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Partial Differential Equations

Organised by
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August 7th – August 13th, 2011

Abstract. The workshop dealt with partial differential equations in geometry and technical applications. The main topics were the combination of nonlinear partial differential equations and geometric problems, regularity of free boundaries, conformal invariance and the Willmore functional.

Mathematics Subject Classification (2000): 35 J 60, 35 J 35, 58 J 05, 53 A 30, 49 Q 15.

Introduction by the Organisers

The workshop *Partial differential equations*, organised by Luigi Ambrosio (SNS Pisa), Alice Chang (Princeton), Reiner Schätzle (Universität Tübingen), and Georg S. Weiss (University of Tokyo) was held August 7-13, 2011. This meeting was well attended by 52 participants, including 5 females, with broad geographic representation. The program consisted of 17 talks and 6 shorter contributions and left sufficient time for discussions.

New results were presented in geometric measure theory, for example a striking lower bound for the density of singular minimal cones and the regularity of stationary, stable, integral varifolds in codimension 1. Also there were results on singular cones and uniqueness of tangent cones for certain free boundary problems.

Also there were several contributions to regularity of solutions of partial differential equations and to mean curvature flow. We mention a well-posedness result for a critical nonlinear wave equation in two space dimensions.

A major part of the leading experts of partial differential equations with conformal invariance attended the workshop. Here new results were presented in conformal geometry, for the Yamabe problem and the Paneitz operator. For the

Willmore functional, it was established that the Clifford torus minimizes the Willmore energy in an open neighbourhood of conformal classes.

The organisers and the participants are grateful to the Oberwolfach Institute for presenting the opportunity and the resources to arrange this interesting meeting.

Workshop: Partial Differential Equations

Table of Contents

Abstracts

Proof of the Fundamental Gap Conjecture

Julie Clutterbuck
(joint work with Ben Andrews)

Consider the Dirichlet eigenvalue problem for a Schrödinger operator on a bounded domain $\Omega \subset \mathbb{R}^n$, with eigenvalues and associated eigenfunctions satisfying

$$\Delta\phi_i - V(\phi_i) + \lambda_i \phi_i \equiv 0 \quad \text{in } \Omega,$$
$$\phi_i \equiv 0 \quad \text{on } \partial\Omega.$$

The spectrum is

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,$$

and in the special case that $V \equiv 0$, it is a geometric invariant of Ω related to other geometric invariants (such as diameter, volume, perimeter measure) in interesting ways. In statistical physics, the *fundamental gap* between the first two eigenvalues, $\lambda_2 - \lambda_1$, represents the excitation energy of a quantum system.

In 1983, van den Berg **conjectured** that for convex domains Ω and convex potentials V the gap is bounded below by $3\pi^2/D^2$, where $D := \text{diameter}(\Omega)$ [8].

Previous results: Singer, Wong, Yau and Yau [7] used a gradient estimate on the ratio of eigenfunctions ϕ_2/ϕ_1 (satisfying an elliptic equation with Neumann boundary conditions) to find a lower bound $\lambda_2 - \lambda_1 \geq \frac{\pi^2}{4D^2}$. Yu and Zhong, using essentially the same methods and some delicate analysis of the asymmetrical situation when $\max\phi_2 \neq -\min\phi_2$, improved this to $\lambda_2 - \lambda_1 \geq \frac{\pi^2}{D^2}$ [9].

In the one-dimensional case, after significant progress on the problem by Ashbaugh-Benguria and Horváth, it was proved by Lavine in 1994 [3, 5, 6].

In recent work with Ben Andrews, we used estimates for parabolic equations to settle the conjecture completely [2]:

Theorem 1 (Optimal gap bound)**.** *Let $\Omega \subset \mathbb{R}^n$ be convex with diameter D, and let $V : \Omega \to \mathbb{R}$ be convex. Then the fundamental gap satisfies*

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{D^2}.$$

A sketch of the proof: Note that in the one-dimensional case, the domain is an interval and the problem becomes

$$\tilde\phi_i'' - \tilde V(\tilde\phi_i) + \mu_i \tilde\phi_i \equiv 0 \quad \text{in } \left(-\tfrac{D}{2}, \tfrac{D}{2}\right)$$
$$\tilde\phi_i \equiv 0 \quad \text{on } \left\{-\tfrac{D}{2}, \tfrac{D}{2}\right\}.$$

When the potential is constant there is a simple solution with $\mu_i = i^2\pi^2/D^2 + \tilde V$, so that $\mu_2 - \mu_1 = 3\pi^2/D^2$, thus attaining the gap's lower bound. We use this solution as a comparison.

When making estimates on the ratio of eigenfunctions ϕ_2/ϕ_1, Singer-Wong-Yau-Yau employed a result of Brascamp–Lieb [4]: the first eigenfunction ϕ_1 is log-concave, $D^2(\log\phi_1) \leq 0$. In fact, one can improve this – the first eigenfunction ϕ_1 is *at least as log-concave* as $\tilde{\phi}_1$, the first eigenfunction in the one-dimensional case:

Theorem 2 (Improved log-concavity of the first eigenfunction)**.**

$$(\nabla \log\phi_1(y) - \nabla\log\phi_1(x)) \cdot \frac{(y-x)}{|y-x|} \leq 2(\log\tilde{\phi}_1)' \left(\frac{|y-x|}{2}\right).$$

The term on the right is simply $-\frac{2\pi}{D}\tan\left(\frac{\pi|y-x|}{2D}\right)$.

Our next step is to make a parabolic version of the eigenfunction ratio, $v(x,t) := \frac{e^{-\lambda_2 t}\phi_2(x)}{e^{-\lambda_1 t}\phi_1(x)}$. This satisfies a heat equation with drift term and Neumann boundary data

$$(1) \quad v_t = \Delta v + 2\,\nabla v \cdot \nabla\log\phi_1 \quad \text{in } \Omega\times[0,\infty), \quad D_\nu v = 0 \quad \text{on } \partial\Omega\times[0,\infty).$$

For this equation, we use a technique originally developed to find short-term gradient estimates for mean curvature flow with rough initial data [1] to find:

Theorem 3 (Oscillation bound for heat equation with drift)**.** *Let v satisfy* (1)*. Then*

$$v(y,t) - v(x,t) \leq C\,e^{-(\mu_2-\mu_1)\,t}$$

for any $y, x \in \Omega$ and any $t \in [0,\infty)$.

The improved log-concavity estimate is essential here.

The final step is to note that this oscillation estimate for $v = e^{-(\lambda_2-\lambda_1)\,t}\left(\frac{\phi_2}{\phi_1}\right)$ may be rearranged as

$$\operatorname{osc}_\Omega\left(\frac{\phi_2}{\phi_1}\right) \leq Ce^{[(\lambda_2-\lambda_1)-(\mu_2-\mu_1)]\,t}.$$

If it were the case that $(\lambda_2-\lambda_1)-(\mu_2-\mu_1) < 0$, then letting $t\to\infty$ would imply that $\operatorname{osc}_\Omega(\phi_2/\phi_1) \equiv 0$, an absurdity as these are distinct eigenfunctions. Hence, we must have

$$\lambda_2 - \lambda_1 \geq \mu_2 - \mu_1 = \frac{3\pi^2}{D^2}.$$

This completes the proof of the conjecture.

The method also gives sharp results in cases where the potential is either non-convex, or satisfies a stronger convexity condition. In these cases the one-dimensional problem includes a potential $\tilde{V}$ which depends explicitly on V.

Theorem 4 (Gap bound for general Schrödinger operators)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain of diameter D. Then the Dirichlet eigenvalues of the Schrödinger operator satisfy*

$$\lambda_2 - \lambda_1 \geq \mu_2 - \mu_1,$$

where μ_1, μ_2 are the Dirichlet eigenvalues of the one-dimensional Schrödinger operator on $(-D/2, D/2)$ with potential $\tilde{V}$ satisfying

$$(\nabla V(y) - \nabla V(x)) \cdot \frac{(y-x)}{|y-x|} \geq 2\tilde{V}'\left(\frac{|y-x|}{2}\right).$$

References

[1] Ben Andrews, Julie Clutterbuck: Time-interior gradient estimates for quasilinear parabolic equations. *Indiana Univ. Math. J.*, 58, 351–380, 2009.

[2] Ben Andrews, Julie Clutterbuck: Proof of the fundamental gap conjecture. *J. Amer. Math. Soc.*, 24(3), 899–916, 2011.

[3] Mark S. Ashbaugh, Rafael Benguria: Optimal lower bound for the gap between the first two eigenvalues of one-dimensional Schrödinger operators with symmetric single-well potentials. *Proc. Amer. Math. Soc.*, 105(2), 419–424, 1989.

[4] Herm Jan Brascamp, Elliott H. Lieb: On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4), 366–389, 1976.

[5] Miklós Horváth: On the first two eigenvalues of Sturm-Liouville operators. *Proc. Amer. Math. Soc.*, 131(4), 1215–1224, 2003.

[6] Richard Lavine: The eigenvalue gap for one-dimensional convex potentials. *Proc. Amer. Math. Soc.*, 121(3), 815–821, 1994.

[7] I. M. Singer, Bun Wong, Shing-Tung Yau, Stephen S.-T. Yau: An estimate of the gap of the first two eigenvalues in the Schrödinger operator. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(2), 319–333, 1985.

[8] M. van den Berg: On condensation in the free-boson gas and the spectrum of the Laplacian. *J. Statist. Phys.*, 31(3), 623–637, 1983.

[9] Qi Huang Yu, Jia Qing Zhong: Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator. *Trans. Amer. Math. Soc.*, 294(1), 341–349, 1986.

Applications of Di Perna-Lions' theory to semiclassical limits for the Schrödinger equation

Alessio Figalli

In their seminal paper [5], DiPerna and Lions studied the connection between the well-posedness of transport equations and the associated ODEs. Their main result states that the continuity equation

$$\partial_t \mu_t + div(b_t \mu_t) = 0$$

is well-posed in $L^1_{t,x} \cap L^\infty_{t,x}$ provided $b_t : \mathbb{R}^n \to \mathbb{R}^n$ is Sobolev (uniformly with respect to t) and satisfies suitable global conditions. Moreover, from this result they deduce that, roughly speaking, the associated ODE

$$\left\{ \begin{array}{l} \dot{X}(t,x) = b_t(X(t,x)), \\ X(0,x) = x \end{array} \right\}$$

has a unique solution for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ (the precise result is actually slightly different, and we refer to [3] for more details). This result has then been extended by Ambrosio to BV vector fields [1].

In [2, 4] we investigated this theory in a more general setting, which allows us to show the convergence as $\epsilon \to 0$ of the quantum dynamics

$$i\epsilon\,\partial_t \psi^\epsilon = -\epsilon^2 \Delta \psi^\epsilon + U(\psi^\epsilon)$$

to the Liouville dynamics under very weak regularity assumptions on the potential U. This setting includes, for instance, the treatment of the Born-Oppenheimer potential energy surface in molecular dynamics, see [4] and also [6]. In analogy to the classical DiPerna-Lions' theory, the price to pay for allowing singular potentials is that the convergence result holds true only for "a.e. initial data", where "a.e." means with respect to some suitable family of reference measures within the space of all initial data.

References

[1] L. Ambrosio: Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, 158 (2004), no. 2, 227-260.

[2] L. Ambrosio, A. Figalli: Almost everywhere well-posedness of continuity equations with measure initial data. *C. R. Math. Acad. Sci. Paris*, 348 (2010), no. 5-6, 249-252.

[3] L. Ambrosio: Transport equation and Cauchy problem for non-smooth vector fields. Calculus of variations and nonlinear partial differential equations, 1-41, *Lecture Notes in Math.*, 1927, Springer, Berlin, 2008.

[4] L. Ambrosio, A. Figalli, G. Friesecke, J. Giannoulis, T. Paul: Semiclassical limit of quantum dynamics with rough potentials and well-posedness of transport equations with measure initial data. *Comm. Pure Appl. Math.*, to appear.

[5] R. J. DiPerna, P.-L. Lions: Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98 (1989), no. 3, 511-547.

[6] A. Figalli, M. Ligabò, T. Paul: Semiclassical limit for mixed states with singular and rough potentials. Preprint, 2011.

$C^\alpha-$regularity for a class of non-linear elliptic systems with $p-$growth

Jens Frehse

(joint work with Miroslav Bulicek)

We consider Euler equations to variational integrals of the type

$$\int_\Omega F(x, Du) - f \cdot u\, dx$$

with $p-$growth and with boundary conditions. Under certain structure conditions we achieve $C^\alpha-$a-priori-estimates and existence of $C^\alpha-$extremals. These structure conditions have a rather different form than the "Uhlenbeck-case" $\int_\Omega F(x, \mid Du \mid^2) - f\, u\, dx$. We are able to treat integrals of the type

$$\int_\Omega \sum_{i=1}^m \mid Q_i(Du) \mid^{p_i} - f \cdot u\, dx,$$

where $Q_i(Du) := \sum_{\mu,\nu=1}^m \nabla u^\mu A^i_{\mu\nu} \nabla u^\nu$ with positive definite symmetric matrices $A^i_{\mu\nu}$. Since the $A^i_{\mu\nu}$ need not commute, the structures of the examined functionals differ from the structures of Uhlenbeck-type functionals. Non-convex integrands, for example $F(x, Du) := \Pi_{i=1}^{m} \mid Q_i(Du) \mid^{p_i}$ can also be treated in the sense, that weak solutions of the Noether-equations are of regularity class C^α.

A new conformal invariant from generalized scalar curvature

Yuxin Ge
(joint work with Guofang Wang)

In this talk we describe some new conformal invariants related to an inequality proved recently by De Lellis and Topping. In particular, we prove that the De Lellis-Topping inequality is true on 3-dimensional and 4-dimensional Riemannian manifolds of nonnegative scalar curvature. More precisely, if (M^n, g) is a 3-dimensional or 4-dimensional closed Riemannian manifold with non-negative scalar curvature, then

$$\int_M |Ric - \frac{\overline{R}}{n} g|^2 \, dv(g) \leq \frac{n^2}{(n-2)^2} \int_M |Ric - \frac{R}{n} g|^2 \, dv(g),$$

where $\overline{R} = vol(g)^{-1} \int_M R \, dv(g)$ is the average of the scalar curvature R of g. Equality holds if and only if (M, g) is an Einstein manifold. We in fact study the following new conformal invariant

$$\widetilde{Y}([g_0]) := \sup_{g \in \mathcal{C}_1([g_0])} \frac{vol(g) \int_M \sigma_2(g) \, dv(g)}{\left(\int_M \sigma_1(g) \, dv(g)\right)^2},$$

where $\mathcal{C}_1([g_0]) := \{g = e^{-2u} g_0 \mid R > 0\}$. By improving the analysis developed in the study of the σ_k-Yamabe problem, we prove that $\widetilde{Y}([g_0]) \leq 1/3$ when $n = 3$ and $\widetilde{Y}([g_0]) \leq 3/8$ when $n = 4$, which particularly implies the above inequality. Some related results in high dimensions $n > 4$ are also described.

Fractional Yamabe problems

Maria del Mar Gonzalez Nogueras

Based on the relations between scattering operators of asymptotically hyperbolic metrics and Dirichlet-to-Neumann operators of uniformly degenerate elliptic boundary value problems observed, we formulate fractional Yamabe problems that include the boundary Yamabe problem. We observe an interesting Hopf type maximum principle together with interplays between analysis of weighted trace Sobolev inequalities and conformal structure of the underlying manifolds, which extend the phenomena displayed in the classic Yamabe problem and boundary Yamabe problem.

Existence of smooth solutions of degenerate partial differential equations

Qing Han

I am interested in degenerate partial differential equations for which degeneracy occurs in a controlled way. One particular problem is the eigenvalue problem for the Monge-Ampère equation on strictly convex domains with zero Dirichlet boundary values, which results in degeneracy along the boundary. Another problem is the global isometric embedding of the torus into 3-dimensional Euclidean space, for which the degeneracy occurs in those points where the Gauss curvature vanishes. The first problem is degenerately elliptic, while the second one is of mixed type. The goal is to prove the existence of (global) smooth solutions. In a recent work I proved that $C^{1,1}-$solutions are always smooth if the set of degeneracy has a simple geometry and if the Hessian matrix of the solutions have at most one zero eigenvalue. Different from previous works, we do not assume asymptotic behaviours of degenerate functions.

Rigidity estimates for mean curvature flow

Gerhard Huisken
(joint work with C. Sinestrari)

We consider ancient convex solutions of mean curvature flow, that is a one–parameter family of smooth immersions $F : \mathbb{S}^n \times (-\infty, T) \to \mathbb{R}^{n+1}$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = -H(p,t)\,\nu(p,t), \qquad p \in \mathbb{S}^n, t > -\infty, \tag{1}$$

$$F(\,\cdot\,,0) = F_0, \tag{2}$$

where $H(p,t)$ and $\nu(p,t)$ are the mean curvature and the outer normal respectively at the point $F(p,t)$ of the surface $\mathcal{M}_t := F(\,\cdot\,,t)(\mathbb{S}^n)$. The signs are chosen such that $-H\,\nu = \vec{H}$ is the mean curvature vector and the mean curvature of a convex surface is positive.

