(D \cdot D) = 0$ which however does not move in a family.

References

[1] O. Biquard, V. Minerbe, *A Kummer construction for gravitational instantons*, Comm. Math. Phys., to appear.

[2] J. Cheeger, G. Tian, *Curvature and injectivity radius estimates for Einstein 4-manifolds*, J. Amer. Math. Soc. **19** (2006), 487–525.

[3] M. Gross, P. Wilson, *Large complex structure limits of K3 surfaces*, J. Differential Geom. **55** (2000), 475–546.

[4] H.-J. Hein, *Complete Calabi-Yau metrics from* $\mathbb{P}^2 \# 9\bar{\mathbb{P}}^2$, preprint, arXiv:1003.2646.

[5] V. Minerbe, *Rigidity for Multi-Taub-NUT metrics*, J. Reine Angew. Math. **656** (2011), 47–58.

[6] G. Tian, *Aspects of metric geometry of four manifolds*, Inspired by S. S. Chern, 381–397, Nankai Tracts Math. 11, World Scientific Publishing, Hackensack, NJ, 2006.

[7] G. Tian, S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature*, I, J. Amer. Math. Soc. **3** (1990), 579–609.

[8] E. Wright, *Quotients of gravitational instantons*, Ann. Glob. Anal. Geom., to appear.

Smoothing problem for locally CAT(0) metrics

Tadeusz Januszkiewicz

Negatively and nonpositively curved manifolds are in the center of Riemannian geometry and are studied in considerable depth. Perhaps part of the reason for this is the amazing confluence of methods coming from geometry, topology, group theory, dynamical systems...

Despite their importance we know relatively few examples of smooth nonpositively curved metrics. This contrasts sharply with the situation for nonsmooth metrics (on large class of spaces including smoothable topological manifolds), which are nonpositively curved in the comparison sense: satisfy CAT0 condition locally.

There are tools for constructing CAT0 metrics including the reflection group method and (strict) hyperbolization ([2], [5], [3], [1]) which work well for manifolds and give a plethora of nonpositively curved manifolds, sometimes with interesting additional properties.

It seems reasonable to expect that very few of these manifolds carry smooth Riemannian nonpositively curved metrics. We present here some results in this direction, all based on various forms of the Cartan Hadamard theorem.

A. The standard Riemannian version of Cartan Hadamard theorem asserts that if M is a closed nonpositively curved manifold, then the exponential map $exp : T_p \to \tilde{M}$ is a diffeomorphism between the tangent space and the universal cover of M. For closed locally CAT0 manifolds all we can conclude is contractibility of the universal cover. Hence a manifold with universal cover not homeomorphic to R^n is not smoothable.

Examples of such M with CAT0 metrics can be construct as follows. Let Σ^n be a nonsimply connected homology sphere which bound D^{n+1}: a manifold which is a homology disc, such that the induced map of fundamental groups $\pi_1(\Sigma) \to \pi_1(D)$ is injective.

Let $X = cone(D \cup_\Sigma cone(\Sigma))$, $Y = X \cup_{cone\Sigma} X$ and $Z = Y \cup_B Y$, where B is the "boundary" of Y, that is $D \cup_\Sigma D$.

It follows from a deep recognition theorem of R.D. Edwards that Z is a topological manifold. The natural triangulation on Z is not piecewise linear.

Hyperbolizing such a triangulation gives a CAT0 manifold we are after. The construction can be done so that the resulting manifold carry a smooth structure.

B. A slightly less standard version of Cartan Hadamard theorem asserts that if M is a closed nonpositively curved manifold, then the visual compactification of $\tilde{M}$ is homeomorphic to a closed disc. In particular, if $\pi_1 M$ is Gromov hyperbolic, then its Gromov boundary is homeomorphic to a sphere S^{n-1}. Hence a manifold such that the Gromov boundary of $\tilde{M}$ is not homeomorphic to a sphere is not smoothable.

Examples of such M with CAT0 metrics can be construct as follows. Let Σ^n be a nonsimply connected homology sphere. Let Σ^{n+2} be the double suspension of Σ.

The recognition theorem asserts that Σ^{n+2} is a topological manifold, in fact that it is homeomorphic to a sphere. The natural triangulation on Z is not piecewise linear.

Hyperbolizing such a triangulation gives a CAT0 manifold we are after. The construction can be done so that the resulting manifold carry a smooth structure.

