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Arbeitsgemeinschaft: Quasiperiodic Schrödinger Operators

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Abstract. This Arbeitsgemeinschaft discussed the spectral properties of quasi-periodic Schrödinger operators in one space dimension. After presenting background material on Schrödinger operators with dynamically defined potentials and some results about certain classes of dynamical systems, the recently developed global theory of analytic one-frequency potentials was discussed in detail. This was supplemented by presentations on an important special case, the almost Mathieu operator, and results showing phenomena exhibited outside the analytic category.

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Introduction by the Organisers

This Arbeitsgemeinschaft was organized by Artur Avila (Paris 7 and IMPA Rio de Janeiro), David Damanik (Rice), and Svetlana Jitomirskaya (UC Irvine) and held April 1–7, 2012. There were 31 participants in total, of whom 26 gave talks.

The objective was to discuss quasiperiodic Schrödinger operators of the form

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$

with potential $V : \mathbb{Z} \to \mathbb{R}$ given by $V(n) = f(\omega + n\alpha)$, where $\omega, \alpha \in \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, $\lambda \in \mathbb{R}$ and $\alpha = (\alpha_1, \dots, \alpha_k)$ is such that $1, \alpha_1, \dots, \alpha_k$ are independent over the rational numbers, and $f : \mathbb{T}^k \to \mathbb{R}$ is assumed to be at least continuous.

H acts on the Hilbert space $\ell^2(\mathbb{Z})$ as a bounded self-adjoint operator. The associated unitary group $\{e^{-itH}\}_{t\in\mathbb{R}}$ describes the evolution of a quantum particle subjected to the quasiperiodic environment given by V. For any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, $|\langle \delta_n, e^{-itH}\psi\rangle|^2$ is the probability of finding the particle, whose initial state at time zero is given by the ℓ^2-normalized ψ, at time t at site n. The long-time behavior of these probabilities is of interest and many relevant questions about them can be studied by means of spectral theory (i.e., by "diagonalizing" the operator H). The spectral theorem for self-adjoint operators associates a measure μ_ψ with the initial state ψ. Roughly speaking, the more continuous the measure μ_ψ is, the faster $e^{-itH}\psi$ spreads. For this reason, one wants to determine the spectral type of H. For example, H is said to have purely absolutely continuous (resp., purely singular continuous or pure point) spectrum if every μ_ψ is purely absolutely continuous (resp., purely singular continuous or pure point). Again roughly speaking, the absolutely continuous case corresponds to transport, whereas the pure point case typically corresponds to the absence of transport ("dynamical localization"), while the singular continuous case corresponds to intermediate transport behavior. In the case where not all spectral measures have the same type, one collects all those states whose measures have the same type in a single subspace, restricts the operator to the resulting three subspaces, and the spectra of these three restrictions are then called the absolutely continuous, singular continuous, and pure point spectrum of H, respectively.

In the recent past, the spectral analysis of quasiperiodic Schrödinger operators has seen great advances. It was the goal of this Arbeitsgemeinschaft to present many of these advances.

It is useful to regard quasiperiodic potentials as being dynamically defined in the sense that they are obtained by sampling along the orbit of an ergodic transformation with a real-valued sampling function. Concretely, if we consider the map $T : \mathbb{T}^k \to \mathbb{T}^k$, $\omega \mapsto \omega + \alpha$, it is invertible and has normalized Lebesgue measure as its unique invariant Borel probability measure. Then, the potential V may be obtained as $V(n) = f(T^n\omega)$. More generally, whenever we have such a dynamically defined situation with an ergodic (Ω, T, μ) and a (bounded) measurable $f : \Omega \to \mathbb{R}$, several fundamental results hold. Namely, the spectrum, as well as the absolutely continuous spectrum, the singular continuous spectrum, and the point spectrum, of H are μ-almost surely independent of ω and are denoted by $\Sigma, \Sigma_{\mathrm{ac}}, \Sigma_{\mathrm{sc}}, \Sigma_{\mathrm{pp}}$. Moreover, the density of states dk, given by

$$\int_\Omega \langle \delta_0, g(H)\delta_0 \rangle \, d\mu(\omega) = \int_{\mathbb{R}} g(E)\, dk(E)$$

and the Lyapunov exponent

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|A_E^n(\omega)\| \, d\mu(\omega),$$

where

$$A_E^n(\omega) = \begin{pmatrix} E - f(T^{n-1}\omega) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

are defined and determine Σ and Σ_{ac} as follows,

$$\Sigma = \operatorname{supp} dk, \qquad \Sigma_{\mathrm{ac}} = \overline{\{E \in \mathbb{R} : L(E) = 0\}}^{\mathrm{ess}}.$$

These two fundamental results can be considered classical and have been known since the early 1980's.

Note that $L(E) \geq 0$ and hence one naturally distinguishes between the two cases $L(E) = 0$ and $L(E) > 0$. The result quoted above shows that the first case is connected to the absolutely continuous part, whereas the general tendency is for the second case to be connected to the pure point part. Indeed, among the major recent advances are ways to go from positive Lyapunov exponents to pure point spectral measures (and more, such as exponentially decaying eigenfunctions, dynamical localization, etc.) in the quasiperiodic case with sufficiently regular sampling function f and for most α. Another major recent development is that in the case $k = 1$ (i.e., $\alpha, \omega \in \Omega = \mathbb{T}$), the regime of zero Lyapunov exponents has been studied in a global sense and it has been shown for analytic f, that one typically has purely absolutely continuous spectrum there. As a consequence, one now understands the typical spectral type of a one-frequency quasiperiodic Schrödinger operator with analytic sampling function.

Arbeitsgemeinschaft: Quasiperiodic Schrödinger Operators

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Abstracts

Schrödinger operators and quantum dynamics

Ilya Kachkovskiy

1. Quantum Dynamics. The main object of study during this Arbeitsgemeinschaft is the 1-dimensional discrete Schrödinger operator $H = \Delta + V$ defined by

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$

in a Hilbert space $\mathcal{H} = l^2(\mathbb{Z})$. Here $V : \mathbb{Z} \to \mathbb{R}$ is a bounded function. The operator H is a bounded self-adjoint operator in $\mathcal{H}$ and is the energy operator of a quantum particle with wave function ψ.

The time-evolution of the wave function is described by the Schrödinger equation

$$\begin{cases} i\partial_t \psi(t) = H\psi(t) \\ \psi(0) = \psi_0. \end{cases}$$

It can be immediately solved by $\psi(t,n) = (e^{-itH}\psi_0)(n)$. In the first part of the talk we discuss the relations between the behaviour of e^{-itH} and spectral properties of H.

By spectral theorem, we have

$$e^{-itH} = \int_{\mathbb{R}} e^{-it\lambda}\, dE_H(\lambda),$$

where E_H is the projector-valued spectral measure of H. The support of this measure is the spectrum $\sigma(H)$. The Hilbert space $\mathcal{H}$ can be uniquely decomposed into $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}$ such that the corresponding projectors commute with H, and for each $f \in \mathcal{H}_\alpha$ the Borel measure $\delta \mapsto (E_H(\delta)f, f)$ has the following type — pure point, absolutely continuous or singular continuous. This gives us the classification of types of spectra: $\sigma_\alpha(H) = \sigma(H|_{\mathcal{H}_\alpha})$, $\alpha = pp$, ac or sc, see [3]. The set $\sigma_{pp}(H)$ admits a direct characterization as the closure of the set of all eigenvalues

$$\sigma_{pp}(H) = \overline{\{\lambda \in \mathbb{R} \colon \exists \psi \in \mathcal{H}, \psi \neq 0 \text{ such that } H\psi = \lambda\psi\}}.$$

We now proceed to the dynamical characterization of spectra. The first result of this kind is RAGE Theorem, see [4] Let

$$(\chi_L\psi)(n) = \begin{cases} \psi(n), & |n| \le L \\ 0, & |n| > L \end{cases}$$

Theorem 1 (RAGE).

$$\mathcal{H}_c = \{\psi \in \mathcal{H} \mid \lim_{L\to\infty} \frac{1}{T} \lim_{T\to\infty} \int_0^T \|\chi_L e^{-itH}\psi\|\, dt = 0\}$$

$$\mathcal{H}_{pp} = \{\psi \in \mathcal{H} \mid \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T \|(1-\chi_L)e^{-itH}\psi\|\, dt = 0\}.$$

This result is a characterization of the type of the spectrum in terms of the "average" behaviour of e^{-iHt}. From this theorem, it follows that the projector P_c onto $\mathcal{H}_c = \mathcal{H}_{ac} + \mathcal{H}_{sc}$ can be expressed as the following weak limit,

$$(P_c\varphi, \psi) = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T ((1-\chi_L)e^{-iHt}\varphi, e^{-iHt}\psi)\, dt.$$

The next few results are regarding the asymptotic behaviour of $e^{-iHt}\psi$. If $\psi \in \mathcal{H}_{pp}$, then the scalar product $(e^{-iHt}\psi, \psi)$ gets close to $\|\psi\|^2$ infinitely many times for arbitrarily large t, which is an example of *dynamical localization.* On the contrary, if $\psi \in \mathcal{H}_{ac}$, then $(e^{-iHt}\psi, \psi) \to 0$ as $t \to \infty$ which simply follows from Riemann-Lebesgue lemma. Another important notion here is the phenomenon of *ballistic transport behaviour* which happens when

$$t^{-2} \sum_{n\in\mathbb{Z}} |n|^2 |\psi(t,n)|^2 \not\to 0, \quad t \to \infty.$$

This physically corresponds to the particle moving to infinity with constant speed.
Theorem (Simon, 1990). *Let $\sigma(H) = \sigma_{pp}(H)$. Let ψ_0 be a localized state (i. e. finitely supported on $\mathbb{Z}$). Then*

$$\lim_{t\to\infty} t^{-2} \sum_{n\in\mathbb{Z}} |n|^2 |\psi(t,n)|^2 = 0,$$

therefore there is no ballistic transport for this initial data. The power of t cannot be improved.

The last set of results is related to the behaviour of the solutions of the eigenfunction equation

$$\psi(n+1) + \psi(n-1) + V(n)\psi(n) = E\psi(n). \tag{1}$$

Theorem (Schnol, see [1, Section 2.4]). *Let there exist a polynomially bounded solution of* (1). *Then $E \in \sigma(H)$.*

The last result concerning the characterization of spectra is as follows. For simplicity, let us study the solutions of (1) for $n \in \mathbb{N}$. A solution ψ is called *subordinate*, if

$$\lim_{L\to\infty} \frac{\|\chi_L\psi\|}{\|\chi_L\varphi\|} = 0$$

for any other solution φ. For details, see, for example, [2].
Theorem (Gilbert-Pearson). *For the discrete Schrödinger operator H on $\mathbb{N}$, the measure $E_{ac}(H)$ is essentially supported on the set of energies for which there exist no subordinate solutions.*

2. Ergodic operators. We are going to study a special class of potentials related to dynamical systems. See, for example, [1, Ch. 9] for details. Let (Ω, P) be a probability space. A measure-preserving bijection $T\colon \Omega \to \Omega$ is called *ergodic*, if any T-invariant measurable set $A \subset \Omega$ has either $P(A) = 1$ or $P(A) = 0$. By a *dynamically defined* potential we denote a family $V_\omega(n) = f(T^n\omega)$, $\omega \in \Omega$, where $f\colon \Omega \to \mathbb{R}$ is a measurable function. The corresponding family of operators H_ω is called an *ergodic family.*

A very important case of such families is *almost periodic operators.* Here $\Omega = S^1$ — a unit circle, the map T is a rotation over an angle $\alpha \notin 2\pi\mathbb{Q}$, so

$$(H_\omega\psi)(n) = \psi(n+1) + \psi(n-1) + V(n\alpha + \omega)\psi(n),$$

where $v\colon \mathbb{R} \to \mathbb{R}$ is a continuous 2π-periodic function.

Even though the family H_ω depends on a random parameter, it makes sense to speak about certain typical behaviour. The most famous result of this type is Pastur's theorem which, informally speaking, says that the spectra of the operators H_ω are not random.

Theorem (Pastur). *There exists a set $\Omega_0 \subset \Omega$ and $\Sigma, \Sigma_{pp}, \Sigma_{ac}, \Sigma_{sc} \subset \mathbb{R}$ such that for all $\omega \in \Omega_0$ we have $\sigma(H_\omega) = \Sigma$, $\sigma_\alpha(H_\omega) = \Sigma_\alpha$, $\alpha = pp, ac, sc$.*

The idea of the proof is based on the fact that if $f(T\omega) = f(\omega)$ a. e. on Ω for a measurable f, then f is a constant almost surely. A slightly more strong result holds for almost periodic operators — for Σ, we may take $\Omega_0 = \Omega$ by a simple continuity argument (which, however, does not work for Σ_α). It can also be shown that a single E is an eigenvalue of H_ω with zero probability, but it does not imply that the point spectrum may not be typical.

The last notion we want to introduce is *density of states.* Informally, it is a continuous analogue of the normalized eigenvalue counting function. In general, it's a measure $dK(E)$ which may be defined as

$$\int\limits_\Delta dk(E) = \lim_{L\to\infty} \frac{1}{2L+1} \dim \operatorname{Ran}(\chi_L E_{H_\omega}(\Delta)\chi_L).$$

in the weak sense. The limit exists and is equal to $\int f(E)\,dk(E) = \mathbb{E}(f(H_\omega)e_0, e_0)$ The function $k(E) = \int_{(-\infty;R)} dk(E)$ is called *integrated density of states.*

Theorem (Avron-Simon). *Almost surely* $\operatorname{supp}(dk) = \sigma(H_\omega)$.

Theorem (Craig-Simon ($d = 1$), Avron-Souillard ($d \geq 2$)). *The integrated density of states $k(E)$ is a continuous function of E.*

The last result cannot be significantly improved.

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Basic properties of $SL(2,\mathbb{R})$ cocycles and connections to spectral theory

William N. Yessen

All of what we discuss below can be followed in detail in [1, 2] and in [3, Chapters V and VI].

The setup. Let S^1 denote the unit circle, and for $\alpha \in \mathbb{R}$, let $f_\alpha : S^1 \to S^1$ be rotation by α; that is,

$$f_\alpha(x) = x + \alpha \mod 1.$$

Notice that if $\alpha \notin \mathbb{Q}$, then f_α is uniquely ergodic, with the ergodic measure being the Lebesgue measure on S^1; moreover, the dynamical system (f_α, S^1) is minimal (i.e. the orbit under f_α of each $x \in S^1$ is dense in S^1). We shall need these facts later, either in this presentation or in those that follow, or both.

Now let $A : S^1 \to SL(2,\mathbb{R})$ of class C^r, with $r \in \mathbb{N} \cup \{0, \infty, \omega\}$. The $SL(2,\mathbb{R})$ *cocycle (of class C^r) over f_α*, $(\alpha, A) : S^1 \times \mathbb{R}^2 \hookleftarrow$, is given by

$$(\alpha, A)(x, v) = (f_\alpha(x), A(x)v),$$

with $(\alpha, A)^n(x, v)$ given by, respectively for $n > 0$ and $n < 0$,

$$\left(f_\alpha^n(x), \left[\prod_{k=n-1}^{0} A(f_\alpha^k(x))\right] v\right) \text{ and } \left(f_\alpha^n(x), \left[\prod_{k=n+1}^{0} A(f_\alpha^k(x))\right] v\right)$$

(and $f_\alpha^0(x) = x$, $A^0 = \mathbb{I}$, the 2×2 identity matrix, $(\alpha, A)^0(x, v) = (x, v)$).

Thus (α, A) defines an invertible dynamical system on $S^1 \times \mathbb{R}$. Notice that if $\alpha \in \mathbb{Q}$, then the corresponding cocycle (α, A) is periodic and the dynamics of the cocycle is trivial.

By abuse of notation and terminology, we shall denote the cocycle (α, A) simply by A (keeping dependence on α implicit), and define $A_n(x, v) := (\alpha, A)^n(x, v)$ (or simply $A_n(v)$ when emphasis on x isn't necessary). In the literature, α is called *the frequency*, and x is referred to as *the phase*.

Motivation. Let $V : S^1 \to \mathbb{R}$ be of class C^r, $r \in \mathbb{N} \cup \{0, \infty, \omega\}$. Define *the potential* $\{V_n\}$, for $x \in S^1$ and $\alpha \in \mathbb{R}$, by $V_n = V(f_\alpha^n(x))$ for $n \in \mathbb{Z}$. Now for $\lambda > 0$, the so-called *coupling constant*, define the Schrödinger operator $H_{\alpha,x,\lambda} : \ell^2 \hookleftarrow$ by

$$(H_{\alpha,x,\lambda}\phi)_n = \phi_{n-1} + \phi_{n+1} + V_n\phi_n.$$

This operator is called *periodic* provided that $\alpha \in \mathbb{Q}$, and *quasiperiodic* otherwise (notice that if $\alpha \in \mathbb{Q}$, then the sequence $\{V_n\}_{n\in\mathbb{Z}}$ is q–periodic, where $\alpha = p/q$, $\gcd(p, q) = 1$). Obviously $H_{\alpha,x,\lambda}$ is bounded and self-adjoint. It is also known that the spectrum of this operator consists of those *energies* $E \subset \mathbb{R}$, for which

$$H_{\alpha,x,\lambda}\theta = E\theta$$

is satisfied, with $\{\theta_n\} \in \ell^2$ not diverging exponentially at $\pm\infty$. On the other hand one can easily verify that the last equation is equivalent to

$$\begin{pmatrix} \theta_{n+1} \\ \theta_n \end{pmatrix} = A_n^E \begin{pmatrix} \theta_1 \\ \theta_0 \end{pmatrix}, \quad \text{where } A^E = \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}$$

is the *Schrödinger cocycle* (the recursion above is given for $n > 0$; a similar recursion holds with $n < 0$ by inverting the cocycle A^E as in the previous section). Notice that A^E is of the same smoothness class as V, and $A^E \in SL(2, \mathbb{R})$. If $\alpha \in \mathbb{Q}$, then the cocyle is periodic (as in the previous section). By Floquet theory, in this case the spectrum of $H_{\alpha,x,\lambda}$, $\sigma(H_{\alpha,x,\lambda})$, is a finite union of compact intervals. In general, as should now be apparent, dynamics of A^E plays a central role in determination of the spectrum of $H_{\alpha,x,\lambda}$ (and even spectral type, as discussed briefly below, and in detail in the other presentations).

Notions of hyperbolicity and Lyapunov exponents. We shall say that a cocycle A is *uniformly hyperbolic* provided the following.

- There exist sections $S^\pm : S^1 \to \mathbb{PR}^2$, with $S^+ \oplus S^- = \mathbb{R}^2$, such that for any $x \in S^1$ and $v \in S^\pm(x)$, $A(x, v) \in S^\pm(f_\alpha(x))$ and $A^{-1}(x, v) \in S^\pm(f_\alpha^{-1}(x))$;
- there exist constants $C > 0$ and $0 < \gamma < 1$ such that for any $x \in S^1$ and $v \in S^\pm(x)$, we have, for $n \in \mathbb{N}$, $\|A^{\pm n}(x, v)\| \leq C\gamma^n \|v\|$.

A theorem of Johnson from 1986 gives

Theorem. *The Schrödinger cocycle A^E is uniformly hyperbolic if and only if E is not in the spectrum of the corresponding Schrödinger operator.*

Define

$$\mathcal{UH} := \{E \in \mathbb{R} : A^E \text{ is uniformly hyperbolic}\}.$$

Then we have $\sigma^c(H_{\alpha,x,\lambda}) = \mathcal{UH}$. Next, let us give a characterization of σ in terms of the dynamics of the cocycle A^E.

For a given cocycle A, we define the *Lyapunov exponent* of A by

$$\mathcal{L}(A) := \inf_n \frac{1}{n} \int_{S^1} \log \|A_n(x, \cdot)\| d\mu = \lim_{n\to\infty} \frac{1}{n} \int_{S^1} \log \|A_n(x, \cdot)\| d\mu \in [0, \infty)$$

(equality follows by subadditivity, and the inclusion on the right follows since $A \in SL(2, \mathbb{R})$; μ is the Lebesgue measure). Moreover (assuming $\alpha \notin \mathbb{Q}$), by the subadditive ergodic theorem, we have $\mathcal{L}(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A_n\|$ almost everywhere in $x \in S^1$. Now define

$$\mathcal{NUH} := \{E \in \mathbb{R} : \mathcal{L}(A^E) > 0 \text{ and } E \notin \mathcal{UH}\} \text{ and } \mathcal{Z} := \{E \in \mathbb{R} : \mathcal{L}(A^E) = 0\}.$$

Obviously $\sigma = \mathcal{NUH} \bigcup \mathcal{Z}$. $\mathcal{NUH}$ stands for *non-uniformly hyperbolic.*

Kotani theory, $\sigma_{ac} = \overline{\mathcal{Z}}^{\text{ess}}$ (this is discussed in more detail in later presentations).

A fundamental relation between the spectrum (or, more precisely, the integrated density of states measure on the spectrum) and the Lyapunov exponent is given by the *Thouless formula* (first rigorously proved by Avron and Simon):

Thoerem. *For every $z \in \mathbb{C}$,*

$$\mathcal{L}(A^E) = \int_{\mathbb{R}} \log |E - z| dk(E)$$

where k is the density of states measure.

Conjugacy and reducibility. As is apparent from the previous section, dynamics of A^E completely determines $\sigma(H_{\alpha,x,\lambda})$. To investigate A^E, it is desirable to find another cocyle, which is easier to analyze, whose dynamical properties concide with those of A^E. This brings us to the concept of conjugacy and reducibility.

We say that two C^r cocycles A_1, A_2 are $C^{r'}$ *conjugate* (with $r' \leq r$) if there exists $B : S^1 \to SL(2, \mathbb{R})$ of class $C^{r'}$ such that $B(f_\alpha(x))A_1 = A_2 B(x)$. Then all dynamical information of A_1 (in the $C^{r'}$ category) is contained in A_2. We say that a cocycle A is *reducible* if it is conjugate to a constant cocycle.

Remark. Sometimes it is required to take $B \in PSL(2, \mathbb{R})$. For example, there are cocycles which are reducible with $B \in PSL$ but not with $B \in SL$.

The question of conjugacy, reducibility and in general dynamics of cocycles is a very deep one, and this area of research has enjoyed tremendous progress in recent years, owing to contributions of Avila, Damanik, Krikorian, Jitomirskaya, Bourgain, and many others (this is discussed in detail in the other presentations, and the reader should follow the references therein).

The rotation number. Assume that A is a continuous cocycle homotopic to the identity (for example, the Schrödinger cocycle A^E); then the same holds for the map

$$F : (x, v) \mapsto \left(f_\alpha(x), \frac{A(x)v}{\|A(x)v\|} \right).$$

Hence F above can be lifted to a continuous map $\widetilde{F} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ of the form $\widetilde{F}(x, v) = (f_\alpha(x), x + g(x, v))$ with $g(x, v + 1) = g(x, v)$ and $\pi(v + g(x, v)) = A(x)\pi(v)/\|A(x)\pi(v)\|$, where $\pi(v) = e^{i2\pi v}$. Since f_α is uniquely ergodic when $\alpha \notin \mathbb{Q}$, for such α we can employ a theorem of Herman and Johnson-Moser:

Theorem. *For every $(x, v) \in S^1 \times \mathbb{R}$, the limit*

$$\lim_{n \to \pm\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\widetilde{F}^k(x, v))$$

exists, is independent of (x, v) and the convergence is uniform in (x, v). We call this limit the rotation number of A. This limit, modulo $\mathbb{Z}$, is independent of the choice of the lift.

We conclude with the following relationship between the rotation number, call it ρ, and the density of states measure.

Theorem. *With k denoting the density of states for $H_{\alpha,x,\lambda}$, we have $k(E) = 1 - 2\rho(A^E)$.*

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Kotani theory: Lyapunov exponents and the absolutely continuous spectrum

Florian Metzger

1. Introduction

The aim of this talk is to discuss the link between the absolutely continuous spectrum and the Lyapunov exponents associated to the eigenvalue problems $H_\omega \psi = E\psi$ of an ergodic family of Schrödinger operators $(H_\omega)_{\omega\in\Omega}$ for $\Omega = \mathbb{T}$ endowed with the measure $\mu = \text{Leb}$ and a μ-invariant ergodic transformation T (namely an irrational rotation $x \mapsto x + \alpha$), and to see which consequences on the Lebesgue measure of the spectrum can be obtained.