Ancient solutions of mean curvature flow appear as blow-up limits from singularities of the flow [3], [4] and are of independent interest in the theory of renormalisation group flows [2]. It is wellknown that apart from the homothetically shrinking sphere there are other examples of embedded ancient compact solutions that are strictly convex and degenerate in some way as $t \to -\infty$. An example is the "Angenent oval", a convex ancient solution of the curve shortening flow discovered by Angenent that decomposes into two translating solutions of the flow as $t \to -\infty$. Daskalopoulos, Hamilton and Sesum showed that apart from the homothetically shrinking circle this is the only other embedded convex compact ancient solution of the curve shortening flow, [5].

The lecture explains new estimates that show that the homothetically shrinking sphere is rigid in the class of ancient convex solutions that satisfy certain uniformity estimates as $t \to -\infty$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the principal curvatures.

Theorem 1. *Suppose $F : \mathbb{S}^n \times (-\infty, T) \to \mathbb{R}^{n+1}$ is a smooth convex solution of mean curvature flow satisfying the estimate*

$$\lambda_1 \geq \epsilon\, H \tag{3}$$

or

$$\mathrm{diam}(\mathcal{M}_t) \leq C\,(1 + \sqrt{-t}) \tag{4}$$

uniformly on $\mathbb{S}^n \times (-\infty, T)$. Then the solution $\mathcal{M}_t$ is a homothetically shrinking sphere.

The proof of the first result relies on the fact that certain integrals of powers of the tracefree part of the second fundamental form are decaying at supercritical rate under mean curvature flow, allowing the conclusion that the tracefree part has to vanish on ancient solutions satisfying the first assumption. The second part of the theorem can be reduced to the first part via a contradiction argument, exploiting Hamilton's Harnack inequality in [2].

This result is reminiscent of a similar result for Ricci flow obtained by Brendle, Huisken and Sinestrari in [1]. There it is shown that ancient solutions of Ricci flow that are positively curved and sufficiently pinched must be shrinking spherical space-forms.

The lecture continues to explain partial results for the case of convex, uniformly 2-convex ancient solutions of the flow. It also explains work in progress on how these estimates can be extended to asymptotically flat Riemannian 3-manifolds of positive mass in order to construct unique ancient solutions that are asymptotic to the center of mass in backward time.

References

[1] S. Brendle, G. Huisken, C. Sinestrari: Ancient solutions to the Ricci flow with pinched curvature. Duke Math. J. Volume 158, Number 3 (2011), 537-551.

[2] V. A. Fateev: The sigma model (dual) representation for a two-parameter family of integrable quantum field theories. Nuclear Phys. B 473 (1996), 509–538.

[3] R. Hamilton: Harnack estimate for the mean curvature flow. J. Differential Geom., 41 (1995), 215-226.

[4] G. Huisken, C. Sinestrari: Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math., 183 (1999), 45-70.

[5] P. Daskalopoulos, R. Hamilton, N. Sesum: Classification of compact ancient solutions to the curve shortening flow. J. Differential Geometry 84 (2010), 455-464.

$W^{2,2}$-conformal immersions of closed Riemann surfaces

Ernst Kuwert
(joint work with Yuxiang Li)

We study sequences $f_k : \Sigma_k \to \mathbb{R}^n$ of conformally immersed, compact Riemann surfaces with fixed genus p and Willmore energy

$$\mathcal{W}(f_k) = \frac{1}{4} \int_\Sigma |\vec{H}_{f_k}|^2 \, d\mu_{f_k} \leq \Lambda.$$

By the Gauß equations and the Gauß-Bonnet theorem, the bound is equivalent to an L^2-bound for the second fundamental forms A_k. We rely on results of Müller & Šverák [7] and Hélein [2] about regularity of conformal parametrizations in this context.

Theorem 1. *Under the assumptions above, suppose the Σ_k converge to a Riemann surface Σ in moduli space, i.e. there exist orientation-preserving diffeomorphisms $\phi_k : \Sigma \to \Sigma_k$ such that $\phi_k^*(\Sigma_k) \to \Sigma$ as complex structures. Then for a subsequence there exist Möbius transformations σ_k and a finite set $S \subset \Sigma$ such that*

$$\sigma_k \circ f_k \circ \phi_k \to f \qquad \textit{weakly in } W^{2,2}_{\mathrm{loc}}(\Sigma \backslash S, \mathbb{R}^n),$$

where f is a $W^{2,2}-$ branched conformal immersion. Moreover if $\Lambda < 8\pi$, then f is unbranched.

To define the notion of branched conformal immersions in the $W^{2,2}-$context, we use a local expansion following from [7]. Our second theorem deals with the case when the surfaces Σ_k degenerate.

Theorem 2. *Under the above assumptions, assume now that the Σ_k diverge in moduli space. Then we have*

$$\liminf_{k\to\infty} \mathcal{W}(f_k) \geq \begin{cases} 8\pi & \textit{for } p = 1, \\ \min(8\pi, \omega_p^n) & \textit{for } p \geq 2. \end{cases}$$

Let β_p^n be the infimum of the Willmore energy among genus p immersions into $\mathbb{R}^n$. Then the constants ω_p^n are defined by

$$\omega_p^n = \min \Big\{ 4\pi + \sum_i (\beta_{p_i}^n - 4\pi) : p = \sum_i p_i, \, 1 \leq p_i < p \Big\}.$$

We know from [1] *that $\omega_p^n > \beta_p^n$, and in fact $\omega_p^n > 8\pi$ for sufficiently large p by* [4].

We refer to our paper [3] for further information. Previous work by Kuwert & Schätzle relates as follows to the presented results: Theorem 1 was proved in [6], if $n = 3$ and $\Lambda < \min(8\pi, \omega_p^3)$, and also if $n = 4$ and $\Lambda < \min(8\pi, \omega_p^4, \beta_p^4 + \frac{8\pi}{3})$. This was applied to construct immersions which minimize in a given conformal class. The existence result is restricted by the above bounds, while the regularity is general. Theorem 2 was proved in [5] for $n = 3$, and for $n = 4$ with the lower bound $\min(8\pi, \omega_p^4, \beta_p^4 + \frac{8\pi}{3})$.

By Theorem 1 we get existence of conformally constrained minimizers in any codimension below 8π. This is obtained independently in [8] by Rivière, including regularity under a nondegeneracy assumption. The existence result follows also from recent work by Schätzle [10]. Theorem 2 was announced by Rivière in [8]; the proof is given in a recent preprint [9]. We finally mention the preprint [11] of M. Schmidt, which motivated some of the research.

References

[1] Bauer, M., Kuwert, E.: *Existence of minimizing Willmore surfaces of precribed genus*, Int. Math. Res. Notices **10** (2003), 553–576.

[2] Hélein, F.: *Harmonic maps, conservation laws and moving frames*, second edition, Cambridge tracts in Mathematics **150**, Cambridge University Press, Cambridge 2002.

[3] Kuwert, E., Li, Yuxiang: *$W^{2,2}$-conformal immersions of a closed Riemann surface into $\mathbb{R}^n$*, arXiv:1007.3967 (2010).

[4] Kuwert, E., Schätzle, R. & Li, Yuxiang: *The large genus limit of the infimum of the Willmore energy*, Amer. J. Math. **132** (2010), 37–51.

[5] Kuwert, E., Schätzle, R.: *Closed surfaces with bounds on their Willmore energy*, arXiv:1009.5286 (2010), to appear in Annali Scuola Norm. Sup. Pisa.

[6] Kuwert, E., Schätzle, R.: *Minimizers of the Willmore functional under fixed conformal class*, arXiv:1009.6168 (2010).

[7] Müller, S., Šverák, V.: *On surfaces of finite total curvature*, J. Differential Geometry **42** (1995), 229–258.

[8] Rivière, T.: *Variational principles for immersed surfaces with L^2-bounded second fundamental form*, arXiv:1007.2997 (2010).

[9] Rivière, T.: *Lipschitz conformal immersions from degenerating surfaces with L^2-bounded second fundamental form*, Preprint.

[10] Schätzle, R., *Estimation of the conformal factor under bounded Willmore energy*, Preprint (2011).

[11] Schmidt, M. U.: *Existence of minimizing Willmore surfaces of prescribed conformal class*, arXiv:math/0403301 (2004).

On the log determinant functional of the Paneitz operator

Andrea Malchiodi

(joint work with Matthew Gursky)

Let (M^n, g) be a closed Riemannian manifold. Let $\Delta = \Delta_g$ denote the Laplace-Beltrami operator, and label the eigenvalues of $(-\Delta_g)$ by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, counting multiplicities. The *spectral zeta function* of (M^n, g) is

$$\zeta(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}. \tag{1}$$

By Weyl's asymptotic law, (1) defines an analytic function for $\mathrm{Re}(s) > n/2$.

Note that formally–that is, if we were to take the definition in (1) literally–then

$$\zeta'(0) = -\sum_{j=1}^{\infty} \log \lambda_j = -\log \det(-\Delta_g), \tag{2}$$

although of course the series (1) does not define an analytic function near $s = 0$. However, one can meromorphically extend so that ζ becomes regular at $s = 0$ (see [5]), and in view of (2) define the regularized determinant by

$$\det(-\Delta_g) = e^{-\zeta'(0)}. \tag{3}$$

For compact surfaces Polyakov was able to write a local formula for the ratio of the determinants for two conformal metrics (see [4]). Suppose $\hat{g} = e^{2w}g$, then

$$\log \frac{\det(-\Delta_{\hat{g}})}{\det(-\Delta_g)} = -\frac{1}{12\pi} \int_M (|\nabla w|^2 + 2Kw)\, dA, \tag{4}$$

where $K = K_g$ is the Gauss curvature of g.

In deriving (4) Polyakov exploited a crucial property of the Laplacian in two-dimensions, namely, its conformal covariance: if $\hat{g} = e^{2w}g$, then $\Delta_{\hat{g}} = e^{-2w}\Delta_g$. In general, we say that the metric-dependent differential operator $A = A_g$ is *conformally covariant of bi-degree* (a, b) if $\hat{g} = e^{2w}g$ implies

$$A_{\hat{g}}(\psi) = e^{-bw}A_g(e^{aw}\psi) \tag{5}$$

for each smooth function ψ. In four dimensions, the Paneitz operator

$$P = (-\Delta)^2 + \delta\left(\frac{2}{3}Rg - 2Ric\right) \circ \nabla, \tag{6}$$

satisfies (5) with $a = 0$ and $b = 4$. In [2], [1] it was proved that, analogously to (4), one has that

$$\log \frac{\det(P_{\hat{g}})}{\det(P_g)} = -\frac{1}{4}I - 14II + \frac{8}{3}III, \tag{7}$$

where

$$I[w] = 4\int w|W|^2\, dv - \left(\int |W|^2\, dv\right) \log \int e^{4w}\, dv, \tag{8}$$

$$II[w] = \int wP(w)\, dv - \left(\int Q\, dv\right) \log \int e^{4(w-\overline{w})}\, dv, \tag{9}$$

$$III[w] = 12\int (\Delta w + |\nabla w|^2)^2\, dv - 4\int (w\Delta R + R|\nabla w|^2)\, dv. \tag{10}$$

Here W stands for the Weyl curvature, R for the scalar curvature and Q for the Q-curvature, see e.g. [2]. The main result in [3] is the following.

Theorem 1. *Let $\mathbb{S}^4$ be the 4-sphere, and g_0 the round metric it inherits as a submanifold of $\mathbb{R}^5$. Then there is some critical point $u \in C^\infty(\mathbb{S}^4)$ of F_P such that u is rotationally symmetric and even, and moreover the metric $g = e^{2u}g_0$ is not conformally equivalent to g_0.*

References

[1] T. Branson: *An anomaly associated with 4-dimensional quantum gravity*, Comm. Math. Phys. **178** (1996), 301–309.
[2] T. Branson, B. Oersted: *Explicit functional determinants in four dimensions*, Proc. Amer. Math. Soc. **113** (1991), 669–682.
[3] M. Gursky, A. Malchiodi: *The functional determinant of the Paneitz operator*, preprint, 2010.
[4] A. Polyakov: *Quantum geometry of fermionic strings*, Phys. Lett. B **103** (1981), 211–213.
[5] D. Ray, I. Singer: *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210.

The role of Wolff-Potentials in the analysis of degenerate parabolic equations

Giuseppe Mingione

The aim of this talk is to present the recent discovery, made in [7], [8], [9], [10], of the fact that, provided a natural intrinsic formulation is considered, Wolff potentials play a fundamental role in the regularity analysis of non-homogeneous degenerate parabolic equations of p-Laplacean type, i.e., those modeled by

$$u_t - div(|Du|^{p-2}Du) = \mu. \tag{1}$$

Wolff potentials ([4]) play a fundamental role in the analysis of nonlinear elliptic equations and in the fine properties of solutions to boundary value problems. In particular, basic results in [5], [14], [2], [3] yield L^∞−bounds for solutions (and their derivatives) to equations as $-div(|Du|^{p-2}Du) = \mu$ with p-growth. In particular, the gradient estimates in [2] and [3], whose proofs allow to recover the pointwise estimates for u, show – for basic model problems – several of the integrability results known for measure data problems. Apart from the case $p = 2$, the problem of finding potential estimates for the parabolic case was still open. Even the definition of suitable nonlinear potentials was unclear. In particular, as crystalized in [1], it is impossible to analyze the behaviour of solutions to equations as (1) without using the concept of intrinsic geometry, that is, studying the behaviour of u on “intrinsic cylinders” of the type $Q_r^\lambda(x_0, t_0) := B_r(x_0) \times (t_0 - \lambda^{2-p}r^2, t_0)$ whose sizes depend on the solutions itself in the following intrinsic way:

$$\left(\fint_{Q_r^\lambda} |Du|^{p-1}\, dxdt \right)^{1/(p-1)} := \left(\frac{1}{|Q_r^\lambda|} \int_{Q_r^\lambda} |Du|^{p-1}\, dxdt \right)^{1/(p-1)} \approx \lambda.$$

This, in turn, makes the usual definition of nonlinear Wolff potentials

$$\tilde{\mathbf{W}}^\mu_{\beta,p}(x_0, t_0, r) := \int_0^r \left(\frac{|\mu|(Q_\rho(x_0, t_0))}{\rho^{N-\beta p}} \right)^{1/(p-1)} \frac{d\rho}{\rho}, \qquad \beta \in (0, N/p]$$

constructed by means of standard parabolic cylinders $Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0)$, unuseful for this setting. The approach of [7], [8], [9], [10] proposes to adopt the intrinsic geometry approach in the context of nonlinear potential estimates. This provides some class of intrinsic Wolff potentials that turn out to be the natural objects to be considered, as their structures allow to recast the behaviour of the Barenblatt solution - the so-called nonlinear fundamental solution. For this reason we introduce the following intrinsic Wolff potential

$$\mathbf{W}^{\mu}_{\lambda}(x_0,t_0,r) := \int_0^r \left(\frac{|\mu|(Q^{\lambda}_{\rho}(x_0,t_0))}{\lambda^{2-p}\rho^{N-1}}\right)^{1/(p-1)} \frac{d\rho}{\rho}, \qquad N := n+2,$$

defined by means of intrinsic cylinders, where N is the usual parabolic dimension. The key result is the following which holds for properly defined solutions to measure data problems:

Theorem 1. *Let u be a solution to (1) with $p \geq 2$. For almost every $(x_0,t_0) \in \Omega_T := \Omega \times (0,T)$ there exists some constant $c \geq 1$, depending only on n, p, μ, such that whenever $Q^{\lambda}_r := B_r(x_0) \times (t_0 - \lambda^{2-p}r^2, t_0) \subset \Omega_T$ is an intrinsic cylinder with vertex at (x_0,t_0), such that*

$$c\,\mathbf{W}^{\mu}_{\lambda}(x_0,t_0,r) + c\left(\fint_{Q^{\lambda}_r} (|Du|+s)^{p-1}\,dxdt\right)^{1/(p-1)} \leq \lambda$$

holds, then

$$|Du(x_0,t_0)| \leq \lambda.$$

Theorem 1, which in fact extends to general quasilinear parabolic equations, in turn, gives back the classical $L^{\infty}-$bound due to DiBenedetto [1] who indeed proved that

$$c\left(\fint_{Q^{\lambda}_r} (|Du|+s)^{p-1}\,dxdt\right)^{1/(p-1)} \leq \lambda \quad \Longrightarrow \quad |Du(x_0,t_0)| \leq \lambda.$$

Moreover, Theorem 1 is in a way universal in that it allows

- To recast in a sharp way the asymptotic behaviour of the Barenblatt- (i.e. fundamental) solution when applied to the equation $u_t - div(|Du|^{p-2}\,Du) = \delta$, where δ is the Dirac measure charging the origin; such an estimate is then found to hold for every quasilinear parabolic equation of the type $u_t - div(a(Du)) = \delta$;
- To formulate a non-intrinsic a priori estimate on standard parabolic cylinders, which in fact exhibits the natural anisotropic structure typical for parabolic problems:
$$|Du(x_0,t_0)| \leq c\,\tilde{\mathbf{W}}^{\mu}_{1/p,p}(x_0,t_0,r) + c\fint_{Q_r} (|Du|+s+1)^{p-1}\,dxdt$$
holds whenever $Q_{2r} \equiv B_{2r}(x_0) \times (t_0 - 4r^2, t_0) \subset \Omega_T$;
- To recast the known elliptic gradient Wolff potential estimates in the stationary case;
- To have an a-priori estimate which involves standard elliptic Wolff potentials in those cases when μ is time-independent or admits a favourable space/time decomposition.

