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Convex Geometry and its Applications

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9 December – 15 December 2012

Abstract. The geometry of convex domains in Euclidean space plays a central role in several branches of mathematics: functional and harmonic analysis, the theory of PDE, linear programming and, increasingly, in the study of other algorithms in computer science. High-dimensional geometry is an extremely active area of research: the participation of a considerable number of talented young mathematicians at this meeting is testament to the fact that the field is flourishing.

Mathematics Subject Classification (2000): 52A, 68Q25, 60D05.

Introduction by the Organisers

The meeting *Convex Geometry and its Applications* organised by Keith Ball, Martin Henk and Monika Ludwig, was held from December 9 to December 15, 2012. It was attended by some 55 participants working in all areas of high-dimensional geometry. Of these 20% were female and about one third were younger participants. The programme involved 10 plenary lectures of one hour's duration and about 20 shorter lectures. Some highlights of the program were as follows.

Assaf Naor spoke about joint work with Lafforgue on a striking new isoperimetric principle for maps from the Heisenberg group into uniformly convex spaces, which has applications to embedding problems and algorithms. Among other things it shows that the ball of radius n in the Heisenberg group cannot be embedded in Hilbert space with a distortion less than $\sqrt{\log n}$.

Emanuel Milman described delicate new isoperimetric principles for manifolds with more general conditions on curvature than the standard positivity condition. The isoperimetric principles are sharp and are expressed by means of model spaces in which the worst behaviour occurs. The main subtlety occurs because each situation now requires a one parameter family of model spaces rather than a single space.

Alexander Barvinok gave a delightful and entertaining talk about a magical new way to approximate convex domains by polytopes. The proof depends upon using Fritz John's theorem on the ellipsoid of minimal volume for a convex set in a high-dimensional tensor power of the original space and then applying the conclusion to linear forms on this space built from Chebyshev polynomials.

Alexander Litvak gave a remarkable lecture based on joint work with Adamczak, Guédon, Latała, Oleszkiewicz, Pajor and Tomczak-Jaegermann which provides a short and very clear proof of Paouris' Theorem on the distribution of mass in a convex domain. The crucial ingredient is Gordon's Minimax Theorem for Gaussian processes. Paouris's Theorem is one of the most striking results in the area to have been found during the last half dozen years: so a clear new argument can be expected to have a wide variety of applications.

There were several excellent talks by young researchers. Susanna Dann gave an interesting plenary lecture on her solution of the Busemann-Petty problem in complex hyperbolic space building on the existing results in real and complex Hilbert spaces. Luis Rademacher gave a clear and very surprising talk on extremal domains for the slicing problem showing that they must possess a rich symmetry: the key was to find a usable modification of a domain so as to exploit the extremality. Louise Jottrand was one of several student participants. She gave a very poised lecture on her recent joint work with Larman and Mani proving a conjecture of McMullen concerning the finiteness of shadow boundaries. The proof depends principally upon using geometric mesaure theory to generalise the Cauchy formula. Daniel Dadush spoke about a new integer programming algorithm found jointly with Peikert and Vempala. Integer programming is provably a computationally hard problem but existing algorithms do not even run in exponential time (as a function of the dimension). The new algorithm reduces the run time in n dimensions from n^{2n} to n^n and would achieve almost exponential time (namely $(\log n)^n$) subject to the conjecture of Lovász et al. on the spectral gap for convex domains. Peter Pivovarov discussed his joint work with Paouris and Zinn on a central limit theorem for random projections of the cube. This is in the spirit of central limit theorems for Gaussian chaos because the volume is a multilinear expression in the projection map but the proof involves geometric considerations as well. Manuel Weberndorfer presented remarkable new results on polar bodies of asymmetric bodies, which contain results of Lutwak, Yang and Zhang as well as Campi and Gronchi for origin-symmetric bodies as special cases. Judit Abardia spoke about her complete classification of Minkowski valuations in complex vector spaces. Lukas Parapatits discussed his joint work with Haberl on the classifaction of $\mathrm{SL}(n)$ invariant valuations. Whereas previously only homogeneous valuations were classified, they succeeded in dropping this assumption completely and obtain as an application a general affine Hadwiger theorem.

Workshop: Convex Geometry and its Applications

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Abstracts

Difference bodies in a complex vector space

Judit Abardia

Let V denote a real vector space of dimension n and $\mathcal{K}(V)$ the space of compact convex bodies in V. An operator $Z : \mathcal{K}(V) \to (A, +)$ with $(A, +)$ an abelian semi-group is called a *valuation* if it satisfies the following additivity property

$$Z(K \cup L) + Z(K \cap L) = Z(K) + Z(L),$$

for all $K, L \in \mathcal{K}(V)$ such that $K \cup L \in \mathcal{K}(V)$. If $(A, +)$ is the set of convex bodies with addition the Minkowski sum, then Z is called *Minkowski valuation*. They have been largely studied, see for instance [4, 5, 9, 10, 11, 12, 14, 15].

Two important properties of Minkowski valuations are the covariance and the contravariance with respect to the special linear group $\mathrm{SL}(V, \mathbb{R})$. A valuation $Z : \mathcal{K}(V) \to \mathcal{K}(V^*)$ is $\mathrm{SL}(V, \mathbb{R})$-*contravariant* if

$$Z(gK) = g^{-*}Z(K), \quad \forall g \in \mathrm{SL}(V, \mathbb{R}),$$

where V^* denotes the dual space of V and g^{-*} denotes the inverse of the dual map of g.

A valuation $Z : \mathcal{K}(V) \to \mathcal{K}(V)$ is $\mathrm{SL}(V, \mathbb{R})$-*covariant* if

$$Z(gK) = gZ(K), \quad \forall g \in \mathrm{SL}(V, \mathbb{R}).$$

An example of a continuous, translation invariant Minkowski valuations which is $\mathrm{SL}(V, \mathbb{R})$-contravariant is the projection body operator. For $K \in \mathcal{K}(V)$ the *projection body* ΠK *of* K has support function

$$h(\Pi K, u) = \frac{n}{2} V(K, \dots, K, [-u, u]), \quad u \in V,$$

where $V(K, \dots, K, [-u, u])$ denotes the mixed volume with $(n-1)$ copies of K and one copy of the segment joining u and $-u$. Ludwig proved in [9, 10] that the projection body operator is the only (up to a constant factor) continuous, translation invariant $\mathrm{SL}(V, \mathbb{R})$-contravariant Minkowski valuation.