2. The main theorem

The main result is the Ishii-Pastur-Kotani theorem:

Theorem 2.1. $\Sigma_{\mathrm{ac}} = \overline{\{E \in \mathbb{R} : L(E) = 0\}}^{\mathrm{ess}} =: \overline{Z}^{\mathrm{ess}}$ *where* Σ_{ac} *denotes the almost surely constant absolutely continuous part of the spectrum of* (H_ω) *and* $L(E) = L(E,\omega)$ *for* μ*-a.e.* ω*, the Lyapunov exponent for* $H_\omega\psi = E\psi$*.*

2.1. The inclusion $\Sigma_{\mathrm{ac}} \subseteq \overline{Z}^{\mathrm{ess}}$.

It is a result of Ishii and Pastur and is based on classical arguments. The main proposition which gives the result is the following:

Proposition 2.1. *Suppose that* $L(E) > 0$ *for Lebesgue almost every* $E \in (a,b)$*, then* $\mathrm{E}^{\mathrm{ac}}_{(a,b)}(H) = 0$ *where* $\mathrm{E}^{\mathrm{ac}}_{(a,b)}(H)$ *denotes the spectral projection on the absolutely continuous part.*

The Fubini theorem ensures that for μ-a.e. ω and Leb-a.e. E, $L(E,\omega) > 0$. Then, using Berezansky's theorem which states that ρ-a.e. $E \in \mathbb{R}$ – where ρ denotes a spectral measure for the operator – is a generalized eigenvalue, and Oseledets' theorem, one can see that the only generalized eigenfunctions are exponentially localized, so in ℓ^2, but there are only countably many such E's since $\ell^2(\mathbb{Z})$ is separable. So, splitting (a,b) into parts where H has generalized eigenfunctions or where $L(E,\omega) = 0$ for a.e. ω, $\mathrm{E}^{\mathrm{ac}}_{(a,b)}(H_\omega) = 0$ can be easily proved for a.e. ω.

A simple argument then gives: $(\overline{Z}^{\mathrm{ess}})^c \subseteq (\mathrm{supp}\,\mathrm{E}^{\mathrm{ac}}(H_\omega))^c = (\mathrm{supp}\,\rho^{\mathrm{ac}})^c = {\Sigma_{\mathrm{ac}}}^c$.

2.2. **The converse inclusion** $\Sigma_{ac} \supseteq \overline{Z}^{\text{ess}}$**.** This one is due to Kotani for the continuous case and has been generalized for the discrete one by Simon. It requires a deeper analysis and uses results on Herglotz functions, i.e., analytic functions mapping the semi upper plane onto itself. The main proposition is the following:

Proposition 2.2. *For $\Im z > 0$, denote $m_\pm(z,\omega)$ the Weyl functions (invariant sections associated to the cocycle), $b(z,\omega) = m_+(z,\omega) + m_-(z,\omega) + z - V_\omega(0)$ and $n_\pm(z,\omega) = \Im m_\pm + \frac{1}{2}\Im z$. Then, if $\mathbb{E}$ denotes the expectation on ω:*

- $\mathbb{E}\left\{\dfrac{1}{n_\pm(z,\omega)}\right\} \leq \dfrac{2L(z)}{\Im z}$
- $\mathbb{E}\left\{\left(\dfrac{1}{n_+} + \dfrac{1}{n_-}\right)\dfrac{(n_+ - n_-)^2 + (\Re b)^2}{|b|^2}\right\} \leq 4\left(\dfrac{L(z)}{\Im z} - \dfrac{\partial L(z)}{\partial \Im z}\right)$

The key lies in the fact that, due to the Thouless formula:

$$\forall \Im z > 0 \qquad L(z) = \int_\Sigma \log|z-t|dk(t)$$

the opposite of the derivative of the Lyapunov exponent is the Borel transform of the integrated density of states, a particular case of Herglotz function. From that and Herglotz theory these major theorems can be deduced:

Theorem 2.2. (1) *If $L(E) = 0$ for $E \in A \subset \mathbb{R}$ with $\text{Leb}(A) > 0$ then for a.e. ω, $\text{E}^{\text{ac}}_A(H_\omega) \neq 0$.*
(2) *If $L = 0$ a.e. on $I = (a,b) \subset \mathbb{R}$, then for a.e. ω the spectral measures ν^0_ω (associated to δ_0) are absolutely continuous on I.*

Then it is straightforward to prove $(\overline{Z}^{\text{ess}})^c \subseteq {\Sigma_{ac}}^c$, which is the desired inclusion.

3. Consequences on the Lebesgue measure of the a.c. spectrum

3.1. **Inequalities.** Then, once this major result is set up one can prove that

Theorem 3.1. *For all $a < b \in \mathbb{R}$, $\text{Leb}\{E \in [a,b] \,:\, L(E) = 0\} \leq 4$.*

The conclusion on the Lebesgue measure of the a.c. spectrum is then similar. This result is an easy consequence of the following main estimation:

Proposition 3.1. *For a.e. $E \in Z$, $\quad -4\sin\big(2\pi\rho(E)\big)\rho'(E) \geq 1$*

where $\rho(E)$ is the fibered rotation number of the Schrödinger cocycle on $\Omega \times \mathbb{R}^2$

$$(T, S_{E,V}) : (\omega, v) \longmapsto (T\omega, S_{E,V_\omega}v) \quad \text{where} \quad S_{E,V} = \begin{pmatrix} E-V & -1 \\ 1 & 0 \end{pmatrix}$$

ρ is related to the I.D.S. via Johnson-Moser's theorem : $k(-\infty, E) = 1 - 2\rho(E)$, so it's decreasing from $\frac{1}{2}$ at $-\infty$ to 0 at $+\infty$ (and so its derivative exists for a.e. $E \in \mathbb{R}$). The above proposition can be established using some properties of $\zeta(z) = \int_\Sigma \log(z-t)dk(t)$ for $\Im z > 0$ and the behavior of the complex rotation number $\beta(z) = \Im\zeta(z)$ as $\Im z \to 0^+$, and a fundamental lemma from harmonic analysis:

Lemma 3.1. *For all $\Im z > 0$, $2\sin\{\beta(z)\}\sinh\{L(z)\} \geq \Im z$.*

3.2. **Dealing with equalities.** Interestingly, equalities in most of the inequalities stated before imply that the potential is constant. The first useful result is that if the I.D.S. verifies $k = k_0 = k_{V=0}$, then $V = 0$. Using

$$\lim_{\varepsilon \to 0} \beta(E + i\varepsilon) = 2\pi\rho(E) \qquad \text{for a.e. } E \in \mathbb{R}$$

just one $\Im z_0 > 0$ such that $2\sin\{\beta(z_0)\}\sinh\{L(z_0)\} = \Im z_0$ is needed for the potential to be constant: indeed, harmonic function theory shows that there exists $E_0 \in \mathbb{R}$ s.t. if we set $\tilde{\zeta} = \zeta(\cdot + E_0)$ then $e^{\tilde{\zeta}}$ solves the caracteristic equation for the free cocycle, namely $X^2 - zX + 1 = 0$. Consequently $\tilde{\zeta} = \zeta_{V=0}$ is the free one, which implies the result with Johnson-Moser's theorem and the first result above.

Thanks to more Herglotz theory it is possible to prove the following :

Corollary 3.1. *If* $4\sin\big(2\pi\rho(E)\big)\rho'(E) = -1$ *for all* $E \in S \subset \mathbb{R}$ *with* $\mathrm{Leb}(S) > 0$ *s.t.* $L(E) = 0$*, then* V *is constant.*

Conversely, if $V = c$ is constant then it is straightforward with a Fourier transform to see that the spectrum of H_V is purely absolutely continuous with a spectrum $\Sigma = [-2 + c, 2 + c]$, which is obviously of Lebesgue measure 4.

4. **Constancy of the a.c. spectrum**

Finally, we mentioned the results of Last and Simon which give explicit supports for the absolutely continuous part of the spectral measure of almost periodic Schrödinger operators. Let H^+ be the restricted operator on $\ell^2\{n > 0\}$ with boundary condition $u(0) = 0$, $T_E(m, n)$ its the transfer matrix from n to m and

$$S := \left\{ E \in \mathbb{R} \ : \ \liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \|T_E(k, 0)\|^2 < +\infty \right\}$$

then S is an essential support of the a.c. part of the spectral measure for H^+ and $\mu_{\mathrm{s}}(S) = 0$ where μ_{s} is the singular part of the measure. Additionnaly, if $(m_i), (k_j)$ are sequences in $\mathbb{N}^*$ going to infinity, then another support of the a.c. part of the spectral measure is $S_1 := \left\{ E \in \mathbb{R} \ : \ \liminf_{i \to +\infty} \|T_E(m_i, k_i)\| < +\infty \right\}$.

This result is a first step to establish the constancy of the absolutely continuous spectrum not only for almost every ω, but for all such parameter in this particular context of almost periodic potentials. The precise result of Simon and Last is :

Theorem 4.1. *For* V *an almost periodic potential on* $\mathbb{Z}$*, let us consider* H *the operator on* $\ell^2(\mathbb{Z})$ *and* H_ω *the operator for* V_ω *in the hull of* V*. Then* $\Sigma_{\mathrm{ac}}(H_\omega)$*, an essential support of the a.c. part of the spectral measure, is independent of* $\omega \in \Omega$*.*

Thus, Pastur's theorem of almost surely constant a.c. spectrum is extended.

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Positivity of the Lyapunov exponent at large coupling

Nils Schippkus

This talk covered two results by Herman and Sorets-Spencer regarding estimates for the Lyapunov Exponent $L(E)$ of Schrödinger cocycles in terms of a coupling constant λ. It introduced some foundational work of the field and provided the important method of subharmonicity estimates that became important in future works and which saw use in various later talks of this conference. Consider as usual the discrete Schrödinger operator $(H\psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda V_n(\omega)\psi_n$ on $l^2(\mathbb{Z})$, where $V_n(\omega) = V(\omega + n\alpha)$ and $V : \mathbb{T} \to \mathbb{R}$ is a real analytic potential on the 1-d Torus, $\omega \in \mathbb{T}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational number and $\lambda \in \mathbb{R}^+$. Let $A_n^E(\omega)$ denote the associated n-th iterate of the cocycle. The Lyapunov Exponent $L(E)$ is then defined as $L(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log||A_n^E(\omega)||d\omega$.

The first result presented was of Herman and concerned the special case $V_n(\omega) = 2\cos(2\pi(\omega + n\alpha))$, the Almost-Matthieu Operator (AMO). It reads as follows.

Theorem: For the supercritical AMO ($\lambda > 1$) , and every energy E, we have:

$$L(E) \geq \log(\lambda)$$

independent of α.

The proof uses the analycity of the potential. V has a complex analytic extension to $\mathbb{C}$ and consequently $A_n^E(\omega)$ extends to an analytic function $A_n^E(z)$ on $\mathbb{C}$ as well. Using subharmonicity of $\log||A_n^E(z)||$, one can estimate the integral over $\mathbb{T} = S^1 \subset \mathbb{C}$ by the value at the center of the disc, proving the theorem.

Sorets-Spencer generalized this result to arbitrary real analytic potentials.
Theorem: Let $V : \mathbb{T} \to \mathbb{R}$ real analytic. Then there exists $\lambda_0 := \lambda_0(V)$, such that for every $\lambda > \lambda_0$ and every energy E, we have the estimate:

$$L(E) \geq \frac{1}{2}\log(\lambda)$$

independent of α.

The proof is a generalization of the subharmonicity argument. For general real analytic potentials V, one will have a complex analytic extension not to the disc, but merely some annulus in the complex plane $\mathbb{C}$, determined by the regularity of V. The subharmonicity estimate can therefore only be done on that annulus. A generalized Jensen's formula provides this estimate. The main technical difficulty of the proof is to determine a *good* circle of radius less than one on which the estimate will yield positivity of $L(E)$. To do so, one has to avoid the zero set of the potential. The talk covered this argument in full detail and concluded with the mention of an analogous result by Bourgain for the higher dimensional case.

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Regularity of the Lyapunov exponent I

Christian Seifert

We present preliminary results for proving Hölder continuity of the Lyapunov in energy for Diophantine frequency and joint continuity of the Lyapunov exponent in energy and frequency (which will be done in the second part).

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For $\alpha, \omega \in \mathbb{T}$ and $v\colon \mathbb{T} \to \mathbb{R}$ we consider the one-frequency Schrödinger operator

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + v(\omega + n\alpha) \quad (n \in \mathbb{Z})$$

on $\ell^2(\mathbb{Z})$. For $E \in \mathbb{R}$ we consider the eigenvalue equation $H\psi = E\psi$ and the associated N-step transfer matrix

$$A_E^N(\omega) := \prod_{j=N}^{1} \begin{pmatrix} v(\omega + j\alpha) - E & -1 \\ 1 & 0 \end{pmatrix}.$$

We prove the following large deviation theorem estimating the probability that

$$u_N(\omega) := \frac{1}{N} \log \|A_E^N(\omega)\|$$

is "far away" from its average $L_N(E) = \int_{\mathbb{T}} u(\omega)\, d\omega$ (which actually is the N-step Lyapunov exponent).

Theorem 0.2 ([1, Theorem 5.1]). *Assume $\alpha \in \mathbb{T}$ satisfies Diophantine condition*

$$\|k\alpha\| := \operatorname{dist}(k\alpha, \mathbb{Z}) > c\frac{1}{|k(|\log(1+|k|))^3} \quad (k \in \mathbb{Z} \setminus \{0\}.$$

Let $v\colon \mathbb{T}\to\mathbb{R}$ be real analytic, $A_E^N(\omega)$ the N-step transfer matrix and $L_N(E)$ the N-step Lyapunov exponent. Then, for $\kappa > N^{-1/10}$ and N large

$$\operatorname{mes}\left\{\omega\in\mathbb{T};\ \left|\frac{1}{N}\log\|A_E^N(\omega)\| - L_N(E)\right| > \kappa\right\} < Ce^{-c\kappa^2 N}.$$

The theorem actually states that $\left|u_N - \int_{\mathbb{T}} u_N\,dx\right| > \kappa$ will be a rare event (and the probability decays exponentially). The proof of this theorem rests on a bound of the Fourier coefficients $(\hat{u}(k))_{k\in\mathbb{Z}}$ of a bounded one-periodic function with bounded subharmonic extension to some strip $\mathbb{R}\times(-\varrho,\varrho)$ of the form

$$|\hat{u}(k)| \le \frac{C}{|k|} \quad (k\in\mathbb{Z}\setminus\{0\}),$$

see [1, Corollary 4.7].

There are different versions of large deviation theorems in the literature. One can weaken the Diophantine condition and still obtains exponential decay for suitable chosen κ and N.

Lemma 0.1 ([2, Lemma 4]). *Let*

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}, \quad (a,q) = 1$$

Let $\kappa\in(0,1)$. Then, for $N > C\kappa^{-2}q$,

$$\operatorname{mes}\left\{\omega\in\mathbb{T};\ \left|\frac{1}{N}\log\|A_E^N(\omega)\| - L_N(E)\right| > \kappa\right\} < e^{-c\kappa q}.$$

The second main ingredient for proving the continuity properties is the so-called Avalanche principle. It relates norms of n-fold products of unimodular matrices with the product of the norms of these matrices. This will later (i.e., in part II) be applied to obtain bounds on the Lyapunov exponents on different scales of N. Before actually stating the Avalanche principle, we need some definitions.

Definition 0.1. *Let K be a unimodular 2×2-matrix. Let u_K^+, u_K^- be the normalized eigenvectors of $\sqrt{K^*K} = |K|$. Then $Ku_K^+ = \|K\|v_K^+$ and $Ku_K^- = \|K\|^{-1}v_K^-$, where $v_K^+\cdot v_K^- = 0$ and $\|v_K^+\| = \|v_K^-\| = 1$.*

Let M, K be unimodular 2×2-matrices. Let

$$b^{(+,+)}(K,M) := v_K^+\cdot u_M^+,$$

where the quantity is certainly defined only up to a sign.

Proposition 0.1 (Avalanche Principle, see [3, Proposition 2.2]). *Let $A_1,\dots,A_n$ be unimodular 2×2-matrices satisfying*

(i) $\min_{1\le j\le n}\|A_j\| \ge \mu > n$,
(ii) $\max_{1\le j<n}\left(\log\|A_{j+1}\| + \log\|A_j\| - \log\|A_{j+1}A_j\|\right) < \frac{1}{2}\log\mu$.

Then

$$\left|\log\|A_n\cdots A_1\| - \sum_{j=1}^{n}\log\|A_j\| - \sum_{j=1}^{n-1}\log|b^{(+,+)}(A_j,A_{j+1})|\right| < C\frac{n}{\mu},$$

$$\left|\log\|A_n\cdots A_1\| + \sum_{j=2}^{n-1}\log\|A_j\| - \sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\right| < C\frac{n}{\mu}.$$

The Avalanche principle is related to the methods developped in [4]. Thinking of $\|A_j\| \approx \lambda$ for all j Young obtained $\frac{1}{n}\log\|A_n\cdots A_1\| \approx \log\lambda$.

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Regularity of the Lyapunov exponent II

Mircea Voda

The object of this talk was to sketch the proofs of the following two regularity results for the Lyapunov exponent. Note that in what follows the potential is real-analytic and the underlying dynamics are given by the shift on $\mathbb{T}$.

Theorem 1. *(Goldstein, Schlag [4]) If α satisfies a Diophantine condition of the form*

$$\|n\alpha\| > \frac{C_\alpha}{n\,(1+\log n)^a},\ a>1,$$

and $L(E) > \gamma > 0$ for $E \in I = (E', E'')$, then

$$|L(E_1) - L(E_2)| \le C\,|E_1 - E_2|^\sigma,\ E_1, E_2 \in I,$$

where $\sigma = \sigma(\gamma)$.

Theorem 2. *(Bourgain, Jitomirskaya [1]) $L(\alpha, E)$ is jointly continuous at every (α_0, E_0) with irrational α_0.*

Note. If α_0 is rational then $L(\alpha_0, E)$ is Hölder-1/2 continuous in E, but we don't necessarily have joint continuity.

Using the Mean Value Theorem it is easy to see that $L_N(\alpha, E)$ is jointly continuous, in fact

$$|L_N(\alpha_1, E_1) - L_N(\alpha_2, E_2)| \le C^N (|\alpha_1 - \alpha_2| + |E_1 - E_2|).$$

Since $L(\alpha, E) = \inf_N L_N(\alpha, E)$, we automatically get that L is jointly upper semi-continuous. To obtain better regularity for L we need to compare it to L_N in a controlled way. This can be done by first using the Avalanche Principle to obtain estimates comparing L_N's at different scales, and then by iterating this estimates.

We proceed by giving a brief overview of the proof of the Hölder continuity, following [2, Chapter 7]. Using the factorization $A_E^N(\omega) = \prod_{j=0}^{m-1} A_E^{N_0}(\omega + jN_0\alpha)$ we can apply the Avalanche Principle, with $A_j = A_E^{N_0}(\omega + jN_0\alpha)$, to get

$$|L_N + L_{N_0} - 2L_{2N_0}| < Ce^{-c_0\gamma^2 N_0},$$

where $N = mN_0$, $m \in \mathbb{N} \cap \left[\frac{1}{4}e^{c_1\gamma^2 N_0}, e^{c_1\gamma^2 N_0}\right]$, and N_0 satisfies certain restrictions. To make sure the hypotheses of the Avalanche Principle are satisfied (up to a small set in ω) one uses a large deviations estimate from the previous talk and the assumption that $L > \gamma > 0$. The Diophantine restriction on the frequency ensures that as the scale increases the estimates become better (due to the large deviations estimate). Iterating the previous estimate yields

$$|L + L_{N_0} - 2L_{2N_0}| < Ce^{-c\gamma^2 N_0}.$$

The Hölder continuity follows from the fact that we can write the above estimate for $E_1, E_2 \in I$ with $N_0 \sim -\sigma \log|E_1 - E_2|$, for an appropriate $\sigma \in (0,1)$.

Since L is upper semi-continuous, we get continuity for free at points (α_0, E_0) such that $L(\alpha_0, E_0) = 0$. Indeed

$$0 \le \liminf_n L(\alpha_n, E_n) \le \limsup_n L(\alpha_n, E_n) \le L(\alpha_0, E_0) = 0.$$

Hence, to prove continuity one can restrict to the case $L(\alpha_0, E_0) > \gamma > 0$, and try to use the same approach as in the proof of Hölder continuity. The first step is almost the same. The only difference comes from the fact that one needs to use a modified large deviations estimate (see [1, Lemma 4]) since the frequency isn't necessarily Diophantine. More precisely, if q is an approximant of the frequency then

$$|L_N + L_{N_0} - 2L_{2N_0}| < C\exp(-c_0\gamma q),$$

where $N = mN_0$, $m \in \mathbb{N} \cap \left[\frac{1}{4}e^{c_1\gamma q}, e^{c_1\gamma q}\right]$, and $N_0 \gtrsim q$ satisfies certain restrictions. Since this estimate doesn't improve as N increases, we can only iterate it a finite number of times before we lose the control. However, the next approximant of the frequency, call it q', can be arbitrarily far from q. Dealing with the situation when $q' \gg e^q$ is the main difference between the proofs for continuity and for Hölder continuity. For this one needs an extension of the Avalanche Principle (see [1, Lemma 5]) that allows for the comparison of two scales $N = mN_0$, with m arbitrarily large. Furthermore, the hypotheses of the Avalanche Principle are satisfied only up to an exceptional set coming from the large deviations estimate, so another problem when m is large, is to control the size of the exceptional set. This is done by taking advantage of the assumption that $q' \gg e^q$ (through $|\alpha - p/q| < 1/(qq')$) and by using a complexity bound for the exceptional sets (see the proof of [1, Lemma 8]). In the end one is able to build a sequence of scales and approximants $q_0 < N_0 < q_1 < \ldots < N_s < q_{s+1} < N_{s+1} < \ldots$, starting with any sufficiently large approximant (recall that we are assuming that α_0 is irrational), so that, by the same iteration procedure as before, one gets

$$|L(\alpha_0, E_0) + L_{N_0}(\alpha_0, E_0) - 2L_{2N_0}(\alpha_0, E_0)| < Ce^{-c\gamma q_0}.$$

Making sure that the above holds for any (α_n, E_n) sufficiently close to (α_0, E_0) (with the same q_0 and N_0), one immediately gets that

$$\limsup_n |L(\alpha_n, E_n) - L(\alpha_0, E_0)| < Ce^{-c\gamma q_0}.$$

Letting $q_0 \to \infty$ it follows that $L(\alpha_0, E_0) = \lim_n L(\alpha_n, E_n)$, as desired.

Finally, we mention what is known in the case when the underlying dynamics are given by the shift on $\mathbb{T}^d$, $d > 1$. Instead of Hölder continuity one gets a weaker result:

$$|L(E_1) - L(E_2)| < C \exp\left(-c\left(\log|E_1 - E_2|\right)^{\sigma}\right)$$

under the assumption that $\|\boldsymbol{\alpha} \cdot \mathbf{k}\| > C_\epsilon |\mathbf{k}|^{-(d+\epsilon)}$, $\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ (see [4]). However, the continuity still holds. More precisely, continuity in E holds for any $\boldsymbol{\alpha}_0 \in \mathbb{T}^d$, and joint continuity holds provided $\boldsymbol{\alpha}_0 \cdot \mathbf{k} \neq 0$, $\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ (see [3]).

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Classical KAM theory

Peter Gmeiner and Marcello Seri

1. Hamiltonian Setting

Consider integrable systems on a symplectic $2n$-dimensional manifold. Due to the Liouville-Arnold Theorem it is enough to consider such systems on a n-dimensional torus $\mathbb{T}^n := (\mathbb{R}^n/\mathbb{Z}^n)$.