References

[1] DiBenedetto, E.: Degenerate parabolic equations. Universitext. Springer-Verlag, New York, 1993.
[2] Duzaar F., Mingione G.: Gradient estimates via nonlinear potentials. Amer. J. Math., to appear.
[3] Duzaar F., Mingione G.: Gradient estimates via linear and nonlinear potentials. J. Funct. Anal. 259 (2010), 2961–2998.
[4] Hedberg L.I., Wolff T.: Thin sets in nonlinear potential theory. Ann. Inst. Fourier (Grenoble) 33 (1983), 161–187.
[5] Kilpelainen T., Maly.: The Wiener test and potential estimates for quasilinear elliptic equations. Acta Math. 172 (1994), 137–161.
[6] Kuusi T., Mingione G.: Universal potential estimates. Preprint.
[7] Kuusi T., Mingione G.: Nonlinear potential estimates in parabolic problems. Rendiconti Lincei – Matematica e Applicazioni 22 (2011).
[8] Kuusi T., Mingione G.: The Wolff gradient bound for degenerate parabolic equations. Preprint 2010.
[9] Kuusi T., Mingione G.: Gradient regularity for nonlinear parabolic equations. In preparation.
[10] Kuusi T., Mingione G.: Universal intrinsic geometries and the regularity of degenerate parabolic equations via Wolff potentials. In preparation.
[11] Mingione G.: Gradient estimates below the duality exponent. Math. Ann. 346 (2010), 571–627.
[12] Mingione G.: Gradient potential estimates. J. Europ. Math. Soc. 13 (2011), 459–486.
[13] Mingione G.: Nonlinear aspects of Calderon-Zygmund theory. Jahresbericht der DMV 112 (2010), 159–191.
[14] Trudinger N.S., Wang X.J.: On the weak continuity of elliptic operators and applications to potential theory. Amer. J. Math. 124 (2002), 369–410.

New explicit examples of constrained Willmore minimizers

Cheikh Birahim Ndiaye
(joint work with Reiner Michael Schätzle)

In this talk, we present a convergence procedure which improves weak convergence of conformally constrained Willmore immersions to smooth convergence and apply this to get new explicit examples of constrained Willmore minimizers. In fact, by estimates of Li-Yau in [4] and Montiel-Ros in [5], the Clifford torus $T_{Cliff} := \frac{1}{\sqrt{2}}(\mathbb{S}^1 \times \mathbb{S}^1) \subset \mathbb{S}^3$ minimizes the Willmore energy within its conformal class. Using our convergence procedure, we extend this to the constant mean curvature surfaces $T_r := r\,\mathbb{S}^1 \times \sqrt{1-r^2}\,\mathbb{S}^1 \subset \mathbb{S}^3$ for $r \approx \frac{1}{\sqrt{2}}$. Furthermore, as a by-product of our arguments, we give a result which improves the region of validity of the Willmore conjecture.

In order to describe our results more precisely, we first recall that, given an immersion $f : \Sigma \longrightarrow \mathbb{R}^n$ of a closed orientable surface, the Willmore energy of f is defined by

$$\mathcal{W}(f) := \frac{1}{4}\int_{\Sigma} |\vec{H}|^2 d\mu_g,$$

where $\vec{H}$ is the mean curvature vector of f, $g := f^* g_{euc}$ is the pull-back by f of the standard metric g_{euc} of $\mathbb{R}^n$ and μ_g the induced area measure on Σ. Critical points of $\mathcal{W}$ under compactly supported variations are called Willmore surfaces. They satisfy the following Euler-Lagrange equation:

$$\Delta_g \vec{H} + Q(A^0)\vec{H} = 0, \tag{1}$$

where Δ_g is the Laplacian of the normal bundle of f, $A^0 := A - \frac{1}{2} g \otimes \vec{H}$ is the trace free part of the second fundamental form A of f, and $Q(A^0)$ acts linearly on normal vectors along f by

$$Q(A^0)(\phi) := g^{ik} g^{jl} A^0_{ij} < A^0_{kl}, \phi > .$$

Critical points of $\mathcal{W}$ under fixed conformal class are called constrained Willmore surfaces. They satisfy the following Euler-Lagrange equation:

$$\Delta_g \vec{H} + Q(A^0)\vec{H} = g^{ik} g^{jl} A^0_{ij} q_{kl}, \tag{2}$$

where q is a smooth transverse traceless symmetric 2-covariant tensor with respect to g, i.e.

$$\begin{aligned} q_{kl} &= q_{lk}, \\ tr_g q = g^{kl} q_{kl} &= 0, \\ g^{ij} \nabla_i q_{jk} &= 0. \end{aligned} \tag{3}$$

For $\Sigma \not\cong \mathbb{S}^2$, $\mathcal{M}_0$ the set of metrics on Σ, $\mathcal{T}$ its Teichmüller space, $\pi : \mathcal{M}_0 \longrightarrow \mathcal{T}$ the projection, $f : \Sigma \longrightarrow \mathbb{R}^n$, and $V \in C^\infty(\Sigma, \mathbb{R}^n)$ we set

$$\delta \pi_f . V := \frac{d}{dt} \pi \left((f + tV)^* g_{euc} \right) |_{t=0},$$

and for a chart $\psi : U(\pi(g)) \longrightarrow \mathbb{R}^{dim \mathcal{T}}$, we put $\hat{\pi} := \psi \circ \pi$. We call an immersion f weakly conformally constrained Willmore, if

$$\delta \mathcal{W}(f) \in span\{\delta \hat{\pi}^s_f |\ s = 1, \ldots, d\} \tag{4}$$

for some $d \leq dim \mathcal{T}$. We say that such an f is of full rank for the weak constraint, if there exist variations $V_1, \ldots, V_d \in C^\infty(\Sigma, \mathbb{R}^n)$ such that

$$(\delta \hat{\pi}_f . V^s_r)_{r,s=1,\ldots,d} \in \mathbb{R}^{d \times d} \ \text{ is non-singular.}$$

Now, having fixed the needed notation and definitions, we are ready to state our convergence result which reads as follows:

Theorem 1. *Let $f_m : \Sigma \not\cong \mathbb{S}^2 \longrightarrow \mathbb{R}^n$ with $g_m := f_m^* g_{euc} = e^{2u_m} g_{poin,m}$ for some unit constant curvature metric $g_{poin,m}$ and*

$$\begin{aligned} f_m &\longrightarrow f \quad \textit{weakly in } W^{2,2}(\Sigma), \ \textit{weakly}^* \ \textit{in } W^{1,\infty}(\Sigma), \\ u_m &\longrightarrow u \quad \textit{weakly in } W^{1,2}(\Sigma), \ \textit{weakly}^* \ \textit{in } L^\infty(\Sigma), \\ g_{poin,m} &\longrightarrow g_{poin} \ \textit{smoothly.} \end{aligned} \tag{5}$$

Assume further that f_m are weakly conformally constrained Willmore and f is of full rank for the weak constraint. Then for a subsequence

$$f_m \longrightarrow f \;\text{ is smooth on }\; \Sigma \setminus \{p_1, \dots, p_N\},$$

outside finitely many points $p_1, \dots, p_N \in \Sigma$, and if $f_m \longrightarrow f$ strongly in $W^{2,2}(\Sigma)$, then

$$f_m \longrightarrow f \;\text{ smoothly on }\; \Sigma.$$

The proof of Theorem 1 is based on the definition of weak conformal constraint, of full rank for the weak conformal constraint, the interior estimate of Kuwert-Schätzle [3], the work of Kuwert-Schätzle [2] and standard elliptic regularity theory. In order to present the second result of this talk, we first recall that any torus is conformally equivalent to a quotient $\mathcal{T}_\omega := \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ with g_{euc} and

$$\omega \in \mathbb{M} = \{a + ib \in \mathbb{C} : \;\; b > 0, \;\; 0 \leq a \leq \frac{1}{2}, \;\; a^2 + b^2 \geq 1\}.$$

Obviously, we have $T_{Cliff} \simeq \mathcal{T}_i$ and for $r \neq \frac{1}{\sqrt{2}}$, $T_r \simeq \mathcal{T}_{ib_r}$ with $b_r = \frac{\sqrt{1-r^2}}{r}$. On the other hand, we have that T_r reach each rectangular structure exactly once for $r_b = \frac{1}{\sqrt{1+b^2}}$. Now, setting

$$\mathcal{M}(\omega) = \inf\{\mathcal{W}(f) |\; f : \mathcal{T}_\omega \longrightarrow \mathbb{R}^3 \;\text{ conformal}\}$$

for $\omega \in \mathbb{M}$, the second presented result reads as follows:

Theorem 2. *T_r minimizes the Willmore energy within its conformal class for $r \simeq \frac{1}{\sqrt{2}}$ and $n = 3$. Actually, we have*

$$\mathcal{W}(T_{r_b}) = \mathcal{M}(ib) = \min_{a \in \mathbb{R}} \mathcal{M}(a + ib), \quad \textit{for } b \simeq 1, b \geq 1.$$

The proof of Theorem 2 uses the results of Kuwert-Schätzle in [1] and [2], our convergence theorem (Theorem 1), the uniqueness result of Li-Yau [4] and the stability result of Weiner [6]. A direct corollary of Theorem 2 is the following result which improves the region of validity of the Willmore conjecture.

Corollary 1. *The Clifford torus T_{Cliff} is the unique minimizer in $\mathbb{R}^3$ – up to Möbius transformations – of the Willmore energy in an open neighbourhood in moduli space of its conformal structure.*

References

[1] Kuwert E., Schätzle R.: *Closed surfaces with bounds on their Willmore energy*, to appear.
[2] Kuwert E., Schätzle R.: *Minimizers of the Willmore functional under fixed conformal class*, submitted.
[3] Kuwert E., Schätzle R.: *Removability of point singularities of Willmore surfaces*, Ann. of. Math. 160, no. 1, (2004), 315-357.
[4] Li P., Yau S. T.: *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math 69, (1982), 269-291.
[5] Montiel S., Ros A.: *Minimal immersions of surfaces by the first eigenvalue and conformal area*, Invent. Math. 83, (1986), 153-166.
[6] Weiner J.: *On a problem of Chen, Willmore, et al*, Indiana University Mathematical Journal 27, (1978), 18-35.

Normalized Ricci flows and conformally compact Einstein metrics

Jie Qing
(joint work with Yuguang Shi and Jie Wu)

In our recent work we investigated the behaviour of the normalized Ricci flow on asymptotically hyperbolic manifolds. We showed that the normalized Ricci flow exists globally and converges to an Einstein metric when starting from a non-degenerate and sufficiently Ricci pinched metric. More importantly, we used maximum principles to establish the regularity of conformal compactness along the normalized Ricci flow including that of the limit metric at time infinity. Therefore we were able to recover the existence results by Robin Graham, John Lee, Oliver Biquard on conformally compact Einstein metrics with conformal infinities which are perturbations of that of given non-degenerate conformally compact Einstein metrics.

On the Dirichlet problem for variational integrals in BV

Thomas Schmidt

A model problem in the multi-dimensional calculus of variations is the minimization problem for the integral

$$E_1[w] := \int_\Omega \sqrt{1+|\nabla w(x)|^2}\,\mathrm{d}x \qquad \text{among functions} \quad w : \Omega \to \mathbb{R}^N$$

with prescribed Dirichlet boundary data. Here, $n, N \in \mathbb{N}$ are arbitrary dimensions, Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ ($n \geq 2$), the derivative $\nabla w(x)$ is understood as an element of $\mathbb{R}^{Nn}$, and $|\nabla w(x)|$ denotes its Euclidean norm.

In codimension $N = 1$ the quantity $E_1[w]$ measures the area of the graph of w and the minimization problem is the non-parametric version of Plateau's famous problem; see [9] for a survey and references. In the following the focus is on codimensions $N > 1$, where $E_1[w]$ is quite different from the area of the graph of w.

Existence results for minimizers of E_1 are available in the space $\mathrm{BV}(\Omega, \mathbb{R}^N)$ of functions of bounded variation: fixing boundary values $u_0 \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ one extends E_1 from the Dirichlet class $\mathcal{D} := u_0 + \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N)$ to $\mathrm{BV}(\Omega, \mathbb{R}^N)$ by letting

$$(1) \quad \mathcal{E}_1^{\mathcal{D}}[w] := \int_\Omega \sqrt{1+|\nabla w(x)|^2}\,\mathrm{d}x + |\mathrm{D}^s w|(\Omega) + \int_{\partial\Omega} |u_0(x) - w(x)|\,\mathrm{d}\mathcal{H}^{n-1}(x),$$

where $\mathrm{D}w = \nabla w \cdot \mathrm{d}x + \mathrm{D}^s w$ is the Lebesgue decomposition of the $\mathbb{R}^{Nn}$-valued gradient measure $\mathrm{D}w$ into its absolutely continuous part with density ∇w and its singular part $\mathrm{D}^s w$, $|\mathrm{D}^s w|(\Omega)$ denotes the total variation of $\mathrm{D}^s w$, and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. It is then well-known [8, 1] that $\mathcal{E}_1^{\mathcal{D}}$ is the reasonable extension of E_1 from $\mathcal{D}$ to $\mathrm{BV}(\Omega, \mathbb{R}^N)$ satisfying

$$\min_{\mathrm{BV}(\Omega,\mathbb{R}^N)} \mathcal{E}_1^{\mathcal{D}} = \inf_{\mathcal{D}} E_1\,.$$

In particular, the minimum on the left-hand side exists, and those functions achieving it are called generalized minimizers of E_1 in $\mathcal{D}$ – even though they need not coincide with u_0 on $\partial\Omega$ and need not be contained in $\mathcal{D}$.

In a joint work [3] with L. Beck we showed that *every* generalized minimizer u of E_1 in $\mathcal{D}$ is actually more regular than BV, namely

$$(*) \qquad u \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N) \qquad \text{and} \qquad |\nabla u| \log(1 + |\nabla u|^2) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)\,.$$

Our proof of $(*)$ is based on an $\mathrm{L}\log\mathrm{L}$-estimate for "good" minimizing sequences, which are selected by multiple regularizations (including a device of [7]) and an application of Ekeland's variational principle in the negative Sobolev space $\mathrm{W}^{-1,1}$. We hereby improve some previous work [4, 5] of M. Bildhauer, who established $(*)$ for *only one* generalized minimizer and under slightly stronger assumptions. A consequence of having $(*)$ for *every* generalized minimizer is uniqueness of $\mathrm{D}u$, which implies uniqueness of generalized minimizers u *up to additive constants* $c \in \mathbb{R}^N$. At this stage it should be noted that, even for $n = 2$, $N = 1$ and the non-parametric area, attainment of the boundary values and full uniqueness of generalized minimizers u only hold in particular situations [10], while in general non-attainment may occur and generalized minimizers need only coincide up to constants [11, 2].

Uniqueness up to constants of $\mathbb{R}^N$ means that the set $\mathcal{M}_1^{\mathcal{D}}$ of generalized minimizers of E_1 in $\mathcal{D}$ is an N-parameter-family. Improving this assertion we show that $\mathcal{M}_1^{\mathcal{D}}$ is indeed a 1-parameter-family. Moreover, we provide a quite explicit description of the boundary behaviour of generalized minimizers in case of non-uniqueness, we formulate versions of our theorems for more general integrals with linear growth, we give several related examples, and we discuss connections with Bernstein's genre, Serrin's classification of non-uniformly elliptic equations, and the μ-ellipticity condition of [6].

References

[1] G. Anzellotti, M. Giaquinta: Convex functionals and partial regularity. *Arch. Ration. Mech. Anal.* **102**, 243–272 (1988).

[2] S. Baldo, L. Modica: Non uniqueness of minimal graphs. *Indiana Univ. Math. J.* **40**, 975–983 (1991).

[3] L. Beck, T. Schmidt: On the Dirichlet problem for variational integrals in BV. To appear in *J. Reine Angew. Math.* **63** (2011). Available at `http://www.mi.uni-erlangen.de/~schmidt/papers/BVintegrals.pdf`.

[4] M. Bildhauer: A priori gradient estimates for bounded generalized solutions of a class of variational problems with linear growth. *J. Convex Anal.* **9**, 117–137 (2002).

[5] M. Bildhauer: Two dimensional variational problems with linear growth. *Manuscr. Math.* **110**, 325–342 (2003).

[6] M. Bildhauer, M. Fuchs: On a class of variational integrals with linear growth satisfying the condition of μ-ellipticity. *Rend. Mat. Appl., VII. Ser.* **22**, 249–274 (2002).

[7] M. Carozza, J. Kristensen, A. Passarelli di Napoli: Higher differentiability for minimizers of convex variational integrals. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **28**, 395–411 (2011).

[8] M. Giaquinta, G. Modica, J. Souček: Functionals with linear growth in the calculus of variations. *Commentat. Math. Univ. Carol.* **20**, 143–156 & 157–172 (1979).

[9] E. Giusti: *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, Basel, 1984.

[10] M. Miranda: Un principio di massimo forte per le frontiere minimali e una sua applicazione alla risoluzione del problema al contorno per l'equazione delle superfici di area minima. *Rend. Sem. Mat. Univ. Padova* **45**, 355–366 (1971).
[11] E. Santi: Sul problema al contorno per l'equazione delle superfici di area minima su domini limitati qualunque. *Ann. Univ. Ferrara, N. Ser., Sez. VII* **17**, 13–26 (1972).

