Mathematisches Forschungsinstitut Oberwolfach

Report No. 53/2011

DOI: 10.4171/OWR/2011/53

Combinatorial Optimization

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November 13th – November 19th, 2011

Abstract. Combinatorial Optimization is a very active field that benefits from bringing together ideas from different areas, e.g., graph theory and combinatorics, matroids and submodularity, connectivity and network flows, approximation algorithms and mathematical programming, discrete and computational geometry, discrete and continuous problems, algebraic and geometric methods, and applications. We continued the long tradition of triannual Oberwolfach workshops, bringing together the best researchers from the above areas, discovering new connections, and establishing new and deepening existing international collaborations.

Mathematics Subject Classification (2000): 90C27; secondary: 90C57, 90C10, 90C11, 90C22, 90C06, 90C59, 90C90.

Introduction by the Organisers

The triannual Oberwolfach workshops on Combinatorial Optimization play a key role for our field. No other workshop manages to bring together the best researchers from all over the world. This success is due to the outstanding research conditions at Oberwolfach, to the unique format of Oberwolfach workshops, and above all to their reputation of excellence.

Continuing the tradition, the program consisted of pre-arranged one-hour focus lectures - one each morning - followed by 24 thirty-minute presentations that were scheduled during the week. As a new feature for the combinatorial optimization workshops, all participants were asked to give a short presentation (four minutes plus one minute discussion) on their recent or current work at the beginning of the workshop. There were more than 50 of these short presentations on Monday and Tuesday; a list of titles appear in this report. The goal was to foster deeper discussions and more collaborations between the participants, and indeed this is what we observed during the week. The feedback was extremely positive. Moreover, the traditional open problem session on Wednesday evening was again very active. In this session, ten interesting problems were presented and discussed, and more had been presented already in some of the 4-minute presentations.

The focus lectures featured new approaches and the state-of-the-art in several fundamental areas of combinatorial optimization:

- Bruce Shepherd (McGill, Montréal) — Minimum congestion versus maximum throughput: connections and distinctions.
- Bertrand Guenin (Waterloo) — Flows in matroids and related results.
- Frank Vallentin (Delft) — Applications of semidefinite programming and harmonic analysis
- Gérard Cornuéjols (CMU, Pittsburgh) — Multi-row cuts and integer lifting.
- Pablo Parrilo (MIT, Cambridge) — From stable sets to sums of squares.

During the workshop we were excited to see great advances in classical topics such as multi-commodity flows, the traveling salesman problem, cutting plane closures, extended linear formulations of polytopes, or matroid matching. Very recent solutions of important open problems, partly proving long-standing conjectures, have been presented. At the same time, a large part of the workshop was devoted to new directions, such as geometric and algebraic techniques, semidefinite relaxations, symmetry reduction, and harmonic analysis for geometric packing and coloring in the continuous setting. The combination of classical hard problems and a variety of new techniques makes our growing field more interesting than ever.

We feel that this workshop was very successful in bringing together the best researchers from our field and in stimulating new cooperations. In many cases, two or three participants who hardly knew each other before began to work together. In one case, two participants discovered that they had just solved the same long-standing open problem, but with completely different techniques. We concluded the workshop with their lectures on Friday afternoon.

We would like to thank all participants for their carefully prepared contributions and the many exciting discussions. Last but not least, we thank the Oberwolfach research institute for providing outstanding meeting and working conditions and the unique inspiring Oberwolfach atmosphere.

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(2) Bansal, Nikhil: *On a signed sum of real numbers*
(3) Conforti, Michele: *Mixed-integer representability of sets*
(4) Cook, William: *Riven's system for the subtour polytope*
(5) Cunningham, William: *Augmenting path algorithms for matroid problems*
(6) de Oliveira, Fernando: *Grothendieck inequalities with rank constraints*

(7) Dunkel, Juliane: *The Gomory-Chvátal closure of a non-rational polytope is a polytope*
(8) Eisenbrand, Friedrich: *Closest vector*
(9) Fiorini, Samuel: *The minimum generating set problem*
(10) Fujishige, Satoru: *Dual consistency of systems of linear inequalities*
(11) Gijswijt, Dion: *SDP with symmetries*
(12) Goemans, Michel: *Integrality gap for hypergraphic Steiner tree relaxations*
(13) Guenin, Bertrand: *Two questions on the set covering polyhedron*
(14) Heismann, Olga: *A special hypergraph minimum cut problem*
(15) Held, Stephan: *Maximum weight stable sets and graph coloring*
(16) Hirai, Hiroshi: *Tractability of μ-weighted maximum integer multiflow*
(17) Hougardy, Stefan: *Linear time strip packing algorithms*
(18) Iwata, Satoru: *Approximating max-min weighted T-join*
(19) Jordán, Tibor: *Globally or universally rigid frameworks and graphs*
(20) Jünger, Michael: *Separation of Kuratowski constraints*
(21) Kaibel, Volker: *Extended formulations for alternahedra*
(22) Király, Tamás: *The nine dragon tree conjecture*
(23) Kobayashi, Yusuke: *Restricted t-matchings*
(24) Korte, Bernhard: *Combinatorial optimization and chip design*
(25) Laurent, Monique: *Gram dimension of graphs*
(26) Liebling, Thomas: *Dependence on edge sharpness of spherotetrahedra packings*
(27) Martin, Alexander: *Towards globally optimal solutions for MinLPs by discretization techniques*
(28) Möhring, Rolf: *Optimizing ship traffic on the Kiel canal*
(29) Murota, Kazuo: *Discrete convex duality in matrix pencils*
(30) Olver, Neil: *Approximation algorithms for scheduling from games*
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Abstracts

Min-Max Graph Partitioning

Nikhil Bansal
(joint work with Uriel Feige, Robert Krauthgamer, Konstantin Makarychev, Viswanath Nagarajan, Joseph Naor, Roy Schwartz)

We study graph partitioning problems from a *min-max* perspective, in which an input graph on n vertices should be partitioned into k parts, and the objective is to minimize the maximum number of edges leaving a single part. The two main versions we consider are where the k parts need to be of equal-size, and where they must separate a set of k given terminals. We consider a common generalization of these two problems, and design for it an $O(\sqrt{\log n \log k})$-approximation algorithm. This improves over an $O(\log^2 n)$ approximation for the second version due to Svitkina and Tardos [ST04], and roughly $O(k \log n)$ approximation for the first version that follows from other previous work. We also give an improved $O(1)$-approximation algorithm for graphs that exclude any fixed minor.

The main tool we use is a new approximation algorithm for ρ-Unbalanced Cut, the problem o finding in an input graph $G = (V; E)$ a set $S \subset V$ of size $|S| = \rho n$ that minimizes the number of edges leaving S. We provide a bicriteria approximation of $O(\sqrt{\log n \log(1/\rho)})$; when the input graph excludes a fixed-minor we improve this guarantee to $O(1)$. Note that the special case $\rho = 1/2$ is just the Minimum Bisection problem, and indeed our bound generalizes that of Arora, Rao and Vazirani [ARV08]. Our algorithms also work for the closely related Small Set Expansion problem, which asks for a set $S \subset V$ of size $|S| \leq \rho n$ with minimum conductance (edge-expansion), and was suggested recently by Raghavendra and Steurer [RS10]. In fact, our algorithm handles more general, weighted, versions of both problems. Previously, an $O(\log n)$ true approximation for both ρ-Unbalanced Cut and Small Set Expansion follows from Räcke [Rac08].

At a high level, our algorithm for min-max partitioning works by writing a configuration LP for which the dual separation problem is ρ-Unbalanced Cut. This LP produces a fractional covering of the graph using about ρ size sets with small cut sizes. To convert this into a partition, we introduce a certain randomized uncrossing procedure which be feel could also be useful in other contexts. A conference version of this work appears in [BFK], and the journal version can be found here [BFK+].

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Split, Mixed Interger Rounding and Gomory inequalities

Michele Conforti
(joint work with Giacomo Zambelli)

1. Split inequalities

Let $P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : \; Ax + Gy \leq b\}$ and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. A *split* is a disjunction $(\pi x \leq \pi_0) \vee (\pi x \geq \pi_0 + 1)$ where $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$. We will also say that (π, π_0) defines a split.

An inequality is a *split inequality* if, for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, it is valid for both sets

$$\Pi_1 := P \cap \{(x, y) : \; \pi x \leq \pi_0\} \quad \Pi_2 := P \cap \{(x, y) : \; \pi x \geq \pi_0 + 1\}.$$

Split inequalities were introduced by Cook, Kannan and Schrijver [3]. Clearly an inequality is a split inequality if and only if it is valid for the polyhedron $P^{(\pi,\pi_0)} := \text{conv}(\Pi_1 \cup \Pi_2)$ for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. Since $S \subseteq P^{(\pi,\pi_0)} \subseteq P$, $P^{(\pi,\pi_0)}$ in general provides a formulation for S that is better than P and a split inequality is valid for S. The *split closure* of P is the set defined by

$$P^{\text{split}} := \bigcap_{(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}} P^{(\pi,\pi_0)}$$

2. Mixed Integer Rounding inequalities

It is easy to show that the convex hull of the 2-dimensional mixed-integer set $\{(\xi, v) \in \mathbb{Z} \times \mathbb{R}_+ : \; \xi - v \leq \beta\}$ is defined by the original inequalities $v \geq 0$, $\xi - v \leq \beta$ and the *simple rounding inequality*

$$\xi - \frac{1}{1 - f_0} v \leq \lfloor \beta \rfloor,$$

where $f_0 = \beta - \lfloor \beta \rfloor$. The simple rounding inequality is a split inequality, relative to the split $(\xi \leq \lfloor \beta \rfloor) \vee (\xi \geq \lceil \beta \rceil)$.

The Mixed-Integer Rounding (MIR) inequalities were introduced by Nemhauser and Wolsey [5], [6]. Later Wolsey [7] and Marchand and Wolsey [4]) revisited MIR inequalities. MIR inequalities are derived from the simple rounding inequality using variable aggregation.

Let $Ax + Gy \leq b$ be a system of m linear constraints that defines P. Given $u \in \mathbb{R}^m$ such that $uA \in \mathbb{Z}^n$ and $uG = 0$, consider inequality $u^+ Ax + u^+ Gy \leq u^+ b$, which is valid for P. (We define u^+ to be the vector whose components are $\max\{0, u_j\}$ and $u^- := -(u)^+$, so $u = u^+ - u^-$.)

Such inequality can be re-written in the form

$$uAx - u^-(b - Ax - Gy) \leq ub.$$

Since uA is an integral vector and $u^-(b - Ax - Gy) \geq 0$ is a valid inequality for P, by substituting $\xi = uAx$, $\upsilon = u^-(b - Ax - Gy)$ and $\beta = ub$, we derive from the simple rounding

$$uAx - \frac{u^-}{1 - f_0}(b - Ax - Gy) \leq \lfloor ub \rfloor \tag{1}$$

where $f_0 = ub - \lfloor ub \rfloor$ his is the Mixed Integer Rounding (MIR) inequality. This definition is more restrictive than the original one given by Nemhauser and Wolsey [5].

Note that, if $u \geq 0$, then the above is just the Chvàtal inequality $uAx \leq \lfloor ub \rfloor$ derived from the valid inequality $uAx \leq ub$.

The *MIR closure* of P is the set P^{MIR} defined as the intersection of P with all the MIR inequalities. Since MIR are split inequalities, it follows that $P^{\text{split}} \subseteq P^{MIR}$. The next theorem implies that the reverse containment holds as well.

3. The main result

Theorem 1. *Let $P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p :\ Ax + Gy \leq b\}$ be a polyhedron and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. Given $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, let B_π be the set of basic solutions to the linear system $uA = \pi$, $uG = 0$. Then*

$$P^{(\pi,\pi_0)} = P \cap \bigcap_{\substack{u \in B_\pi: \\ \pi_0 < ub < \pi_0 + 1}} \left\{(x, y) :\ \pi x - \frac{u^-(b - Ax - Gy)}{1 - (ub - \pi_0)} \leq \pi_0\right\}. \tag{2}$$

Corollary 1 (Nemhauser and Wolsey [5])**.** *Let $P \subseteq \mathbb{R}^n \times \mathbb{R}^p$ be a polyhedron and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. Then*

$$P^{\text{split}} = P^{MIR} = \{(x, y) \in P :\ (x, y) \text{ satisfies } (1)\ \forall u \text{ s.t. } uA \in \mathbb{Z}^n,\ uG = 0\}.$$

Corollary 2 (Andersen, Cornuéjols, and Li [1])**.** *Let $P = \{(x, y) \in \mathbb{R}^{n+p} :\ Ax + Gy \leq b\}$ be a polyhedron and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. Let $k = \text{rank}(A, G)$, and let $\mathcal{B}$ be the family of subsets B of $\{1, \ldots, m\}$ such that $|B| = k$ and the vectors (a^i, g^i), $i \in B$, are linearly independent. For every $B \in \mathcal{B}$, let $P_B := \{(x, y) : a^i x + g^i y \leq b_i,\ i \in B\}$. Then*

$$P^{\text{split}} = \bigcap_{B \in \mathcal{B}} P_B^{\text{split}}.$$

We prove that when the polyhedron P is defined by a system of inequalities in nonnegative variables, the strengthening procedure of Balas and Jeroslow [2] applied to the MIR inequalities produces the Gomory inequalities. We then discuss the separation problem and indicate how a most violated MIR inequality can found by solving an LP in the original space.

Using our framework, we give a short proof of the following theorem.

Theorem 2 (Cook, Kannan and Schrijver [3]). *Let $P \subseteq \mathbb{R}^n \times \mathbb{R}^p$ be a rational polyhedron and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. Then P^{split} is a rational polyhedron.*

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Multi-row cuts and integer lifting

Gérard Cornuéjols

Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^n$ such that $\dim(S) = n$. We consider the following infinite relaxation to a general Mixed Integer Linear Program

$$\begin{aligned} x = \quad & f + \sum_{r \in \mathbb{R}^n} r s_r + \sum_{r \in \mathbb{R}^n} r y_r \\ & x \in S \\ & s_r \geq 0, \qquad\qquad r \in \mathbb{R}^n \\ & y_r \geq 0,\, y_r \in \mathbb{Z}, \quad r \in \mathbb{R}^n \\ & s, y \text{ have finite support.} \end{aligned} \tag{1}$$

Given two functions ψ and π from $\mathbb{R}^n$ to $\mathbb{R}$, the inequality

$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1 \tag{2}$$

is *valid* for (1) if it holds for every (x, s, y) satisfying (1). If (2) is valid, we say that the function (ψ, π) is *valid* for (1). A valid function (ψ, π) is *minimal* if there is no valid function (ψ', π') distinct from (ψ, π) such that $\psi'(r) \leq \psi(r)$, $\pi'(r) \leq \pi(r)$ for all $r \in \mathbb{R}^n$.

The following simpler model has been studied recently [7]

$$(3)\qquad \begin{aligned} x = \quad & f + \sum_{r\in\mathbb{R}^n} r s_r \\ & x \in S \\ & s_r \geq 0, \quad r \in \mathbb{R}^n \\ & s \text{ has finite support.} \end{aligned}$$

We refer to this model as the *continuous infinite relaxation relative to* f. Given a valid function ψ for (3), the function π is a *lifting* of ψ if (ψ, π) is valid for (1).

Minimal valid inequalities for (3) are well understood in terms of maximal S-free convex sets [4, 8, 2]. We are interested in characterizing liftings of minimal valid inequalities for (3).

If ψ is a minimal valid function for (3) and π is a lifting of ψ such that (ψ, π) is minimal, we say that π is a *minimal lifting* of ψ.

We remark that, given any valid function ψ for (3) and a lifting π of ψ, the function π' defined by $\pi'(r) = \min\{\psi(r), \pi(r)\}$ is also a lifting for ψ. In particular, if ψ is a minimal valid function for (3) and π is a minimal lifting of ψ, then $\pi \leq \psi$. Furthermore, there is always a ball centered at the origin where $\pi = \psi$.

We first concentrate on deriving the best possible lifting coefficient of one single integer variable. Namely, given $d \in \mathbb{R}^n$, we consider the model

$$(4)\qquad \begin{aligned} x = \quad & f + \sum_{r\in\mathbb{R}^n} r s_r + dz \\ & x \in S \\ & s_r \geq 0, \qquad\qquad r \in \mathbb{R}^n \\ & z \geq 0,\ z \in \mathbb{Z}, \\ & s \text{ has finite support.} \end{aligned}$$

Given a minimal valid function ψ for (3), we want to determine the minimum scalar λ such that the inequality

$$\sum_{r\in\mathbb{R}^n} \psi(r) s_r + \lambda z \geq 1$$

is valid for (4). Given $d \in \mathbb{R}^n$, let $\pi_\ell(d)$ be such minimum λ. By definition, $\pi_\ell \leq \pi$ for every lifting π of ψ. In general, the function (ψ, π_ℓ) is not valid for (1). However, when (ψ, π_ℓ) is valid, π_ℓ is the unique minimal lifting for ψ.

In this talk we give a geometric characterization of the function π_ℓ, and use this characterization to analyze specific functions ψ in which π_ℓ is the unique minimal lifting [1, 3, 5]. The motivation for this work was provided by the results of Dey and Wolsey [6] on lifting 2-row cuts.

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The Gomory-Chvátal Closure of a Non-Rational Polytope is a Rational Polytope

Juliane Dunkel
(joint work with Andreas S. Schulz)

The question as to whether the Gomory-Chvátal closure of a non-rational polytope is a polytope has been a longstanding open problem in integer programming. In this paper, we answer this question in the affirmative, by combining ideas from polyhedral theory and the geometry of numbers. This result is part of the dissertation "The Gomory-Chvátal closure: Polyhedrality, Complexity, and Extension", which revolves around theoretical aspects of Gomory-Chvátal cutting planes.

Cutting-plane methods, when combined with branch and bound, are among the most successful techniques for solving integer programming problems in practice; numerous types of cutting planes have been studied in the literature and several of them are used in commercial solvers (see, e.g., [2] and the references therein). Cutting planes also give rise to a rich theory (see again [2]). In general, a cutting plane for a polyhedron P is an inequality that is satisfied by all integer points in P, and, when added to the polyhedron P, it typically yields a stronger relaxation of its integer hull. A Gomory-Chvátal cutting plane [8, 1] is an inequality of the form $cx \leq \lfloor \delta \rfloor$, where c is an integral vector and $cx \leq \delta$ is valid for P. The Gomory-Chvátal closure of P is the intersection of all half-spaces defined by such inequalities; it is usually denoted by P'. Even though the Gomory-Chvátal closure is defined as the intersection of an infinite number of half-spaces, the Gomory-Chvátal closure of a rational polyhedron is again a rational polyhedron. Namely, Schrijver [10] showed that, for a rational polyhedron P, the Gomory-Chvátal cuts corresponding to a totally dual integral system of linear inequalities describing P specify its closure P' fully. For polyhedra that cannot be described by rational data the situation is different. It is well-known that the integer hull P_I of an unbounded non-rational polyhedron P may not be a polyhedron (see, e.g., [9]). In fact, the integer hull may not be a closed set, and the Gomory-Chvátal closure may not be rational polyhedron. On the other hand, in the case of a non-rational polytope, P_I is the convex hull of a finite set of integer points and, therefore, a rational polytope. Yet, there is no notion of total dual integrality for non-rational systems of linear inequalities. In fact, it was unknown whether the Gomory-Chvátal closure of an arbitrary polytope is a rational polytope. We show that this is indeed the case: also the Gomory-Chvátal closure of a non-rational polytope is again a rational polytope, that is, it can be described by a finite set of rational inequalities.