Strictly speaking to obtain M with Gromov hyperbolic fundamental group, one needs to use the strict hyperbolization of Charney and Davis. If one uses nonstrict hyperbolization the result is still not smoothable, but one needs a slightly more involved argument, which is not in the literature.

The examples above were constructed in [3] long time ago. All the triangulations we used were not PL. In fact hyperbolizations of PL triangulations always lead to manifolds covered by R^n whose ideal boundary is a sphere. So to find obstructions in PL context we need to do something more delicate. This was done recently in [4] along the following lines.

C. The third version of Cartan Hadamard theorem we use asserts that if M is a closed nonpositively curved manifold, and N is a totally geodesic submanifold, then the exponential map provides a diffeomorphism between pairs (R^n, R^k) and $(\tilde{M}, \tilde{N})$. An extension of this statement to the ideal boundaries gives a homeomorphism between pairs $(\partial\tilde{M}, \partial\tilde{N})$ and the standard sphere pair.

We want to use this together with the Flat Torus Theorem, which asserts that a copy of Z^n in the fundamental group of M is carried by an immersed totally geodesic torus. Hence the ideal boundary of the universal cover of the torus is unknotted sphere. Hence if M contains a totally geodesic torus with knotted infinity, M is not smoothable.

A problem with this "argument" that the notion of boundary is not an invariant of the fundamental group. However the knottedness or unknottedness of a boundary of a flat is.

In [4] we have used reflection group method to provide slightly complicated construction of examples as above with an additional property that M has "isolated flats property". This makes the boundary into an invariant of fundamental group.

Alternative approach is: take a non locally flat torus $T^n \in V^{n+2}$, and hyperbolize the pair, so that the metric on T^n remains unchanged and torus becomes totally geodesic. Such a "relative, metric" hyperbolization is not documented in the literature, but can be obtained from one of hyperbolizations of [3].

D. I don't know any obstruction to smoothing a PL metric of negetive curvature.

On the other hand Pedro Ontaneda has recently announced a smooth variant of strict hyperbolization, that is a hyperbolization procedure in the sense of [3], for which the singular metric can be smoothed out.

References

[1] Charney, Ruth M.; Davis, Michael W., Strict hyperbolization, Topology 34 (1995), no. 2, 329350. 57Q05 (53C23)

[2] Davis, Michael W., The geometry and topology of Coxeter groups, London Mathematical Society Monographs Series, 32. Princeton University Press, Princeton, NJ, 2008. xvi+584 pp. ISBN: 978-0-691-13138-2; 0-691-13138-4 20F55 (05B45 05C25 51-02 57M07)

[3] Davis, Michael W.(1-OHS); Januszkiewicz, Tadeusz(PL-WROC), Hyperbolization of polyhedra, J. Differential Geom. 34 (1991), no. 2, 347388. 57Q05 (53C20)

[4] M. W. Davis, T. Januszkiewicz and J. Lafont, 4-dimensional CAT(0)-manifolds with no Riemannian smoothings, to appear in Duke Math. J., 2011 pdf, arXiv

[5] Gromov, M.(F-IHES), Hyperbolic groups. Essays in group theory, 75263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987. 20F32 (20F06 20F10 22E40 53C20 57R75 58F17)

Reporter: Stephan Stadler

Participants

Prof. Dr. Bernd Ammann
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg

Hugues Auvray
DMA UMR 8553 CNRS
Ecole Normale Superieure
45 rue d'Ulm
F-75230 Paris Cedex 05

Prof. Dr. Werner Ballmann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Dr. Richard Bamler
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

HDoz. Dr. Oliver Baues
Institut für Algebra u. Geometrie
Fakultät für Mathematik
KIT
Kaiserstr. 89-93
76133 Karlsruhe

Tillmann Berg
Fachbereich Mathematik
Humboldt-Universität Berlin
Unter den Linden 6
10117 Berlin

Prof. Dr. Gerard Besson
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Olivier Biquard
Dept. de Mathematiques et Applications
Ecole Normale Superieure
45, rue d'Ulm
F-75005 Paris Cedex

Prof. Dr. Mario Bonk
Department of Mathematics
University of California, Los Angeles
Box 951555
Los Angeles CA 90095-1555
USA

Prof. Dr. Gilles Carron
Dept. de Mathematiques
Universite de Nantes
2, rue de la Houssiniere
F-44322 Nantes Cedex 03

Prof. Dr. Xiuxiong Chen
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook , NY 11794-3651
USA