On the singularities of a free boundary through Fourier expansion

Henrik Shahgholian
(joint work with John Andersson and Georg S. Weiss)

In this paper we are concerned with singular points of solutions to the unstable free boundary problem

$$\triangle u = -\chi_{u>0} \qquad \text{in} \quad B_1(0).$$

The problem arises in applications such as solid combustion, composite membranes, climatology and fluid dynamics. It is known that solutions to the above problem may exhibit singularities - that is points at which the second derivatives of the solution are unbounded - as well as degenerate points. This causes breakdown of by-now classical techniques. Here we introduce new ideas based on Fourier expansion of the nonlinearity $\chi_{u>0}$. The method turns out to have enough momentum to accomplish a complete description of the structure of the singular set in $\mathbb{R}^3$. A surprising fact in $\mathbb{R}^3$ is that although

$$\frac{u(rx)}{sup_{B_1(0)}|u(rx)|}$$

can converge at singularities to each of the harmonic polynomials

$$xy,\ \frac{x^2+y^2}{2} - z^2 \text{ and } z^2 - \frac{x^2+y^2}{2},$$

it may not converge to any of the non-axially-symmetric harmonic polynomials $\alpha((1+\delta)\,x^2 + (1-\delta)\,y^2 - 2z^2)$ with $\delta \neq 1/2$. We also prove the existence of stable singularities in $\mathbb{R}^3$.

Partial regularity for fully nonlinear elliptic PDE

Luis Silvestre
(joint work with Scott Armstrong and Charles Smart)

We prove that viscosity solutions to a fully nonlinear elliptic equation $F(D^2u) = 0$ are smooth, i.e. of class $C^{2,\alpha}$, outside of a set of Hausdorff-dimension at most $n-\epsilon$, where n is the dimension and ϵ a small constant depending on the ellipticity bounds of F and on n. We do not make any convexity assumption on the equation F, but we assume that it is of differentiability-class C^1 in addition to uniform ellipticity. We also discuss the relationship of this partial regularity result with the question of unique continuation of solutions.

Intrinsic Flat Convergence and Stability of the Positive Mass Theorem

Christina Sormani
(joint work with S. Wenger)

In 1991, Gromov introduced the Gromov-Hausdorff distance between compact Riemannian manifolds. Applying the Bishop-Gromov Volume Comparison Theorem, he proved that sequences of Riemannian manifolds, M^m, with uniform upper bounds on their diameter and lower bounds on their Ricci curvature have subsequences which converge in the Gromov-Hausdorff sense to compact geodesic metric spaces. Fukaya refined this notion, defining metric measure convergence, in which the measures converge as well. Cheeger-Colding proved that manifolds with lower Ricci curvature bounds converge in the metric measure sense to rectifiable metric measure spaces satisfying the Bishop-Gromov Volume Comparison Theorem. One key consequence was the convergence of the Laplace spectrum.

In 2004, Ilmanen proposed the necessity of a new form of convergence for Riemannian manifolds, one for which sequences of three dimensional spheres with increasingly many splines and positive scalar curvature would converge. Such sequences do not converge in the Gromov-Hausdorff sense. Recently Sormani-Wenger introduced the Intrinsic Flat Distance between compact oriented Riemannian manifolds applying Ambrosio-Kirchheim's notion of integral currents on metric spaces [1]:

$$d_{\mathcal{F}}(M_1^m, M_2^m) = \inf\left\{ d_F^Z(\varphi_{1\#}[M_1], \varphi_{2\#}[M_2]) : \varphi_i : M_i \to Z \right\},$$

where the infimum is again taken over all metric spaces, Z, and all isometric embeddings, $\varphi_i : M_i^m \to Z$, and where d_F^Z is the Flat Distance between the submanifolds $\varphi_i(M_i)$ viewed as integral currents $\varphi_{i\#}[M_i]$ in the metric space Z [11]. Here, as in Gromov, an isometric embedding is a map $\varphi : X \to Z$ such that

$$d_Z(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \ \ \forall x_1, x_2 \in X.$$

Recall that the Flat Distance between integral currents on Euclidean space was first introduced by Federer-Flemming based on work of Whitney. Intuitively, it measures the amount of volume between the two submanifolds.

To estimate the Intrinsic Flat distance between a pair of oriented Riemannian manifolds one needs only find a pair of isometric embeddings, $\varphi_i : M_i^m \to Z$, into a common complete metric space, Z. When one finds a filling submanifold, $B^{m+1} \subset Z$, and an excess boundary submanifold, $A^m \subset Z$, such that

$$\int_{\varphi_1(M_1)} \omega - \int_{\varphi_2(M_2)} \omega = \int_B d\omega + \int_A \omega,$$

then the Intrinsic Flat distance is bounded by

$$d_{\mathcal{F}}(M_1^m, M_2^m) \le \mathrm{Vol}_{m+1}(B^{m+1}) + \mathrm{Vol}_m(A^m). \tag{1}$$

Generally the filling manifold can have corners and the excess boundary manifold may have many components. One can easily see that Ilmanen's Example converges to a sphere using these estimates. Techniques for estimating the Intrinsic Flat distance appear in the Appendix to [11], in [5] and in [3].

More generally, the Intrinsic Flat distance is defined between pairs of integral current spaces [11]. An integral current space (X, d, T) is a countably $\mathcal{H}^m-$rectifiable metric space (X, d) with an integral current structure, $T \in \mathbf{I}_m(\bar{X})$, such that set(T) = X where set(T) denotes the set of positive density as in Ambrosio-Kirchheim [1]. The integral current structure encodes both an orientation and a measure $||T||$. The fact that $X = \text{set(T)}$ guarantees that X is countably $\mathcal{H}^m-$rectifiable, has the correct dimension and does not include cusp singularities [11].

When M_j^m are a noncollapsing sequence of compact Riemannian manifolds with nonnegative Ricci curvature, $\text{diam}(M_j) \le D$ and $\text{Vol}(M_j) \ge V_0$ then the Gromov-Hausdorff and Intrinsic Flat limits agree [10]. More generally the Gromov-Hausdorff limit (if it exists) may contain the Intrinsic Flat limit as a proper subset. The Intrinsic Flat limit may also be the 0 space. This occurs, for example, when we have a collapsing sequence with $\text{Vol}(M_j) \to 0$ or due to cancellation [11].

Wenger has proven a compactness theorem: sequences of compact oriented manifolds, M_j^m, with $\text{diam}(M_j) \le D$, $\text{Vol}_m(M_j) \le V$ and $\text{Vol}_{m-1}(\partial M_j) \le A$, have subsequences which converge in the Intrinsic Flat sense to an integral current space [12]. We conjecture that *if M_j are three dimensional, with positive scalar curvature and no interior closed minimal surfaces, then there is no cancellation; so the limit space is not the* 0 *space unless* $\text{Vol}(M_j) \to 0$ [11]. One approach to proving this conjecture would involve estimating the filling volumes of spheres [10].

There are a number of consequences of Intrinsic Flat convergence following immediately from the work of Ambrosio-Kirchheim: including, in particular, the lower semicontinuity of mass (or volume). These are explored in [6]. When one also assumes that the mass converges, one has convergence of the measures and additional consequences [7].

Applications of the Intrinsic Flat convergence are explored in work of Lakzian-Sormani [3] and Lee-Sormani [5] [4]. In [3] we study smooth convergence away from singular sets using Intrinsic Flat convergence to understand the Gromov-Hausdorff limits.

In the work of Lee-Sormani, we explore the stability of the Positive Mass Theorem. Recall that the Schoen-Yau Positive Mass Theorem states that an asymptotically flat Riemannian manifold, M^3, with nonnegative scalar curvature has nonnegative ADM mass, and if $m_{ADM}(M^3) = 0$ then M^3 is Euclidean space [SY]. We propose the following conjecture: *If a sequence of asymptotically flat manifolds, M_j^3, has nonnegative scalar curvature and no interior closed minimal surfaces and has $m_{ADM}(M_j^3) \to 0$ then it converges in the pointed Intrinsic Flat sense to Euclidean space if the points are chosen on CMC-surfaces of constant area.* We proved this theorem in the rotationally symmetric case in [5]. In fact, we have volume convergence in this case. We examined a similar theorem concerning the Penrose Inequality in [4]. One may also wish to extend the results in [8] using the Intrinsic Flat Distance.

Other possible notions of distances between Riemannian manifolds and convergence of Riemannian manifolds are proposed in [9]. A scalable Intrinsic Flat distance is being developed by Basilio. A notion of area convergence is being developed by Burago-Ivanov [2]. One may speculate on how to define an intrinsic varifold convergence for manifolds [9]. For all of these notions of convergence, one may wish to understand sequences of conformal Riemannian manifolds: *examining which forms of convergence of* $(M, e^{f_j} g) \to (M, e^f g)$ *correspond with which forms of convergence of* $f_j \to f$ *as functions on* (M, g).

References

[1] L. Ambrosio and B. Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1), 1–80, 2000.
[2] D. Burago and S. Ivanov. Area Spaces: First Steps, with Appendix by Nigel Higson. *Geometric and Functional Analysis*, 19(3), 662–677, 2009.
[3] S. Lakzian and C. Sormani. Smooth Convergence away from Singular Sets. *preprint in progress.*
[4] D. A. Lee and C. Sormani. Almost Inequality in the Penrose Equality for Rotationally Symmetric Riemannian Manifolds. *preprint in progress.*
[5] D. A. Lee and C. Sormani. Stability of the Positive Mass Theorem for Rotationally Symmetric Riemannian Manifolds. *preprint on arxiv*, 2011.
[6] C. Sormani. Properties of the Intrinsic Flat Convergence. *preprint in progress.*
[7] C. Sormani. Volume Preserving Intrinsic Flat Convergence and Scalar Curvature. *preprint in progress.*
[8] C. Sormani. Friedmann Cosmology and Almost Isotropy. *Geom. Funct. Anal.*, 14(4), 853–912, 2004.
[9] C. Sormani. How Riemannian Manifolds Converge: a Survey. *Progress in Mathematics*, 2010.
[10] C. Sormani and S. Wenger. Weak convergence and cancellation, Appendix by Raanan Schul and Stefan Wenger. *Calculus of Variations and Partial Differential Equations*, 38(1-2), 2010.
[11] C. Sormani and S. Wenger. Intrinsic flat convergence of manifolds and other integral current spaces. *Journal of Differential Geometry*, 87, 2011.
[12] S. Wenger. Compactness for manifolds and integral currents with bounded diameter and volume. *to appear in Calculus of Variations and Partial Differential Equations*, 2010.

On Almgren's center manifold and regularity of minimal surfaces

Emanuele Nunzio Spadaro
(joint work with Camillo De Lellis)

In this talk I presented an $\epsilon-$regularity result for minimal surfaces leading to their higher (i.e. $C^{3,\alpha}-$) regularity without the use of non-parametric techniques. Our model statement is as follows:

Theorem 1. *There exists some* $\epsilon > 0$ *with the following property:*
Let $\Sigma \subset \mathbf{B}^n_1(0) \subset \mathbb{R}^n$ *be an* m*-dimensional area-minimizing surface without boundary such that* $0 \in \Sigma$ *and* $\mathrm{Vol}_m(\Sigma) \leq \omega_m + \epsilon$*, where* ω_m *denotes* $\mathrm{Vol}_m(\mathbf{B}^m_1(0))$.
Then $\Sigma \cap \mathbf{B}^n_{\frac{1}{2}}(0)$ *is a graph of some* $C^{3,\alpha}-$*function, for some* $\alpha \in (0,1)$.

This result applies to more general situations, leading e.g. to $C^{3,\alpha}-$regularity of supports of area-minimizing integral currents (in GMT).

Although the $\epsilon-$regularity result of De Giorgi-Allard joint with non-parametric theory implies the above theorem, our proof is different and applies to the regularity of *Almgren's center manifold* which is an "approximate average" of higher codimensional area-minimizing integral currents.

Instability of Ginzburg-Landau vortices on a 2-manifold

Peter Sternberg

In this talk we consider stable critical points of the Ginzburg-Landau energy without magnetic effects, posed on a smooth, compact, simply connected 2-manifold $\mathcal{M}$ without boundary, endowed with metric g:

$$E_\varepsilon(u) := \frac{1}{2}\int_{\mathcal{M}} ||\nabla_g u||_g^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} dvol_g,$$

where $u : \mathcal{M} \to \mathbb{C}$. The goal is to explore the existence/non-existence of stable solutions with vortices. In particular we ask: Can there exist stable "geometrically induced vortices" based on curvature properties of the manifold ? The main result presented, due to Ko-Shin Chen, is that a sequence of critical points $\{u_{\varepsilon_i}\}$ possessing vortices any of whose asymptotic location (as $\varepsilon_i \to 0$) is at a point of positive Gaussian curvature, must be unstable. A second result discussed is that for the special case of a surface of revolution, there are no stable vortex solutions provided the Gauss curvature of the north and south poles are non-zero.

The critical nonlinear wave equation in 2 space dimensions

Michael Struwe

1. Results

Consider the equation

$$u_{tt} - \Delta u + ue^{u^2} = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^2 \tag{1}$$

with smooth Cauchy data

$$(u, u_t)_{|t=0} = (u_0, u_1) \in C^\infty(\mathbb{R}^2)\,. \tag{2}$$

Upon multiplying (1) by u_t we obtain the conservation law

$$0 = \frac{d}{dt}e(u) - div(\nabla u \cdot u_t) \tag{3}$$

for the energy density

$$e(u) = \frac{1}{2}\big(|u_t|^2 + |\nabla u|^2 + e^{u^2} - 1\big) \tag{4}$$

and the density of momentum, given by $\nabla u \cdot u_t$. In particular, for compactly supported data the support of any solution u to (1), (2) grows at most with unit speed and we may integrate (3) to see that

$$E(u(t)) = \int_{\mathbb{R}^2} e(u(t))dx = E(u(0)) \tag{5}$$

for all $t \in \mathbb{R}$.

Equation (1) is closely related to the critical Sobolev embedding in two space dimensions, given by the Moser-Trudinger inequality

$$\sup_{u \in H_0^1(\Omega); ||\nabla u||^2_{L^2(\Omega)} \leq 1} \int_\Omega e^{\alpha u^2} dx = \sup_{u \in H_0^1(\Omega); ||\nabla u||^2_{L^2(\Omega)} \leq \alpha} \int_\Omega e^{u^2} dx \leq C(\alpha)|\Omega| \tag{6}$$

for any bounded domain $\Omega \subset \mathbb{R}^2$ having 2-dimensional Lebesgue measure $|\Omega|$ and any $\alpha \leq 4\pi$, with a constant $C(\alpha) < \infty$ independent of Ω; see [6], [12]. For $\alpha > 4\pi$ the above supremum is infinite.

In [2], [4] Ibrahim, Majdoub, Masmoudi, and Nakanishi demonstrated that the initial value problem for equation (1) is well-posed for initial data with $E(u(0)) \leq 2\pi$, where (5) together with (6) allow to control the nonlinear term. However, when $E(u(0)) > 2\pi$ not even a locally uniform spatial L^1-bound is available for the term ue^{u^2}. In analogy with nonlinear wave equations

$$u_{tt} - \Delta u + u|u|^{p-2} = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^n \tag{7}$$

with $p > \frac{2n}{n-2}$ in $n \geq 3$ space dimensions, where the nonlinear term cannot be bounded in the dual space of H^1 in terms of the Dirichlet energy, the Cauchy problem for equation (1) was therefore termed "supercritical" for initial data with energy $E(u(0)) > 2\pi$. The recent results [1], [3] of Ibrahim, Jrad, Majdoub, and Masmoudi, showing that the local solution of the Cauchy problem (1), (2) does not depend on the initial data in a locally uniformly continuous fashion when $E(u(0)) > 2\pi$, seemed to further justify this classification.

However, in contrast with these results, in [9], [10] we were able to show that the Cauchy problem (1), (2) is well-posed regardless of the size of the data.

Theorem 1. *For any $u_0, u_1 \in C^\infty(\mathbb{R}^2)$ there exists a unique, smooth solution u to the Cauchy problem* (1), (2)*, defined for all time.*

2. Some key ingredients in the proof of Theorem 1

We argue indirectly; that is, we suppose that the local solution u to (1), (2) for certain Cauchy data $u_0, u_1 \in C^\infty(\mathbb{R}^2)$ cannot be smoothly extended to a neighborhood of some point (T_0, x_0) where $T_0 \geq 0$. By finiteness of propagation speed for (1), we may assume that u_0, u_1 are compactly supported, $T_0 > 0$, and that $u \in C^\infty([0, T_0[\times \mathbb{R}^2)$. After translating the origin of our coordinate system to the point x_0, reversing the arrow of time, and shifting time by T_0, in the following we may assume that we have a compactly supported solution $u \in C^\infty(]0, T_0] \times \mathbb{R}^2)$ of (1) blowing up at $(0, 0)$.

The work of Ibrahim, Majdoub, and Masmoudi [2] gives rise to the following characterization of blow-up through concentration of energy. (The short proof of Lemma 1 given in [9] also works in the non-symmetric case.)

