Mathematisches Forschungsinstitut Oberwolfach

Report No. 36/2012

DOI: 10.4171/OWR/2012/36

Calculus of Variations

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22 July – 28 July 2012

Abstract. Since its invention, the calculus of variations has been a central field of mathematics and physics, providing tools and techniques to study problems in geometry, physics and partial differential equations. On the one hand, steady progress is made on long-standing questions concerning minimal surfaces, curvature flows and related objects. On the other hand, new questions emerge, driven by applications to diverse areas of mathematics and science. The July 2012 Oberwolfach workshop on the Calculus of Variations witnessed the solutions of famous conjectures and the emerging of exciting new lines of research.

Mathematics Subject Classification (2000): 49-xx, 35Jxx, 53Cxx, 58Exx.

Introduction by the Organisers

The workshop has been attended by 50 participants, 14 of which have completed their PhD studies recently or were still completing them at the time of the workshop. The Calculus of Variations is at the same time a classical subject, with long-standing open questions which have generated exciting discoveries in recent decades, and a modern subject in which new types of questions arise, driven by mathematical developments and by emergent applications. This edition of the conference saw the recent solution of important famous conjectures and recorded big progresses in areas which have been stuck for a long time.

The main themes of the workshop can be roughly divided into five groups:

- The theory of area-minimizing and minimal surfaces;
- Optimal transportation and applications;
- Geometric variational problems;

- Variational problems from mathematical physics;
- Geometric flows.

The first group contains one highlight of the workshop: Simon Brendle's talk on his recent proof of the Lawson Conjecture, which characterizes the Clifford torus as the unique minimal embedded torus in the standard $\mathbb{S}^3$. The proof hinges on a virtuoso application of a technique, combining variable-doubling and a maximum principle argument, introduced by G. Huisken in the quite different setting of distance-comparison theorems for the curve-shortening flow, and further developed by B. Andrews in recent work on non-collapsing in mean curvature flow. The theory of minimal surfaces in Riemannian 3-manifolds was also the topic of Theodora Bourni's joint work with Baris Coskunuzer, where the authors study the genus of area-minimizing surfaces in compact orientable 3-manifolds M with a mean-convex smooth boundary ∂M. The main tool of this work is a suitable bridge principle for absolutely minimizing surfaces.

As is well known, minimal surfaces are in general not regular, not even if they are absolute minimizer. Geometric measure theory provides several objects to study minimizers and critical points even in the presence of singularities. In his talk Bob Hardt described a general theory of flat chains in metric spaces with coefficients in general groups. The theory developed by Hardt in collaboration with DePauw and Pfeffer unifies previous approaches by Brian White, Ambrosio and Kirchheim, which extended the classical theory of Federer and Fleming. A fundamental result in the regularity for area minimizing currents is Almgren's estimate of the size of the singular set, originally a preprint of 1728 pages. The understanding of Almgren's theory is a long standing issue and Emanuele Spadaro reported on the successful conclusion (jointly with Camillo De Lellis) of a program to reduce and simplify Almgren's proof, combining new methods in geometric measure theory with some new insightful ideas. Two other talks were strongly linked to the regularity theory of measure theoretic generalizations of minimal surfaces. The one of Nicola Fusco described a sharp quantitative version of Almgren's general isoperimetric inequality. Any closed m-dimensional surface Σ in $\mathbb{R}^N$ bounds a generalized surface (more precisely an integer rectifiable current) with volume at most $C(n)\,[\mathrm{Vol}^{\mathrm{m}}(\Sigma)]^{\frac{m+1}{m}}$. The optimal constant is achieved only by boundaries of flat $m+1$-dimensional disks. In a recent work with Verena Bögelein and Frank Duzaar, Fusco has studied, in a quantitative form, how a small deviation from the optimal constant forces the surface to be close to the boundary of a flat $m+1$-dimensional disk. A major tool in the regularity of minimal surfaces are excess decay estimates and corresponding regularity results for n-dimensional integral varifolds with generalized mean curvature in L^p. Ulrich Menne described in his talk a series of results where this question is investigated for the critical power $p = n$ and the subcritical powers $1 \leq p < n$, generalizing previous fundamental works by Allard and Brakke.

The Allen-Cahn equation has long been known to be strongly tied to the theory of minimal surfaces. Manuel Del Pino described in his talk the existence of entire solutions to the Allen-Cahn equation which are not one-dimensional and related results (joint work M. Kowalczyk, M. Musso, F. Pacard and J. Wei). The theorem of Del Pino and collaborators gives a counterexample, in high dimension, to a famous conjecture of De Giorgi.

Optimal transport is a rapidly-developing subfield of the calculus of variations that touches on a huge number of other areas, including geometry, physics, functional inequalities, applications in other areas such as economics, to name only some. The workshop featured three talks on these topics. A recent breakthrough in the regularity theory of the Monge-Ampère equation has been obtained by Guido De Philippis and Alessio Figalli. In his talk Figalli reported on their recent papers where they succeeded in proving Caffarelli's classical $W^{2,p}$ estimates without any smallness assumption on the L^∞ norm of the density, answering therefore a long-standing question in the field. In a first paper De Philippis and Figalli estimate the $L\log L$ norm of D^2u. The tools of the paper were then used in one joint work with Ovidiu Savin to achieve L^p estimates: the same result has been proved independently by Thomas Schmidt, still relying on the first work by De Philippis and Figalli.

Other work on the Monge-Ampere equation included new regularity results for a class of optimal transportation problems arising in economics (by Young-Heon Kim, reporting on a joint work with Alessio Figalli and Robert McCann), as well as very recent work by Neil Trudinger in collaboration with Wei Zhang on Hessian measures in Heisenberg groups. The work of Trudinger and Zhang also resolves an outstanding problem for Heisenberg groups of dimension larger than two.

Geometric variational problems have provided the impetus for the development of much of the deepest theory in the calculus of variations, and this theory is continually being expanded, refined, clarified, while also finding new applications. Two talks focused on recent developments in the theory of Willmore surfaces, i.e. immersed surfaces Σ in Riemannian manifolds which are critical points of the Willmore functional $\int_\Sigma |A|^2\, d\mathrm{vol}$. Mondino surveyed several results obtained in collaboration with Kuwert, Rivière and Schygulla about the existence of Willmore surfaces in Riemannian 3-manifolds and of surfaces minimizing more general curvature functionals. The work of Kuwert, Mondino and Schygulla was described more in detail in the conference of Kuwert. Kuwert also reported on a theorem of Schygulla about the existence of minimizers of the Willmore energy in the class of 2-dimensional embedded surfaces in $\mathbb{R}^3$ having a fixed isoperimetric ratio.

A classical question in differential geometry concerns which smooth functions f can arise as Gauss curvature of a conformal metric on a 2-dimensional Riemannian manifold M. This amounts to solve a partial differential equation which is the Euler-Lagrange equation of an energy functional. Michael Struwe described a theorem with Franziska Borer and Luca Galimberti about the existence of a second critical point when the functional admits a strong minimizer. In their work the authors face the important difficulty that a Palais-Smale condition does not seem to hold for the relevant energy functional.

Guido De Philippis and Aldo Pratelli reported on recent progress in the study of variational problems involving eigenvalues of the Laplacian. In his talk Pratelli described the last available results and the most important open problems concerning shape minimizers of spectral problems, i.e. about domains Ω which minimize functionals of the eigenvalues of the Laplacian relative to suitable boundary conditions. Most of the talk focused on regularity properties of the minimizers, which were proved proved to exist by Buttazzo and Dal Maso in a suitable weak sense. In particular two theorems by Bucur and by Pratelli and Mazzoleni show the boundedness of the minimizers under very general assumptions. De Philippis reported on a joint work with Lorenzo Brasco and Bernardo Ruffini about the second eigenvalue of the Stekloff Laplacian. It is well known that the second eigenvalue of the Stekloff Laplacian achieves its maximum when the domain is a ball and the authors describe with a suitable inequality how much the domain is far from a ball if the second eigenvalue is close to such maximum.

Physics is a perennial source of problems in the calculus of variations. Talks this year on problem of physical origin were notably diverse, in terms of both the mathematical content and the physical models considered.

Deep new existence results for variational problems coming from gauge theory and high-energy physics have been proved recently using arguments developed over the past 10-12 years in foundational work on geometric measure theory in general metric spaces (see the talk of Tristan Rivière, joint work with Micea Petrache). Recent progress has been made on micromagnetics and related issues such as the Aviles-Giga functional through very thorough exploitation of entropy methods, as described in the talk by Ignat Radu.

Filip Rindler described new lower semicontinuity results for integral functionals connected to nonlinear elasticity which rely on new refinements, combining rigidity arguments with iterated blow-up constructions, of classical Young measure arguments. Stefan Müller reported on a rigorous proof of conjectural scaling laws for thin elastic films developing conical singularities (joint work with Heiner Olbermann).

Curvature-driven flows were addressed in the talks of Jörg Enders, Robert Haslhofer and Didier Smets. Enders described a joint work with Reto Müller and Peter Topping about blow-up points of type I for the Ricci flow, i.e. points p where the curvature tensor blows up at a rate $C(T-t)^{-1}$. Using the curvature control stemming from Perelman's pseudolocality theorem they show that the rescaled Ricci flow converges to a nontrivial soliton, thus answering positively a conjecture of Hamilton. The novelty is that in the rescaling of Enders, Müller and Topping the focal point is p itself and not some carefully chosen nearby point, as was customary in the previous literature. Haslhofer reported on a theorem obtained with Jeff Cheeger and Aaron Naber about the stratification of the singular set in Brakke's mean curvature flow. Cheeger, Haslhofer and Naber recover previous results by White and obtain new curvature estimates near the singular set exploiting a quantitative version of the usual strata subdivison in geometric measure theory.

Didier Smets described a new measure-theoretic approach to curves flowing by binormal curvature. In a joint work with Robert Jerrard the authors prove a global existence result, weak-strong uniqueness and some stability theorems for their generalized flow. Unlike the standard parametrized approach, which always yields smooth solutions, the flow defined by Jerrard and Smets captures the singularities which are motivated by the underlying physical models.

Workshop: Calculus of Variations

Table of Contents

Abstracts

The Lawson Conjecture

Simon Brendle

In 1966, Almgren [1] showed that any immersed minimal surface in S^3 of genus 0 is totally geodesic, hence congruent to the equator. In 1970, Lawson [3] constructed many examples of minimal surfaces in S^3 of higher genus; he also constructed numerous examples of immersed minimal tori. Motivated by these results, Lawson [4] conjectured that any embedded minimal surface in S^3 of genus 1 is congruent to the Clifford torus.

In this talk, we describe our recent proof of Lawson's conjecture (cf. [2]). The key idea is to consider the quantity

$$\kappa = \sup_{x,y \in \Sigma} \sqrt{2} \, \frac{|\langle \nu(x), F(y) \rangle|}{|A(x)| \, (1 - \langle F(x), F(y) \rangle)}.$$

Here, $F : T^2 \to S^3$ is a minimal embedding of a two-dimensional torus into S^3; $\nu : T^2 \to S^3$ denotes its Gauss map; and $|A|$ denotes the norm of the second fundamental form. Using the maximum principle and the Simons identity [5], we show that $\kappa \leq 1$. This is an intricate calculation which uses special identities and inequalities which hold at the maximum point. On the other hand, we can perform an asymptotic expansion of the expression

$$\sqrt{2} \, \frac{|\langle \nu(x), F(y) \rangle|}{|A(x)| \, (1 - \langle F(x), F(y) \rangle)}$$

for $y \to x$. This shows that $\kappa > 1$ whenever the gradient of the function $|A|$ is nonzero somewhere. Since $\kappa \leq 1$, it follows that $|A|$ is constant and F is congruent to the Clifford torus.

References

[1] F.J. Almgren, Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem,* Ann. of Math. 84, 277–292 (1966)

[2] S. Brendle, *Embedded minimal tori in S^3 and the Lawson conjecture,* arxiv:1203.6597

[3] H.B. Lawson, Jr., *Complete minimal surfaces in S^3,* Ann. of Math. 92, 335–374 (1970)

[4] H.B. Lawson, Jr., *The unknottedness of minimal embeddings,* Invent. Math. 11, 183–187 (1970)

[5] J. Simons, *Minimal varities in Riemannian manifolds,* Ann. of Math. 88, 62–105 (1968)

A sharp quantitative isoperimetric inequality in higher codimension

Nicola Fusco
(joint work with Verena Bögelein, Frank Duzaar)

In a beautiful paper [1] published in 1986 Almgren proved in the context of currents the higher codimension counterpart of the classical isoperimetric inequality established by De Giorgi in [6]. In the particular case of smooth $(n-1)$-dimensional manifolds $\Gamma \subset \mathbb{R}^{n+k}$ without boundary, spanning an area minimizing smooth surface M, his inequality states that

$$\mathcal{H}^{n-1}(\Gamma) \geq \mathcal{H}^{n-1}(\partial D), \tag{1}$$

where D is an n-dimensional flat disk with the same area as M and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional surface measure. Moreover equality occurs if and only if Γ is the boundary of a flat disk.

A natural question is the stability of inequality (1). For the classical isoperimetric inequality, this stability issue was raised by Bernstein and Bonnesen in the particular case of planar convex sets [2, 4]. Later on the first results in higher dimensions were established in [8] by Fuglede in the case of convex or nearly spherical sets. His main result states that if $E \subset \mathbb{R}^n$ is a nearly spherical set in the sense that

$$\partial E = \big\{(1+u(x))x : x \in S^{n-1}\big\}$$

for some $u\colon S^{n-1} \to \mathbb{R}$ with small C^1-norm, whose volume is equal to the volume of the unit ball $B \subset \mathbb{R}^n$ and whose barycenter is at the origin, then

$$\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(\partial B) \geq c(n)\|u\|^2_{W^{1,2}(S^{n-1})}.$$

In particular, this inequality implies that the isoperimetric gap on the left-hand side controls the square of the measure of the symmetric difference $E\Delta B$. The extension of Fuglede's result to general sets of finite perimeter was first obtained in [9] (see also [10, 11] for a similar, but non optimal inequality). The result proved in [9] states that there exists a constant C depending only on the dimension n such that if E is a set of finite perimeter with $|E| = |B_r|$, then

$$D(E) \geq C(n)\,\alpha^2(E). \tag{2}$$

Here, $D(E)$ stands for the (normalized) *isoperimetric gap*

$$D(E) := \frac{\mathcal{H}^{n-1}(\partial E) - n\omega_n r^{n-1}}{r^{n-1}},$$

$\alpha(E)$ is the so-called *Fraenkel asymmetry*

$$\alpha(E) := \min_x \left\{\frac{|E\Delta B_r(x)|}{r^n}\right\}$$

and ω_n denotes the volume of the n-dimensional unit ball.

While the proof in [9] used mainly symmetrization arguments, in [7] a new proof based on arguments from the theory of optimal mass transport appeared. These arguments allowed an extension of (2) also to anisotropic perimeter functionals. Moreover, recently Cicalese and Leonardi [5] gave a shorter proof of the quantitative isoperimetric inequality via a selection principle based on a suitable penalization of the functional $E \mapsto \frac{D(E)}{\alpha^2(E)}$ and the use of the regularity theory for minimal surfaces. Their approach has also inspired the proof of our main result, stated below.

Some basic notation are needed in order to state Almgren's inequality in the proper current setting. Fix two integers $n \geq 2$, $k \geq 0$. We denote by $\mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ the space of *locally rectifiable integer multiplicity* $(n-1)$*-currents* in $\mathbb{R}^{n+k}$. As usual, given a current T, its mass is denoted by $\mathbf{M}(T)$ and ∂T stands for the boundary current. Note that if $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ is a current with no boundary, i.e. $\partial T = 0$, there exists a mass minimizing current $Q(T) \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with boundary $\partial Q(T) = T$. The mass of $Q(T)$ is denoted by $\mathbf{m}(T)$, i.e

$$\mathbf{m}(T) := \mathbf{M}(Q(T)) \equiv \inf_{\substack{P \in \mathcal{R}_n(\mathbb{R}^{n+k}),\\ \partial P = T}} \mathbf{M}(P).$$

In general mass minimizers are not unique.

Let us now denote by $[\![D_r]\!]$ the n-dimensional current associated to an n-dimensional flat disk of radius r. Almgren's isoperimetric inequality states that given any current $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$, if the disk D_r has the same mass of $Q(T)$, that is $\omega_n r^n = \mathbf{m}(T) = \mathbf{M}(Q(T))$, then

$$\mathbf{M}(T) \geq n\omega_n r^{n-1}, \tag{3}$$

with equality holding if and only if T is the boundary of a flat disk.

In order to state the quantitative counterpart of the inequality (3), proved in [3], we introduce the *isoperimetric gap* of a current $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ without boundary, which is defined as

$$\mathbf{D}(T) := \frac{\mathbf{M}(T) - n\omega_n r^{n-1}}{r^{n-1}},$$

where r is the radius of an n-dimensional flat disk D_r such that $\mathcal{H}^n(D_r) = \mathbf{m}(T)$, the minimal area spanned by T. Thus, the isoperimetric gap of T is just the difference between the two sides of inequality (3), normalized in such a way that the resulting quantity is invariant with respect to translations, rotations and dilations.

Next, we observe that, given a flat disk D_r with r as before, the mass of the minimal current spanned by $T - \partial[\![D_r]\!]$, i.e. $\mathbf{m}(T - \partial[\![D_r]\!])$, may be regarded as a 'natural' measures of how close T and $\partial[\![D_r]\!]$ are. Of course, when taking an arbitrary disk of radius r this distance can be very large. Therefore, in order to measure the deviation of the surface from round spheres of radius r we shall take the infimum over all such spheres. This quantity we call the *asymmetry index* of T, and it is a measure for the deviation of T from being a round sphere. Hence, for $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T = 0$ we define

$$\mathbf{d}(T) := \inf_{[\![D_r]\!]} \frac{\mathbf{m}(T - \partial[\![D_r]\!])}{r^n},$$

where now the infimum is taken over all flat n-dimensional disks $[\![D_r]\!]$ of radius r, i.e. about those disks with mass equal to the minimal mass $\mathbf{m}(T)$ spanned by T.

Note that also $\mathbf{d}(T)$ is invariant under translations, rotations and dilations. Now we are in the position to state our result.

Theorem. *Let $n \geq 2$ and $k \geq 0$. Then, there exists a constant $C > 0$ depending only on n and k such that for any $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T = 0$ the sharp quantitative isoperimetric inequality holds*

$$\mathbf{D}(T) \geq C\,\mathbf{d}^2(T).$$

References

[1] F. Almgren, Optimal isoperimetric inequalities. *Indiana Univ. Math. J.*, 35(3):451–547, 1986.

[2] F. Bernstein, Über die isoperimetriche Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene. *Math. Ann.*, 60:117–136, 1905.

[3] V. Bögelein, F. Duzaar and N. Fusco, A sharp quantitative isoperimetric inequality in higher codimension. *preprint*, 2012.

[4] T. Bonnesen, Über das isoperimetrische Defizit ebener Figuren. *Math. Ann.*, 91:252–268, 1924.

[5] M. Cicalese and G. P. Leonardi, A selection principle for the sharp quantitative isoperimetric inequality. *preprint*, 2011.

[6] E. De Giorgi, Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita. *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I*, 8:33–44, 1958.

[7] A. Figalli, F. Maggi and A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.* 182(1):167–211, 2010.

[8] B. Fuglede, Stability in the isoperimetric problem for convex of nearly spherical domains in $\mathbb{R}^n$. *Trans. Amer. Math. Soc.*, 314(2):619–638, 1989.

[9] N. Fusco, F. Maggi and A. Pratelli, The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.

[10] R. R. Hall, A quantitative isoperimetric inequality in n-dimensional space. *J. Reine Angew. Math.*, 428:161–176, 1992.

[11] R. R. Hall, W. K. Hayman and A. W. Weitsman, On asymmetry and capacity. *J. d'Analyse Math.*, 56:87–123, 1991.