For the covariant case, Ludwig proved in [10] that the difference body is the unique (up to a positive constant) continuous Minkowski valuation which is translation invariant and $\mathrm{SL}(V, \mathbb{R})$-covariant. In fact, she classified the continuous, $\mathrm{SL}(V, \mathbb{R})$-covariant Minkowski valuations (not necessarily translation invariant). The *difference body of a convex body* $K \in \mathcal{K}(V)$ is defined by

$$\mathrm{D}K = K + (-K),$$

where $-K$ denotes the reflection of K at the origin.

In this work, we are interested in obtaining a classification result for the Minkowski valuations in a complex vector space W which are continuous, translation invariant and $\mathrm{SL}(W, \mathbb{C})$-covariant. A classification result for the $\mathrm{SL}(W, \mathbb{C})$-contravariant case was given in [2]. Some other results concerning convex bodies in a complex vector space as ambient space can be found in [6, 7, 8].

The classification result we have proved is the following.

Let W be a complex vector space of complex dimension $m \geq 3$. A map $Z : \mathcal{K}(W) \to \mathcal{K}(W)$ is a continuous, translation invariant and $\mathrm{SL}(W, \mathbb{C})$-covariant Minkowski valuation if and only if there exists a convex body $C \subset \mathbb{C}$ such that $Z = \mathrm{D}_C$, where $\mathrm{D}_C K \in \mathcal{K}(W)$ is the convex body with support function

$$h(\mathrm{D}_C K, \xi) = \int_{S^1} h(\alpha K, \xi) dS(C, \alpha), \quad \forall \xi \in W^*,$$

where $dS(C, \cdot)$ denotes the area measure of C, and $\alpha K = \{\alpha k : k \in K \subset W\}$ with $\alpha \in S^1 \subset \mathbb{C}$. Moreover, C is unique up to translations.

In the case $\dim_{\mathbb{C}} W = 2$ the previous result is not true, since there exists also Minkowski valuations satisfying those properties which are not homogenouos of degree 1 but 3.

The main techniques used to prove the classification result are based in using the McMullen decomposition theorem for real-valued valuations [13] and then study each degree of homogeneity by itself. Using the $\mathrm{SL}(W, \mathbb{C})$-covariance, it can be proved that, if $\dim_{\mathbb{C}} W \geq 3$ then Z has homogeneity degree 1. In order to compute its support function, it is used a characterization result of Goodey and Weil [3] which, as a particular case, expresses all 1-homogeneous real-valued valuations in terms of distributions in a certain class of functions. From this result and using the three properties that Z has to satisfy, we get the result.

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Random polytopes, Orlicz norms and the central limit theorem

David Alonso-Gutiérrez
(joint work with Joscha Prochno)

A convex body $K \subseteq \mathbb{R}^n$ is called isotropic if it has volume 1 and verifies the following two conditions:

- $\int_K \langle x, \theta \rangle dx = 0$ for every $\theta \in S^{n-1}$
- $\int_K \langle x, \theta \rangle^2 dx = L_K^2$ for every $\theta \in S^{n-1}$.

The number L_K, independent of the direction $\theta \in S^{n-1}$, is called the isotropic constant of the body K and it is a major problem in high-dimensional convex geometry to know whether it is or bounded from above by an absolute constant independent of the convex body and the dimension or it is not.

One possible approach to this problem consists in relating the value of L_K to the value of some parameters of a random polytope in K so that getting good upper bounds for such parameters would imply upper bounds for L_K.

By Steiner's formula, for every convex body $L \subseteq \mathbb{R}^n$, the volume of $L + tB_2^n$ is a polynomial in t of degree n

$$|L + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_k(L) t^k.$$

The coefficients $W_k(L)$ in this polynomial are the quermaßintegrals of the body L. Calling $Q_k(L) = \left(\frac{W_{n-k}(L)}{|B_2^n|} \right)^{\frac{1}{k}}$ we have, by Alexandrov-Fenchel inequalities, that these normalized quermaßintegrals $Q_k(L)$ are decreasing in k.

In [1] the authors studied the value of these quermaßintegrals for random polytopes in an isotropic convex body $K \subseteq \mathbb{R}^n$. Denoting by

$$K_N = \mathrm{conv}\{\pm X_1, \dots, \pm X_N\},$$

where $X_1, \dots, X_N$ are independent random vectors uniformly distributed in an isotropic convex body $K \subseteq \mathbb{R}^n$ the authors showed that if $cn \leq N \leq e^{\sqrt{n}}$

$$c_1 \sqrt{\log \frac{N}{n}} L_K \leq \mathbb{E} Q_n(K_N) \leq \mathbb{E} Q_1(K_N) \leq c_2 \sqrt{\log N} L_K,$$

where c_1, c_2 are absolute constants. Consequently, if $n^2 \leq N \leq e^{\sqrt{n}}$, we have that

$$\mathbb{E} Q_k(K_N) \sim \sqrt{\log N} L_K$$

for every $1 \leq k \leq n$.

We close the gap left when $n \leq N \leq n^2$ in the study of $\mathbb{E} Q_1(K_N) = \frac{1}{2} w(K_N)$, where $w(K_N)$ denotes the mean width of K_N, showing that also in this range of N

$$\mathbb{E} w(K_N) \sim \sqrt{\log N} L_K.$$

The main tools we use to do that are the central limit theorem for convex bodies, proved by Klartag in [2], that shows that for many directions $\theta \in S^{n-1}$ the one dimensional marginals of a random vector uniformly distributed in K behave like Gaussian in some interval, and the representation of the support function of K_N in each direction θ as an Orlicz norm of the vector $(1, \dots, 1)$, which follows from [3]:

$$h_{K_N}(\theta) = \max_{1 \le i \le N} |\langle X_i, \theta \rangle| = \inf\{s \,:\, M_\theta\left(\frac{1}{s}\right) \le \frac{1}{N}\},$$

where

$$M(s) = \int_0^s \int_{\{\frac{1}{t} \le |X_1|\}} |X_1| d\mathbb{P} dt.$$

The use of this Orlicz function M also allows us to prove that the average of the distribution function of the 1-dimensional marginals of a random vector uniformly distributed in K behave in a subgaussian and in a supergaussian way in a larger interval than what it was known. Namely, if

$$F(t) = \int_{S^{n-1}} |\{x \in K \,:\, |\langle x, \theta \rangle| \ge t L_K\}| d\mu(\theta),$$