Lemma 1. *There exists* $\varepsilon_0 > 0$ *such that*

$$E(u(T), B_T(0)) := \int_{B_T(0)} e(u(T))dx \geq \varepsilon_0 \quad \text{for all} \quad 0 < T \leq T_0 \,. \tag{8}$$

In order to complete the proof it suffices to derive a contradiction to (8). We first show only partial energy decay. Introduce polar coordinates (r, ϕ), and now let

$$e = e(u) = \frac{1}{2}(u_t^2 + u_r^2 + r^{-2}u_\phi^2 + e^{u^2}),\ m = m(u) = u_t u_r \,.$$

For $0 < T \leq T_0$ also denote as $v(y) = u(|y|, y)$ the restriction of u to the lateral boundary of the truncated forward light cone

$$K^T = \{z = (t, x); 0 < t \leq T, |x| \leq t\}$$

with vertex at $z = (0, 0)$, and let

$$\bar{v} = \bar{v}(t) = \frac{1}{2\pi} \int_0^{2\pi} v(te^{i\phi})\, d\phi$$

be its spherical average. Multiplying (1) by $\frac{x}{t} \cdot \nabla u \pm \frac{r}{t} u_t + \frac{1}{2t}(u - \bar{v})$, and using the structure of the resulting terms in a suitable fashion, we find the following result.

Lemma 2. *For any* $0 < \varepsilon < 1$ *there exists* $0 < T_\varepsilon \leq T_0$ *such that*

$$\int_{K^{T_\varepsilon}} \big((1 \pm \frac{r}{t})(e \pm m) + \frac{|u - \bar{v}|^2}{2t^2} + \frac{1}{4}|u - \bar{v}|^2 e^{u^2}\big)\frac{dx\, dt}{t} \leq \varepsilon \,.$$

Note that in characteristic coordinates

$$\xi = t + r\,, \quad \eta = t - r$$

by Lemma 2 for any $0 < \xi_0 < T_\varepsilon$ we have control of

$$\int_{\Gamma(\xi)} \big((1 \pm \frac{r}{t})(e \pm m) + \frac{|u - \bar{v}|^2}{t^2}\big)do$$

in average over the dyadic interval $\xi_0/2 < \xi < \xi_0$, where for any $0 < \xi_1 < T_\varepsilon$ we let

$$\Gamma(\xi_1) = \{(t, x) \in K^{T_\varepsilon};\ \xi = t + |x| = \xi_1\}$$

with area element $do = r d\eta d\phi$. Observe that

$$2u_\xi^2 + \frac{1}{2r^2}u_\phi^2 = \frac{1}{2}\big((u_t + u_r)^2 + r^{-2}u_\phi^2\big) \leq e + m\ ,\ 2u_\eta^2 = \frac{1}{2}(u_t - u_r)^2 \leq e - m\ .$$

Another application of the multiplier estimate leading to Lemma 2 then gives the bound

$$\sup_{0<\xi<\xi_\varepsilon} \int_{\Gamma(\xi)} \big((1 - \frac{r}{t})u_\eta^2 + r^{-2}u_\phi^2 + \frac{|u - \bar{v}|^2}{t^2}\big)do < \varepsilon \tag{9}$$

for any $0 < \varepsilon < 1$ and suitably small $0 < \xi_\varepsilon < T_\varepsilon$.

Estimate (9) together with the Moser-Trudinger inequality (6) allows to bound the nonlinear term in equation (1) in any L^p-norm away from the light cone. Near the light cone (9) only yields smallness of the angular derivative, while the energy inequality

$$2\int_{\Gamma(\xi)} u_\eta^2 do \le \int_{\Gamma(\xi)} (e-m) do \le E(u(\xi/2), B_{\xi/2}(0)) \le E_0 \tag{10}$$

provides a (possibly huge) uniform L^2-bound for u_η for all $0<\xi<T_0$. A key role now is played by the following improvement of the Moser-Trudinger inequality (6).

Lemma 3. *For any $E>0$, any $p<\infty$ there exists a number $\varepsilon = \frac{4\pi^2}{p^2E} > 0$ and a constant $C>0$ such that for any $\xi_0>0$, any $v \in H_0^1([0,1]^2)$ with*

$$\int_0^1\int_0^1 \left(\xi_0|v_y|^2 + \xi_0^{-1}|v_x|^2\right) dx\, dy \le E,\ \int_0^1\int_0^1 \xi_0^{-1}|v_x|^2\, dx\, dy \le \varepsilon$$

there holds

$$\int_0^1\int_0^1 e^{pv^2} dx\, dy \le C\ .$$

Hence for suitably small $\varepsilon>0$ from (9), (10) for any $0<\xi_0<\xi_\varepsilon$ with a constant C independent of ξ_0 we find

$$\int_{\Gamma(\xi)} e^{4u^2}\, do \le C\ . \tag{11}$$

To finish the proof, for $0<T\le\xi_\varepsilon$ let $u^{(0)}$ be the solution to the homogeneous wave equation $u_{tt}^{(0)} - \Delta u^{(0)} = 0$ in K^T with initial data $u^{(0)}(T) = u(T)$, $u_t^{(0)}(T) = u_t(T)$ at $t=T$. With the notation $D = (\partial_t, \nabla)$, from (11) we obtain

$$\sup_{0<t\le T}\int_{B_t(0)} |D(u-u^{(0)})(t)|^2\, dx \le 4T\int_{K^T(0)} e^{4u^2}\, dx\, dt \le CT^2 < \varepsilon_0\ ,$$

provided $0<T\le\xi_\varepsilon/2$ is sufficiently small, where $\varepsilon_0>0$ is the constant defined in Lemma 1. Since

$$\lim_{t\downarrow 0}\int_{B_t(0)} |Du^{(0)}(t)|^2\, dx = 0\ ,$$

and since by (11) we also have

$$\liminf_{t\downarrow 0}\int_{B_t(0)} e^{u(t)^2}\, dx = 0\ ,$$

we then find that

$$\liminf_{t\downarrow 0} E(u(t), B_t(0)) \le \frac{1}{2}\limsup_{t\downarrow 0}\int_{B_t(0)} |D(u-u^{(0)})(t)|^2\, dx \le \varepsilon_0/2 < \varepsilon_0\ ,$$

which contradicts Lemma 1 and completes the argument.

3. Note

Note that no weighted energy estimates are required in the proof, as would be expected in a truely "super-critical" context. It thus appears that problem (1), (2) still belongs to the realm of "critical" equations. More generally, it seems that this may be true for all problems where smallness of the energy implies regularity, as in the present case. See [3], [5], [8], [11] for further results on supercritical wave equations, and [7] for background material on nonlinear wave equations in general.

References

[1] S. Ibrahim, R. Jrad: *Strichartz type estimates and the wellposedness of an energy critical 2d wave equation in a bounded domain,* J. Differential Equations 250 (2011), 3740–3771.

[2] S. Ibrahim, M. Majdoub, N. Masmoudi: *Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity*, Comm. Pure Appl. Math. 59 (2006), no. 11, 1639–1658.

[3] S. Ibrahim, M. Majdoub, N. Masmoudi: *Well- and ill-posedness issues for energy super-critical waves*, preprint (2009).

[4] S. Ibrahim, M. Majdoub, N. Masmoudi, K. Nakanishi: *Scattering for the two-dimensional energy-critical wave equation*, Duke Math. J. 150.2 (2009), 287–329.

[5] G. Lebeau: *Non linear optic and supercritical wave equation*, Bull. Soc. Roy. Sci. Liège 70 (2001), no. 4-6, 267–306.

[6] J. Moser: *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (1971), 1077–1092.

[7] J. Shatah, M. Struwe: *Geometric wave equations*, Courant Lecture Notes 2, New York University (1998), 2nd edition: AMS (2000).

[8] M. Struwe: *On uniqueness and stability for supercritical nonlinear wave and Schrödinger equations.* Int. Math. Res. Not. 2006, Art. ID 76737, 14 pp.

[9] M. Struwe: *Global well-posedness of the Cauchy problem for a super-critical nonlinear wave equation in 2 space dimensions*, Math. Annalen 350 (2011), 707-719.

[10] M. Struwe: *The critical nonlinear wave equation in 2 space dimensions*, preprint (2010).

[11] T. Tao: *Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data*, J. Hyperbolic Differ. Equ. 4 (2007), 259–265.

[12] N.S. Trudinger: *On embeddings into Orlicz spaces and some applications*, J. Math. Mech. 17 (1967), 473–484.

A new proof of Brakke's partial regularity for MCF

Yoshihiro Tonegawa

(joint work with Kota Kasai)

A family $\{M_t\}_{t\geq 0}$ of k-dimensional surfaces in $\mathbb{R}^n$ is called the mean curvature flow (MCF) if the velocity v of M_t is equal to its mean curvature vector H at each point and time. As one of the most fundamental geometric evolution problems, the MCF has been the subject of intensive research since the 1980's. The earliest study of MCF goes back to the seminal work of Brakke [2] who used the notion of varifold [1] to show the existence of weak solutions for general initial surfaces. More precisely, given any k-dimensional integral varifold V_0 with some mild finiteness assumptions, he showed the existence of a family of varifolds $\{V_t\}_{t\geq 0}$ each of which satisfies the MCF equation in a distributional sense for all $t \geq 0$. For a.e. time, Brakke additionally proved that V_t is integral. Assuming that the density function is equal to 1 almost everywhere, Brakke also demonstrated that the weak MCF varifold solutions are supported by smooth k-dimensional surfaces almost everywhere. The hypothesis is called the "unit density hypothesis" and it is a natural assumption since even the time-independent case of Allard's theory has essentially the same hypothesis. The regularity proof is remarkably ingeneous with novel propositions such as the "clearing-out lemma" and the 'popping soap films lemma', and it has a significant influence on the analysis of the related singular perturbation problems such as the Allen-Cahn equation and the parabolic Ginzburg-Landau equation. However, the details of Brakke's regularity theorem are technically involved and a clear accessible proof is desired due to its importance.

The result presented in the seminar has two aspects. The first is to present a new and accessible proof of Brakke's local regularity theorem. Brakke's proof relies on a long chain of graphical approximations which is complicated and hard to follow (see [2, Section 6.9, "Flattening out"]). By fully utilizing the parabolic monotonicity formula [4, 3] we replace this part by Allard-like Lipschitz approximations. The second aspect is to generalize the result so that the velocity may have an additional transport term which belongs to a certain integrability class. The existence of such flows has been recently studied by Liu-Sato-Tonegawa [5] and it motivated us to investigate the generalization of Brakke's theorem. The result reduces essentially to Allard's well-known regularity theorem [1] in the special case of time-independence. We note that simple modifications of Brakke's original proof do not seem to yield our theorem if one puts the general transport term.

The major difference of our proof compared to [2] is the use of the "height lemma". It shows that the smallness of measurement of "height" of MCF in the L^2-sense guarantees that the whole support of MCF lies in a narrow region close to a k-dimensional plane. In a sense, we obtain an interior $L^\infty-$estimate of the height of the graph in terms of the L^2-norm. The height lemma may be considered as a robust version of Brakke's clearing-out lemma in the sense that the former may accommodate general transport terms while the latter apparently has some limitations doing so. With this new input the proof may be outlined as follows. We make a full use of the popping soap films lemma with some modifications from [2, Section 6.6], which gives controls of the uniform-in-time tilt-excess bound and of space-time L^2-norm of the mean curvature in terms of the smallness of L^2-height. Then the rest of the proof proceeds more or less like Allard's regularity proof with parabolic modifications. Namely we approximate the supports of moving varifolds by Lipschitz graphs (with respect to the parabolic metric) utilizing parabolic monotonicity formulae. Then through a contradiction argument, we carry out the well-known blow-up procedures. While the blow-up limit is a harmonic function in Allard's theorem, it is a solution of the heat equation in our case. It turned out to be essential to have the $L^\infty-$estimate, so that we need to use the signed test function in Brakke's formulation. It is also interesting to note that we need to utilize the certain monotone time-decreasing property of L^2-norm of blow-up sequences to ensure the strong space-time L^2-convergence. The blow-up analysis gives a decay estimate of L^2-height with respect to a slightly tilted k-dimensional plane in a smaller scale. An iteration gives a Hölder-estimate of the spatial gradient of the graph just as in Allard's case. Though technically involved, we expect that researchers familiar with Allard's proof would find the outline of our proof more tractable and natural than Brakke's original one.

References

[1] W. Allard: *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417-491.

[2] K. Brakke: *The Motion of a Surface by its Mean Curvature*, Math. Notes **20**, Princeton Univ. Press, Princeton, NJ, 1978.

[3] Y. Chen, M. Struwe: *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. **201** (1989), 83-103.

[4] G. Huisken: *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), 285-299.

[5] C. Liu, N. Sato, Y. Tonegawa: *On the existence of mean curvature flow with transport term.* Interfaces Free Bound. **12** (2010), no. 2, 251-277.

On generated prescribed Jacobian equations

Neil S. Trudinger

(joint work with J-K Liu and X-J Wang)

This report describes one aspect of ongoing investigations in regularity in optimal transportation and extensions to geometric optics by J-K Liu, X-J Wang and myself.

Let Ω be a domain in Euclidean n-space, $\mathbb{R}^n$, and Y a mapping from $\Omega\times\mathbb{R}\times\mathbb{R}^n$ into $\mathbb{R}^n$. The *prescribed Jacobian equation* is a partial differential equation of the form

$$\det DY(\,\cdot\,,u,Du) = \psi(\,\cdot\,,u,Du), \tag{1}$$

where ψ is a given scalar function on $\Omega\times\mathbb{R}\times\mathbb{R}^n$ and Du denotes the gradient of the function $u:\Omega\to\mathbb{R}$. We will always assume that the matrix Y_p is invertible, that is $\det Y_p \neq 0$, whence we may write (1) as a general equation of Monge-Ampère type,

$$\det[D^2u - A(\,\cdot\,,u,Du)] = B(\,\cdot\,,u,Du), \tag{2}$$

where

$$A = -Y_p^{-1}(Y_x + Y_z\otimes p), \qquad B = (\det Y_p)^{-1}\psi. \tag{3}$$

A function $u\in C^2(\Omega)$ is degenerate elliptic, (elliptic), for equation (2), henceforth called *admissible*, whenever

$$D^2u - A(\,\cdot\,,u,Du)\geq 0, \quad (>0), \tag{4}$$

in Ω. If u is admissible, then the function $B(\cdot, u, Du) \geq 0$. The *second boundary value problem* for the prescribed Jacobian equation is to prescribe the image

$$Tu(\Omega) := Y(\cdot, u, Du)(\Omega) = \Omega^*, \tag{5}$$

where Ω^* is another given domain in $\mathbb{R}^n$. When ψ is separable, in the sense that

$$|\psi(x, z, p)| = f(x)/g \circ Y(x, z, p), \tag{6}$$

for positive $f \in L^1(\Omega)$, $g \in L^1(\Omega^*)$ respectively, then a necessary condition for the existence of an admissible solution, for which the mapping Tu is a diffeomorphism, to the second boundary value problem (1), (5) is the *mass balance* condition

$$\int_\Omega f = \int_{\Omega^*} g. \tag{7}$$

The class of prescribed Jacobian equations embraced by this report are defined in terms of a smooth *generating function* $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Let $\mathcal{U}$ be an open set in $\mathbb{R}^n \times \mathbb{R}^n$ and I an open interval in $\mathbb{R}$. For points $(x, y) \in \mathcal{U}$, we denote their corresponding projections by $\mathcal{U}^*_x := \{y \in \mathbb{R}^n \,|\, (x, y) \in \mathcal{U}\}$, $\mathcal{U}_y := \{x \in \mathbb{R}^n \,|\, (x, y) \in \mathcal{U}\}$ and write $\mathcal{U}^{(1)} := \bigcup\{\mathcal{U}_y \,|\, y \in \mathbb{R}^n\}$ and $\mathcal{U}^{(2)} := \bigcup\{\mathcal{U}^*_x \,|\, x \in \mathbb{R}^n\}$. Denoting points in I by z, we assume that G is smooth in $\mathcal{U} \times I$, $G_z \neq 0$ and

- **G1**: For each $(x, y) \in \mathcal{U}$, there exists an open interval $I(x, y) \subset I$ such that the mapping $(G_x, G)(x, \cdot, \cdot)$ is one-to-one in $y \in \mathcal{U}^*_x, z \in I(x, y)$, for each $x \in \mathcal{U}^{(1)}$.
- **G2**: For each $(x, y) \in \mathcal{U}, z \in I(x, y)$, $\det E(x, y, z) \neq 0$, where E is the $n \times n$ matrix given by

$$E = [E_{x,y}] = [G_{x,y} - (G_z)^{-1} G_{x,z} \otimes G_y]. \tag{8}$$

From G1 and G2, the vector field Y, together with a scalar function Z, are generated by G through the equations

$$G_x(x, Y, Z) = p, \quad G(x, Y, Z) = u. \tag{9}$$

Note that the Jacobian determinant of the mapping $(y, z) \mapsto (G_x, G)(x, y, z)$ is $G_z \det E \neq 0$ by G2, so that Y and Z are accordingly smooth. Also by differentiating (9) with respect to p, we obtain $Y_p = E^{-1}$. Next using (3) or differentiating (9) for $p = Du$, with respect to x, we obtain that the resultant prescribed Jacobian equation (1) is a Monge-Ampère equation of the form (2) with

$$\begin{aligned} A(x, u, p) &= G_{xx}[x, Y(x, u, p), Z(x, u, p)], \\ B(x, u, p) &= \det E(x, Y, Z)\, \psi(x, u, p) \end{aligned} \tag{10}$$

and is well defined in domains $\Omega \subset \mathcal{U}^{(1)}$ for $Y \in \mathcal{U}^*_x, Z \in I(x,Y), x \in \Omega$. Note that the latter restrictions may automatically place constraints on u and Du. For the special case of optimal transportation with cost function $c : \mathcal{U} \to \mathbb{R}$ we take

$$(11) \qquad G(x,y,z) = c(x,y) - z, \qquad G_z = -1, \qquad I = I(x,y) = \mathbb{R},$$

and conditions G1 and G2 correspond to A1 and A2 in [5], [7]. To develop the underlying convexity theory and consequent regularity, we need a dual condition, which in the optimal transportation case is obtained by interchanging x and y. Using the property that the generating function G is strictly monotone with respect to z, we introduce a dual function H on $\mathcal{U} \times \mathbb{R}$ by

$$(12) \qquad G[x, y, H(x, y, u)] = u.$$

Clearly H is well defined whenever $(x,y) \in \mathcal{U}$ and $u \in G(x,y,\cdot)(I)$. Furthermore we have the relations

$$(13) \qquad H_x = -G_x/G_z, H_y = -G_y/G_z, H_u = 1/G_z.$$

This motivates the following dual condition:

- **G1***: The mapping $Q := -G_y/G_z$ is one-to-one in x, for all $y \in \mathcal{U}^{(2)}, z \in I(x,y)$.