Hölder continuity of optimal transport maps

Young-Heon Kim

(joint work with Alessio Figalli and Robert J. McCann)

Let $\Omega, \bar{\Omega}$ be two open and bounded domains in an n-dimensional Riemannian manifold M, equipped with the transportation cost $c(x, \bar{x}) = dist^2(x, \bar{x})/2$. This choice of cost function is for simplicity of presentation and in fact, the following discussion applies to a more general class of cost functions. We assume the distance function $dist$ is smooth on $\Omega \times \bar{\Omega}$, i.e. exp is invertible on Ω, $\bar{\Omega}$, or in other words, $\Omega \times \bar{\Omega}$ is outside the cut locus. Consider two L^∞ density functions ρ, $\bar{\rho}$ that are supported on Ω and $\bar{\Omega}$, respectively. We assume $\int_\Omega \rho dx = \int_{\bar{\Omega}} \bar{\rho} d\bar{x} = 1$ as well as $\lambda_0 \leq \rho, \bar{\rho} \leq \frac{1}{\lambda_0}$, for a fixed constant $\lambda_0 > 0$, on Ω, $\bar{\Omega}$, respectively. A basic question of optimal transportation is to understand the minimizer T, called optimal map, of the functional $F \mapsto\ int_M c(x, F(x))\rho(x)dx$, among all maps $F : \Omega \to \bar{\Omega}$ with the condition that F pushes forward ρ to $\bar\rho$, i.e. $F_{\#}\rho = \bar\rho$ for the measures $\rho = \rho(x)dx$, $\bar\rho = \bar\rho(\bar x)d\bar x$. By the result of Brenier and McCann, we have the existence and uniqueness of such optimal map T and this measurable map is given as

$$T(x) = \exp_x \nabla\phi(x) \quad \text{for a.e. } x$$

where ϕ is a c-convex function given by

$$\phi(\cdot) = \sup_{\bar x\in\bar\Omega} -c(\cdot,\bar x) - \bar\phi(\bar x).$$

where, $\bar\phi(\cdot) = \sup_{x\in\Omega} -c(x,\cdot) - \phi(x)$. The condition $T_{\#}\rho = \bar\rho$ implies that ϕ solves the following Monge-Ampère type equation a.e.,

$$|\det D_x D_{\bar x} c(x,T(x))|^{-1} \det(D^2_{xx}\phi(x) + D^2_{xx}c(x,T(x))) = \frac{\rho(x)}{\bar\rho(T(x))}.$$

Define the c-subdifferetial

$$\partial^c\phi(x) = \{\bar x \in \bar\Omega \mid \phi(x) + \bar\phi(\bar x) = -c(x,\bar x)\}$$

Then, the function ϕ solves a weak form of the above Monge-Ampère type equation, namely, for a constant $\lambda > 0$,

$$\lambda|A| \le |\partial^c\phi(A)| \le \frac{1}{\lambda}|A| \quad \text{for each Borel subset } A \subset \Omega, \tag{1}$$

where $|A|$ denotes the Lebesgue volume.

We address regularity of ϕ. First, it is crucial to modify c and ϕ in an appropriate coordinate system. Namely, fix $\bar x_0 \in \bar\Omega$. Use $\exp_{\bar x_0} : T_{\bar x_0} \to M$ to give parametrization $x = x(q) = \exp_{\bar x_0} q$, $q \in T_{x_0}M$. Now, let

$$\tilde c(q,\bar x) = c(x(q),\bar x) - c(x(x),\bar x_0),$$
$$\tilde\phi(q) = \phi(x(q)) + c(x(q),\bar x_0)$$

Note that $\tilde\phi$ still satisfies (1), where c is replaced with $\tilde c$ and λ is replaced with a different constant depending on λ and c. The usefulness of this modification can be seen easily for $c = |x-\bar x|^2/2$ on $\mathbb{R}^n$. Even though a c-convex function $\phi(x) = \sup_{\bar x} -|x-\bar x|^2 - \bar\phi(\bar x)$ may not be convex, but its modification $\tilde\phi(x) = \sup_{\bar x} x\cdot\bar x - \bar\phi(\bar x) + c(x,\bar x_0)$ is convex. Such convexity of $\tilde\phi$ will be very useful, but unfortunately, it does not hold for a general Riemannian manifold. This leads to the following crucial assumption, which is a geometric manifestation of the so-called Ma, Trudinger and Wang's curvature condition:

$$\forall \bar x_0 \in \bar\Omega,\ \forall a \in \mathbb{R},\ \textit{the sub-level set } \{\tilde\phi \le a\} \subset T_{\bar x_0}M \textit{ is convex.} \tag{2}$$

The Ma, Trudiner and Wang's condition (MTW condition) is known to hold on various spaces such as the round sphere and its products, quotients, and perturbation, but it does not hold whenever there is a negative curvature point. Moreover, there are positively curved domains that do not satisfy MTW conditon. (See references in [1].) This condition is *necessary* for regularity of optimal maps, since Loeper verified that if MTW condition is violated then there are smooth ρ and $\bar\rho$ on nice domains $\Omega, \bar\Omega$, but, with discontinuous optimal maps.

Under MTW condition, we may exploit the sub-level set convexity of $\tilde{\phi}$, to use some tools from the classical Monge-Ampère equation in the Euclidean case. Still, we have to overcome serious difficulties, caused by the nonlinearity of $\tilde{c}$ in the general case.

By developing some technical tools, handling the nonlinearity, we could prove the following result, which extends Caffarelli's result in the Euclidean case to domains with MTW condition (2):

Theorem 1 [Continuity and injectivity of T] (see [1]). *Assume* (1) *and* (2)*, moreover, that there exists open set* Ω' *with* $\Omega \subset \Omega'$ *such that* $\exp_x^{-1}(\bar{\Omega})$, $\exp_{\bar{x}}^{-1}(\Omega')$ *are strongly convex for each* $x \in \Omega'$, $\bar{x} \in \bar{\Omega}$*, respectively. (Here, note that there is no geometric restriction of* Ω *inside* Ω'*.) Assume that* ϕ *satisfies* (1)*. Then,* $\phi \in C^1_{loc}(\Omega)$ *and is strictly c-convex, i.e. for each* $\bar{x} \in \bar{\Omega}$*, the set* $\{x \in \Omega \mid \phi(x) + \bar{\phi}(\bar{x}) = -c(x, \bar{x})\}$ *is singleton. Thus,* $T \in C^1_{loc}(\Omega)$ *and* T *is injective on* Ω*.*

Such continuity result of T was obtained by Figalli and Loeper in the two dimensional case (in fact using a result in [1]). In general dimensions, the continuity and injectivity of T was obtained in a previous version of [1] (see [3]) under a stronger condition, called nonnegative cross curvature condition, where $\tilde{\phi}$ is convex in q, not only sub-level-set convex. We remark that after posting our revision [1], Jerome Vétois [4] communicated us his preprint showing the same result in Theorem 1, based on our method in [3].

On the round sphere, or more generally on positively curved domains with a strengthened MTW condition (called A3 in the original Ma, Trudinger and Wang's paper), Loeper obtained Hölder regularity of ϕ, and Liu showed that the sharp Hölder exponent is $\frac{1}{2n-1}$. However, for more general case, especially allowing zero sectional curvature, such Hölder regularity result was not known beyond Caffarelli's result on the Euclidean domain. In the following theorem we extend his result to general case, under the MTW condition.

Theorem 2 [Hölder continuity of T](see [1]) *Assume* (1) *and* (2)*. Further assume that* ϕ *is strictly c-convex. Then,* $\phi \in C^{1,\alpha}_{loc}(\Omega)$*, i.e.* $T \in C^{\alpha}_{loc}(\Omega)$*, for* $0 < \alpha = \alpha(\lambda, n) < 1$*.*

In fact, in [1], the Hölder continuity of T is stated under the same assumptions on Ω, $\bar{\Omega}$, as in Theorem 1, but, one can check that the result can be stated separately from such assumptions, once one has the strict c-convexity of ϕ as assumed here (and proved in Theorem 1).

Let us describe the method of the proof of Theorem 2, along the way, describing our key estimate. In the following we make the same assumptions as in Theorem 2. Consider $x \in M$ where we want to show $\phi \in C^{1,\alpha}$. Pick $\bar{x}_0 \in \partial^c \phi(x)$. Perform the modification to $\tilde{c}$ and $\tilde{\phi}$ using $T_{\bar{x}_0}M$. By strict c-convexity of ϕ, one sees that $\tilde{\phi}$ has a unique minimum at $q_0 = \exp_{\bar{x}_0}^{-1} x_0$. Define the sections $S_h = \{q \mid \tilde{\phi}(q) - \tilde{\phi}(q_0) \leq h\}$. By the MTW condition (2), each S_h is convex. The geometry of S_h as $h \to 0$, controls the regularity of $\tilde{\phi}$. Namely, the following statements are equivalent:

A. $\tilde{\phi} \in C^{1,\alpha}$ at q_0

B. $\exists\ 1/2 < k_1 < k_2 < 1$, independent of h, with $k_1 S_h \subset S_{h/2} \subset k_2 S_h$ for $h \ll 1$.

Here, kS_h is the dilation with respect to the center of maximal ellipsoid contained in S_h. This useful equivalence was observed by Caffarelli, and later was used by Guiterrez, Huang, Forzani and Maldonado among others, for regularity theory of the (classical) Monge-Ampère equation.

A key tool for Theorem 2 can now be phrased as the following Alexandrov type estimate, which we showed in [1] for $\tilde{c}$-convex functions, with nonaffine $\tilde{c}$, extending the classical estimate in the affine case $\tilde{c} = -x \cdot \bar{x}$:

Lemma 3 (see [1]) Assume S_h is sufficiently small (which can be achieved for small h due to strict c-convexity of ϕ). For $q_t \in t\partial S_h$, if $0 < 1 - t \ll 1$, then

$$|\tilde{\phi}(q_t) - h| \leq C(c, n, \lambda)(1-t)^{2^{1-n}/n} h.$$

For example, one can easily see using this estimate that for $q \in S_{h/2} \cap t\partial S_h$, it holds $t < k_2 < 1$ for some constant k_2, thus, $S_{h/2} \subset k_2 S_h$, showing one of the inclusions in the statement B. The other inclusion can also be obtained from a more involved argument.

The above lemma can be used to show so-called the engulfing property of the sections of ϕ (see [1]). Lemma 3 presented here is a combination of two Alexandrov type estimates proved in [1], one of which uses a geometric result about supporting hyperplanes to convex bodies proved in [2].

References

[1] A. Figalli, Y.-H. Kim and R.J. McCann, *Hölder continuity and injectivity of optimal maps*, Preprint, arXiv:1107.1014.

[2] A. Figalli, Y.-H. Kim and R.J. McCann, *On supporting hyperplanes to convex bodies*, Preprint, arXiv:1107.1016.

[3] A. Figalli, Y.-H. Kim and R.J. McCann, *Continuity and injectivity of optimal maps for nonnegatively cross-curved costs*, Preprint, arXiv:0911.3952.

[4] J. Vétois, *Continuity and injectivity of optimal maps*, Preprint.

Stability of optimal shapes for the Stekloff Laplacian

Guido De Philippis

(joint work with Lorenzo Brasco and Berardo Ruffini)

We are concerned with the following spectral optimization problem with volume constraint

$$\max\Big\{\sigma_2(\Omega): \quad \Omega \subset \mathbb{R}^n \quad |\Omega| = |B_1|\Big\}. \tag{1}$$

Here $\sigma_2(\Omega)$ denotes the first non trivial Stekloff eigenvalue of the Laplacian, i.e.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = \sigma_2(\Omega) u & \text{on } \partial\Omega, \end{cases}$$

with u not identically constant. Notice that through the Rayleigh quotient, σ_2 is naturally associated with the best constant of a Poincaré type trace inequality:

$$\sigma_2(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2} : \quad u \in W^{1,2}(\Omega) \quad \int_{\partial\Omega} u = 0 \right\}.$$

For more reference on the Stekloff Laplacian and on spectral optimization problems we refer to [2, Section 4] and [5, 6].

Solution to (1) are provided by balls as it has been showed by Weinstock for $n = 2$ ([7]) and by Brock in every dimension ([4]). The proof of this fact is based on the following isoperimetric property of the ball:

$$P_2(\Omega) \geq P_2(B_1) \quad \forall \Omega : \ |\Omega| = |B_1|, \tag{2}$$

where

$$P_2(\Omega) := \int_{\partial\Omega} |x|^2.$$

The above isoperimetric type inequality has been proved by Betta, Brock, Mercaldo, Posteraro in [1] through a symmetrization technique.

We enforce (1) in a quantitative way, namely we prove that there exists a positive (and computable) constant c_n such that

$$\sigma_2(\Omega) \leq \sigma_2\big(B\big)(1 - c_n \mathcal{A}^2(\Omega)\big) \quad \forall \Omega : \ |\Omega| = |B_1| \tag{3}$$

where we have introduced the asymmetry of Ω

$$\mathcal{A}(\Omega) := \min\left\{ \frac{|B \Delta \Omega|}{|B|} \quad B \text{ ball}, \ |B| = |\Omega| \right\}.$$

To prove (3) we had to show a quantitative version of (2), that reads as

$$P_2(B_1)\big(1 + \tilde{c}_n |\Omega \Delta B_1|^2\big) \leq P_2(\Omega) \quad \forall \Omega : \ |\Omega| = |B_1|. \tag{4}$$

In order to do this we give a simpler proof of (2) through calibrations which allows to take care of all the reminders in order to obtain (4).

Showing that (3) is optimal, i.e. that there exists a sequence of sets Ω_ε converging to B_1 such that

$$\sigma_2(\Omega_\varepsilon) - \sigma_2(B_1) \approx \mathcal{A}^2(\Omega_\varepsilon),$$

requires some fine construction due to the fact the $\sigma_2(B_1)$ is a multiple eigenvalue. This kind of phenomenon has been first observed by Brasco and Pratelli for the case of the Neumann Laplacian in [3].

References

[1] M. F. Betta, F. Brock, A. Mercaldo, M. R. Posteraro, A weighted isoperimetric inequality and applications to symmetrization, J. of Inequal. & Appl., **4** (1999), 215–240.

[2] L. Brasco, G. De Philippis, B. Ruffini Spectral optimization for the Stekloff Laplacian: the stability issue, accepted for publication on J. Funct. Anal. (2012), available at `http://cvgmt.sns.it/paper/1694/`

[3] L. Brasco, A. Pratelli, Sharp stability of some spectral inequalities, accepted for publication on Geom. Funct. Anal. (2011), available at `http://cvgmt.sns.it/paper/1034/`

[4] F. Brock, An isoperimetric inequality for eigenvalues of the Stekloff problem, ZAMM Z. Angew. Math. Mech., **81** (2001), 69-71.
[5] G. Buttazzo, Spectral optimization problems, Rev. Mat. Complut., **24** (2011), 277–322.
[6] A. Henrot, Extremum problems for eigenvalues of elliptic operators. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
[7] R. Weinstock, Inequalities for a classical eigenvalue problem, J. Rational Mech. Anal., **3** (1954), 745–753.

Regularity of area-minimizing integer rectifiable currents in higher codimension

Emanuele Spadaro
(joint work with Camillo De Lellis)

In this talk I discuss the main steps of a new proof of the partial regularity of area-minimizing integer rectifiable currents in the Euclidean space as proven in Almgren's big regularity paper [1].

Theorem 1. *Let T be an integer rectifiable area-minimizing m-dimensional current in $\mathbb{R}^{m+n}$. Then, there exists a (possibly empty) closed set $\mathrm{Sing}(T)$ with Hausdorff dimension at most $m-2$ such that $\mathrm{supp}(T) \setminus (\mathrm{Sing}(T) \cup \mathrm{supp}(\partial T))$ is an analytical m-dimensional submanifold of $\mathbb{R}^{m+n}$.*

The estimate on the singular set is optimal. Indeed, by a classical result due to Wirtinger and Federer, every complex variety induces by integration an area-minimizing integer rectifiable current, which can have a singular set of branch points of dimension $m-2$: for example, the origin in the complex curve $\mathcal{V} = \{(z,w) \in \mathbb{C}^2 : z^2 = w^3\}$.

The understanding of possible branch points is the main novelty with respect to the regularity of minimizing hypersurfaces. Due to the problem of varying multiplicities, Allard's regularity result does not apply and no decay of the excess holds true. Nevertheless, a direct blow-up analysis is possible, reducing the problem to the regularity of multiple valued (unavoidable because of branch points!) "harmonic" functions, in this context called *Dirichlet minimizing.*

Several problems of geometric and analytical nature need to be faced in this blow-up analysis. In our works we give new solutions for most of them, considerably simplifying the original arguments by Almgren. The following are the main steps.

1. Center manifold. Differently from the case of hypersurfaces, in higher codimension the singularities can appear as "higher order perturbations" of a smooth surface. For example, for the complex curve $\mathcal{W} = \{(z,w) \in \mathbb{C}^2 : (z-w^2)^2 = w^5\}$ any reasonable blow-up would converge to a smooth surface, e.g. the smooth complex curve $\mathcal{W}_\infty = \{z = w^2\}$ counted with multiplicity 2, and the singularity at the origin is due to the higher order branching $\pm w^{\frac{5}{2}}$. For this reason a fundamental step in the proof is the construction of an averaging manifold (in this case $\mathcal{W}_\infty$ itself) which is sufficiently regular (in general only $C^{3,\alpha}$) and represents the regular part on which the minimizing current branchs (in the case above, the branching is given by $w \mapsto \pm w^{\frac{5}{2}}$). This submanifold $\mathcal{M}$ is called *center manifold* and is constructed by a suitable approximation of the minimizing current on scales which are prescribed by a Whitney-type decomposition.

2. Normal approximation. The current T can be approximated by the graph of a multiple valued function defined on the center manifold and taking values in the normal bundle:

$$N : \mathcal{M} \to \mathcal{A}_Q(\mathcal{N}\mathcal{M}).$$

A fundamental feature of the approximation N is that all the errors made with respect to the original current (in particular the estimates on the average of the Q values of N) are superlinear in the excess. This result, as well as the construction of the center manifold, is possible thanks to a L^p-gradient estimate on the excess density of a minimizing current which is proven in [3].

3. Flat tangent cones. In the blow-up analysis around singular points, it is necessary to ensure the convergence to a multiple valued function defined on a flat space. Using Almgren's stratification theorem together with an induction argument on the dimension m of the current and the number Q of the covering, it is possible to guarantee the existence of a common subsequence of radii for which there exists convergence to a flat tangent cone and the measure of the singular set is preserved positive.

4. Tilting of tangent cones. In general, it is not possible to construct a unique center manifold approximating the average of the current's sheets up to the singular point. This is due to the Whitney-type decomposition at the base of the construction and is connected to the lack of knowledge on the uniqueness of tangent cones (one of the major problems in the field), which can in principle tilt on different scales. For this reason, we introduce a stopping condition for the center manifold to be a good approximation and construct a family of center manifolds $\{\mathcal{M}_l\}_{l\in\mathbb{N}}$ and normal approximation $\{N_l\}_{l\in\mathbb{N}}$ defined on annulii, $\{(\theta, r) : s_l \leq r \leq t_l\}$ in polar coordinates.

5. Finite order of contact. The blow-up analysis in now performed on rescaling of the normal approximations N_l. From the analytical point of view, it is essential to exclude that N_l has an infinite order of contact with $\mathcal{M}_l$, because this would imply the converge to 0 and the loss of the singular set. To this aim, for what concerns the single approximation N_l, we show a frequency function-type estimate. Setting,

$$I_{N_l}(r) := \frac{\int_{\mathcal{M}_l} \phi\left(\frac{d(p,q)}{r}\right) |\nabla N_l|^2(p)\, d\mathcal{H}^m(p)}{-\int_{\mathcal{M}_l} \phi'\left(\frac{d(p,q)}{r}\right) \frac{|N_l|^2(p)}{d(p,q)}\, d\mathcal{H}^m(p)}$$

we indeed prove that

$$I_{N_l}(r) \leq C(1 + I_{N_l}(t_l)) \quad \forall\, s_l \leq r \leq t_l, \tag{1}$$

where ϕ is a suitable Lipschitz cut-off function and $C > 0$ a constant. It is simple to see that (1) implies a bound on the order of contact of N_l with $\mathcal{M}_l$.

6. Splitting before tilting. On the other hand, it is necessary to bound the frequency $I_{N_l}(t_l)$ on the different approximations N_l. To this regard, we show

$$\limsup_{l \to +\infty} I_{N_l}(t_l) < +\infty, \tag{2}$$

thus concluding the analysis on the order of contact of the normal approximations. The proof of (2) is done showing a property of "splitting before tilting" first proved by Rivière [6] for 2-dimensional pseudeholomorphic cycles: before the tangent plane can tilt (thus inducing a change of center manifold in our context), the current has to splitt, controlling the L^2-norm of the approximation from below by the energy, i.e. (2).

7. Persistence of singularities. Thanks to the previous steps, suitable rescalings of the normal approximations converge to a nontrivial Dir-minimizing multiple valued functions. In principle, there is no reason to argue that the normal approximations N_l retain the singularities of the minimizing current. Nevertheless, a capacitary argument is used to prove that, under the hypothesis that the singular set has positive $\mathcal{H}^{m-2+\alpha}$ measure, a subset of positive measure of singular points is preserved in the limit function, thus contradiction the regularity of Dir-minimizing Q-valued functions (see our previous paper [2] for a proof).

Detailed statements and proofs of the above results will be presented in [3, 4, 5].

References

[1] F. J. Almgren, *Almgren's big regularity paper*, World Scientific Publishing Co. Inc. 2000.

[2] C. De Lellis and E. Spadaro, *Q-valued functions revisited*, Memoirs of AMS **211** (2011) vi-79.

[3] C. De Lellis and E. Spadaro, *Regularity of area-minimizing currents I: L^p-gradient estimates*, in preparation (2012).

[4] C. De Lellis and E. Spadaro, *Regularity of area-minimizing currents II: center manifold*, in preparation (2012).

[5] C. De Lellis and E. Spadaro, *Regularity of area-minimizing currents III: blow-up*, in preparation (2012).

[6] T. Rivière, *A lower-epiperimetric inequality for area minimizing surfaces*, Comm. Pure and App. Math. **57** (2004), 1673–1685.

Families of Chains and Cochains on a Metric Space

Robert Hardt

(joint work with Thierry De Pauw and Washek Pfeffer)

To handle a higher dimensional least mass Plateau problem, H. Federer and W. Fleming in 1960 [4] introduced integer-multiplicity rectifiable currents and proved important closure and compactness properties. Flat chains having coefficients in a finite group G were also studied by W. Fleming in [5]. This allowed for the modeling of nonorientable least-area surfaces including a minimal Mobius band in 3-space. These properties were optimally extended by Brian White [7] to any complete normed abelian group which contains no nonconstant Lipschitz curves.