For $G \subset \mathbb{R}^m$ open and bounded let $H_0 \in C^\omega(G)$ an integrable Hamiltonian function and define a perturbed Hamiltonian function

$$H_\epsilon(I, \varphi) := H_0(I) + \epsilon H_1(I, \varphi),$$

for some $|\epsilon| < \epsilon_0$, $\epsilon_0 > 0$ and $H_1 \in C^\omega(G \times \mathbb{T}^n)$. As usual we get the Hamiltonian differential equations

$$\text{(1)} \qquad \begin{cases} \dot{I} &= -\epsilon D_2 H_1(I,\varphi) \\ \dot{\varphi} &= \omega(I) + \epsilon D_1 H_1(I,\varphi) \end{cases}$$

with a frequency $\omega := DH_0 : G \to \mathbb{R}^n$. In general such a perturbed system is no more integrable, but we will show that under some conditions such systems are actually integrable. To make this more precise we need to define the frequencies of diophantine type.

Definition 1.1. *A frequency $\alpha \in \mathbb{R}^n$ is called* **diophantine***, if there exists a $\beta \geq 0$ and a constant $C_\beta > 0$ such that for all $k \in \mathbb{Z}\backslash\{0\}$*

$$|\langle k, \alpha\rangle| \geq \frac{C_\beta}{|k|^{n+\beta}},$$

where $\langle k, \alpha\rangle := \sum_{i=1}^{n} k_i\alpha_i$, $|k| := \sup_i |k_i|$ and $\|x\| := \inf_{p\in\mathbb{Z}} |\tilde{x}+p|$, $\tilde{x}$ is a lift of x on $\mathbb{R}$.

We need some further notation to formulate a first version of the KAM-Theorem. Let $P_\mathbb{C} := \mathbb{C}^n \times \mathbb{T}^n_\mathbb{C}$, where $\mathbb{T}^n_\mathbb{C} := (\mathbb{C}/\mathbb{Z})^n$. For some $s > 0$ we define complex phase spaces

$$P_s := \{(I,\varphi) \in P_\mathbb{C} : \|(I,\varphi)\|_P \leq s\},$$

with $\|(I,\varphi)\|_P := \max_{1\leq k\leq n} \max(|I_k|, |\mathrm{Im}\varphi_k|)$. For $U, V \subset \mathbb{C}^n$ the space of real analytic functions $g \in C(U,V)$ is denoted by $\mathcal{A}(U,V)$. The space of Hamiltonian-functions is denoted by $\mathcal{H}_s := \mathcal{A}(P_s,\mathbb{C})$ which is a Banach-space with the norm $|H|_s := \sup_{(I,\varphi)\in P_s} |H(I,\varphi)|$. $\mathcal{K}_{s,\omega} := \{H \in \mathcal{H}_s : DH|_{P_0} = (\omega, 0)\}$ is the space of integrable Hamiltonian functions.

Theorem 1.1 (KAM, Kolmogorov 1954, [5])**.** *Let ω be a diophantine frequency, $H_0 \in \mathcal{K}_{t,\omega}$ and*

$$\int_{\mathbb{T}^n} D_1^2 H_0(0,\varphi)d\varphi \in \mathrm{Mat}(n,\mathbb{R})$$

regular. Then all Hamiltonian functions $H \in \mathcal{H}_t$ in a small neighborhood of H_0 have an invariant torus with frequency ω (i.e. H_ϵ is integrable).

2. Measure of KAM-tori

There is a more global version of the KAM-Theorem. Define $\Omega := \omega(G)$ and with $\Psi : \Omega \times \mathbb{T}^n \to G \times \mathbb{T}^n$ given by $(\hat{w},\varphi) \mapsto (w^{-1}(\hat{w}),\varphi)$ we write $\mathcal{H}_\epsilon := H_\epsilon \circ \Psi : \Omega \times \mathbb{T}^n \to \mathbb{R}$. Furthermore we denote with Ω_{C_β} the set of all diophantine frequencies in Ω.

Theorem 2.1. *With the assumptions of Theorem 1 there exists a diffeomorphism $\mathcal{T}_\epsilon : \Omega \times \mathbb{T}^n \to \Omega \times \mathbb{T}^n$ which transforms (1) on $\Omega_{\sqrt{\epsilon}} \times \mathbb{T}^n$ into a system*

$$\frac{d}{dt}\hat{w}(t) = 0, \qquad \frac{d}{dt}\varphi(t) = \hat{w}(t),$$

with $\lambda^n(\Omega_{\sqrt{\epsilon},n}\cap\Omega) = \lambda^n(\Omega)(1-O(\sqrt{\epsilon})$ *(where* $\lambda^n(.)$ *is the* n*-dimensional Lebesgue measure).*

For a more specific version of the KAM-Theorem we need some notation. For $r \geq 0$, $r = \infty$ or $r = \omega$ we denote the space of r-times differentiable functions with $C^r(\mathbb{T}^n)$ if r is an integer and the space of Hölder-continuous functions if it is not an integer. $\mathrm{Diff}^r(\mathbb{T}^n)$ is the group of diffeomorphisms on the torus of class C^r and $\mathrm{Diff}^r_+(\mathbb{T}^n) := \{f \in \mathrm{Diff}^r(\mathbb{T}^n) : f$ is C^r isotope to the identity$\}$. Furthermore we define $D^r(\mathbb{T}^n) := \{f \in \mathrm{Diff}^r(\mathbb{T}^n) : f = Id + \varphi, \varphi \in C^r(\mathbb{T}^n, \mathbb{R}^n)\}$ and $D^r(\mathbb{T}^n, 0) := \{f \in D^r(\mathbb{T}^n) : f(0) = 0\}$.

Theorem 2.2 (Herman, 1983)**.** *Let* $\alpha \in \mathbb{R}^n$ *be diophantine. Let*

$$\Theta = \begin{cases} n+\beta & \text{if } \beta \neq 0 \text{ and } \beta \notin \mathbb{Z} \\ n+\beta+\epsilon & \text{if } \beta = 0 \text{ or } \beta \in \mathbb{Z},\ \epsilon > 0 \end{cases}$$

Then there exists a neighborhood $V^{2\Theta}_{R_\alpha}$ *of a rotation* $R_\alpha : \mathbb{T}^n \to \mathbb{R}^n$ *in* $D^{2\Theta}(\mathbb{T}^n)$ *and a mapping* $s : V^{2\Theta}_{R_\alpha} \to \mathbb{R}^n \times D^{\Theta}_+(\mathbb{T}^n, 0)$ *such that* s *is continuous and if* $s(f) = (\lambda, g)$ *then* $f = R_\lambda \circ g \circ R_\alpha \circ g^{-1}$*.*

3. KAM Theorem corollaries for cocycles close to a constant

Let $A : \mathbb{T}^{n-1} \to \mathrm{SL}(2,\mathbb{R})$ be a continuous application homotopic to the identity, then the same can be said for the linear cocycle associated to A

$$(\alpha, A) := F : \mathbb{T}^{n-1} \times \mathbb{S}^1 \to \mathbb{T}^{n-1} \times \mathbb{S}^1, \quad F(\theta, w) := \left(\theta + \alpha, \frac{A(\theta)w}{\|A(\theta)w\|}\right)$$

and thus it admits a continuous lift

$$\widetilde{F} : \mathbb{T}^{n-1} \times \mathbb{R} \to \mathbb{T}^{n-1} \times \mathbb{R}, \quad \widetilde{F}(\theta, x) := (\theta + \alpha, x + f(\theta, x))$$

with $f(\theta, x+1) = f(\theta, x)$ and $\pi(x + f(\theta, x)) = A(\theta)\pi(x)/\|A(\theta)\pi(x)\|$ where π is just the canonical projection on the circle $\pi : \mathbb{R} \to \mathbb{S}^1$. Such a map is independent from the choice of the lift up to the addition of an integer.

Definition 3.1. *The* ***fibered rotation number*** [4, Section 5] *is the topological invariant*

$$\rho_f(A) := \limsup_{n\to\infty} \frac{1}{n} f\left(\widetilde{F}^n(\theta, x)\right) (\mathrm{mod} 1) \in \mathbb{T}^1. \tag{2}$$

If α is irrational, $\rho_f(A)$ is independent on (θ, x) and converge uniformly with respect to (θ, x), moreover ρ_f is a continuous non-increasing function of $A \in C^0\left(\mathbb{T}^{n-1}, \mathrm{SL}(2,\mathbb{R})\right)$.

Definition 3.2. *We will say that two cocycles maps associated to* A*,* A_1 *are* ***conjugated*** *if there exists a continuous map* $B : \mathbb{T}^{n-1} \to \mathrm{SL}(2,\mathbb{R})$ *(****conjugacy matrix****) such that*

$$A_1(\theta) = B(\theta + \alpha)A(\theta)B^{-1}(\theta), \qquad \forall \theta \in \mathbb{T}^{n-1}.$$

Definition 3.3. *A cocycle is said* ***reducible*** *if it is* PSL*-conjugated to a rotation.*

The iterations of F are also linear cocycles $(\alpha, A)^n = (n\alpha, A_n)$ with $A_n(\theta) = A(\theta + (n-1)\alpha) \cdots A(\theta)$. If, moreover, (α, A) is reducible to some constant matrix C, then $A_n(\theta) = B(\theta + n\alpha)C^n B^{-1}(\theta)$ where B is the conjugacy matrix.

In [4, Sections 5.11, 5.12 and 5.14] it is proved that the KAM-Theorem implies the reducibility for cocycles (α, A) close to a constant fulfilling a diophantine condition on α and on the fibered rotation number of A.

Theorem 3.1. *Let $(\alpha, \beta) \in \mathbb{T}^{n-1} \times \mathbb{T}^1$ be diophantine. Then there exist a neighborhood $\mathcal{V}_{\alpha,\beta} \subset C^\infty\left(\mathbb{T}^{n-1}, \mathrm{SL}(2,\mathbb{R})\right)$ of R_β such that if $A \in \mathcal{V}_{\alpha,\beta}$ is close to a constant with $\rho_f(A) = \beta$, then (α, A) is C^∞ PSL-conjugate to a constant rotation (α, R_β).*

In the proof of this theorem the main step is to show the existence of a constant $C > 1$ such that for all $x \in \mathbb{T}^{n-1}$ and for all $k \geq 0$

$$\|A_k(\theta)\| \leq C. \tag{3}$$

4. Application to Schrödinger cocycle

Given a potential $V \in C^0(\mathbb{T}^1, \mathbb{R})$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\theta \in \mathbb{T}^1$, we consider the operator

$$(H_\theta \psi)_n = \psi_{n+1} + \psi_{n-1} + V(\theta + n\alpha)\psi_n \tag{4}$$

in $\ell^2(\mathbb{Z})$. Then we can study the spectrum of (4) through the fact that u is a solution of the difference equation $u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n = Eu_n$ iff

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A^n_{E,V} \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}, \qquad A_{E,V}(\theta) := \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Moreover the spectrum Σ is independent of $\theta \in \mathbb{T}^1$. $A_{E,V}$ is called *Schrödinger cocycles*. If $A_{E,V}$ is close to a constant, then it is possible to use Theorem 3.1 to prove the presence of absolutely continuous spectrum for some Schrödinger operators similarly as in [1, Theorem 8].

Theorem 4.1. *Let $\alpha \in \mathbb{T}^1$ be diophantine. Then there exist a set $\mathcal{A} \subset \mathbb{R}$ of positive Lebesgue measure such that if $V \in C^\infty(\mathbb{T}^1, \mathbb{R})$ and $E_0 \in \mathbb{R}$ are such that $(\alpha, A_{E_0,V})$ is C^∞ PSL-conjugate to a rotation (α, R_β) for some $\beta \in \mathcal{A}$, then the associated Schrödinger operator presents absolutely continuous spectrum.*

The key ideas are setting $\mathcal{A} := \{\beta \in \mathbb{T}^1 \mid (\alpha, \beta) \text{ diophantine}\}$ and $\mathcal{R} := \{E \in \mathbb{R} \mid (\alpha, A_{E,V}) \text{ conjugate to a constant rotation}\}$ and use the following theorem of Last and Simon [6, Theorem 1] with the remark that being A conjugated to a rotation, we can apply the estimate (3).

Theorem 4.2. *Let H_+ be the operator (4) on $\ell^2(\mathbb{N})$ with $u(0) = 0$ boundary conditions. Let*

$$S = \left\{ E \;\middle|\; \liminf_{L \to \infty} \frac{1}{L} \sum_{k=1}^{L} \left\| A^k_{E,V} \right\|^2 < \infty \right\}.$$

Then S is an essential support of the absolutely continuous part of the spectral measure for H_+ and S has zero measure with respect to the singular part of the spectral measure.

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KAM theory for Schrödinger operators I

Xuanji Hou

In this talk, we consider a quasi-periodic Schrödinger operator

$$(\mathcal{L}y)(t) = -y''(t) + q(\theta + \omega t)y(t), \tag{1}$$

where the potential $q : \mathbb{T}^d \to \mathbb{R}$ is analytic. The Schrödinger equations

$$(\mathcal{L}y)(t) = -y''(t) + q(\theta + \omega t)y(t) = Ey(t) \tag{2}$$

($E \in \mathbb{R}$ is called energy) is closely related to systems ($x \in \mathbb{R}^2$)

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ q(\theta) - E & 0 \end{pmatrix} x \\ \dot{\theta} = \omega \end{cases} \tag{3}$$

More generally, we consider $Sys.(\omega, A)$ defined as

$$\begin{cases} \dot{x} = A(\theta)x \\ \dot{\theta} = \omega \end{cases}, \tag{4}$$

where $A : \mathbb{T}^d \to sl(2, \mathbb{R})$ is analytic, and the frequency $\omega \in \mathbb{R}^d$ is irrational. Let $\Phi^t_{\omega,A}(\theta_0)$ denotes the basic solution of (4) begin with $\theta = \theta_0$, i.e.,

$$\frac{d}{dt}\Phi^t_{\omega,A}(\theta_0) = A(\theta_0 + \omega t)\Phi^t_{\omega,A}(\theta_0), \quad with \quad \Phi^0_{\omega,A}(\theta_0) = Id. \tag{5}$$

We define the rotation number of $Sys.(\omega, A)$ as ($0 \neq x \in \mathbb{R}^2$)

$$\rho = \lim_{t \to +\infty} \frac{arg(\Phi^t_{\omega,A}(\theta_0)x)}{t}, \tag{6}$$

where arg denotes the angle. It is well-defined and is independent of θ_0 and x. ρ is said to be rational w.r.t. ω if $\rho = \frac{1}{2} < k_0, \omega >$ for some $k_0 \in \mathbb{Z}^d$. Otherwise, we call it irrational w.r.t. ω.

We call that $Sys.(\omega, A)$ and $Sys.(\omega, \widetilde{A})$ are conjugate if there exists analytic $B : \mathbb{T}^d \to sl(2, \mathbb{R})$, s.t.

$$\partial_\omega B = AB - B\widetilde{A}. \tag{7}$$

If $Sys.(\omega, A)$ and $Sys.(\omega, C)$ are conjugate, with C being a constant, we call that it is reducible. Usually, we also say a system is reducible if it can be conjugated to a constant one via a B defined on $2\mathbb{T}^d = \mathbb{R}^d/2\mathbb{Z}^d$. It is a basic problem that when a system is reducible. Naturally, one can consider such a problem in local sense, i.e., a systems close to a constant (in analytic sense). As for Schrödinger case, the local usually denotes the small potential or high energy. For one can always conjugate such a Schrödinger system to a local one via a constant B.

Even for local case, the reducibility is not trivial. The frequency and the rotation number also play an important role. In [4], Dinaburg and Sinai proved that a system is reducible if the perturbation is small enough (in analytic sense) provided that ω is Diophantine and the rotation number ρ is Diophantine w.r.t. ω, i.e.,

$$|\langle k, \omega\rangle| \geq \frac{\nu^{-1}}{|k|^\sigma}, \quad |\langle k, \omega\rangle \pm 2\rho| \geq \frac{\gamma^{-1}}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^d \tag{8}$$

for some constants $\nu, \sigma, \gamma, \tau$. Such a reducibility result indicates the existence of absolutely spectrum for quasi-periodic Shrödinger operator with small analytic potential (or analytic potential together with high energy). Note that here the smallness of the perturbation depends on $\nu, \sigma, \gamma, \tau$. Some years later, in [8], Moser and Poschel extended such a result to rational rotation number case.

Note that all results in [4, 8] holds only for positive measure of rotation number. It is H.Eliasson, who firstly proved in [5] that: in fact, for analytic local systems with Diophantine fixed frequency, reducibility holds for full measure rotation number. In other words, he proved that a system is reducible if the perturbation is small enough provided that ω is Diophantine and the rotation number ρ is Diophantine or rational w.r.t. ω, but the smallness dose not depends on the rotation number. Furthermore, by using the full-measure reducibility result together with some detailed estimates, Eliasson has proved that for quasi-periodic Shrödinger operator with small analytic potential (or analytic potential together with high energy), the spectrum is purely absolutely continuous. It is rather a nice result! Besides these, in [5], Eliasson also proved that the cantor spectrum holds for generic small analytic potential (or generic analytic potential with enough high energy), and the smallness does not depend on the rotation number.

Eliasson's reducibility result is proved through a non-standard KAM iteration. We give a rough introduction here. Let us consider a system $Sys.(\omega, A_0 + F_0)$ with A_0 being constant and F_0 being analytic and small in $|Im\theta| < h_0$ ($h_0 > 0$). As the first trial, we ant to find $B = e^Y$ with $Y : \mathbb{T}^d \to sl(2, \mathbb{R})$ being small conjugate

$Sys.(\omega, A_0 + F_0)$ to a constant one, i.e.,

$$\partial_\omega e^Y = (A_0 + F_0)e^Y - e^Y C \tag{9}$$

with $C \in sl(2,\mathbb{R})$. However, we usually solve a linearized equation

$$\partial_\omega Y - [A_0, Y] = -(F_0 - [F_0]) \tag{10}$$

instead. It can be represented by using Fourier expansion as

$$\sum_{0\neq k\in\mathbb{Z}^d} i\langle k,\omega\rangle \widehat{Y}(k) - [A_0, \widehat{Y}(k)\rangle = -\widehat{F_0}(k). \tag{11}$$

As we solve such a linear equation, there is small divisor of the form:

$$\langle k,\omega\rangle \quad and \quad \langle k,\omega\rangle \pm 2\rho_0 \tag{12}$$

with the assumption that $A_0 = \begin{pmatrix} 0 & \rho_0 \\ -\rho_0 & 0 \end{pmatrix}$ (without lose of generality).

Firstly, with the Diophantine assumption of ω, after a conjugation closing to identity, one can remove all terms with $|k| \le N_0 = C \ln \frac{1}{\varepsilon_0}$, where $\varepsilon_0 = \sup_{|Im\theta|<h_0} \|F_0(\theta)\|$ is small enough, except the non-diagonal part of k_0 term as $\langle k_0,\omega\rangle \pm 2\rho_0$ is small (one can prove that there is only one such k_0 at most until $|k| > N_0$).

Secondly, we use a rotation $Q(\theta) = \begin{pmatrix} \cos\frac{\langle k_0,\theta\rangle}{2} & -\sin\frac{\langle k_0,\theta\rangle}{2} \\ \sin\frac{\langle k_0,\theta\rangle}{2} & \cos\frac{\langle k_0,\theta\rangle}{2} \end{pmatrix}$ defined on $2\mathbb{T}^d$ to remove the non-diagonal part of k_0 term. Note that the rotation is large in analytic sense but it does not impact the smallness of the perturbation after a diminishment of h_0. Now there is no small divisors until $|k| > N_0$. We then use a conjugation closing to identity to remove all terms with $|k| \le N_0$.

Thus we complete a step of KAM to obtain a system $Sys.(\omega, A_1 + F_1)$ with the new perturbation much more smaller (one also needs to decrease h_0). More precisely, let $\varepsilon_0 = \sup_{|Im\theta|<h_0} \|F_0(\theta)\|$, there is $\varepsilon_1 = \sup_{|Im\theta|<h_1} \|F_1(\theta)\| \le \varepsilon_0^{1+\sigma}$ with some $\sigma \in (0, 1)$. As we repeat the process, we obtain a sequence of $Sys.(\omega, A_n+F_n)$ with a decrease sequence of h_n and $\varepsilon_n = \sup_{|Im\theta|<h_n} \|F_n(\theta)\| \le \varepsilon_0^{(1+\sigma)^n}$. However, if we furthermore assume that the rotation number is Diophantine or rational w.r.t. ω, one will not need to use rotations and all conjugation is close to identity after finite step of iterations. For such cases, the composition of all conjugation converges together with h_n decrease to a positive number and the reducibility follows. As the composition fails to converge, one obtains the so-called almost reducibility.

Eliasson's KAM scheme gives a unified method to deduce all rotation numbers when ω is Diophantine. Such a scheme also works for discrete case [2, 6]. In the end, we remark that one can also apply KAM to some systems with non-Diophantine frequency [1, 3, 7], etc..

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KAM theory for Schrödinger operators II

Darren C. Ong

Avila, Fayad and Krikorian have developed a new KAM scheme that applies to $\mathrm{SL}(2,\mathbb{R})$ cocycles with one frequency, irrespective of any Diophantine condition on the base dynamics. The precise statement of the main theorem is as follows:

Theorem 0.3. *For every $\tau > 0$, $0 < \nu < 1/2$, $\epsilon > 0$, $h_* > 0$. there exists $\eta > 0$ depending on τ, ν, ϵ with the following property. Let $h > h_*$ and let A be a real-symmetric analytic $\mathrm{SL}(2,\mathbb{R})$ valued function from the torus $\mathbb{T}$ whose domain is the complex strip $\Delta_h = \{|\Im z| < h\}$. Furthermore, assume $\sup_{\Delta_h} ||A - R|| < \eta$ for some rotation matrix R, and that the fibered rotation number ρ satisfies a certain positive-measure condition depending on τ, ν, ϵ and α. Then there must exist real symmetric $B : \Delta_{h-h^*} \to \mathrm{SL}(2,\mathbb{C})$ and $\phi : \Delta_{h-h_*} \to \mathbb{C}$ such that B is ϵ-close to the identity, and ϕ is ϵ-close to ρ on the strip Δ_{h-h_*}, and $B(x+\alpha)A(x)B(x)^{-1}$ is equal to rotation by $\phi(x)$.*

We note that since the Almost Reducibility Conjecture has been established, it is possible to replace the positive measure condition on ρ with a full measure condition.

In the special case of Schrödinger cocycles, this theorem has the following two implications:

Theorem 0.4. *Let the potential $v : \mathbb{T} \to \mathbb{R}$ be analytic and close to constant. For every $\alpha \in \mathbb{R}$, the set $X(v,\alpha)$ of all energies $E \in \mathbb{R}$ for which the corresponding Schrödinger cocycle is conjugate to a cocycle of rotations has positive Lebesgue measure. Indeed the Lebesgue measure of this set $X(v,\alpha)$ converges to 4 as v converges to a constant.*

Theorem 0.5. *Let α be irrational and $v : \mathbb{T} \to \mathbb{R}$ be analytic. Then for almost every energy $E \in \mathbb{R}$ the corresponding Schrödinger cocycle has either Lyapunov exponent 0, or is analytically conjugate to a cocycle of rotations.*

The proof of the first theorem begins by establishing a subsequence $\{Q_k\}$ of the sequence of denominators of the fractional approximants of α. This subsequence has the property that if the cocycle (α, A) is close to a constant rotation, then the iterated cocycle $(Q_k\alpha, A^{Q_k})$ is close to a constant rotation as well. We then apply a series of inductive estimates on $(Q_k\alpha, A^{Q_k})$ and use the fact that $(Q_k\alpha, A^{Q_k})$ and (α, A) commute to reach conclusions about the dynamical properties of (α, A) and arrive at the desired conjugacy.