- If $1 \le t \le \frac{\sqrt{n}}{c}$
$$F(t) \ge e^{-c^2 t^2}.$$
- If $1 \le t \le n^{\frac{1}{4}}$
$$F(t) \le e^{-c'^2 t^2}.$$

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Thrifty approximations of convex bodies by polytopes

Alexander Barvinok

Let $B \subset \mathbb{R}^d$ be a convex body containing the origin in its interior and let $\tau > 1$ be a real number. We want to construct a polytope P with as few vertices as possible such that $P \subset B \subset \tau P$. Our first result concerns bodies B symmetric about the origin, $B = -B$.

Theorem 1. *Let k be a positive integer such that*

$$\left(\tau - \sqrt{\tau^2 - 1}\right)^k + \left(\tau + \sqrt{\tau^2 - 1}\right)^k \ge 6\binom{d+k}{k}^{1/2}.$$

Then, for any convex body $B \subset \mathbb{R}^d$ such that $B = -B$ there is a polytope P with at most $8\binom{d+k}{k}$ vertices such that $P = -P$ and $P \subset B \subset \tau P$.

Letting $\tau = 1 + \epsilon$ for a small $\epsilon > 0$ we conclude that for any origin-symmetric convex body $B \subset \mathbb{R}^d$ there is an origin-symmetric polytope $P \subset \mathbb{R}^d$ with at most

$$\left(\frac{\gamma}{\sqrt{\epsilon}} \ln \frac{1}{\epsilon}\right)^d$$

vertices such that $P \subset B \subset (1+\epsilon)P$, where $\gamma > 0$ is an absolute constant. In fact, one can choose any

$$\gamma > \frac{e}{4\sqrt{2}} \approx 0.48$$

for all sufficiently small $0 < \epsilon < \epsilon_0(\gamma)$. This is the first improvement, uniform over all dimensions d and all origin-symmetric convex bodies B, of the classical volumetric bound of $(3/\epsilon)^d$ for the number of vertices of the approximating polytope. We note that results of Dudley [4] and of Bronshtein and Ivanov [3] on approximations convex bodies in the Hausdorff metric imply that one can choose P with at most $\gamma(d)\epsilon^{-(d-1)/2}$ vertices with $\gamma(d)$ of the order of $d^{d/4}$. For C^2-smooth origin-symmetric convex bodies Gruber obtains an approximating polytope with at most $(\gamma/\epsilon)^{(d-1)/2}$ vertices, where $\gamma > 0$ is an absolute constant, and $\epsilon < \epsilon_0(B)$ for some $\epsilon_0(B) > 0$ depending on the approximated convex body B [5]. Moreover, it is proved in [5], see also [2], that the Euclidean ball requires the largest number of vertices of the approximating polytope as $\epsilon \longrightarrow 0+$ in the class of origin symmetric C^2-smooth convex bodies. No such results appear to be known for non-smooth convex bodies.

Choosing k in Theorem 1 to be constant, we conclude that for any fixed $\epsilon > 0$ and any origin-symmetric convex body $B \subset \mathbb{R}^d$ of a sufficiently high dimension $d > d_0(\epsilon)$, one can find an origin-symmetric polytope $P \subset \mathbb{R}^d$ with at most $d^{\gamma/\epsilon}$ vertices such that $P \subset B \subset \sqrt{\epsilon d}P$, where $\gamma > 0$ is an absolute constant. In other words, to achieve $\tau = \sqrt{\epsilon d}$ for any fixed $\epsilon > 0$ in Theorem 1, one can use approximating polytopes with the number of vertices polynomial in the dimension. The example of the Euclidean ball shows that if we want to keep the number of vertices of P polynomial in the dimension, we cannot improve the bound for τ by more than a logarithmic in d factor.

In the case of a general (that is, not necessarily origin-symmetric) convex body, the quality of approximation depends on the symmetry coefficient, that is, on the smallest $\mu \geq 1$ such that $-B \subset \mu B$.

Theorem 2. *For $\tau, \mu \geq 1$ let us define*

$$\lambda = \lambda(\tau, \mu) = \frac{2\tau}{\mu + 1} + \frac{\mu - 1}{\mu + 1} \geq 1.$$

Let k be a positive integer such that

$$\left(\lambda - \sqrt{\lambda^2 - 1}\right)^k + \left(\lambda + \sqrt{\lambda^2 - 1}\right)^k \geq 6\binom{d+k}{k}^{1/2}.$$

Then, for any convex body $B \subset \mathbb{R}^d$ containing the origin in its interior and such that $-B \subset \mu B$, there is a polytope P with at most $8\binom{d+k}{k}$ vertices such that $P \subset B \subset \tau P$.

As a function of the symmetry coefficient μ, the number of vertices of P grows roughly as $\mu^{d/2}$ as long as the ratio τ/μ stays small enough. This is the first improvement, uniform over all dimensions d and all convex bodies B, over the straightforward μ^d estimate. We note that results of Gruber [5] imply that if B is C^2-smooth then for all sufficiently small $0 < \epsilon < \epsilon_0(B)$ one can construct a polytope P with not more than $\mu^{d/2}(\gamma/\epsilon)^{(d-1)/2}$ vertices approximating B within a factor of $\tau = 1 + \epsilon$, where $\gamma > 0$ is an absolute constant. No such results appear to be known for non-smooth convex bodies. It is an interesting question how the bounds of Theorem 2 can be improved if we are allowed to choose the origin inside B. Approximating B by the inscribed simplex of the maximum volume produces $\tau = d$ and P with $d + 1$ vertices, but not much is known beyond that.

The proofs of Theorems 1 and 2 are found in [1].