The new proofs of basic theorems from Geometric Measure Theory involved slicing to reduce to questions about 0 dimensional chains (which are finite or countable sums of weighted point masses). Independently L. Ambrosio and B. Kirchheim [1] also generalized some basic rectifiability theorems of Federer and Fleming to currents in a general metric space. The work [2] with T. De Pauw shares features and results with all these works, includes new definitions of a flat G chains in a metric space, and a proof that such a chain is determined by its 0 dimensional slices. An m dimensional rectifiable chain is the Lipschitz push-forward of a region in $\mathbb{R}^m$ equipped with a measurable G-valued density. Flat chains are then obtained by completion using a certain Whitney flat norm [6] on polyhedral or Lipschitz chain approximations. Numerous basic results of geometric measure theory for these chains are derived including the rectifiability of finite mass flat chains provided that G contains no nonconstant Lipschitz curves. We also discuss another current work [3] with De Pauw and Pfeffer. Here we first establish a "normal chains" compactness theorem for flat G chains in a compact metric space that have bounded masses and bounded boundary masses. One readily obtains solutions to a fixed boundary Plateau problem. One may also minimize mass in a fixed homology class provided the space enjoys some fixed growth estimate on an isoperimetric ratio. For the case $G = \mathbb{R}$, we also study dual cochains, called "charges", that are topologized variationally and that give a geometric cohomology. In case of compact spaces satisfy a *linear* isoperimetric inequality, we obtain a continuous duality between the (possibly infinite dimensional) homology of normal chains and cohomology of charges. There are potentially interesting applications with the ambient metric space being a singular variety or a fractal or a metric limit of Riemannian manifolds.

References

[1] L. Ambrosio and B. Kirchheim, *Currents in metric spaces*, Acta Math. **185** (2000), 1–80.
[2] T. De Pauw and R. Hardt, *Rectifiable and flat G chains in a metric space*, American J. Math. **134**, no.1, (2012), 1–69.
[3] T. De Pauw, R. Hardt, and W. Pfeffer, *Homology of normal chains and cohomology of charges* In preparation.
[4] H. Federer and W.H. Fleming, *Normal and integral currents*, **72** no.2, (1960), 458–520.
[5] W. Fleming, *Flat chains over a finite coefficient group*, Trans. Amer. Math. Soc, **121**, (1966), 160–186.
[6] H. Whitney, *Geometric integration theory*, Princeton University Press, (1957).
[7] B. White, *Rectifiability of flat chains*, Ann. of Math. **150**, no.2, (1999), 165–184.

Static and dynamic issues for the Allen Cahn equation

Manuel del Pino
(joint work with M. Kowalczyk, M. Musso, F. Pacard and J. Wei)

We consider the problem of finding entire solutions to the Allen-Cahn equation

$$\Delta u + (1-u^2)u = 0 \quad \text{in } \mathbb{R}^N. \tag{1}$$

E. De Giorgi [3] formulated in 1978 the following celebrated conjecture:

Let u be a bounded solution of equation (1) such that $\partial_{x_N} u > 0$. Then the level sets $[u=\lambda]$ are hyperplanes, at least for dimension $N \le 8$.

Equivalently, there exist $p \in \mathbb{R}^N$, $\nu \in \mathbb{R}^N$, $|\nu| = 1$ such that u has the form

$$u(x) = w(\nu \cdot (x-p))$$

where $w(t)$ is the unique solution of

$$w'' + (1-w^2)w = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1,$$

namely $w(t) = \tanh(t/\sqrt{2})$. De Giorgi's conjecture was proven in dimensions $N = 2$ by Ghoussoub and Gui [8] and for $N = 3$ by Ambrosio and Cabré [1]. Savin [9] proved its validity for $4 \le N \le 8$ under the additional assumption $\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1$.

We discuss and infinite dimensional Lyapunov-Schmidt reduction method such that for a given minimal surface Γ, embedded that separates the space into two components, a solution whose zero level set is close to $\varepsilon^{-1}\Gamma$ exists. More precisely we want $u_\varepsilon(x) = w(z) + o(1)$ for $x = y + z\nu(\varepsilon y)$, $y \in \varepsilon^{-1}\Gamma$ where ν is a normal vector field on Γ. We discuss the result in [4] that such a solution exists if Γ is chosen to be a non-flat minimal entire graph in dimension $N = 9$, as found by Bombieri, de Giorgi and Giusti [2]. This solution turns out to be a "counterexample" to De Giorgi's statement for dimensions 9 and higher. We discuss extensions of this construction to general embedded, curvature finite total curvature minimal surfaces in $\mathbb{R}^3$, finding similar results under nondegeneracy, in particular an axially symmetric solution associated to a large dilation of the catenoid, and a solution whose zero level set is close to a large dilation of the Costa-Hoffmann-Meeks minimal surface [5]. We can also deal with the case of a zero set being exactly a helicoid when one looks for solutions invariant under screw-motion [7]. These ideas also apply to disprove a known conjecture on semilinear overdetermined problems, and to finding small-speed travelling front solutions to the parabolic Allen Cahn equation,

$$u_t = \Delta u + (1-u^2)u \quad \text{in } (-\infty,\infty) \times \mathbb{R}^N. \tag{2}$$

whose zero set lies close to a self-translating, largely dilated axially symmetric graphical solution of the mean curvature flow [6].

References

[1] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in R3 and a conjecture of De Giorgi,* Journal Amer. Math. Soc. 13 (2000), 725–739.

[2] E. Bombieri, E. De Giorgi, E. Giusti, *Minimal cones and the Bernstein problem.* Invent. Math. 7 1969 243–268.

[3] E. De Giorgi, *Convergence problems for functionals and operators,* Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna. (1979).

[4] del Pino, Manuel; Kowalczyk, Michal; Wei, Juncheng *On De Giorgi's conjecture in dimension $N \geq 9$.* Ann. of Math. (2) **174** (2011), no. 3, 1485-1569.

[5] del Pino, Manuel; Kowalczyk, Michal; Wei, Juncheng *Entire solutions of the Allen Cahn equation and complete embedded minimal surfaces of finite curvature in R^3.* Preprint arXiv:0902.2047v1.

[6] del Pino, Manuel; Kowalczyk, Michal; Wei, Juncheng *Traveling waves with multiple and non-convex fronts for a bistable semilinear parabolic equation.* Preprint.

[7] M. del Pino, M. Musso and F. Pacard. *Solutions of the Allen-Cahn equation which are invariant under screw motion.* Manuscripta Mathematica, 138, (2012), 273–286

[8] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems,* Math. Ann. 311 (1998), 481–491.

[9] O. Savin, *Regularity of flat level sets in phase transitions.* Ann. of Math. (2) 169 (2009), no. 1, 41-78.

Sobolev regularity for the Monge-Ampère equation with applications

Alessio Figalli

The Monge-Ampr̀e equation arises in connections with several problems from geometry and analysis (regularity for optimal transport maps, the Minkowski problem, the affine sphere problem, etc.) The regularity theory for this equation has been widely studied. In particular, Caffarelli developed in [3, 5, 4] a regularity theory for Alexandrov/viscosity solutions, showing that convex solutions of

$$\begin{cases} det(D^2u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

are locally $C^{1,\alpha}$ provided $0 < \lambda \leq f \leq \Lambda$ for some $\lambda, \Lambda \in \mathbb{R}$. Moreover, for any $p > 1$ there exists $\delta > 0$ such that $u \in W^{2,p}_{loc}(\Omega)$ provided $|f - 1| \leq \delta$.

Then, few years later, Wang [9] showed that for any $p > 1$ there exists a function f satisfying $0 < \lambda \leq f \leq \Lambda$ such that $u \notin W^{2,p}_{loc}(\Omega)$. This counterexample shows that the results of Caffarelli were more or less optimal. However, an important question which remained open was whether solutions of (1) with $0 < \lambda \leq f \leq \Lambda$ could be at least $W^{2,1}_{loc}$, or even $W^{2,1+\epsilon}_{loc}$ for some $\epsilon = \epsilon(n, \lambda, \Lambda) > 0$.

The reason for being interested in this $W^{2,1}$ regularity comes from the semigeostrophic equations: The semigeostrophic equations are a simple model used in meteorology to describe large scale atmospheric flows, and they can be derived from the 3-d incompressible Euler equations, with Boussinesq and hydrostatic approximations, subject to a strong Coriolis force. It has been clear for several years that $W^{2,1}$ estimates for the Monge-Ampère would have been a key tool to obtain global existence of distributional solutions to these equations.

Recently, we proved in [6] that this Sobolev regularity holds: indeed, not only solutions are $W^{2,1}_{loc}$, but actually for any $k > 0$

$$\int_{\Omega'} |D^2u| \log^k(2 + |D^2u|) < \infty \qquad \forall\, \Omega' \subset\subset \Omega.$$

The proof of this result strongly exploits the affine invariance of Monge-Ampère, and can actually be pushed forward to show that solutions are $W^{2,1+\delta}_{loc}$ for some $\delta > 0$ [7, 8].

As an application of this result, in [1] and [2] we prove existence of distributional solutions to the semigeostrophic equations on the 2-dimensional torus and on $\mathbb{R}^3$, respectively, under some suitable assumptions on the initial data.

References

[1] Ambrosio L., Colombo M., De Philippis G., Figalli A., *Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case*, Comm. Partial Differential Equations, to appear.

[2] Ambrosio L., Colombo M., De Philippis G., Figalli A., *A global existence result for the semigeostrophic equations in three dimensional convex domains*, preprint (2012).

[3] L.Caffarelli: *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity.* Ann. of Math. (2), **131** (1990), no. 1, 129–134.

[4] L.Caffarelli: *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation.* Ann. of Math. (2), **131** (1990), no. 1, 135–150.

[5] L.Caffarelli: *Some regularity properties of solutions to Monge-Ampére equations.* Comm. Pure Appl. Math., **44** (1991), 965–969.

[6] G. De Philippis, A. Figalli, *$W^{2,1}$ regularity for solutions of the Monge-Ampère equation*, Invent. Math., to appear.

[7] G. De Philippis, A. Figalli, O. Savin, *A note on interior $W^{2,1+\epsilon}$ estimates for the Monge-Ampère equation*, preprint (2012).

[8] T. Schmidt, *$W^{2,1+\epsilon}$ estimates for the Monge-Ampère equation*, preprint (2012).

[9] X.-J Wang., *Some counterexamples to the regularity of Monge-Ampère equations*, Proc. Amer. Math. Soc., **123** (1995), no. 3, 841–845.

The Willmore and other L^p-curvature functionals in Riemannian manifolds

Andrea Mondino

An important problem in Geometric Analysis concerning the *intrinsic* geometry of manifolds sounds roughly as follows: given an n-dimensional smooth manifold find the **"best metrics"** on it, where with "best metric" we mean a metric whose curvature tensors satisfy special conditions (for example some traces of the Riemann curvature tensor are null or constant or prescribed, or minimize some functional; think of the Yamabe Problem, the Uniformization Theorem, etc.).

The analogous problem concerning the *extrinsic* geometry of surfaces sounds roughly as follows: given an abstract 2-dimensional surface Σ (we will always consider Σ closed: compact and without boundary) and a Riemannian n-dimensional manifold (M^n, h), find the **"best immersions"** $f : \Sigma \hookrightarrow M^n$ of Σ into M^n. Here with "best immersion" we mean an immersion whose curvature, i.e. second fundamental form, satisfies special conditions: for example if the second fundamental form is null the immersion is totally geodesic, if the mean curvature is null the immersion is minimal, if the trace-free second fundamental form is null the immersion is totally umbilical, etc.

Before proceeding let us introduce some notation. Given an immersion $f : \Sigma \hookrightarrow (M^n, h)$ let us denote by $g = f^*h$ the pull back metric on Σ (i.e. the metric on Σ induced by the immersion f); the area form $\sqrt{\det g}$ is denoted with $d\mu_g$; the second fundamental form is denoted with A and half of its trace $H := \frac{1}{2} g^{ij} A_{ij}$ is called mean curvature (notice that we use the convention that, in the codimension one case, the mean curvature is the arithmetic mean of the principal curvatures), finally $A^\circ := A - Hg$ is called trace-free second fundamental form.

As explained above, classically the "best immersions" are the ones for which the quantities A, H, A° are null or constant (i.e. parallel) but in many cases such immersions **do not exist**: for example if Σ is a closed surface and $(M, h) = (\mathbb{R}^3, \mathrm{eucl})$ is the Euclidean three dimensional space, by maximum principle there exist no minimal, and in particular totally geodesic, immersion of Σ into $(\mathbb{R}^3, \mathrm{eucl})$; moreover if the ambient manifold is the Heisenberg group or a non constant curvature Berger sphere then there exists no totally umbilic-and a fortiori totally geodesic-immersion.

If such classical special submanifolds do not exist it is interesting to study the minimization of natural integral functionals associated to A, H, A° of the type

$$\int_\Sigma |A|^p d\mu_g, \quad \int_\Sigma |H|^p d\mu_g, \quad \int_\Sigma |A^\circ|^p d\mu_g, \quad \text{for some } p \geq 1.$$

A global minimizer, if it exists, can be seen respectively as a generalized totally geodesic, minimal, or totally umbilic immersion in a natural integral sense. For some recent results regarding the existence of minimizers of such functionals in Riemannian manifolds (in the class of integral m-varifolds and in case $p > m$) see [4] and the references therein.

An important particular case of such functionals is given by the Willmore functional

$$W(f) := \int_\Sigma H^2 d\mu_g.$$

The topic is classical and goes back to the 1920-'30 when Blaschke and Thomsen, looking for a conformally invariant theory which included the minimal surfaces, discovered the functional and proved its invariance under Moebius transformations of $\mathbb{R}^n$. The functional relative to immersions in $\mathbb{R}^n$, $\mathbb{S}^n$ and more generally in space forms has been deeply studied with remarkable results (for a matter of space we avoid the long citation of articles, let us just quote some authors: Bauer, Bérnard, Chen, Kuwert, Y. Li, P. Li, Marques, Montiel, Neves, Rivière, Ros, Schätzle, Schygulla, Simon, Weiner, Yau).

Let us recall that the Willmore functional has lots of applications: biology (Hellfrich energy), general relativity (Hawking mass), string theory (Polyakov extrinsic action), elasticity theory and optics.

While, as we remarked, there is an extensive literature for immersions into $\mathbb{R}^n$ or $\mathbb{S}^n$, up to five years ago very little was known for general ambient manifolds (apart from the case of minimal surfaces).
The first result regarding the existence of Willmore surfaces in non constantly curved spaces is in [2] where we studied the Willmore functional in a perturbative setting: endowed $\mathbb{R}^3$ with the perturbed metric $\delta_{\mu\nu} + \epsilon h_{\mu\nu}$ (where $\delta_{\mu\nu}$ is the Euclidean metric), under generic conditions on the scalar curvature of $(\mathbb{R}^3, \delta_{\mu\nu} + \epsilon h_{\mu\nu})$ and a fast decreasing assumption at infinity on the perturbation $h_{\mu\nu}$ we proved existence and multiplicity of embeddings of $\mathbb{S}^2$ which are critical points for the functional $\int H^2 d\mu_g$. The method was perturbative and the proof relied on a Lyapunov-Schmidt reduction. Using a similar technique, in [3] we studied the conformal Willmore functional $\frac{1}{2}\int |A^\circ|^2 d\mu_g$, which is conformally invariant in Riemannian manifolds, in the same perturbative setting $(\mathbb{R}^3, \delta_{\mu\nu} + \epsilon h_{\mu\nu})$ under generic conditions on the trace-free Ricci tensor $S_{\mu\nu} := Ric_{\mu\nu} - \frac{1}{3} R g_{\mu\nu}$.
The case of Willmore spheres under area constraint in a perturbative setting has been analyzed by Lamm-Metzger and Schulze.

Then using more global techniques coming on one hand from geometric measure theory (the so called "ambient approach" of Simon, involving mainly varifolds as weak objects), and functional analysis on the other hand (the so called "parametric approach" of Rivière involving Sobolev immersions) we investigated the existence of Willmore spheres in Riemannian manifolds.

Adopting the first point of view, together with E. Kuwert and J. Schygulla (see [1]) we proved the existence of a smooth immersion of $\mathbb{S}^2$ into a compact Riemannian 3-manifold, with positive sectional curvature, minimizing the L^2 norm of the second fundamental form.

Using the same approach, in [7] together with Schygulla we extended the above existence theorem to similar L^2 curvature functionals (as $\int |H|^2 + 1$ and $\int |A|^2 + 1$) on immersions of $\mathbb{S}^2$ in non compact Riemannian 3-manifolds satisfying asymptotic conditions which are natural in general relativity (as asymptotically Euclidean or Hyperbolic).

Since in higher codimension it is natural to expect existence of branched immersions minimizing the Willmore functional (this follows a fortiori from the existence of branched area minimizing surfaces), together with Rivière in [5] we introduced the notion of weak, possibly branched, immersion: a Lipschitz quasi-conformal map having at most finitely many branched points and finite total curvature. In the same paper we proved compactness results in this framework: in order to have $W^{2,2}$-weak compactness away the branched points it is enough to have uniform bounds on the areas and on the total curvatures and a uniform positive lower bound on the diameter of the images (related compactness results have been obtained independently by Chen-Li).

In the second paper [6] in collaboration with Rivière, we proved the differentiability of the Willmore functional on this space of weak immersions and we showed the regularity of the *critical points*. We then applied the theory to get existence of Willmore surfaces in homotopy groups (this result in particular shows how the Willmore functional can be useful to complete the theory of minimal surfaces, indeed our result complete the classical result of Sacks-Uhlembeck regarding area minimizing spheres in homotopy groups) and under other various conditions and constraints.

Let us stress that the ambient approach of Simon uses in a crucial way that we are dealing with a minimization problem; on the contrary, the regularity theory developed by Rivière for immersions in Euclidean space, and in [5]-[6] for immersions in manifolds uses just that the Willmore PDE is satisfied. It is therefore more appropriate for attacking min-max problems in the future.

References

[1] E. Kuwert, A. Mondino, J. Schygulla, *Existence of immersed spheres minimizing curvature functionals in compact 3-manifolds,* arXiv:1111.4893, submitted, (2011).
[2] A. Mondino, *Some results about the existence of critical points for the Willmore functional,* Math. Zeit., Vol. 266, Num. 3, (2010), 583–622.
[3] A. Mondino, *The conformal Willmore Functional: a perturbative approach,* Journal of Geometric Analysis (24 September 2011), pp. 1-48, (Online First).
[4] A. Mondino, *Existence of Integral m-Varifolds minimizing $\int |A|^p$ and $\int |H|^p$ in Riemannian Manifolds*, arXiv:1010.4514, submitted, (2010).
[5] A. Mondino, T. Rivière, *Immersed Spheres of Finite Total Curvature into Manifolds,* submitted, (2011).
[6] A. Mondino, T. Rivière, *Willmore spheres in compact Riemannian manifolds,* arXiv:1203.5501, submitted, (2012).
[7] A. Mondino, J. Schygulla *Existence of immersed spheres minimizing curvature functionals in non-compact 3-manifolds,* arXiv:1201.2165, submitted, (2012).

On the flow of a curve by its binormal curvature

Didier Smets
(joint work with Robert L. Jerrard)

The binormal curvature flow equation for a smooth family $(\gamma_t)_{t\in I}$ of curves in $\mathbb{R}^3$ is traditionally written in terms of an arc-length parametrization $\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\partial_t\gamma = \partial_s\gamma \times \partial_{ss}\gamma \tag{1}$$

where $t \in I$ is the time variable, $s \in \mathbb{R}$ is the arc-length parameter, and $\times$ denotes the vector product in $\mathbb{R}^3$. The arc-length parametrization condition

$$|\partial_s\gamma(t,s)|^2 = 1 \tag{2}$$

is indeed compatible with equation (1), since

$$\partial_t\big(|\partial_s\gamma|^2\big) = 2\partial_s\gamma \cdot \partial_{st}\gamma = 2\partial_s\gamma \cdot \big(\partial_s\gamma \times \partial_{sss}\gamma\big) = 0$$

whenever (1) is satisfied, at least for sufficiently smooth solutions. In particular, closed curves evolved by the binormal curvature flow equation (1) all have the same length. In more geometric terms, equation (1) takes its name from its equivalent form

$$\partial_t \gamma = \kappa b$$

where κ and b are the curvature function and the binormal vector field along γ_t respectively.

It seems that equation (1) first appeared in the 1906 Ph.D. thesis of L.S. Da Rios [1], whose work was promoted in a series of lectures in 1931 in Paris by its advisor T. Levi-Civita [6]. The problem considered by Da Rios and Levi-Civita goes back to the celebrated 1858 paper of H. Helmholtz [2] on the motion of a three dimensional incompressible fluid in rotation.

Formulation (1) for binormal curvature flows has at least two limitations which we wish to address. First, by essence this formulation is tailored for parametrized curves. In particular, and since it involves derivatives with respect to the parameters only, it is necessarily insensitive to self-intersections in the curves γ_t. This property is surely unsatisfactory if one believes that such flows arise as limits from three dimensional fluid dynamics. Instead, it would be desirable for a formulation to be able to detect such self-intersections, as well as possible collisions between elements of disconnected vortex filaments and changes of topology. Second, there are presumably important configurations of curves which are too singular to be considered under formulation (1). Indeed, invoking distributional derivatives on can give a meaning to equation (1) in a variety of spaces, but those spaces just fail to include the case of curves which are barely Lipschitz. On the other hand, in numerical simulations of the Euler equation or the Gross-Pitaevskii equation for quantum fluids, it is observed (see e.g. [4] and [5]) that vortex-filaments often tend to recombine by exchanging strands in cases of collisions or self-intersections. Those recombinations, when the intersections are transverse, inevitably create discontinuities of the tangent vector.