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Renormalization I

Qi Zhou

In this talk, we consider the global reducibility results for one frequency analytic quasiperiodic Schrödinger cocycles $(\alpha, S_{v,E})$ where

$$S_{v,E}(\theta) = \begin{pmatrix} E - v(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{R}).$$

If the cocycle is close to constant, reducibility results are both verified in Diophantine and Liouvillean context [2, 5]. If the cocycle is not close to constants, we will talk about the following results, which are due to Avila-Krikorian [3] and Avila-Fayad-Krikorian [2]:

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a C^ω potential. Then we have the following: If α is recurrent Diophantine, then for Lebesgue almost every E, the Schrödinger cocycle $(\alpha, S_{v,E})$ is either nonuniformly hyperbolic or C^ω reducible; If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then for Lebesgue almost every $E \in \{E : |L(\alpha, S_{v,E}) = 0\}$, the Schrödinger cocycle $(\alpha, S_{v,E})$ is rotations reducible.

We will follow the approach of Herman's celebrated work on linearization to prove the main result. We define a renormalization operator, under some assumptions, the renormalization admits a "linear attractor" (constant cocycle), this allows one to obtain "global" results by reducing to the "local case" of nearly constant cocycle, then we apply local reducibility results to finish the proof.

The concept of renormalization arises in many forms through mathematics and physics, a great reference from the dynamical side is [1]. The renormalization operator consists two steps: inducing and scaling. Inducing usually means we consider the first return map of the given dynamics, after inducing, we need a scaling procedure, so that the dynamics considered occur at a fixed spatial scale. Simple example is the renormalization of rigid rotation of the circle, it can be shown that renormalization just means the Gauss map.

If we apply renormalization principle directly to one frequency C^r cocycle, the questions arises in the following: if we consider the first return map to the annulus

$[x_0, x_0 + q_{n_k}] \times \mathbb{R}/\mathbb{Z}$, where we identify the boundary circles via $(x, u) \mapsto (x + q_n\alpha, A(x) \cdot u)$, then we cannot get a C^r cocycle, since we lose the smoothness the cocycle. A possible way to deal with this question is that we can look at the cocycle in another way, considering

$$(\alpha, A) \qquad \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2$$
$$(x, u) \mapsto (x + \alpha, A(x)u).$$

then if (α, A) commutes with $(1, id)$, then (α, A) is a cocycle. This observation allows us to introduce the concept of $\mathbb{Z}^2$ action Φ, a homomorphism from $\mathbb{Z}^2$ to Ω^r(subgroup of $Diff^r(\mathbb{R} \times \mathbb{R}^2)$). There is nice relationship between $\mathbb{Z}^2$ action and cocycle, if $\Phi(1, 0) = (1, id)$, then $\Phi(0, 1) = (\alpha, A)$ can be viewed as a C^r-cocycle. Given a C^r-cocycle (α, A), $\alpha \in [0, 1]$, we associate a $\mathbb{Z}^2$ action $\Phi_{\alpha,A}$ by setting $\Phi_{\alpha,A}(1, 0) = (1, id)$, $\Phi_{\alpha,A}(0, 1) = (\alpha, A)$.

We can do the renormalization operations for $\mathbb{Z}^2$ action other than for cocycles. The operation consists two steps: base change and rescaling, base change just means we consider the iteration of the cocycle. It can be shown that reducibility is invariant under renormalization, and $\Phi_{\alpha,A}$ is C^r-reducible (resp. rotations reducible) if and only if (α, A) is C^r-reducible (resp. rotations reducible). This gives us some convince when we are deal with reducibility results.

The renormalization sequence takes the form

$$\mathfrak{R}^n(\Phi)(1, 0) = (1, A_{(-1)^{n-1}q_{n-1}}(\beta_{n-1}x)),$$
$$\mathfrak{R}^n(\Phi)(0, 1) = (\alpha_n, A_{(-1)^{n-1}q_n}(\beta_{n-1}x)),$$

where $\Phi = \Phi_{\alpha,A}$. However, $\mathfrak{R}^n(\Phi)(0, 1)$ is not a cocycle. So we need the normalization step, which means for any $\mathbb{Z}^2$ action $\Phi : \mathbb{Z}^2 \to \Omega^r$ with $\Pi_1(\Phi(1, 0)) = 1$, then it can be C^r conjugated to $\Phi(1, 0) = (1, id)$, $r \in N \cup \infty, \omega$.

The next question arises, when does the renormalization sequences converges? Usually it is called "a prior estimate". If (α, A) has positive Lyapunov exponent, then it is easy to see that the renormalization sequence diverges. What is in fact the basin of the renormalization attractor? Of course it has zero Lyapunov exponent, by Bourgain-Jitomirskaya's result [4], it is closed, however, the basin of a local attractor is by nature open. In fact, the main result shows that for Lebesgue almost every $E \in \{E : |L(\alpha, S_{v,E}) = 0\}$, the Schrödinger cocycle $(\alpha, S_{v,E})$ can be renormalized to near constant cocycle.

The good starting point is Kotani's theory, for Lebesgue almost every $E \in \{E : |L(\alpha, S_{v,E}) = 0\}$, the cocycle $(\alpha, S_{v,E})$ is L^2 conjugated to a cocycle in $SO(2, \mathbb{R})$, then an explicit estimate allows us to control the derivatives of iterates of the cocycle restricted to certain small intervals. This is just the "prior estimate" what we needed.

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Renormalization II

Raphaël Krikorian

As was explained in the previous talk by Qi Zhou, renormalization is a useful tool to prove "global" reducibility results (i.e. non perturbative ones) by reducing them to local (i.e. perturbative) ones. A typical application of this technique for $SL(2,\mathbb{R})$-valued cocycles is the a.e. dichotomy obtained in [3]: given a recurrent diophantine $\alpha \in \mathbb{T}$ (which means that infinitely many iterates of α under the Gauss map satisfy a *fixed* diophantine condition), and $V : \mathbb{T} \to \mathbb{R}$ a smooth or real analytic potential, then for Lebesgue a.e. $E \in \mathbb{R}$, either $LE(\alpha, S_{E-V}) > 0$ or the cocycle (α, S_{E-V}) is smoothly or real analytically reducible (conjugated to a constant elliptic matrix in $SL(2,\mathbb{R})$). Since Kotani's theory tells us that for a.e E where $LE(\alpha, S_{E-V}) = 0$, the cocycle (α, S_{E-V}) is L^2-rotations-reducible (which means conjugated to an $SO(2,\mathbb{R})$-valued cocycle by an L^2 conjugation), it is indeed enough to prove the following *differentiable rigidity* theorem: if a smooth or real analytic cocycle with a recurrent diophantine frequency is L^2-rotations-reducible, and has a fibered rotation number which is diophantine with respect to α, then it is smoothly or real analytically reducible. The proof of the previous result relies on the *convergence* of the renormalization scheme to $SO(2,\mathbb{R})$-valued cocycles ($\mathbb{Z}^2$-actions) which are not necessarily constant, convergence which in turn heavily relies on *a priori* estimates. Let us describe what these *a priori* estimates are designed for. Roughly speaking, to understand the n-th iterate of a cocycle (α, A) under the renormalization operator it is enough to understand the iterates $(\alpha, A)^{q_n}$ on some interval $[x_*, x_* + q_n\alpha] \subset \mathbb{T}$ (as ususal q_n is the n-th denominator of the continuous fraction expansion of α). One has to keep in mind that the n-th renormalized cocycle is related to the dilated cocycle $(\alpha_n, A^{(q_n)}(\beta_{n-1}^{-1}(\cdot - x_*)))$ on the interval $[0, 1]$, where x_* is some point at which dilation is made. In order to be able to prove that renormalization "converges", a first step is to prove that the preceding cocycles have reasonable C^k-norms, which in our case means bounded. A first step in that direction is to prove that these cocycles have bounded Lispchitz norms. The main technical tool is then the following: Let $A(\cdot) : \mathbb{T} \to SL(2,\mathbb{R})$ be Lipschitz, $B(\cdot) : \mathbb{T} \to SL(2,\mathbb{R})$ such that $\int_{\mathbb{T}} \|B(x)\|^2 dx < \infty$ and $B(\cdot + \alpha)^{-1}A(\cdot)B(\cdot)$ is $SO(2,\mathbb{R})$-valued. Denote by $\phi : \mathbb{T} \to \mathbb{R}$ the L^1-function $\phi(\cdot) := \|B(\cdot)\|^2$ and by $S(x) := \sup_{n\geq 1} \frac{1}{n}\sum_{k=0}^{n-1} \phi(x + k\alpha)$. By the Maximal Ergodic Theorem, for a.e. $x_0 \in \mathbb{T}$ one has $S(x_0) < \infty$ (and in fact S satisfies a weak-type inequality). We can now state the main Lemma: for every $x_0 \in \mathbb{T}$

where $S(x_0) < \infty$ one has

$$\|A_n(x_0)^{-1}(A_n(x) - A_n(x_0))\| \leq e^{n|x-x_0|\|A\|_{C^0} Lip(A) S(x_0)\phi(x_0)} - 1$$

and

$$\|A_n(x)\| \leq e^{n|x-x_0|\|A\|_{C^0} Lip(A) S(x_0)\phi(x_0)}(\phi(x_0)\phi(x_0+n\alpha))^{1/2}$$

Using this lemma one can prove a higher differentiability version: if A is of class C^k

$$\|(\partial^r A_n)(x)\| \leq C^r n^r \phi(x_0+n\alpha)^{1/2}\left(c_1(x_0)e^{nc_2(x_0)|x-x_0|}\right)^{r+\frac{1}{2}}\|\partial^r A\|_{C^0}$$

where $c_1(x_0) = \phi(x_0)S(x_0)\|A\|^2_{C^0}$, $c_2(x_0) = 2S(x_0)\phi(x_0)\|A\|_{C^0}\|\partial A\|_{C^0}$ and C is a universal constant. This last lemma joined with the L^2-rotation reducibility assumption is the key to proving the convergence of renormalization to $SO(2,\mathbb{R})$-valued cocycles. More precisely, using a measurable continuity argument, one can prove the following : with the same assumptions as before, for a.e. $x_* \in \mathbb{T}$, there exists $K > 0$ such that for every $d > 0$, $\epsilon > 0$ and for every $n > n_0(d,\epsilon)$, if $\|\alpha n\|_{\mathbb{T}} \leq d/n$ then $\|\partial^r A_n(x)\| \leq K^{r+1}n^r\|A\|_{C^r}$ provided $|x - x_*| \leq d/n$; furthermore, the matrix $B(x_*)^{-1}A_n(x)B(x_*)$ is ϵ-close to $SO(2,\mathbb{R})$ for $|x - x_*| \leq d/n$. This result, applied to A_{q_n} instead of A_n, shows that there exists a point x_* such that the n-th iterates of (α, A) under the renormalization operator (centered at x_*) become closer and closer to $SO(2,\mathbb{R})$ and their C^k -norms stay bounded. In case α is recurrent diophantine, infinitely many times, α_n, the n-th iterate of α under the Gauss map, will be in some fixed diophantine class. Then, naturally associated to (α, A) one gets a $SO(2,\mathbb{R})$-valued smooth or real analytic cocycle $(\alpha_\infty, \tilde{A}^{(\infty)})$ which can easily be conjugated to a constant one (α_∞, R) (α_∞ being diophantine). Stopping the renormalization procedure at some large step n, one is reduced to the following: decide whether a cocycle of the form $(\alpha_n, Re^{F_n(\cdot)})$ is reducible, where R is a constant, F_n is small in C^k (or real analytic) norm and α_n is in a fixed diophantine class. But this is a local situation and in that case one has a local theorem that gives the answer, namely Eliasson's Theorem: the cocycle $(\alpha_n, Re^{F_n(\cdot)})$ is reducible provided its fibered rotation number is diophantine with respect to α_n. By Kotani theory, this last condition is fulfilled for a.e. E such that $LE(\alpha, S_{E-V}) = 0$.

Let us mention that the preceding a.e. dichotomy can be extended to the case where α is *any* irrational number on $\mathbb{T}$ (see [2]) provided the notion of reducibility is replaced by the more general one of rotations-reducibility (meaning conjugation to $SO(2,\mathbb{R})$-valued cocycle): Given any irrational $\alpha \in \mathbb{T}$, and $V : \mathbb{T} \to \mathbb{R}$ a smooth or real analytic potential, then for Lebesgue a.e. $E \in \mathbb{R}$, either $LE(\alpha, S_{E-V}) > 0$ or the cocycle (α, S_{E-V}) is smoothly or real analytically rotations-reducible i.e. conjugated to an $SO(2,\mathbb{R})$-valued cocycle.The proof uses as before Kotani's theory to reduce the result to an L^2-differentiable rigidity result but here it is important to prove the convergence of L^2-rotations-reducible cocycles (homotopic to the identity) to *constant* cocycles. We are thus reduced to the following situation: given $(\alpha_n, Re^{F_n(\cdot)})$, where R is an elliptic constant in $SO(2,\mathbb{R})$, F_n is small in C^k (or real analytic) norm, what can be said about its rotations-reducibility? To answer this question an important tool is the so-called "cheap trick" introduced in [6] and [2]. The basic idea underlying this technique is that when α_n is too small to give informations from a classical KAM point of view, *algebraic* conjugation of $Re^{F_n(\cdot)}$ to a cocycle of rotations (or equivalently rotations-reducibility of $(0, Re^{F_n(\cdot)})$) translates to a *dynamical* conjugation, conjugating $(\alpha_n, Re^{F_n(\cdot)})$ to a cocycle which is closer to a cocycle of rotations, how close now depending on the size of α_n; notice that this procedure can be iterated a finite number of time which allows usable estimates. The implementation of this idea gives a (semi-) local result which can be seen as an extension of the classical Dinaburg-Sinai result: there exists a smallness condition, depending on some arithmetic condition imposed to the fibered rotation number of $(\alpha_n, Re^{F_n(\cdot)})$ (with respect to α_n), but *not* on the arithmetic properties of α_n, such that if F_n satisfies this smallness condition, the cocycle $(\alpha_n, Re^{F_n(\cdot)})$ is rotations-reducible.

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The metal-insulator transition for the almost Mathieu operator

Roman Schubert

The topic of this talk is to discuss Anderson localisation for the almost Mathieu operator (AMO) in the supercritical regime, following [1]. The almost Mathieu operator on $l^2(\mathbf{Z})$ is given by

(1) $$[H\psi](n) = \psi(n+1) + \psi(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))\psi(n)$$

and we are interested in properties of eigenfunctions of H, i.e. $H\psi(n) = E\psi(n)$. On says that the operator displays *Anderson localization* if it has pure point spectrum and any eigenfunction decays exponentially for large $|n|$.

We will assume that the *frequency* α satisfies a Diophantine condition

(2) $$\alpha \in DC \leftrightarrow \quad |\sin(\pi j\alpha)| \geq \frac{c(\alpha)}{|j|^{r(\alpha)}}\,, \quad \text{for some} \quad c(\alpha) > 0\,, \quad 1 < r(\alpha) < \infty$$

and the *phase* ω is non-resonant:

(3) $$\omega \notin \mathcal{R} := \{\omega |\, |\sin(2\pi(\omega + k\alpha/2))| \leq \exp(-k^{\frac{1}{2r(\alpha)}}) \text{ for infinitely many } k \in \mathbf{Z}\}$$

The main result we discuss is then

Theorem [1] *Assume $\lambda > 1$, $\alpha \in DC$ and $\omega \notin \mathcal{R}$. Then H displays Anderson localization.*

Remarks:

(a) For a history of the many partial results known before we refer to [1].

(b) The AMO is Aubry dual to itself with λ replaced by $1/\lambda$, see the following talk. Therefore the theorem implies that for $\lambda < 1$ the spectrum is absolutely continuous. Hence we have at $\lambda = 1$ a transition from absolutely continuous to pure point spectrum, i.e, a transition from a conducting metal to an insulator.

(c) The conditions on α and ω are close to being sharp, in more recent papers by Avila and Jitomirskaya it was studied what happens if one relaxes them.

- One says that ω is ε-resonant ($\varepsilon > 0$) if there exist infinitely many integers n_j, $j = 1, 2, \cdots$, such that $||2\omega - n_j\alpha||_{\mathbf{R}/\mathbf{Z}} \leq e^{-|n_j|\varepsilon}$. In [4] it is shown that if $\alpha \in DC$ and ω is ε-resonant then eigenfunctions localise around the resonances n_j, but are exponentially decaying away from them. This is called *almost localisation*.
- In [3] the case of more general α is considered. Let p_n/q_n be the continued fractions approximates for α and set

$$\beta := \limsup \ln(q_{n+1})/q_n \ .$$

If $\alpha \in DC$, then $\beta = 0$. If α is irrational, $\beta < \infty$ and ω is non-resonant then eigenfunctions are localised for $\lambda > e^{16\beta/9}$. It is expected that for non-resonant ω Anderson localisation holds for $\lambda > e^{\beta}$ but not for $\lambda < e^{\beta}$, see [3].

Let us now describe some elements of the setup of the proof. One says that ψ is a generalised eigenfunction if $\psi(n)$ satisfies $H\psi(n) = E\psi(n)$ and $|\psi(n)| \leq C(1+|n|)$, and to show Anderson localisation it is enough to show that every generalised eigenfunction is exponentially decaying.

The main tool will be estimates on Greens functions. Let $I = [x_1, x_2] \subset \mathbf{Z}$ be an interval and let $(H_I - E)$ be the restriction of $H - E$ to I with Dirichlet boundary conditions $\psi(x_1 - 1) = \psi(x_2 + 1) = 0$. This is a band matrix of the form

$$(H_I - E) = \begin{pmatrix} V_E(x_1) & 1 & 0 & 0 & \cdots & 0 \\ 1 & V_E(x_1+1) & 1 & 0 & \cdots & 0 \\ 0 & 1 & V_E(x_1+2) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & \cdots & \cdots & 0 & 1 & V_E(x_2) \end{pmatrix} \tag{4}$$

where $V_E(x) = 2\lambda\cos(2\pi(\omega + x\alpha)) - E$ and the Greens function is defined as its inverse

$$G_I(x, y) = (H_I - E)^{-1}(x, y) \ .$$

Let

$$P_k(\omega) := \det(H_{[0,k-1]} - E) ,$$

and $A(k,\omega)$ be the standard cocycle associated with the AMO, then we have the following identities

(d) $A(k,\omega) = \begin{pmatrix} P_k(\omega) & -P_{k-1}(\omega+\alpha) \\ P_{k-1}(\omega) & -P_{k-2}(\omega+\alpha) \end{pmatrix}$

(e) $|G_{[x_1,x_2]}(x_1,y)| = \frac{|P_{x_2-y}(\omega+(y+1)\alpha)|}{|P_k(\omega+x_1\alpha)|}$ and $|G_{[x_1,x_2]}(y,x_2)| = \frac{|P_{y-x_1}(\omega+x_1\alpha)|}{|P_k(\omega+x_1\alpha)|}$ where $k = x_2 - x_1$

(f) If $H\psi(n) = E\psi(n)$ and $x \in I$, then

$$\psi(x) = -G_I(x_1,x)\psi(x_1-1) - G_I(x_2,x)\psi(x_2+1) .$$

The first relation follows, e.g. by induction from $P_{k+1}(\omega) = V(k)P_k(\omega) - P_{k-1}(\omega)$, which in turns follows by Laplace expansion in the last row of $H_{[0,k-1]} - E$. The second one follows by Cramer's rule and the last one from $(H_{[0,k-1]} - E)\psi(n) = -\psi(x_1-1)\delta_{n,x_1} - \psi(x_2+1)\delta_{n,x_2}$.

Now if $\lambda > 1$ then

(5) $$L := \lim_{k\to\infty} \int \frac{\ln ||A(k,\omega)||}{\ln k} d\omega > 0 ,$$

in fact $L = \ln\lambda$ if α is irrational [2]. Therefore we expect that for large enough k and most ω we have $||A(k,\omega)|| \sim e^{Lk}$ and hence by (d)

(6) $$P_k(\omega) \sim e^{Lk} .$$

This would imply $|G_{[x_1,x_2]}(x_1,y)| \sim e^{-L|x_1-y|}$ (by (e)) and therefore any eigenfunction which satisfies $|\psi(n)| \leq C(1+|n|)$ has to be exponentially decaying (by (f)). So localisation follows if (6) holds, and this is where the Diophantine condition on α and the non-resonance condition on ω are used. Let us give some indication how these conditions come into play.

A point y is called (m,k)-*regular* if there exist an interval $I = [x_1,x_2]$ with $y \in I$ and $k = x_2 - x_1 + 1$ such that for $i = 1,2$

$$|G_{[x_1,x_2]}(y,x_i)| \leq e^{-m|y-x_i|} , \quad \text{dist}(y,x_i) \geq k/5$$

Otherwise y is called (m,k)-*singular*.

(g) We have by (4) that $P_k(\omega) = Q_k(\cos(\omega + (k-1)\alpha/2))$ where $Q_k(z)$ is a polynomial of degree k.

(h) There is a uniform (in ω) upper bound $|P_k(\omega)| = |Q_k(z)| \leq e^{(L+\epsilon)k}$ for k large enough, see [1].

(i) If y is $(L-\epsilon,k)$-singular, and k is sufficiently large, then one can produce a sequence of $\sim k/2$ points

$$z_j = \cos(\omega + (x+j+(k-1)/2)\alpha) ,$$

where $x \sim y - 3k/4$, such that $Q_k(z_j)$ is small, i.e, $\leq e^{(L-\epsilon')k}$. This follows from the definition of $(L-\epsilon,k)$-singular together with (e) and the upper bound in (h) on $Q_k(z)$.

(j) If both 0 and y are $(L-\epsilon,k)$-singular and a distance k apart, then the previous observation applied to 0 and y produces altogether k points z_j on which Q_k is small. Now by (5) there exist an z_0 with $Q_k(z_0) \geq e^{kL}$, applying Lagrange interpolation to the polynomial Q_k gives then

$$Q_k(z_0) = \sum_{j=1}^{k} Q_k(z_j) \prod_{l \neq j} \frac{z_0 - z_l}{z_j - z_l} \ .$$

Now by assumption $Q_k(z_0)$ is large, but $Q_k(z_j)$ is small for $j = 1, \cdots, k$ if 0 and y are $(L-\epsilon,k)$-singular. The Diophantine conditions on α and the non-resonance conditions can now be used to analyse the size of the terms

$$\prod_{l \neq j} \frac{z_0 - z_l}{z_j - z_l} \ .$$

It turns out that these are small enough to produce a contradiction (the distribution of the z_j is sufficiently *uniform*). Hence 0 and y cannot be both be $(L-\epsilon,k)$-singular, and this implies that eigenfunctions are localised.

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Duality and the almost Mathieu operator

Christoph A. Marx

The special form of the almost Mathieu operator (AMO) implies a symmetry with respect to Fourier transform, known as Aubry duality [1]. Since, taking a Fourier transform physically corresponds to a change to "momentum states", Aubry duality is heuristically understood as a correspondence between localized and "extended states".

Several approaches that formulate this correspondence rigorously exist in the literature, ranging from the very first one [5] to the most recent one [13], Section 8 therein. In this talk we introduce duality from two view points: a "classical" spectral theoretic approach and a more recent dynamical formulation in terms of Schrödinger cocycles. Even though originally discovered for the AMO, the concept is general to any Schrödinger operator $H_{\alpha,\theta}$,

$$(H_{\alpha,\theta}\psi)_n := \psi_{n+1} + \psi_{n-1} + v(\alpha n + \theta)\psi_n \ , \tag{1}$$

where $v \in \mathcal{C}(\mathbb{T},\mathbb{R})$ and $\alpha \in \mathbb{T}$ is allowed to be *both rational or irrational.*

The following spectral theoretic approach is based on an elegant formulation for the AMO due to [7], further developed to the general setting given by (1) in

[13]. Within this framework duality is formulated as a unitary equivalence on the constant fiber direct integral, $\mathcal{H}' := \int_{\mathbb{T}}^{\oplus} l^2(\mathbb{Z})\mathrm{d}\theta$. $H_{\alpha,\theta}$ naturally lifts to a decomposable operator on $\mathcal{H}'$ given by $H'_\alpha := \int_{\mathbb{T}}^{\oplus} H_{\alpha,\theta}\mathrm{d}\theta$.

Consideration of $\mathcal{H}'$ is not unexpected from a physics point of view. In solid state physics, quasi-periodic operators arise from the description of two dimensional crystal layers immersed in a magnetic field of flux β acting perpendicular to the lattice plane. An interpretation of duality in terms of the corresponding two-dimensional magnetic model can be found in [14], Sec. 2-3 therein.