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The variance conjecture on some polytopes

Jesús Bastero

(joint work with David Alonso-Gutiérrez)

This talk is based on the paper [1] that will appear in the the Proceedings of the Asymptotic Geometric Analysis Program in the Fields Institute.

The variance conjecture states that there exists an absolute constant C such that for every isotropic log-concave vector X, if we denote by $|X|$ its Euclidean norm, Var $|X|^2 \leq Cn$.

This conjecture was considered by Bobkov and Koldobsky (see [4]) in the context of the Central Limit Problem for isotropic convex bodies. It was conjectured before by Antilla, Ball and Perissinaki (see [2]) that for an isotropic log-concave vector X, $|X|$ is highly concentrated in a “thin shell” and in this paper the authors proved that a vector X uniformly distributed on the unit balls B_p^n, $1 \leq p \leq \infty$ satisfies the variance conjecture. Wojtaszczyk extended in [11] the result the the Orlicz balls. Klartag proved the conjecture for random vectors uniformly distributed on isotropic unconditional convex bodies (see, [9]). The best known (dimension dependent) bound for general log-concave isotropic random vectors was proved by Guédon and Milman with a factor $n^{1/3}$ instead of C (see [7]). Very recently Eldan, ([5]) obtained a breakthrough showing that the variance conjecture implies the stronger Kannan, Lovász and Simonovits spectral gap conjecture (see [8]) with an extra polylogarithmic factor. It is also known (see [3], [6]) that these conjectures are stronger than the hyperplane conjecture, which states that every convex body of volume 1 has a hyperplane section of volume greater than some absolute constant.

Our purpose in this lecture is to consider the general variance conjecture: there exists an absolute constant C such that for every centered log-concave vector X

$$\mathrm{Var}\ |X|^2 \leq C\lambda_X^2 \mathbb{E}|X|^2$$

(given a centered log-concave random vector X in $\mathbb{R}^n$, we will denote by λ_X the highest eigenvalue of its covariance matrix and if the random vector is isotropic the covariance matrix is the identity).

The main results are the following

Theorem 1. *Let $\theta \in S^{n-1}$ and let $K = P_H B_\infty^n$ be the projection of B_∞^n on the hyperplane $H = \theta^\perp$. If X is a random vector uniformly distributed on K then, for any two orthonormal vectors $\eta_1, \eta_2 \in H$, we have*

$$\mathbb{E}\langle X, \eta_1\rangle^2 \langle X, \eta_2\rangle^2 \leq \mathbb{E}\langle X, \eta_1\rangle^2 \mathbb{E}\langle X, \eta_2\rangle^2.$$

Consequently, X satisfies the negative square correlation property with respect to any orthonormal basis in H and, moreover, the vectors X and TX for all $T \in GL(n)$ satisfy the variance conjecture.

Theorem 2. *There exists an absolute constant C such that for every hyperplane H, if X is a random vector uniformly distributed on $P_H B_1^n$, X verifies the variance conjecture with constant C, i.e.*

$$Var\ |X|^2 \leq C\lambda_X^2 \mathbb{E}|X|^2.$$

Notice that since the general variance conjecture is not invariant under linear maps it cannot easily be reduced to the isotropic case. We will study how this conjecture behaves under linear transformations presenting two more results. We will sketch the proof of the second one here since it does not appear in the aforementioned paper [1].

Proposition 3. *Let X be a centered isotropic, log-concave random vector in $\mathbb{R}^n$ verifying the variance conjecture with constant C_1. Let $T \in GL(n)$ be any linear transformation. If U is a random matrix uniformly distributed in $O(n)$ then*

$$\mathbb{E}_U\, Var\ |T \circ U(X)|^2 \leq CC_1 \|T\|_{op}^2 \|T\|_{HS}^2 = CC_1 \lambda_{T\circ u(X)}^2 \mathbb{E}|T \circ u(X)|^2$$

for any $u \in O(n)$, where C is an absolute constant.

For a non-singular linear map T we introduce the parameter

$$\alpha(T) = \frac{\left(\sum_{i=1}^n \lambda_i^4\right)^{1/4}}{\left(\sum_{i=1}^n \lambda_i^2\right)^{1/2}} = \frac{\|T\|_4}{\|T\|_{HS}}$$

where the $\{\lambda_i\}_1^n$ are the singular values and $\|T\|_4$ is the Schatten norm $\|T\|_4 = \left(\sum_{i=1}^n \lambda_i^4\right)^{1/4}$. It is clear that $n^{-1/4} \leq \alpha(T) < 1$. The next proposition will prove that, whenever $n^{-\alpha} \leq \alpha(T) < 1$, for $0 < \alpha < 1/4$, or even more, $\alpha(T)^{-1} = o(n^{1/4})$, the corollary above is true with high probability

Proposition 4. *Let X be an isotropic log-concave random vector in $\mathbb{R}^n$ verifying the variance conjecture with constant C_1. There exists an absolut constant C such that the measure of the set of orthogonal operators U for which the random vector $T \circ U(X)$ verifies the variance conjecture with constant CC_1 is greater than*

$$1 - C_2 \exp(-C_3 n \alpha(T)^4)$$

whenever $\alpha(T)^4 n \geq 2$.

Proof. We consider the function $F(U) = \mathrm{Var}|T \circ U(X)|^2$, the Lipschitz costant is bounded for above by

$$L \leq C \left(\sum_i \lambda_i^2\right) \left(\sum_j \lambda_j^4\right)^{1/2} = C\alpha(T)^2 \|T\|_{HS}^4.$$

Also we bound from below the expected value

$$\mathbb{E}F(U) \geq 2\frac{\|T\|_{HS}^4}{n+2}\left(n\alpha(T)^4 - 1\right)$$

So

$$\frac{\mathbb{E}F(U)}{L} \geq C\alpha(T)^2 \left(1 - \frac{1}{n\alpha(T)^4}\right) > C\alpha(T)^2$$

whenever $\alpha(T)^4 n \geq 2$. The result follows by using the concentration of measure phenomena on the group $SO(n)$ (see [10])

$$\mathbb{P}\left\{U \in SO(n); |F(U) - \mathbb{E}F(U)| > \frac{1}{2}\mathbb{E}F(U)\right\} \leq C_1 \exp\left(-C_2 n \left(\frac{\mathbb{E}F}{L}\right)^2\right)$$

and we get the result (we note that computing $\mathbb{E}_U F(U)$ in $O(n)$ or in $SO(n)$ gives the same result by symmetry). □

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A duality of isoperimetric problems

Andreas Bernig

A *definition of volume* gives a way to measure volumes on Finsler manifolds. Roughly speaking, a definition of volume μ associates to each finite-dimensional normed space $(V, \|\cdot\|)$ a norm on $\Lambda^n V$, where $n = \dim V$. We refer to [1, 7] for more information. The best known examples of definitions of volume are the *Busemann volume* μ^b which equals the Hausdorff measure, the *Holmes-Thompson volume* μ^{ht} which is related to symplectic geometric and *Gromov's mass** μ^{m*}, which thanks to its convexity properties is often used in Geometric Measure Theory.