Even though Gomory-Chvátal cuts were originally introduced for polyhedra, they have lately been applied to other convex sets as well. Of particular relevance is the work by Dey and Vielma [5] who showed that the Gomory-Chvátal closure of a full-dimensional ellipsoid described by rational data is a polytope. Dadush, Dey, and Vielma [3] recently extended this result to strictly convex bodies and to the intersection of strictly convex bodies with rational polyhedra. Since the original proof of Schrijver for rational polyhedra relies strongly on polyhedral properties, Dadush et al. had to develop a new proof technique, which can roughly be described as follows: One first shows that there exists a finite set of Gomory-Chvátal cuts that separate every non-integral point on the boundary of the strictly convex body. In a second step, one proves that if the intersection of the boundary of a convex body with a finite set of Gomory-Chvátal cuts is contained in the Gomory-Chvátal closure, only a finite set of additional inequalities is needed to fully describe the Gomory-Chvátal closure of the body. Our general proof strategy for showing the polyhedrality of the Gomory-Chvátal closure of a non-rational polytope is inspired by [3]. Yet, the key argument is very different and new, since the proof in [3] relies on properties of strictly convex bodies that do not extend to polytopes. More precisely, strictly convex bodies do not have any higher-dimensional "flat faces", and therein lies the main difficulty in establishing the polyhedrality of the elementary closure for non-rational polytopes. Our proof is geometrically motivated and uses ideas from convex analysis, polyhedral theory, and the geometry of numbers. In particular, the underlying geometric idea relies on properties of integer lattices and reduced lattice bases. For the complete proof, we refer the reader to [7] and [6]. Simultaneously and independently from this work, Dadush, Dey, and Vielma [4] proved that the Gomory-Chvátal closure of any compact convex set is a rational polytope.

Basics and Notations. For any vector $a \in \mathbb{R}^n$, we define $a_P := \max\{ax | x \in P\}$. We denote the hyperplane $\{x \in \mathbb{R}^n | ax = a_0\}$ by $(ax = a_0)$ and, similarly, $(ax \leq a_0)$ denotes the half-space of all points satisfying the inequality $ax \leq a_0$. For any set $S \subseteq \mathbb{Z}^n$, $C_S(P) := \bigcap_{a \in S} \left(ax \leq \lfloor a_P \rfloor\right)$ denotes the intersection of all half-spaces corresponding to Gomory-Chvátal cuts for P with normal vector in S.

General Proof Idea. Our general strategy for proving that for any polytope a finite number of Gomory-Chvátal cuts is sufficient to describe the polytope's closure is a modification of the two-step technique in [3] for a strictly convex body: We first show that one can find a finite set S of integral vectors such that

(P1) $C_S(P) \subseteq P$,

(P2) $C_S(P) \cap \text{rbd}(P) \subseteq P'$.

(Here, $\text{rbd}(P)$ denotes the relative boundary of P.) We then argue that, given the polytope $C_S(P)$, no more than a finite number of additional Gomory-Chvátal cuts are necessary to describe the closure P'. The main challenge of this proof strategy lies in showing the existence of a set S satisfying property (P1). This is due to the presence of higher-dimensional faces with non-rational affine hulls. The other steps of the proof require modification compared to the strictly convex body case. However, these adjustments come along quite naturally. Therefore, in this abstract we will only provide some intuition of how we can construct a subset of P from a finite number of Gomory-Chvátal cuts.

Suppose that we can find a set $S \subseteq \mathbb{Z}^n$ with $C_S(P) \subseteq P$ for some polytope $P \subseteq \mathbb{R}^n$ for which a non-rational inequality $ax \leq a_P$ is facet-defining. As this inequality cannot be facet-defining for the *rational* polytope $C_S(P)$, there must exist a finite set of Gomory-Chvátal cuts that dominate $ax \leq a_P$. More formally, there must exist a subset $S_a \subseteq S$ such that $C_{S_a}(P) \subseteq (ax \leq a_P)$. If V_R denotes the maximal rational affine subspace of $(ax = a_P)$, that is, the affine hull of all rational points in $(ax = a_P)$, then the Gomory-Chvátal cuts associated with the vectors in S_a have to separate every point in $(ax = a_P) \setminus V_R$. Indeed, our strategy for the first step of the proof is to show that for each non-rational facet-defining inequality $ax \leq a_P$ for P there exists a finite set of integral vectors S_a that satisfies $C_{S_a}(P) \subseteq (ax \leq a_P)$. This fact is proven in a series of steps. First, we establish the existence of a sequence of integral vectors satisfying a specific list of properties. These vectors are based on Diophantine approximations of the non-rational normal vector a, and they give rise to Gomory-Chvátal cuts that separate all points in the non-rational facet $F = P \cap (ax = a_P)$ that are not contained in the maximal rational affine subspace V_R of $(ax = a_P)$. The number of Gomory-Chvátal cuts needed in our construction for separating the points in $(ax = a_P) \setminus V_R$ depends only on the dimension of V_R. If $\dim(V_R) = n-2$, that is, the hyperplane $(ax = a_P)$ has a single "non-rational direction", then only two cuts are necessary. One can visualize these cuts to form a kind of "tent" in the half-space $(ax \leq a_P)$, with the ridge being V_R. With each decrease in the dimension of V_R by 1, the number of necessary cuts is doubled. Hence, at most 2^{n-1} Gomory-Chvátal cuts are required to separate the non-rational parts of a non-rational facet of the polytope. The core argument for the construction of these cuts relies on integral lattices and, in particular, on properties of reduced lattice bases.

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On sub-determinants and the diameter of polyhedra

Friedrich Eisenbrand
(joint work with Nicolas Bonifas, Marco Di Summa, Nicolai Hähnle, Martin Niemeier)

One of the fundamental open problems in optimization and discrete geometry is the question whether the diameter of a polyhedron can be bounded by a polynomial in the dimension and the number of its defining inequalities. The problem is readily explained: A *polyhedron* is a set of the form $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$, where $A \in \mathbb{R}^{m\times n}$ is a matrix and $b \in \mathbb{R}^m$ is an m-dimensional vector. A *vertex* of P is a point x^* such that there exist n linearly independent rows of A whose corresponding inequalities of $Ax \leq b$ are satisfied by x^* with equality. Two different vertices x^* and y^* are *neighbors* if there exist $n-1$ linearly independent rows of A whose corresponding inequalities of $Ax \leq b$ are satisfied with equality both by x^* and y^*. In this way, we obtain the undirected *polyhedral graph* with edges being pairs of neighboring vertices of P. This graph is connected. The *diameter* of P is the smallest natural number that bounds the length of a shortest path between any pair of vertices in this graph. The question is now as follows.

> Can the diameter of a polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be bounded by a polynomial in m and n?

Despite a lot of research effort during the last 50 years, the gap between lower and upper bounds on the diameter remains huge. While, when the dimension n is fixed, the diameter can be bound by a linear function of m [9, 2], for the general case the best upper bound, due to Kalai and Kleitman [7], is $O(m^{1+\log n})$. The best lower bound is of the form $(1+\varepsilon)\cdot m$ for some $\varepsilon > 0$ in fixed and sufficiently large dimension n. This is due to a celebrated recent result of Santos [12] who disproved the until then longstanding *Hirsch conjecture*. The Hirsch conjecture stated that the diameter of a bounded polyhedron is at most $m-n$. Interestingly, this huge gap (polynomial versus quasi-polynomial) is also not closed in a very simple combinatorial abstraction of polyhedral graphs [6]. However, it was shown by Vershynin [13] that every polyhedron can be perturbed by a small random amount so that the expected diameter of the resulting polyhedron is bounded by a polynomial in m and n. See Kim and Santos [8] for a recent survey.

In light of the importance and apparent difficulty of the open question above, many researchers have shown that it can be answered in an affirmative way in some special cases. Naddef [10] has proven that the Hirsch conjecture holds true for 0/1-polytopes. Orlin [11] provided a quadratic upper bound for flow-polytopes. Brightwell et al. [3] have shown that the diameter of the transportation polytope is linear in m and n, and a similar result holds for the dual of a transportation polytope [1] and the axial 3-way transportation polytope [4].

The results on flow polytopes and classical transportation polytopes concern polyhedra that are defined by *totally unimodular matrices*, i.e., integer matrices whose sub-determinants are $0, \pm 1$. For such polyhedra Dyer and Frieze [5] had previously shown that the diameter is bounded by a polynomial in n and m. Their bound is $O(m^{16} n^3 (\log mn)^3)$. Their result is also algorithmic: they show that there exists a randomized simplex-algorithm that solves linear programs defined by totally unimodular matrices in polynomial time.

Our main result is a generalization and considerable improvement of the diameter bound of Dyer and Frieze. We show that the diameter of a polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$, with $A \in \mathbb{Z}^{m \times n}$ is bounded by $O\left(\Delta^2 n^4 \log n\Delta\right)$. Here, Δ denotes the largest absolute value of a *sub-determinant* of A. If P is bounded, i.e., a *polytope*, then we can show that the diameter of P is at most $O\left(\Delta^2 n^{3.5} \log n\Delta\right)$. To compare our bound with the one of Dyer and Frieze one has to set Δ above to one and obtains $O\left(n^4 \log n\right)$ and $O\left(n^{3.5} \log n\right)$ respectively. Notice that our bound is independent of m, i.e., the number of rows of A.

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Linear vs. Semidefinite Extended Formulations: Exponential Separation and Strong Lower Bounds

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(joint work with Serge Massar, Sebastian Pokutta, Hans Raj Tiwary and Ronald de Wolf)

In 1986–1987 there were attempts to prove P = NP by giving a polynomial-size linear program (LP) that would solve the traveling salesman problem (TSP). Due to the large size and complicated structure of the proposed LP for the TSP, it was difficult to show directly that the LP was erroneous. In a groundbreaking effort to prevent such attempts, Yannakakis [9] proved that every *symmetric* LP for the TSP has exponential size. Because the proposed LP for the TSP was symmetric, it could not possibly be correct.

In his paper, Yannakakis left as a main open problem the question of proving that the TSP admits no polynomial-size LP, *symmetric or not*. We answer this question by proving a super-polynomial lower bound on the number of inequalities in *every* LP for the TSP. Moreover, we also prove super-polynomial lower bounds for the maximum cut and maximum stable set problems. Therefore, it is impossible to prove P = NP by giving a polynomial-size LP for any of these problems.

These results follow from a new connection that we make between one-way quantum communication protocols and semidefinite programming reformulations of LPs. We summarize our results here. For a full version, see [2].

1. State of the Art

An *extended formulation* (EF) or *extension* of a polytope $P \subseteq \mathbb{R}^d$ is a polytope $Q \subseteq \mathbb{R}^e$ along with a linear map that projects Q onto P. Optimizing a linear function f over P amounts to optimizing the linear function $f \circ \pi$ over its EF Q, where $\pi : \mathbb{R}^d \to \mathbb{R}^e$ linearly projects P onto Q. We define the *size* of an EF Q as the number of facets of Q, and the *extension-complexity* of P as the minimum size of an EF of P.

Yannakakis [9] proved a $2^{\Omega(n)}$ lower bound on the size of any *symmetric* EF of the TSP polytope TSP(n). Although he remarked that he did "not think that asymmetry helps much", it was recently shown by Kaibel *et al.* [5] that symmetry is a restriction in the sense that there exist polytopes that have polynomial-size EFs but no polynomial-size symmetric EF.

The strongest unconditional lower bounds so far were obtained by Rothvoß [7]. Via a counting argument inspired by Shannon's theorem, he proved that there exist 0/1-polytopes in $\mathbb{R}^d$ whose extension-complexity is at least $2^{d/2-o(1)}$. However, Rothvoß's technique does not provide *explicit* 0/1-polytopes with an exponential extension-complexity.

Yannakakis [9] discovered that the extension-complexity of a polytope P is determined by certain factorizations of an associated matrix, called the *slack matrix* of P, that records for each pair (F, v) where F is a facet and v is a vertex the algebraic distance of v to a hyperplane supporting F. Defining the *nonnegative rank* of a nonnegative matrix M as the smallest natural number r such that M can be expressed as $M = UV$ where U and V are nonnegative matrices with r columns and r rows, respectively, it turns out that the extension-complexity of every polytope P is exactly the nonnegative rank of its slack matrix.

This *factorization theorem* led Yannakakis to explore connections between EFs and communication complexity. Let $S = S(P)$ denote the slack matrix of the polytope P. He observed that: (i) every deterministic protocol of complexity k computing S gives rise to an EF of P of size at most 2^k, provided S is a 0/1-matrix; (ii) the nondeterministic communication complexity of the support matrix of S is a lower bound on the extension-complexity of P.

Recently, Faenza *et al.* [1] proved that the base-2 logarithm of the nonnegative rank of a matrix equals, up to a small additive constant, the minimum complexity of a randomized communication protocol (with nonnegative outputs) that computes the matrix *in expectation.* In particular, every EF of size r can be regarded as such a protocol of complexity $\log r + O(1)$ that computes the slack matrix in expectation.

2. Contribution

Our contribution is three-fold.

- First, we generalize the factorization theorem to *conic* EFs, that allow reformulating an LP through a conic program. In particular, this implies a factorization theorem for *semidefinite* EFs: the *semidefinite extension-complexity* of a polytope equals the *positive semidefinite rank* (shortly: *PSD rank*) of its slack matrix.
- Second, we generalize the tight connection between (linear) EFs and classical communication complexity found by Faenza *et al.* [1] to a tight connection between *semidefinite* EFs and *quantum* communication complexity. We show that any *rank-r PSD factorization* of a nonnegative matrix M gives rise to a one-way quantum protocol computing M in expectation that uses $\log r + O(1)$ qubits and, *vice versa*, that any one-way quantum protocol computing M in expectation that uses q qubits results in a PSD factorization of M of rank 2^q. Via the semidefinite factorization theorem, this yields a characterization of the semidefinite extension-complexity of a polytope in terms of the minimum

complexity of quantum protocols that compute the corresponding slack matrix in expectation.

Then, we give a complexity $\log r + O(1)$ quantum protocol for computing a nonnegative matrix M in expectation, whenever there exists a rank-r matrix N such that M is the entry-wise square of N. This result implies in particular that every d-dimensional polytope with 0/1 slacks has a semidefinite EF of size $O(d)$.

Finally, inspired by earlier work [8], we construct a $2^n \times 2^n$ matrix $M = M(n)$ that provides an exponential separation between classical and quantum protocols that compute M in expectation. On the one hand, our quantum protocol gives a rank-$O(n)$ PSD factorization of M. On the other hand, the nonnegative rank of M is $2^{\Omega(n)}$ because the nondeterministic communication complexity of the support matrix of M is $\Omega(n)$. This second part follows from an adaptation of the well-known result of Razborov [6] on the disjointness problem, see de Wolf [8].

- Third, we use the matrix $M = M(n)$ and a small-rank PSD factorization of M to prove a $2^{\Omega(n)}$ lower bound on the extension-complexity of the cut polytope $\mathrm{CUT}(n)$. That is, *every* (linear) EF of the cut polytope has an exponential number of inequalities. Via reductions, we infer from this: (i) an infinite family of graphs G such that the extension-complexity of the corresponding stable set polytope $\mathrm{STAB}(G)$ is $2^{\Omega(n^{1/2})}$, where n denotes the number of vertices of G; (ii) that the extension-complexity of the TSP polytope $\mathrm{TSP}(n)$ is $2^{\Omega(n^{1/4})}$.

  In addition to settling simultaneously the open problems of Yannakakis [9] and Rothvoß [7] described above, our results provide a lower bound on the extension-complexity of stable set polytopes that goes well beyond what is implied by a conjecture of Huang and Sudakov [4].

  Finally, we remark that although our lower bounds are strong, unconditional and apply to explicit polytopes that are well-known in combinatorial optimization, they have very accessible proofs.

We would like to point out that some of our results were also obtained (see [2] for a detailed account) by J. Gouveia, P. Parillo and R. Thomas [3]. However, this does not apply to our main results.

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Simple Push-Relabel Algorithms for Matroids and Submodular Flows

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(joint work with Zoltán Miklós)

1. Introduction

Push-relabel algorithms, unlike augmenting path type algorithms, use only small, local steps. In selecting the current element where the next local step is to be performed, they use a control parameter $\Theta : S \to \{0, 1, 2, \dots\}$ called a **level** (or distance) **function**. In the present work simple push-relabel algorithms are developed for matroid partition, for membership in a matroid polytope, and for submodular flow feasibility. The previous algorithms relied on a selection rule based on a consistent ordering of the elements which is a counterpart of the lexicographic rule of Schönsleben [5]. The new push-relabel algorithms do not use the consistency rule and the proof of strong polynomiality becomes much simpler. The true role of the consistency rule is that, though not needed for strong polynomiality, it improves the complexity of the algorithm by one order of magnitude.

Acknowledgements The authors received a grant (no. CK 80124) from the National Development Agency of Hungary, based on a source from the Research and Technology Innovation Fund.

2. Matroid partition

Let $M_1 = (S, \mathcal{B}_1), M_2 = (S, \mathcal{B}_2), \dots, M_k = (S, \mathcal{B}_k)$ be k matroids on an n-element ground-set S. We say that a subset $F \subseteq S$ is **coverable** if $F \subseteq B_1 \cup \cdots \cup B_k$ for some $B_i \in \mathcal{B}_i$ $(i = 1, \dots, 2)$. We construct a push-relabel algorithm for finding a largest coverable subset. Previously, Edmonds and Fulkerson [2] developed an augmenting path type algorithm for this purpose and proved that the largest cardinality of a coverable subset of S is equal to $\min\{\sum_i r_i(Z) + |S - Z| : Z \subseteq S\}$. The optimality criteria are as follows.

$$S - Z \subseteq \cup_i B_i \tag{1}$$

$$B_i \cap B_j \cap Z = \emptyset \text{ for } 1 \le i < j \le k \tag{2}$$

$$B_i \cap Z \text{ spans } Z \text{ in } M_i \text{ for } i = 1, \dots, k. \tag{3}$$

We show how the push-relabel technique can be used for finding a subset Z and k basis $B_i \in \mathcal{B}_i$ satisfying the three optimality criteria. At the beginning, Θ

is identically 0. At an intermediate stage of the algorithm, we are given M_i-bases B_i for $i = 1, \dots, k$ and a level function $\Theta : S \to \{0, 1, \dots, n = |S|\}$ for which the following level properties hold.

(L1) $\Theta(u) = 0$ holds for every $u \in S$ covered by more than one of the bases B_i.
(L2) $\Theta_{\min}(C_i(B_i, u)) \geq \Theta(u) - 1$ holds for every $u \in S - B_i$.

The algorithm terminates when one of the following **stopping rules** occurs.

(A) $S = B_1 \cup \cdots \cup B_k$.
(B) There is an empty level set L_j so that every element under j is covered.

There are two basic operations at an uncovered element s. Lifting s means that $\Theta(s)$ is increased by 1. A **basis-change** at s means that we take a basis B_i along with an element $t \in C_i(B_i, s) - s$ and replace B_i with $B_i - t + s$.

The algorithm runs as follows. At a general step, assuming that neither of the stopping rules holds, we select an uncovered element s for which $\Theta(s) \leq n - 1$. If there is a basis B_i $(1 \leq i \leq k)$ and an element $t \in C_i(B_i, s) - s$ for which $\Theta(t) = \Theta(s) - 1$, perform a basis-change by replacing B_i with $B_i - t + s$. If no such a B_i and t exist anymore, lift s.