Our starting point in trying to address these two important limitations is the following identity for smooth solutions of (1), which was remarked R. L. Jerrard in [3] in a more general context.

Lemma 1 ([3]). *If γ is a smooth solution of* (1) *on $I \times T^1$, where $I \subset \mathbb{R}$ is some open interval and $T^1 = \mathbb{R}/\ell\mathbb{Z}$ for some $\ell > 0$, then for every vector field $X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ and every $t \in I$*

$$\frac{d}{dt}\int_{\gamma_t} X \cdot \tau_t \, d\mathcal{H}^1 = -\int_{\gamma_t} D\,(\mathrm{curl} X) : (\tau_t \otimes \tau_t)\, d\mathcal{H}^1, \tag{3}$$

where $\gamma_t \equiv \gamma(t, \cdot)$ and τ_t is the oriented tangent vector along γ_t.

We next propose definitions for generalized and weak binormal curvature flows. Generalized flows have common features with measured-valued solutions of Euler equations. Weak binormal curvature flows are a restricted class of generalized flows which will satisfy both the existence and the weak-strong uniqueness properties.

Definition 1. *A measurable family $(V_t)_{t\in I}$ of Radon measures on $\mathbb{R}^3\times S^2$ is called a generalized binormal curvature flow on I if for any $X\in\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$ the function $t\mapsto V_t(X\cdot\xi)$ is Lipschitz on I and satisfies*

$$\frac{d}{dt}\int X\cdot\xi\, dV_t = -\int D(\mathrm{curl}(X))\,:\,\xi\otimes\xi\, dV_t \tag{4}$$

for almost every $t\in I$.

The first moment of a Radon measure V on $\mathbb{R}^3\times S^2$ is the 1-current T_V on $\mathbb{R}^3$ defined by

$$T_V\;:\;\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)\to\mathbb{R},\;\; X\mapsto\int X(x)\cdot\xi dV(x,\xi).$$

Definition 2. *A continuous family $(T_t)_{t\in I}$ of integral 1-currents in $\mathbb{R}^3$ without boundary is called a weak binormal curvature flow on I with initial datum T_0 if there exists a generalized binormal curvature flow $(V_t)_{t\in I}$ on I such that*

1. *The first moment T_{V_t} of V_t coincides with T_t for every $t\in I$.*
2. *The mass $\|V_t\|$ satisfies $\|V_t\|\le\|T_0\|$ for every $t\in I$.*

Our two main results are :

Theorem 1 (Global existence)**.** *For any integral 1-current in $\mathbb{R}^3$ without boundary T_0, there exist a weak binormal curvature flow $(T_t)_{t\in\mathbb{R}}$ on $\mathbb{R}$ with initial datum T_0.*

and

Theorem 2 (Weak-strong uniqueness)**.** *Let $\ell>0$ and $\gamma\;:\;I\times(\mathbb{R}/\ell\mathbb{Z})\to\mathbb{R}^3$ denote a smooth classical solution of the binormal curvature flow equation (1), and assume that for any $t\in I$, the curve $\gamma_t:=\gamma(t,\cdot)$ is without self-intersection. Then the weak binormal curvature flow $(T_{\gamma,t})_{t\in I}$ induced by γ is the unique weak binormal curvature flow on I with initial datum $T_{\gamma,0}$.*

As a matter of fact, we deduce Theorem 2 from a stronger quantitative estimate. To that purpose, consider a compact subset $J\subset I$ containing 0 and set

$$r\equiv r(\gamma,J):=\frac{1}{2}\min_{t\in J}\min\big(\|\partial_{ss}\gamma(t,\cdot)\|_\infty^{-1},r_s(t)\big)>0,$$

where the security radius $r_s(t)$ is defined as the largest positive real number with the property that every point x satisfying $d(x,\gamma_t)<r_s(t)$ has a unique closest point $P_t(x)$ on γ_t. Define then the vector field $X_{\gamma,r}$ on $\mathbb{R}^3\times J$ by

$$X_{\gamma,r}(x,t)=f(d^2(x,\gamma_t))\tau_t(P_t(x)) \tag{5}$$

where τ_t is the oriented unit tangent vector along γ_t and

$$f(d^2)=\begin{cases}\big(1-\big(\frac{d}{r}\big)^2\big)^3, & \text{for } 0\le d^2\le r^2,\\ 0, & \text{for } d^2\ge r^2.\end{cases}$$

Theorem 3 (Control of instability)**.** *Let $T_0 \in \mathcal{T}$ and let $(T_t)_{t\in J}$ be a weak binormal curvature flow on J with initial datum T_0. Define the non-negative functions F an G on J by*

$$G(t) := \|T_0\| - \int X_{\gamma,r}(x,t)\cdot\xi \, dV_t(x,\xi) \geq F(t) := \int \big(1 - X_{\gamma,r}(x,t)\cdot\xi\big)\, dV_t(x,\xi) \geq 0.$$

Then G is Lipschitzian on J and

$$\left|\frac{d}{dt}G(t)\right| \leq KF(t) \leq KG(t)$$

almost everywhere on J, where $K \equiv K\big(r(\gamma,J),\ \|\partial_{sss}\gamma\|_{L^\infty(J\times T^1)}\big)$.

References

[1] L. S. Da Rios, *Sul moto di un filetto vorticoso di forma qualunque*, Rend. del Circolo Mat. di Palermo **22** (1906), 117–135 and **29** (1910), 354–368.

[2] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen*, J. Reine Angew. Math. **55** (1858), 25–55.

[3] R. L. Jerrard, *Vortex filament dynamics for Gross-Pitaevsky type equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), no. 4, 733–768.

[4] S. Kida and M. Takaoka, *Vortex reconnection*, Annual review of fluid mechanics, Vol. 26, 169–189, Annual Reviews, Palo Alto, CA, 1994.

[5] J. Koplik and H. Levine, *Vortex reconnection in superfluid helium*, Phys. Rev. Lett., **71** (1993), 1375-1378.

[6] T. Levi-Civita, *Attrazione newtoniana dei tubi sottili e vortici filiformi*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **1** (1932), no. 1-2, 1–33 and no. 3, 229–250.

Energy scaling for conical singularities in thin elastic sheets

Stefan Müller

(joint work with Heiner Olbermann)

1. Main result

Let $B_1 \subset \mathbb{R}^2$ denote the open unit disc and let $\gamma : \partial B_1 \to \mathbb{R}^3$ be a C^3 curve with

$$|\gamma| = |\gamma'| = 1 \quad \text{on } \partial B_1\,, \tag{1}$$

let

$$V_\gamma = \big\{y \in W^{2,2}(B_1,\mathbb{R}^3) : y|_{\partial B_1} = \gamma,\ y(0) = 0\big\}$$

and consider the functional

$$E_h(y) = \int_{B_1} \big(|\nabla y^T \nabla y - Id|^2 + h^2|\nabla^2 y|^2\big)\, \mathrm{d}x\,.$$

Theorem 1. *Suppose that the curve γ does not lie in a plane. Then there exists constants C_2 and C_3 which only depend on γ such that for sufficiently small $h > 0$*

$$C_1 \ln \frac{1}{h} - 4C_1 \ln\left(\ln \frac{1}{h}\right) - C_2 \leq \frac{1}{h^2} \min_{y \in V_\gamma} E_h(y) \leq C_1 \ln \frac{1}{h} + C_3,$$

where

$$C_1 = C_1(\gamma) = \frac{1}{\ln 2} \int_{B_1 \setminus B_{1/2}} |\nabla^2 \tilde{y}|^2 \mathrm{d}x\,.$$

This improves work of Brandman, Kohn and Nguyen [1] who showed that $\liminf_{h\to 0} \frac{1}{h^2 \ln(1/h)} \min E_h \geq C_1/2$ and $\limsup_{h\to 0} \frac{1}{h^2 \ln(1/h)} \min E_h \leq C_1$ and very recently in parallel to our work that $\lim_{h\to 0} \frac{1}{h^2 \ln(1/h)} \min E_h = C_1$.

2. Motivation

The functional E_h is a prototypical energy for the deformation of a thin elastic sheet of thickness h. The map y is the deformation of the midplane of the sheet and the first term reflects the stretching energy (per unit height) while the second term reflects the bending energy. Condition (1) implies that the cone over γ defined by the one-homogeneous extension

$$\tilde{y}(x) := |x|\gamma(\frac{x}{|x|})$$

for $x \in B_1 \setminus \{0\}$ and $y(0) = 0$ is an isometric immersion of $B_1 \setminus \{0\}$, i.e. $(\nabla\tilde{y})^T \nabla\tilde{y} = Id$. Indeed, (1) is also necessary for $\tilde{y}$ to be an isometric immersion. In the physics literature $\tilde{y}$ is referred to as a developable cone, or d-cone for short.

For small h we expect a minimizer y of E_h to be close to $\tilde{y}$ (except on scale h) which leads to logarithmic divergence of E_h. Theorem 1 makes this precise on the level of the energy. There are also estimates for $y - \tilde{y}$, see Lemma 2 below.

Conical singularities and more generally energy concentration in elastic sheets have recently been extensively discussed in the physics literature. Conical singularities arise e.g. in the following experiment. Put an elastic sheet of radius 1 concentrically on top of a hollow cylinder of radius $R < 1$ and push the sheet down at its centre. It has been observed that the sheet assumes (to a high degree of approximation) the shape of a developable cone and detaches from the boundary of the cylinder on a segment with a well defined angle (of approximately 140 degrees). There are convincing explanations of these effects in the physics literature [2, 3] but there is also some discussion about finer details, see e.g. the survey [5].

To make progress on rigorous mathematical results on the effect of the singular perturbation by the bending energy Brandman, Kohn and Nguyen [1] suggested to replace the specific problem with the special obstacle boundary condition by the general Dirichlet boundary condition $y_{|\partial B_1} = \gamma$. They also introduced condition (1) on the boundary curve. This condition is natural (since it is necessary and sufficient for $\tilde{y}$ to be an isometric immersion), but it also plays a crucial role in the proof of the lower bound, both in their and in our argument (see (4)). Indeed one would expect that the minimal energy is even large if $|\gamma| \leq 1 - \delta$, with $\delta > 0$ but this is not known rigorously. See [4] for results and conjectures about compressed elastic sheets.

3. Sketch of proof

For the upper bound it suffices to consider $y_h(x) = h\phi(\frac{|x|}{h})\gamma(\frac{x}{|x|})$ where $\phi \in C^2(\mathbb{R})$ and $\phi(t) = t$ for $t > 1$ and $\phi(t) = 0$ for $t \le 1/2$.

The proof of the lower bound is based on the following three lemmata. We fix γ and may and will assume that

$$E_h(y) \le C_1 \frac{1}{h^2} \ln \frac{1}{h}\,. \tag{2}$$

Lemma 1. $\sup_{B_h} |y| \le Ch \ln \frac{1}{h}$.

Lemma 2. *Assume that* $h \ln \frac{1}{h} \le r_0 \le 1$ *and set* $A_{r_0} = B_{r_0} \setminus \overline{B_{r_0/2}}$. *Then*

$$\int_{A_{r_0}} |y - \tilde{y}|^2 \,\mathrm{d}x \le C r_0^3 h \ln \frac{1}{h}\,.$$

Lemma 3. *Assume that* $h \ln \frac{1}{h} \le r_0 \le 1$. *Then*

$$\left| \int_{A_{r_0}} \nabla^2 (y - \tilde{y}) : \nabla^2 \tilde{y} \,\mathrm{d}x \right| \le C \left(\frac{r_0}{h} \right)^{1/8} \left(\ln \frac{1}{h} \right)^{1/2}\,.$$

Lemma 1 follows from the scale invariant estimate

$$\sup_{x \in B_h} \left| y(x) - y(0) - x \cdot \frac{1}{|B_h|} \int_{B_h} \nabla y \right| \le C \|\nabla^2 y\|_{L^2(B_h)}\,,$$

the estimate $\left| \int_{B_1} \nabla y \right| = \left| \int_{\partial B_1} y \otimes \nu \right| \le 2\pi$ and the BMO-type estimate

$$\left| \frac{1}{|B_h|} \int_{B_h} \nabla y - \frac{1}{|B_1|} \int_{B_1} \nabla y \right| \le C \left(\ln \frac{1}{h} \right)^{1/2} \|\nabla^2 y\|_{L^2(B)}$$

(to get the optimal exponent $1/2$ in the logarithm one can use e.g. the Trudinger-Moser inequality).

To prove Lemma 2 set $e = y - \tilde{y}$ and denote by $e' = \partial_r e$ the derivative in the radial direction. On a.e. segment $r \mapsto (r\cos\theta, r\sin\theta)$ we have

$$|e(r,\theta) - e(h,\theta)|^2 \le r \int_h^r |e'|^2(\rho,\theta) \,\mathrm{d}\rho \tag{3}$$

and using that $y(1,\theta) \cdot \gamma(\theta) = |\gamma(\theta)|^2 = 1$ we get

$$\int_h^1 |e'(\rho,\theta)|^2 \,\mathrm{d}\rho = \int_h^1 \left(|\partial_r y|^2 - 1 \right) \,\mathrm{d}\rho - 2h + 2y(h,\theta) \cdot \gamma(\theta)\,. \tag{4}$$

To finish the proof we integrate (3) with respect to $r \,\mathrm{d}r \,\mathrm{d}\theta$, use the pointwise estimate $|(\nabla y)^T \nabla y - Id|^2 \ge (|\partial_r y|^2 - 1)^2$, the Cauchy-Schwarz inequality with respect to $\mathrm{d}\rho \,\mathrm{d}\theta$ and Lemma 1.

To prove Lemma 3 we use integration by parts, Lemma 2, the simple estimate $\|\nabla^2 e\|^2_{L^2(A_r)} \leq \ln \frac{1}{h}$ and standard interpolation estimates for $\|\nabla e\|_{L^2(A_r)}$ and $\|\nabla e\|_{L^2(\partial A_r)}$ as well as the homogeneity properties of $\nabla^2 \tilde{y}$ and $\nabla^3 \tilde{y}$.

Proof of Theorem 1. Let $M \in \mathbb{N}$ with $M \approx \log_2 \frac{1}{h} - 4 \log_2 \ln \frac{1}{h}$. Then by Lemma 3

$$\begin{aligned}\int_{B_1 \setminus B_{2^{-M}}} |\nabla y|^2 \,\mathrm{d}x &\geq \int_{B_1 \setminus B_{2^{-M}}} |\nabla \tilde{y}|^2 \,\mathrm{d}x - 2 \int_{B_1 \setminus B_{2^{-M}}} \nabla^2 (y - \tilde{y}) : \nabla^2 \tilde{y} \,\mathrm{d}x \\ &\geq C_1 M \ln 2 - 2C \sum_{k=0}^{M-1} (2^k h)^{1/8} \left(\ln \frac{1}{h} \right)^{1/2} \\ &\geq C_1 M \ln 2 - 2C (2^M h)^{1/8} \left(\ln \frac{1}{h} \right)^{1/2}\end{aligned}$$

and the assertion follows from the choice of M.

Remark 1. *A modification of the final argument which uses Young's inequality shows that the prefactor* 4 *in front of the* $\ln \ln$ *term can be reduced to* 1*.*

References

[1] Jeremy Brandman, Robert V. Kohn, and Hoai-Minh Nguyen, *Energy scaling laws for conically constrained thin elastic sheets*, Preprint, 2012.

[2] E. Cerda and L. Mahadevan, *Conical surfaces and crescent singularities in crumpled sheets*, Phys. Rev. Lett. **80** (1998), 2358–2361.

[3] Enrique Cerda, Sahraoui Chaieb, Francisco Melo, and L. Mahadevan, *Conical dislocations in crumpling*, Nature **401** (1999), 46–49.

[4] Sergio Conti and Francesco Maggi, *Confining thin elastic sheets and folding paper*, Arch. Ration. Mech. Anal. **187** (2008), 1–48.

[5] T. A. Witten, *Stress focusing in elastic sheets*, Rev. Mod. Phys. **79** (2007), 643–675.

Area minimizing surfaces in mean convex 3-manifolds

Theodora Bourni

(joint work with Baris Coskunuzer)

In this talk I present work from a recent paper, which is joint with Baris Coskunuzer. We give several results on area minimizing surfaces in strictly mean convex 3-manifolds. More precisely, we study the genus of absolutely area minimizing surfaces in a compact, orientable, strictly mean convex 3-manifold M bounded by a simple closed curve that lies in ∂M. Our main result is that for any $g \geq 0$, the space of simple closed curves in ∂M where all the absolutely area minimizing surfaces they bound in M has genus $\geq g$ is open and dense in the space $\mathcal{A}$ of nullhomologous simple closed curves in ∂M. For showing this we prove a bridge principle for absolutely area minimizing surfaces. As an application of these results, we further prove some important properties about minimal surfaces bounded by such curves, with the most important being that the simple closed curves in ∂M bounding more than one minimal surface in M is an open and dense subset of $\mathcal{A}$. This allows us to show that for any strictly mean convex 3-manifold M, there exists a simple closed curve Γ in ∂M which bounds a stable minimal surface which is not embedded and this answers a question of Meeks. In what follows we give a more precise description of the presented results.

The Plateau problem concerns the existence of an area minimizing disk with boundary a given curve in an ambient manifold M. This problem was solved in the case when the ambient manifold is $\mathbf{R}^3$ by Douglas [Do], and Rado [Ra] and was later generalized by Morrey [Mo] for Riemannian manifolds. In the 1980s, Meeks and Yau showed that if M is a mean convex 3-manifold, and Γ is a simple closed curve in ∂M, then any area minimizing disk with boundary Γ is embedded [MY1]. Later, White gave a generalization of this result to any genus [Wh2].

In the early 1960s, the same question was studied for absolutely area minimizing surfaces, i.e. for surfaces that minimize area among all orientable surfaces with a given boundary (without restriction on the genus). Using techniques from geometric measure theory, Federer and Fleming [FeF] were able to solve this problem by proving the existence of an absolutely area minimizing integral current (see also [DG, Giu] for the existence of a minimizing Caccioppoli set). In [ASSi] Almgren, Schoen and Simon showed that this current is a smooth embedded surface away from its boundary. Boundary regularity, when the boundary is sufficiently smooth, was proven later by Hardt [H] and Hardt and Simon [HSi].

In the presented paper, we study the genus of absolutely area minimizing surfaces in a strictly mean convex 3-manifold, with boundary a simple closed curve lying in the boundary of the ambient manifold. From now on we let M be the ambient manifold, which is always assumed to be a compact, orientable, strictly mean convex 3-manifold. We let $\mathcal{A}$ be the space of all nullhomologous simple closed curves in ∂M equipped with the $\mathcal{C}^0$-topology. By the results mentioned in the previous paragraph, any $\Gamma \in \mathcal{A}$ bounds an embedded absolutely area minimizing surface Σ in M with $\partial\Sigma = \Gamma$. We define $\mathcal{A}_g$ to be the set of all the curves in $\mathcal{A}$ such that any embedded absolutely area minimizing surface in M that they bound has genus $\geq g$. The main theorem of the paper states that for any $g \geq 0$, $\mathcal{A}_g$ is an open and dense subset of $\mathcal{A}$.

The proof of the openness is an application of a compactness theorem of White [Wh2]. The hard part of the main theorem is to show density. The idea of the proof is as follows. We construct an operation on a given simple closed curve Γ in ∂M, which we call *horn surgery*, that adds a *thin handle* to an absolutely area minimizing surface Σ in M, which is bounded by Γ. In other words, by modifying the boundary curve, this operation increases the genus of an absolutely area minimizing surface. Moreover, the construction is such that this new curve can be made as close as we want to the original curve. Hence by using the horn surgery operation, we show that $\mathcal{A}_g$ is dense in $\mathcal{A}$. We point out that in doing this, we prove *a bridge principle for absolutely area minimizing surfaces.*

The previously mentioned bridge principle and the horn surgery is also used to derive some interesting results on the space $\mathcal{B}_g = \mathcal{A}_g \setminus \mathcal{A}_{g+1}$; the space of all the curves in $\mathcal{A}$, such that the minimum genus of the embedded absolutely area minimizing surfaces in M that they bound is exactly g. In particular we prove that for any $g \geq 0$, $\mathcal{B}_g$ is not empty and is nowhere dense in $\mathcal{A}$. Furthermore we show that there exist simple closed curves in ∂M which bound two absolutely area minimizing surfaces of different genus.

Finally and as an application of our results, we further derive several interesting theorems about minimal surfaces in strictly mean convex 3-manifolds. We show that curves that bound more that one minimal surface are generic for a strictly mean convex 3-manifold M. We remark that in [Co1], and [CE], it is proven that a generic nullhomotopic simple closed curve in ∂M bounds a unique area minimizing disk, and similarly, a generic nullhomologous simple closed curve in ∂M bounds a unique absolutely area minimizing surface in M. However, here we show that when we relax the condition of being area minimizing to just minimal, the situation is completely opposite.