To each definition of volume μ may be associated a dual definition of volume μ^*. For instance, the dual of Busemann's volume is Holmes-Thompson volume and vice versa. The dual of Gromov's mass* is Gromov's mass, which however lacks good convexity properties and is less used.

Given a definition of volume and a compact convex set $K \subset V$ in a finite-dimensional normed vector space, we let

$$A_\mu(K) := \int_{\partial K} \mu$$

be the surface area of K, measured with μ. Note that when K has a smooth boundary, then ∂K is a Finsler manifold. More precisely, each tangent space $T_p\partial K \subset V$ carries the induced norm.

The definition of μ is called convex, if for compact convex bodies $K \subset L$, we have $A_\mu(K) \leq A_\mu(L)$. There are many equivalent ways of defining convexity of volume definitions, we refer to [1] for details. The above mentioned three examples are convex.

Given a convex definition of volume and an n-dimensional normed space V with unit ball B, there is a unique (up to translations) compact convex body $\mathbb{I}_\mu B$ such that

$$A_\mu(K) = nV(K[n-1], \mathbb{I}_\mu B), \quad K \in \mathcal{K}(V).$$

Here V denotes the mixed volume and $\mathcal{K}(V)$ stands for the space of compact convex bodies in V. $\mathbb{I}_\mu B$ is called the isoperimetrix [1].

As its name indicates, the isoperimetrix is related to isoperimetric problems. More precisely, among all compact convex bodies of a given, fixed volume, a homothet of the isoperimetrix has minimal μ-surface area.

The isoperimetrices of the above mentioned examples of definitions of volumes are related to important concepts from convex geometry. For Busemann's definition of volume, we have $\mathbb{I}_{\mu^b} B = \omega_n (IB)^\circ$, where $IB \subset V^*$ is the *intersection body* of B, $(IB)^\circ \subset V$ its polar body, and ω_n the volume of the (Euclidean) unit ball. For the Holmes-Thompson volume, we have $\mathbb{I}_{\mu^{ht}} B = \frac{1}{\omega_n}\Pi(B^\circ)$, where Π denotes the *projection body*. The isoperimetrix for Gromov's mass* is a dilate of the *wedge body* of B.

We introduce a *dual isoperimetrix* which belongs to the dual Brunn-Minkowski theory. Recall that in the dual Brunn-Minkowski theory, the natural setting is that of star bodies (i.e. compact, star shaped bodies, with continuous radial function). The Minkowski sum of convex bodies is replaced by the radial sum and mixed volumes by dual mixed volumes. The dual Brunn-Minkowski theory was developed by Lutwak [4, 5] and plays a prominent role in modern convexity.

Let us describe our main results. The space of star bodies with smooth radial function in a vector space V is denoted by $\mathcal{S}^\infty(V)$. Let $S \in \mathcal{S}^\infty(V)$, where V is a finite-dimensional normed space V. Given a point $p \in \partial S$, the tangent space $T_p\partial S$ is naturally identified with the quotient space $V/\langle p\rangle$ and inherits the quotient norm. Therefore ∂S is in a natural way a Finsler manifold. This Finsler metric was (to the best of our knowledge) first studied in a recent paper by Faifman [3].

We denote by

$$\tilde{A}_\mu(S) := \int_{\partial S} \mu$$

the surface area of S with respect to a definition of volume μ.

Let us now formulate our main theorem.

Theorem. *Let μ be a definition of volume, and V an n-dimensional normed space with unit ball B.*

(1) *There exists a star body $\tilde{\mathbb{I}}_\mu B \subset V$ such that*

$$\tilde{A}_\mu(S) = n\tilde{V}(S[n-1], \tilde{\mathbb{I}}_\mu B), \quad S \in \mathcal{S}^\infty(V).$$

Here $\tilde{V}$ denotes the dual mixed volume [4]*. We call $\tilde{\mathbb{I}}_\mu B$ the* dual isoperimetrix.

(2) *Dual isoperimetric problem: Among all smooth star bodies of a given volume, a dilate of the dual isoperimetrix has* maximal *surface area.*

(3) *Suppose that the dual definition of volume μ^* is convex. Then the usual isoperimetrix for the dual definition of volume μ^* and the dual isoperimetrix are related by*

$$\tilde{\mathbb{I}}_\mu B = (\mathbb{I}_{\mu^*} B^\circ)^\circ .$$

Corollary. *The dual isoperimetrix for Busemann's definition of volume μ^b is*

$$\tilde{\mathbb{I}}_{\mu^b}(B) = \omega_{n-1}\Pi^\circ(B).$$

while for the Holmes-Thompson volume μ^{ht}, we have

$$\tilde{\mathbb{I}}_{\mu^{ht}}(B) = \frac{1}{\omega_{n-1}} I(B^\circ).$$

Our second main theorem is an affinely invariant inequality.

Theorem (Surface area of the unit sphere)**.** *Let μ^b be Busemann's definition of volume, μ^{ht} the Holmes-Thompson definition of volume. Let (V, B) be an n-dimensional normed space. Then*

$$\sqrt{n}\omega_n < \tilde{A}_{\mu^{ht}}(B) \leq \tilde{A}_{\mu^b}(B) \leq n\omega_n, \tag{1}$$

Equality on the right hand side is attained precisely for centered ellipsoids.