The algorithm terminates when lifting s leaves an empty level set such that all elements under s is covered. In this case, **(B)** holds. The other way of termination occurs when after the current basis-change every element is covered in which case **(A)** holds.

3. Testing membership in a matroid polytope

Let $M = (S, r)$ be a matroid. The matroid (or independence) polytope $P(r)$ of M is the convex hull of the characteristic vectors of independent sets of M. The base polytope $B(r)$ of M is the convex hull of the characteristic vectors of bases of M. Edmonds proved the following polyhedral descriptions:

$$P(r) = \{x \in \mathbf{R}^S : x \geq 0 \text{ and } \widetilde{x}(Z) \leq r(Z) \text{ for every } Z \subseteq S\}. \tag{4}$$

$$B(r) = \{x \in \mathbf{R}^S : x \geq 0 \text{ and } \widetilde{x}(Z) \leq r(Z) \text{ for every } Z \subseteq S,\ \widetilde{x}(S) = r(S)\}. \tag{5}$$

$P(r)$ and $B(r)$ are often called the matroid (or independence) polyhedron and the base polyhedron of M, respectively.

Cunningham [1] developed a strongly polynomial algorithm to test if a given vector g belongs to $P(r)$. He also solved the more general problem when g is not in $P(r)$ and one is interested in finding a subset most violating (4) along with an element $x \leq g$ of $P(r)$ for which $\widetilde{x}(S)$ is maximum. His approach uses shortest augmenting paths and also the technique of lexicographic selection rule introduced by Schönsleben [5]. The lexicographic rule turned out to be an unavoidable device in all combinatorial algorithms concerning submodular frameworks. It was adapted to push-relabel algorithms as well.

We describe a simple push-relabel algorithm for the matroid membership problem that does not use the lexicographic rule. The algorithm works for the slightly more general problem when a specified upper bound $g : S \to \mathbf{R}_+$ is given and we are interested in finding a member $x \in P(r)$ for which $x \leq g$ and $\widetilde{x}(S)$ is maximum. Clearly, g belongs to $P(r)$ if and only if this maximum is $\widetilde{g}(S)$.

Theorem 3. *Let $M = (S, r)$ be a matroid and $g : S \to \mathbf{R}_+$ function. Then*

$$\max\{\widetilde{x}(S) : x \leq g,\ x \in P(r)\} = \min\{r(Z) + \widetilde{g}(S - Z)\}. \tag{6}$$

Proof. We call an element $x \in P(r)$ **feasible** if $x \leq g$. For a subset $Z \subseteq S$ and for a feasible $x \in \mathbf{R}$, one has

$$\widetilde{x}(S) = \widetilde{x}(Z) + \widetilde{x}(S - Z) \leq r(Z) + \widetilde{g}(S - Z) \tag{7}$$

from which max $\leq$ min follows. In the estimation (7), equality holds if and only if the following optimality criteria are met.

$$\widetilde{x}(Z) = r(Z) \tag{8}$$

$$\widetilde{x}(S - Z) = \widetilde{g}(S - Z). \tag{9}$$

We shall prove the theorem by developing an algorithm that computes a feasible x and a subset $Z \subseteq S$ satisfying the optimality criteria. A convex combination of bases of M will be described by a coefficient function $\lambda : \mathcal{B} \to \mathbf{R}_+$ for which $\sum[\lambda(B) : B \in \mathcal{B}] = 1$. The element of $B(r)$ defined by λ is $x_\lambda = \sum[\lambda(B)\underline{\chi}_B : B \in \mathcal{B}]$. Clearly, a non-negative vector x belongs to $P(r)$ if and only if there is a convex combination x_λ of bases such that x_λ covers x in the sense that $x_\lambda \geq x$. We say that a basis B is **λ-active** (or simply **active**) in the convex combination x_λ if $\lambda(B) > 0$. By a theorem of Charathodory, every element of $B(r)$ can be expressed as a convex combination of at most n bases. An element $s \in S$ is **g-larger, g-smaller** or **neutral** according to whether $g(s) > x_\lambda(s)$, $g(s) < x_\lambda(s)$ or $g(s) = x_\lambda(s)$.

Beside a convex combination x_λ of bases, the algorithm maintains a level function $\Theta : S \to \{0, 1, \ldots, n\}$. In the following **level properties** we use again the notation $\Theta_{\min}(X) := \min\{\Theta(v) : v \in X\}$ for $X \subseteq S$.

(L1) $\Theta(u) = 0$ holds for every g-smaller $u \in S$.
(L2) $\Theta_{\min}(C(B, u)) \geq \Theta(u) - 1$ holds for every λ-active basis B and for every $u \in S - B$.

The algorithm terminates when one of the following **stopping rules** occurs.

(A) There is no g-larger element of S.
(B) There is an empty level set L_j so that there is no g-larger element under j.

Basic operations: push and lift Let s be a g-larger element for which $\Theta(z) \leq n-1$. Lifting s means again that we increase $\Theta(s)$ by 1. Push is performed at s when there is an active basis B not containing s for which $\Theta_{\min}(C(B, s)) = \Theta(s) - 1$. Let $t \in C(B, s)$ for which $\Theta(t) = \Theta(s) - 1$, let $B' = B - t + s$, and define $\Delta := \min\{g(s) - x_\lambda(s), \lambda(B)\}$. A **push** decreases $\lambda(B)$ by Δ and increases $\lambda(B')$

by Δ. A push is called **neutralizing** if $\Delta = g(s) - x_\lambda(s)$. In this case s becomes neutral. A non-neutralizing push does not change the number of active bases while a neutralizing push either preserves or increases this number by 1. Note that the only element that may become g-larger after a push operation is t and t is under s. This observation will be used in estimating the number of steps.

Treating a g-larger element s with $\Theta(s) \leq n-1$ means that we apply push operations at s as long as possible. No more push is possible at s when either the last push at s was neutralizing or else when there is no more active basis B not containing s for which $\Theta_{\min}(C(B,s)) = \Theta(s) - 1$. In the latter case, lift s.

The algorithms runs as follows. As long as neither of the stopping rules holds select a g-larger node s for which $\Theta(s) \leq n-1$ and $\Theta(s)$ is maximum (the highest level rule), and treat s. The algorithm terminates either when after a push there are no more g-larger elements, that is, **(A)** holds, or else, when after a lift every g-larger element is in L_n, in which case **(B)** holds.

The algorithm teminates after at most $O(n^6)$ basic operations. If the complexity of the required subroutine that determines for a basis B and an element $s \in S - B$ if there is an element $t \in C(B,s) - s$ for which $\Theta(t) = \Theta(s) - 1$ is γ, then the overall complexity of the algorithm is $O(\gamma n^6)$.

A similar approach gives rise to a simple push-relabel algorithm for polymatroid intersection and for submodular flow. (For the full report, please contact A. Frank.)

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Polyhedra with the Integer Carathéodory Property

Dion Gijswijt

(joint work with Guus Regts)

In this talk, we show the following. Any nonnegative integer combination of (incidence vectors of) bases of a matroid can be written as such a combination using at most n bases, where n is the size of the ground set. This answers a question raised by Cunningham [5] in his paper on testing membership in base polyhedra, see also [11, 7]. For earlier work on this problem, see [9, 3].

The proof uses polyhedral methods and extends to other classes of integer polyhedra, such as: polymatroids, polyhedra defined by TU systems and projections of these, see [8] for full proofs.

A polyhedron $P \subseteq \mathbb{R}^n$ has the *integer decomposition property*, introduced by Baum and Trotter [1], if for every positive integer k, every integer vector in kP is the sum of k integer vectors in P. Equivalently, every $\frac{1}{k}$-integer vector $x \in P$ is a convex combination

$$x = \lambda_1 x_1 + \cdots + \lambda_t x_t, \quad x_i \in P \cap \mathbb{Z}^n, \lambda_i \in \tfrac{1}{k}\mathbb{Z}. \tag{1}$$

Examples of such polyhedra include: stable set polytopes of perfect graphs, polyhedra defined by totally unimodular matrices and (poly)matroid base polytopes.

It is worth pointing out the relation with Hilbert bases. Recall that a finite set of integer vectors H is called a *Hilbert base* if every integer vector in the convex cone generated by H, is an integer sum of elements from H. Hence if P is an integer polytope and $H := \{\binom{1}{x} \mid x \in P \text{ integer}\}$, then P has the integer decomposition property, if and only if H is a Hilbert base.

Let P be a polyhedron with the integer decomposition property. It is natural to ask for the smallest number T, such that we can take $t \leq T$ in (1) for every k and every $\frac{1}{k}$-integer vector $x \in P$. We denote this number by $\mathrm{cr}(P)$, the *Carathéodory rank* of P. Clearly, if P is a polytope, $\mathrm{cr}(P) \geq \dim(P) + 1$ holds, since P is not contained in the union of the finitely many affine spaces spanned by at most $\dim(P)$ integer vectors in P.

Cook et al. [4] showed that when H is a Hilbert base generating a pointed cone C of dimension $n \geq 1$, every integer vector in C is the integer linear combination of at most $2n - 1$ different elements from H. For $n > 1$, this bound was improved to $2n - 2$ by Sebő [11]. By the above remark, this implies that $\mathrm{cr}(P) \leq 2\dim(P)$ holds for any polytope P of positive dimension.

Bruns et al.[2] gave an example of a Hilbert base H generating a pointed cone C of dimension 6, together with an integer vector in C that cannot be written as a nonnegative integer combination of less than 7 elements from H. Their example yields a $0-1$ polytope with the integer decomposition property of dimension 5 but with Carathéodory rank 7, showing that $\mathrm{cr}(P) = \dim(P) + 1$ does not always hold.

In this talk we prove that if P is a (poly)matroid base polytope or if P is a polyhedron defined by a totally unimodular matrix, then P and projections of P satisfy the inequality $\mathrm{cr}(P) \leq \dim(P)+1$. For matroid base polytopes this answers a question of Cunningham [5] whether a sum of bases in a matroid can always be written as a sum using at most n bases, where n is the cardinality of the ground set (see also [11, 7]).

In our proof we use the following strengthening of the integer decomposition property, inspired by Carathéodory's theorem from convex geometry. We say that a polyhedron $P \subseteq \mathbb{R}^n$ has the *Integer Carathéodory Property* (notation: ICP) if for every positive integer k and every integer vector $w \in kP$ there exist affinely independent $x_1, \ldots, x_t \in P \cap \mathbb{Z}^n$ and $n_1, \ldots, n_t \in \mathbb{Z}_{\geq 0}$ such that $n_1 + \cdots + n_t = k$

and $w = \sum_i n_i x_i$. Equivalently, the vectors x_i in (1) can be taken to be affinely independent. In particular, if P has the ICP, then $\mathrm{cr}(P) \leq \dim P + 1$.

It is implicit in [4, 11] that the stable set polytope of a perfect graph has the ICP since a 'greedy' decomposition can be found, where the x_i are in the interior of faces of strictly decreasing dimension, and hence are affinely independent.

Consider the class $\mathcal{P}$ of rational polyhedra $P \subseteq \mathbb{R}^n$ (for some n) satisfying the following condition:

$$\text{For any } a, b \in \mathbb{Z}_{\geq 0} \text{ and } w \in \mathbb{Z}^n \quad \text{the intersection } aP \cap (w - bP) \text{ is box-integer.} \tag{2}$$

It is not hard to see that every $P \in \mathcal{P}$ has the integer decomposition property and that the class $\mathcal{P}$ is closed under taking faces and taking the intersection with an integral box. The main result is the following theorem.

Theorem 4. *If $P \in \mathcal{P}$, then any projection of P has the Integer Carathéodory Property.*

Examples of polyhedra that belong to $\mathcal{P}$ are the following: polyhedra defined by a TU-matrix A (since then $[A^\mathsf{T} - A^\mathsf{T} I - I]^\mathsf{T}$ is TU as well the intersection is box-integer), and polymatroid base polytopes (by Edmonds matroid partitioning theorem [6]).

Gammoids form a subclass of so-called strongly base orderable matroids. It is known that for any two strongly base orderable matroids, the common base polytope has the integer decomposition property (see [10]). Since gammoids are projections of flow polyhedra, they have the ICP. It is an open question whether this generalizes to intersections of other matroids.

Question 1. *Does the intersection of two base polytopes of strongly base orderable matroids have the ICP?*

In [11] Sebő asks whether the Carathéodory rank of the r-arborescence polytope can be bounded by the cardinality of the ground set. An r-arborescence is a common base of a partition matroid and a graphic matroid.

Question 2. *Does the r-arborescence polytope have the ICP?*

Finally, we wonder whether for matroids a more intuitive greedy approach for finding a decomposition into affinely independent vectors could work. We say that polytope P admits *greedy decomposition* if for every positive integer k and every $w \in kP$ integral, there is an integral $x \in P$ such that

$$\max\{\lambda \geq 0 \mid w - \lambda x \in (k - \lambda)P\}$$

is integer.

Question 3. *Does the spanning tree polytope admit greedy decomposition?*

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Flows in matroids and related problems

Bertrand Guenin

1. Flows in graphs and matroids.

Let G be a graph with a set of *demand* edges $\Sigma \subseteq E(G)$. Edges $E(G) \setminus \Sigma$ are the *capacity* edges. Every edge e is assigned a non-negative weight w_e. The weight of a demand edge represents the amount of flow that needs to be carried between the endpoint of that edge while the weight of a capacity edge indicates the total flow that can be carried by that edge. Let $\mathcal{C}$ be the set of all circuits of G that contain exactly one edge of Σ. A (G,Σ,w)-*flow* is an assignment $y_C \geq 0$ for every $C \in \mathcal{C}$ that satisfies the following conditions:

$$\sum (y_C : e \in C \in \mathcal{C}) = w_e \qquad (e \in \Sigma), \tag{1}$$

$$\sum (y_C : e \in C \in \mathcal{C}) \leq w_e \qquad (e \notin \Sigma), \tag{2}$$

where (1) guarantees that all demands are met while (2) ensures that no capacity is exceeded. A necessary condition for the existence of a (G,Σ,w)-flow is that for every cut B the total demand across the cut does not exceed the total capacity across the cut, i.e. $w(C \cap \Sigma) \leq w(B \setminus \Sigma)$. This is known as the *cut condition.* Weights w of G are said to be *Eulerian* if $w(B)$ is even for every cut B of G.

Consider a signed graph (G,Σ) with edge weights w, and assume that the cut-condition holds. We may then ask,

(Q1) Does there exist an integer flow?
(Q2) Does there exist a fractional flow?
(Q3) Assuming the weights w are Eulerian, does there exist an integer flow?

We may replace circuits C of G with circuits of a binary matroid M in the aforementioned definition of flow. Then any assignment $y_C \geq 0$ for every $C \in \mathcal{C}$ that satisfies (1) and (2) is an (M,Σ,w)-flow. We extend the cut-condition and the Eulerian condition to M by replacing cuts of G by cocycles of M. Thus we may also ask (Q1)-(Q3) in the context of binary matroids.

A signed graph (G,Σ) admits three possible minor operations: deleting an edge, contracting an edge not in Σ, and replacing Σ by $\Sigma\triangle B$ for some cut B of G. An *odd-K_n* is the signed graph $\big(K_n, E(K_n)\big)$. A signed matroid is a pair (M,Σ) where M is a binary matroid and $\Sigma \subseteq E(M)$. The definition of minors for signed graphs extend naturally to signed matroids.

2. Survey of results and conjectures

Consider a signed graph (G,Σ) with weights w, and assume that the cut-condition holds. If (G,Σ) has no odd-K_4 minor, then there exists an integer flow [9]. If (G,Σ) has no odd-K_5 minor, then there exists a fractional flow [4]. This was further strengthened by showing that if in addition the weights are Eulerian then there exists an integer flow [3]. Hence, (Q1)-(Q3) are well understood in the case of signed graphs.

Consider a signed matroid (M,Σ) with weights w, and assume that the cut-condition holds. Seymour [9] showed that if (M,Σ) does not have $\big(M(K_4),E(K_4)\big)$ as a signed minor, then there exists an integer flow. For a short proof see [6]. Seymour's *flowing* conjecture predicts that if (M,Σ) does not contain any of three special signed minors, then there exists a fractional flow [10]. Seymour's *cycling* conjecture predicts that if (M,Σ) does not contain any of four special signed minors and the weights are Eulerian, then there exists an integer flow [10]. The flowing and cycling conjectures remain open even for very special cases. Hence, (Q2) and (Q3) are unresolved in the case of signed matroids. It is straightforward to show that it suffices to consider single commodity flows for both the flowing and cycling conjectures. Hence, in the following discussion we will assume that we are restricting ourself to this special case.

A cycle $C \subseteq E(G)$ of a signed graph (G,Σ) is even if $|C\cap\Sigma|$ is even. A cut $\delta(U)$ of a graft (G,T) is even if $|U\cap T|$ is even. The set of all even-cycles of a signed graph (resp. even-cut of a graft) forms the cycles of a binary matroid, called the even-cycle matroid (resp. even-cut matroid) [11]. Denote by $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\mathcal{M}_4$ the class of even-cycle, even-cut, duals of even-cycles, and duals of even-cut matroids respectively. It is shown in [5] that the flowing conjecture holds for each of $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\mathcal{M}_4$. A special case of the cycling conjecture for $\mathcal{M}_1$ was proved recently [1]. The cycling conjecture remains open for each of $\mathcal{M}_2,\mathcal{M}_3$ and $\mathcal{M}_4$. The cycling conjecture for these last two classes both imply the 4-color theorem.

3. Proof strategy for the 1-flowing conjecture

As the cycling conjecture contains several wide ranging generalization of the 4-coloring theorem, we focus our attention on the flowing conjecture. As previously mentioned it suffices to consider the case where we have a single commodity. We can weaken the conjecture by requesting that the max-flow min-cut relation hold for every commodity (rather than a fixed commodity). The resulting conjecture is known as the 1-flowing conjecture. The 1-flowing conjecture predicts that the aforementioned minimax relation holds for all binary matroids that have no minor in $\mathcal{S} := \{AG(3,2), T_{11}, T_{11}^*\}$. Together with Cornuéjols [2] we showed that any counterexample to the flowing conjecture is "highly connected".

Our working hypothesis for the flowing conjecture is that a sufficiently connected binary matroid M that has no minor in $\mathcal{S}$ must be in one of the classes $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ or belongs to a "thin" (highly structured) class of matroids. To prove such a result we will require an excluded minor characterization of even-cycle and even-cut matroids. Alas this in itself appears to be a very challenging problem. In the remainder of the abstract we discuss the easier, yet still open, problem of recognizing if a binary matroid (given by its matrix representation) is an even-cycle matroid.

4. The recognition problem

By a *representation* of an even-cycle matroid M, we mean a signed graph whose even-cycles are equal to the cycles of M. We say that N is a *stabilizer* if for every connected even-cycle matroid M that contains N as a minor, a representation of N extends to at most one representation of M (up to equivalence). Therefore, if N is a stabilizer, then the representations of N completely determine the representations of M. In [8] we gave sufficient conditions for a matroid to be a stabilizer. In particular, we show that any matroid that is not "close to being graphic" is a stabilizer.