Note that the Jacobian matrix of the mapping $x \mapsto Q(x,y,z)$ is $-E/G_z$, so its determinant is automatically non-zero when condition G2 holds. Our main conditions extend the conditions A3 and A3w introduced for regularity in [5], [7] and are expressed in terms of the matrix function A in (2):

$$\textbf{G3}\,(\textbf{G3w}) \qquad A^{kl}_{ij}\xi_i\xi_j\eta_k\eta_l := (D_{p_k p_l} A_{ij})\xi_i\xi_j\eta_k\eta_l >, (\geq)\ 0,$$

for all $(x,Y) \in \mathcal{U}, Z \in I(x,Y), \xi, \eta \in \mathbb{R}^n$ such that $\xi.\eta = 0$. We illustrate the above conditions with two examples from near field geometric optics. The first models the reflection of a parallel beam to a flat target, [4]. For $\mathcal{U} = \mathbb{R}^n \times \mathbb{R}^n$ and $I = (0, \infty)$, we define

$$(14) \qquad G(x,y,z) := \frac{1}{2z} - \frac{z}{2}\,|x-y|^2.$$

Then G satisfies G1, G2, G1*, G3 with

$$I(x,y) = (0, \frac{1}{|x-y|}), \quad J(x,y) := G(x,y,\cdot)(I(x,y)) = (0,\infty),$$

and the corresponding Monge-Ampère equation

$$(15) \qquad \det\big\{D^2u + \frac{(1-|Du|^2)}{2u}I\big\} = \frac{(1-|Du|^2)^{n+1}}{(1+|Du|^2)(2u)^n}\,\psi,$$

is well defined for $u > 0$ and $|Du| < 1$. The second example comes from the reflection of a point source beam to a flat target. Using polar coordinates and the reciprocal function as in [4], we may take $\mathcal{U} = B_1(0) \times \mathbb{R}^n$, $I = (0, \infty)$ and

$$(16) \qquad G(x,y,z) := \frac{(|y|^2 + z)^{1/2} - x.y}{z}.$$

Then G satisfies G1, G2, G1*, G3w ($A = 0$), with $I(x, y) = J(x, y) = (0, \infty)$, and the corresponding Monge-Ampère equation

$$\text{(17)} \qquad \det D^2 u = \frac{[(u^*)^2 - |Du|^2]^{n+1}}{(u^*)^2 + |Du|^2 - 2\,u^*(x.Du)}\,\psi$$

is well defined for $|Du| < -u^*$, where $u^* := x.Du - u < 0$. The corresponding mappings are given respectively by

$$\text{(18)} \qquad Tu = I + \frac{2\,u\,Du}{1 - |Du|^2} \qquad \text{and} \qquad Tu = \frac{Du}{(u^*)^2 - |Du|^2} \quad .$$

In the rest of this report, we briefly indicate some results in the ensuing theory. First we point out that a convexity theory, with respect to generating functions, replicating usual convexity, and more generally the optimal transportation case (see for example [9]) can be built under conditions G1, G2, G1*, G3w. Here, functions of the form $u_0 = G(\,\cdot\,, y, z)$ for fixed y, z are called *G-affine* and $u \in C^0(\Omega)$ is called *G-convex* in Ω if there exists a G-affine support from below at each point of its graph. An admissible function $u \in C^2(\Omega)$ is G-convex if additionally the matrix function A is non-decreasing in u and the images $Q(\,\cdot\,, y, z)(\Omega)$ are convex for all $(y, z) \in Y \times Z(\,\cdot\,, u, Du)(\Omega)$. As in the usual convexity theory, the mapping Tu (for G-convex functions u) is extended to a multi-valued *G-normal* mapping which is determined by local subdifferentials. Appropriately defined sections of G-convex functions will also be G-convex. A *generalized solution* of the second boundary value problem (2)-(7) under condition G1, with initial and target domains satisfying $\Omega \times \Omega^* \subset \mathcal{U}$, may be defined by

$$\text{(19)} \qquad \int_{(Tu)^{-1}(\omega)} f = \int_\omega g,$$

for all Borel sets $\omega \subset \Omega^*$. This extends the notion of potential in optimal transportation. The existence of generalized solutions follows by piecewise G-affine approximation under appropriate conditions to control their gradients. Extending the optimal transportation case we could assume for example that there exist constants $m_0 \geq -\infty, K_0 \geq 0$, such that $(m_0, \infty) \subset J(x, y)$ and $|G_x(x, y, z)| \leq K_0$, for all $x \in \Omega, y \in \Omega^*, G(x, y, z) \geq m_0$. In example (14), we may take $m_0 = 0, K_0 = 1$. More generally we can also replace G by $\mu(G)$ for any $\mu \in C^1(m_0, \infty), \mu' \neq 0, \mu' \in C^0[m_0, \infty)$, which enables us to cover example (16), as in [2]. To develop the necesssary properties of generalized solutions, paralleling the optimal transportation case in [5], we introduce the dual concept of the G-transform v of a G-convex function u, given by

$$\text{(20)} \qquad v(y) = \sup_\Omega H(\,\cdot\,, y, u).$$

When $u \in C^2(\Omega)$ and the mapping Tu is one-to-one, it follows that

$$\text{(21)} \qquad v = Z(\,\cdot\,, u, Du) \circ (Tu)^{-1},$$

which also shows the significance of the scalar function Z. It also follows that conditions G3 and G3w are invariant under duality.

Following [5], we can then prove local regularity of generalized solutions u under conditions G1, G1*, G2 and G3. The target domain Ω^* is assumed to be Y^*-*convex* with respect to u as in [6], that is the sets $\{p \mid Y(x, u(x), p) \in \Omega^*\}$ are convex in $\mathbb{R}^n$, for each $x \in \Omega$. For $f \in C^2(\Omega), g \in C^2(\Omega^*), \inf\{f, g\} > 0, \sup\{f, g\} < \infty$, we obtain $u \in C^3(\Omega)$, as in [5], while if we drop the smoothness of f, g, then $u \in C^1(\Omega)$, as in [3], [8]. Note that example (14) satisfies these hypotheses. In the case of example (16), the same conclusions already follow from [1],[4]. Finally we remark that, as in the optimal transportation case [3], [5], the Y^*-convexity of Ω^* and condition G3w are necessary for regularity.

References

[1] L. Caffarelli: *The regularity of mappings with a convex potential.* J. Amer. Math. Soc. 5, (1992), 99–104.

[2] A. Karakhanyan, X.-J. Wang: *On the reflector shape design.* J. Diff. Geom. 84, (2010), 561–610.

[3] G. Loeper: *On the regularity of solutions of optimal transportation problems.* Acta Math. 202 (2009), 241–283.

[4] J.-K. Liu, N. S. Trudinger: *On classical solutions of near field reflection problems.* (in preparation).

[5] X.-N. Ma, N.S. Trudinger, X.-J. Wang: *Regularity of potential functions of the optimal transportation problem.* Arch. Rat. Mech. Anal. 177, (2005), 151–183.

[6] N.S. Trudinger: *On the prescribed Jacobian equation.* Gakuto Intl. Series, Math. Sci. Appl. Vol. XX, (2008), Proc. Intl. Conf. for the 25th Anniversary of Viscosity Solutions, 243–255.

[7] N.S. Trudinger, X.-J. Wang: *On the second boundary value problem for Monge-Ampère type equations and optimal transportation.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., Vol. VIII, (2009), 143–174.

[8] N.S. Trudinger, X.-J. Wang: *On strict convexity and continuous differentiability of potential functions in optimal transportation.* Arch. Rat. Mech. Anal. 192, (2009), 403–418.

[9] C.Villani: *Optimal transportation, old and new.* Springer, 2008.

Regularity results for optimal transportation problems when standard conditions fail

Micah Warren

At this workshop in 2007, Loeper presented his regularity proof [8] for optimal transport when the A3 condition (see Ma-Trudinger-Wang, [9]) is satisfied, and McCann presented joint work with Kim [6] supplementing the picture of Loeper and Ma-Trudinger-Wang. There was much rejoicing, as it seemed that the questions of regularity were for the most part understood. However, work on regularity under the weak MTW assumption has continued even until this summer, culminating with the preprint of Figalli, Kim, and McCann [3]. What we know now is that, given the MTW condition and convexity conditions on the domains, one can conclude that any two smooth densities are paired smoothly, and conversely, if these conditions fail, then one can find smooth densities which do not have a smooth optimal pairing. On manifolds, these conditions have been studied as the Transport Continuity Property [4] and significant work has gone to showing when this condition holds.

On the other hand, there is another line of questioning: What if any of the standard conditions do not hold ? Can we give conditions on the densities which are sufficient to conclude that the map is regular ?

These questions do not appear to have elegant geometric answers.

One situation is that the MTW condition fails, but fails on a set which one can expect the map to avoid under certain conditions. Delanoe and Ge [2] studied an example of this situation on perturbations of spherical metrics. A further complication occurs when the twist (A1) condition fails. For example, on the sphere with (external) Euclidean distance squared cost, (A1) fails, and it is shown by Gangbo and McCann [5] that optimal maps often split. With this cost, the MTW condition holds half-the-time: whenever two points are within $\pi/2$ away from each other. With Kitigawa [7] we show that given enough Lipschitz control over the logs of the densities, the map is one-to-one and smooth. The Lipschitz control gives a strong enough gradient estimate, so that the map stays within the MTW region. The Lipschitz condition is "life-sized" and is the same order of magnitude of counterexamples. We can refine this result to rougher data: If the measures are close to each other, we can use a "stay-close-by" argument to control the gradient. The constraints in this case are small but tractable numbers. These results also hold for uniformly convex boundaries, where the constants are computable in terms of the largest and smallest curvatures.

Another case is when the MTW condition fails on an unavoidably large set. In [11] we show that if the densities are C^3-close to restricted Gaussians, and the cost is C^4-close to Euclidean cost, then the optimal pairing is regular, regardless of whether or not the MTW condition holds. The main estimate in the proof follows the paper of Trudinger and Wang [10] using an idea of Caffarelli [1], the latter of which shows very strong regularity in the Euclidean case when the densities are Gaussians. The required closeness can be made explicit with some effort (which we do not put forth).

References

[1] L.A. Caffarelli. Monotonicity properties of optimal transportation and the FKG and related inequalities. *Comm. Math. Phys.*, 214(3), 547-563, 2000.

[2] P. Delanoe, Y. Ge. Regularity of optimal transport on compact, locally nearly spherical manifolds. *J. Reine Angew. Math.*, 646, 65-115, 2010.

[3] A. Figalli, Y.-H. Kim, R. McCann. Hölder continuity and injectivity of optimal maps. Preprint.

[4] A. Figalli, L. Rifford. Continuity of optimal transport maps and convexity of injectivity domains on small deformations of $\mathbb{S}^2$. *Comm. Pure Appl. Math.*, 62(12), 1670-1706, 2009.

[5] W. Gangbo, R.J. McCann. Shape recognition via Wasserstein distance. *Quart. Appl. Math.*, 58(4), 705-737, 2000.

[6] Y.-H. Kim, R.J. McCann. Continuity, curvature, and the general covariance of optimal transportation. *J. Eur. Math. Soc. (JEMS)*, 12(4), 1009-1040, 2010.

[7] J. Kitagawa, M. Warren. Regularity of optimal transport with Euclidean distance squared cost on the embedded sphere. Preprint.

[8] Gregoire Loeper. On the regularity of solutions of optimal transportation problems. *Acta Math.*, 202(2), 241-283, 2009.

[9] Xi-Nan Ma, N.S. Trudinger, Xu-Jia Wang. Regularity of potential functions of the optimal transportation problem. *Arch. Rat. Mech. Anal.*, 177(2), 151-183, 2005.
[10] N.S. Trudinger, Xu-Jia Wang. On the second boundary value problem for Monge-Ampère type equations and optimal transportation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 8(1), 143-174, 2009.
[11] M. Warren. Regularity for a log-concave to log-concave mass transfer problem with near Euclidean cost. *Comm. Anal. Geom.*, to appear.

Densities of Minimal Cones

Brian White
(joint work with Tom Ilmanen)

If C is an m-dimensional cone with vertex p, the **density** of the cone is

$$(*) \qquad \Theta(C) := \frac{\text{area}(C \cap \mathbf{B}(p,r))}{\omega_m r^m}$$

where ω_m is the area of the unit m-dimensional ball. (Note that (*) is independent of r since C is a cone.) In this lecture, I describe joint work with Tom Ilmanen on densities of minimal cones. I will call a cone "simple" provided it has exactly one singularity (the vertex of the cone).

Theorem 1. *Suppose C is a simple, area-minimizing hypercone in $\mathbb{R}^{m+1}$. Suppose also that C is topologically nontrivial, i.e., that at least one of the two components of $\mathbb{R}^{m+1} \setminus C$ is not contractible. Then $\Theta(C) > \sqrt{2}$.*

The constant $\sqrt{2}$ is sharp because the Simons Cone

$$C_{m,m} := \{(x,y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} : |x| = |y|\}$$

is area-minimizing for $m \geq 3$, and $\Theta(C_{m,m}) \to \sqrt{2}$ as $m \to \infty$.

If C is topologically nontrivial, then one of the components of the complement must have nontrivial k^{th} homotopy for some k. We get a better bound in terms of k:

Theorem 2. *Suppose C is a simple, area-minimizing hypercone in $\mathbb{R}^{m+1}$. Suppose also that one of the components of $\mathbb{R}^{m+1} \setminus C$ has nontrivial k^{th} homotopy. Then*

$$\Theta(C) > d_k = \left(\frac{k}{2\pi e}\right)^{k/2} \sigma_k$$

where σ_k is the area of the unit k-dimensional sphere.

Theorem 2 is also sharp: the cone

$$C_{k,n} := \{(x,y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n+1} : n|x|^2 = k|y|^2\}$$

satisfies the hypotheses for all sufficiently large n (namely $n \geq 6 - k$ if $k \geq 2$ and $n \geq 6$ if $k = 1$), and $\Theta(C_{k,n}) \to d_k$ as $n \to \infty$.

The proofs use mean curvature flow, and the constants d_k in Theorem 2 have a simple interpretation in terms of mean curvature flow: d_k is the Gaussian density of a shrinking round k-sphere in $\mathbb{R}^{k+1}$ or, equivalently, the Gaussian density of a shrinking cylinder $\mathbb{S}^k \times \mathbb{R}^{m-k}$ in $\mathbb{R}^{m+1}$.

Theorems 1 and 2 say nothing about simple minimal hypercones of dimension < 7, since no such cones are area-minimizing. However, for low dimensional cones we have

Theorem 3. *Suppose C is a simple, m-dimensional minimal cone in $\mathbb{R}^{m+1}$ with $m < 7$. Suppose $C \cap \partial\mathbf{B}$ has nontrivial k^{th} homology. Then*

$$\Theta(C) > \min\{d_k, d_{m-1-k}\}.$$

Now let C be a simple minimal hypercone C in $\mathbb{R}^4$. Then the link $C \cap \partial\mathbf{B}$ is a smooth, compact embedded surface in the 3-sphere. By a theorem of Almgren (also proved by Calabi), the only minimal 2-sphere in $\mathbb{S}^3$ is the totally geodesic one. Hence (since C is singular), $C \cap \partial\mathbf{B}$ must not be a sphere, and so must have genus ≥ 1. Thus $H_1(C \cap \partial\mathbf{B}) \neq 0$, so by Theorem 3:

$$\Theta(C) > \max\{d_1, d_{3-1-1}\} = d_1.$$

Thus we have

Corollary 1. *If C is a simple, 3-dimensional minimal cone in $\mathbb{R}^4$, then*

$$\Theta(C) > d_1 = 1.52.$$

Using the corollary together with standard dimension reducing, it is easy to prove the following theorem about arbitrary (i.e., not necessarily simple) minimal cones:

Theorem 4. *Let C be a singular, m-dimensional minimal cone in $\mathbb{R}^{m+1}$, with $m \leq 3$. Then*

$$\Theta(C) \geq \frac{3}{2},$$

with equality if and only if C is a triple junction, i.e, the union of three half-planes meeting at equal angles along their common edge.