The upper bound of this theorem was conjectured (in the two-dimensional case) by Faifman [3], who gave the non-optimal upper bound of 8. He also gave a lower bound of 4 and conjectured that $8 \log 2$ is the optimal lower bound. Our lower bound is $\sqrt{2}\pi \approx 4.4$, which is not optimal.

As a corollary, we prove an upper bound for the quotient girth. Recall that the *girth* of a normed space is the length of the shortest symmetric curve on the unit sphere, measured with the Finsler metric induced by the norm. Analogously, the quotient girth is the length of the shortest symmetric curve on the unit sphere, measured with the quotient Finsler metric.

Corollary. *In any dimension, the quotient girth is bounded from above by 2π, with equality precisely for ellipsoids.*

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Convergence in shape of Steiner symmetrizations

Gabriele Bianchi
(joint work with A. Burchard, P. Gronchi and A. Volčič)

Steiner symmetrization is often used to identify the ball as the solution to geometric optimization problems. Starting from any given body, one can find sequences of iterated Steiner symmetrals that converge to the centered ball of the same volume as the initial body. If the objective functional improves along the sequence, the ball must be optimal.

Recently, several authors have studied how a sequence of Steiner symmetrizations can fail to converge to the ball. This may happen, even if the sequence of directions is dense in S^{n-1}. Steiner symmetrizations of a convex body along any dense sequence of directions can be made to converge or diverge just by reordering [3], and any given sequence of Steiner symmetrizations (convergent or not) can be realized as a subsequence of a non-convergent sequence [4, Proposition 5.2].

In contrast, a sequence of Steiner symmetrizations that uses only finitely many distinct directions always converges [6]. The limit may be symmetric under all rotations or under a non-trivial subgroup, depending on the algebraic properties of those directions that appear infinitely often.

A number of authors have studied Steiner symmetrizations along random sequences of directions. If the directions are chosen independently, uniformly at random on the unit sphere, then the corresponding sequence of Steiner symmetrals converges almost surely to the ball simultaneously for all choices of the initial set [8, 9, 10]. Others have investigated the rate of convergence of random and non-random sequences [2, 4, 5, 7].

In this talk we address several questions that were raised in these recent papers. The examples of non-convergence presented there use sequences of Steiner symmetrizations along directions where the differences between successive angles are square summable. Our main result says that such sequences will converge if the Steiner symmetrizations are followed by suitable isometries. We call this *convergence in shape.*

Theorem 1 ([1])**.** *Let (u_m) be a sequence in S^{n-1} with $u_{m-1}\cdot u_m = \cos\alpha_m$, where (α_m) is a sequence in $(0,\pi/2)$ that satisfies $\sum_{m=1}^{\infty}\alpha_m^2<\infty$. There exists a sequence of rotations $(\mathcal{R}_m)$ such that for every non-empty compact set $K\subset\mathbb{R}^n$, the rotated symmetrals*

$$K_m=\mathcal{R}_m\mathcal{S}_{u_m}\dots\mathcal{S}_{u_1}K \tag{1}$$

converge in Hausdorff distance and in symmetric difference to a compact set L.

The proof of the theorem poses two technical challenges: to show convergence of a sequence of symmetrals to an unknown limit, rather than a ball; and to show convergence in Hausdorff distance for an arbitrary compact initial set. This is more delicate than convergence in symmetric difference, because Steiner symmetrization is not continuous and Lebesgue measure is only upper semicontinuous on compact sets.

Next lemma may be of independent interest. The symbols λ_n, Δ and L_δ denote respectively Lebesgue measure, symmetric difference and the Minkowski sum of L and of a ball centered at o and of radius δ.

Lemma 2 ([1]). *Let L and K_m, $m \geq 1$, be non-empty compact sets.*

(1) *The sequence (K_m) converges in Hausdorff distance to L if and only if*

$$\lim_{m\to\infty} \lambda_n((K_m)_\delta \triangle L_\delta) = 0$$

for each $\delta > 0$.

(2) *If (K_m) converges in Hausdorff distance to L and each K_m is obtained from a compact set K via finitely many Steiner symmetrizations and Euclidean isometries, then*

$$\lim_{m\to\infty} \lambda_n(K_m \triangle L) = 0\,.$$

In particular, $\lambda_n(L) = \lambda_n(K)$.

To address the geometric problem of identifying the limits of convergent subsequences, we use functionals of the form

$$\mathcal{I}_p(K) = \int_{\mathbb{R}^n} \phi(|x-p|)\,\psi(\mathrm{dist}(x,K))\,dx\,,$$

where K is a compact set, p a point in $\mathbb{R}^n$, the function ϕ is increasing, and ψ is decreasing. Then $\mathcal{I}_p(K)$ decreases under simultaneous Steiner symmetrization of K and p. Note that by setting $p = o$, $\phi(t) = t^2$ for all t, $\psi(0) = 1$, and $\psi(t) = 0$ for $t > 0$, we recover the classical inequality for the moment of inertia.

We consider the special case where ϕ and ψ are the characteristic functions of $[r, \infty)$ and $[0, \delta]$, respectively. Lemma 3.2 in [1] implies that for every pair of strictly monotone functions ϕ and ψ, the functional decreases strictly unless K and p agree with their Steiner symmetrals up to a common translation. By allowing $p \neq o$, we obtain information about the intersection of the limiting shape with a family of non-centered balls and half-spaces. Lemma 3.3 in [1] implies that these intersections uniquely determine the shape.

The previous lemmas are also useful for establishing convergence of Steiner symmetrals in Hausdorff distance in other situations, without the customary convexity assumption on the initial set. We illustrate this with two more examples.

We consider Steiner symmetrization in the plane along non-random sequences of directions that are uniformly distributed (in the sense of Weyl) on S^1, a property more restrictive than being dense. Theorem 5.1 in [1] shows that a sequence of Steiner symmetrals along a Kronecker sequence of direction always converges to a ball. In the opposite direction, we give examples where convergence to a ball fails for certain uniformly distributed sequences.

Finally, we strengthen a recent result of Klain [6] on Steiner symmetrization along sequences chosen from a finite set of directions. Klain proves that when K is a convex body the sequence of Steiner symmetrals always converges. We extend this result to compact sets.