Furthermore, we generalize Peter Hall's results [Ha], which answers Meeks' questions, to any strictly mean convex 3-manifold M. As previously mentioned, in [MY1], Meeks and Yau proved that any area minimizing disk in a mean convex 3-manifold bounded by a simple closed curve in ∂M must be embedded. After establishing this result, Meeks posed the question of whether or not the same holds for stable minimal surfaces. Then, Hall constructed an example of a simple closed curve Γ in $S^2 = \partial \mathbf{B}^3$, where $\mathbf{B}^3$ is the unit 3-ball in $\mathbf{R}^3$, such that Γ bounds a stable minimal disk M in B^3 which is not embedded. This shows that if we relax the area minimizing condition to just being minimal again, the embeddedness result of Meeks and Yau is no longer valid. We generalize Hall's example by showing that for any strictly mean convex 3-manifold, there exist simple closed curves in the boundary that bound non embedded stable minimal surfaces.

References

[All1] W. K. Allard *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.

[All2] W. K. Allard *On the first variation of a varifold: boundary behavior*, Ann. of Math. (2) **101** (1975), 418–446.

[AS] F.J. Almgren, L. Simon, *Existence of embedded solutions of Plateau's problem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **6** (1979) no. 3, 447–495.

[AT] F.J. Almgren, W.P. Thurston, *Examples of unknotted curves which bound only surfaces of high genus within their convex hulls*, Ann. of Math. (2) **105** (1977), no. 3, 527–538.

[ASSi] F.J. Almgren, R. Schoen, L. Simon *Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals*, Acta Math. **139** (1977) 217–265.

[Bo] T. Bourni, *$C^{1,\alpha}$ Theory for the Prescribed Mean Curvature Equation with Dirichlet Data*, J. Geom. Anal. **21** (2011) 982–1035.

[Co1] B. Coskunuzer, *Generic uniqueness of area minimizing disks for extreme curves*, Amer. J. Math. **132** (2010) 1091–1104.

[CE] B. Coskunuzer, T. Etgu, *Generic uniqueness of area minimizing disks for extreme curves II*, in preparation.

[DG] E. De Giorgi, *Frontiere orientate di misura minima*, Sem. Mat. Scuola Norm. Sup. Pisa, 1960-61.

[Do] J. Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931) 263–321.

[EWW] T. Ekholm, B. White, D. Wienholtz, *Embeddedness of minimal surfaces with total boundary curvature at most 4π*, Ann. of Math. (2) **155** (2002), no. 1, 209–234.

[Fe] H. Federer, *Geometric measure theory*, Springer-Verlag, New York 1969.

[FeF] H. Federer, W. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960) 458-520.
[Giu] E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhauser, Basel, 1984.
[Ha] P. Hall, *Two topological examples in minimal surface theory*, J. Differential Geom. **19** (1984) 475–481.
[H] R. Hardt *On boundary regularity for integral currents or flat chains modulo two minimizing the integral of an elliptic integrand*, Comm. Partial Differential Equations **2** (1977), no. 12, 1163–1232.
[HSi] R. Hardt, L. Simon, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Ann. of Math. (2) **110** (1979), no. 3, 439–486.
[HS] J. Hass, P. Scott, *The existence of least area surfaces in* 3*-manifolds*, Trans. Amer. Math. Soc. **310** (1988) no. 1, 87–114.
[Jo] J. Jost, *Conformal mappings and the Plateau-Douglas problem in Riemannian manifolds*, J. Reine Angew. Math. 359 (1985) 37–54.
[Mo] C.B. Morrey, *The problem of Plateau on a Riemannian manifold*, Ann. of Math. (2) **49** (1948) 807–851.
[Me] W. Meeks, *Uniqueness theorems for minimal surfaces*, Illinois J. Math. **25** (1981) 318–336.
[MY1] W. Meeks and S.T. Yau, *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. (2) **112** (1980) 441–484.
[MY2] W. Meeks and S.T. Yau, *The classical Plateau problem and the topology of three manifolds*, Topology **21** (1982) 409–442.
[MY3] W. Meeks and S.T. Yau, *The existence of embedded minimal surfaces and the problem of uniqueness*, Math. Z. **179** (1982) no. 2, 151–168.
[Ra] T. Rado, *On Plateau's problem*, Ann. of Math. (2) **31** (1930) no. 3, 457–469.
[Sim] L. Simon *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Canberra, 1983.
[Wh1] B. White, *The space of m-dimensional surfaces that are stationary for a parametric elliptic functional* Indiana Univ. Math. J. **36** (1987), no. 3, 567–602.
[Wh2] B. White, *Existence of smooth embedded surfaces of prescribed genus that minimize parametric even elliptic functionals on* 3*-manifolds*, J. Diff. Geom. **33** (1991) no. 2, 413–443.
[Wh3] B. White, *On the topological type of minimal submanifolds*, Topology **31** (1992) 445–448.
[Wh4] B. White, *The bridge principle for stable minimal surfaces*, Calc. Var. PDE **2** (1994) no. 4, 405–425.

On the existence and regularity of minimizers for spectral problems

Aldo Pratelli
(joint work with Dario Mazzoleni)

This talk is devoted to describe the last available results and the most important open problems concerning shape minimizers of spectral problems.
We are concerned in the following question: given natural numbers k and N, and given a function $F : \mathbb{R}^k \to \mathbb{R}$, lower semi-continuous and non-decreasing in each variable, we aim to minimize the functional

$$\mathcal{F}(\Omega) := F\big(\lambda_1(\Omega),\, \lambda_2(\Omega),\, \dots\, ,\, \lambda_k(\Omega)\big)$$

among open sets $\Omega \subseteq \mathbb{R}^N$ of fixed (say, unite) volume, being $\{\lambda_i(\Omega)\}_{i\in\mathbb{N}}$ the spectrum of the Dirichlet eigenvalues of the standard Laplacian.
The main difficulties of the problem are two: first of all, as for all the shape optimization problems, one has to face the lack of a good topology for sets; second,

having the whole $\mathbb{R}^N$ as ambient space gives as usual concentration-compactness problems.
The first difficulty has been settled by Buttazzo and Dal Maso in 1993, using the $\gamma-$convergence for sets. Their result, which is very well-known, can be written as follows.

Theorem 1 (Buttazzo–Dal Maso, [4]). *Let $D \subseteq \mathbb{R}^N$ be a bounded open set. Then, there exists a quasi-open minimizer for the functional $\mathcal{F}$ among all the sets of unit volume contained in D.*

Of course, one would like also to consider the problem in the whole $\mathbb{R}^N$. For this general case, only few results were available up to one year ago. Namely, the existence is classically known for a minimizer of λ_1 and λ_2, and in particular the first one is any unit ball, while the second one is any disjoint union of two balls of volume 1/2. More recently, it has been shown the existence for a minimizer of λ_3, even if it is not known which set is it, see [2]. Finally, the question of the existence has been completely solved last year, thanks to two contemporary papers.

Theorem 2 (Mazzoleni–Pratelli, [6]). *There exists a bounded minimizer for the functional $\mathcal{F}$ among all the quasi-open subsets of $\mathbb{R}^N$ of unit volume.*

Theorem 3 (Bucur, [1]). *There exists a minimizer for λ_k among all the quasi-open subsets of $\mathbb{R}^N$ of unit volume. Moreover, any minimizer is bounded and has finite perimeter.*

Notice that the two results give different informations: indeed, the second one is only concerned with the particular case of the k-th eigenvalue, instead of the general functional $\mathcal{F}$, but on the other hand it gives the important additional information of the finiteness of the perimeter, which is particularly interesting since one is working within the very general setting of the quasi-open sets. A last comment has to be done concerning the boundedness of the minimizers: the first result only stated the boundedness for some minimizer, but in fact it can be proved that *any* minimizer has to be bounded, see [5].

We conclude by briefly commenting the question of the regularity of the minimizers. In fact, all the above existence results are valid in the very general framework of the quasi-open sets, and this is needed in order to use the γ-convergence for sets. But on the other hand, this setting appears quite unnatural and it is not very satisfactory, and actually one would imagine that the minimizers have to be smooth. It would be extremely important to show that the minimizers are at least open: indeed, this would be a first huge step towards the regularity (just keep in mind that the eigenfunctions for open sets are analytic, while eigenfunctions for quasi-open sets need not to be continuous). Up to now, it is known that minimizers of the first eigenvalue in a non-necessarily bounded domain $D \subseteq \mathbb{R}^N$ are open (of course, if the domain contains a unit ball, then the ball is the minimizer!), and this is a simple result; moreover, also minimizers of Lipschitz functionals involving the first two eigenvalues, corresponding to our general case for $k = 2$ and F Lipschitz, are open, as it is shown in [3]. The general case seems much more demanding, and it is presently studied by Bucur, Mazzoleni, Pratelli and Velichkov, up to now only with some partial results.

References

[1] D. Bucur, *Minimiziation of the k-th eigenvalue of the Dirichlet Laplacian*, preprint (2011).
[2] D. Bucur & A. Henrot, *Minimization of the third eigenvalue of the Dirichlet Laplacian*, Proc. Roy. Soc. London, **456** (2000), 985–996.
[3] D. Bucur & B. Velichkov, in preparation (2012).
[4] G. Buttazzo & G. Dal Maso, *An existence result for a class of shape optimization problems*, Arch. Rational Mech. Anal. **122** (1993), 183–195.
[5] D. Mazzoleni, in preparation (2012).
[6] D. Mazzoleni & A. Pratelli, *Existence of minimizers for spectral problems*, preprint (2011).

Hessian Measures on the Heisenberg group

Neil S. Trudinger
(joint work with Wei Zhang)

We report on a recent discovery [6] with my colleague Wei Zhang of new measures in the Heisenberg groups $\mathbb{H}^m$, $m > 1$, which extend the Monge-Ampère measure of Aleksandrov and the Hessian measures of Trudinger and Wang [4] in Euclidean space $\mathbb{R}^n$.

We begin with a general system of smooth vector fields $\{X_1, ...X_m\}$ in $\mathbb{R}^n$, $m \leq n$, that is first order differential operators of the form, $X_i = b^{ij} D_j$, with smooth coefficients, $b^{ij}, i = 1, ...m, j = 1, ...n$ defined on a domain Ω in $\mathbb{R}^n$. The Hessian and symmetrised Hessian of a function $u \in C^2(\Omega)$ are defined respectively by $X^2u = [X_i X_j u]$ and $X^2{}_s u = [\frac{1}{2}(X_i X_j u + X_j X_i u)]$ and the function u is called k-convex, $k = 1, ...m$, in Ω with respect to $\{X_1, ...X_m\}$ if

$$F_l[u] = S_l(X^2{}_s u) \geq 0, \quad l = 1, ...k,$$

where $S_l(r)$ denotes the sum of the $l \times l$ principal minors of the matrix $r = [r_{ij}]$. The fully nonlinear operator F_k will be degenerate elliptic with respect to a k-convex function u, that is the matrix

$$[F^{ij}] = [\frac{\partial S_k}{\partial r_{ij}}(X_s^2 u)] \geq 0,$$

but the converse is only true when $k = m$, in which case we also call u convex with respect to $\{X_1, ...X_m\}$. For our purposes here, a function $u \in C(\Omega)$ is called k-convex, if there exists a sequence $\{u_m\} \subset C^2(\Omega)$ such that in any subdomain $\Omega' \Subset \Omega$, u_m is k-convex for sufficiently large m and converges uniformly to u. In the Euclidean case $X_i = D_i, i = 1, ...n$, these notions coincide with those introduced in [4], with n-convexity equivalent to local convexity in the classical sense. We denote by $\Phi^k(\Omega)$ the class of k-convex functions in $C(\Omega)$.

The Heisenberg group $\mathbb{H}^n$ is generated by the system of 2n vector fields in $\mathbb{R}^{2n+1}(x, y, t)$ given by

$$X_i = D_{x_i} - \frac{y_i}{2} D_t, \qquad X_{i+n} = D_{y_i} + \frac{x_i}{2} D_t, \qquad i = 1, ...n.$$

Note that the only nonvanishing commutators are $[X_i, X_{i+n}] = T = D_t$, for $i = 1, ...n$ and their negatives. For $k \leq n$, we now introduce the augmented even Hessian operators:

$$\mathcal{F}_{2k}[u] = \sum_{l=0}^{k} a_{2l} S_{2k,2l}(X_s^2 u)(Tu)^{2l}$$

where $S_{2k,2l}$ is the sum of minors given by

$$S_{2k,2l} = \sum_{i_1 i_2 \cdots i_l = 1}^{n} \sum_{j_1 j_2 \cdots j_l = 1}^{n} \frac{\partial^{2l} S_{2k}}{\partial r_{i_1 j_1} \partial r_{n+i_1, n+j_1} \cdots \partial r_{i_l j_l} \partial r_{n+i_l, n+j_l}}$$

and the sequence $\{a_{2l}\}$ is defined by

$$a_{2l} = \frac{(2l+1)}{[(2l)!!]^2}, \qquad l = 0, 1, \cdots, k.$$

When u is $2k$-convex , it follows that the minor sums $S_{2k,2l}(X_s^2) \geq 0$ so that $\mathcal{F}_{2k}[u]$ becomes an even polynomial in the commutator Tu with non-negative coefficients. Our main result concerns the extension of $\mathcal{F}_{2k}[u]$ to $\Phi^{2k}(\Omega), \Omega \subset \mathbb{R}^{2n+1}$, as a weakly continuous Borel measure.

Theorem 1. *For any $u \in \Phi^{2k}(\Omega)$, there exists a Borel measure $\mu_{2k}[u]$ such that when $u \in C^2(\Omega)$,*

$$\mu_{2k}[u](e) = \int_e \mathcal{F}_{2k}[u] dg,$$

for any Borel set $e \subset \Omega$. Moreover, if $\{u_m\} \subset \Phi^{2k}(\Omega)$ and $u_m \to u$ locally uniformly in Ω, then the corresponding measures $\mu_{2k}[u_m] \to \mu_{2k}[u]$ weakly, that is

$$\int_\Omega f d\mu_{2k}[u_m] \to \int_\Omega f d\mu_{2k}[u],$$

for any $f \in C(\Omega)$ with compact support in Ω.

In the special case k=n=1, $\mathcal{F}_2[u] = F_2[u] + \frac{3}{4}(Tu)^2$, and Theorem 1 was found by Gutierrez and Montanari [2], by combining the approach of [4] with an appropriate variational identity for $\mathcal{F}_2[u]$. Using the approach from [5], the result was extended in [3] to $n > 1$ and almost everywhere convergence as well as to more general systems of vector fields. Note that for $k > (2n+1)/4$ almost everywhere convergence coincides with local uniform convergence [3].

Our proof of Theorem 1 also adapts the approach in [4], the key ingredient being a monotonicity formula whose proof is equivalent to establishing a null Lagrangian property for the augmented Hessian $\mathcal{F}_{2k}[u]$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^{2n+1}$ be a C^1 bounded domain. Let $u, v \in \Phi^k(\Omega) \cap C^2(\bar{\Omega})$ satisfy $u \geq v$ in Ω and $u = v$ on $\partial\Omega$. Then*

$$\int_\Omega \mathcal{F}_{2k}[u] \leq \int_\Omega \mathcal{F}_{2k}[v].$$

We remark that the case k=n=2 of Theorem 2 was discovered by Garofalo and Tournier in [1] by an amazing brute force computation of individual terms. However they did conjecture that the result extended to $n > 2$ and in particular that there existed an even polynomial in Tu with coefficients depending on minors of X_s^2 satisfying the above monotonicity property.

In our paper [6] we also prove versions of Theorems 1 and 2 for odd Hessians, that is for corresponding operators $\mathcal{F}_{2k-1}, k = 1, ...n$, given by

$$\mathcal{F}_{2k-1}[u] = \sum_{l=0}^{k-1} a_{2l} S_{2k-1,2l}(X_s^2 u)(Tu)^{2l}.$$

Finally we remark that for the Heisenberg groups, as well as more general Carnot groups, the continuous k -convex functions coincide with the continuous subharmonic functions with respect to the operator F_k.

References

[1] Garofalo, Nicola and Tournier, Federico, *New properties of convex functions in the Heisenberg group.* Trans. Amer. Math. Soc. **358** (2006), 2011–2055.

[2] Gutiérrez, Cristian E. and Montanari, Annamaria, *Maximum and comparison principles for convex functions on the Heisenberg group*, Comm. Partial Differential Equations **29** (2004), 1305–1334.

[3] Trudinger, Neil S., *On Hessian measures for non-commuting vector fields*, Pure Appl. Math. Q. **2** (2006), 147–161.

[4] Trudinger, Neil S. and Wang, Xu-Jia, *Hessian measures I. Dedicated to Olga Ladyzhenskaya*, Topol. Methods Nonlinear Anal. **10** (1997), 225–239.

[5] Trudinger, Neil S. and Wang, Xu-Jia *Hessian measures II*, Ann. of Math. (2) **50** (1999), 579–604.

[6] Trudinger, Neil S. and Zhang, Wei, *Hessian measures on the Heisenberg group*, preprint,(2012).

Blow-ups of Type I Ricci flows

Joerg Enders

(joint work with Reto Müller and Peter M. Topping)

We consider families of complete, not necessarily compact, Riemannian manifolds $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, of arbitrary dimension $n \geq 2$ solving Hamilton's *Ricci flow*

$$\frac{\partial}{\partial t} g = -2Ric_{g(t)}. \tag{1}$$

Due to the diffeomorphism and scaling invariance of this quasilinear weakly parabolic partial differential equation, self-similar ancient solution called *gradient shrinking solitons* arise. After normalizations, those satisfy the equation

$$Ric_{g(t)} + \nabla^{g(t)}\nabla f(t) - \frac{1}{2(T-t)}g(t) = 0,$$

for potential functions $f(\cdot,t) : \mathcal{M} \to \mathbb{R}$, and have *Gaussian volume*

$$\tilde{V}(\mathcal{M}, g(t), f(t)) = \int_{\mathcal{M}} \big(4\pi(T-t)\big)^{-\frac{n}{2}} e^{-f(q,t)} dvol_{g(t)}(q)$$

which is constant in t and equal to 1 precisely on the trivial solution of flat Euclidean space with potential function $f(q,t) = \frac{|q|^2}{4(T-t)}$, the *Gaussian soliton.*

R. Hamilton [3] conjectured that singularities of Ricci flow should be modeled on nontrivial gradient shrinking solitons, at least under the assumption that the curvature tensor blows up at the *Type I rate*, i.e. if there exist $C \geq c > 0$ such that for all $t \in [0,T)$

$$\frac{c}{T-t} \leq \sup_{\mathcal{M}} |Rm_{g(t)}|_{g(t)} \leq \frac{C}{T-t}. \tag{2}$$

If a singularity occurs at time T, this assumption is 'natural' as the left inequality is automatically satisfied for bounded curvature complete Ricci flows due to the maximum principle. Otherwise, we call a solution to (1) satisfying only the right inequality in (2) a *Type I Ricci flow.* In [2] we prove the conjecture for parabolic rescalings around points p in the singular set Σ, the set of points in $\mathcal{M}$ for which there exists no neighborhood in which the curvature stays bounded as t approaches T:

Main Theorem. *Let $(\mathcal{M}^n, g(t), p)$, $t \in [0,T)$, $p \in \Sigma$, be a complete Type I Ricci flow. For any $\lambda_j \nearrow \infty$ the rescaled Ricci flows $(\mathcal{M}, g_j(t), p)$ defined on $[-\lambda_j T, 0)$ by*

$$g_j(t) := \lambda_j g(T + \frac{t}{\lambda_j})$$

subconverge to a Cheeger-Gromov-Hamilton ancient limit flow $(\mathcal{M}^n_\infty, g_\infty(t), p_\infty)$ on $(-\infty, 0)$, which is a nontrivial gradient shrinking soliton.

The soliton structure of the limit flow (for arbitrary points $p \in \mathcal{M}$) in this setting has been proved by Naber [5]. We use the curvature control stemming from Perelman's pseudolocality theorem [7] to show the nontriviality of the blow-ups, which in fact does not require their soliton structure. After our work, this theorem has also been obtained in [4] for compact $\mathcal{M}$ using an extension of Perelman's entropy to singular time, similar to the extension to singular time of Perelman's reduced volume used here.

It is well known (see e.g. [8]) that a gradient shrinking soliton has strictly positive scalar curvature unless it is the Gaussian soliton. This rigidity allows us to show that in the case of a Type I Ricci flow the singular set Σ coincides with the set of points in $\mathcal{M}$ where the scalar curvature $R_{g(t)}$ blows up at the Type I rate as t approaches T. We therefore obtain the following

Corollary.

(1) *In the presence of a Type I singularity, all singular points have the same curvature blow-up behavior: All curvature quantities blow up at the Type I rate.*

(2) *If $\mathcal{M}^n$ is compact and $\sup_{\mathcal{M}} |R_{g(t)}|_{g(t)} < K$ for all $t \in [0,T)$, then the Type I Ricci flow can be extended smoothly past time T.*

Moreover, the scalar curvature blow-up rate at the singularity can be used to prove that the volume of an initially finite volume singular set will vanish as the singular time is approached.