Duality is mediated by the unitary [7],

$$(\mathcal{U}\psi)(\eta, m) := \sum_{n\in\mathbb{Z}} \int_{\mathbb{T}} \mathrm{d}\theta \mathrm{e}^{2\pi i m\theta} \mathrm{e}^{2\pi i n(m\alpha+\eta)} \psi(\theta, n) \ , \tag{2}$$

on $\mathcal{H}'$, which defines the *dual* operator of H'_α by $\mathcal{U}^{-1}H'_\alpha\mathcal{U} =: \widetilde{H}'_\alpha$. $\widetilde{H}'_\alpha$ is itself decomposable and acts on the fibers according to[1]

$$(\widetilde{H}_{\alpha,\theta}\psi)_n = 2\cos(2\pi(\theta + \alpha n))\psi_n + (\check{v} * \psi)_n \ . \tag{3}$$

A speciality of the AMO is that up to energy rescaling, its dual lies within the almost Mathieu family, i.e. $\widetilde{H}_{\alpha,\theta,\lambda} = \lambda H_{\alpha,\theta,\lambda^{-1}}$. In particular, duality relates the sub($\lambda < 1$)- and super-critical ($\lambda > 1$) regime. The critical point $\lambda = 1$ is self dual.

When formulating duality as unitary equivalence between operators, preservation of spectral properties becomes immediate. In this respect, we mention the union spectrum $S_+(\alpha) := \cup_{\theta\in\mathbb{T}}\sigma(H_{\alpha,\theta})$, analogously defined for the dual operator ($\widetilde{S}_+(\alpha)$). Note that for $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $S_+(\alpha) = \sigma(H_{\alpha,\theta}) =: \Sigma(\alpha)$ for all $\theta \in \mathbb{T}$.

Theorem 0.6. *For any $\alpha \in \mathbb{T}$, we have $S_+(\alpha) = \widetilde{S}_+(\alpha)$. Moreover, if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, the density of states is invariant under duality i.e.*

$$k(\{H_{\alpha,\theta}\};\Delta) = k(\{\widetilde{H}_{\alpha,\theta}\};\Delta) \ , \tag{4}$$

for $\Delta \subseteq \mathbb{R}$ Borel.

Invariance of S_+ for general, continuous $v(x)$ is proven in [13]. A first rigorous proof of the moreover statement was obtained in [14]; within the present framework a proof can be found in [10] .

As an application of invariance of the density of states, we mention an alternative proof of Herman's bound for the Lyapunov exponent (LE) [14] : Using Thouless formula, one obtains

$$L(\alpha, A^E_\lambda) = L(\alpha, A^{E/\lambda}_{\lambda^{-1}}) + \log\lambda \ , \tag{5}$$

relating the LE in the dual regimes. In particular, $L(\alpha, A^E_\lambda) \geq \log\lambda$, which yields positivity for $\lambda > 1$.

Whereas above approach easily yields the spectral invariances formulated in Theorem 0.6, the following dynamical point of view very transparently formulates the heuristic correspondence between localized and extended states.

[1]We denote by $\hat{\phi}(x) := \sum_{n\mathbb{Z}} \phi_n \mathrm{e}^{2\pi i n x}$ the Fourier-transform on $l^2(\mathbb{Z})$ with inverse written as $\check{f}$, for $f \in L^2(\mathbb{T})$.

To this end, suppose $0 \neq \psi \in l^2(\mathbb{Z})$ such that $\widetilde{H}_{\alpha,\theta}\psi = E\psi$, for some θ. As shown in [2] this produces an L^2-semi-conjugacy of the Schrödinger cocycle (α, A^E) to the constant rotation $R_\theta := \left(\begin{smallmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{smallmatrix}\right)$,

$$(6) \quad A^E(x)C(x) = R_\theta C(x+\alpha) \ , \ C(x) = \begin{pmatrix} \hat{\psi}(x) & \overline{\hat{\psi}(x)} \\ \hat{\psi}(x-\alpha)e^{-2\pi i\theta} & \overline{\hat{\psi}(x-\alpha)e^{-2\pi i\theta}} \end{pmatrix} .$$

For all but countably many phases, this semi-conjugacy is in fact a conjugacy:

Proposition 0.1. [2, 3] *Suppose $\theta \in \mathbb{T}$ such that $\alpha\mathbb{Z} + 2\theta \cap \mathbb{Z} = \emptyset$, then for some $c \in \mathbb{R} \setminus \{0\}$, $\det C(x) = ic$ a.e. x.*

In particular, we conclude $L(\alpha, A^E) = 0$ if $\alpha\mathbb{Z} + 2\theta \cap \mathbb{Z} = \emptyset$ [2]. The fact that point spectrum for the dual operator implies zero LE was first proven in [8] by non-dynamical means. There, the argument was used to conclude absence of point-spectrum for the subcritical AMO, using positivity of the LE in the supercritical regime.

The L^2-conjugacy obtained in Proposition 0.1 already implies a bound of the n-step transfer matrices in L^1-norm. This bound can be improved to one holding *uniformly* on $\mathbb{T}$, if the eigenvector ψ decays sufficiently fast.

For example, for the AMO, Anderson-localization in the supercritical regime is known:

Theorem 0.7. [12] *Let $\lambda > 1$ and α Diophantine with $\|\alpha k\| \geq \frac{\kappa(\alpha)}{|k|^{r(\alpha)}}$ for $k \in \mathbb{Z} \setminus \{0\}$ and $\kappa(\alpha) > 0, r(\alpha) > 1$. If $\theta \notin \mathcal{R}$, $H_{\alpha,\theta,\lambda}$ has only pure point spectrum with exponentially localized eigenfunctions. Here,*

$$(7) \qquad \mathcal{R} = \{\theta \in \mathbb{T} : \|\alpha k + 2\theta\| < e^{-k^{\frac{1}{2r(\alpha)}}} \ i.o.\} .$$

Thus for $E \in \cup_{\theta \in \mathbb{T}'} \sigma_{\mathrm{pt}}(\lambda H_{\alpha,\lambda^{-1}}) =: \Sigma_0(\lambda)$, one concludes $\|A_n^E\|_\infty = \mathcal{O}(1)$, for all $n \in \mathbb{N}$. Here, $\mathbb{T}'$ is the full-measure set of non-resonant phases, in addition satisfying $\alpha\mathbb{Z} + 2\theta \cap \mathbb{Z} = \emptyset$.

Since $\overline{\Sigma_0(\lambda)} = \Sigma(\alpha, \lambda)$, *joint* continuity of the LE in E and α [6] yields $L(\alpha, A_\lambda^E) = 0$ for all *irrational* α and $E \in \Sigma(\alpha, \lambda)$. Together with (5), this proves the Aubry-André formula for the LE of the AMO, $L(\alpha, A_\lambda^E) = \log_+(\lambda)$, for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and energies in the spectrum.

It is interesting that this dynamical formulation of duality can even be used to obtain statements about the self-dual point:

Theorem 0.8. [3] *If θ satisfies $\alpha\mathbb{Z} + 2\theta \cap \mathbb{Z} = \emptyset$, $H_{\alpha,\theta,1}$ has empty point spectrum.*

Zero measure spectrum at $\lambda = 1$ for all irrational α [4] in particular yields,

Corollary 0.1. *For all but countably many θ, the spectrum of $H_{\alpha,\theta,1}$ is purely singular continuous.*

[2]We mention that a dynamical description for the remaining countable set of θ not covered by (0.1) can be given [2].

We conclude this talk by giving two results illustrating the necessity of the arithmetic conditions on α and θ in Theorem 0.7. In view of necessity of the Diophantine condition on the frequency, we call $\alpha \in \mathbb{T}$ Liouville with constant $c > 0$, if there exists a sequence of rationals $\frac{p_n}{q_n} \to \alpha$ such that $|\alpha - \frac{p_n}{q_n}| < \mathrm{e}^{-cq_n}$. For every $c > 0$, Liouville numbers with constant c form a dense G_δ.

Theorem 0.9. [9] *Let $v \in Lip^r(\mathbb{T}, \mathbb{R})$. There exists $c = c(v) > 0$ such that for every Liouville α with constant c, $H_{\alpha,\theta}$ has empty point spectrum for all $\theta \in \mathbb{T}$.*

To state necessity of the condition on the phase, assume $v \in Lip^r(\mathbb{T}, \mathbb{R})$ is reflection symmetric about some point expressed by $v(R\theta) = v(\theta)$, $\theta \in \mathbb{T}$. For $c > 0$, define the set of c-resonances, $\mathcal{R}(c) := \{\theta : \|T^{2k}\theta - R\theta\| < \mathrm{e}^{-ck}\ , \text{i.o.}\}$, where $T : \theta \mapsto \theta + \alpha$ on $\mathbb{T}$. As before, given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, for each fixed $c > 0$, $\mathcal{R}(c)$ forms a dense G_δ.

Theorem 0.10. [11] *There exists $c = c(v) > 0$ such that for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $H_{\alpha,\theta}$ has empty point spectrum for all $\theta \in \mathcal{R}(c)$.*

We mention that in [11] this Theorem is proven for the general almost periodic setup.

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Measure of the spectrum of the almost Mathieu operator

Rajinder Mavi

We continue the discussion of the almost Mathieu operator,

$$H_{\alpha,\omega,\lambda}u(n) = u(n+1) + u(n-1) + 2\lambda\cos\left(2\pi(n\alpha+\theta)\right) \tag{1}$$

for $\alpha, \theta, \lambda \in \mathbb{R}$, begun in the previous two lectures. Our concern in this talk is the spectrum $\sigma\left(H_{\alpha,\omega,\lambda}\right)$, which is invariant under the reflection of the coupling parameter $\lambda \to -\lambda$ so we will assume $\lambda \geq 0$. In particular we discuss the Lebesgue measure of the spectrum $S(\alpha,\lambda) = \left|\sigma\left(H_{\alpha,\omega,\lambda}\right)\right|$ for irrational α, (the spectrum does not depend on the phase ω for irrational frequency α). It is useful to specify related sets for all α, $\sigma_+(\alpha,\lambda) = \cup_\omega \sigma\left(H_{\alpha,\omega,\lambda}\right)$; $\sigma_-(\alpha,\lambda) = \cap_\omega \sigma\left(H_{\alpha,\omega,\lambda}\right)$; and $S_\pm(\alpha,\omega,\lambda) = |\sigma_+(\alpha,\omega,\lambda)|$. A conjecture of Aubry and André's[1] states that the Lebesgue measure of the spectrum at irrational frequencies is given by the formula

$$S(\alpha,\lambda) = 4\left|1-\lambda\right|.$$

In this talk we will demonstrate the proof of this conjecture. The duality properties of (1), discussed in the previous lecture, simplify the proof of the Aubry-André conjecture. We use the formula which follows from duality of the integrated density of states, Avron and Simon [4], $S_+(\alpha,\lambda) = \lambda S_+(\alpha,\lambda^{-1})$ and for irrational frequencies α, $S(\alpha,\lambda) = \lambda S(\alpha,\lambda^{-1})$. The problem therefore divides into dual subcritical $0<\lambda<1$ and supercritical regimes $\lambda > 1$, where proving the conjecture in one case implies the other; the critical self dual point $\lambda = 1$ is a special case.

In all but a zero measure set of frequencies at the critical coupling the method of proof is by approximation of the measure of spectra at rational frequencies and then approximating spectra at irrational frequencies by rational frequencies. Continuity of $\sigma_+(\cdot,\lambda)$ in the Hausdorff metric is shown in [4, 9], however, $S_+(\cdot,\lambda)$ is discontinuous at rational values. It is therefore necessary to obtain stronger statements of continuity and use numerical properties of irrational numbers, particularly the continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \equiv [a_0; a_1, a_2, \ldots]$$

with $a_i \in \mathbb{N}$, $a_i > 0$. A truncation $[a_0; a_1, a_2, \ldots, a_n] \equiv \frac{p_n}{q_n}$, with $p_n, q_n \in \mathbb{N}$ and $(p_n, q_n) = 1$ is known as a rational approximant.

Avron, Mouche and Simon [3] computed the measure of the spectra at rational frequency and obtained $S_-(\alpha,\lambda) = 4|1-\lambda|$ for $0<\lambda<1$ directly. S_+ is bounded above in [3] by controlling the difference $\sigma_+(\alpha,\lambda)\backslash\sigma_-(\alpha,\lambda)$. $S_-(\alpha,\lambda)$ is calculated using cocycles of the transfer matrices, which were introduced in the second talk, and truncated Hamiltonians. Chamber's formula [6] is used at rational $\alpha = \frac{p}{q}$ to obtain, a q degree polynomial in E to be the trace of the q length cocycle. The formula for a fixed rational α is $\frac{1}{2}\mathrm{Tr}_{(\alpha,\omega,\lambda)}(E) = \Delta_{\alpha,\lambda}(E) + \lambda^q\cos(q\omega)$. Where $\Delta_{\alpha,\lambda}$ is a polynomial of degree q. Therefore, $\sigma_-(\alpha,\lambda)$ is $\Delta^{-1}_{\alpha,\lambda}([-1+\lambda^q, 1-\lambda^q])$, thus it suffices to compute the total size of these intervals. For this the periodic, corresponding to $\Delta_{\alpha,\lambda} = 1-\lambda^q$, and antiperiodic, corresponding to $\Delta_{\alpha,\lambda} = -1+\lambda^q$, truncated Hamiltonians are used to compute the total length of these intervals. The energies in both cases are ordered to alternately correspond to even and odd vector subspaces of the truncated space, starting with even on top. The total length therefore is the sum of the even periodic trace and odd antiperiodic trace less the sum of the odd periodic trace and the even antiperiodic trace. To control $\sigma_+\backslash\sigma_-$, we turn to cocycle dynamics and use the formula $\frac{1}{2}\mathrm{Tr}_{\lambda,\omega=0} = (-1)^q \cos(qr)$ for the rotation number $r(E)$ of the cocycles on the spectrum. Avron, Mouche and Simon use the formula of Deift and Simon [8] that on the spectrum $\frac{dr}{dE} \geq 1/2$, this obtains $S_+ \leq S_- + C\lambda^{q/2}$ for some $C > 0$.

In the case of critical λ the bound of S_+ above is not effective. Instead the following formula shown by Last [15] is used $\sum_{1\leq\nu\leq q} \frac{1}{\Delta'_{p/q,1}(E_\nu)} = \frac{1}{q}$ where E_ν enumerate the zeros of $\Delta_{p/q,1}$. To bound S_+, we bound above the distance of a point in $\Delta^{-1}_{p/q,1}(\pm 2)$ to the nearest point in $\Delta^{-1}_{p/q,1}(0)$ by its trigonometric behavior and the above formula obtains $S_+ < Cq^{-1}$ for some $C < \infty$. Using duality, we have for all α, λ the limit $\lim_{p/q\to\alpha} S_+(p/q,\lambda) = 4|1-\lambda|$; thus, we require only continuity for some sequence of rationals $p/q \to \alpha$ to conclude the proof.

In [3] the authors show $\frac{1}{2}-$Hölder continuity of $\sigma_+(\cdot,\lambda)$ in the frequency. The method for any $E \in \sigma_+(\alpha,\lambda)$ is construction of a sequence $f_n \in \ell^2(\mathbb{Z})$ for any nearby frequency α' and properly chosen θ' so that $\|(E - H_{\alpha',\theta',\lambda})f_n\| < C|\alpha - \alpha'|^{1/2}$. The variation principle implies there is an $E' \in \sigma(\alpha',\theta',\lambda)$ so that $|E - E'| < C|\alpha-\alpha'|^{1/2}$. The sequence for the new frequency is constructed out of the Weyl sequence of the original operator at E by multiplying each function by a properly centered cutoff function. The $\frac{1}{2}-$Hölder is an improvement over $\frac{1}{3}-$Hölder continuity shown by Chui, Elliot and Yui [7] made possible by an improved choice of cutoff function. As briefly discussed above, $\sigma_+(p_n/q_n,\lambda)$ consists of at most q_n intervals, the edges of which have $1/2-$Hölder continuity. The continuity of S_+ was demonstrated by Last [14] as a corollary of the results in [3], using the Hölder continuity of the bands of the spectrum and the general property $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_{n+1}}$. Because $q_{n+1} = a_n q_n + q_{n-1}$ it is exactly the irrationals with an unbounded sequence of continued fraction coefficients a_i for which we can conclude the conjecture. Finally, for irrationals so that a_i are bounded Jitomirskaya and Krasovsky [13] show almost Lipshitz continuity of the spectra in the Hausdorff metric when $\lambda > 1$, (in fact this result holds for analytic potentials and Diophantine irrationals).

The completion of the Aubry-Andre conjecture is listed as problem five in Simon [16], at that time it only remained to solve the problem at critical coupling with frequencies α with continued fraction coefficients a_i bounded. This was solved by Avila and Krikorian in [2] (the method in this paper applies to a full measure set of Diophantine frequencies). From Bourgain and Jitomirskaya [5] it is known that the Lyapunov exponent is zero at the critical coupling, and thus by a result in [2] the transfer matrix is reducible, that is analytically conjugate to a constant cocycle, for almost every energy in the spectrum. But a result of Elliasson [10] states that cocycles near a reducible cocycle are uniformly hyperbolic or have zero Lyapunov exponent. By continuity of the spectrum, a reducible cocycle at $\lambda = 1$ in the spectrum would imply existence of energies in the spectrum at coupling $\lambda > 1$ with zero Lyapunov exponent, but this contradicts positivity of the Lyapunov exponent in this case, as discussed in the previous lecture. It follows that the spectrum in this case has measure zero.

Beyond the conjecture we remark that approximation of the measure of the spectrum at irrational frequencies by rational frequencies extends to analytic functions [11] and even $1/2 + \epsilon-$Hölder continuous functions in the case of positive Lyapunov exponent [12].

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Non-perturbative localization for analytic potentials

Michael Goldstein and Kai Tao

We consider the Schrödinger equations

$$[H(x,\omega)\varphi](n) \equiv -\varphi(n-1) - \varphi(n+1) + V(T_\omega^n(x))\varphi(n) = E\varphi(n), \quad n \in \mathbb{Z} \tag{1}$$

where T_ω is a shift ,$T_\omega : \mathbb{T}^\nu \to \mathbb{T}^\nu$, $T_\omega(x) = x + \omega$, $x, \omega \in \mathbb{T}^\nu$. In the regime of *of positive Lyapunov exponents*. The goal of this talk is to explain how the large deviations theorems for the norm of the transfer-matrix can be combined with the fundamental ideas of localization of eigenfunctions introduced by Frölich and Spencer. The implementation of these ideas goes through the elimination of the spectral parameter from the double resonance events. The major technical tool for this elimination process consists of the analysis of "thin" semi-algebraic sets coming from the large deviation theorem.

We always assume that $V(x)$ is a real-analytic function on $\mathbb{T}^\nu$. Let $H_{[a,b]}(x,\omega)$ be the restriction of $H(x,\omega)$ to the finite interval $[a,b]$ with zero boundary conditions $\psi(a-1) = 0, \psi(b+1) = 0$. By $f_{[a,b]}(x,\omega,E)$ we denote the characteristic polynomial $\det(H_{[a,b]}(x,\omega) - E)$. Let $M_{[a,b]}(x,\omega,E)$ be the transfer-matrix of the equation and $L(\omega,E)$ be the the Lyapunov exponent. Recall that

$$M_{[1,N]}(x,\omega,E) = \begin{bmatrix} f_{[1,N]}(x,\omega,E) & -f_{[2,N]}(x,\omega,E) \\ f_{[1,N-1]}(x,\omega,E) & -f_{[2,N-2]}(x,\omega,E) \end{bmatrix} \tag{2}$$

The first fundamental idea of Frölich and Spencer on the localization is as follows. Any solution of the equation

$$-\psi(n+1) - \psi(n-1) + v(n)\psi(n) = E\psi(n) , \quad n \in \mathbb{Z} , \tag{3}$$

obeys the relation

$$\psi(m) = G_{[a,b]}(E)(m,a-1)\psi(a-1) + G_{[a,b]}(E)(m,b+1)\psi(b+1), \quad m \in [a,b]. \tag{4}$$

where $G_{[a,b]}(E) = \left(H_{[a,b]} - E\right)^{-1}$ is the Green's function, $H_{[a,b]}$ is the linear operator defined by (3) for $n \in [a,b]$ with zero boundary conditions.

By Cramer's rule

$$\left|\left(H_{[a+1,a+N]}(x,\omega) - E\right)^{-1}(k,m)\right| = \frac{\left|f_{[a+1,k]}(x,\omega,E)\right| \left|f_{[m+1,a+N]}(x,\omega,E)\right|}{\left|f_{[a+1,a+N]}(x,\omega,E)\right|} \tag{5}$$

$a+1 \le k < m \le a+N$. Heuristically,

$$\begin{aligned} \log|f_{[a+1,a+n]}(x,\omega,E)| &\approx \log\|M_{[1,n]}(x,\omega,E)\| \approx nL(\omega,E), \\ \log\left|\left(H_{[a+1,a+N]}(x,\omega) - E\right)^{-1}(k,m)\right| &\approx -(m-k)L(\omega,E) \end{aligned} \tag{6}$$

which leads to exponential decay of the eigenfunctions due to (4). Obviously, given x, ω the previous argument does not apply to some intervals $[a+1, a+N]$. The collection of the ideas one borrows from Fröhlich and Spencer to address this issue is as follows.

- The heuristic (6) fails if and only if E is extremely close to the spectrum of $H_{[a+1,a+N]}(x,\omega)$. The latter event is called a simple resonance or just a resonance.

• For a fixed E the chances of a resonance to happen are very small. In other words the total measure of corresponding x, ω is very small. In the context of random Scrödinger operator this statement is the fundamental Wegner estimate. We explain what is the quasi-periodic counterpart of this estimate.

• The chances that some E is very close to the spectrum of $H_{[a+1,a+N]}(x, \omega)$ and to the spectrum $H_{[a+N+T+1,a+T+2N]}(x, \omega)$ with $T \gg N$ are so small that E can be eliminated from such events via crude partition of the domain of the parameter E. The latter event is called double resonance.

In contrast with the random case the estimate for the double resonance to occur in quasi-periodic setting comes not from statistical independence of the two simple resonances involved but on the "transversality" of the corresponding two sets in the space of x, ω. The large deviation theorem says that the measure that the $\log |f_{[a+1,a+n]}(x, \omega, E)| \approx \log \|M_{[1,n]}(x, \omega, E)\| \approx nL(\omega, E)$ fails is small. So, for any given E the resonance sets are "thin". A very important feature here is that one can replace these sets by some "thin" sets of the following form

$$P(x, \omega, E) > 0 \tag{7}$$

where P is a polynomial which degree is not to high. This is due to the assumption that the potential is analytic. Sets which can be defined via inequalities like (7) and more general ones including several polynomials are called semi-algebraic. The most important feature of the semi-algebraic sets in $\mathbb{R}^n$ is the estimate on the number of connected components which comes from the Bezout theorem combined with implicit function theorem from multivariate calculus. This combination allows to analyze semi-algebraic sets by covering them via a union of smooth images of "nice sets" in $\mathbb{R}^k$.

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The absence of non-perturbative results in higher dimensions

Yiqian Wang

Recently non-KAM methods have been developed to obtain non-perturbative results in quasi-periodic Schrodinger operators.
In the study of Almost Mathieu operator

$$(H(x)\phi)_n = \lambda \cos 2\pi(x + n\omega)\phi_n + \phi_{n-1} + \phi_{n+1}, \quad \phi \in l^2(\mathbb{Z}),$$

Jitomirskaya (1999, [7]) proved that for any Diophantine ω and *a.e.* x, it holds that

(i) if $\lambda > 2$, then H has only pure point spectrum with Anderson localization;
(ii) if $\lambda = 2$, then H has purely singular continuous spectrum;
(iii) if $\lambda < 2$, then H has purely absolutely continuous spectrum.