Theorem 3 ([1]). *Let (u_m) be a sequence of vectors chosen from a finite set $F = \{v_1, \dots, v_k\} \subset S^{n-1}$. Then, for every compact set $K \subset \mathbb{R}^n$, the symmetrals*

$$K_m = \mathcal{S}_{u_m} \dots \mathcal{S}_{u_1} K$$

converge in Hausdorff distance and in symmetric difference to a compact set L. Furthermore, L is symmetric under reflection in each of the directions $v \in F$ that appear in the sequence infinitely often.

We conclude with two open problems. Here C^* denotes the centered ball of the same volume as C.

Problem 4. *Do iterated Steiner symmetrals $\mathcal{S}_{u_m} \dots \mathcal{S}_{u_1} K$ always converge in shape, without any assumptions on the sequence of directions?*

Problem 5. *Assume that a sequence of directions (u_m) is such that $(\mathcal{S}_{u_m} \dots \mathcal{S}_{u_1} C)$ converges to C^* for each convex body C. Is it true that $(\mathcal{S}_{u_m} \dots \mathcal{S}_{u_1} K)$ converges to K^* for each compact set K?*

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Towards a logarithmic Brunn-Minkowski theory

Károly J. Böröczky

(joint work with Erwin Lutwak, Deane Yang, Gaoyong Zhang)

The fundamental Brunn-Minkowski inequality for convex bodies (compact convex subsets with nonempty interiors) states that for convex bodies K, C in Euclidean n-space, $\mathbb{R}^n$, and for $\lambda \in (0,1)$, the volume of the bodies and of their Minkowski combination $(1-\lambda)K + \lambda C = \{(1-\lambda)x + \lambda y : x \in K \text{ and } y \in C\}$, are related by

$$V((1-\lambda)K + \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda}, \tag{1}$$

with equality if and only if K and C are homothetic. The Brunn-Minkowski inequality exposes the crucial log-concavity property of the volume functional (see Gardner [6], Gruber [7] or Schneider [16] for in-depth discussion).

We write h_K to denote the support function of K in $\mathbb{R}^n$, and S_K to denote the surface area measure of K on S^{n-1}. An equivalent formulation of the Brunn-Minkowski inequality is the Minkowski inequality for mixed volumes, which we write now in the form

$$\text{if } V(K)=V(C), \text{ then } \int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_k\, dS_K, \tag{2}$$

with equality if and only if K and L are translates.

According to the solution of the Minkowski problem, a non-trivial Borel measure μ is the surface area measure of some convex body K if and only if

$$\begin{aligned} \int_{S^{n-1}} u\, d\mu(u) &= o, \quad \text{and} \\ \mu(L\cap S^{n-1}) &< \mu(S^{n-1}) \quad \text{for any linear } (n-1)\text{-subspace } L. \end{aligned} \tag{3}$$

The solution is homothetic to a convex body minimizing $\int_{S^{n-1}} h_C\, d\mu$ under the condition $V(C)=1$. It follows from the equality conditions for the Minkowski inequality (2) that $S_K=S_C$ implies that K and L are translates. We observe that if μ is even, then the only condition is (3) on μ, and there is unique o-symmetric solution.

We note that an L_p version of the Brunn-Minkowski theory has been developed by Lutwak, Yang and Zhang, where $p=1$ is the classical case above. Here we discuss only the case of o-symmetric convex bodies. The L_p surface area measure is defined by $dS_{K,p}=h_K^{1-p}dS_K$ for $p>0$ by Lutwak [12], and the even L_p Minkowski problem is solved by Lutwak, Yang and Zhang [13]. For $p>0$ and $\lambda\in(0,1)$, the L_p Minkowski combination is defined by Firey in the 1960's (see [12]) by

$$(1-\lambda)K+_p\lambda C=\{x\in\mathbb{R}^n : \langle x,u\rangle^p \leq (1-\lambda)h_K(u)^p+\lambda h_C(u)^p\ \forall u\in S^{n-1}\}.$$

For $p>1$, the L_p Brunn-Minkowski inequality follows from the classical one.

Recently, Böröczky, Lutwak, Yang and Zhang considered an L_0, or logarithmic Brunn-Minkowski theory in a series of papers. The L_0 surface area measure of an o-symmetric convex body K is the cone-volume measure V_K on S^{n-1}, which is defined by $dV_K=(h_K/n)dS_K$. Its study was initiated by Gromov and Milman [8] (see for example Naor [14], and Paouris and Werner [15] for recent results).

THEOREM 1 (Even logarithmic Minkowski problem, BLYZ [3])**.** *For an even Borel measure μ on S^{n-1}, $\mu=V_K$ for some o-symmetric convex body K iff*

(1) $\mu(L\cap S^{n-1})\leq \frac{\dim L}{n}\,\mu(S^{n-1})$ *for any linear subspace* $L\neq\{o\},\mathbb{R}^n$
(2) *in the case of equality for some L, we have* $\operatorname{supp}\mu\subset L\cup L'$ *for some complementary L'.*

The solution is obtained via minimizing $\int_{S^{n-1}}\log h_C\, d\mu$ assuming $V(C)=1$. In the case of polytopes, the planar case of Theorem 1 was obtained by Stancu [17] and [18], and the necessity of (1) is proved independently by Henk, Schürmann and Wills [10] and He, Leng and Li [9]. The properties in Theorem 1 are linked to an isotropic position of a measure.

THEOREM 2 (BLYZ). *For a non-trivial Borel measure μ on S^{n-1}, (1) and (2) in Theorem 1 hold iff there exists $A \in \mathrm{GL}(n)$ such that*

$$\int_{S^{n-1}} \frac{Au}{\|Au\|} \otimes \frac{Au}{\|Au\|}\, d\mu(u) = \tfrac{1}{n}\, \mathrm{Id}_n.$$

Theorem 2 for discrete measures is due to Carlen, Lieb and Loss [4] in their study of the equality case of the Brascamp-Lieb inequality. The sufficiency when strict inequality holds in (1) for any linear subspace L is due to Klartag [11].