Prof. Dr. Fuquan Fang
Department of Mathematics
Capital Normal University
Xisanhuan North Road 105
Beijing 100048
P.R. CHINA

Dr. Joel Fine
Service Geometrie Differentielle
Universite Libre de Bruxelles
CP 218
Boulevard du Triomphe 2
B-1050 Bruxelles

Prof. Dr. Akito Futaki
Department of Mathematics
Faculty of Science
Tokyo Institute of Technology
Ohokayama, Meguro-ku
Tokyo 152-8551
JAPAN

Dr. Nicolas Ginoux
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Dr. Nadine Große
Mathematisches Institut
Universität Leipzig
Johannisgasse 26
04109 Leipzig

Dr. Hans-Joachim Hein
Department of Mathematics
Imperial College London
South Kensington Campus
GB-London SW7 2AZ

Prof. Dr. Tadeusz Januszkiewicz
Department of Topology
Institute of Mathematics of the Polish
Academy of Sciences
ul. Sniadeckich 8 - P.O.Box 21
00-956 Warszawa
POLAND

Dr. Martin Kerin
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Robert Kremser
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Urs Lang
Departement Mathematik
ETH Zürich
Rämistr. 101
CH-8092 Zürich

Enrico Le Donne
ETH Zuerich
Departement Mathematik
HG G 68.1
Raemistr. 101
CH-8092 Zuerich

Prof. Dr. Claude LeBrun
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook , NY 11794-3651
USA

Prof. Dr. Bernhard Leeb
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. John Lott
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley CA 94720-3840
USA

Dr. Alexander Lytchak
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Toshiki Mabuchi
Department of Mathematics
Osaka University
Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Jan Metzger
Fachbereich Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam

Dr. Vincent Minerbe
Institut de Mathematiques
Universite Paris VI
UFR 929
4, Place Jussieu
F-75005 Paris Cedex

Dr. Gregoire Montcouquiol
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Dr. Aaron Naber
Department of Mathematics
Massachusetts Institute of
Technology
77 Massachusetts Avenue
Cambridge , MA 02139-4307
USA

Dr. Duc-Manh Nguyen
Institut de Mathematiques de Bordeaux
Universite de Bordeaux I
351, cours de la Liberation
F-33405 Talence Cedex

Prof. Dr. Frank Pacard
Centre de Mathematiques
Faculte de Sciences et Technologie
Universite Paris-Est - Creteil
61, Ave. du General de Gaulle
F-94010 Creteil Cedex

Prof. Dr. Anton Petrunin
Department of Mathematics
Pennsylvania State University
University Park , PA 16802
USA

Dr. Frank Pfäffle
Institut für Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam

Prof. Dr. Joan Porti
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus Universitari
E-08193 Bellaterra

Carlos Ramos-Cuevas
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Viktor Schroeder
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Prof. Dr. Lorenz Schwachhöfer
Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

Stephan Stadler
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Manuel Streil
NWF I - Mathematik
Universität Regensburg
93040 Regensburg

Dr. Jan Swoboda
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Prof. Dr. Iskander A. Taimanov
Sobolev Institute of Mathematics
Siberian Branch of the Russian Academy of Sciences
Koptyuga Prospect N4
630 090 Novosibirsk
RUSSIA

Prof. Dr. Gudlaugur Thorbergsson
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln

Prof. Dr. Gang Tian
Department of Mathematics
Princeton University
609 Fine Hall
Washington Road
Princeton , NJ 08544
USA

Prof. Dr. Jeff A. Viaclovsky
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison , WI 53706-1388
USA

Dr. Thomas Vogel
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Prof. Dr. Bing Wang
Department of Mathematics
Princeton University
609 Fine Hall
Washington Road
Princeton , NJ 08544
USA

Prof. Dr. Gregor Weingart
Instituto de Matematicas
Universidad Nacional Autonoma de Mexico
Avenida Universidad s/n
62210 Cuernavaca , Morelos
MEXICO

Dr. Hartmut Weiss
Mathematisches Institut
Ludwig-Maximilians-Universität München
Theresienstr. 39
80333 München

Dr. Frederik Witt
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Bin Zhou
Beijing International Center for Mathematical Research
Beijing University
Beijing 100871
P.R. CHINA

Prof. Dr. Xiaohua Zhu
Department of Mathematics
Peking University
100871 Beijing
P.R.CHINA