A strategy for recognizing if a binary matroid M is an even-cycle matroid is as follows. Suppose M is a binary matroid. If it is graphic, then it is an even-cycle matroid. Otherwise we can find a minimally non-graphic minor N of M. If N were a stabilizer, then we could construct all the representations of M from the representations of N. If any such representation exists, then M is an even-cycle matroid, otherwise it is not. Unfortunately, minimally non-graphic matroids are not stabilizers under our current definition of equivalence. We plan to rectify this by modifying our definition of equivalence. The key difficulty is the presence of *blocking pairs* in some of the representations of N. (A blocking pair in a signed graph is a pair of vertices that intersect every odd circuit.) Of help should be a recent result that characterizes the set of all possible blocking pairs in a signed graph [7].

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Combinatorial Optimization in Chip Design

Stephan Held
(joint work with Berit Braun)

Chip design has a long history as an application field for combinatorial optimization [4]. A central problem, which has to be solved several million times per chip, is the distribution of an electrical signal from a root r to a set S of sinks by a buffered interconnect. In the repeater tree problem a topology for the interconnect is to be found that is embedded in the plane and that will later be filled with repeaters.

Following the definition in [1, 2], an instance consists of a *source* r and a finite non-empty set S of *sinks* with locations $Pl : \{r\} \cup S \to \mathbb{R}^2$, a *delay bound* $a_s \in \mathbb{R}$ for every sink $s \in S$, and two numbers $c, d \in \mathbb{R}_{>0}$. A *feasible solution* is a rooted tree $T = (V(T), E(T))$ with vertex set $\{r\} \cup S \cup I$ where I is a set of $|S|-1$ vertices, with embedding $Pl : I \to \mathbb{R}^2$, such that r is the root of T and has exactly one child, the elements of I are the internal vertices of T and have exactly two children each, and the elements of S are the leaves of T. Associated with a solution are its length

$$l(T) := \sum_{(u,v)\in E(T)} ||u - v||,$$

where $||u-v|| := ||Pl(u) - Pl(v)||_1$, and the delay $\delta_T(s)$ from r to each sink $s \in S$:

$$\delta_T(s) := \sum_{(u,v)\in E[r,s]} d||u - v|| - c(|E[r, s]| - 1).$$

The delay grows linearly with the path-length through the tree and with every bifurcation on the path, which adds an electrical capacitance and, thus, delay.

Two conflicting objectives in the computation of topologies are the minimization of the length $l(T)$ and the maximization of the slack $\sigma(T) := \min\{0, \min_{s\in S} \sigma(T,s)\}$, where $\sigma(T,s) := a_s - \delta_T(s)$.

The minimization of the length is equivalent to the Steiner tree problem. In [2] we describe a generic greedy algorithm that works like a PRIM-heuristic for the length minimization, achieving a 3/2-approximation in the ℓ_1-norm due to Hwang's theorem [6]. Furthermore, it solves the slack maximization problem neglecting the length. However, in practice neither of these extreme cases is useful and a variant of the greedy algorithm serves just as an effective heuristic for fast and light topologies [1].

We present a bicriteria approximation algorithm that, given a topology T_σ maximizing $\sigma(T)$ and a constant $\alpha \geq 1 + \frac{4\cdot c}{\min_{s\in S}(d(T_\sigma))}$, generates a new topology T_α with

$$\delta_{T_\alpha}(s) \leq \alpha\, \delta_{T_\sigma}(s), \text{ for all } s \in S, \tag{1}$$

and

$$l(T_\alpha) \leq \left(1 + O\left(\frac{1}{\alpha}\right)\right)\frac{3}{2}\text{Steiner}(\{r\}\cup S) + \frac{c}{d}O\left(\frac{n}{\alpha}\right) \tag{2}$$

with a running time bound of $O(|S|\log|S|)$, where Steiner$(\{r\}\cup S)$ is the length of a minimum Steiner tree for $\{r\}\cup S$ with respect to the ℓ_1-norm. It is an extension of an algorithm for the special case $c = 0$ that was given by Khuller et al. [7].

First, we compute a minimum spanning tree T_l for $\{r\}\cup S$ with a degree bound of four. This can be done in $O(|S|\log|S|)$ with respect to the ℓ_1-norm using a Delaunay-triangulation and point perturbation. Then we traverse T_l in a depth-first-search (DFS) order. Thereby, T_α is constructed by tentatively connecting each traversed vertex $s \in S$ to its predecessor through a new internal vertex. Let $K(s) := \min\{|E(T_\sigma(r,s'))| - 1 \;:\; s' \in \text{ subtree containing } s\}$ be the minimum number of internal vertices on a r-s'-path in T_σ to a sink s' in the subtree of T_α containing s, and $\alpha' := \alpha - \frac{4\cdot c}{\min_{s\in S}(d(T_\sigma))}$. If the tightened delay constraint $\delta_{T_\alpha}(pred(s)) + d \cdot ||s - pred(s)|| + c \cdot K(s) > \tilde{\alpha} \cdot \delta_{T_\sigma}(s)$ is violated for s, we disconnect it before continuing the traversal and mark s as a root of a new subtree that will be connected later. The use of α' instead of α makes sure that we will be able to connect all subtree roots obeying (1).

During a backward step in the DFS-traversal, we will reconnect to a direct successor v of s if $\delta_{T_\alpha}(s) + c \cdot K(s) > \delta_{T_\alpha}(v) + d \cdot ||s - v|| + 4 \cdot c + c \cdot K(v)$. This way the delay to s can be improved and s might loose its status as a subtree root.

Finally, we build a balanced tree at $Pl(r)$ connecting all remaining subtree roots s by Huffman-Coding [5] leading to a tree T_α fulfilling (1). The length of T_α equals the length of T_l plus the length for connecting all subtree roots to r. An analysis of these extra lengths shows that (2) holds.

By refining repeater trees in conjunction with the global optimization of layer-assignments through a time-cost tradeoff algorithm using as costs the dual variables of a global routing solution, we were able to reduce the cycle time of a currently developed microprocessor by 18%.

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Weight classification in multiflow problems

Hiroshi Hirai

Let $G = (V, E)$ be an undirected graph with possible parallel edges and loops, and $S \subseteq V$ a set of terminals. An S-path is a path P whose ends s_P, t_P are distinct terminals in S. A (fractional) *multiflow* is a pair $(\mathcal{P}, \lambda)$ of a set P of S-paths and a nonnegative function $\lambda : \mathcal{P} \to \mathbf{Q}_+$ satisfying the capacity constraint $\sum_{e \in P \in \mathcal{P}} \lambda(P) \leq 1$ for $e \in E$. An *integer multiflow* is a multiflow f for which λ is integer-valued.

We are given a weight function $\mu : \binom{S}{2} \to \mathbf{Q}_+$ on the set $\binom{S}{2}$ of pairs on terminal set S. For a multiflow $f = (\mathcal{P}, \lambda)$, the *$\mu$-flow-value* $\mu \cdot f$ is defined by $\sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P)$. We consider the following weighted fractional and integer multiflow problems:

μ-FMP: Maximize $\mu \cdot f$ over all multiflows f in G, S.

μ-IMP: Maximize $\mu \cdot f$ over all integer multiflows f in G, S.

Our interest is to classify weight functions μ for which μ-FMP or μ-IMP has *nice* properties. For example, let $S = \{s, t\}$. Then μ-FMP is the maximum single commodity flow problem. Ford-Fulkerson's max-flow min-cut theorem [4] says that μ-FMP always has an integral optimal solution, and μ-IMP is also polynomial time solvable. Consider the case where $S = \{s, t, s', t'\}$, and $\mu(s, t) = \mu(s', t') = 1$ and the other weights are zero. Then μ-FMP is the 2-commodity flow problem. In this case, μ-FMP may not have an integral optimal solution, and the integer version μ-IMP is NP-hard [3]. Hu [10] proved that there exists a half-integral optimal solution in μ-FMP. On the other hand, the 3-commodity flow problem does not have such a half-integrality, more strongly, $1/k$-integrality for every k. The other interesting case is the all-one weight $\mu = \mathbf{1}$, which corresponds to the fractional/integral S-path packing problem. Lovász [18] and Cherkassky [1] independently proved that $\mathbf{1}$-FMP always has a half-integer optimal solution. Mader [20] established a min-max theorem for $\mathbf{1}$-IMP, and then extended it to a node-disjoint version [21]. Lovász [19] derived it from the matroid matching theory. Schrijver [22] explicitly formulated it as a linear matroid matching problem. So **1**-IMP is polynomial time solvable. These results motivate us to consider the following classification problems:

Fractionality problem: Classify weights μ for which μ-FMP has an optimal solution with bounded denominator.

Tractability problem: Classify weights μ for which μ-IMP is polynomial time solvable.

These classification problems have been considered by Karzanov [12, 13, 15] for (mainly) 0-1 weights.

Recent works [5, 7, 8, 9] settled both classification problems. We first describe the result on the fractionality problem. Define the *fractionality* of μ by the smallest positive integer k such that μ-FMP has a $1/k$-integral optimal solution for every graph G having terminal set S. If such a k does not exist, then the fractionality is defined to be infinity. Let us define a polyhedral set in $\mathbf{R}_+^S$ associated with μ:

$$T_\mu := \text{the set of minimal elements of } \{p \in \mathbf{R}_+^S \mid p(s) + p(t) \geq \mu(s,t)\ s,t \in S\}.$$

This polyhedral set T_μ is known as the tight span of μ, which was introduced by Isbell [11] and Dress [2]. The finiteness of the fractionality is determined by the dimension $\dim T_\mu$ of T_μ:

Theorem 5.
(1) [5] *If* $\dim T_\mu > 2$*, then the fractionality of* μ *is infinity.*
(2) [8] *If* $\dim T_\mu \leq 2$*, then the fractionality of* μ *is at most* 24.

A connection between tight spans and multiflows was discovered by Karzanov [16, 17] for metric weights μ. Based on it, [5] showed, for general weights, that the linear program dual to μ-FMP reduces to a certain location problem on the metric space (T_μ, l_∞), and that if $\dim T_\mu \leq 2$, then this location problem can be discretized, and gives a combinatorial min-max relation. The existence of such a combinatorial min-max relation is a necessary condition for the finiteness of the fractionality, which implies Theorem 1 (1). [7] further investigates this multiflow combinatorial duality, and [8] proved Theorem 1 (2) by a fractional version of the splitting-off, which was originally devised by [6] for a special case.

We next describe the result on the tractability problem. A weight μ is said to be a *tree metric* if there exist a tree Γ and a map $\phi : S \to V(\Gamma)$ such that $\mu(s,t) = d_\Gamma(\phi(s), \phi(t))$ for $s,t \in S$, where d_Γ is the shortest path metric of Γ. Also, μ is said to be a *truncated tree metric* if there exist a tree metric μ and $r : S \to \mathbf{Q}_+$ such that $\mu(s,t) = \max\{0, \mu(s,t) - r(s) - r(t)\}$ for $s,t \in S$.

Theorem 6 ([9])**.**
(1) *If* μ *is a truncated tree metric, then* μ*-IMP is in P.*
(2) *If* μ *is not a truncated tree metric, then* μ*-IMP is NP-hard.*

Actually Theorem 1 (1) follows from the polynomial time solvability of the minimum cost version of μ-IMP with tree metric μ. [9] showed that this problem has a combinatorial min-max relation extending Mader's edge-disjoint S-path theorem [20] and its mincost generalization by Karzanov [14, 15].

Acknowledgments

The author is partially supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan, by Aihara Project, the FIRST program from JSPS, and by the Global COE "The Research and Training Center for New Development in Mathematics."

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Weighted Linear Matroid Parity

Satoru Iwata

The matroid parity problem [3] was introduced as a common generalization of matching and matroid intersection problems. In the worst case, it requires an exponential number of independence oracle calls [2, 5]. However, Lovász [4, 5, 6] showed that the problem is solvable in polynomial time if the matroid in question is represented by a matrix. Subsequently, efficient algorithms have been developed for this linear matroid parity problem [1, 8].

In this talk, we present a combinatorial, deterministic, strongly polynomial algorithm for its weighted version. The algorithm builds on a polynomial matrix formulation of the problem using Pfaffian and an augmenting path algorithm for the unweighted version by Gabow and Stallmann [1].

Independently of this work, Gyula Pap obtains the same result based on a different approach, which is also presented in this meeting.

Let A be a matrix of row-full rank with row set U and column set V. Assume that both $r = |U|$ and $n = |V|$ are even. The column set V is partitioned into pairs, called lines. Each $v \in V$ has its mate $\bar{v}$ such that $\{v, \bar{v}\}$ is a line. We denote by L the set of lines, and suppose that each line $\ell \in L$ has a weight $w_\ell \in \mathbb{R}$.

The linear dependence of the column vectors naturally defines a matroid $\mathbf{M}(A)$ on V. Let $\mathcal{B}$ denote its base family. A base $B \in \mathcal{B}$ is called a parity base if it consists of lines. As a weighted version of the linear matroid parity problem, we will consider the problem of finding a parity base of minimum weight, where the weight of a parity base is the sum of the weights of lines in it. We denote the optimal value by $\zeta(A, \Pi, w)$. This problem generalizes finding a minimum-weight perfect matching in graphs and a minimum-weight common base of a pair of linear matroids on the same ground set.

Associated with this minimum-weight parity base problem, we consider a skew-symmetric polynomial matrix $\hat{A}(\theta)$ in θ defined by

$$\hat{A}(\theta) = \begin{pmatrix} O & A \\ -A^\top & D(\theta) \end{pmatrix},$$

where $D(\theta)$ is a block-diagonal matrix in which each block is a 2×2 skew-symmetric polynomial matrix $D_\ell(\theta) = \begin{pmatrix} 0 & -\alpha_\ell \theta^{w_\ell} \\ \alpha_\ell \theta^{w_\ell} & 0 \end{pmatrix}$ corresponding to a line $\ell \in L$.

Assume that the coefficients α_ℓ are independent parameters (or transcendental indeterminates).

Then $\hat{A}(\theta)$ is nonsingular if and only if there is a parity base in $\mathbf{M}(A)$. The optimal value of the minimum-weight parity base problem is given by

$$\zeta(A, \Pi, w) = \sum_{\ell \in L} w_\ell - \deg_\theta \operatorname{Pf} \hat{A}(\theta).$$

For a base $B \in \mathcal{B}$, we construct an auxiliary graph $G_B = (V, F \cup L)$ with vertex set V and edge set $F \cup L$, where $F = \{(u,v) \mid u \in B,\ v \in V \setminus B,\ B - \{u\} \cup \{v\} \in \mathcal{B}\}$. A line $\ell = \{v, \bar{v}\}$ is called a source line if exactly one of its two vertices belongs to B. The fundamental circuit matrix C with respect to the base B is a matrix with row set B and column set $V \setminus B$ obtained by $C = A[U, B]^{-1} A[U, V \setminus B]$. The edges in F correspond to the nonzero entries in C.

The algorithm works on a matrix $C^\sharp$ obtained from C by attaching some columns and rows called transforms. It also uses an augmented graph $G^\sharp = (V^\sharp, F^\sharp \cup L)$ with vertex set $V^\sharp \supseteq V$, which includes some transforms and the vertices in V. The edges in $F^\sharp$ correspond to the nonzero entries in $C^\sharp$. The row set of $C^\sharp$ is denoted by $B^\sharp$, and the column set is $V^\sharp \setminus B^\sharp$.

In addition to the graph $G^\sharp$, the algorithm keeps a nested (laminar) collection $\Lambda = \{H_1, \dots, H_k\}$ of vertices in $V^\sharp$. Each member in Λ is called a blossom. It consists of lines and transforms. Each blossom H_i has its bud $b_i \in V \setminus H_i$ if it does not contain a source line. The vertices in H_i adjacent to b_i in $G^\sharp$ are called tips of H_i. The set of the tips of H_i is denoted by T_i.

A transform $v = \tau(b, s, t)$ is defined with reference to the bud b of a blossom H and a pair of its tips s and t. If $b \in B^\sharp$, then $s, t \in V \setminus B$, and $v \in V^\sharp \setminus B^\sharp$. The column vector of C corresponding to v is a linear combination of the columns indexed by s and t such that $C^\sharp_{bv} = 0$. Similarly, if $b \in V^\sharp \setminus B^\sharp$, then $s, t \in B$ and $v \in B^\sharp$. The row vectors of C corresponding to v is a linear combination of the rows indexed by s and t such that $C^\sharp_{vb} = 0$. The transform thus defined belongs to the blossom H.

For a blossom H and an edge $(u, v) \in F^\sharp$, we define $\eta(H, u, v)$ as follows. Let T denote the set of the tips of H. If one of u and v is a tip of H and the other one is outside H, then $\eta(H, u, v) = -1$. If one of u and v is a vertex in $H \setminus T$ and the other one is outside H, then $\eta(H, u, v) = 1$. Otherwise (if $|H \cap \{u, v\}| \neq 1$), we put $\eta(H, u, v) = 0$.

The algorithm maintains a potential $p : V \to \mathbb{R}$ and a nonnegative variable $q : \Lambda \to \mathbb{R}_+$. These are collectively called dual variables. For each edge $(u, v) \in F^\sharp$, we denote

$$Q_{uv} = \sum_{H \in \Lambda} \eta(H, u, v) q(H).$$

The dual variables are called feasible if they satisfy the following conditions.

(DF1): $p(v) + p(\bar{v}) = w_\ell$ for every line $\ell = \{v, \bar{v}\} \in L$,
(DF2): $p(v) - p(u) \geq Q_{uv}$ for every $(u, v) \in F^\sharp$.
(DF3): $p(v) = p(s)$ for any transform $v = \tau(b, s, t)$.

Given feasible dual variables p and q, we say that an edge $(u,v) \in F^\sharp$ is tight if $p(v) - p(u) = Q_{uv}$. Let F° denote the set of tight edges. Then an augmenting path in $G^\circ = (V^\sharp, F^\circ \cup L)$ is defined as follows. For a vertex $v \in V^\sharp$, let $v^\natural$ denote s if v is a transform $\tau(b,s,t)$, and v otherwise. A parity path is a sequence $v_0, v_1, \ldots, v_h$ of vertices in $V^\sharp$ such that $\{v_{i-1}^\natural, v_i^\natural\} \in L$ for odd i, and $(v_{i-1}, v_i) \in F^\circ$ or $(v_i, v_{i-1}) \in F^\circ$ for even i. A parity path $P = v_0 v_1 \cdots v_h$ is an augmenting path if it satisfies the following conditions.

- **(AP1):** The length h is odd.
- **(AP2):** The vertices v_0 and v_h belong to distinct source lines. No other vertices in P belong to source lines.
- **(AP3):** If a transform $\tau(b,s,t)$ is in P, then t is a vertex in P, but s is not.
- **(AP4):** For any blossom H that intersects P, the intersection forms an interval $v_i \cdots v_j$ in P. In addition, if H does not have a source line, then either v_{i-1} or v_{j+1} is the bud of H.
- **(AP5):** The subgraph $G^*[P]$ of $G^* = (V^\sharp, F^\circ)$ induced by P has a unique perfect matching.

Note that an augmenting path P does not necessarily form a path in G°.

The algorithm starts with splitting the weight w_ℓ into $p(v)$ and $p(\bar{v})$ for each line $\ell = \{v, \bar{v}\} \in L$. Then it executes the greedy algorithm for finding a base $B \in \mathcal{B}$ with minimum value of $p(B) = \sum_{u \in B} p(u)$. If B is a parity base, then B is obviously a minimum weight parity base. Otherwise, the algorithm proceeds iterations of primal and dual updates as follows. If an augmenting path P is found, then the algorithm updates the base B to the symmetric difference $B \triangle P^\natural$, where $P^\natural = \{v^\natural \mid v \in P\}$. This will reduce the number of source lines by two. Otherwise, the algorithm updates the dual variables so that new edges will be tight. The algorithm repeats this process until B becomes a parity base. Then B will be a minimum weight parity base, which is proved via the polynomial matrix formulation with a technique from combinatorial relaxation [7].