Based on the main theorem, we now prove an ε-regularity theorem. We first use the monotonicity of the *reduced volume based at singular time* $\tilde{V}_{p,T}(t)$ (see [1], [5], or also [2]) to define the *Gaussian density* $\theta_{p,T}$ at a point $p \in \mathcal{M}$ at the singular time T in a Type I Ricci flow as

$$\theta_{p,T} := \lim_{t \nearrow T} \tilde{V}_{p,T}(t) \in (0,1].$$

Using the fact that the Gaussian density corresponds to the Gaussian volume of the limiting gradient shrinking soliton, the gap theorem [10] for the Gaussian volume of solitons implies the following

Theorem. *Let $(\mathcal{M}^n, g(t))$ be a complete Type I Ricci flow on $[0,T)$ with singular time T and singular set Σ. Then $\theta_{p,T} = 1$ if and only if $p \in \mathcal{M} \setminus \Sigma$.*

In fact, there exists $\varepsilon = \varepsilon(n) > 0$ such that if $\theta_{p,T} > 1 - \varepsilon$ for a Ricci flow as above, then $p \in \mathcal{M} \setminus \Sigma$.

This is an analogue to B. White's partial regularity result for mean curvature flow [9] using Huisken's monotonicity and L. Ni's result in Ricci flow [6] using a localized reduced volume monotonicity.

References

[1] J. Enders, *Generalizations of the reduced distance in the Ricci flow – monotonicity and applications*, Ph.D. thesis, Michigan State University (2008).

[2] J. Enders, R. Müller and P. Topping, *On Type I singularities in Ricci flow*, Comm. Anal. Geom. **19** (2011), 905–922.

[3] R.S. Hamilton, *The formation of singularities in the Ricci flow.* Surveys in differential geometry, Vol. II (Cambridge, MA, 1993) 7–136, International Press, Cambridge, MA, 1995.

[4] C. Mantegazza and R. Müller, *Perelman's entropy functional at Type I singularities of the Ricci flow*, Preprint (2012), arXiv:1205.4143v1.

[5] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. **645** (2010), 125–153.

[6] L. Ni, *Mean value theorem on manifolds*, Asian J. Math. **11** (2007), 277–304.

[7] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, (2002), arXiv:math/0211159v1

[8] S. Pigola, M. Rimoldi and A. Setti, *Remarks on non-compact gradient Ricci solitons*, Prepring (2009), arXiv:0905.2868v3

[9] B. White, *A local regularity theorem for mean curvature flow*, Ann. Math. **161**(3) (2005), 1487–1519.

[10] T. Yokota, *Perelman's reduced volume and a gap theorem for the Ricci flow*, Comm. Anal. Geom. **17** (2009), 227–263.

Yang-Mills Fields in supercritical dimension, a variational approach

Tristan Rivière
(joint work with Mircea Petrache)

The existence of a $W^{1,p}$-controlled gauge to L^p curvatures in critical or subcritical $[2p]$-dimension [1] has been proved by K.Uhlenbeck in the early eightees. It is the key step for studying the variations of the p-Yang-Mills lagrangian - the L^p norm of the curvature - over manifolds of dimension less or equal to $[2p]$.

In contrast to the critical and subcritical cases, in dimension larger than 4 for instance - the supercritical case for the 2-Yang Mills energy - it is easy to construct L^2 curvatures which do not possess, even locally, $W^{1,2}$ gauges. This is also related to the fact that the 2-Yang-Mills energy in dimension larger than 4 does not control the topology of the bundle anymore. This illustrates the difficulty to proceed to the calculus of variations of p-Yang-Mills Lagrangians in super-critical dimension. In particular the approach used in the eightees in $[2p]$-dimension for producing p-Yang-Mills minimizers and consisting in minimizing the Yang-Mills energy over connections of a fixed bundles over $W^{1,p}$ connections is a-priori not well posed In dimension larger than $[2p]$.

In order to remedy to this difficulty we introduce the notion of "L^p curvature over weak bundles". In the abelian case in 3 dimensions this consists in looking at L^p 2-forms whose integral over almost every sphere equals an integer. In [1] we proved that, for $p > 1$, this class is weakly sequentially closed. We can then produce abelian curvatures minimizing of the L^p norm, the p-Yang-Mills energy, for smooth boundary data as long as $p > 1$.

We then present a recent result of Mircea Petrache ([2]) asserting that p-Yang-Mills Minimizers in 3 dimension, for $p > 1$ and smooth boundary data, have at most isolated singularities.

In the last part of the talk we discuss briefly the possibility to extend these results to the non-abelian supercritical case.

References

[1] M. Petrache, T. Rivière, *Weak closure of singular abelian L^p-bundles in 3-dimensions* Geom. Funct. Anal. 21 (2011), 1419-1442.

[2] M. Petrache, *Regularity up to the boundary for minimizing L^p-vector fields with integer fluxes* In preparation, 2012.

[1] $[x]$ denotes the largest integer smaller than x

Lower semicontinuity and Young measures in the space BD of functions of bounded deformation

Filip Rindler

Consider the task of finding a minimiser of the integral functional

$$F(u) := \int_\Omega f(x, \mathcal{E}u(x))\,\mathrm{d}x + \int_\Omega f^\infty\biggl(x, \frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}(x)\biggr)\,\mathrm{d}|E^s u|(x),$$

in the class of all functions $u \in \mathrm{BD}(\Omega)$. It turns out that the above functional is the "natural" extension of the functional

$$\int_\Omega f(x, \mathcal{E}u(x))\,\mathrm{d}x, \qquad u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^d).$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded open domain with Lipschitz-continuous boundary, the integrand $f\colon \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}$ has linear growth at infinity (i.e. $|f(x,A)| \leq C(1+|A|)$), and f^∞ is its recession function,

$$f^\infty(x,A) := \lim_{\substack{x' \to x\\ t\to\infty}} \frac{f(x',tA)}{t} \qquad x \in \overline{\Omega},\ A \in \mathbb{R}^{d\times d}_{\mathrm{sym}},$$

which describes the behaviour of f "at infinity" and which we assume to exist in the above sense. Moreover,

$$E^s u = \mathcal{E}u\,\mathcal{L}^d + \frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}\,|E^s u|, \qquad E^s u \text{ singular to } \mathcal{L}^d,$$

is the Lebesgue–Radon–Nikodým decomposition of the symmetrised derivative Eu, which is first defined as a distribution by duality with the symmetrised gradient

$$\mathcal{E}u := \frac{1}{2}\bigl(\nabla u + \nabla u^T\bigr),$$

but which for functions in $\mathrm{BD}(\Omega)$ can be represented as a finite Radon measure on Ω with values in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$.

The main result of this talk, published in [15], asserts weak* lower semicontinuity of the above integral functional on the whole space $\mathrm{BD}(\Omega)$ under the natural assumption that f is symmetric-quasiconvex, i.e.

$$f(x,A) \leq \int_\omega f\bigl(x, A + \mathcal{E}\psi(z)\bigr)\,\mathrm{d}z$$

for all $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and all $\psi \in C_c^\infty(\omega;\mathbb{R}^d)$, where $\omega \subset \mathbb{R}^d$ is an arbitrary Lipschitz domain. This condition is natural since it is also necessary for lower semicontinuity in BD. The result is the precise BD-analogue of the classical BV-lower semicontinuity theorem by Ambrosio & Dal Maso [2] and Fonseca & Müller [8]. The main novelty in comparison to previously known results in BD (see for example [4, 3, 6, 9]) is that it does not need any further assumptions to prohibit any (fractal) Cantor part in the symmetrized derivative Eu.

The proof combines the well-known "blow-up technique" (see [7]) with rigidity arguments (see e.g. [13]). The pivotal idea here is to show, by various techniques (Harmonic Analysis, slicing, ...), that we can construct "good" blow-ups at a singular point that only take certain “rigid” shapes, for example, in BD they always are either a sum of one-directional functions, or even affine. As a nice side-result, this strategy also allows to considerably simplify the proof of the classical lower semicontinuity theorem in BV, circumventing the use of the difficult Alberti’s Rank-One Theorem [1], see [16].

To explain one case of this strategy in more detail, consider the differential inclusion

$$\mathcal{E}u \in \{\lambda(a \odot b) : \lambda \in \mathbb{R}\},$$

where $a \odot b := (ab^T + ba^T)/2$ for $a, b \in \mathbb{R}^d$. What can be said about u (assuming enough regularity, which can as usual be achieved by mollification)? It is well-known that almost all blow-ups satisfy either this differential inclusion or a similar one, with $a \odot b$ replaced by a matrix that cannot be written in the form $a \odot b$. In the following we consider the first case; in the second case it can be shown that there always exists an affine blow-up by a similar strategy.

For the corresponding inclusion $\nabla u \in \{\lambda(a \otimes b) : \lambda \in \mathbb{R}\}$ one can show that u has the form $u(x) = u_0 + h(x \cdot b)a$ for a scalar function h and a fixed vector u_0. However, in the above situation it is not possible to conclude the analogous result that

$$u(x) = u_0 + h_1(x \cdot b)a + h_2(x \cdot a)b + Rx$$

for scalar functions h_1, h_2, a fixed vector u_0 and a skew-symmetric matrix R. But this rigid form is needed in order to average the function by a staircase construction in the proof of lower semicontinuity. However, it turns out to be possible to show enough structure of a function satisfying the above differential inclusion such that a *second* blow-up (of the first blow-up) will indeed have the form we want. Thus, the combination of rigidity arguments together with an iteration of the blow-up construction yields one “good” blow-up.

The proof of the lower semicontinuity result is organised using (generalized) Young measures, introduced in [5], which provides a convenient framework for the blow-up constructions. Part of these techniques were developed together with Jan Kristensen in [12], but new localization principles can be found in [15, 16]. It should be mentioned that while for the BV-lower semicontinuity without Alberti’s Theorem one can write a proof without Young measures, for the BD-lower semicontinuity result they are an integral component of the strategy and cannot easily be removed.

The final result presented in the talk is a characterisation theorem for Young measures generated by sequences in BD [14], which describes the asymptotic oscillations and concentration effects. This characterisation theorem puts generalised Young measures in duality with quasiconvex functions with linear growth at infinity via Jensen-type inequalities. THis generalises the well-known Kinderlehrer–Pedregal Theorem for classical Young measures [10, 11]. The proof proceeds via a Hahn–Banach argument and a localization technique for (generalized) Young measures. This also provides a second proof for the characterisation of BV-Young measures, first proved together with Jan Kristensen [12].

References

[1] G. Alberti, *Rank one property for derivatives of functions with bounded variation*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), 239–274.

[2] L. Ambrosio and G. Dal Maso, *On the relaxation in* $\mathrm{BV}(\Omega;\mathbf{R}^m)$ *of quasi-convex integrals*, J. Funct. Anal. **109** (1992), 76–97.

[3] A. C. Barroso, I. Fonseca, and R. Toader, *A relaxation theorem in the space of functions of bounded deformation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **29** (2000), 19–49.

[4] G. Bellettini, A. Coscia, and G. Dal Maso, *Compactness and lower semicontinuity properties in* $\mathrm{SBD}(\Omega)$, Math. Z. **228** (1998), 337–351.

[5] R. J. DiPerna and A. J. Majda, *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Comm. Math. Phys. **108** (1987), 667–689.

[6] F. Ebobisse, *A lower semicontinuity result for some integral functionals in the space SBD*, Nonlinear Anal. **62** (2005), 1333–1351.

[7] I. Fonseca and S. Müller, *Quasi-convex integrands and lower semicontinuity in* L^1, SIAM J. Math. Anal. **23** (1992), 1081–1098.

[8] ——, *Relaxation of quasiconvex functionals in* $\mathrm{BV}(\Omega,\mathbf{R}^p)$ *for integrands* $f(x,u,\nabla u)$, Arch. Ration. Mech. Anal. **123** (1993), 1–49.

[9] G. Gargiulo and E. Zappale, *A lower semicontinuity result in SBD*, J. Convex Anal. **15** (2008), 191–200.

[10] D. Kinderlehrer and P. Pedregal, *Characterizations of Young measures generated by gradients*, Arch. Ration. Mech. Anal. **115** (1991), 329–365.

[11] ——, *Gradient Young measures generated by sequences in Sobolev spaces*, J. Geom. Anal. **4** (1994), 59–90.

[12] J. Kristensen and F. Rindler, *Characterization of generalized gradient Young measures generated by sequences in* $W^{1,1}$ *and BV*, Arch. Ration. Mech. Anal. **197** (2010), 539–598, Erratum: Volume 203 (2012), 693-700.

[13] S. Müller, *Variational models for microstructure and phase transitions*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Lecture Notes in Mathematics, vol. 1713, Springer, 1999, pp. 85–210.

[14] F. Rindler, *Characterization of young measures generated by sequences in bv an bd*, submitted (December 2011), arXiv:1112.5613, 2011.

[15] ——, *Lower semicontinuity for integral functionals in the space of functions of bounded deformation via rigidity and Young measures*, Arch. Ration. Mech. Anal. **202** (2011), 63–113.

[16] ——, *Lower semicontinuity and Young measures in BV without Alberti's Rank-One Theorem*, Adv. Calc. Var. **5** (2012), 127–159.

Pattern formation in micromagnetics

Radu Ignat

We present several non-local variational models leading to rich pattern formation. These models arise mainly in micromagnetics and we are interested in developing an asymptotic analysis based on an entropy method coming from scalar conservation laws.

1. The Aviles-Giga model. Let $\Omega \subset \mathbf{R}^2$ be an open domain. For vector fields $u \in H^1_{div}(\Omega,\mathbf{R}^2)$ of vanishing divergence $\nabla \cdot u = 0$ in Ω, the following energy functional is defined:

$$AG_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2\,dx + \frac{1}{\varepsilon}\int_\Omega (1-|u|^2)^2\,dx,$$

for a small parameter $\varepsilon > 0$. The question of $\Gamma-$convergence of $\{AG_\varepsilon\}_{\varepsilon\downarrow 0}$ was intensively studied. The compactness of configurations $\{u_\varepsilon\}_{\varepsilon\downarrow 0}$ of uniformly bounded energy $AG_\varepsilon(u_\varepsilon) \leq C$ was proved (in strong L^2-topology) by Ambrosio, De Lellis and Mantegazza [2] and DeSimone, Kohn, Müller and Otto [7]. The limiting configurations u_0 satisfy

$$|u_0| = 1 \quad \text{and } \nabla \cdot u_0 = 0 \quad \text{in } \Omega. \tag{1}$$

Moreover, De Lellis and Otto [6] proved the $\mathcal{H}^1-$rectifiability of the jump set J of u_0, even if u_0 is in general not BV (see [2]). It is expected that the limit energy of $\{AG_\varepsilon(u_\varepsilon)\}_{\varepsilon\downarrow 0}$ concentrates on the jump set J and has the following form (first stated by Aviles and Giga [3]):

$$AG_0(u_0) = \frac{1}{3}\int_J |u_0^+(x) - u_0^-(x)|^3 \, d\mathcal{H}^1.$$

In fact, AG_0 is a lower-bound of $\{AG_\varepsilon\}_{\varepsilon\downarrow 0}$ (see Aviles and Giga [4], Jin and Kohn [11]). The difficulty consists in the upper bound construction for limiting configurations u_0: recovery sequences have been constructed *only* for BV configurations u_0 (see Conti and De Lellis [5] and Poliakovsky [12]).

Entropies. One of the main tool of this study consists in the concept of entropies coming from the scalar conservation law hidden in (1). Indeed, writing $u_0 = (v, h(v))$ for the flux $h(v) = \pm\sqrt{1-v^2}$, the divergence-free condition in u_0 turns into the nonlinear transport equation:

$$\partial_t v + \partial_s[h(v)] = 0, \tag{2}$$

where $(t, s) := (x_1, x_2)$ correspond to (time, space) variables. The notion of entropy solution is introduced via the pair (entropy, entropy-flux), i.e., a couple of scalar functions (η, q) such that $\frac{dq}{dv} = \frac{dh}{dv}\frac{d\eta}{dv}$ which entails that every smooth solution v of (2) has vanishing entropy production, i.e.,

$$\partial_t[\eta(v)] + \partial_s[q(v)] = 0. \tag{3}$$

More general, an entropy solution v has the property that for every pair (η, q), the entropy production is a (signed) measure that concentrates on lines (corresponding to "shocks" of v). It suggests the interest of using "global" quantities $\Phi(u_0) := (\eta(v), q(v))$ to detect "local" line-singularities of u_0. Indeed, we will say that $\Phi \in C^\infty(\mathbf{R}^2, \mathbf{R}^2)$ is a $DKMO-$entropy (see [7]) if

$$\Phi(0) = 0,\ D\Phi(0) = 0 \text{ and } z \cdot D\Phi(z)z^\perp = 0 \text{ holds for all } z \in \mathbf{R}^2.$$

In particular, if u_0 is a smooth vector field satisfying (1), then $\nabla \cdot [\Phi(u_0)] = 0$ (similarly to (3)). More general, the family of entropy productions $\{\nabla \cdot [\Phi(u_\varepsilon)]\}_{\varepsilon\downarrow 0}$ is asymptotically bounded as measure for every family $\{u_\varepsilon\}_{\varepsilon\downarrow 0} \subset H^1_{div}(\Omega, \mathbf{R}^2)$ of uniformly bounded energy: there exists a constant $C_\Phi > 0$ such that

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega \nabla \cdot [\Phi(u_\varepsilon)]\zeta \, dx\right| \leq C_\Phi\|\zeta\|_\infty \limsup_{\varepsilon\to 0} AG_\varepsilon(u_\varepsilon), \quad \text{for every } \zeta \in C_c^\infty(\Omega).$$

This is the starting point in proving the L^2-compactness result and the fine structure of the limiting configurations u_0 (see [7, 6]).

2. The Bloch wall model. Let us now discuss a rather more "geometric" and non-convex model coming from micromagnetics: For S^2-valued vector fields $m = (u, m_3) \in H^1_{div}(\Omega, S^2)$ with $\nabla \cdot u = 0$ in $\Omega \subset \mathbf{R}^2$, we define the functional:

$$E_\varepsilon(m) = \varepsilon \int_\Omega |\nabla m|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega m_3^2 \, dx,$$

for a small parameter $\varepsilon > 0$. As before, the aim is to analyze the asymptotic behavior of E_ε as $\varepsilon \to 0$. First, note that E_ε dominates the Aviles-Giga energy AG_ε, i.e., $AG_\varepsilon(u) \le E_\varepsilon(m)$, since $|\nabla u| \le |\nabla m|$ and $(1 - |u|^2)^2 = m_3^4 \le m_3^2$. Therefore, the L^2-strong compactness holds for uniformly bounded energy configurations $E_\varepsilon(m_\varepsilon) \le C$; the limiting configurations m_0 are in-plane, i.e., $m_0 = (u_0, 0)$ with (1) and a $\mathcal{H}^1-$rectifiable jump set J of u_0 can be defined. It is conjectured that the transition layers (at level $\varepsilon > 0$) corresponding to a jump (u_0^-, u_0^+) are one-dimensional and that the Γ-limit of $\{E_\varepsilon\}_{\varepsilon \downarrow 0}$ is given by

$$E_0(m_0) = \int_J |u_0^+(x) - u_0^-(x)|^2 \, d\mathcal{H}^1.$$

In a joint work with Merlet (see [9]), we obtained several partial results. In order to deal with the expected quadratic cost of jumps, we analyze the following class of Lipschitz entropies: $\Phi \in Lip(S^2, \mathbf{R}^2)$ such that for $\varepsilon \downarrow 0$,

$$\nabla \cdot [\Phi(m)] \le \varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} m_3^2 + o(1) \quad \text{in } \Omega, \quad \forall m \in C^\infty_{div}(\Omega, S^2) \tag{4}$$

with the condition that $[\Phi(u_0^+) - \Phi(u_0^-)] \cdot \nu = |u_0^+ - u_0^-|^2$ for jumps (u_0^-, u_0^+) of normal direction $\nu := \mathbf{e}_1$. We find such an entropy for the biggest jump $(0, \pm 1, 0)$ proving that the one-dimensional layer is optimal in this case. Even if we find entropies for each jump (u_0^-, u_0^+) satisfying (4) but in a restricted class of configurations m, we prove that the entropy method doesn't work in general for small angles. However, we show in a second paper [8] that E_0 is lower semicontinuous (in L^2 topology), enforcing the expectation that no microstructure appears for the Bloch wall model.