Then Bourgain and Goldstein (2000, [1]) proved that with $d \geq 1$ and a general nonconstant analytic potential $v(x)$, $x \in \mathbb{T}^d$, for the operator

$$(H(x)\phi)_n = \lambda v(x+n\omega)\phi_n + \phi_{n-1} + \phi_{n+1}, \quad \phi \in l^2(\mathbb{Z}), \tag{1}$$

it holds that for fixed x_0, there exists $\lambda_1(v) > 0$ such that for $\lambda > \lambda_1$ and *a.e.* ω, H has Anderson localization.
Later for $d = 1$ Bourgain and Jitomirskaya (2002, [3]) proved that there exists $\lambda_2(v) > 0$ such that if $\lambda < \lambda_2$, then for *a.e.* x and any Diophantine ω, H in (1) has purely absolute continuous spectrum.

The common feature of non-perturbative results is that they holds true for ω in full measure and λ is independent on ω (see the constants $\lambda = 2, \lambda_1, \lambda_2$ as above).

For the case $d > 1$, we have the following classical *perturbative* results on the existence of absolute continuous spectrum.
Theorem 1 (Eliasson 1992 [6], Chulaevsky and Sinai 1993 [5], Bourgain 2002 [2]) *For $d \geq 1$ and $\epsilon > 0$, there exists $\lambda_3(v,\epsilon) > 0$ such that if $\lambda < \lambda_3$, then there is a set Ω_1 with a measure $> 1-\epsilon$ such that for any $\omega \in \Omega_1$ and a.e. x, the operator H in (2.2) has purely absolute continuous spectrum.*

Then a natural question is: can we have a nonperturbative result of the existence of purely absolute continuous spectrum when $d > 1$?

Bourgain gave a negative answer for the question with $d = 2$, thus the nonperturbative result of [3] has no a multi-frequency counterpart.
Theorem 2 (Bougain 2002, [2]) *Let $v = v(x_1, v_2)$ be a trigonometric polynomial on $\mathbb{T}^2$ with a nondegenerate local maximum. Fix (x_1, x_2). For any $\lambda \neq 0$, there exists a set Ω_2 of positive measure such that if $\omega \in \Omega_2$, then H in (1) has some point spectrum and $mes(\Sigma_{pp}(H)) > 0$.*

Remark 1 ω in Theorem 2 will be chosen too small for Theorem 1 to apply, so there is no contradiction.
Remark 2 In [2], Bourgain also constructed examples of the form

$$(H(x)\phi)_n = (\lambda_1 \cos(x_1 + n\omega_1) + \lambda_2 \cos(x_2 + n\omega_2))\phi_n + \phi_{n-1} + \phi_{n+1},$$

such that for ω in a set of positive measure, $\Sigma_{ac}(H) \neq \emptyset$ and $mes(\Sigma_{pp}(H)) > 0$.

For simplicity, we will focus on the special potential $v = \cos x_1 + \cos x_2$.

Using the standard duality technique [5] based on Fourier transform, the model

$$(H(x)\phi)_n = \lambda(\cos(x_1 + n\omega_1) + \cos(x_2 + n\omega_2))\phi_n + \phi_{n-1} + \phi_{n+1}, \quad \phi \in l^2(\mathbb{Z}) \tag{2}$$

is dual to

$$\tilde{H}(\theta) = 2\cos(\theta + n \cdot \omega)\delta_{nn'} + \frac{\lambda}{2}\Delta, \tag{3}$$

where Δ is the 2-dimensional discrete Laplacian. We have the following dual results for $\tilde{H}$ from Theorem 1 and 2.

Theorem 3 (Eliasson 1992 [6], Chulaevsky and Sinai 1993 [5], Bourgain 2002 [2]) *For $d \geq 1$ and $\epsilon > 0$, there exists $\tilde{\lambda}_3(v, \epsilon) > 0$ such that if $\lambda < \tilde{\lambda}_3$, then there is a set $\tilde{\Omega}_1$ with a measure larger than $1 - \epsilon$ such that for any $\omega \in \tilde{\Omega}_1$ and a.e. x, the operator $\tilde{H}$ in (3) has Anderson localization.*

Theorem 4 (Bougain 2002, [2]) *Fix θ. For any $\lambda \neq 0$, there exists a set $\tilde{\Omega}_2$ of positive measure such that if $\omega \in \tilde{\Omega}_2$, then $\tilde{H}$ in (3) has some absolute continuous spectrum.*

Remark 3. Theorem 3 implies that the $1D$ non-perturbative results in [7, 1] have no $\mathbb{Z}^2$ counterpart.

Remark 4. For the case $x \in \mathbb{T}^2$ and $n \in \mathbb{Z}^2$, Bourgain, Goldstein and Schlag obtained a perturbative result for the occurrence of Anderson Localization, see [4].

Fix $\lambda \neq 0$. Let $\omega = (\omega_1, \omega_2)$ be Diophantine with $|\omega| \ll 1$. The sketch of the proof for Theorem 2 is as follows.

First, we will find an interval I of length $c_0\lambda > 0$ near the edge of the spectrum set($\subset [-2 - |v|, 2 + |v|]$) such that

$$mes(I \bigcap spec(H)) \approx |I| \tag{4}$$

and

$$mes(I \bigcap L_+(H)) \approx |I|, \tag{5}$$

where $L_+(H) = \{E | L(E) > 0\}$ with $L(E)$ the Lyapunov exponent of cocycles corresponding to the operator in (2) and the energy E.

Obviously, (4) and (5) imply that

$$mes(\Sigma_{sc}(H) \bigcup \Sigma_{pp}(H)) > mes(I \bigcap spec(H) \bigcap L_+(H)) > 0. \tag{6}$$

Then we will prove Anderson localization if $L(E) > 0$ and E corresponds to an extended state, which combining with (6), implies

$$mes(\Sigma_{pp}(H)) > 0. \tag{7}$$

The proof of (4)–(7) will be inductively obtained from exponentially decay estimate on Green functions and eigenfunctions. It is worthy to note that the proof essentially depends on the fact that for multi-frequency situation, we have more freedom to choose suitable phases than in 1-frequency situation. For example, for $d = 1$ we cannot find similar interval satisfying (4) and (5) if λ is small. On the contrary, in multi-frequency situation, we can prove the existence of such an interval by choosing suitable frequencies from higher dimensional base space.

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Global theory of one-frequency Schrödinger operators I

Julie Déserti

Let us consider one-dimensional Schrödinger operators with an analytic one-frequency potential that is

$$H = H_{\alpha,v} \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \quad \text{given by} \quad (Hu)_n = u_{n+1} + u_{n-1} + v(n\alpha)u_n$$

where $v\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is an analytic function (the potential) and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (the frequency). Denote by $\Sigma = \Sigma_{\alpha,v}$ the spectrum of H. For any energy E in $\mathbb{R}$ let us define

$$(1)\quad A(x) = A^{(E-v)}(x) = \begin{bmatrix} E - v(x) & -1 \\ 1 & 0 \end{bmatrix}, \quad A_n(x) = A\big(x+(n-1)\alpha\big)\dots A(x)$$

which are analytic functions with values in $\mathrm{SL}(2,\mathbb{R})$. They are relevant to the analysis of H because a formal solution of $Hu = Eu$ satisfies

$$\begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} = A_n(0) \begin{bmatrix} u_0 \\ u_{-1} \end{bmatrix}.$$

The *Lyapunov exponent* at energy E is denoted by $L(E)$ and given by

$$(2)\qquad \lim_{n\to+\infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln ||A_n(x)||\, dx \geq 0.$$

Energies $E \in \Sigma$ can be

- *supercritical* if $L(E) > 0$;
- *subcritical* if there is a uniform subexponential bound $\ln ||A_n(x)|| = o(n)$ through some band $|\Im z| < \varepsilon$;
- *critical* otherwise.

Progress has been made mainly into the the understanding of the behavior in regions of the spectrum belonging to two regimes with behavior characteristic of "large", resp. "small" potentials; but until [1, 2] there was no global theory of such operators and the transition between the two regimes was not understood. In general subcritical and supercritical regimes can coexist in the spectrum of the same operator ([5]). However to go from one regime to the other it may not be necessary to pass through the critical regime since we usually expect the spectrum to be a Cantor set.

Lyapunov exponent of $\mathrm{SL}(2,\mathbb{C})$ cocycles. In the dynamical approach the understanding of the Schrödinger operator is obtained through the detailed description of a certain family of dynamical systems.

A (one-frequency, analytic) *quasiperiodic* $\mathrm{SL}(2,\mathbb{C})$ *cocycle* is a pair (α, A), where

$$\alpha \in \mathbb{R} \qquad A\colon \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C}) \text{ is analytic,}$$

understood as defining a linear skew product acting on $\mathbb{R}/\mathbb{Z} \times \mathbb{C}^2$ by $(x, w) \mapsto (x + \alpha, A(x) \cdot w)$. The iterates of the cocyles are given by $(n\alpha, A_n)$ where A_n is given by (1). The *Lyapunov exponent* $L(\alpha, A)$ of the cocycle (α, A) is given by the left hand side of (2). Let $\mathcal{UH} \subset C^\omega\big(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C})\big)$ denote the set of A such that (α, A) is uniformly hyperbolic. If (α, A) belongs to $\mathcal{UH}$ then $L(\alpha, A) > 0$. Uniform hyperbolicity is a stable property: $\mathcal{UH}$ is open and $A \mapsto L(\alpha, A)$ is analytic over $\mathcal{UH}$ (regularity properties of the Lyapunov exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables). If $L(\alpha, A) > 0$ but $(\alpha, A) \notin \mathcal{UH}$ then we say that (α, A) is *nonuniformly hyperbolic* and denote it by $\mathcal{NUH}$.

Most important examples are Schrödinger cocycles and $L(E) = L\big(\alpha, A^{(E-v)}\big)$. One of the most basic aspects of the connection between spectral and dynamical properties is that $E \notin \Sigma_{\alpha,v}$ if and only if $\big(\alpha, A^{(E-v)}\big)$ is $\mathcal{UH}$.

If $A \in C^\omega\big(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C})\big)$ admits a holomorphic extension to $|\Im z| < \delta$ then for $|\varepsilon| < \delta$ we can define $A_\varepsilon \in C^\omega\big(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C})\big)$ by $A_\varepsilon(x) = A(x + \mathbf{i}\varepsilon)$. The Lyapunov exponent $L(\alpha, A_\varepsilon)$ is a convex function of ε. We can thus introduce the function *acceleration* defined by

$$\omega(\alpha, A) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi\varepsilon}\big(L(\alpha, A_\varepsilon) - L(\alpha, A)\big).$$

Since the Lyapunov exponent is a convex and continuous function the acceleration is an upper semi-continuous function in $\mathbb{R} \setminus \mathbb{Q} \times C^\omega\big(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C})\big)$. The acceleration is quantized ([1]):

(♣) *If (α, A) is $\mathrm{SL}(2,\mathbb{C})$-cocycle with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\omega(\alpha, A)$ is an integer.*

A direct consequence is the following:

The function $\varepsilon \mapsto L(\alpha, A_\varepsilon)$ is a piecewise affine function of ε.

It is thus natural to introduce the notion of regularity. A cocycle (α, A) in $\mathbb{R} \setminus \mathbb{Q} \times C^\omega\big(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C})\big)$ is *regular* if $L(\alpha, A_\varepsilon)$ is affine for ε in a neighborhood of 0. In other words (α, A) is regular if the equality $L(\alpha, A_\varepsilon) - L(\alpha, A) = 2\pi\varepsilon\omega(\alpha, A)$

holds for all ε small, and not only for the positive ones. Regularity is equivalent to the accceleration being locally constant near (α, A). It is an open condition in $\mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$. The following statement gives a characterization of the dynamics of regular cocycles with positive Lyapunov exponent ([1]):

Assume that $L(\alpha, A) > 0$. Then (α, A) is regular if and only if (α, A) is $\mathcal{UH}$.

It thus follows that:

(♠) *for any (α, A) in $\mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ there exists ε_0 such that*

- *$L(\alpha, A_\varepsilon) = 0$ (and $\omega(\alpha, A) = 0$) for every $0 < \varepsilon < \varepsilon_0$,*
- *or $(\alpha, A_\varepsilon) \in \mathcal{UH}$ for every $0 < \varepsilon < \varepsilon_0$.*

Stratified analyticity of the Lyapunov exponent (consequence of the quantization). Let $(\alpha, A) \in \mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ such that $\omega(\alpha, A) = j > 0$. According to (♠) there exists a small deformation (α, A_ε) of (α, A) that belongs to $\mathcal{UH}$ and satisfies $\omega(\alpha, A_\varepsilon) = j$. The Lyapunov exponent is analytic in that deformation; moreover using (♣) we obtain that $L(\alpha, A) = L(\alpha, A_\varepsilon) - 2\pi j\varepsilon$. Hence the Lyapunov exponent is analytic in "the stratum" of cocycles (α, A) with acceleration j. In other words Avila establishes the following result ([1]):

Let α be in $\mathbb{R} \setminus \mathbb{Q}$ and let v be any real analytic function. Then the Lyapunov exponent is a C^∞-stratified function of the energy.

Boundary of nonuniform hyperbolicity & acriticality. Let $C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \times \mathbb{R}$ be the real Banach space of analytic functions from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$ admitting a holomorphic extension to $|\Im z| < \delta$.

Let $(\alpha, A) \in \mathbb{R} \setminus \mathbb{Q} \times C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ be a critical cocycle with acceleration j. Then there exists $0 < \delta' < \delta$ such that

$$(\alpha, A_{\delta'}) \in \mathcal{UH}, \qquad \omega(\alpha, A_{\delta'}) = j \qquad \text{and} \qquad L_{\delta,j}(\alpha, A) = 0$$

where $L_{\delta,j}(\alpha, A) = L(\alpha, A_{\delta'}) - 2\pi j\delta'$. Moreover if A is $\mathrm{SL}(2,\mathbb{R})$-valued, criticality implies that the acceleration is positive. So the locus of critical $\mathrm{SL}(2,\mathbb{R})$-valued cocycles is covered by countably many analytic sets $L^{-1}_{\delta,j}(0)$. The $L_{\delta,j}$ are non-degenerate hence one has the following statement ([1]):

($\star$) *For any α in $\mathbb{R} \setminus \mathbb{Q}$ the set of potentials and energies (v, E) such that E is a critical energy for $H_{\alpha,v}$ is contained in a countable union of codimension-one analytic submanifolds of $C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \times \mathbb{R}$.*
In particular a typical operator H has at most countably many critical energies.

This yields to the following conjecture ([1]):

(♢) **Conjecture** — *For a typical operator H, the spectrum measures have no singular continuous component.*

According to ($\star$) a typical operator H has at most countably many critical energies: this is the starting point of the proof of the following statement which says that the critical set is typically empty ([2]). Let us say that H is acritical if no energy E in Σ is critical.

($\triangle$) *Let α be in $\mathbb{R} \setminus \mathbb{Q}$; then for a typical v in $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ the operator $H_{\alpha,v}$ is acritical.*

Let us give an idea of the proof. Given a Schrödinger operator $H_{\alpha,v}$ the acceleration ω at energy $E \in \mathbb{R}$ is given by $\omega(E) = \omega\big(\alpha, A^{(E-v)}\big)$. Then E is critical if and only if $L(E) = 0$ and $\omega(E) > 0$; if $\omega(E) = k$ we say that E is a *critical point of degree* k. Avila constructs a foliation by monotonic curves ([4]) in the hypersurface $L_{\delta,k} = 0$. Using renormalization ([4]) he shows that the measure of the critical points of degree k is zero; so these points can be destroyed by a small perturbation of the potential. He then iterates the process.

Stability of the different regimes. Supercritical energies E in Σ are usually called $\mathcal{NUH}$. The $\mathcal{NUH}$ regime is stable: if we perturb α in $\mathbb{R}\backslash\mathbb{Q}$, v in $C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ and E in $\mathbb{R}$ (but still belonging to the perturbed spectrum) we stay in the same regime (continuity of the Lyapunov exponent [7]).

Critical regime is unstable (consequence of ($\star$)). More precisely the critical regime equals to the boundary of the $\mathcal{NUH}$ regime ([2]).

Subcritical regime is also stable ([1]): regularity is an open condition in $\mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z},\mathbb{R})$ (and subcriticality is equivalent to regularity and being in the spectrum). The subcritical energies are also said to be away from $\mathcal{NUH}$.

The role of ($\triangle$) in the Spectral Dichotomy Program. Statement ($\triangle$) reduces the spectral theory of a typical one-frequency Schrödinger operator H to the separate "local theories" of (uniform) supercriticality and subcriticality so it is a key step to establish the spectral dichotomy: the decomposition of a typical operator in a direct sum of operators with the spectral type of "large-like" and "small-like" operators (large potentials fall into the supercritical regime [9]; small potentials fall into the subcritical regime [7, 8]).

Almost Reducibility Conjecture and some consequences. Statements ($\star$) and ($\triangle$) give further motivation to the research on set of regular cocycles with zero Lyapunov exponent, *i.e.* on subcritical cocyles. In that direction let us mention the Almost Reducibility Conjecture ([1]):

Regularity with zero Lyapunov exponents implies almost reducibility. More precisely assume that $L(\alpha, A_\varepsilon) = 0$ for $a < \varepsilon < b$. Then for every n there exists a holomorphic map $B_n\colon \big\{a + 1/n < |\Im z| < b - 1/n\big\} \to \mathrm{SL}(2,\mathbb{C})$ such that

$$\big|\big|\, B_n(z+\alpha)A(z)B(z)^{-1} - \mathrm{id}\,\big|\big| < 1/n \quad \text{for } a + 1/n < \Im z < b - 1/n.$$

Subcriticality is strongly related to the concept of almost reducibility which by definition generalizes the scope of applicability of the theory of small potentials. Almost Reducibility Conjecture would at once provide a precise understanding of subcriticality. The results of [3, 6] and a proof of the Almost Reducibility Conjecture would give a proof of ($\diamondsuit$) (*see* [2]).

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Global theory of one-frequency Schrödinger operators II

Paul Munger

By a one-frequency Schrödinger operator, we mean an operator on $\ell^2(\mathbf{Z})$ of the form $(H\psi)(n) = \psi(n-1)+\psi(n+1)+v(n\alpha)\psi(n)$, where v is an analytic function on the unit circle and α is the irrational frequency. In [1], Artur Avila shows that the Lyapunov exponent of such an operator, which is poorly behaved in some ways, is quite regular in a *stratified* way: if we break the space of possible potentials v into certain natural strata, the Lyapunov exponent is in fact analytic on each stratum (a stratification of a topological space is a decomposition into nested closed subsets with empty intersection). This constitutes a new point of view on the Lyapunov exponent.

The previous talk, Global Theory of One-Frequency Schrödinger Operators I, exposited this point of view and some of the ramifications of the results in [1], while this talk gave detailed proofs of two theorems from [1].

The first of these is Theorem 3, which states that if X is any real analytic manifold and $v : X \longrightarrow C^\omega(\mathbf{R}/\mathbf{Z}, \mathbf{R})$ is an analytic function (giving a family of potentials $v(\lambda)$) then the Lyapunov exponent is C^ω-stratified in λ and the energy E. We stratify the parameter space $\mathbf{R} \times X$ using a function ω, as follows. At this point it becomes more natural to deal with $SL(2, \mathbf{C})$ cocycles; their correspondance with Schrödinger operators is well explained in many previous talks. We then say the the acceleration $\omega(A, \alpha)$ of a uniformly hyperbolic cocycle A at frequency α is

$$\lim_{\epsilon \to 0+} \frac{1}{2\pi\epsilon}(L(\alpha, A_\epsilon)),$$

where L is the Lyapunov exponent and A_ϵ is the cocycle $A(x + i\epsilon)$.

It happens that (Theorem 5 of [1]) ω is always an integer. Therefore $X \times \mathbf{R}$ is stratified according to whether the cocycle is uniformly hyperbolic or not, and if not, according to increasing ω. By considering how L looks when restricted to cocycles with a fixed acceleration, [1] shows L is analytic on each stratum. A little more precisely, define $\Omega_{\delta,j}$ to be the set of cocycles that extend analytically to $|\Im(z)| < \delta$, and such that there is a $\delta' < \delta$ such that $(\alpha, A_{\delta'})$ is uniformly hyperbolic with acceleration j. Then we may define the function $L_{\delta,j} : \Omega_{\delta,j} \longrightarrow \mathbf{R}$ by $L_{\delta,j}(\alpha, j) = L(\alpha, A_{\delta'}) - 2\pi j\delta'$. For uniformly hyperbolic cocycles, this is the same as the Lyapunov exponent. Each $L_{\delta,j}$ is C^{∞}. The proof is finished by relating the sets $\Omega_{\delta,j}$ to the strata.

We say a cocycle is *critical* if it has $L = 0$ and it is non-regular, in that $L(\alpha, A_\epsilon)$ is not affine at 0. The second theorem the talk dealt with says that the set of critical $SL(2, \mathbf{R})$ cocycles is covered by finitely many sets $L^{-}_{\delta,j}1(0)$, and these are analytic submanifolds of the space of cocycles (the previous talk discussed the implications of this theorem). The only thing to prove is the analyticity, i.e. that $L_{\delta,j}$ is a submersion. To prove this, Avila introduces the *derivative of the Lyapunov exponent* for uniformly hyperbolic cocycles: by considering the stable and unstable directions of the cocycle, one gets a formula for $\frac{d}{dt}L(\alpha, Ae^{tw})$ for $w \in sl(2, \mathbf{C})$. This is used to prove by contradiction that $L_{\delta,j}$ must be a submersion (by looking at what the implications for the stable and unstable directions must be if the derivative vanishes).

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Global theory of one-frequency Schrödinger operators III

Nikolaos Karaliolios

This talk was the third part of a three-part talk on the global theory of one-dimensional Schödinger operators, as it was founded in [1] and [2], and our interest was centered in the explication of the setting, the importance, and the basic points of the proof, of the main theorem obtained in the second article. We refer to these articles for a more extended bibliography on the subject and the results that preceded the formation of this theory.

Introduction We therefore consider a one-frequency quasiperiodic Schrödinger operator $H : \ell^2 \hookleftarrow$ given by

$$(Hu)_n = u_{n-1} + u_{n+1} + v(n\alpha)u_n$$

where $\alpha \in \mathbb{T}\backslash\mathbb{Q}$ and $v \in C^{\omega}(\mathbb{T}, \mathbb{C})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus. We denote by $\Sigma = \Sigma_{\alpha,v} \subset \mathbb{R}$ its spectrum.

The study of the spectral properties of such operators is closely related to the dynamics of $SL(2, \mathbb{R})$ cocycles

$$\begin{array}{rccc} (\alpha, A(\cdot)) : & \mathbb{T} \times \mathbb{R}^2 & \to & \mathbb{T} \times \mathbb{R}^2 \\ & (x, y) & \mapsto & (x + \alpha, A(x)y) \end{array}$$

where, for any energy $E \in \mathbb{R}$,

$$A(\cdot) = A^{(E-v)}(\cdot) = \begin{pmatrix} E - v(\cdot) & -1 \\ 1 & 0 \end{pmatrix} \in C^{\omega}(\mathbb{T}, SL(2,\mathbb{R}))$$

The connection of the two objects is due to the fact that a formal solution of the equation $Hu = Eu$, where u is not necessarily in ℓ^2, satisfies

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A_n(0) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}$$

where $A_n(\cdot)$ is the mapping defining the n-th iterate of $(\alpha, A(\cdot))$:

$$\begin{aligned} (\alpha, A(\cdot))^n &= (n\alpha, A_n(\cdot)) \\ A_n(\cdot) &= A(\cdot + (n-1)\alpha)...A(\cdot) \end{aligned}$$

An advantage of this point of view is that, given the analyticity of the potential, one can consider complex values of the dynamical variable x : for a given potential $v \in C^{\omega}$ there exists $\delta > 0$ such that v is defined and analytic in $|Imz| < \delta$. One can therefore consider analytic deformations of a general analytic $SL(2,\mathbb{R})$ cocycle $(\alpha, A(\cdot))$ defined by $(\alpha, A^{(\varepsilon)}(\cdot)) = (\alpha, A(\cdot + \varepsilon i))$ for ε small enough. The corresponding cocycles in $\mathbb{T} \times \mathbb{C}^2$ have nice properties already exploited in, say, [3], and the passage to the limit $\varepsilon \to 0$ is the key to the study of certain aspects of the dynamics of real-analytic cocycles in $\mathbb{T} \times SL(2,\mathbb{R})$.