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Algorithms for Finding a Maximum Non-k-linked Graph

Yusuke Kobayashi
(joint work with Yuichi Yoshida)

A graph is said to be *k-linked* if it has at least $2k$ vertices and for any ordered k-tuples $(s_1, \ldots, s_k)$ and $(t_1, \ldots, t_k)$ of $2k$ distinct vertices, there exist pairwise vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i connects s_i and t_i for $i = 1, \ldots, k$. The k-linkedness has been well-studied by many graph theorists, and there are many results on relationships between the k-linkedness and the vertex-connectivity of graphs. From the algorithmic point of view, the k-linkedness has attracted attention because of similarities with the vertex-disjoint paths problem, which is one of the most important problems in computer science and algorithmic graph theory. In the *vertex-disjoint paths problem*, we are given a graph G and $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ called *terminals*, and the objective is to find pairwise vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i connects s_i and t_i for $i = 1, \ldots, k$. With the terminology of the vertex-disjoint paths problem, a graph is k-linked if and only if the vertex-disjoint paths problem has a solution for any choice of $2k$ terminals. In this talk, we consider the problem of finding a minimum number of vertices whose removal makes the graph non-k-linked, which can be stated as follows.

Max Non-k-Linked Induced Subgraph

Input: A graph $G = (V, E)$.
Problem: Find a vertex set $V_0 \subseteq V$ with maximum cardinality such that $G[V_0]$ (the subgraph induced by V_0) is not k-linked.

We mainly discuss the case of $k = 2$, which is interesting because of its relation to the problem of finding a maximum planar induced subgraph. By a classical result on the 2 vertex-disjoint paths problem [3], it is well-known that the graph is not 2-linked if and only if it cannot be embedded in a plane up to "3-separations". That is, the non-2-linkedness is a similar concept to the planarity. The problem of finding a maximum planar induced subgraph is an important problem in theoretical computer science, because it amounts to computing a measure for non-planarity of graphs. Max Non-2-Linked Induced Subgraph can also be regarded as a problem of computing a measure for non-planarity of graphs, which is one of our motivations for studying Max Non-2-Linked Induced Subgraph. In this talk, we show that Max Non-2-Linked Induced Subgraph can be solved in polynomial time. This result is surprising because most of all natural problems of computing measures for non-planarity, such as finding a maximum planar (induced) subgraph or computing the minimum number of crossings in an embedding in a plane, are known to be NP-hard (see [2]).

We now give some remarks on proof techniques for the polynomial solvability of Max Non-2-Linked Induced Subgraph. A natural approach to solve Max Non-2-Linked Induced Subgraph is to consider the problem of finding a maximum vertex set whose inducing subgraph contains no two vertex-disjoint paths connecting fixed terminal pairs. We call it Max 2-VDP-free Induced Subgraph, whose formal description is as follows.

Max 2-VDP-free Induced Subgraph

Input: A graph $G = (V, E)$ and distinct terminals $s_1, t_1, s_2, t_2 \in V$.
Problem: Find a vertex set $V_0 \subseteq V$ with maximum cardinality such that $\{s_1, t_1, s_2, t_2\} \subseteq V_0$ and the vertex-disjoint paths problem with terminal pairs (s_1, t_1) and (s_2, t_2) has no solution in $G[V_0]$.

We can easily see that by solving Max 2-VDP-free Induced Subgraph for every choice of the terminals s_1, t_1, s_2, t_2, we obtain a solution of Max Non-2-Linked Induced Subgraph. However, we show that Max 2-VDP-free Induced Subgraph is NP-hard. This result suggests that the above reduction does not work for solving Max 2-VDP-free Induced Subgraph. Therefore, we need another approach to Max Non-2-Linked Induced Subgraph.

An extended abstract of this talk appears in ESA 2011 and a preprint is [1].

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Grothendieck inequalities with rank constraint

Fernando Mário de Oliveira Filho

(joint work with Jop Briët and Frank Vallentin)

For an integer $r \geq 1$ and a matrix $A \in \mathbb{R}^{m \times n}$ consider the following optimization problem:

$$\mathrm{SDP}_r(A) = \max\left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_i \cdot y_j : x_i,\, y_i \in S^{r-1} \right\},$$

where $x \cdot y$ denotes the Euclidean inner product and $S^{r-1} = \{ x \in \mathbb{R}^r : x \cdot x = 1 \}$ is the $(r-1)$-dimensional unit sphere. We also allow $r = \infty$, in which case S^∞ is the set of all square-summable sequences of norm 1.

This is a semidefinite programming problem with an added rank constraint; setting $r = \infty$ amounts to removing the rank constraint. Notice then that if $A \in \mathbb{R}^{m \times n}$ then $\mathrm{SDP}_\infty(A) = \mathrm{SDP}_{m+n}(A)$ and this number can be efficiently approximated by means of semidefinite programming.

For each $r \geq 1$, let $K(r)$ be the smallest constant such that for all real matrices A we have

$$\mathrm{SDP}_r(A) \leq \mathrm{SDP}_\infty(A) \leq K(r)\, \mathrm{SDP}_r(A).$$

Grothendieck [4] proved that $K(1) < \infty$ and since then there have been many attempts to compute the exact value of $K(1)$. Up until recently, the best lower and upper bounds known for $K(1)$ were

$$1.676956\ldots \leq K(1) \leq 1.782213\ldots,$$

where the lower bounds is due to Davie [2] and Reeds [7] and the upper bound is due to Krivine [6]. Recently, Braverman, Makarychev, Makarychev, and Naor have announced that the upper bound of Krivine is actually strict.

Krivine's upper bound could be roughly sketched as follows: from an optimal solution x_1^*, $\ldots$, x_m^*, y_1^*, $\ldots$, y_n^* of $\mathrm{SDP}_\infty(A)$ one constructs new vectors $\tilde{x}_1$, $\ldots$, $\tilde{x}_m$, $\tilde{y}_1$, $\ldots$, $\tilde{y}_n$ having some prescribed inner products, and then one rounds these vectors to $+1$, -1 (that is, to vectors in S^0) using the random hyperplane rounding technique also used by Goemans and Williamson [3] in their approximation algorithm for the maximum-cut problem.

The random hyperplane technique itself can be thus described. One generates a vector $z \in \mathbb{R}^n$ by picking each entry of z independently from a normal distribution with mean 0 and variance 1. This can be seen as equivalent to picking a vector from S^{n-1} at random according to the uniform distribution. Then the result of rounding a vector $x \in S^{n-1}$ is

$$\frac{z \cdot x}{|z \cdot x|}$$

which is just the sign of $z \cdot x$.

In analyzing this procedure, one makes use of *Grothendieck's identity*: if x, $y \in S^{n-1}$ and $z \in \mathbb{R}^n$ is picked at random as above, then

$$\mathbb{E}\left[\frac{z \cdot x}{|z \cdot x|}\frac{z \cdot y}{|z \cdot y|}\right] = \frac{2}{\pi}\arcsin(x \cdot y).$$

In our work (cf. Brïet, Oliveira, and Vallentin [1]) we extend the idea of Krivine to be able to compute upper bounds for $K(r)$ for $r \geq 1$. Our main technical contribution is a matrix version of Grothendieck's identity, which can be so stated. Let $Z \in \mathbb{R}^{r \times n}$ be such that each of its entries if picked independently from a normal distribution with mean 0 and variance 1. Then if x, $y \in S^{n-1}$ we have

$$\mathbb{E}\left[\frac{Zx}{\|Zx\|} \cdot \frac{Zy}{\|Zy\|}\right] = \frac{2}{r}\left(\frac{\Gamma((r+1)/2)}{\Gamma(r/2)}\right)^2 (x \cdot y)\, {}_2F_1\left(\begin{matrix} 1/2, 1/2 \\ r/2+1 \end{matrix}; (x \cdot y)^2\right),$$

where ${}_2F_1$ is the hypergeometric series.

From our approach we obtain Krivine's upper bound for $r = 1$, and an upper bound due to Haagerup [5] for $r = 2$. For $r = 3$ we obtain

$$K(3) \leq 1.280812\ldots,$$

a result with applications in physics.

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On the integrality gap of hypergraphic Steiner tree relaxations

Neil Olver
(joint work with Michel X. Goemans, Thomas Rothvoß and Rico Zenklusen)

Let $G = (V, E)$ be a graph with a set $R \subseteq V$ of *terminals*, and edge costs $c : E \to \mathbb{R}_+$. The **NP**-hard Steiner tree problem asks for a connected subgraph of minimum cost that spans all the terminals.

Until recently, LP relaxations have only played a very limited role in the design of approximation algorithms for this problem. For the well-studied bidirected cut relaxation, no bound on the integrality gap better than 2 is currently known. But in a breakthrough result, Byrka, Grandoni, Rothvoß and Sanità showed that a different component-based LP, one of a family of equivalent "hypergraphic" relaxations, could be used to obtain improved approximation algorithms. They presented a $\ln(4) + \epsilon \approx 1.39$ approximation based on this relaxation. Curiously however, their analysis compares against the integral optimum, and they did not show a matching bound on the integrality gap of the LP (though they did show a bound of 1.55 using other methods).

The hypergraphic LP we use is the undirected component-based relaxation due to Warme [4]:

$$\text{(LP)}\qquad \begin{aligned} \min \quad & \sum_{C \in \mathcal{K}} x_C \operatorname{cost}(C) \\ & \sum_{C \in \mathcal{K}} x_C (|S \cap R(C)| - 1)^+ \leq |S| - 1 && \forall\, \emptyset \neq S \subseteq R \\ & \sum_{C \in \mathcal{K}} x_C (|R(C)| - 1) = |R| - 1 \\ & x_C \geq 0 && \forall\, C \in \mathcal{K}. \end{aligned}$$

Here, $\mathcal{K}$ is the set of all *full components* of the instance; these are subgraphs of G forming trees where all leaves are terminals and all internal nodes are non-terminals. For $C \in \mathcal{K}$, $\operatorname{cost}(C)$ is the sum of the edge costs of the component, and $R(C)$ is the set of terminals in C. This LP is easily seen to be a relaxation of the Steiner tree problem; notice in particular that if $R = V$, every full component is simply an edge, and the relaxation reduces to a standard formulation of the spanning tree polytope. It can be approximately solved (to within a $1 + \epsilon$ factor for any positive ϵ) in polynomial time, by restricting to full components of bounded size [1].

We show that indeed the integrality gap of (LP) (and hence any of the equivalent hypergraphic relaxations [3]) is bounded by $\ln(4)$. Our approach relies heavily on the theory of matroids and submodular functions, and in the process we obtain a number of structural insights into the hypergraphic LPs.

To obtain our bounds, we provide an iterative algorithm along the lines of Byrka et al. [2]. In each iteration, a component is chosen to be part of the integral solution, and is contracted for the purpose of the next iteration. However, whereas in [2] the LP is re-solved after each contraction, instead we update the LP solution of the previous iteration to be feasible in the new contracted instance. Once all of the terminals have been contracted together, the union of the contracted components will be a Steiner tree for the original instance. Roughly speaking, the goal is to show that the component to contract, as well as the modification to the LP solution, can be chosen in such a way that the decrease in cost of the LP solution is comparable to the cost of the contracted component.

In order to do this, it is necessary to answer the question: how can a solution x to (LP) be modified after a contraction in order to retain feasibility? Consider the projection π of a (feasible or not) solution x to (LP) onto the edges: $\pi(x) = z$ where $z_e = \sum_{C\in\mathcal{K}:e\in C} x_C$. Then the cost of solution x can be written as $\sum_{e\in E} c_e \pi(x)_e$; we can think of $\pi(x)_e$ as the capacity that x induces on edge e. Then for two vectors $x, x' \in \mathbb{R}^{\mathcal{K}}_{+}$ and a vector $y \in \mathbb{R}^{E}_{+}$, we say that x' *is a reduction of* x *by* y if $\pi(x') = \pi(x) - y$. For some x that is infeasible after a contraction step, we will only consider modifications that can be obtained by reducing x. The relationship between x and x' is somewhat complicated; reducing the capacity on an edge can cause components to split up.

Let x be a feasible solution to (LP), which will then be infeasible after contracting a component Q. Call a vector $y \in \mathbb{R}^{E}_{+}$ a *feasible capacity removal* if x can be reduced by y to obtain a feasible solution. Now consider $\mathcal{B}_Q$, the set of all feasible capacity removals which in addition are *minimal* (with respect to the standard partial order on vectors). Crucially, we show that these vectors form the base polytope of a polymatroid. We also precisely describe the rank function, and show that the polymatroid has a gammoid description (implying that the rank of a given set can be determined efficiently with a maximum flow computation). Requiring minimality is important in the definition of $\mathcal{B}_Q$; the set of all feasible capacity removals does not seem to be as well structured. Note however that after removing a vector in $\mathcal{B}_Q$, it may still be possible to remove more capacity and remain feasible; we consider this as a kind of cleanup step, and it is important for the analysis. Exploiting this structural understanding, as well as other tools involving submodular functions, we are able to show the following. Let K be a maximal subset of E such that each connected component of $(V, E \setminus K)$ contains precisely one terminal. Now let Q be a randomly chosen component, such that the probability that any particular component C is chosen is proportional to x_C. Then vectors $y^C \in \mathcal{B}_C$ (i.e., a minimal feasible removal vector upon contraction of C) with $\text{supp}(y^C) \subseteq K$ can be chosen for each component C so that $\mathbb{E}\{y_e^Q\} \geq \mathbb{P}\{e \in Q\}$ for every $e \in K$. For some perspective, notice that if this were possible with $K = E$ it would imply an integrality gap of 1, since then the decrease in cost of the LP solution would be at least the expected cost of the contracted component. Capacity in edges outside of K is removed only as part of a cleanup operation; after a component Q is picked and the LP solution is reduced by y^Q to obtain a solution x' feasible to the new instance, x' may possibly be reduced further by removing capacity in $E \setminus K$, without affecting feasibility. This potential extra decrease in the LP solution is important in obtaining the integrality gap bound, and is accounted for by means of an appropriate potential function. We show that upon choosing a random component Q to contract, the expected change in the potential at each iteration of the algorithm is bounded by the expected cost of the chosen component. By considering the specifics of the potential function, this already yields the integrality gap bound.

Algorithmically, the randomization can be easily avoided by choosing at each iteration a component with cost no larger than the change in potential in that step. The analysis discussed above implies that such a component always exists, and the gammoid structure of the feasible removal polymatroids allows such a component to be found efficiently via flow computations. As well as yielding the $\ln(4)$ integrality gap bound for general instances, our approach also easily yields a bound of 73/60 for quasi-bipartite instances (meaning an instance where every edge contains at least one terminal). Again, this matches the approximation factor obtained by Byrka et al. [2].

Obtaining a near-optimal solution to (LP) is, while polynomial, unfortunately very expensive. An advantage of our approach is that this LP only needs to be solved once, rather than after every iteration. In the quasi-bipartite case, we additionally show how to construct an optimal solution to the hypergraphic LP from an optimal solution to the bidirected cut relaxation (extending a non-constructive equivalence demonstrated by Chakrabarty et al. [3]). So in this case, we obtain a dramatically more efficient algorithm.

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Polynomial Time Graver Bases Methods for Integer Programming

Shmuel Onn

With colleagues Berstein, De Loera, Hemmecke, Lee, Romanchuk, Rothblum, and Weismantel, we developed a theory, described in my monograph [1] and video [2], which uses *Graver bases* to solve linear and nonlinear integer programs in variable dimension in polynomial time. It leads to polynomial time solutions of multidimensional table problems, multicommodity flows, n-fold integer programming, and stochastic integer programming. Moreover, this theory is universal and provides a new, variable dimension, parametrization of all of integer programming, and suggests a simple approximation hierarchy for every integer programming problem.

Our theory can be viewed as a culmination of the line of research on test set iterative methods for integer programming, providing the first polynomial time procedures for computing the Graver basis (which is a universal test set containing the universal Gröbner basis) and using it iteratively for optimization. We believe this is a very important and promising line of research in integer programming.

1. Universality and Parametrization. One way to view the universality and parametrization provided by our theory is via multidimensional tables. Consider linear optimization over three dimensional $3 \times m \times n$ tables with given line sums:

$$\min \left\{ wx \ : \ x \in \mathbb{Z}_+^{3\times m\times n} \, , \ \sum_i x_{i,j,k} = a_{j,k} \, , \ \sum_j x_{i,j,k} = b_{i,k} \, , \ \sum_k x_{i,j,k} = c_{i,j} \right\} \, .$$

Theorem. Every integer program can be put precisely into this form; and for any fixed parameter m, the problem can be solved in polynomial time with variable n.

2. Some Applications. Let us briefly mention a few applications. Precise statements, more details and further examples are in [1] and the references therein.

Corollary. For fixed $m_1, \ldots, m_k$, nonlinear optimization and data security problems over margined $m_1 \times \cdots \times m_k \times n$ tables with variable n are polytime solvable.

Corollary. For fixed l commodities and m suppliers, the separable convex integer multicommodity flow problem with variable n consumers is polytime solvable.

Corollary. Stochastic integer programming with n scenarios is polytime solvable.

3. Improved Complexity. Till lately, the polynomial running times of the algorithms underlying our theory were huge, with the degrees being the *Graver complexities* of the systems, very large and rarely known exactly; for instance, for $3 \times 4 \times n$ table problems, the running time was polynomial but at least $\Omega(n^{27})$.

Fortunately, very recently we found in [3] a drastic improvement, leading to cubic running time for any system. This makes the theory practical and leads to improved hierarchy of approximations for any integer program. Here is one result of [3] on nonlinear n-fold integer programming (see [3] and [4] for more details).

Theorem. The following n-fold integer programming problem defined by bimatrix A with variable entries, with separable convex p-piecewise affine objective function, is solvable in time cubic in n, linear in $\log(f,b,l,u)$, and polynomial in $\max|A_{i,j}|$,

$$\min\left\{f(x)\ :\ A^{(n)}x=b\,,\ l\leq x\leq u\,,\ x\in\mathbb{Z}^{nt}\right\}\ ,$$

where $A^{(n)}$ is the n-fold product of the $(r,s)\times t$ bimatrix $A=A^{(1)}$, defined by

$$A^{(n)}\quad:=\quad\begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_2 \end{pmatrix}\quad .$$

4. Hierarchy of Approximations for Integer Programming. One simple way of using our theory to set a *Graver approximation hierarchy* for any integer programming problem is to use the above universal table problem and the following hierarchy of increasingly better approximations of the true Graver basis $G^{m,n}$,

$$G_1^{m,n}\ \subseteq\ G_2^{m,n}\ \subseteq\ \cdots\ \subseteq\ G_d^{m,n}\ \subseteq\ \cdots\ \subseteq\ G_{g(m)}^{m,n}\ =\ G^{m,n}\ .$$

For fixed d the set $G_d^{m,n}$ can be computed and used in polynomial time with both m,n variable, and for fixed m, the true Graver basis is attained as $G_{g(m)}^{m,n}$ with $g(m)$ the Graver complexity of the bipartite graph $K_{3,m}$, see [1]. Experimentation shows that already $G_3^{m,n}$ seems to yield rapid convergence over random instances. Another Graver approximation hierarchy is suggested by the new algorithm of [3].