An optimal embeddedness criterion for stable codimension 1 integral varifolds

Neshan Wickramasekera

The classical regularity theory for locally area minimizing hypersurfaces, developed in the 1960's with contributions by De Giorgi, Reifenberg, Federer, Fleming, Almgren and Simons [DG, R, F, FF, A, S] says that such a hypersurface in the interior is smooth and embedded away from a closed singular set of codimension at least 7 (which is absent if the dimension of the hypersurface is less than or equal to 6 and discrete if its dimension is 7). In the early 1980's, Schoen and Simon [SS] showed that the same conclusions hold if instead of the area minimizing hypothesis, the hypersurface is assumed to be stationary and stable, viz. to have vanishing first variation and non-negative second variation with respect to area, provided also that its singular set a priori is assumed to have locally finite codimension 2 Hausdorff measure.

A long standing question left open by the work of Schoen and Simon was to find an optimal condition on the singular set of a stable hypersurface guaranteeing the above embeddedness conclusions. In the author's recent work [W2], such a geometrically optimal condition is given. This new theory says that the above embeddedness conclusions can be made for a stationary hypersurface V (i.e. a stationary codimension 1 integral varifold) provided (a) that V is stable on the regular part $\operatorname{reg} V$ ($\equiv$ smooth, embedded part of $\operatorname{spt} V$, the support of V) in the sense that the stability inequality

$$\int_{\operatorname{reg} V} |A|^2 \zeta^2 \, d\mathcal{H}^n \leq \int_{\operatorname{reg} V} |\nabla \zeta|^2 d\mathcal{H}^n$$

holds for every $\zeta \in C^1_c(\operatorname{reg} V)$, where $|A|$ is the length of the (classical) second fundamental form of $\operatorname{reg} V$ and ∇ is the gradient on $\operatorname{reg} V$, and (b) that V satisfies the following Structural Hypothesis: none of the singular points of V ($\equiv$ those points of $\operatorname{spt} V$ not in $\operatorname{reg} V$) has a neighbourhood in which $\operatorname{spt} V$ is the (finite) union of 3 or more embedded hypersurfaces-with-boundary meeting only along a common boundary. More precisely, we have the following:

Theorem 1 (W2). *A stationary stable integral n-varifold V on a smooth $(n+1)$-dimensional Riemannian manifold B corresponds to an embedded smooth hypersurface with no singularities if $n \leq 6$, to an embedded smooth hypersurface away from a set of discrete singular points if $n = 7$ and to an embedded smooth hypersurface away from a closed singular set of Hausdorff dimension $\leq n - 7$ if $n \geq 8$, provided only that V satisfies the following condition for some $\alpha \in (0, 1/2)$:*
STRUCTURAL HYPOTHESIS: *No point of* $\operatorname{spt} V$ *has a neighbourhood in which* $\operatorname{spt} V$ *is the (finite) union of 3 or more $C^{1,\alpha}-$hypersurfaces-with-boundary meeting only along their common $C^{1,\alpha}-$boundary.*

Furthermore, any mass-bounded subset of the class of stable codimension 1 integral varifolds on B satisfying this Structural Hypothesis is compact in the topology of varifold convergence.

Note that as an easy consequence of Allard's regularity theorem, we know that $\operatorname{reg} V$ is non-empty (in fact is a dense open subset of $\operatorname{spt} V$) whenever the integral varifold V is stationary, so the stability hypothesis in this theorem always contains some information. However, since a priori no information is available concerning the size of the singular set of V (so that $\operatorname{reg} V$ could, a priori, have very small positive measure), it is not a priori clear how much influence the above stability inequality might have on the whole varifold. For this reason, it is somewhat surprising that optimal regularity conclusions follow from the (seemingly weak) hypotheses of this theorem.

The Structural Hypothesis in this theorem is of course necessary for the conclusions, and is geometrically optimal, in view of the obvious counterexamples such as a pair of crossed hyperplanes in an Euclidean space. It is in principle a more "checkable" condition than any size hypothesis on the singular set. Note also that the Structural Hypothesis is readily implied by either of the hypotheses that the varifold corresponds to a locally area minimizing integral current or that its singular set has vanishing codimension 1 Hausdorff measure. Thus the above theorem generalizes and combines into a single theory both the classical regularity theory for codimension 1 area minimizers and the Schoen–Simon theory.

New applications of this theorem include the two results described below:
(A) *An optimal strong maximum principle for stationary codimension 1 integral varifolds* [W2]. This generalizes and directly follows from earlier work of Ilmanen [I] and can be stated as follows:

Theorem 2 (W2)**.** *(a) If V_1, V_2 are stationary integral n-varifolds on a smooth Riemannian manifold B with $\mathcal{H}^{n-1}(\operatorname{spt} V_1 \cap \operatorname{spt} V_2) = 0$, then $\operatorname{spt} V_1$ and $\operatorname{spt} V_2$ must be disjoint.*
(b) If Ω_1, Ω_2 are open subsets of B with $\Omega_1 \subset \Omega_2$ and for $i = 1, 2$, the set $V_i \equiv \partial\,\Omega_i$ is connected, $\mathcal{H}^{n-1}(\operatorname{sing} V_i) = 0$ and V_i is stationary in B, then either $V_1 = V_2$ or $V_1 \cap V_2 = \emptyset$.

(B) *Regularity of sharp phase-interfaces arising from sequences of stable critical points of the Allen–Cahn functional with perturbation parameter tending to zero.* This result, joint with Y. Tonegawa [TW], can precisely be described as follows:

For $\epsilon \in (0, 1)$, consider the family of functionals (the Allen–Cahn functionals)

$$E_\epsilon(u) = \int_\Omega \frac{\epsilon |Du|^2}{2} + \frac{W(u)}{\epsilon}\, dx$$

for $u \in H^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $W : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is a "standard" C^3 double-well potential with strict minima at ± 1 with $W(\pm 1) = 0$ (e.g. $W(t) = (1 - t^2)^2$). Suppose

(i) $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ are positive numbers with $\lim_{j\to\infty} \epsilon_j = 0$.
(ii) $u_{\epsilon_j} \in H^1(\Omega)$ and there exists a fixed constant $c \geq 1$ such that $E_{\epsilon_j}(u_{\epsilon_j}) + \sup_\Omega |u_{\epsilon_j}| \leq c$.
(iii) u_{ϵ_j} is a stable critical point of E_{ϵ_j}. Thus $-\epsilon_j \Delta u_{\epsilon_j} + \epsilon_j^{-1} W'(u_{\epsilon_j}) = 0$ weakly on Ω and u_{ϵ_j} satisfies

$$\int_\Omega \epsilon_j |\nabla\phi|^2 + \frac{W''(u_{\epsilon_j})}{\epsilon_j}\phi^2\, dx \geq 0 \quad \text{for each } \phi \in C^1_c(\Omega).$$

(These conditions are equivalent, respectively, to $\frac{d}{dt}\big|_{t=0} E_{\epsilon_j}(u_{\epsilon_j} + t\phi) = 0$ and $\frac{d^2}{dt^2}\Big|_{t=0} E_{\epsilon_j}(u_{\epsilon_j} + t\phi) \geq 0 \;\forall\, \phi \in C^1_c(\Omega)$.)

Then we have the following:

Theorem 3 (TW)**.** *If (i), (ii), (iii) hold, then either*

(a) *$u_{\epsilon_j} \to 1$ or $u_{\epsilon_j} \to -1$ locally uniformly in Ω, or*
(b) *after passing to a subsequence of $\{\epsilon_j\}$ without changing notation, for each fixed $s \in (0, 1)$, the interface regions $\{x \in \Omega : |u_{\epsilon_j}(x)| < s\}$ converge locally in Hausdorff distance to an embedded stable minimal hypersurface*

M of Ω with $\operatorname{sing} M = \emptyset$ *if* $2 \leq n \leq 7$, $\operatorname{sing} M$ *(at most) a discrete set for* $n = 8$ *and* $\dim_{\mathcal{H}}(\operatorname{sing} M) \leq n - 8$ *for* $n \geq 9$.

In the absence of the above Structural Hypothesis, a stable codimension 1 integral varifold can develop branch point singularities. In this case, and whenever the varifold corresponds to an integral current T without boundary, a fairly detailed structure theorem, including an optimal size bound on the branch point set and optimal regularity of the varifold near the branch point set, is available for the (relatively open) part of the support of the varifold where its volume density $\Theta(T, \cdot)$ is less than 3. The precise statement of this result is as follows:

Theorem 4 (W3)**.** *Let T be a stable integral n-current on a smooth Riemannian manifold B with $\partial T = 0$ in B and let*

$$\Sigma := \{X \in \operatorname{spt} T \cap B \,:\, \Theta(T, X) < 3\}.$$

If $\Sigma \neq \emptyset$, then Σ is a smooth immersed hypersurface of B away from a closed set of Hausdorff dimension $\leq n - 2$.

In fact we have the following much more detailed description of Σ:

$$\Sigma = S_e \cup S_t \cup S_c \cup S_s \quad \textit{(disjoint union)} \quad \textit{where}$$

(i) $S_e = \operatorname{reg} T \cap \Sigma$ *and* $\mathcal{H}^{n-1}\left((\Sigma \setminus S_e) \cap K\right) < \infty$ *for each compact subset K of B;*

(ii) *S_t is the set of points of Σ near each of which* $\operatorname{spt} T$ *is the union of two transversely intersecting smooth embedded hypersurfaces; S_t is equal to the set of points of Σ where T has one tangent cone equal to a pair of transversely intersecting multiplicity 1 hyperplanes; if $S_t = \emptyset$, then $\Sigma = S_e \cup S_s$, i.e. $S_c = \emptyset$. (See (iii) and (iv) below for the definitions of S_c and S_s);*

(iii) *S_c is the set of points of $\Sigma \setminus S_e$ at each of which T has a tangent cone equal to a multiplicity 2 hyperplane; this tangent hyperplane is the unique tangent cone to T at that point, and*

$$\dim_{\mathcal{H}}(S_c) \leq n - 2;$$

furthermore,

$$S_c = S_c^r \cup S_c^b, \quad \textit{where}$$

(a) *S_c^r is the set of points $Y \in S_c$ near each of which* $\operatorname{spt} T$ *is equal to the union of two (distinct) smooth, properly embedded, intersecting hypersurfaces, each of which is the graph over the (unique) tangent hyperplane to T at Y of a smooth function with small C^2-norm;*

(b) *$S_c^b \equiv S_c \setminus S_c^r$ is the set of branch point singularities of Σ; thus, for each point $Z \in S_c^b$ there is no neighbourhood of Z in which* $\operatorname{spt} T$ *is equal to the union of two C^1 embedded hypersurfaces; near each $Z \in S_c^b$* $\operatorname{spt} T$ *is the graph over the (unique) tangent hyperplane to T at Z of a 2-valued $C^{1,1/2}$-function with small $C^{1,1/2}$-norm; moreover, we have that either $S_c^b = \emptyset$ or $\mathcal{H}^{n-2}(S_c^b) > 0$;*

(iv) $S_s \equiv \Sigma \setminus (S_e \cup S_t \cup S_c)$ *is the set of "genuine singularities" of* Σ*;* S_s *is empty if* $n \leq 6$*, discrete if* $n = 7$ *and is a relatively closed subset of* Σ *with Hausdorff dimension at most* $n - 7$ *if* $n \geq 8$.

This theorem in part depends on the results and techniques of [W1] and [SW].

References

[A] F. J. Almgren, Jr.: *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem.* Ann. of Math. **84** (1966), 277–292.

[DG] E. De Giorgi: *Frontiere orientate di misura minima.* Sem. Mat. Scuola Norm. Sup. Pisa (1961), 1–56.

[F] H. Federer: *The singular set of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension.* Bull. Amer. Math. Soc. **76** (1970), 767–771.

[FF] H. Federer, W. Fleming: *Normal and integral currents.* Ann. of Math. **72** (1960), 458–520.

[I] T. Ilmanen: *A strong maximum principle for singular minimal hypersurfaces.* Calc. Var. & PDE. **4** (1996), 443–467.

[R] R. E. Reifenberg: *Solution of the Plateau problem for m-dimensional surfaces of varying topological type.* Acta Math. **104** (1960), 1–92.

[SS] R. Schoen, L. Simon: *Regularity of stable minimal hypersurfaces.* Comm. Pure & Appl. Math. **34** (1981), 742–797.

[S] J. Simons: *Minimal varieties in Riemannian manifolds.* Ann. of Math. **88** (1968), 62–105.

[SW] L. Simon, N. Wickramasekera: *A frequency function and singular set bounds for branched minimal immersions.* arXiv: 1012.5028

[TW] Y. Tonegawa, N. Wickramasekera: *Stable phase interfaces in the Van der Waals–Cahn–Hilliard theory.* arXiv: 1007.2060, to appear in J. Reine. Angew. Math.

[W1] N. Wickramasekera: *A regularity and compactness theory for immersed stable minimal hypersurfaces of multiplicity at most 2.* Jour. Diff. Geom. **80** (2008), 79–173.

[W2] N. Wickramasekera: *A general regularity theory for stable codimension 1 integral varifolds.* arXiv: 0911.4883.

[W3] N. Wickramasekera: *The structure of a stable codimension 1 integral current near points of density* < 3. in preparation.

Wellposedness of the two and three dimensional full water wave problem

Sijue Wu

We consider the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing into the upward vertical direction, and at time $t \geq 0$, the free interface is $\Sigma(t)$, and the fluid occupies the region $\Omega(t)$. When the surface tension is zero, the motion of the fluid is described by

$$(1) \qquad \begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\mathbf{k} - \nabla P & \text{on } \Omega(t),\ t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0 & \text{on } \Omega(t),\ t \geq 0, \\ P = 0 \qquad \text{on } \Sigma(t) \\ (1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)), \end{cases}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. It is well-known that when the surface tension is neglected, the water wave motion can be subject to Taylor instability [3, 16, 2]. Assume that the free interface $\Sigma(t)$ is described by $\xi = \xi(\alpha, t)$, where $\alpha \in \mathbb{R}^{n-1}$ is the Lagrangian coordinate, i.e. $\xi_t(\alpha, t) = \mathbf{v}(\xi(\alpha, t), t)$ is the fluid velocity on the interface, $\xi_{tt}(\alpha, t) = (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v})(\xi(\alpha, t), t)$ is the acceleration. Let $\mathbf{n}$ be the unit normal pointing out of $\Omega(t)$. The Taylor sign condition relating to Taylor instability is

$$-\frac{\partial P}{\partial \mathbf{n}} = (\xi_{tt} + \mathbf{k}) \cdot \mathbf{n} \geq c_0 > 0, \tag{2}$$

point-wisely on the interface for some positive constant c_0. In previous works [17, 18], we showed that the Taylor sign condition (2) always holds for the n-dimensional infinite depth water wave problem (1), $n \geq 2$, as long as the interface is non self-intersecting; and the initial value problem of the water wave system (1) is uniquely solvable **locally** in time in Sobolev spaces for arbitrary given data. Earlier work includes Nalimov [13], Yosihara [21] and Craig [6] on local existence and uniqueness for small data in 2D. Notice that if the surface tension is not zero, or if there is a bottom, or nonzero-vorticity, the Taylor sign condition need not hold. Local wellposedness for water wave motion with the effect of surface tension, bottom and a non-zero vorticity, under the assumption (2) can be found in [1, 4, 5, 9, 11, 12, 14, 15, 22].

In order to understand the long time behavior of the water wave motion, we need to understand the nature of the nonlinearity of the water wave equation. In [19, 20], we showed that the nature of the nonlinearity of the water wave equation (1) is of cubic and higher orders. For water waves in two space dimensions, if initially the amplitude of the interface and the kinetic energy (and finitely many of their derivatives) are of size $O(\epsilon)$ and small, then there exists a unique classical solution of the water wave equation (1) for a time period $[0, e^{c/\epsilon}]$; during this time period, the interface remains small and as regular as the initial interface. Here c is a constant independent of ϵ (c.f. Theorem 1, [19]). For water waves in three space dimensions, if initially the steepness of the interface and the fluid velocity on the interface (and finitely many of their derivatives) are small, then there exists a unique classical solution of the water wave equation (1) for all time, and the interface remains to have small steepness and is as regular as the initial interface for all time (c.f. Theorem 2, [20]).

Let's state what we have obtained so far in precise terms. Notice that equation (1) is a nonlinear equation defined on moving domains. It is difficult to obtain results directly from it. One key step in our approach is to rewrite (1) into forms from which results and information can be obtained.

For clarity, we mainly write in terms of the 2D water waves. We regard the 2D space as the complex plane and use the same notation for the complex form $\xi = x + i\,y$ and $\xi = (x, y)$. So $\bar{\xi} = x - i\,y$.