The context of the main theorem Given a Schrödinger operator, we define its Lyapunov exponent at energy E, denoted by $L(E)$, as

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \int \log \|A_n(\cdot)\|$$

where the norm typically chosen is the supremum norm on the coefficients of the matrix. The classification of energies in the spectrum resulting from this dynamic invariant is as follows: the energy $E \in \Sigma$ is

(1) supercritical if $L(E) = L((\alpha, A^{(E-v)}(\cdot))) > 0$, in which case the dynamics of the corresponding cocycle are non-uniformly hyperbolic (NUH)
(2) subcritical if there exists $\epsilon > 0$ such that, for $|\varepsilon| < \epsilon$,

$$\lim_{n\to\infty} \frac{1}{n} \int ln\|A_n^{(v,\varepsilon)}(\cdot)\| = 0$$

(3) critical otherwise

Subcriticality has replaced the notion of almost reducibility and has the advantage of being dynamically defined. The two properties are in fact conjectured to be equivalent, and the case of Diophantine frequencies has recently been settled by A. Avila, as announced in [2]. On the other hand, energies outside the spectrum are known to be associated to cocycles with uniformly hyperbolic (UH) dynamics, so that for $E \notin \Sigma$, $L(E) > 0$. In fact, in [1] it was shown that a cocycle $(\alpha, A^{(v)}(\cdot))$ is UH if, and only if, the Lyapunov exponent of $(\alpha, A^{(v,\varepsilon)}(\cdot))$ is locally constant and positive.

Therefore, critical energies are exactly those satisfying the property that $L(E) = 0$, but $\omega(E) \neq 0$, where

$$\omega(E) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi\varepsilon}(L((\alpha, A^{(E-v,\varepsilon)}(\cdot))) - L((\alpha, A^{(E-v)}(\cdot))))$$

is the acceleration at energy E. In [1] it was proved that $\omega(E) \in \mathbb{N}$. Therefore, if we denote by $\mathcal{C}^k$ the sets of potentials such that $(\alpha, A^{(v)}(\cdot))$ has acceleration $k \in \mathbb{N}^*$, by $\mathcal{A}_0$ the set of non-critical potentials and by $\mathcal{A}_k = \mathcal{A}_0 \cup (\cup_{1 \leq i \leq k} \mathcal{C}^i)$, the union of $\mathcal{A}_k$ equals C^ω. An operator is acritical if $E + v(\cdot) \in \mathcal{A}_0$ and critical of degree k if $\max(i, E + v(\cdot) \in \mathcal{C}^i \text{ for some } E \in \mathbb{R}) = k$.

Since supercriticality, subcriticality and UH are open conditions with respect to perturbations of the potential, it is reasonable to conjecture that the existence of critical energies in the spectrum is rare in C^ω, which is actually the main result of the paper:

Theorem (A. Avila) Let α be an irrational frequency. Then, for a measure theoretically typical potential, the operator $H_{\alpha,v}$ is acritical.

This theorem allows to split the spectrum of a typical Schrödinger operator into a finite number of disjoint pieces $\Sigma_i = \Sigma \cap (a_i, b_i)$, where $a_i < b_i < a_{i+1} < b_{i+1}$ and the energies in Σ_i alternate between sub- and super-critical, as i increases. Moreover, the Lyapunov exponent is uniformly bounded away from 0 in the supercritical regime, while the $\epsilon > 0$ in the definition of subcriticality can be chosen uniformly in the subcritical one. Finally, acriticality is an open condition jointly in the (irrational) frequency and the potential.

The precise definition of what is a typical potential used in the article is based on the notion of prevalence, a substitute for the absence of a translation-invariant measure in non-locally compact spaces. More precisely the compact embedding of $\mathbb{D}^\mathbb{N}$ ($\mathbb{D} \subset \mathbb{C}$ is the unit disk) $\mathbb{D}^\mathbb{N} \ni (t_n) \mapsto \sum \varepsilon(n)\Re(t_n e^{2i\pi n\cdot}) \in C^\omega$, where $\varepsilon : \mathbb{N} \to \mathbb{R}_+^*$ is exponentially decreasing, was considered. This embedding induces a measure on its image $\mathcal{S}$, which is the space of admissible perturbations, by the push forward of the Lebesgue measure, and a property is said to be typical if for every $v \in C^\omega$, and for *a.e.* $w \in \mathcal{S}$, $v + w$ satisfies this property. The notion of typicality used in the statement is related to the choice of a fixed such function ε, although the proof implies a stronger statement.

Idea of the proof In what follows, $\alpha \in \mathbb{T} \setminus \mathbb{Q}$, $v \in \mathcal{C}_k$ and $\delta > 0$ such that v is defined in $|\Im(z)| < \delta$ are to be considered fixed.

The first step in the strategy of the proof is to use the fact that the measure on $\mathcal{S}$ sees all finite dimensional subspaces of C^ω, and most importantly trigonometric polynomials. We denote by $\mathcal{P}^n$ the space of real-symmetric trigonometric polynomials of degree at most n, and by $\mathcal{P}^n(\varepsilon)$ the ball of radius ε centered at 0 in the same space.

Theorem 1 For every $v \in \mathcal{C}^k$ there exist $n \in \mathbb{N}^*$ and $\varepsilon > 0$ for which the following property holds. The $w \in \mathcal{P}^n(\varepsilon)$ such that, $v + z \in \mathcal{C}^{k-1}$ is of $2n-1$-dimensional Hausdorff measure 0.

The first step in the proof of Theorem 1 is the use of the fact that the differential of

$$L_{\xi_0,k}(\alpha, A^{(v)}) = L(\alpha, A^{(v,\xi_0)}) - 2\pi k\xi_0$$

is of rank 1 at $v \in \mathcal{C}^k$, proved in [1], and this gives a trigonometric polynomial w_1 such that

$$DL_{\xi_0,k}(\frac{d}{d\lambda}(\alpha, A^{(v+\lambda w_1)})|_{\lambda=0}) \neq 0$$

which implies that for λ small $v+\lambda w_1 \notin \mathcal{C}^{k-1}$. In order to obtain a second direction $w_2 \in \ker(DL_{\xi_0,k})$ such that $v + \lambda w_2 \notin \mathcal{C}^{k-1}$ for *a.e.* λ small enough (which implies the theorem), the notion of monotonicity with respect to parameters, introduced in [4], was used.

Definition 2 An analytic family of potentials $\lambda \mapsto v_\lambda$ is called monotonous with respect to λ if $(A^{(v_\lambda)})^{-1}\frac{d}{d\lambda}A^{(v_\lambda)}|_{\lambda=0}$ (which is a $sl(2,\mathbb{R})$-valued mapping) has positive determinant for all $x \in \mathbb{T}$.

The positive-determinant matrices in $sl(2,\mathbb{R})$ are the infinitesimal rotations, which means that the projective action of $A^{(v_\lambda)}$ is monotonous in the sense of [4]. The key result which establishes the importance of monotonicity is the following theorem, whose proof uses renormalization and therefore strongly depends on the fact that $(\alpha, A^{(v)})$ is a one-frequency cocycle.

Theorem 3 If v_λ is a monotonic family of potentials, for *a.e.* λ small enough , $\omega(\alpha, A^{(v_\lambda)}) = 0$.

Since monotonicity is not a dynamic invariant, one need only show the existence of an analytic potential $w \in \ker(DL_{\xi_0,k})$ such that the family $(\alpha, A^{(v+\lambda w)})$ can be conjugated to $(\alpha, A^{(v_\lambda)})$ which is monotonous. Since monotonicity is an open condition, one can approximate $\frac{dv_\lambda}{d\lambda}$ by a trigonometric polynomial and obtain Theorem 1.

The existence of such a potential is granted by the following theorem.

Theorem 4 If v is a critical potential, then $DL(\alpha, A^{(v)})$ is not a signed measure.

The proof of this theorem is obtained by using the fact that $(\alpha, A^{(v,\varepsilon)})$ is UH for all $\varepsilon > 0$ and small enough. Then, the formula for the derivative of the Lyapunov exponent, obtained in [1] is used in order to pass to the limit $\varepsilon \to 0^+$.

Finally, a consequence of Theorem 4 and the fact that $DL_{\xi_0,k}$ is of rank 1 is the following:

Theorem 4 If $E - v$ is a critical potential, then there exists a trigonometric polynomial w such that $E - v + tw$ is supercritical for arbitrarily small t.

In other terms, the critical regime is the boundary of the supercritical one.

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Quantitative duality

Igor Krasovsky

The authors consider quasiperiodic Schrödinger operators $H = H_{\lambda v,\alpha,\theta}$ on $\ell^2(\mathbb{Z})$ given by

$$(1) \qquad (Hu)_n = u_{n+1} + u_{n-1} + \lambda v(n\alpha + \theta)u_n, \qquad n = \dots, -1, 0, 1, \dots$$

where $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is analytic, $\theta, \lambda \in \mathbb{R}$, and the frequency α is Diophantine, $\alpha \in \mathrm{DC}$:

$$\mathrm{DC} = \cup_{\kappa,\tau>0}\mathrm{DC}(\kappa,\tau), \qquad \alpha \in \mathrm{DC}(\kappa,\tau) : |\alpha - p/q| \geq \kappa q^{-\tau}, \quad p, q \in \mathbb{Z}, \quad q \neq 0.$$

In [1], the authors obtain extensive results in the nonperturbative metallic regime, i.e., for $0 < |\lambda| < \lambda_0(v)$, where λ_0 is sufficiently small depending on v only.

The main example of such operators is the almost Mathieu operator that corresponds to $v(x) = 2\cos(2\pi x)$. In this case $\lambda_0 = 1$.

Consider the dual to (1) operator on $\ell^2(\mathbb{Z})$ given by

$$(2) \qquad (\hat{H}\hat{u})_n = \sum_k \lambda \hat{v}_k \hat{u}_{n-k} + 2\cos 2\pi(n\alpha + \theta)\hat{u}_n, \qquad n = \dots, -1, 0, 1, \dots$$

where $\hat{v}_k$ are the Fourier coefficients of $v(x) = \sum_k \hat{v}_k e^{2\pi i k x}$.

By Aubry duality, it is known that the localization for $\hat{H}$ (exponentially decaying eigenfunctions) corresponds to the reducibility for H. This is a very useful property, however, it has a serious limitation: it does not hold for all energies in the spectrum. In the paper, the authors overcome this difficulty by showing that for all 'bad' energies a weaker *almost localization* still holds, which by a new quantitative version of Aubry duality, also obtained in the paper, corresponds to the *almost reducibility* for H. This important improvement of the duality allows the authors to obtain immediate consequences for spectral properties of H solving several long-standing problems.

Fix $\alpha \in \mathrm{DC}$, $\theta \in \mathbb{R}$, $\varepsilon > 0$. A number k is called ε-resonance if $||2\theta - k\alpha||_{\mathbb{R}/\mathbb{Z}} \leq e^{-|k|\varepsilon}$. The number θ is called ε-resonant if the set of resonances is infinite. If θ is non-resonant with resonances $n_1, \dots, n_j$, set formally $n_{j+1} = \infty$. The family $\{\hat{H}_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ is said to exhibit almost localization if there exist positive constants $C_0, C_1, \varepsilon_0, \varepsilon_1$ such that for every solution $\hat{u}$ of $\hat{H}\hat{u} = E\hat{u}$ satisfying $\hat{u}_0 = 1$ and $|\hat{u}_k| \leq 1 + |k|$, and for every $C_0(1 + |n_j|) < |k| < C_0^{-1}|n_{j+1}|$, we have $|\hat{u}_k| \leq C_1 e^{-\varepsilon_1|k|}$, where n_j, n_{j+1} are the neighboring ε_0-resonances of θ, and k is between them.

The first main statement of the paper is

Theorem 1 *If $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is analytic and $|\lambda| < \lambda_0(v)$ then $\{\hat{H}_{\lambda v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ is almost localized for every $\alpha \in \mathrm{DC}$. For $v(x) = 2\cos 2\pi x$, we have $\lambda_0 = 1$.*

The method of the proof is based on the earlier papers by Jitomirskaya and Bourgain.

For each energy E in the spectrum of H (the spectrum is independent of θ), one can find $\theta(E)$ such that there exists a solution to $\hat{H}_{\lambda v,\alpha,\theta}\hat{u} = E\hat{u}$ with $\hat{u}_0 = 1$, $|\hat{u}_n| \le 1$. Assume the conditions of Theorem 1, fix $\theta(E)$ as above, and let n_j be its ε_0-resonances. Let

$$A(x) = \begin{pmatrix} E - \lambda v(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

Let C, c be various positive constants independent of E and θ. For a bounded analytic function f on a strip $|\Im z| < \varepsilon$, let $||f||_\varepsilon = \sup_{|\Im z|<\varepsilon} |f(z)|$.

Then the following 2 versions of the quantitative almost reducibility/Aubry duality are proved in the paper.

Theorem 2 *Fix some* $n = |n_j| + 1 < \infty$ *and let* $N = |n_{j+1}|$. *Then there exists* $\Phi : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ *analytic with* $\|\Phi\|_{cn^{-C}} \le Cn^C$ *such that*

$$\Phi(x+\alpha)A(x)\Phi(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} q_1(x) & q(x) \\ q_3(x) & q_4(x) \end{pmatrix},$$

with

$$\|q_1\|_{cn^{-C}}, \|q_3\|_{cn^{-C}}, \|q_4\|_{cn^{-C}} \le Ce^{-cN}$$

and

$$\|q\|_{cn^{-C}} \le Ce^{-cn(\ln(1+n))^{-C}}.$$

Theorem 3 *Fix some* $n = |n_j| + 1 < \infty$ *and let* $N = |n_{j+1}|$. *Let* $L^{-1} = \|2\theta - n_j\alpha\|_{\mathbb{R}/\mathbb{Z}}$, *and assume that* $0 < L^{-1} < c$. *Then there exists* $W : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ *analytic such that* $|\deg W| \le Cn$, $\|W\|_c \le CL^C$ *and* $\|W(x+\alpha)A(x)W(x)^{-1} - R_{\mp\theta}\|_c \le Ce^{-cN}$, *where*

$$R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

The last theorem gives, in the case $N = \infty$, the quantitative version of the usual reducibility.

The proof of the above Theorems 2,3 in the paper uses ingenuous Fourier analysis and Lagrange interpolation arguments.

The authors then obtain the following important corollaries of these results. It follows from Theorem 3 that $\theta(E)$ as defined above is resonant (for some ε) if and only if the rotation number $\rho(\alpha, E)$ is resonant (for a possibly different ε). A consequence is a solution of the Dry Ten Martini problem for the non-critical case $|\lambda| \ne 1$ of the almost Mathieu operator with Diophantine frequencies, i.e. the statement that all possible gaps in the spectrum are open. It is known that the integrated density of states at possible gap boundaries has the form $N(E) = \alpha k \mod 1$, $k \in \mathbb{Z}$. The authors thus prove

Theorem 4 *Let $v(x) = 2\cos 2\pi x$, and let $|\lambda| \neq 0, 1$, $\alpha \in \mathrm{DC}$. If E is in the spectrum Σ of H and such that $N(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ then E belongs to the boundary of a component of $\mathbb{R} \setminus \Sigma$.*

Another important result is a corollary of Theorem 2.

Theorem 5 *Let $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be analytic, $|\lambda| < \lambda_0(v)$, $\alpha \in \mathrm{DC}$. Then the integrated density of states for H is $1/2$-Hölder.*

This result is the best possible in this generality, as is clear from known facts in the case of the almost Mathieu operator.

Furthermore, almost reducibility implies the following characterization of the spectral measures, so-called phase stability.

Theorem 6 *Let $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be analytic, $|\lambda| < \lambda_0(v)$, $\alpha \in \mathrm{DC}$. Then the singular spectrum is the same for all phases $\theta \in \mathbb{R}$ (and empty).*

Theorems 4,5,6 complete or greatly extend many partial results on these questions obtained previously in the literature.

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The absolutely continuous spectrum of the almost Mathieu operator

Zhenghe Zhang

This talk is concerned with the almost Mathieu operator $H = H_{\lambda,\alpha,\theta}$ defined on $l^2(\mathbb{Z})$

$$(Hu)_n = u_{n+1} + u_{n-1} + 2\lambda\cos[2\pi(\theta + n\alpha)]u_n$$

where $\lambda \neq 0$ is the coupling, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is the frequency and $\theta \in \mathbb{R}/\mathbb{Z}$ is the phase.

Let $(\alpha, S_{\lambda,E})$ be the Schrödinger cocycles, $\mu_{\lambda,\alpha,\theta}$ be the spectral measure and $\Sigma = \Sigma_{\lambda,\alpha,\theta}$ be the spectrum.

We would like sketch a rough idea about the proof of the following main result in [1]:

Main Theorem. *The spectral measures of the almost Mathieu operator are absolutely continuous if and only if $|\lambda| < 1$.*

It settles the Problem 6 of Barry Simon's list of Schrödinger operator problems for the twenty-first century.

Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α and let

$$\beta = \beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}.$$

The proof will be divided into two parts: $\beta = 0$ (the subexponential regime) and $\beta > 0$ (the exponential regime).

Let's first consider the subexponential regime, of which the starting point is the following:

Theorem 1. *Let $\mathcal{B}$ be the set of $E \in \mathbb{R}$ such that the cocycle $(\alpha, S_{\lambda,E})$ is bounded. Then $\mu_{\lambda,\alpha,\theta}|\mathcal{B}$ is absolutely continuous for all $\theta \in \mathbb{R}/\mathbb{Z}$.*

We show that $\mu_{\lambda,\alpha,\theta}(\Sigma \setminus \mathcal{B}) = 0$ for all θ. Instead of considering $\mathcal{B}$, we consider $\mathcal{R}$, where $\mathcal{R}$ is the set of E such that $(\alpha, S_{\lambda,E})$ is reducible. It will be sufficient to show that $\mu_{\lambda,\alpha,\theta}(\Sigma \setminus \mathcal{R}) = 0$ for all θ. Because $\mathcal{R} \setminus \mathcal{B}$ is a countable set and there is no eigenvalue in $\mathcal{R}$.

In [2], the authors show that the dual operator, $\hat{H}_{\lambda,\alpha,\theta}$, of $H_{\lambda,\alpha,\theta}$ satisfies some strong localization estimates. Using a quantitative version of Aubry duality, which is first developed in [2], the authors then obtain several sharp estimates for the dynamics of Schrödinger cocycles $(\alpha, S_{\lambda,\alpha})$. These lead to some good estimates for the integrated density $N = N_{\lambda,\alpha}$ and some Lipschitz estimates on spectral measure $\mu_{\lambda,\alpha,\theta}$.

On the other hand, for $E \in \Sigma \setminus \mathcal{R}$, the fibred rotation number $\rho(\alpha, E)$ of $(\alpha, S_{\lambda,E})$ is characterized by some resonant condition in [1]. This means that $\rho(\alpha, E)$ will be sufficient close to $k\alpha$, $k \in \mathbb{Z}$.

Now we consider some suitable cover, $K_m,\ m \in \mathbb{N}$, of $\Sigma \setminus \mathcal{R}$ satisfying

$$\Sigma \setminus \mathcal{R} \subset \limsup K_m.$$

[1] is able to show that $N(K_m)$ can be covered by some open intervals $J_{m,s},\ 1 \leq s \leq N_m$ such that: by resonance, the number N_m and length of each $J_{m,s}$ can be controlled; by the good estimates on N and spectral measures, $N^{-1}(J_{m,s}),\ 1 \leq s \leq N_m$ becomes a desirable cover of K_m. Which finally leads to the estimate that

$$\sum_{m=0}^{\infty} \mu(K_m) < \infty.$$

By Borel-Cantelli lemma, this implies the proof of the subexponential regime.

Now we turn to the exponential case. In this case, the starting point is in the following. Let $\mathbb{H}$ be the upperhalf plane in $\mathbb{C}$, $m_{\lambda,\alpha}(\theta, E) \in \mathbb{H}$ be the invariant section of the dynamics $(\alpha, S_{\lambda,E})$ and $\phi : \mathbb{H} \to \mathbb{R}$ be the function $\phi(z) = \frac{1+|z|^2}{2\Im z}$. Then by Kotani theory, [1] shows that

$$\frac{d}{dE}\mu_{\lambda,\alpha,\theta}(E) = \frac{1}{\pi}\phi(m_{\lambda,\alpha}(\theta, E))$$

for all θ and almost all $E \in \Sigma$.

In [3], by periodic approximation, the authors show that

$$\int_{\Sigma}\int_{\mathbb{R}/\mathbb{Z}} \phi(m_{\lambda,\alpha}(\theta, E))d\theta dE = 2\pi,$$

which is equivalent to say that

$$\int_{\Sigma} \frac{d}{dE}N(E)dE = 1.$$

This obviously implies that N is absolutely continuous. Then by Fubini's Theorem, $\mu_{\lambda,\alpha,\theta}$ is absolute continuous for Lebesgue almost every θ.

To prove the Main Theorem in exponential regime, it's sufficient to improve $\int_\Sigma \frac{d}{dE}N(E)dE = 1$ to

$$\int_\Sigma \phi(m_{\lambda,\alpha}(\theta, E))dE = 2\pi$$

for all $\theta \in \mathbb{R}/\mathbb{Z}$. Combined with the results in [3], the above equality is achieved in [1] by some cancellation lemmas.

Roughly speaking, [1] observes that the following equality holds:

$$\frac{1}{s}\sum_{k=0}^{s-1} \phi(B \cdot R_{rk/s} \cdot z) = \phi(z)\phi(B \cdot i),$$

where $\mathbb{Q} \ni \frac{s}{r} \neq \frac{1}{2}$, $B_0 \in SL(2, \mathbb{R})$, $z \in \mathbb{H}$, $R_\theta \in SO(2, \mathbb{R})$ is rotation by the angle θ and $B \cdot z$ is the Möbius transformation.

Starting from this equality for rational rotation $\frac{s}{r}$, passing to some inequality for irrational rotation and applying the inequality to $\phi(m_{\lambda,\alpha}(\theta, E))$ and $\phi(m_{\lambda,\frac{p_n}{q_n}}(\theta, E))$, [1] proves the following:

If

$$\int_\Sigma \phi(m_{\lambda,\alpha}(\theta, E))dE < 2\pi$$

for some θ, then

$$\frac{1}{b}\sum_{k=0}^{b-1} \int_\Sigma \phi(m_{\lambda,\alpha}(\theta + kq_n\alpha, E))dE > 2\pi,$$

which is obviously a contradiction. Hence this completes the proof of the Main Theorem.

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Progress towards the Almost Reducibility Conjecture

Jake Fillman

We consider one-frequency cocycles of the form $T : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2$ given by $T(\omega, v) = (\omega + \alpha, A(\omega) \cdot v)$, where $\alpha \in \mathbb{T}\backslash\mathbb{Q}$ and $A \in \mathcal{C}^\omega(\mathbb{T}, \mathrm{SL}_2(\mathbb{R}))$. Iterates of this map obey $T^n(\omega, v) = (\omega + n\alpha, A_n^\alpha(\omega) \cdot v)$, with $A_n^\alpha(\omega) = A(\omega + (n-1)\alpha) \ldots A(\omega)$ for $n \in \mathbb{Z}_+$. From the perspective of hyperbolic dynamics, one is interested in the presence or absence of exponential growth of $\|A_n^\alpha(\omega)\|$ as $n \to \infty$, which is measured by the Lyapunov exponent, defined by

$$L(T) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A_n^\alpha(\omega)\| \mathrm{d}\omega.$$

We shall divide cocycles into four distinct regimes: uniformly hyperbolic, nonuniformly hyperbolic, subcritical, and critical. If the cocycle iterates $\|A_n^\alpha(\omega)\|$ grow exponentially in n and uniformly on $\mathbb{T}$, one calls the cocycle T *uniformly hyperbolic.* One notices that uniform hyperbolicity of T immediately implies positivity of the Lyapunov exponent, but the converse of this is not true - there exist cocycles T which are not uniformaly hyperbolic and yet $L(T) > 0$. We refer to such cocycles as *supercritical* or *nonuniformly hyperbolic.* One calls the cocycle T *subcritical* if the cocycle iterates $\|A_n^\alpha(z)\|$ are uniformly subexponentially bounded in some strip $\{z \in \mathbb{C}||Im(z)| < \epsilon\}$ (one identifies $A : \mathbb{T} \to \mathrm{SL}_2(\mathbb{R})$ with the corresponding 1-periodic map $A : \mathbb{R} \to \mathrm{SL}_2(\mathbb{R})$ in the natural manner). One sees that subcriticality implies vanshing of the Lyapunov exponent, but again, the converse fails - those cocycles which are not subcritical and yet have vanishing Lyapunov exponent are called *critical.* Finally, we call T *almost reducible* if there exist $\epsilon > 0$ and a sequence $B^{(n)} \in \mathcal{C}^\omega(\mathbb{T}, \mathrm{PSL}_2(\mathbb{R}))$ such that each $B^{(n)}$ admits a bounded, analytic extension to the common strip $\{z \in \mathbb{C}||Im(z)| < \epsilon\}$ and $B^{(n)}(z+\alpha)A(z)B^{(n)}(z)^{-1}$ converges uniformly to a constant on the same strip.