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Weighted linear matroid matching

Gyula Pap

An approach to construct a polynomial time algorithm is presented in the talk to solve the weighted linear matroid matching problem. A completely different approach for the same problem, proposed at this workshop by Satoru Iwata, has resulted in a polynomial time algorithm as well – both approaches have been discovered independently, and apparently at the same time. The coincidence has been discovered at the Combinatorial Optimization Workshop in Oberwolfach.

But first let us focus on the background, and relevant references in the theory of matroid matching.

Matroid matching has been introduced as an open problem by Lawler in 1976, as a highly non-trivial common generalization of matroid intersection and graph matching (see definition below). Without any assumption on the input matroid, the problem turned out to be of exponential time complexity in the oracle model, as seen by Lovász 1981, and independently by Jensen and Korte 1982. NP-hardness has been established by Schrijver via a direct reduction from stable sets. Thus our attention is towards instances for which the matroid matching problem is tractable. A major breakthourgh came with results of Lovász, 1980 showing the polynomial time solvability of matroid matching in case the matroid has a linear representation – a special case that is called linear matroid matching. Lovász also proposed further combinatorial applications, including Mader's disjoint S-paths, thus providing the first polynomial time algorithm for this problem. Scores of other applications have been investigated, including maximum genus embedding, Nebesky 1981, Furst, Gross, McGeoch 1988; parity-constrained orientations, Frank, Jordán, Szigeti 2001, Király, Szabó 2003; maximum triangle cactus, Szigeti 2003, minimum rigid pinning-down in the plane, Fekete 2007. The fastest algorithm for linear matroid matching to date is given by Gabow, Stallman, 1986, matched by a different algorithm by Orlin 2008. One should also mention a recent result by Lee, Sviridenko, Vondrák 2010, who gave a PTAS for matroid matching in an arbitrary matroid based on local search.

The linear matroid matching problem may be defined in several different ways, we consider the following definition. Consider a vectorspace V. A rank-2 subspace of V is called a *line*. Let E be a set of lines. The input of the problem is given by specifying a basis – a pair of vectors in V – for each line in E. A subset of lines $M \subseteq E$ is called a *matching* if $r(\cup M) = 2|M|$, i.e. if the lines are independent. The (unweighted) *linear matroid matching problem* is to maximize $|M|$ over all matchings with respect to V, E. This problem is solved by the following min-max formula of Lovász.

Theorem 7 (Lovász, 1980)**.** *The maximum cardinality of a matching with respect to V, E is equal to*

$$\min_{K,\pi}\ r(K) + \sum_{F\in\pi} \left\lfloor \frac{1}{2} r_{V/K}(F) \right\rfloor ,$$

where K is a linear subspace in V, π is a partition of E, and we are using the notation of $r_{V/K}(F) = r(K \cup F) - r(K)$.

A *weighted* generalization of the matroid matching problem is defined quite naturally in the following way. Besides V, E, we are also given a weight function $w : E \to \mathbb{R}_+$, and our goal is to find the maximum of $w(M) := \sum_{e\in M} w(e)$ over all matchings M. Given the similarity between Lovász' Theorem, and the Berge-Tutte formula for graph matching, one might expect that also the weighted problem is tractable, like the weighted graph matching problem is solvable due to results of Edmonds. However, the weighted linear matroid matching problem stood open for all those decades, despite all those similarities. A good explanation for the "delay" is that no simple LP description is known for the convex hull of matchings, and still no such description seems to follow from neither of this approach, nor from Iwata's approach.

Results on weighted linear matroid matching have been few and far between. Graph matching and matroid intersection are solvable also in the weighted setting. If the input matroid is a gammoid, then Tong, Lawler, Vazirani 1984 have shown it to be solvable by a reduction to graph matching. By polynomial matrices, Camerini, Galbiati, Maffioli 1992 have derived a randomized weakly polynomial algorithm. A PTAS in case of strongly base orderable matroids is given by Soto 2011. As it seems, these results do not point in the direction of a general approach for linear matroids.

The trouble with finding a linear programming description comes from the difficulty of decomposing the expression in the minimum in Lovász' min-max formula into the linear combination of simpler looking valid inequalities. While Iwata's approach circumvents this difficulty by using a different kind of upper bound (an algebraic one), the approach proposed in this talk is to determine an extended LP description by introducing auxiliary variables to the problem. Thus we would describe a polytope in a higher dimension such that its projection to the space of $\mathbb{R}^E$ would be equal to the convex hull of matchings.

To describe the extended LP we need some definitions. For a matching $M \subseteq E$, we define the matching incidence vector $x^M := \chi_M \in \mathbb{R}^E$, and for a pair of subspaces $L < K < V$ and subset $F \subseteq E$, we define a value for $y^M_{K,L}(F) := r_{V/L}(K \wedge sp(M \cap F))$. Our extended LP is based on valid inequalities for the extended vector x^M, y^M.

A *chain* is a set of subspaces in V that can be indexed by $\mathcal{D} = \{D_i : 0 \leq i \leq k\}$ such that $\{0\} = D_0 < D_1 < \cdots < D_k = V$, where $I = \{i : 1 \leq i \leq k\}$. For all $l \in E$ we introduce a variable $x : E \to \mathbb{R}_+$. For all pairs of subspaces $L < K < V$ and a subset $F \subseteq E$, we introduce a variable $y_{K,L}(F) \in \mathbb{R}_+$. A triple $\mathcal{F}, \mathcal{D}, j$ is called a *subchain triple* if $\mathcal{F}$ is a subpartition of E, $\mathcal{D}$ is a chain, and $j \leq k$. A quadruple $\mathcal{D}, I_1, I_2, F$ is called an *chain component* if $\mathcal{D}$ is a chain, I_1, I_2 is a partition of I, and $F \subseteq E$. The value of a chain component $\mathcal{D}, I_1, I_2, F$ is given by $val(\mathcal{D}, I_1, I_2, F) := \left\lfloor \frac{1}{2} \sum_{i \in I_1} r_{V/D_{i-1}}(D_i \wedge sp(F)) \right\rfloor$. We require the following constraints:

$$(1) \qquad \sum_{i \leq j} \sum_{F \in \mathcal{F}} y_{D_{i-1},D_i}(F) \leq r(D_j) \qquad \text{subchain triple } \mathcal{F}, \mathcal{D}, j$$

$$(2) \qquad x(F) - \sum_{i \in I_2} y_{D_{i-1},D_i}(F) \leq val(\mathcal{D}, I_1, I_2, F) \qquad \text{chain component } \mathcal{D}, I_1, I_2, F$$

$$(3) \qquad 2x(e) - \sum_{i \leq j} y_{D_{i-1},D_i}(e) \leq 0 \qquad \text{chain } \mathcal{D},\ e \subseteq D_j$$

The "necessity" of these inequalities is that for any matching M, x^M, y^M satisfies each of these constraints, which may be proved in a couple of lines of easy computation based on the above definitions. The main result of the talk is that, conversely, the maximum weight of a matching is always determined by the optimum over the above description, and an optimum dual solution of small support always exists. The approach to prove this result is a primal-dual optimum that maintains a nicely structured dual solution and, in each step, constructs an auxiliary unweighted instance that is equivalent with the complementary slackness conditions. Once a "perfect matching" is found in the auxiliary unweighted instance, the algorithm terminates by expanding shrunk blossoms. Otherwise a dual change is preformed based on a dual solution from Lovász' unweighted min-max.

Theorem 8. *For an instance V, E, w of linear weighted matroid matching, the maximum weight of a matching is equal to the maximum of wx over solutions (x, y) of* (1)-(3)*, and the dual optimum attains with support of cardinality no more than $O(r(V)|E|)$. Moreover, there is an algorithm to find this dual optimum and a maximum weight matching by no more than $O(r(V)|E|)$ invokations of an algorithm for unweighted linear matroid matching.*

(Remark: The algorithm uses an algorithm for the unweighted problem merely as a black box. Using the algorithm of Gabow and Stallman of running time $O(r(V)|E|^3)$ we obtain a running time of $O(r(V)^2|E|^4)$ for the weighted problem. This may be improved by using Orlin's unweighted algorithm of the same running instead, and exploiting its details instead of using it merely in a black box. The author conjectures that one could eventually push the running time down to $O(r(V)|E|^3)$, matching that of the unweighted problem.)

From stable sets to sums of squares and conic factorizations

Pablo A. Parrilo
(joint work with Joao Gouveia, Rekha Thomas)

Summary: The theta body $\mathrm{TH}(G)$ is a well-known relaxation of the stable set polytope $\mathrm{STAB}(G)$ that is computable using semidefinite programming. These ideas can be extended via sum of squares techniques to other combinatorial optimization problems, providing a natural generalization to polynomial ideals. We describe these techniques, as well as recent results on lifting of convex sets and their relations with conic factorizations of slack operators. A complete version of these results can be found in [3].

Linear optimization over convex sets plays a central role in optimization. In many instances, a convex set $C \subset \mathbb{R}^n$ may come with a complicated representation that cannot be altered if one is restricted in the number of variables and type of representation that can be used. However, if we are allowed to represent the set C as the *projection* of a higher-dimensional convex set, then much more parsimonious representations may be possible.

In our work, we ask the following basic geometric questions about a given convex set $C \subset \mathbb{R}^n$:

(1) *Given a full-dimensional closed convex cone $K \subset \mathbb{R}^m$, when does there exist an affine subspace $L \subset \mathbb{R}^m$ and a linear map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ such that $C = \pi(K \cap L)$?*
(2) *If the cone K comes from a family $\{K_k\}$ (e.g. $\{\mathbb{R}^k_+\}$ or $\{\mathcal{S}^k_+\}$), then what is the least k for which $C = \pi(K_k \cap L)$ for some π and L?*

If $C = \pi(K \cap L)$, then $K \cap L$ is called a *K-lift* of C. In [5], Yannakakis points out a remarkable connection between the smallest k for which a polytope has a $\mathbb{R}^k_+$-lift and the *nonnegative rank* of its *slack matrix*. In Theorem 10 (see [3] for context and additional results), we prove an extension of Yannakakis' result to the general scenario of K being any closed convex cone and C any convex set, answering Question (1). The main tool is a generalization of nonnegative factorizations of nonnegative matrices to *cone factorizations* of *slack operators* of convex sets.

Stable Set Polytopes An interesting example of polytopes that arise from combinatorial optimization is that of stable set polytopes. Let G be a graph with vertices $V = \{1, \ldots, n\}$ and edge set E. A subset $S \subseteq V$ is *stable* if there are no edges between elements in S. To each stable set S we can associate a vector $\chi_S \in \{0,1\}^n$ where $(\chi_S)_i = 1$ if $i \in S$ and $(\chi_S)_i = 0$ otherwise. The *stable set polytope* of the graph G is the polytope

$$\mathrm{STAB}(G) = \mathrm{conv}\{\chi_S : S \text{ is a stable set of } G\}.$$

Finding the largest stable set in a (possibly vertex-weighted) graph is a classic NP-hard problem in combinatorial optimization that can be formulated as linear optimization over $\mathrm{STAB}(G)$.

The polytopes $\mathrm{STAB}(G)$ also give rise to one of the most celebrated results in semidefinite lifts of polytopes. Recall that a graph is *perfect* if the chromatic number of every induced subgraph equals the size of its largest clique.

Theorem 9. [4] *Let G be a perfect graph with n vertices, then* $\mathrm{STAB}(G)$ *has a $\mathcal{S}^{n+1}_+$-lift.*

Sum of squares and theta bodies A natural class of lift-and-project methods can be obtained using multivariate polynomials and sums of squares techniques. Let I be a polynomial ideal. Given a polynomial $p(x)$, we say that it is a *sum of squares (sos) modulo I*, if there exist polynomials $h_1(x), \ldots, h_s(x)$ such that $p(x) - \sum h_i(x)^2 \in I$. If the degrees of all the h_i are bounded above by k we say that p is *k-sos modulo I*. This is a sufficient condition for nonnegativity over a real variety that has been used often to construct sequences of semidefinite relaxations to convex hulls of real varieties (see [2]). One such hierarchy is the *theta body hierarchy*, defined geometrically by taking the k-th theta body relaxation, denoted by $\mathrm{TH}_k(I)$, to be the intersection of all half-spaces $\{\ell(x) \geq 0\}$ such that $\ell(x)$ is a linear polynomial that is k-sos modulo I.

In the particular case where I is the ideal generated by the polynomials $x_i^2 - 1$ for $i \in V$ and $x_i x_j$ for $(i,j) \in E$, then the corresponding theta body is equal to $\mathrm{TH}(G)$. These ideas can be extended to more general ideals, such as the ones arising from binary matroids [1].

Cone lifts and cone factorizations We have extended Yannakakis' characterization of the existence of a lift to general convex bodies. We sketch these ideas below.

A convex set is called a *convex body* if it is compact and contains the origin in its interior. Recall that the *polar* of a convex set $C \subset \mathbb{R}^n$ is the set

$$C^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \quad \forall x \in C\}.$$

Let $\text{ext}(C)$ denote the set of *extreme points* of C, namely, all points $p \in C$ such that if $p = p_1 + p_2$, with $p_1, p_2 \in C$, then $p = p_1 = p_2$. Since C is compact, it is the convex hull of its extreme points. Consider the operator $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $S(x, y) = 1 - \langle x, y \rangle$. We define the *slack operator* S_C, of the convex set C, to be the restriction of S to $\text{ext}(C) \times \text{ext}(C^\circ)$.

Definition 1. *Let $K \subset \mathbb{R}^m$ be a closed convex cone and $C \subset \mathbb{R}^n$ a full-dimensional convex body. A K-*lift *of C is a set $Q = K \cap L$, where $L \subset \mathbb{R}^m$ is an affine subspace, and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map such that $C = \pi(Q)$. If L intersects the relative interior of K we say that Q is a* proper K-lift *of C.*

Definition 2. *We say that the slack operator S_C is K-*factorizable *if there exist maps (not necessarily linear)*

$$A : \text{ext}(C) \to K \quad \text{and} \quad B : \text{ext}(C^\circ) \to K^*$$

such that $S_C(x, y) = \langle A(x), B(y) \rangle$ for all $(x, y) \in \text{ext}(C) \times \text{ext}(C^\circ)$.

With the above set up, we can now characterize the existence of a K-lift of C.

Theorem 10. *If C has a proper K-lift then S_C is K-factorizable. Conversely, if S_C is K-factorizable then C has a K-lift.*

Cone ranks of slack operators We established earlier necessary and sufficient conditions for the existence of a K-lift of a given convex body $C \subset \mathbb{R}^n$ for a fixed cone K. In many instances, the cone K belongs to a family such as $(\mathbb{R}^i_+)_i$ or $(\mathcal{S}^i_+)_i$. In such cases, it becomes interesting to determine the smallest cone in the family that admits a lift of C. In this section, we study this scenario and develop the notion of *cone rank* of a convex body.

Definition 3. *A* cone family *$\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ is a sequence of closed convex cones K_i indexed by $i \in \mathbb{N}$. The family $\mathcal{K}$ is said to be* closed *if for every $i \in \mathbb{N}$ and every face F of K_i there exists $j \leq i$ such that F is isomorphic to K_j.*

We can then define a natural notion of cone rank for a nonnegative matrix (or operator). By considering the slack operator of a convex body, we have the corresponding notion of cone rank of a convex body.

Definition 4. *Let $\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ be a closed cone family.*

(1) *The $\mathcal{K}$-rank of a nonnegative matrix M, denoted as $\text{rank}_{\mathcal{K}}(M)$, is the smallest i such that M has a K_i-factorization. If such an i does not exist, say that $\text{rank}_{\mathcal{K}}(M) = +\infty$.*

(2) *The $\mathcal{K}$-rank of a convex body $C \subset \mathbb{R}^n$, denoted as* $\operatorname{rank}_{\mathcal{K}}(C)$*, is the smallest i such that the slack operator S_C has a K_i-factorization. If such an i does not exist, say that* $\operatorname{rank}_{\mathcal{K}}(C) = +\infty$.

Theorem 11. *Let $\mathcal{K} = (K_i)_{i \geq 0}$ be a closed cone family and $C \subset \mathbb{R}^n$ a convex body. Then* $\operatorname{rank}_{\mathcal{K}}(C)$ *is the smallest i such that C has a K_i-lift.*

¿From the optimization viewpoint, perhaps the most interesting cases are those when the cone family is the nonnegative orthant, or the positive semidefinite cone. These give rise to the notion of *nonnegative rank* and *PSD rank*, respectively. In the paper [3] we provide a number of results and bounds for these cone ranks.

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Some 0/1 polytopes need exponential size extended formulations

Thomas Rothvoss

Combinatorial optimization deals with finding the best solution out of a finite number of choices $X \subseteq \{0,1\}^n$, e.g. finding the cheapest spanning tree in a graph. If possible one aims of course to design a polynomial time algorithm. However another popular way to study combinatorial problems is to express the convex hull $P = \operatorname{conv}(X)$ by linear inequalities $Ax \leq b$, i.e. describing them as the solutions of a linear program. A drawback of this approach is that in general an exponential number of inequalities is needed. In principle one could use the Ellipsoid method to optimize these systems, if at least the separation problem can be solved in polynomial time. But in practice this method is considered to be not applicable. A more satisfactory approach is to allow polynomially many extra variables in order to reduce the number of necessary inequalities to a polynomial. This is called a *compact formulation* $P = \{x \mid \exists y : Ax + Uy \leq b\}$. Such compact formulations exist for example for the spanning tree polytope [5], the parity polytope and the permutahedron (see [6] for an extensive account).

This naturally leads to the question for which problems such a compact formulation does *not* exist. Yannakakis [8] showed that the TSP polytope P_{TSP} (the convex full of the characteristic vectors of all Hamiltonian cycles in the complete graph on n nodes) does not have a subexponential size *symmetric* formulation. Surprisingly the same result holds true for the matching polytope, though here a complete description of all facets is known due to Edmonds [2] and the problem itself as well as the separation problem are solvable in polynomial time. Kaibel, Pashkovich and Theis [4] demonstrate that symmetric formulations are in some cases more restricted by proving that there is a compact non-symmetric formulation for all $\log n$-size matchings, while symmetric formulations still need size $n^{\Omega(\log n)}$.

1. Our contribution

However, it remains a fundamental open problem to show that the matching polytope or the TSP polytope do not admit any non-symmetric compact formulation. In fact, it was even an open problem to prove that there *exists* any family of 0/1 polytopes without a compact formulation[1]. In this paper we answer this question affirmatively. Let $\mathrm{xc}(P)$ be the minimum number of inequalities that are needed to describe a polytope Q which can be linearly projected on P. We say $\mathrm{xc}(P)$ is the *extension complexity* of P.

Theorem 12. *For any $n \in \mathbb{N}$, there exists a set $X \subseteq \{0,1\}^n$ such that*

$$xc(conv(X)) \geq \Omega(2^{n/2}/\sqrt{n\log(2n)}).$$

In fact, this bound also holds with high probability, if the set $X \subseteq \{0,1\}^n$ is picked at random. More precisely, we have

Corollary 3. *Let $X^{(1)},\dots,X^{(M)} \subseteq \{0,1\}^n$ be distinct subsets and let $0 < \delta < 1$ be a parameter such that $\delta M \geq 2^{n^4}$. Then for at least $(1-\delta)\cdot M$ many indices $j \in \{1,\dots,M\}$ one has $xc(conv(X^{(j)})) \geq \Omega\Big(\sqrt{\frac{\log(\delta M)}{n\log(2n)}}\Big)$.*

Since it is well known that there are doubly-exponentially many matroids on n elements, this implies that there must be a family of matroid polytopes with exponential extension complexity.