Let $\xi = \xi(\alpha, t)$ be the free interface $\Sigma(t)$ at time t in Lagrangian parameter α, $N = i\xi_\alpha$ be the normal vector pointing out of the fluid domain, $\mathbf{n} = \frac{N}{|N|}$ be the unit normal, $\mathfrak{a} = -\frac{1}{|N|} \frac{\partial P}{\partial \mathbf{n}}$. We know from [17, 18] that equation (1) is equivalent to the following system defined on the interface $\Sigma(t)$:

$$\xi_{tt} + i = i\mathfrak{a}\xi_\alpha \tag{3}$$
$$\bar{\xi}_t = \mathfrak{H}\bar{\xi}_t \tag{4}$$

where

$$\mathfrak{H}f(\alpha, t) = \frac{1}{\pi i}\, p.v. \int \frac{f(\beta, t)\xi_\beta(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)}\, d\beta \tag{5}$$

is the Hilbert transform on $\Sigma(t) : \xi = \xi(\alpha, t)$, $\alpha \in \mathbb{R}$. Notice that (3)-(4) is fully nonlinear. To solve (3)-(4) on a (small) time interval $[0, T]$, we furthermore derived the following equation by taking derivative w.r.t. t of (3):

$$\begin{cases} \bar{\xi}_{ttt} + i\mathfrak{a}\bar{\xi}_{t\alpha} = -i\mathfrak{a}_t\bar{\xi}_\alpha \\ \bar{\xi}_t = \mathfrak{H}\bar{\xi}_t \end{cases} \tag{6}$$

Using the fact $\bar{\xi}_t = \mathfrak{H}\bar{\xi}_t$, and $\mathfrak{a}$, $\mathfrak{a}_t$ are real valued, we deduced that

$$\begin{aligned} &(I + \mathfrak{K}^*)(\mathfrak{a}_t|\bar{\xi}_\alpha|) = \\ &- \Re\Big(\frac{i\xi_\alpha}{|\xi_\alpha|}\{2[\xi_{tt}, \mathfrak{H}]\frac{\bar{\xi}_{t\alpha}}{\xi_\alpha} + 2[\xi_t, \mathfrak{H}]\frac{\bar{\xi}_{tt\alpha}}{\xi_\alpha} - \frac{1}{\pi i}\int \Big(\frac{\xi_t(\alpha, t) - \xi_t(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)}\Big)^2 \bar{\xi}_{t\beta}\, d\beta\}\Big). \end{aligned} \tag{7}$$

Here $\Re\xi$ indicates the real part of ξ,

$$\mathfrak{K}^* f(\alpha, t) = \int \Re\{\frac{-1}{\pi i}\frac{\xi_\alpha}{|\xi_\alpha|}\frac{|\xi_\beta(\beta, t)|}{(\xi(\alpha, t) - \xi(\beta, t))}\} f(\beta, t)\, d\beta$$

is the adjoint of the double layered potential operator $\mathfrak{K}$ in $L^2(\Sigma(t), dS)$. Notice that $I + \mathfrak{K}^*$ is invertible in $L^2(\Sigma(t), dS)$. Rewriting

$$-i\mathfrak{a}_t\bar{\xi}_\alpha = -i\frac{\bar{\xi}_\alpha}{|\bar{\xi}_\alpha|}\mathfrak{a}_t|\bar{\xi}_\alpha| = \frac{\bar{\xi}_{tt} - i}{|\xi_{tt} + i|}\mathfrak{a}_t|\bar{\xi}_\alpha|,$$

using (7) for $\mathfrak{a}_t|\bar{\xi}_\alpha|$, (6) is now a quasi-linear system with the right hand side of the first equation in (6) consisting of terms of lower order derivatives of $\bar{\xi}_t$.

Let $\mathfrak{u} = \bar{\xi}_t$. Notice that $i\partial_\alpha\mathfrak{u} = \nabla_{\mathbf{n}}\mathfrak{u}$, and that the Dirichlet-Neumann operator $\nabla_{\mathbf{n}}$ is a positive operator. By furthermore proving $\mathfrak{a} = -\frac{1}{|N|}\frac{\partial P}{\partial \mathbf{n}} > 0$ for non self-intersecting interfaces, we showed that (6)-(7) is a quasi-linear equation of weakly hyperbolic type. The local in time wellposedness of (6)-(7) in Sobolev spaces (with $(\mathfrak{u}, \mathfrak{u}_t) \in C([0, T], H^{s+1/2} \times H^s)$, $s \geq 4$) was then proved by energy estimates and a fixed point iteration argument. By establishing the equivalence of (1) with (6)-(7), we obtained the local in time well-posedness in Sobolev spaces of the full water wave equation (1) (c.f. [17, 18]).

For 3D water waves we introduced the framework of Clifford algebra, or in other words, the algebra of quaternions $\mathcal{C}(V_2)$ [18]. Let $\{1, e_1, e_2, e_3\}$ be the basis of $\mathcal{C}(V_2)$, satisfying $e_i^2 = -1$ for $i = 1, 2, 3$, $e_ie_j = -e_je_i$, $i \neq j$, $e_3 = e_1e_2$. Let $\mathcal{D} = \partial_x e_1 + \partial_y e_2 + \partial_z e_3$. By definition, a Clifford-valued function $F : \Omega \subset \mathbb{R}^3 \to \mathcal{C}(V_2)$ is Clifford analytic with domain Ω iff $\mathcal{D}F = 0$ in Ω. Therefore $F = \sum_{i=1}^3 f_ie_i$ is

Clifford analytic in Ω iff $\text{div}F = 0$ and $\text{curl}F = 0$ in Ω. Furthermore we know that F is Clifford analytic in Ω iff $F = \mathfrak{H}_\Sigma F$, where

$$\mathfrak{H}_\Sigma g(\alpha,\beta) = p.v. \iint K(\eta(\alpha',\beta') - \eta(\alpha,\beta))\,(\eta'_{\alpha'} \times \eta'_{\beta'})\, g(\alpha',\beta')\, d\alpha' d\beta'$$

is the 3D version of the Hilbert transform on $\Sigma = \partial\Omega : \eta = \eta(\alpha,\beta),\ (\alpha,\beta) \in \mathbb{R}^2$, with normal $\eta_\alpha \times \eta_\beta$ pointing out of Ω, and $K(\eta) = -2\mathcal{D}\Gamma(\eta) = -\frac{2}{\omega_3}\frac{\eta}{|\eta|^3}$.

All of this indicated Clifford analysis can be an effective tool for 3D water waves. Indeed, in the framework $\mathcal{C}(V_2)$, we derived the quasi-linear equation (cf. (5.21)-(5.22) of [18]) for the 3D water waves, and the local in time well-posedness of the 3D full water wave equation was therefore obtained from energy estimates and a fixed point iteration argument applied to the quasi-linear equation.

We now turn to the question of the long time behavior of the solutions to the water wave equation (1) for small initial data.

Let's state what we discovered for 2D water waves ($n = 2$) [19]. Let $U_g f = f \circ g = f(g(\cdot,t),t)$, and for $\kappa : \mathbb{R} \to \mathbb{R}$ a diffeomorphism, let

$$\zeta := \xi \circ \kappa^{-1} = \mathfrak{x} + i\mathfrak{y}, \qquad U_\kappa^{-1} D_t := \partial_t U_\kappa^{-1}, \qquad U_\kappa^{-1}\mathcal{P} := (\partial_t^2 - i\mathfrak{a}\partial_\alpha)U_\kappa^{-1}$$
$$b := \kappa_t \circ \kappa^{-1}, \qquad U_\kappa^{-1}\mathcal{A}\partial_\alpha := \mathfrak{a}\partial_\alpha U_\kappa^{-1}, \qquad U_\kappa^{-1}\mathcal{H} = \mathfrak{H} U_\kappa^{-1},$$

so

$$D_t = (\partial_t + b\partial_\alpha), \qquad \mathcal{P} = D_t^2 - i\mathcal{A}\partial_\alpha. \tag{8}$$

In [19], we showed that for any solution $\xi(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$ of (3)-(4), the quantity $\Pi := (I - \mathfrak{H})y$ satisfies the equation

$$\begin{aligned}\mathcal{P}(\Pi \circ \kappa^{-1}) = \frac{2}{\pi i}\int \frac{(D_t\zeta(\alpha,t) - D_t\zeta(\beta,t))(\mathfrak{y}(\alpha,t) - \mathfrak{y}(\beta,t))}{|\zeta(\alpha,t) - \zeta(\beta,t)|^2}\partial_\beta D_t\zeta(\beta,t)\,d\beta \\ + \frac{1}{\pi i}\int \left(\frac{D_t\zeta(\alpha,t) - D_t\zeta(\beta,t)}{\zeta(\alpha,t) - \zeta(\beta,t)}\right)^2 \partial_\beta \mathfrak{y}(\beta,t)\,d\beta.\end{aligned} \tag{9}$$

Notice that the right hand side of (9) is cubicly small if the velocity $D_t\zeta$ and steepness $\partial_\alpha\mathfrak{y}$ (and their derivatives) are small. Furthermore we found a coordinate change $\kappa : \mathbb{R} \to \mathbb{R}$,

$$\kappa(\alpha,t) = \bar{\xi}(\alpha,t) + \frac{1}{2}(I + \mathfrak{H})(I + \mathfrak{K})^{-1}(\xi - \bar{\xi}) \tag{10}$$

so that equation (9) contains no quadratic nonlinear terms if κ is given by (10). [1] Here $\mathfrak{K} = \Re\mathfrak{H}$ is the double layered potential operator. In other words, the projection of the height function y of the interface into the space of holomorphic functions in the air region, under the change of coordinates κ as given in (10): $\pi := \Pi \circ \kappa^{-1} = U_\kappa^{-1}(I - \mathfrak{H})y = (I - \mathcal{H})\mathfrak{y}$ satisfies such an equation

$$(\partial_t^2 - i\partial_\alpha)\pi = G$$

where G contains no quadratic nonlinear terms.

[1] It was shown in [19] that $\kappa : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism if $\xi(\alpha,t) - \alpha$ is small.

For 3D water waves, we use the setting of the quaternions $\mathcal{C}(V_2)$. In this setting, we also found that the projection of the height function of the interface into the space of holomorphic functions in the air region, in an appropriate coordinate system, satisfies an equation containing no quadratic nonlinear terms. (c.f. (1.25) or (1.35) and (1.28) of [20] for the 3D counterparts of the equation (9) and the change of coordinates (10).)

The almost global well-posedness of the 2D water waves and the global well-posedness of the 3D water waves are then obtained by applying the method of vector fields to (9) for 2D and to the equation (1.35) in [20] for 3D. We mention that the method of vector fields was first developed by Klainerman [10] for the nonlinear wave equation. The basic steps involved include the development of a generalized Sobolev inequality that gives an L^∞ decay with rate $1/t^{\frac{n-1}{2}}$ for n-D water waves bounded by the generalized L^2 Sobolev norms defined by the vector fields for the water wave operator $\partial_t^2 - i\partial_\alpha$ [2], an energy estimate and a continuity argument. We state our results:

Theorem 1 (2D water waves, [19])**.** *Let $\xi_0 = (\alpha, y_0(\alpha))$, $\alpha \in \mathbb{R}$, be the initial interface, $\mathbf{v}_0 = \mathbf{v}_0(x, y)$, $(x, y) \in \Omega(0)$, be the initial velocity. Assume $y_0(\alpha) = \epsilon f(\alpha)$, $\mathbf{v}_0(x, y) = \epsilon g(x, y)$, where $f \in L^2(\mathbb{R})$ and $g \in L^2(\Omega(0))$, and that up to 12 derivatives of f and g are in L^2. Then there is some $\epsilon_0 > 0$, such that for $\epsilon \le \epsilon_0$, there exists a unique classical solution of the 2D water wave equation* (1) *for a time period $[0, e^{c/\epsilon}]$. Here c depends on f and g only. During this time period, the solution stays small and has the same regularity as the initial data; and the L^∞ norm of the steepness of the interface $\partial_\alpha y$ and the velocity on the interface ξ_t decay at the rate $1/t^{1/2}$.*

Theorem 2 (3D water waves. [20])**.** *Let $\xi_0 = (\alpha, \beta, z_0(\alpha, \beta))$, $(\alpha, \beta) \in \mathbb{R}^2$, be the initial interface, $\xi_{t,0} = \xi_1(\alpha, \beta)$, $(\alpha, \beta) \in \mathbb{R}^2$, be the initial velocity on the interface. Assume that $|D|^{1/2} z_0 = \epsilon f$, $\xi_1 = \epsilon g$, $f, g \in L^2(\mathbb{R}^2)$, and that 20 derivatives of f and g are in L^2. Then there is some $\epsilon_0 > 0$, such that for $\epsilon \le \epsilon_0$, there exists a unique classical solution of the 3D water wave equation* (1) *for all time $t \in [0, \infty)$. During this time, the solution stays small and is as regular as the initial data; and the L^∞ norm of the steepness of the interface, the acceleration on the interface and the derivative of the velocity on the interface decay at the rate $1/t$.*

References

[1] D. Ambrose, N. Masmoudi: *The zero surface tension limit of two-dimensional water waves.* Comm. Pure Appl. Math. 58 (2005), no. 10, 1287–1315.

[2] T. Beale, T. Hou & J. Lowengrub: *Growth rates for the linearized motion of fluid interfaces away from equilibrium.* Comm. Pure Appl. Math. 46 (1993), no.9, 1269–1301.

[3] G. Birkhoff: *Helmholtz and Taylor instability.* Proc. Symp. in Appl. Math. Vol. XIII, pp.55–76.

[4] D. Christodoulou, H. Lindblad: *On the motion of the free surface of a liquid.* Comm. Pure Appl. Math. 53 (2000), no. 12, 1536–1602.

[2] The water wave operator for 3D is $\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$.

[5] D. Coutand, S. Shkoller: *Wellposedness of the free-surface incompressible Euler equations with or without surface tension.* J. AMS. 20 (2007), no. 3, 829–930.
[6] W. Craig: *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits.* Comm. in P.D.E. 10(8), 1985, 787–1003.
[7] P. Germain, N. Masmoudi, J. Shatah: *Global solutions of the gravity water wave equation in dimension 3.* Preprint July 2009.
[8] J. Gilbert, M. Murray: *Clifford algebras and Dirac operators in harmonic analysis.* Cambridge University Press, 1991.
[9] T. Iguchi: *Well-posedness of the initial value problem for capillary-gravity waves.* Funkcial. Ekvac. 44 (2001), no. 2, 219–241.
[10] S. Klainerman: *The null condition and global existence to nonlinear wave equations.* Lectures in Appl. Math. vol. 23, (1986), 293–325.
[11] D. Lannes: *Well-posedness of the water-wave equations.* J. Amer. Math. Soc. 18 (2005), 605–654.
[12] H. Lindblad: *Well-posedness for the motion of an incompressible liquid with free surface boundary.* Ann. of Math. 162 (2005), no. 1, 109–194.
[13] V. I. Nalimov: *The Cauchy-Poisson Problem* (in Russian). Dynamika Splosh. Sredy 18, 1974, 104–210.
[14] M. Ogawa, A. Tani: *Free boundary problem for an incompressible ideal fluid with surface tension.* Math. Models Methods Appl. Sci. 12, (2002), no. 12, 1725–1740.
[15] J. Shatah, C. Zeng: *Geometry and a priori estimates for free boundary problems of the Euler equation.* Comm. Pure Appl. Math. V. 61., no.5 (2008), 698–744.
[16] G.I. Taylor: *The instability of liquid surfaces when accelerated in a direction perpendicular to their planes I..* Proc. Roy. Soc. London A 201, 1950, 192–196.
[17] S. Wu: *Well-posedness in Sobolev spaces of the full water wave problem in 2-D.* Inventiones Mathematicae 130, 1997, 39–72.
[18] S. Wu: *Well-posedness in Sobolev spaces of the full water wave problem in 3-D.* Journal of the AMS 12, no.2 (1999), 445–495.
[19] S. Wu: *Almost global wellposedness of the 2-D full water wave problem.* Inventiones Mathematicae, Vol. 177, no.1 July 2009., 45–135.
[20] S. Wu: *Global wellposedness of the 3-D full water wave problem.* Inventiones Mathematicae, Vo1. 184, no.1 April 2011.
[21] H. Yosihara: *Gravity waves on the free surface of an incompressible perfect fluid of finite depth.* RIMS Kyoto 18, 1982, 49–96.
[22] P. Zhang, Z. Zhang: *On the free boundary problem of 3-D incompressible Euler equations.* Comm. Pure. Appl. Math. Vol. 61., no.7 (2008), 877–940.

CR-Geometry in 3-D

Paul Yang

In this talk I discussed two problems in three-dimensional "CR-Geometry" that may be understood using some 4th-order operator. The first is the global embedding problem. In dimension 3 there is no local integrability condition for the CR-structure, while previous work of Lempert and Burns-Epstein indicates that the subset of embeddable structures is exceptionally thin within the entire moduli space of CR-structures on a given 3-manifold.

In a joint work with Chamillo and Chin we provided CR-invariant conditions for embeddability:

1.) Positivity of the CR-conformal Laplacian, 2.) Non-negativity of some 4th-order operator that has formal resemblence to the Paneitz-operator in four-dimensional conformal geometry.

These conditions turn out to be the appropriate ones for a positive mass theorem, which is a rigidity result for the three-dimensional Heisenberg space. This positive mass theorem joint with Malchiodi's work on this subject makes it possible to solve for some minimizer of the CR-Yamabe-functional. I also indicate examples of CR-structures that do not satisfy the above mentioned sign conditions and for which the associated masses are actually negative.

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