The hope of the almost reducibility conjecture is that subcriticality implies almost reducibility, so that one could then apply the well-understood dynamical analysis of cocycles close to constant to subcritical cocycles. In this talk, we shall establish the almost reducbility conjecture for an explicit generic set of frequencies, namely exponentially Liouville α, i.e. those α whose continued fraction approximants $\dfrac{p_n}{q_n}$ obey $\lim\limits_{n\to\infty} \dfrac{\log(q_{n+1})}{q_n} > 0$. Once one has proven this, one can deduce as corollaries two significant facts, namely the almost reducibility of cocycles close to constant and the global stability of almost reducibility.

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Beyond analyticity I: C^0-generic singular continuous spectrum

Jacob Stordal Christiansen

Let $d \geq 1$ and suppose $f : \mathbb{T}^d \to \mathbb{R}$ is continuous. Then the potential

$$V(n) = f(\omega + n\alpha), \quad \omega, \alpha \in \mathbb{T}^d$$

is almost periodic (in Bohr/Bochner sense). We consider the Schrödinger operator

$$H_{\omega,\alpha} = \Delta + V$$

on $\ell^2(\mathbb{Z})$. The main result to be presented is the following:

Theorem ([1], [2]). *Suppose the translation $\omega \mapsto \omega + \alpha$ is minimal on $\mathbb{T}^d$. Then there is a dense G_δ set $\mathcal{SC} \subset C^0(\mathbb{T}^d, \mathbb{R})$ such that for every $f \in \mathcal{SC}$ and Lebesgue a.e. $\omega \in \mathbb{T}^d$, the operator $H_{\omega,\alpha}$ has purely singular continuous spectrum.*

Recall that translation by α is minimal if the orbit $n \mapsto \omega + n\alpha$ is dense in $\mathbb{T}^d$ for all ω. This holds if and only if the coordinates of α and 1 are rationally independent. Recall also that a G_δ set is a countable intersection of open sets.

The idea of the proof is to show

i) generic absence of absolutely continuous spectrum
ii) generic absence of point spectrum

To establish *i)*, we follow Avila and Damanik [1]. Let L_f be the Lyapunov exponent and define

$$M(f) := |\{E \in \mathbb{R} : L_f(E) = 0\}|$$

where $|\cdot|$ is Lebesgue measure. Then $H_{\omega,\alpha}$ has empty a.c. spectrum (for a.e. ω) if and only if $M(f) = 0$. We shall show that the set

$$M_\delta = \{f : M(f) < \delta\}$$

is open and dense in $C^0(\mathbb{T}^d, \mathbb{R})$. Hence $\{f : M(f) = 0\} = \bigcap_n M_{1/n}$ is a dense G_δ set. The crucial step is to approximate by sampling functions s that only take finitely many values and for which the potentials $W(n) = s(\omega + n\alpha)$ are not periodic for a.e. ω. As observed by Kotani [4], the operator $\Delta + W$ has no a.c. spectrum (for a.e. ω).

For *ii)*, we follow Boshernitzan and Damanik [2]. Recall that if there are positive integers $q_k \to \infty$ and $C > 0$ such that

$$\max_{1 \le n \le q_k} |V(n) - V(n \pm q_k)| \le C k^{-q_k}$$

then V is called a Gordon potential. As proven in [3], operators with such potentials have no eigenvalues. We shall construct a dense G_δ set $F \subset C^0(\mathbb{T}^d, \mathbb{R})$ such that $f(\omega + n\alpha)$ is a Gordon potential for all $f \in F$ and generic ω (i.e., for all ω in a dense G_δ set $\Omega_f \subset \mathbb{T}^d$). It is more involved to obtain the result for generic f and Lebesgue a.e. ω. The reader is referred to [2] for details.

Generic absence of absolutely continuous spectrum

The first step is a technical lemma that reads

Lemma. *For every $r > 0$, the map $f \mapsto M(f)$ is upper semi-continuous on $L^1(\mathbb{T}^d) \cap \{f \in L^\infty(\mathbb{T}^d) : \|f\|_\infty < r\}$ with respect to $\|\cdot\|_1$.*

As an immediate consequence, the set M_δ is open. We shall merely sketch the proof which goes by contradiction.

Proof. Suppose there were a sequence $\{f_n\}$ of functions such that

1) $f_n \to f$ in L^1 and pointwise a.e.
2) $\|f_n\|_\infty, \|f\|_\infty \le C < \infty$
3) $\liminf M(f_n) \ge M(f) + \varepsilon$ for some $\varepsilon > 0$

We show that 1) − 2) implies $\limsup M(f_n) \le M(f) + \varepsilon/2$, contradicting 3).

As all the potentials are bounded, we can restrict our attention to some bounded interval I. The key is to prove that

$$(*) \qquad \int_I \min\{L_{f_n}(E) - L_f(E), 0\}\, dE \to 0 \text{ as } n \to \infty.$$

With ε as in 3), choose $\delta > 0$ so small that $|\{E \in I : L_f(E) < \delta\}| < M(f) + \varepsilon/4$. By $(*)$, we have $|\{E \in I : L_{f_n}(E) < \delta/2 \text{ and } L_f(E) \ge \delta\}| \to 0$ as $n \to \infty$. So for n large enough, there is a set Y_n of measure $< \varepsilon/4$ such that $L_{f_n}(E) \ge \delta/2$ for all $E \in I \setminus Y_n$ with $L_f(E) \ge \delta$. But then $\limsup M(f_n) \le M(f) + \varepsilon/2$.

It remains to show that $(*)$ holds. By 1) − 2), it follows that L_{f_n} converges pointwise to L_f in the upper half-plane $\mathbb{H}$. Let Φ be a conformal mapping of the unit disk $\mathbb{D}$ onto the interior of the equilateral triangle $T \subset \overline{\mathbb{H}}$ with sides I, J, and K. Then

$$\int_0^{2\pi} \Big[L_{f_n}\big(\Phi(e^{i\theta})\big) - L_f\big(\Phi(e^{i\theta})\big)\Big] \frac{d\theta}{2\pi} = L_{f_n}\big(\Phi(0)\big) - L_f\big(\Phi(0)\big) \to 0$$

as the integrand is bounded and harmonic on $\mathbb{D}$. By a change of variables,

$$\int_{\partial T} \big[L_{f_n}(E) - L_f(E)\big] \Phi'\big(\Phi^{-1}(E)\big)^{-1} dE \to 0$$

and we arrive at $(*)$ by first noting that $\int_{J+K} \cdots$ goes to zero. Then consider the integral over $\{E \in I : L_{f_n}(E) - L_f(E) \ge 0\}$ and use the fact that Φ' has constant sign inside I. □

To establish the denseness of M_δ, we rely on the fact that the set of functions s taking finitely many values and for which $s(\omega + n\alpha)$ is not periodic for a.e. ω is dense in $L^\infty(\mathbb{T}^d)$.

Lemma. *For every $f \in C^0(\mathbb{T}^d, \mathbb{R})$ and for all $\varepsilon, \delta > 0$, there exists $g \in C^0(\mathbb{T}^d, \mathbb{R})$ such that $\|f - g\|_\infty < \varepsilon$ and $M(g) < \delta$.*

Proof. Let s be a finite range function as above with $\|s - f\|_\infty < \varepsilon/2$. Recall that $M(s) = 0$ by Kotani [4]. Choose a sequence $\{f_n\}$ of continuous functions such that $\|f_n - s\|_\infty < \varepsilon/2$ for all n and $\|f_n - s\|_1 \to 0$ as $n \to \infty$. By the technical lemma above, $\limsup M(f_n) \le M(s) = 0$. Take N so large that $M(f_N) < \delta$ and set $g = f_N$. This completes the proof. □

Generic absence of point spectrum

We now explain how to construct a dense G_δ set $F \subset C^0(\mathbb{T}^d, \mathbb{R})$ such that $f(\omega+n\alpha)$ is a Gordon potential for all $f \in F$ and generic ω. Since the shift by α is minimal, there is a sequence $q_k \to \infty$ such that αq_k is closer to 0 than αn for $1 \le n < q_k$. By passing to a subsequence, if necessary, we may assume that

$$\operatorname{dist}(\alpha q_k, 0) < 1/k.$$

For $k \ge 1$, pick a radius r_k so small that

$$B(\alpha, r_k), B(2\alpha, r_k), \ldots, B(4q_k\alpha, r_k)$$

are pairwise disjoint and so that $\bigcup_{l=0}^{3} B\big((j+lq_k)\alpha, r_k\big)$ is contained in a ball of radius $4/k$ for each $j = 1, \ldots, q_k$. Define

$$C_k := \big\{ f : f \text{ is constant on } \textstyle\bigcup_{l=0}^{3} B\big((j+lq_k)\alpha, r_k\big) \text{ for } j = 1, \ldots, q_k \big\}$$

and let F_k be the open k^{-q_k}-neighborhood of C_k in $C^0(\mathbb{T}^d, \mathbb{R})$ (i.e., $g \in F_k$ if and only if $\exists f \in C_k : \|f - g\|_\infty < k^{-q_k}$). Note that $\bigcup_{k \geq m} F_k$ is open and dense in $C^0(\mathbb{T}^d, \mathbb{R})$ for every $m \geq 1$. Hence

$$F := \bigcap_{m \geq 1} \bigcup_{k \geq m} F_k$$

is a dense G_δ set.

Take $f \in F$ and choose a subsequence $k_l \to \infty$ such that $f \in F_{k_l}$ for all l. Then $\bigcup_{l \geq m} \bigcup_{j=1}^{q_{k_l}} B\big((j+q_{k_l})\alpha, r_{k_l}\big)$ is open and dense in $\mathbb{T}^d$ for every $m \geq 1$. So

$$\Omega_f := \bigcap_{m \geq 1} \bigcup_{l \geq m} \bigcup_{j=1}^{q_{k_l}} B\big((j+q_{k_l})\alpha, r_{k_l}\big)$$

is a dense G_δ set.

The final step is now to realize that $f(\omega + n\alpha)$ is a Gordon potential for all $f \in F$ and all $\omega \in \Omega_f$. By passing to a subsequence, if necessary, we may assume that $\omega \in \bigcup_{j=1}^{q_{k_l}} B\big((j+q_{k_l})\alpha, r_{k_l}\big)$ for each l. Hence

$$\max_{1 \leq n \leq q_{k_l}} \big| f(\omega + n\alpha) - f(\omega + (n \pm q_{k_l})\alpha) \big| < 2k_l^{-q_{k_l}}.$$

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Beyond analyticity II: C^0-generic Cantor spectrum

James Tanis

This paper was written by Artur Avila, Jairo Bochi and David Damanik [1]. They consider continuous $SL(2, \mathbb{R})$-cocycles over a generalized skew shift. They prove that for a generic C^0 potential, the spectrum of the corresponding Schrödinger operator is a Cantor set.

Assumptions: X is a compact metric space, $f : X \to X$ is strictly ergodic homeomorphism (so f is minimal and uniquely ergodic) which fibers over an almost periodic dynamical system (generalized skew shift). This means there is an infinite compact abelian group $\mathbb{G}$ and continuous map $h : X \to \mathbb{G}$ such that $h(f(x)) = h(x) + \alpha$. In what follows, we will only consider the case $\mathbb{G} = \mathbb{T}$.

For example, $X = \mathbb{T}^2$, $\mathbb{G} = \mathbb{T}$, $f(x, y) = (x+\alpha, x+y)$, $h(x, y) = x$ is projection onto the first factor.

1. Some Definitions

An $SL(2,\mathbb{R})$ *cocycle* is a continuous map $(f, A)X \times SL(2,\mathbb{R}) \to X \times SL(2,\mathbb{R})$ defined by $(f, A)(x, g) = (f(x), A(x)g)$.

Cocycles $(f, A), (f, \tilde{A})$ are $PSL(2,\mathbb{R})$ *conjugate* if there is $B \in C^0(X, PSL(2,\mathbb{R}))$ such that $\tilde{A}(x) = B(f(x))A(x)B(x)^{-1}$ in $PSL(2,\mathbb{R})$. A cocyle (f, A) is *reducible* if there if it is $PSL(2,\mathbb{R})$ conjugate to a constant cocycle. For example, $A(x) = B(f(x))^{-1}B(x)$ is reducible (by the map $B : X \to PSL(2,\mathbb{R})$). (Note reducibility does not imply conjugate to constant (so $B(x) \in SL(2,\mathbb{R})$, equality in $SL(2,\mathbb{R})$). *Ruth* is the set of all A such that (f, A) is reducible up to homotopy: $A \sim \tilde{A}$ (homotopic), $\tilde{A} \sim C(PSL(2,\mathbb{R})$ conjugate).

Schrodinger cocycles take are those where A takes values in the set

$$S = \{\begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} | t \in \mathbb{R}\}.$$

Given a potential $V \in C^0(X,\mathbb{R})$ and $x \in X$, we consider the operator H_x on $\ell^2(\mathbb{Z})$ defined by

$$(H_x\psi)(n) := \psi_{n+1} + \psi_{n-1}V(f^n x)\psi_n.$$

Solutions u satisfy

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A_{E,V}(x) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

where $A_{E,V}(x) = \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}$.

Notice that Schrodinger cocycles are always in Ruth, so that $A_{E,V}(x)$ is in Ruth.

2. Outline of proof of Cantor spectrum

It is well known that because f is minimal, the spectrum of H_x is independent of x. In this case, [2] gives

$$\mathbb{R}/\Sigma = \{E \in \mathbb{R} | (f, A_{E,V}) \text{ is uniformly hyperbolic } \}.$$

The first major theorem is

Theorem 2.1. *Uniform hyperbolicity is dense in Ruth.*

So S-valued cocycles can be approximated uniformly hyperbolic ones.

Theorem 2.2. *Let P be any conjugacy-invariant property of $SL(2,\mathbb{R})$-valued cocycles over f. If $A \in C^0(X, S)$ can be approximated by $SL(2,\mathbb{R})$-valued cocycles with property P, then A can be approximated by S-valued cocycles with property P.*

Hence, S-valued cocycles can by approximate by S-valued, uniformly hyperbolic cocycles. With this, we can prove the spectrum is a Cantor set.

Corollary 2.1. *For a generic $V \in C^0(X, \mathbb{R})$, we have $\mathbb{R}/\Sigma$ is dense; that is, the associated Schrodinger operators have Cantor spectrum.*

Proof. For $E \in \mathbb{R}$, consider

$$UH_E = \{V \in C^0(X, \mathbb{R}); (f, A_{E,V}) \text{ is uniformly hyperbolic}\}.$$

Because hyperbolicity is an open condition, we know UH_E is open. Next, because S-valued cocycles can be approximated by S-valued uniformly hyperbolic ones, it follows readily that UH_E is dense in $C^0(X, \mathbb{R})$. Now choose any dense countable subset $\{E_n\}_{n \in \mathbb{Z}_+} \subset \mathbb{R}$. Then $\cap_n UH_{E_n}$ is a dense G_δ. Then for any $V \in \cap_n UH_{E_n}$, $\mathbb{R}/\Sigma$ is dense. □

3. Some indication of the proof of Theorem 2.1

From the definition, uniformly hyperbolic cocycles are diagonalizable. Hence, they are contained in Ruth. Now we want to prove that uniformly hyperbolic cocycles are actually dense in Ruth. There are several steps. We reduce the problem to proving the following theorem:

Theorem 3.1. *If $A \in$ Ruth be such that (f, A) is not uniformly hyperbolic. Assume $A_* \in SL(2, \mathbb{R})$ is nonhyperbolic (i.e. $|trA_*| \leq 2$), then (f, A) lies in the closure of the $PSL(2, \mathbb{R})$ -conjugacy class of (f, A_*).*

Proof of Theorem 2.1 *from Theorem* 3.1 : The closure of the uniformly hyperbolic cocycles contain all constant cocycles (f, A_*) with $tr(A_*) = 2$. This set also is invariant under $PSL(2, \mathbb{R})$-conjugacies. So by Theorem 3.1, (f, A) is in the closure of the uniformly hyperbolic cocycles. □

Therefore, it suffices to prove Theorem 3.1. The strategy will be to show that cocycles (f, A) that are not uniformly hyperbolic are the $PSL(2, \mathbb{R})$ closure of cocycles $(f, \tilde{A})$, where $\tilde{A}$ is a constant $SO(2, \mathbb{R})$ valued cocycle. From here it is not hard to conclude Theorem 3.1. An important first step is the following.

Theorem 3.2. *Let $A : X \to SL(2, \mathbb{R})$ be a continuous map such that (f, A) is not uniformly hyperbolic, then there is a continuous $\tilde{A} : X \to SL(2, \mathbb{R})$, arbitrarily C^0-close to A, such that $(f, \tilde{A})$ is conjugate to an $SO(2, \mathbb{R})$ cocycle.*

To prove this, the authors show that they can find an invariant section of a skew-shift $F : X \times Y \to X \times Y$. We say a map : $x \to y(x)$ is an invariant section for F if $F(x, y) = (f(x), y(f(x))$. Such a result allows them to prove the following.

Lemma 3.1. *For every $\phi \in C^0(X, \mathbb{R})$ and every $\delta > 0$, there exists $\tilde{\phi} \in C^0(X, \mathbb{R})$ such that $\|\phi - \tilde{\phi}\|_{C^0} < \delta$ and there exists $w \in C^0(X, \mathbb{R})$ and $a_0 \in \mathbb{R}$ such that $\tilde{\phi} = w \circ f - w + a_0$.*

Finally, we give some indication how Lemma 3.1 and Theorem 3.2 are used to prove Theorem 3.1. Given Theorem 3.2, we can perturb A to some A_ϵ such that A_ϵ is $PSL(2,\mathbb{R})$ conjugate to a $SO(2,\mathbb{R})$ valued cocycle $(f, R_\phi(x))$, where $\phi \in C^0(X,\mathbb{R})$. Now perturb ϕ to $\tilde{\phi} = w \circ f - w + a_0$, and it follows that

$$R_{\tilde{\phi(x)}} = R_{w\circ f(x)} R_{a_0} R^{-1}_{w(x)},$$

which of course is conjugate to the constant, $SO(2,\mathbb{R})$-valued cocycle. from here it is not hard to prove that this constant $SO(2,\mathbb{R})$-valued cocycle can be perturbed then $PSL(2,\mathbb{R})$ conjugated to (f, A_*).

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Beyond analyticity III: Examples of discontinuity of the Lyapunov exponent

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The purpose of this talk is to present results on discontinuity of the Liapunov exponent of quasi-periodic $SL(2,\mathbb{R})$-cocycles in $\mathcal{C}^l (l = 1, 2, \cdots, \infty)$ topology.

The talk firstly reviewed the results in the analytic topology. In [8] Goldstein and Schlag proved that if ω is a Diophantine irrational number and $v(x)$ is analytic, then the Lyapunov exponent $L(E)$ is Hölder continuous. Similar results were proved in [6] by Bourgain, Goldshtein and Schlag and in [8] for underlying dynamics being a shift or skew-shift of a higher dimensional torus. In 2002, Bourgain and Jitomirskaya [5] improved the result of [8] by showing that if ω is an irrational number and the potential $v(x)$ is analytic, then the Lyapunov exponent is jointly continuous on E and ω. Later, Jitomirskaya, Koslover and Schulteis [9] proved that the Lyapunov exponent is a continuous function of general (not necessarily $SL(2,\mathbb{R})$) analytic quasi-periodic cocycles.

On the contrary, Furman [7] proved that $L(A)$ is not continuous at any non-uniformly hyperbolic quasi-periodic cocycles in $\mathcal{C}^0$ topology. More recently, Bochi [2, 3] proved that any non-uniformly hyperbolicity cocycles can be approximated by continuous quasi-periodic cocycles with zero Liapunov exponent. The results show that, the continuity of the Lyapunov exponent of quasi-periodic cocycles in $\mathcal{C}^0$ topology is completely different from that in $\mathcal{C}^\omega$ topology.

Until very recently, it had however been unclear what happens in $\mathcal{C}^l$-topology. The missing link is provided by Wang and You [11], who have recently constructed counter-examples proving discontinuity of the Lyapunov exponent in all of those intermediate spaces, including C^∞. The following is their result.

Theorem 0.3. *Suppose that ω is a fixed irrational number of bounded-type. For any $0 \leq l \leq \infty$, there exist cocycles $D_l \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$ such that the Lyapunov exponent is discontinuous at (ω, D_l) in $\mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$.*

Remark 0.1. *The examples are of the form $\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \begin{pmatrix} \cos\phi(x) & -\sin\phi(x) \\ \sin\phi(x) & \cos\phi(x) \end{pmatrix}$, where $\phi(x)$ is either a 2π-periodic function corresponding to a cocycle homotopic to the identity, or the identity plus a 2π-periodic function corresponding to a cocycle non-homotopic to the identity. The counter-example in the category of Schrödinger cocycles can hopefully be constructed by Theorem 0.3.*

Remark 0.2. *Theorem 0.3 shows that the continuity of Lyapunov exponent in $\mathcal{C}^l$-topology ($l = 1, 2, \cdots, \infty$) and C^ω is different, which illustrated the optimality of the continuity results for analytic potentials. Avila and Krikorian's result* [1] *shows the continuity of Lyapunov exponent in $\mathcal{C}^l$-topology ($l = 1, 2, \cdots, \infty$) and $\mathcal{C}^0$ is also different. Theorem 0.3 also shows that the monotonicity is Avila and Krikorian's paper is necessary.*

Remark 0.3. *Klein* [10] *extended the results in* [8] *to the Gevrey case. More precisely, he proved that the Lyapunov exponent of quasi-periodic Schrödinger cocycles in the Gevrey class is continuous at the potential $v(x)$ satisfying some transversality condition. Hopefully, one can also construct examples of discontinuity in Gevrey topology.*

The examples D_l will be constructed by the limit of C^l sequence of $\{A_n(x), n = N, N+1, \cdots\}$. $\{A_n(x), n = N, N+1, \cdots\}$ possessing degeneracy and some kind of finite hyperbolic property, i.e., $\|A_n^{r_n^+}(x)\| \sim \lambda^{r_n^+}$ for most $x \in \mathbb{S}^1$ where $l \gg 1$ and $r_n^+ \to \infty$ as $n \to \infty$, which gives a lower bound estimate $(1-\varepsilon)\log\lambda$ of the Lyapunov exponent of $(\omega, D_l(x))$. A_n is constructed by modifying Young's method. Then by modifying $\{A_n(x)\}_{n=N}^\infty$, we construct another sequence $\{\tilde{A}_n(x)\}_{n=N}^\infty$ such that $\tilde{A}_n(x) \to D_l(x)$ in $\mathcal{C}^l$-topology as $n \to \infty$. Moreover, for each n, the Lyapunov exponent of $(\omega, \tilde{A}_n(x))$ is less than $(1-\delta)\log\lambda$ with $1 > \delta \gg \varepsilon > 0$ independent on λ, which implies the discontinuity of the Lyapunov exponent at $(\omega, D_l(x))$.

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