2. Proof sketch

Our idea is based on a counting argument similar to Shannon's theorem [7] (see also [1]) for lower bounds on circuit sizes: Let us assume for the sake of contradiction that all n-dimensional 0/1 polytopes have a compact formulation $P = \{x \mid \exists y \geq \mathbf{0} : Ax + Uy = b\}$ of polynomial size $r(n)$. Since there are doubly-exponentially many 0/1 polytopes, there must also be at least that many formulations of size $r(n)$. This would lead to a contradiction under the additional assumption that all coefficients in the system $Ax + Uy = b$ have polynomial encoding length. Unfortunately there is no known result which guarantees that the coefficients of U will even be rational. In our approach, we bypass these difficulties by selecting a linearly independent subsystem of $Ax+Uy = b$ which maximizes the volume of the spanned parallelepiped; then we discretize the entries of U. We thus obtain a subsystem $\bar{A}x + \bar{U}y = \bar{b}$ with the property that $x \in X$ if and only if there is a short certificate y such that $\bar{A}x + \bar{U}y \approx \bar{b}$ for the rounded system. Secondly,

[1]This was posed as an open problem by Volker Kaibel on the 1st Cargese Workshop in Combinatorial Optimization.

all numbers in $\bar{A}, \bar{U}, \bar{b}$ have an encoding length which is bounded by a polynomial in n. In other words, this construction defines an injective map, taking a set X as input and providing $(\bar{A}, \bar{U}, \bar{b})$. Since there are doubly-exponentially many sets $X \subseteq \{0,1\}^n$ and by injectivity, the number of such systems $(\bar{A}, \bar{U}, \bar{b})$ must also be doubly-exponential, which then implies the result.

3. Subsequent work

After publication of this work, Fiorini, Massar, Pokutta, Tiwary and de Wolf [3] were able to prove that the TSP polytope does not admit a compact extended formulation (the same holds true for the stable set polytope, the cut polytope and the correlation polytope).

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From Pyramids to Virtual Private Network Polyhedra

András Sebő
(joint work with Nicola Apollonio, Gianpaolo Oriolo and Fabrizio Grandoni)

Consider a communication network which is represented by an undirected graph $G = (V, E)$ with a set of k *terminals* $W \subseteq V$ and edge costs $c : E \to \mathbb{R}_+$. The terminals want to communicate with each other. However, the exact amount of traffic between pairs of terminals is not known in advance. Instead, each terminal $i \in W$ has an upper bound $b(i) \in \mathbb{Z}_+$ on the cumulative amount of traffic that terminal i can send or receive. The general aim is to install capacities on the edges of the graph supporting any possible communication scenario at minimum cost where the cost for installing one unit of capacity on edge e is $c(e)$. More generally, given a not necessarily linear cost function $f : \mathbb{R}^E \longrightarrow \mathbb{R}$, we may want to minimize the cost $f(u)$ of a capacity vector $u \in \mathbb{R}^E$ that we want to buy.

Inequalities (or equalities) between vectors or functions will be understood coordinatewise (on the entire domain) respectively. If $u \geq v$ we will say that u *majorizes* v.

A *set of traffic demands* $D = \{d_{ij} \mid i, j \in \binom{W}{2}\}$ specifies for each unordered pair of terminals $i, j \in \binom{W}{2}$, denoting by $\binom{W}{2}$ the set of cardinality-two subsets of W, the amount $d_{ij} \in \mathbb{R}_{\geq 0}$ of traffic between i and j. A set D is *valid* if it respects the upper bounds on the traffic of the terminals. That is, (setting $d_{ii} = 0$ for all $i \in W$)

$$\sum_{j \in W} d_{ij} \leq b(i) \qquad \text{for all terminals } i \in W. \tag{$*$}$$

A solution to the *symmetric Virtual Private Network Design* problem defined by G, W, and b, c consists of an i-j-path P_{ij} in G for each unordered pair $i, j \in \binom{W}{2}$, and edge capacities $u(e) \geq 0$, $e \in E$. Such a set of paths $\mathcal{P} := \{P_{ij} : i, j \in \binom{W}{2}\}$ is called a *template.* A template, together with edge capacities $u(e)$, $(e \in E)$, is called a *virtual private network.* We are searching for an optimal one.

A virtual private network is *feasible,* if all valid sets of traffic demands D can be routed without exceeding the installed capacities u where all traffic between terminals i and j is routed along path P_{ij}, that is

$$\text{For all edges } e \in E \qquad \sum_{\{i,j\} \in \binom{W}{2}:\, e \in P_{ij}} d_{ij} \leq u(e).$$

A feasible virtual private network is called *optimal* if the total cost of the capacity reservation $\sum_{e \in E} c(e)\, u(e)$ is minimal (or for more general cost functions $f : \mathbb{R}^E \longrightarrow \mathbb{R}$, the value $f(u)$ has to be minimal).

For any given template $\mathcal{P}$, the vector of optimal capacities $u(\mathcal{P})$ is uniquely determined (and can be efficiently computed as the maximum of a fractional b-matching problem in the complete graph $\binom{W}{2}$ [15]), as follows (irrelevant in the sequel, since we will need it only in situations trivial from scratch):

Indeed, the solutions of (*) are fractional b-matchings in the complete graph $\binom{W}{2}$. The maximum number of paths (with multiplicity, that may be fractional as well) containing e is equal to the maximum weight of a b-matching with the weight function 1 on pairs $i, j \in \binom{W}{2}$ for which $e \in P_{ij}$ and 0 otherwise. This maximum is exactly the capacity $u_{\mathcal{P}}(e)$ that has to be allocated to e when template $\mathcal{P}$ is used.

We define a *multiflow* with a pair $(\mathcal{P}, D)$ where $\mathcal{P}$ is a template and D a (not necessarily valid) traffic matrix. A multiflow is a multiset of paths where the (possibly fractional) multiplicities - in other words *coefficients* are given in the matrix D, the coefficient of path P_{ij} being d_{ij}. The number of paths (with multiplicity) containing an edge (that is, the sum of coefficients containing it) will be called the *flow value* of the edge. Now $u_{\mathcal{P}}(e)$ can be rephrased as the maximum flow value of edge e where the maximum ranges over the multiflows with feasible traffic matrices.

We have just seen that the object that has to be designed is a template, and only a template. The *capacity function* $u_{\mathcal{P}}$ associated with template $\mathcal{P}$ is in $\mathbb{R}^E$ –

that is, the capacity $u_{\mathcal{P}}(e) \in \mathbb{R}$ of edge e for all $e \in E$ according to template $\mathcal{P}$ –, is then uniquely determined. This is reassuring to know, furthermore important for our intuition, but will not be explicitly used in the sequel.

A feasible virtual private network is a *tree solution* if the subgraph of G spanned by the support of u (that is, edges $e \in E$ with $u(e) > 0$) is a tree.

The problem was defined by Fingerhut, Suri and Turner [2] and by Gupta, Kleinberg, Kumar, Rastogi, and Yener [8]. A question that has been open for a while is whether for each instance of the problem there exists an optimal virtual private network that is a tree solution. The following conjecture has been formally stated by Italiano, Leonardi, and Oriolo [10] (see also [8], Erlebach and Rüegg [1]):

Conjecture 1 (The VPN Tree Routing Conjecture)**.** *For each SVPND instance* (G, W, b, c) *there exists an optimal virtual private network which is a tree solution.*

The conjecture was recently proved by Goyal, Olver and Shepherd [4], and it follows that the optimal solution is equal to the optimal tree-solution, where the latter has already been observed to be found in polynomial time by Gupta, Kleinberg, Kumar, Rastogi, and Yener [8]. In this note we provide a polyhedral light and a simple proof of this result:

Given an undirected graph $G = (V, E)$ and $W \subseteq V$, a *Steiner-tree* (in (G, W)) is a tree (U, H) such that $W \subseteq U \subseteq V$, $H \subseteq E$, and is (inclusionwise) minimal with respect to these properties, that is, the vertices of degree 1 of (U, H) are in W. A *tree-template* is a template equal to the set of paths between pairs of terminals of a Steiner-tree of (G, W). The tree-template whose members are the the paths between terminals of a particular Steinet-tree F of (G, W) will be denoted by TF. Define the *capacity polyhedron*

$$C(G, W) := \text{conv}(u_{\mathcal{P}} : \mathcal{P} \text{ is a template in } (G, W)\ ) + \mathbb{R}_+^E.$$

For basic notions of polyhedral combinatorics and linear programming we refer to Schrijver [14], for instance conv(.) is the convex hull of vectors that figure in the argument. The main point of this paper is a simplified proof, and deduction of sharper consequences, of the following version of the VPN-tree conjecture:

Theorem 13. *The set of vertices of the polyhedron C(G,W) equals* $\{u_{TF} : F$ *is a Steiner-tree*$\}$*, that is, the set of capacity vectors of tree-templates.*

It follows that conv(u_{TF} : F is a Steiner tree) $+ \mathbb{R}_+^E$ defines the same polyhedron (its minimal faces are vertices, since the defined polyhedron is full dimensional and its elements are non-negative) as $C(G, W)$. Note that the support of $u = u_{TF}$ is F, and any template different from TF has a different capacity vector since even its support is different from that of u.

Three direct new consequences of this theorem can then be deduced:

- The VPN tree conjecture, i.e. Goyal, Olver and Shepherd's theorem [4] and its polyhedral extension.
- An optimal VPN for convex functions $f : \mathbb{R}^E \longrightarrow \mathbb{R}$ can be found in polynomial time with the ellipsoid method.

– The VPN tree conjecture is true for concave cost functions $f : \mathbb{R}^E \longrightarrow \mathbb{R}$ as well (even if the optimal solution is NP-hard to find), extending the result for separable concave functions in [6].

Besides the results themselves, the main content of the talk may be the demystification of the subject:

The mysterious "existence of tree solutions" turns out to be a special case of the "existence of basic solutions" in linear programming. Indeed, the main result of this work is that the vertices of a corresponding polyhedron are trees.

Furthermore, the "pyramidal" weight function [6] turns out to be simply the *capacity function of particular templates, called rooted templates* (see next section). No doubt, this function is pyramidal, but instead of this property we will have to care about rooted and tree templates and their place on the capacity polyhedron.

Thereby the conceptually difficult proof clears up to a proof of the above polyhedral theorem in a polyhedral way. It uses a slight sharpening of Padberg and Rao's theorem about the relation of Gomory-Hu trees and minimum weight odd cuts, and stimulates research on a possible elementary proof. A combinatorial polynomial algorithm for finding an optimal template is then at hand without further effort.

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Minimum Congestion versus Maximum Throughput: Connections and Distinctions

Bruce Shepherd

We discuss two well-known problems in the theory of routing in graphs. The first is the *nminimum congestion problem*, which given an (capacitated if you like) undirected graph $G = (V, E)$ and pairwise demands D_{ij} asks to route all the demands (fractionally if you like) so that the maximum load (congestion when there are capacities) on any edge is minimized. This gives rise to the well-studied notion of flow-cut gap, and the question of *How much does the cut condition buy you?* One outstanding question asks if every planar instance has a flow-cut gap of size $O(1)$. This question (and similar questions for general graphs where it is known the flow-cut gap is $\Theta(\log n)$ [4]) have tended to be studied from the dual perspective. In particular, the companion conjecture of whether every planar graph (or indeed any minor-closed class) admits a low-distortion embedding into ℓ_1 has attracted considerable attention from the metric embedding side. We present several results and open questions with a more "primal" flavour. In particular we discuss a conjecture about *integral flow cut gaps* [3].

The second part of the talk examines the *maximum disjoint path problem.* Here we are only asked to find a feasible routing for a **subset** of the demands D_{ij}. However, they must route within the given capacitated graph (i.e., with no congestion). This problem is harder than the congestion (flow-cut gap) version in the sense that we must route integrally and also because any sensible linear (or convex) relaxation only suggests which demands to route. E.g., it may say to route .015 of the demand D_{ij} and the problem is to determine which subset of the demands to choose (round up); we call this the subset selection aspect. It is easier than flow-cut gap problems however in a critical way. We do not need to route all the demands and can sacrifice some for the sake of routing others. This is used to great benefit in all of the constant approximation algorithms for maximum disjoint paths to date (at least for the maximum cardinality version). The maximum disjoint path problem is already interesting on a tree. It includes for instance the maximum matching problem in general graphs! It also becomes APX-hard as soon as tree edges may have capacities 1 and 2. This shows that subset selection is one key hurdle. We discuss a weighted 4-approximation for a tree [1] before examining a procedure which establishes a constant integrality gap for planar graphs with edge capacities at least 2 [2, 5] (if capacities are 1, the gap may be as large as $\sqrt{n}$). Along the way we highlight many open problems for both classes of routing problems: minimizing congestion and maximizing throughput.

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Improving Christofides' Algorithm for the s-t Path TSP

David Shmoys

(joint work with Hyung-Chan An, Robert Kleinberg)

We present a deterministic $\left(\frac{1+\sqrt{5}}{2}\right)$-approximation algorithm for the s-t path TSP for an arbitrary metric. Given a symmetric metric cost between n vertices including two prespecified endpoints, the problem is to find a shortest Hamiltonian path between the two endpoints; Hoogeveen [2] showed that the natural variant of the classic TSP algorithm of Christofides [1] is a 5/3-approximation algorithm for this problem, and this asymptotically tight bound in fact has been the best approximation ratio known until now. We modify this algorithm so that it chooses the initial spanning tree based on an optimal solution to the Held-Karp relaxation rather than a minimum spanning tree; we prove this simple but crucial modification leads to an improved approximation ratio, surpassing the 20-year-old barrier set by the natural Christofides' algorithm variant. Our algorithm also proves an upper bound of $\frac{1+\sqrt{5}}{2}$ on the integrality gap of the path-variant Held-Karp relaxation.

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A randomized rounding approach to the Traveling Salesman Problem

Mohit Singh
(joint work with Shayan O. Gharan, Amin Saberi)

For some positive constant ϵ_0, we give a $(\frac{3}{2} - \epsilon_0)$-approximation algorithm for the following problem: given a graph $G_0 = (V, E_0)$, find the shortest tour that visits every vertex at least once. This is a special case of the metric traveling salesman problem when the underlying metric is defined by shortest path distances in G_0. The result improves on the $\frac{3}{2}$-approximation algorithm due to Christofides [5] for this special case.

Similar to Christofides, our algorithm finds a spanning tree whose cost is upper bounded by the optimum, then it finds the minimum cost Eulerian augmentation (or T-join) of that tree. The main difference is in the selection of the spanning tree. Except in certain cases where the solution of LP is nearly integral, we select the spanning tree randomly by sampling from a maximum entropy distribution defined by the linear programming relaxation. Despite the simplicity of the algorithm, the analysis builds on a variety of ideas such as properties of strongly Rayleigh measures from probability theory [4], graph theoretical results on the structure of near minimum cuts [1, 2], and the integrality of the T-join polytope from polyhedral theory. Also, as a byproduct of our result, we show new properties of the near minimum cuts of any graph, which may be of independent interest.

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Semidefinite Programming Hierarchies and the Unique Games Conjecture

David Steurer

We survey recent results about semidefinite programming (SDP) hierarchies in the context of the Unique Games Conjecture. This conjecture has emerged as a unifying approach towards settling many central open problems in the theory of approximation algorithms. It posits the hardness of a certain constraint satisfaction problem, called Unique Games.

We give a subexponential-time approximation algorithm for this problem using SDP hierarchies (based on joint work with Sanjeev Arora and Boaz Barak [ABS10] and joint work with Boaz Barak and Prasad Raghavendra [BRS11]).

On the other hand, we show that certain SDP hierarchies cannot solve the Unique Games problem in a quasi-polynomial number of rounds (joint work with Boaz Barak, Parikshit Gopalan, Johan Håstad, Raghu Meka, and Prasad Raghavendra [BGH+11]).

Both works rely on novel connections between the spectrum of graphs and the expansion of small sets. For example, we show that any regular n-vertex graph with at least $n^{O(\varepsilon)}$ eigenvalues larger than $1-\varepsilon$ contains a set with cardinality at most $n^{1-\Omega(\varepsilon)}$ and expansion at most $1/100$ [ABS10]. (Here, we consider eigenvalues of the normalized adjacency matrix of the graph.)

Finally, we demonstrate that all known instances of Unique Games can be solved in a constant number of rounds of a stronger SDP hierarchy based on sum-of-squares proofs (joint work Boaz Barak, Aram Harrow, Jonathan Kelner, Yuan Zhou [BHK+11]). This result establishes a strong separation between SDP hierarchies based on sum-of-squares proofs and other hierarchies.

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An Excluded Minor Characterization of Seymour Graphs

Zoltán Szigeti

(joint work with Alexander Ageev, Yohann Benchetrit, András Sebő)

A graph G is said to be a *Seymour graph* if for any set F of edges of G, there exists a *complete packing of cuts*, that is $|F|$ pairwise edge disjoint cuts each containing exactly one element of F, provided that F is a *join*, that is for every circuit C of G the necessary condition $|C \cap F| \leq |C \setminus F|$ is satisfied.

Several particular cases of Seymour-graphs have been exhibited by Seymour [5] [6], Gerards [3] and Szigeti [7], while the existence of complete packing of cuts in graphs has been proved to be NP-hard [4].

A first coNP characterization of Seymour graphs has been shown by Ageev, Kostochka and Szigeti [2]. A circuit is called *tight* if equality holds in the above inequality. A graph G is called an *odd* K_4 (respectively, *odd prism*) if it is a subdivision of K_4 (respectively, prism) such that the length of each circuit remains of the same parity as before.

Theorem 14 (Ageev, Kostochka and Szigeti [2]). *A graph is not a Seymour graph if and only if it has a join and two tight circuits whose union forms an odd* K_4 *or an odd prism.*

We show new minor-producing operations that keep this property, and prove excluded minor characterization of Seymour graphs: the operations are the contraction of full stars and that of odd circuits.

We will say that a graph is the *stoc-minor* of G if it arises from G by a series of star and odd circuit contractions. A subdivision of a graph G is said to be *even* if the number of new vertices inserted in every edge of G is even (possibly zero).

Theorem 15 (Ageev, Benchetrit, Sebő, Szigeti [1]). *A graph is not a Seymour graph if and only if it has a stoc-minor containing an even subdivision of* K_4 *as a subgraph.*

Stoc-minors generated an immediate simplification in the characterization: prisms disappeared! Indeed, K_4 is a stoc-minor of the prism. This sharpens the previous results, providing at the same time a simpler and self-contained algorithmic proof of the existing characterizations as well, still using methods of matching theory and its generalizations.

It is an open problem to find an NP characterization of Seymour graphs.