3. A zigzag wall model. We study now the following energy functional:

$$F_\epsilon(m) = \int_\Omega \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} m_2^2 \right) dx + \frac{1}{\varepsilon^s} \|\nabla \cdot m\|^2_{\dot{H}^{-1}(\Omega)}$$

for $m = (m_1, m_2, m_3) \in H^1(\Omega, S^2)$ where the constraint $\nabla \cdot (m_1, m_2) \neq 0$ is penalized in $\dot{H}^{-1}$-seminorm by the energy where $s \in (1, 2)$. The penalization of m_2 (instead of m_3 as previously) generates loss of coercivity of F_ε: configurations of uniformly bounded energy are in general no longer compact in strong L^2-topology due to possible oscillations in x_2-direction. The main idea of a joint work with Moser [10] is to study the quantity

$$\psi = \sin \vartheta - \vartheta \cos \vartheta,$$

where $\vartheta := \arctan \frac{m_3}{m_1}$ in the hemisphere where $|\vartheta| \leq \frac{\pi}{2}$. We show that as long as ϑ remains sufficiently small, the functional

$$F_0(\psi) = 2 \int_\Omega \left| \frac{\partial \psi}{\partial x_1} \right| \, dx$$

is the Γ-limit energy of $\{F_\varepsilon\}_{\varepsilon \downarrow 0}$. In general, the wall energy given by F_0 is *not* achieved by a one-dimensional transition between two limiting states $m^\pm = (\cos\theta, 0, \pm \sin\theta)$ of normal direction $\nu := \mathbf{e}_1$. Instead, in order to obtain the optimal limiting energy given by F_0, a transition with an additional zigzag structure is required. The matching with the upper bound (coming from the zigzag wall construction) is fulfilled via a lower bound based on generalized entropies. More precisely, as in (4), we study the entropies $\Phi \in Lip(S^2, \mathbf{R}^2)$ such that for $\varepsilon \downarrow 0$,

$$\nabla \cdot [\Phi(m)] \leq \varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} m_2^2 \quad \text{in } \Omega, \quad \forall m \in C^\infty_{div}(\Omega, S^2)$$

with the condition that $[\Phi(m^+) - \Phi(m^-)] \cdot \nu = 4\psi(\theta)$ for jumps $m^\pm = (\cos\theta, 0, \pm \sin\theta)$ of normal direction $\nu := \mathbf{e}_1$. In contrast with the Bloch wall model, we succeed to find such entropies for small angles θ and we prove that no entropy exists for the biggest jump $(0, 0, \pm 1)$. There is another situation where the Γ-limit is explicitly known for a problem involving similar microstructures: the problem leading to cross-tie walls in thin ferromagnetic films [13, 14, 1]. The cross-tie wall consists in a mixture of vortices and Néel walls (one-dimensional transition layers similar to Bloch walls, but taking values only in S^1). Remarkably, the function $\psi(\theta) = \sin\theta - \theta \cos\theta$ plays an important role in that context as well, although this may be a mere coincidence.

References

[1] F. Alouges, T. Rivière, S. Serfaty, *Néel and cross-tie wall energies for planar micromagnetic configurations*, ESAIM Control Optim. Calc. Var. **8** (2002), 31–68.

[2] L. Ambrosio, C. De Lellis, C. Mantegazza, *Line energies for gradient vector fields in the plane*, Calc. Var. Partial Differential Equations **9** (1999), 327–255.

[3] P. Aviles, Y. Giga, *A mathematical problem related to the physical theory of liquid crystal configurations*, Proc. Centre Math. Anal. Austral. Nat. Univ. **12** (1987), 1–16.

[4] P. Aviles, Y. Giga, *On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), 1–17.

[5] S. Conti, C. De Lellis, *Sharp upper bounds for a variational problem with singular perturbation*, Math. Ann. **338** (2007), 119–146.

[6] C. De Lellis, F. Otto, *Structure of entropy solutions to the eikonal equation*, J. Eur. Math. Soc. (JEMS) **5** (2003), 107–145.

[7] A. DeSimone, R.V. Kohn, S. Müller, F. Otto, *A compactness result in the gradient theory of phase transitions*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), 833–844.

[8] R. Ignat, B. Merlet, *Entropy method for line energies*, Calc. Var. Partial Differential Equations **44** (2012), 375-418.

[9] R. Ignat, B. Merlet, *Lower bound for the energy of Bloch walls in micromagnetics*, Arch. Ration. Mech. Anal. **199** (2011), 369–406.

[10] R. Ignat, R. Moser, *A zigzag pattern in micromagnetics*, J. Math. Pures Appl. **98** (2012), 139-159.

[11] W. Jin, R.V. Kohn, *Singular perturbation and the energy of folds*, J. Nonlinear Sci. **10** (2000), 355–390.
[12] A. Poliakovsky, *Upper bounds for singular perturbation problems involving gradient fields*, J. Eur. Math. Soc. (JEMS) **9** (2007), 1–43.
[13] T. Rivière, S. Serfaty, *Limiting domain wall energy for a problem related to micromagnetics*, Comm. Pure Appl. Math. **54** (2001), 294–338.
[14] T. Rivière, S. Serfaty, *Compactness, kinetic formulation, and entropies for a problem related to micromagnetics*, Comm. Partial Differential Equations **28** (2003), 249–269.

A sharp lower bound on the mean curvature integral with critical power for integral varifolds

Ulrich Menne

This is an announcement of the principal results of [12] using the notation of [11, §1, §2] which is based on Federer [5] and Allard [1]. To describe the results, some additional terminology from [12, 5.1, 5.4, 5.6] is needed.

The space of nonempty closed subsets of a metric space X is topologised by its injection into $\mathbf{R}^X$ associating to each set its distance function, cp. [5, 2.10.21]. Convergence in this topology is termed *locally in Hausdorff distance.*

If $a \in S \subset \mathbf{R}^n$ and S is closed, then S is called *differentiable at* a if and only if $\operatorname{Tan}(S,a)$ is a linear subspace of $\mathbf{R}^n$ and

$$\boldsymbol{\mu}_{1/r} \circ \boldsymbol{\tau}_{-a}[S] \to \operatorname{Tan}(S,a) \quad \text{locally in Hausdorff distance as } r \to 0+.$$

If $a \in S \subset \mathbf{R}^n$ and S is closed, then S is called *twice differentiable at* a if and only if S is differentiable at a and, in case $0 < m = \dim \operatorname{Tan}(S,a) < n$, there exists a homogeneous polynomial function $Q : \operatorname{Tan}(S,a) \to \operatorname{Nor}(S,a)$ of degree 2 such that with $\tau : \operatorname{Tan}(S,a) \times \operatorname{Nor}(S,a) \to \mathbf{R}^n$,

$$\tau(v,w) = v + w \quad \text{for } v \in \operatorname{Tan}(S,a),\ w \in \operatorname{Nor}(S,a),$$
$$\phi_r = r^{-1} \operatorname{Tan}(S,a)_\natural + r^{-2} \operatorname{Nor}(S,a)_\natural \quad \text{for } 0 < r < \infty$$

there holds

$$\phi_r \circ \boldsymbol{\tau}_{-a}[S] \to \tau[Q] \quad \text{locally in Hausdorff distance as } r \to 0+.$$

Note Q is uniquely determined by S and a, hence the *second fundamental form* $\mathbf{b}(S;a)$ and the *mean curvature vector* $\mathbf{h}(S;a)$ of S at a may be defined by $\mathbf{b}(S;a) = D^2Q(0)$ and $\mathbf{h}(S;a) = \operatorname{trace}\mathbf{b}(S;a)$ respectively; here the notion of trace of [5, 1.7.10] is extended in the obvious way.

Suppose m and n are positive integers, $m < n$, $1 \le p \le \infty$, V is an m dimensional integral varifold in $\mathbf{R}^n$, $\|\delta V\|$ is a Radon measure, and, if $p > 1$,

$$(H_p) \qquad \begin{aligned} &\delta V(g) = -\textstyle\int g(z) \bullet \mathbf{h}(V;z)\, \mathrm{d}\|V\| z \quad \text{for } g \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n), \\ &\mathbf{h}(V;\cdot) \in \mathbf{L}_p(\|V\| \llcorner K, \mathbf{R}^n) \quad \text{whenever } K \text{ is a compact subset of } \mathbf{R}^n. \end{aligned}$$

Instructive examples are constructed in Allard [1, 8.1 (2)], Brakke [3, 6.1], and [9, 1.2]. If $p = m$, then $\mathscr{H}^m \llcorner \operatorname{spt}\|V\| \le \|V\|$ by Allard [1, 8.3]. If $p > m$ and $p \ge 2$, then there exists a relatively open and dense subset G of $\operatorname{spt}\|V\|$ such that G is an m dimensional submanifold of class 1 of $\mathbf{R}^n$ by Allard [1, 8.1 (1)].

The condition (H_1) is sufficient to establish second order differentiability properties in an approximate sense:

Theorem 1 (cf. [10, 4.8])**.** *If V satisfies (H_1), then there exists a countable collection C of m dimensional submanifolds of class 2 of $\mathbf{R}^n$ with*

$$\|V\|(\mathbf{R}^n \sim \textstyle\bigcup C) = 0.$$

Moreover, for every member of M of C there holds

$$\mathbf{h}(M; z) = \mathbf{h}(V; z) \quad \textit{for } \|V\| \textit{ almost all } z \in M.$$

Using different methods, this theorem extends previous results of Schätzle in [13, Theorem 6.1] for the case $n = m + 1$, $p > m$, $p \geq 2$.

If $p = m$, the differentiability properties may be sharpened as follows.

Corollary 2 (cf. [12, 5.11], [10, 4.8])**.** *If V satisfies (H_m) and $S = \operatorname{spt} \|V\|$, then:*

(1) *For $\mathscr{H}^m$ almost all $a \in S$ the closed set S is twice differentiable at a with $\dim \operatorname{Tan}(S, a) = m$ and $\mathbf{h}(S; a) = \mathbf{h}(V; a)$.*

(2) *For $\|V\|$ almost all a there holds*

$$r^{-m} \int_{\mathbf{B}(a,r)} (|R(z) - R(a) - (\|V\|, m) \operatorname{ap} DR(a)(z - a)|/|z - a|)^2 \, \mathrm{d}\|V\| z \to 0$$

as $r \to 0+$, where R maps $w \in S$ such that S is differentiable at w onto $\operatorname{Tan}(S, w)_\natural \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$.

To prove the corollary, first, the necessary flatness properties are deduced from the preceding theorem by means of subsolution properties of the distance function associated to a plane. This step utilises ideas from Ecker [4, 1.6, 1.7], Allard [1, 7.5 (6)], and [10, 5.2 (2)]. Second, the differentiability properties are deduced using techniques from [9, §3]. Finally, the relation of $\mathbf{h}(S; \cdot)$ and $\mathbf{h}(V; \cdot)$ is established similarly as in Schätzle [14, Theorem 4.1].

The next theorem for $m = n - 1$ generalises the area formula for the Gauss map from oriented m dimensional submanifolds of class 2 of $\mathbf{R}^n$ to supports of m dimensional integral varifolds satisfying (H_m) with $m \geq 2$.

Theorem 3 (cf. [12, 7.34])**.** *If V satisfies (H_m), $2 \leq m = n - 1$, $S = \operatorname{spt} \|V\|$,*

$$C = (S \times \mathbf{S}^m) \cap \{(a, u) : \mathbf{U}(a - su, s) \cap S = \emptyset \textit{ for some } 0 < s < \infty\},$$

and B is an $\mathscr{H}^m$ measurable subset of C, then

$$\int_{\mathbf{S}^m} \mathscr{H}^0 \{a : (a, u) \in B\} \, \mathrm{d}\mathscr{H}^m u = \int_S \int_{\mathbf{S}^m \cap \{u \,:\, (a,u) \in B\}} |\operatorname{discr}(\mathbf{b}(S; a) \bullet u)| \, \mathrm{d}\mathscr{H}^0 u \, \mathrm{d}\mathscr{H}^m a,$$

where $\mathbf{b}(S; a) \bullet u : \operatorname{Tan}(S, a) \times \operatorname{Tan}(S, a) \to \mathbf{R}$ denotes the symmetric bilinear function mapping $(v, w) \in \operatorname{Tan}(S, a) \times \operatorname{Tan}(S, a)$ onto $\mathbf{b}(S; a)(v, w) \bullet u \in \mathbf{R}$.

Note $\mathscr{H}^m(S \sim \operatorname{dmn} C) = 0$ by part (1) of Corollary 2.

In view of Theorem 1, the proof of Theorem 3 readily reduces to establishing the following *Lusin property*:

$$\mathscr{H}^m(C[E]) = 0 \quad \text{whenever } E \subset \operatorname{dmn} C \text{ and } \mathscr{H}^m(E) = 0.$$

If $E \subset \{z : \mathbf{\Theta}^m_*(\|V\|, z) < \infty\}$, then the key is to establish a suitable version of a weak Harnack estimate for Lipschitzian real valued functions on S. In this respect inspiration is taken from Bombieri and Giusti [2], Hutchinson [7], and Stampacchia [16, §4, §5]. To treat the case $E \subset \{z : \mathbf{\Theta}^m(\|V\|, z) = \infty\}$, consider $z \in \mathbf{R}^n$ with $\mathbf{\Theta}^m(\|V\|, z) = \infty$. Then the modified monotonicity identity of Kuwert and Schätzle [8, Appendix] (which employs Brakke [3, 5.8]) may be used to estimate barycentres of $\|V\|$ on balls centred at z with suitable radii. In both cases the deduction of the Lusin property from the estimates is carried out analogously to the use of the *Rado-Reichelderfer condition* of Hencl in [6, Theorems 5.1 and 3.5].

Corollary 4 (cf. [12, 7.35]). *If V satisfies (H_m), $2 \leq m = n-1$, and $S = \operatorname{spt} \|V\|$ is nonempty and compact, then*

$$\textstyle\int_{\mathbf{S}^m} |\mathbf{h}(\mathbf{S}^m; z)|^m \, \mathrm{d}\mathscr{H}^m z \leq \int_S |\mathbf{h}(S; z)|^m \, \mathrm{d}\mathscr{H}^m z$$

The weaker estimate resulting from replacing $\mathscr{H}^m$ by $\|V\|$ in the last integral was previously obtained by Kuwert and Schätzle in [8, Appendix] for the case $m = 2$ and certain particular varifolds satisfying (H_p) with $p > m$ by Schulze in [15, Proposition 6.6].

Taking $B = (S \times \mathbf{S}^m) \cap \{(a, u) : (z - a) \bullet u \leq 0 \text{ for } z \in S\}$, Corollary 4 may be deduced from Theorem 3 similarly to Schulze [15, §2].

AEI publication number: AEI-2013-012

References

[1] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[2] E. Bombieri and E. Giusti. Harnack's inequality for elliptic differential equations on minimal surfaces. *Invent. Math.*, 15:24–46, 1972.

[3] Kenneth A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.

[4] Klaus Ecker. On regularity for mean curvature flow of hypersurfaces. *Calc. Var. Partial Differential Equations*, 3(1):107–126, 1995.

[5] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[6] Stanislav Hencl. On the notions of absolute continuity for functions of several variables. *Fund. Math.*, 173(2):175–189, 2002.

[7] John Hutchinson. Poincaré-Sobolev and related inequalities for submanifolds of $\mathbf{R}^N$. *Pacific J. Math.*, 145(1):59–69, 1990.

[8] Ernst Kuwert and Reiner Schätzle. Removability of point singularities of Willmore surfaces. *Ann. of Math. (2)*, 160(1):315–357, 2004.

[9] Ulrich Menne. Some applications of the isoperimetric inequality for integral varifolds. *Adv. Calc. Var.*, 2(3):247–269, 2009.

[10] Ulrich Menne. Second order rectifiability of integral varifolds of locally bounded first variation. *J. Geom. Anal.*, DOI:10.1007/s12220-011-9261-5, 2011.

[11] Ulrich Menne. Decay estimates for the quadratic tilt-excess of integral varifolds. *Arch. Ration. Mech. Anal.*, 204(1):1–83, 2012.

[12] Ulrich Menne. A sharp lower bound on the mean curvature integral with critical power for integral varifolds, 2012. In preparation.

[13] Reiner Schätzle. Quadratic tilt-excess decay and strong maximum principle for varifolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 3(1):171–231, 2004.

[14] Reiner Schätzle. Lower semicontinuity of the Willmore functional for currents. *J. Differential Geom.*, 81(2):437–456, 2009.
[15] Felix Schulze. Nonlinear evolution by mean curvature and isoperimetric inequalities. *J. Differential Geom.*, 79(2):197–241, 2008.
[16] Guido Stampacchia. *Èquations elliptiques du second ordre à coefficients discontinus.* Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965). Les Presses de l'Université de Montréal, Montreal, Que., 1966.

Existence of immersed spheres minimizing curvature functionals

Ernst Kuwert
(joint work with Andrea Mondino and Johannes Schygulla)

We consider variational problems for minimizers of Willmore functionals having the topological type of the 2-sphere. Let $[\mathbb{S}^2, \mathbb{R}^3]$ be the space of immersed 2-spheres in $\mathbb{R}^3$. The Willmore functional is given by

$$\mathcal{W}(f) = \frac{1}{4} \int_{\mathbb{S}^2} |\vec{H}|^2 \, d\mu,$$

where $\vec{H}$ is the mean curvature vector and $d\mu$ is the area element. Willmore (1965) proved that $\mathcal{W}(f) \geq 4\pi$ for any closed surface, with equality only for the round spheres. In the talk, we discussed an existence and regularity theorem proved by J. Schygulla in his Ph.D. thesis, see [Schy11]. For embedded surfaces f, one defines the isoperimetric ratio by

$$I(f) = \sqrt{36\pi}\, \frac{V(f)}{A(f)^{3/2}} \in (0, 1],$$

where $A(f)$ is the area and $V(f)$ is the volume enclosed by f.

Theorem 1 (Schygulla [Schy11])**.** *For any $\sigma \in (0, 1]$, there exists a minimizer of the Willmore functional in the class of smooth embeddings $f : \mathbb{S}^2 \to \mathbb{R}^3$ with prescribed isoperimetric ratio $I(f) = \sigma$. As a function of σ, the corresponding minimum $\beta(\sigma)$ is strictly decreasing with*

$$\beta(1) = 4\pi \quad \text{and} \quad \lim_{\sigma \searrow 0} \beta(\sigma) = 8\pi.$$

Moreover, the minimizers converge as $\sigma \searrow 0$ to a round sphere of multiplicity two in the sense of varifolds.

The theorem is partially motivated by a model for cell membranes due to Helfrich (1973). In that model, the energy contains an extra parameter called the spontaneous curvature, and both the area and the enclosed volume are prescribed. The theorem corresponds to the special case of spontaneous curvature zero, where the two conditions reduce to the isoperimetric ratio as single constraint by the scale invariance of the Willmore functional. We refer to [SeBeLip91] for numerical experiments. Under the assumption of axial symmetry, existence of minimizers for any spontaneous curvature have been constructed recently in [ChoVe12].

As second subject we let M be a 3-dimensional, compact Riemannian manifold, and consider the problem of minimizing the energy

$$E(f) = \frac{1}{2} \int_{\mathbb{S}^2} |A|^2 \, d\mu$$

in the class $[\mathbb{S}^2, M]$ of immersions $f : \mathbb{S}^2 \to M$.

Theorem 2 ([KuMoSchy11]). *For M as above, assume that*

(1) *There exists an $f \in [\mathbb{S}^2, M]$ with $E(f) < 4\pi$,*
(2) *There exists a minimizing sequence $f_k \in [\mathbb{S}^2, M]$ with $A(f_k) \leq C$.*

Then there exists an E-minimizer in $[\mathbb{S}^2, M]$.

By comparing with shrinking geodesic spheres, one gets $\inf_{[\mathbb{S}^2,M]} E \leq 4\pi$ for any M. In manifolds with sectional curvature $K^M < 0$ one has in fact always $E(f) > 4\pi$, so that the infimum is not attained. Condition (1) is also important in ruling out that the minimizing sequence develops branchpoints. Both (1) and (2) are satisfied if M has strictly positive sectional curvature.

Willmore-type surfaces given as pertubations of round spheres have been constructed by Mondino [Mo10] and Lamm, Metzger and Schulze [LaMeSchu09]. Global techniques are used by Chen and Li [CheLi11] and by Mondino and Rivière [MoRi12].

References

[CheLi11] J. Chen, Y.. Li, *Bubble tree of a class of conformal mappings and applications to Willmore functional*, arXiv:1112.1818 (2011).

[ChoVe12] R. Choksi, M. Veneroni, *Global minimizers for the doubly-constrained Helfrich energy: the axisymmetric case*, arXiv:1202.1979v2 (2012).

[KuMoSchy11] E. Kuwert, A. Mondino and J. Schygulla, *Existence of immersed spheres minimizing curvature functionals in compact 3-manifolds*, arXiv:1111.4893v1 (2011).

[LaMeSchu09] T. Lamm, J. Metzger,F. Schulze, *Foliations of asymptotically flat manifolds by surfaces of Willmore type*, Math. Ann. **350** (2010), 1–78.

[Mo10] A. Mondino, *Some results about the existence of critical points for the Willmore functional*, Math. Z. **266** (2010), 583–622.

[MoRi12] A. Mondino, T. Rivière, *Willmore spheres in compact Riemannian manifolds*, arXiv:1203.5501 (2012).

[Schy11] J. Schygulla, *Willmore minimizers with prescribed isoperimetric ratio*, Arch. Rational Mech. Anal. **203** (2012), 901–941.

[SeBeLip91] U. Seifert, K. Berndl, R. Lipowsky, *Shape transformations of vesicles: phase diagram for spontaneous-curvature bilayer-coupling models*, Phys. Rev. A **44** (1991), 1182–1202.

Quantitative stratification and the regularity of mean curvature flow

Robert Haslhofer
(joint work with Jeff Cheeger and Aaron Naber)

The aim here is to report on our recent estimates and quantitative regularity results for the mean curvature flow of n-dimensional surfaces in $\mathbb{R}^N$. For full details please see [2], for related results for the harmonic map flow please see [3].

Recall first that smooth solutions of the mean curvature flow are given by a smooth family of submanifolds $M_t^n \subset \mathbb{R}^N$ satisfying the evolution equation,

$$\partial_t x = H(x), \qquad x \in M_t\,. \tag{1}$$

More generally though, M_t is a family of Radon-measures that is integer n-rectifiable for almost all times and satisfies (1) in the weak sense of Brakke, i.e.

$$\overline{D}_t \int \varphi dM_t \leq \int \left(-\varphi H^2 + \nabla\varphi \cdot H\right) dM_t \tag{2}$$

for all nonnegative test functions φ, where $\overline{D}_t$ is the limsup of difference quotients. Brakke flows enjoy wonderful existence and compactness properties, see the fundamental work of Brakke and Ilmanen [1, 8]. The main problem is then to investigate their regularity.