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Applications of semidefinite programming and harmonic analysis

Frank Vallentin

Geometric packing and coloring problems

Many, often notoriously difficult, problems in combinatorics and geometry can be modeled as packing and coloring problems of graphs $G = (V, E)$ where the vertex set V can be an infinite or even a continuous set. Packing problems correspond to finding the independence number $\alpha(G)$ and coloring problems correspond to finding the chromatic number $\chi(G)$. Both are standard problems in combinatorial optimization. Examples:

- Error correcting q-ary codes:
$$V = \mathbb{F}_q^n, \quad x \sim y \Longleftrightarrow 0 < \|x - y\| < d.$$
- Kissing numbers:
$$V = S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}, \quad x \sim y \Longleftrightarrow 0 < \angle(x, y) < \pi/3.$$
- Body packing: Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex, compact body.
$$V = \mathbb{R}^n \times \mathrm{SO}(n), \quad (x, A) \sim (y, B) \Longleftrightarrow x + A\mathcal{K}^o \cap y + B\mathcal{K}^o \neq \emptyset.$$
- Coloring Euclidean space:
$$V = \mathbb{R}^n, \quad x \sim y \Longleftrightarrow \|x - y\| = 1.$$

The combination of semidefinite programming and harmonic analysis often gives the best known upper bounds for these packing and the best known lower bounds for these coloring problems.

n-point bounds for packing problems

To compute upper bounds for $\alpha(G)$ for finite graphs the t-th step in Lasserre's hierarchy is a useful tool:

$$\mathrm{las}^{(t)}(G) = \max\Big\{\sum_{x \in V} y_{\{x\}} : \big(y_{I\cup J}\big)_{I,J \in \binom{V}{\leq t}} \succeq 0, y_\emptyset = 1, y_{I \cup J} = 0,\ I \cup J \text{ not indep.}\Big\}.$$

It is known that the first step in Lasserre's hierarchy coincides with Lovász' ϑ-number and that the hierarchy converges to α in α steps:

$$\vartheta(G) = \mathrm{las}^{(1)}(G) \geq \mathrm{las}^{(2)}(G) \geq \ldots \geq \mathrm{las}^{(\alpha(G))}(G) = \alpha(G).$$

Many variations are possible to set up an SDP hierarchy: For instance one can consider only "interesting" principal submatrices to simplify the computation and one can also add more constraints. If one strengthens the Lasserre hierarchy by adding the nonnegativity constraints $y_{I\cup J} \geq 0$ one speaks about $\mathrm{las}'^{(t)}(G)$. A rough classification for all these variations can be given in terms of *n-point bounds*. This refers to all variations which make use of variables $y_{I \cup J}$ with $|I \cup J| \leq n$.

For defining a hierarchy for infinite graphs, it is convenient to work with the dual of $\mathrm{las}'^{(t)}(G)$ as it uses positive semidefinite continuous Hilbert-Schmidt kernels as optimization variables (and not Borel measures):

$$\mathrm{las}'^{(t)}(G) = \inf\Big\{\langle M_\emptyset, K\rangle : K \in \mathcal{C}\left(\tbinom{V}{\le t}\times\tbinom{V}{\le t}\right)_{\succeq 0},$$
$$\langle M_{\{x\}}, K\rangle \le -1 \text{ for all } x\in V, \langle M_I, K\rangle \le 0 \text{ if } 2\le |I|\le 2t,\ I \text{ indep..}\Big\},$$

where $\langle M_I, K\rangle = \sum_{J\cup J'=I} K(J,J')$. Then,

$$\vartheta'(G) = \mathrm{las}'^{(1)}(G) \ge \mathrm{las}'^{(2)}(G) \ge \ldots \ge \mathrm{las}'^{(\alpha(G))}(G) \stackrel{?}{=} \alpha(G).$$

The above problem is in general an ∞-dimensional SDP. However, only feasible solutions are needed to prove upper bounds. With help of harmonic analysis and polynomial optimization one can perform explicit, finite-dimensional (of course) computations.

Explicit computations of n-point bounds have been done in a variety of situations. The following table provides a first guide to the relevant literature:

packing	**space**	**2-point**	**3-point**	**4-point**
binary codes	$\mathbb{F}_2^n$	Delsarte (1973)	Schrijver (2005)	Gijswijt, Mittelmann, Schrijver (2011)
q-ary codes	$\mathbb{F}_q^n$	Delsarte (1973)	Gijswijt, Schrijver, Tanaka (2006)	
constant weight codes	$\{x\in\mathbb{F}_q^n : \lvert x\rvert = w\}$	Delsarte (1973)	Schrijver (2005), Regts (2009)	
spherical caps	S^{n-1}	Delsarte, Goethals, Seidel (1977)	Bachoc, V. (2008)	
real projective space	$\mathbb{R}\mathrm{P}^{n-1}$	Kabatiansky, Levenshtein (1978)	Cohn, Woo (2011)	
sphere packing	$\mathbb{R}^n$	Cohn, Elkies (2003)		
body packing	$\mathbb{R}^n\times\mathrm{SO}(n)$	Oliveira, V. (2011+)		

Symmetry reduction and harmonic analysis

Now I illustrate how to apply harmonic analysis in order to be able to perform the calculations of $\mathrm{las}'^{(t)}(G)$. To simplify the notation I consider the case when (V,μ) is a compact measure space and when $t=1$. Then,

$$\vartheta'(G) = \inf\{\ \lambda\ : K\in\mathcal{C}(V\times V)_{\succeq 0}, K(x,x)=\lambda-1 \text{ for } x\in V, K(x,y)\le -1\ \{x,y\}\notin E\}$$

Suppose the graph G has Γ as its symmetry group. If $K\in\mathcal{C}(V\times V)_{\succeq 0}$ is feasible for ϑ', then also its *group average* $\overline{K}$ is:

$$\overline{K}(x,y) = \int_\Gamma K(\gamma x,\gamma y)d\gamma.$$

So it suffices to consider only the Γ-invariant cone

$$\mathcal{C}(V\times V)_{\succeq 0}^{\Gamma} = \{K : \forall\gamma\in\Gamma : K(\gamma x,\gamma y) = K(x,y)\}$$

and a theorem of Bochner (1941) gives an explicit parametrization of $\mathcal{C}(V\times V)_{\succeq 0}^{\Gamma}$.

We state Bochner's theorem now which requires a bit of technical vocabulary. The group Γ acts on $\mathcal{C}(V)$ by $(\gamma f)(x) = f(\gamma^{-1}x)$. So one can speak about Γ-invariant and Γ-irreducible subspaces of $\mathcal{C}(V)$. The Peter-Weyl theorem (1927) says that one can decompose $\mathcal{C}(V)$ orthogonally (using the inner product from $L^2(V)$)

$$\mathcal{C}(V) = (H_{0,1} \perp \ldots \perp H_{0,m_0}) \perp (H_{1,1} \perp \ldots \perp H_{1,m_1}) \perp \ldots,$$

where $H_{k,l}$ is Γ-irreducible and $\dim H_{k,l} < \infty$ and where $H_{k,l} \sim H_{k',l'}$ iff $k = k'$. We fix an orthonormal basis $e_{k,1,1}, \ldots, e_{k,1,\dim H_{k,1}}$ and Γ-isomorphisms $\varphi_{k,l} : H_{k,1} \to H_{k,l}$ and set $e_{k,l,1} = \varphi_{k,l}(e_{k,1,1}), \ldots, e_{k,l,\dim H_{k,1}} = \varphi_{k,l}(e_{k,1,\dim H_{k,1}})$. Bochner's theorem:

$$\mathcal{C}(V \times V)^{\Gamma}_{\succeq 0} = \Big\{K(x,y) = \sum_{k=0}^{\infty} \Big\langle F_k, \Big(\sum_{i=1}^{\dim H_{k,1}} e_{k,l,i}(x)\overline{e_{k,l',i}(y)}\Big)_{l,l'=1,\ldots,m_k} \Big\rangle : F_k \in \mathbb{R}^{m_k \times m_k}_{\succeq 0}\Big\}.$$

This means that instead of optimizing over the cone $\mathcal{C}(V \times V)_{\succeq 0}$ we can optimize over the direct product of the semidefinite cones $S^{m_0}_{\succeq 0} \times S^{m_1}_{\succeq 0} \times \ldots$ and since we are interested only in feasible solutions we can set $F_k = 0$ for large enough k so that the final SDP becomes finite-dimensional.

2-point bounds for coloring problems

To get lower bounds for coloring problems we make use of Lovász' sandwich theorem

$$\alpha(G) \leq \vartheta'(G) \leq \chi(\overline{G}),$$

where a definition of $\vartheta'(\overline{G})$ which generalizes to infinite graphs is

$$\vartheta'(\overline{G}) = \max\Big\{1 - \frac{M(A)}{m(A)} : \; A \in \mathbb{R}^{V \times V}, A \geq 0, A(v,w) = 0 \; \forall \{v,w\} \notin E\Big\}.$$

Here, $M(A) = \sup\{(Af,f) : \; \|f\| = 1\}$ is the largest eigenvalue of A and $m(A) = \inf\{(Af,f) : \; \|f\| = 1\}$ is the smallest eigenvalue of A

Instead of symmetric matrices A we can also consider Hermitian, bounded operators $A : L^2(V) \to L^2(V)$. We say that a measurable set $I \subseteq V$ is independent for A if for all $f \in L^2(I)$ the equality $(Af, f) = 0$ holds. If A is the adjacency matrix of a finite graph, then this definition coincides with the usual one. The (measurable) chromatic number of A is

$$\chi_m(A) = \inf\{k : \exists \; k\text{-partition of } V \text{ into independent sets}\}.$$

Let ω be the normalized Haar measure on S^{n-1} and define the adjacency operator of the unit distance graph by $A_\omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with $(A_\omega f)(x) = (\omega * f)(x) = \int_{S^{n-1}} f(x-y)d\omega(y)$. Then, $\chi_m(\mathbb{R}^n) \geq \chi_m(A_\omega) \geq 1 - \frac{M(A_\omega)}{m(A_\omega)} = 1 - \left(\min_{u \in \mathbb{R}^n} \int_{S^{n-1}} e^{iu \cdot x} d\omega(x)\right)^{-1}$, which can be explicitly computed with the help of elementary properties of Bessel functions. This is a result from Oliveira, V. (2010).

A strongly polynomial algorithm for flows with separable convex objectives

László A. Végh

A well-studied nonlinear extension of minimum-cost flows is to minimize the objective $\sum_{ij\in E} C_{ij}(f_{ij})$ over feasible flows f, where on each arc ij of the network, C_{ij} is a convex function. The model has several applications, see [1, Chapter 14] for further references. For this problem, multiple polynomial-time combinatorial algorithms were given, by Minoux [8], by Hochbaum and Shantikumar [7], and by Karzanov and McCormick [9].

These algorithms are weakly polynomial. Finding a strongly polynomial algorithm seems impossible by the very nature of the problem: the optimal solution might be irrational, and thus the exact optimum cannot be achieved. Beyond irrationality, the result of Hochbaum [6] shows that it is impossible to find an ε-approximate solution in strongly polynomial time even with the C_{ij}'s being polynomials of degree at least three.

The remaining class of polynomial objectives with hope of strongly polynomial algorithms is convex quadratic. For these functions, the existence of a rational optimal solution is always guaranteed. Strongly polynomial algorithms were obtained for special cases, for example, fixed number of suppliers (Cosares and Hochbaum, [2]), and series-parallel graphs (Tamir [13]).

Our paper [15] provides a strongly polynomial algorithm for a broad class of objectives, characterized by some natural assumptions; the most important characteristic of this class is that an optimal solution can be computed exactly provided its support. The class contains separable convex quadratic objectives, providing an algorithm with running time $O(m^4 \log m)$. Besides quadratic objectives, the class also includes certain market equilibrium settings, such as linear Fisher markets, described below.

For linear minimum cost flows, Orlin [10] developed a strongly polynomial version of the scaling algorithm of Edmonds and Karp [4]. This is achieved by maintaining and gradually extending the set of edges which are guaranteed to carry positive flow amount in a certain optimal solution. The algorithm of Minoux [8] is a natural extension of the Edmonds-Karp scaling method to separable convex objectives. We give a strongly polynomial version of Minoux's algorithm in the spirit of Orlin's technique.

The *linear Fisher market model*, is one of the most fundamental models in equilibrium theory. We are given a set B of buyers and a set G of goods. Buyer i has a budget m_i, and there is one divisible unit of each good to be sold. For each buyer $i \in B$ and good $j \in G$, $U_{ij} \geq 0$ is the utility accrued by buyer i for one unit of good j.

An equilibrium solution consist of prices p_i on the goods and an allocation x_{ij}, so that (i) all goods are sold, (ii) all money of the buyers is spent, and (iii) each buyers i buys a best bundle of goods, that is, goods j maximizing U_{ij}/p_j. The first combinatorial algorithm for this problem was given by Devanur et al. [3]; a strongly polynomial algorithm was obtained by Orlin [11].

The equilibrium solutions for linear Fisher markets were described by two different convex programs, by Eisenberg and Gale [5] in 1959 and an entirely different one by Shmyrev [12]. In this latter model, the basic f_{ij} variable represent the money payed by buyer i for product j ($f_{ij} = p_j x_{ij}$).

$$\min \sum_{i\in G} p_j(\log p_j - 1) - \sum_{ij\in E} f_{ij}\log U_{ij}$$

$$\sum_{j\in G} f_{ij} = m_i \qquad \forall i \in B$$

$$\sum_{i\in B} f_{ij} = p_j \qquad \forall j \in G$$

$$f \geq 0$$

The Shmyrev program can be interpreted as an instance of the flows with separable convex objective model. Indeed, orient the edges from the goods to the buyers, and add a new source s, connected to every good. The assumptions of our model hold and thus we obtain a strongly polynomial algorithm for linear Fisher markets (the first such algorithm is due to Orlin [11]). This also extends to the more general setting of spending constraint utilities, defined by Vazirani [14]. We obtain the first strongly polynomial algorithm for this problem.

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A proof of the Boyd-Carr conjecture

Anke van Zuylen
(joint work with Frans Schalekamp, David P. Williamson)

The traveling salesman problem (TSP) is the most famous problem in discrete optimization. Given a set of n cities and the costs $c(i,j)$ of traveling from city i to city j for all i,j, the goal of the problem is to find the least expensive tour that visits each city exactly once and returns to its starting point. An instance of the TSP is called *symmetric* if $c(i,j) = c(j,i)$ for all i,j; it is *asymmetric* otherwise. Costs obey the *triangle inequality* if $c(i,j) \leq c(i,k)+c(k,j)$ for all i,j,k. The TSP is known to be NP-hard, even in the case that instances are symmetric and obey the triangle inequality. From now on we consider only these instances.

Because of the NP-hardness of the traveling salesman problem, researchers have considered approximation algorithms for the problem. The best approximation algorithm currently known is a $\frac{3}{2}$-approximation algorithm given by Christofides in 1976 [4]. Better approximation algorithms are known for special cases. Exciting progress has been made recently in the case of graph-TSP, in which costs $c(i,j)$ are given by shortest path distances in an unweighted graph (see [10, 8, 9, 3]). However, to date, Christofides' algorithm has the best known performance guarantee for the general case.

There is a well-known, natural direction for making progress which has also defied improvement for nearly thirty years. The following linear programming relaxation of the traveling salesman problem was used by Dantzig, Fulkerson, and Johnson [5] in 1954. For simplicity of notation, we let $G = (V,E)$ be a complete undirected graph on n nodes. In the LP relaxation, we have a variable $x(e)$ for all $e = (i,j)$ that denotes whether we travel directly between cities i and j on our tour. Let $c(e) = c(i,j)$, and let $\delta(S)$ denote the set of all edges with exactly one endpoint in $S \subseteq V$. Then the relaxation is

$$\text{Min} \sum_{e \in E} c(e)x(e)$$

subject to:

$$(1) \qquad \sum_{e \in \delta(i)} x(e) = 2, \qquad \forall i \in V, \qquad (SUBT)$$

$$(2) \qquad \sum_{e \in \delta(S)} x(e) \geq 2, \qquad \forall S \subset V,\ 3 \leq |S| \leq |V| - 3,$$

$$(3) \qquad 0 \leq x(e) \leq 1, \qquad \forall e \in E.$$

The first set of constraints (1) are called the *degree constraints.* The second set of constraints (2) are sometimes called *subtour elimination constraints* or sometimes just *subtour constraints*, since they prevent solutions in which there is a subtour of just the nodes in S. As a result, the linear program is sometimes called the *subtour LP*.

The LP is known to give excellent lower bounds on TSP instances in practice, coming within a percent or two of the length of the optimal tour (see, for instance, Johnson and McGeoch [7]). However, its theoretical worst-case is not well understood. In 1980, Wolsey [12] showed that Christofides' algorithm produces a solution whose value is at most $\frac{3}{2}$ times the value of the subtour LP (also shown later by Shmoys and Williamson [11]). This proves that the *integrality gap* of the subtour LP is at most $\frac{3}{2}$; the integrality gap is the worst-case ratio, taken over all instances of the problem, of the value of the optimal tour to the value of the subtour LP, or the ratio of the optimal integer solution to the optimal fractional solution. The integrality gap of the LP is known to be at least $\frac{4}{3}$ via a specific class of instances. However, no instance is known that has integrality gap worse than this, and it has been conjectured for some time that the integrality gap is at most $\frac{4}{3}$ (see, for instance, Goemans [6]).

Not only do we not know the integrality gap of the subtour LP, Boyd and Carr have observed that we don't even know the worst-case ratio of the optimal 2-matching to the value of the subtour LP, which is surprising because 2-matchings are well understood and well characterized. A *2-matching* is an integer solution to the subtour LP obeying only the degree constraints (1) and the bounds constraints (3).[1] A *fractional 2-matching* is a 2-matching without the integrality constraints. Boyd and Carr make the following conjecture.

Conjecture 2 (Boyd and Carr [1]). *The worst-case ratio of an optimal 2-matching to an optimal solution to the subtour LP is at most* $\frac{10}{9}$.

It is known that there are cases for which the cost of an optimal 2-matching is at least $\frac{10}{9}$ times the optimal solution to the subtour LP. In the general case, the only bound on this ratio we know is one of Boyd and Carr [2], who show that the integrality gap of 2-matchings is at most $\frac{4}{3}$; since the constraints of the subtour

[1] We note that what we refer to here as 2-matchings are also sometimes called 2-factors.

LP are a superset of the fractional 2-matching constraints, this implies the ratio is at most $\frac{4}{3}$.

The contribution of this talk is to improve our state of knowledge for the subtour LP by proving Conjecture 2. We start by showing that in some cases the cost of an optimal 2-matching is at most $\frac{10}{9}$ the cost of a fractional 2-matching, which is a stronger statement than Conjecture 2; in particular, we show this is true whenever the support graph of the fractional 2-matching is biconnected. As the first step in this proof, we give a simplification of the Boyd and Carr result bounding the integrality gap for 2-matchings by $\frac{4}{3}$. In the case that the support of an optimal fractional 2-matching is biconnected, the proof becomes quite simple. The perfect matching polytope plays a crucial role in the proof: we use the matching edges to show us which edges to remove from the solution in addition to showing us which edges to add. We note that this idea was independently developed in the recent work of Mömke and Svensson [8]. We also use a notion from Boyd and Carr [2] of a *graphical* 2-matching: in a graphical 2-matching, each node has degree either 2 or 4, each edge has 0, 1, or 2 copies, and each component has size at least three. Given the triangle inequality, we can shortcut any graphical 2-matching to a 2-matching of no greater cost.

To obtain our proof of the Boyd-Carr conjecture, we give a polyhedral formulation of the graphical 2-matching problem, and use it to prove Conjecture 2. If x is a feasible solution for the subtour LP, then, roughly speaking, we show that $\frac{10}{9}x$ is feasible for the graphical 2-matching polytope. Our previous results give us intuition for the precise mapping of variables that we need. Using the graphical 2-matching polytope allows us to overcome the issues with the degree constraints faced in trying to use Goemans' results.

We conclude by posing a new conjecture, namely that the worst-case integrality gap is achieved for solutions to the subtour LP that are fractional 2-matchings (that is, for instances such that adding the subtour constraints to the degree constraints and the bounds on the variables does not change the objective function value).

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