Our results build upon the deep regularity theory of Brian White [9, 10, 11, 12], which we briefly recall now: Given a Brakke flow $\mathcal{M} = \{(M_t, t)\}$ he considered the stratification of the singular set $\mathcal{S} \subset \mathcal{M}$,

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{n+1} \subset \mathcal{S}\,, \tag{3}$$

where by definition $X = (x, t) \in \mathcal{S}^j$ if and only if no tangent flow at X has more than j symmetries. For general Brakke flows White first proved the (parabolic) Hausdorff dimension estimate

$$\dim \mathcal{S}^j \leq j\,. \tag{4}$$

For the flow of mean-convex hypersurfaces, he then proved the deep result

$$\mathcal{S} = \mathcal{S}^{n-1}\,, \tag{5}$$

and thus that the singular set has (parabolic) Hausdorff dimension at most $n-1$. This is based on many clever arguments, ruling out in particular higher multiplicities. He also gives a precise description of the singularities in this mean-convex case: all tangent flows are spheres, cylinders or planes of multiplicity one.

Let us now come to the general idea of quantitative stratification: Recall first that the standard stratification / dimension reduction method, a method to prove Hausdorff dimension estimates like (4), was introduced first in the context of geometric measure theory and later applied successfully in all kind of situations in geometric analysis. In the quantitative stratification, introduced recently by Cheeger-Naber in the elliptic setting [4, 5] and developed and applied now in the parabolic setting by Cheeger-Haslhofer-Naber [2, 3], we replace the singular strata $\mathcal{S}^j$ by quantitative singular strata $\mathcal{S}^j_{\eta,r}$ ($\eta > 0$, $0 < r < 1$). We then show that tubular neighborhoods of $\mathcal{S}^j_{\eta,r}$ have small volume, and that away from a bad set of small volume we get definite estimates on balls of definite size.

Concretely, let $\mathcal{M}_{X,s}$ be the flow obtained by shifting X to the origin and rescaling parabolically by $1/s$, and let d be a suitable distance function on the space of Brakke flows on the unit ball. Then

$$\begin{aligned}(6)\quad \mathcal{S}^j_{\eta,r} := \{X \in \mathcal{M} : d(\mathcal{M}_{X,s}, \mathcal{N}) > \eta \text{ for all } r \leq s \leq 1 \\ \text{and all selfsimilar } \mathcal{N} \text{ with more than } j \text{ symmetries}\}\,.\end{aligned}$$

Theorem 1. *For all $\varepsilon, \eta > 0$, $\Lambda < \infty$ and N, there exists $C = C(\varepsilon, \eta, \Lambda, N) < \infty$ such that: If $\mathcal{M}$ is a Brakke flow, defined on a space-time ball $B_2 \subset \mathbb{R}^{N,1}$ and with mass at most Λ, then its j-th quantitative singular stratum satisfies*

$$\text{Vol}\left(T_r(\mathcal{S}^j_{\eta,r}) \cap B_1\right) \leq C r^{N+2-j-\varepsilon} \qquad (0 < r < 1)\,. \tag{7}$$

By virtue of $\cup_{\eta>0} \cap_{r>0} \mathcal{S}^j_{\eta,r} = \mathcal{S}^j$, we recover the standard Hausdorff dimension estimate (4), but of course our theorem contains much more quantitative information about the singular set than just its dimension.

Coming to applications, we focus on Brakke flows starting at hypersurfaces $M_0 \subset \mathbb{R}^{n+1}$ (smooth, compact, embedded) that are k-convex, i.e. $\lambda_1 + \ldots \lambda_k \geq 0$ where $\lambda_1 \leq \ldots \leq \lambda_n$ denote the principal curvatures. Special instances are the convex case ($k = 1$) with Huisken's classical result, the 2-convex case where Huisken-Sinestrari constructed a mean curvature flow with surgery, and the general mean-convex case ($k = n$) with White's regularity theory. Building on the work of White via elliptic regularization we prove:

Theorem 2. *Let $\mathcal{M}$ be a Brakke flow starting at a k-convex hypersurface. Then any selfsimilar limit flow $\mathcal{N} = \lim \mathcal{M}_{X_\alpha, r_\alpha}$ with at least k symmetries is in fact a static multiplicity one plane. In particular, for every singular point $X \in \mathcal{S}$ all tangent flows are shrinking spheres or cylinders*

$$\mathbb{R}^j \times S^{n-j} \qquad \text{with} \qquad 0 \leq j < k\,. \tag{8}$$

The idea is now to combine Theorem 1 and Theorem 2, to obtain our main regularity result for k-convex mean curvature flows. To state it, for $X = (x,t) \in \mathcal{M}$ we define the regularity scale $r_{\mathcal{M}}(X)$ as the supremum of $0 \leq r \leq 1$ such that $M_{t'} \cap B_r(x)$ is a smooth graph for all $t - r^2 < t' < t + r^2$ and such that

$$\sup_{X' \in \mathcal{M} \cap B_r(X)} r\,|A|(X') \leq 1\,, \tag{9}$$

where A is the second fundamental form. For $0 < r < 1$ we then define the r-bad set

$$\mathcal{B}_r := \{X = (x,t) \in \mathcal{M} \,|\, r_{\mathcal{M}}(X) < r\}\,. \tag{10}$$

Theorem 3. *Let $\mathcal{M}$ be a Brakke flow starting at a k-convex hypersurface $M_0 \subset \mathbb{R}^{n+1}$ and $\varepsilon > 0$. Then there exists a constant $C = C(M_0, \varepsilon) < \infty$ such that we have the volume estimate*

$$\text{Vol}(T_r(\mathcal{B}_r)) \leq C r^{n+4-k-\varepsilon} \qquad (0 < r < 1)\,, \tag{11}$$

for the r-tubular neighborhood of the bad set $\mathcal{B}_r$. In particular, the (parabolic) Minkowski dimension of the singular set is at most $k-1$.

As a consequence, we obtain L^p-estimates for the inverse regularity scale, and thus in particular L^p-estimates for the second fundamental form and its derivatives.

Corollary 4. *Let $\mathcal{M}$ be a Brakke flow starting at a k-convex hypersurface $M_0 \subset \mathbb{R}^{n+1}$. Then for every $0 < p < n+1-k$ there exists a constant $C = C(M_0, p) < \infty$ such that*

$$\int r_{\mathcal{M}}^{-p} dM_t \leq C \qquad \text{and} \qquad \int_0^\infty \int r_{\mathcal{M}}^{-(p+2)} dM_t \, dt \leq C \,. \tag{12}$$

In particular, we have L^p-estimates for the second fundamental form,

$$\int |A|^p dM_t \leq C \qquad \text{and} \qquad \int_0^\infty \int |A|^{p+2} dM_t \, dt \leq C \,, \tag{13}$$

and also L^p-estimates for the derivatives of the second fundamental form,

$$\int |\nabla^\ell A|^{\frac{p}{\ell+1}} dM_t \leq C_\ell \qquad \text{and} \qquad \int_0^\infty \int |\nabla^\ell A|^{\frac{p+2}{\ell+1}} dM_t \, dt \leq C_\ell \,, \tag{14}$$

for some constants $C_\ell = C_\ell(M_0, p) < \infty$ ($\ell = 1, 2, \ldots$).

Using very different techniques, some special cases of this corollary ($k = 2$, first singular time) have been obtained previously by Head [7] and Ecker [6].

References

[1] K. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes 20, Princeton University Press (1978).

[2] J. Cheeger, R. Haslhofer, A. Naber, *Quantitative stratification and the regularity of mean curvature flow*, arXiv:1207.3619 (2012).

[3] J. Cheeger, R. Haslhofer, A. Naber, *Quantitative stratification and the regularity of harmonic map flow*, in preparation.

[4] J. Cheeger, A. Naber, *Lower bounds on Ricci curvature and quantitative behavior of singular sets*, arXiv:1103.1819 (2011).

[5] J. Cheeger, A. Naber, *Quantitative stratification and the regularity of harmonic maps and minimal currents*, arXiv:1107.3097 (2011).

[6] K. Ecker, *Partial regularity at the first singular time for hypersurfaces evolving by mean curvature*, Math. Ann. (to appear).

[7] J. Head, *The surgery and level-set approaches to mean curvature flow*, PhD-thesis, FU Berlin and AEI Potsdam (2011).

[8] T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. 108(520) (1994).

[9] B. White, *Stratification of minimal surfaces, mean curvature flows, and harmonic maps*, J. Reine Angew. Math. 448:1–35 (1997).

[10] B. White, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. 13(3):665–695 (2000).

[11] B. White, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. 16(1):123–138 (2003).

[12] B. White, *Subsequent singularities in mean-convex mean curvature flow*, arXiv:1103.1469 (2011).

Prescribed Gauss curvature on closed surfaces of higher genus

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(joint work with Franziska Borer and Luca Galimberti)

1. Main result

Let (M, g_0) be a closed Riemann surface endowed with a smooth background metric g_0. By the uniformization theorem we may assume that g_0 has constant Gauss curvature $K_{g_0} \equiv k_0$. Moreover, we normalize the volume of (M, g_0) to unity.

A classical problem in differential geometry is the question which smooth functions $f\colon M \to \mathbb{R}$ arise as the Gauss curvature of a conformal metric $g = e^{2u}g_0$ on M. Even when (M, g_0) is closed, this question so far has not been completely settled, aside from the case when the genus $g(M)$ of M is one [10], or when (M, g_0) is the projective plane [12]. In particular, when $g(M) > 1$ so far only partial results are known. Here we focus on this case. Clearly, by passing to the oriented double cover, if necessary, we may assume throughout that M is orientable.

Recall that the Gauss curvature of a conformal metric $g = e^{2u}g_0$ on M is given by

$$K_g = e^{-2u}(-\Delta_{g_0}u + k_0)\ .$$

Given a function $f \in C^\infty(M)$, the problem of finding a conformal metric of prescribed Gauss curvature f then amounts to solving the equation

$$-\Delta_{g_0} u + k_0 = fe^{2u} \quad \text{on } M. \tag{1}$$

Solutions u of (1) can be characterized as critical points of the functional

$$E_f(u) = \frac{1}{2}\int_M \left(|\nabla u|^2_{g_0} + 2k_0u - fe^{2u}\right)\, d\mu_{g_0},\ u \in H^1(M, g_0)\ .$$

Note that E_f is strictly convex and coercive on $H^1(M, g_0)$ when $f \le 0$ does not vanish identically. Hence for such f the functional E_f admits a strict absolute minimizer $u \in H^1(M, g_0)$ which is the unique solution of (1). A stability result of Aubin [1] then shows that for non-constant functions f with a not too large positive maximum the corresponding functional E_f still admits critical points which can be characterized as relative minimizers of E_f; see [2], [4].

For sign-changing functions f the functional E_f is no longer bounded from below, as can be seen by choosing a comparison function $v \ge 0$ supported in the set where $f > 0$ and looking at $E_f(sv)$ for large $s > 0$. Thus, for such functions the functional E_f exhibits a mountain-pass geomtery and we may expect the existence of a second solution of (1) which is not of minimum type. However, standard existence proofs for saddle points rely on the Palais-Smale condition which does not seem to hold for E_f. In [5] we overcome this difficulty by means of the "entropy method" introduced in [15] and [16], allowing us to show the following main result.

Theorem 1.1. *Let (M, g_0) be closed with $g(M) > 1$. Assume that for a given sign-changing function $f \in C^\infty(M)$ the functional E_f admits a relative minimizer*

$u_f \in H^1(M, g_0)$. Then E_f also admits a critical point $u^f \neq u_f$ which is not of minimum-type.

2. Strategy of the proof

An important ingredient in our proof is to show that relative minimizers of E_f always are strict in the sense of (2) below.

Proposition 2.1. *Suppose that for some $f \in C^\infty(M, g_0)$ the functional E_f admits a relative minimizer $u_f \in H^1(M, g_0)$. Then u_f is a non-degenerate critical point of E_f in the sense that*

$$d^2E_f(u_f)(h,h) = \int_M \left(|\nabla h|^2_{g_0} - 2fe^{2u_f}h^2\right) d\mu_{g_0} \geq c_0||h||^2_{H^1} \tag{2}$$

for all $h \in H^1(M, g_0)$.

In order to overcome the apparent lack of compactness we embed equation (1) into the 1-parameter family of problems

$$-\Delta_{g_0} u + k_0 = f_\lambda e^{2u} , \tag{3}$$

where for a given $f \in C^\infty(M)$ and any $\lambda \in \mathbb{R}$ we let $f_\lambda = f + \lambda$. Solutions u of (3) then can be characterized as critical points of the functional $E_\lambda = E_{f_\lambda}$.

Note that E_λ for $\lambda \leq \lambda_1 := -\max_M f$ is strictly convex and coercive on $H^1(M, g_0)$. Hence for such λ there is a strict absolute minimizer $u_\lambda \in H^1(M, g_0)$ of E_λ, uniquely solving (3) and depending smoothly on λ. By Proposition 2.1, moreover, this C^1-branch of *absolute* minimizers $(u_\lambda)_{\lambda \leq \lambda_1}$ extends as a C^1-curve of *relative* minimizers beyond the threshold $\lambda = \lambda_1$; however, since $k_0 < 0$ equation (1) implies that this branch must end before λ attains the value $\lambda_2 := -\int_M f \, d\mu_{g_0}$. On a heuristic level, thus our results indicate that the branch of relative minimizers will be met at some point $\lambda_1 < \lambda^* < \lambda_2$ by a branch of "large" solutions u^λ of saddle-type, bifurcating from infinity at $\lambda = \lambda_1$, a result vaguely reminiscent of "Rellich's conjecture" on the structure of the set of surfaces of prescribed constant mean curvature spanning a curve, which was proved independently by Brezis-Coron and Struwe in 1982; see [6],[7], [13], [14].

Now suppose that for some $\lambda \in]\lambda_1, \lambda_2[$ the functional E_λ admits a relative minimizer $u_\lambda \in H^1(M, g_0)$. By Proposition 2.1 then u_λ is strict and there exists $\rho > 0$ such that

$$E_\lambda(u_\lambda) = \inf_{||u-u_\lambda||_{H^1} < \rho} E_\lambda(u) < \beta_\lambda := \inf_{\rho/2 < ||u-u_\lambda||_{H^1} < \rho} E_\lambda(u). \tag{4}$$

Recalling that for $\lambda > \lambda_1$ the functional E_λ is unbounded from below, we can also fix a function $v_\lambda \in H^1(M, g_0)$ such that $E_\lambda(v_\lambda) < E_\lambda(u_\lambda)$ and hence are able to define

$$c_\lambda = \inf_{p \in P} \max_{t \in [0,1]} E_\lambda(p(t)) \geq \beta_\lambda > E_\lambda(u_\lambda),$$

where

$$P = \{p \in C\left([0,1]; H^1(M, g_0)\right); \; p(0) = u_\lambda, \, p(1) = v_\lambda\}. \tag{5}$$

The value λ will be fixed throughout the following. Again by Proposition 2.1, we may fix an open neighborhood Λ of λ such that for each $\mu \in \Lambda$ there exists a strict relative minimizer $u_\mu \in H^1(M, g_0)$ of E_μ, smoothly depending on $\mu \in \Lambda$; moreover, by continuity we may assume that there holds

$$E_\mu(v_\lambda) < E_\mu(u_\mu) \leq \sup_{\nu \in \Lambda} E_\mu(u_\nu) < \beta_\mu := \inf_{\rho/2 < ||u - u_\lambda||_{H^1} < \rho} E_\mu(u) \leq c_\mu \tag{6}$$

for every $\mu \in \Lambda$, where

$$c_\mu := \inf_{p \in P} \max_{t \in [0,1]} E_\mu(p(t)), \;\; \mu \in \Lambda \, . \tag{7}$$

Note that there holds

$$\frac{d}{d\mu} E_\mu(u) = -\frac{1}{2} \int_M e^{2u} \, d\mu_{g_0} < 0 \tag{8}$$

for every $u \in H^1(M, g_0)$ and every $\mu \in \mathbb{R}$. It follows that the function

$$\Lambda \ni \mu \mapsto c_\mu$$

is non-increasing in μ, and therefore differentiable at almost every $\mu \in \Lambda$. We now have the following result.

Proposition 2.2. *Suppose the map $\Lambda \ni \mu \mapsto c_\mu$ is differentiable at some $\mu > \lambda$. Then there exists a sequence $(p_n)_{n \in \mathbb{N}}$ in P and a corresponding sequence of points $u_n = p_n(t_n) \in H^1(M, g_0)$, $n \in \mathbb{N}$, such that*

$$E_\mu(u_n) \to c_\mu, \; \sup_{0 \leq t \leq 1} E_\mu(p_n(t)) \to c_\mu, \; dE_\mu(u_n) \to 0 \text{ in } H^{-1} \text{ as } n \to \infty, \tag{9}$$

and with $(u_n)_{n \in \mathbb{N}}$ satisfying, in addition, the "entropy bound"

$$\int_M e^{2u_n} \, d\mu_{g_0} = 2 \Big| \frac{d}{d\mu} E_\mu(u_n) \Big| \leq C = C(\mu), \text{ uniformly in } n. \tag{10}$$

The energy bound (9) in Proposition 2.2 together with (10) imply a uniform bound

$$||u_n||^2_{H^1} + \int_M e^{2u_n} \, d\mu_{g_0} \leq C = C(\mu), \tag{11}$$

which is sufficient for proving convergence of a suitable subsequence of $(u_n)_{n \in \mathbb{N}}$ to a critical point u^μ of E_μ. By Proposition 2.1 and (9) the limit u^μ cannot be a relative minimizer of E_μ.

A similar argument gives convergence $u^{\mu_k} \to u^\lambda$ for a suitable sequence $\mu_k \downarrow \lambda$, where the bound (10) now is replaced by – essentially – the geometric bound

$$\int_M f d\mu_g = \int_M f e^{2u} d\mu_{g_0} = 2\pi \chi(M)$$

from the Gauss-Bonnet Theorem. Again by Proposition 2.1 and (9) the limit u^λ cannot be a relative minimizer of E_λ.

References

[1] Aubin, Thierry: *Sur le problème de la courbure scalaire prescrite.* (French. English, French summary) [On the problem of prescribed scalar curvature] Bull. Sci. Math. 118 (1994), no. 5, 465-474.

[2] Aubin, Thierry; Bismuth, Sophie: *Courbure scalaire prescrite sur les variétés riemanniennes compactes dans le cas négatif.* (French) [Prescribed scalar curvature on compact Riemannian manifolds in the negative case] J. Funct. Anal. 143 (1997), no. 2, 529-541.

[3] Berger, Melvyn S.: *Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds.* J. Differential Geometry 5 (1971), 325-332.

[4] Bismuth, Sophie: *Prescribed scalar curvature on a C^∞ compact Riemannian manifold of dimension two.* Bull. Sci. Math. 124 (2000), no. 3, 239-248.

[5] Borer, Franziska; Galimberti, Luca; Struwe, Michael: *Prescribed Gauss curvature on closed surfaces of higher genus*, preprint 2011.

[6] Brezis, Haim; Coron, Jean-Michel: *Sur la conjecture de Rellich pour les surfaces à courbure moyenne prescrite.* [On Rellich's conjecture for surfaces with prescribed mean curvature] C. R. Acad. Sci. Paris Sér. I Math. 295 (1982), no. 11, 615-618.

[7] Brezis, Haim; Coron, Jean-Michel: *Multiple solutions of H-systems and Rellich's conjecture.* Comm. Pure Appl. Math. 37 (1984), no. 2, 149-187.

[8] Chang, Sun-Yung Alice; Yang, Paul C.: *Prescribing Gaussian curvature on S^2*. Acta Math. 159 (1987), no. 3-4, 215-259.

[9] Chang, Kung Ching; Liu, Jia Quan: *A Morse-theoretic approach to the prescribing Gaussian curvature problem.* Variational methods in nonlinear analysis (Erice, 1992), 55-62, Gordon and Breach, Basel, 1995.

[10] Kazdan, Jerry L.; Warner, F. W.: *Curvature functions for compact 2-manifolds.* Ann. of Math. (2) 99 (1974), 14-47.

[11] Kazdan, Jerry L.; Warner, F. W.: *Scalar curvature and conformal deformation of Riemannian structure.* J. Differential Geometry 10 (1975), 113-134.

[12] Moser, J.: *On a nonlinear problem in differential geometry.* Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), pp. 273-280. Academic Press, New York, 1973.

[13] Struwe, Michael: *Nonuniqueness in the Plateau problem for surfaces of constant mean curvature.* Arch. Rational Mech. Anal. 93 (1986), no. 2, 135-157 (received August 13, 1982).

[14] Struwe, Michael: *Large H-surfaces via the mountain-pass-lemma.* Math. Ann. 270 (1985), no. 3, 441-459.

[15] Struwe, Michael: *Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces.* Ann. Inst. H. Poincaré Anal. Non Linéaire 5 (1988), no. 5, 425-464.

[16] Struwe, Michael: *The existence of surfaces of constant mean curvature with free boundaries.* Acta Math. 160 (1988), no. 1-2, 19-64.

[17] Struwe, Michael; Tarantello, Gabriella: *On multivortex solutions in Chern-Simons gauge theory.* Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), no. 1, 109-121.

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