Concerning uniqueness of the cone volume measure, we conjecture that $V_K = V_C$ for o-symmetric convex bodies K and C with $V(K) = V(C)$ iff K and C have dilated direct summands; namely, $K = K_1 \oplus \ldots \oplus K_m$ and $C = C_1 \oplus \ldots \oplus C_m$ with $K_i = \lambda_i C_i$ for $\lambda_1, \ldots, \lambda_m > 0$. This conjecture would follow from

Conjecture 1 (Logarithmic Minkowski inequality, BLYZ [2]). *If K and C are o-symmetric convex bodies with $V(K) = V(C)$, then*

$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K,$$

with equality iff K and C have dilated direct summands.

To logarithmic analogue of the Minkowski linear combination for $\lambda \in (0,1)$ is

$$(1-\lambda)K +_0 \lambda C = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u)^{1-\lambda} h_C(u)^\lambda \ \forall u \in S^{n-1}\}.$$

The following conjecture is equivalent with Conjecture 1.

Conjecture 2 (Logarithmic Brunn-Minkowski inequality, BLYZ [2]). *If $\lambda \in (0,1)$, and K, C are o-symmetric convex bodies, then*

$$V((1-\lambda)K +_0 \lambda C) \geq V(K)^{1-\lambda} V(C)^\lambda,$$

with equality iff K and C have dilated direct summands.

Conjecture 2 is naturally stronger than the Brunn-Minkowski inequality for o-symmetric convex bodies. It is proved in the plane by [2], and for unconditional convex bodies (without the case of equality) by Bollobás, I. Leader [1]. Conjecture 2 (without the case of equality) also makes sense if the Lebesgue measure in $\mathbb{R}^n$ is replaced by any even log-concave measure. For unconditional convex bodies, this generalized conjecture was proved by Cordero-Erausquin, Fradelizi and Maurey [5]. If the log-concave measure is the Gaussian measure, and K and C are dilates, then the conjectured inequality was popularized as (B)-conjecture, and proved in [5].

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Integer Programming via Thin Lattice Projections

Daniel Dadush

(joint work with Christopher Peikert, Santosh Vempala)

Given a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, the Integer Programming Problem (IP) is to decide whether $P \cap \mathbb{Z}^n = \emptyset$, where $\mathbb{Z}^n$ is the n-dimensional integer lattice.

The IP problem has found many applications in Computer Science, and is considered a fundamental problem in Operations Research. The first theoretically efficient algorithm for IP was developed by Lenstra [Len83], who gave a $2^{O(n^3)}$ time and poly(n) space algorithm for the problem based on ideas from the geometry of numbers. Lenstra's result was greatly improved by Kannan [Kan87], who reduced the complexity to $2^{O(n)}n^{2.5n}$, and most recently by Hildebrand and Köppe [HK10], who further improved the complexity to $2^{O(n)}n^{2n}$.

Improving on previous algorithms, we give a new $2^{O(n)}n^n$ time and 2^n space algorithm. Our main technical contribution is a new method for decomposing an IP into subproblems based on thin lattice projections. We also show that assuming a conjecture of Kannan and Lovász [KL88], our algorithmic strategy could yield a $2^{O(n)}(\log n)^n$ time algorithm for IP, providing a way to overcome the $n^{O(n)}$ barrier of previous approaches.

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The Busemann-Petty Problem in the Complex Hyperbolic Space

Susanna Dann

The Busemann-Petty problem asks the following question. Given two origin symmetric convex bodies K and L in $\mathbb{R}^n$ such that

$$\mathrm{Vol}_{n-1}(K \cap H) \leq \mathrm{Vol}_{n-1}(L \cap H)$$

for every hyperplane H in $\mathbb{R}^n$ containing the origin, does it follow that

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)?$$

The answer is affirmative for $n \leq 4$ and negative for $n \geq 5$. This problem, posed in 1956 in [1], was solved in the late 90's as a result of a sequence of papers [2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 12, 10], see [15], p. 3-5, for the history of the solution. Since then the Busemann-Petty problem was studied on other spaces as were its numerous generalizations.

In this talk we discuss the Busemann-Petty problem in the complex hyperbolic n-space. For $\xi \in \mathbb{C}^n$ with $|\xi| = 1$, denote by

$$H_\xi := \{z \in \mathbb{C}^n \,:\, (z, \xi) = \sum_{k=1}^{n} z_k \overline{\xi_k} = 0\}$$

the complex hyperplane through the origin perpendicular to ξ. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the mapping

$$(\xi_{11} + i\xi_{12}, \ldots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \,. \tag{1}$$

Under this mapping the hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \text{ and } \xi_\perp = (-\xi_{12}, \xi_{11}, \ldots, -\xi_{n2}, \xi_{n1}) \,.$$

A convex body K in $\mathbb{R}^{2n}$ is called *R_θ-invariant*, if for every $\theta \in [0, 2\pi]$ and every $\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), \ldots, R_\theta(\xi_{n1}, \xi_{n2})\|_K \,,$$

where R_θ stands for the counterclockwise rotation by an angle θ around the origin in $\mathbb{R}^2$.

An origin symmetric body K in $\mathbb{H}^n_{\mathbb{C}}$ is called *convex* if under the mapping (1) it corresponds to an R_θ-invariant body in $\mathbb{R}^{2n}$ contained in the unit ball such that for any pair of points in $K \subset \mathbb{R}^{2n}$ the geodesic segment with respect to the Bergman metric on $\mathbb{H}^n_{\mathbb{C}}$ joining them also belongs to K. Bodies in $\mathbb{R}^{2n}$ contained in the unit ball and satisfying the latter condition will be called *h-convex.* While it is not true in general that a convex body contained in the unit ball is h-convex, one can dilate a convex body with strictly positive curvature to make it h-convex. We denote the volume element on $\mathbb{H}^n_{\mathbb{C}}$ by $d\mu_n$ and the volume of a body K in $\mathbb{R}^{2n}$ with respect to this volume element by $\mathrm{HVol}_{2n}(K)$.

Now the Busemann-Petty problem in $\mathbb{H}^n_{\mathbb{C}}$ can be posed as follows. Given two R_θ-invariant h-convex bodies K and L in $\mathbb{R}^{2n}$ such that

$$\mathrm{HVol}_{2n-2}(K\cap H_\xi) \le \mathrm{HVol}_{2n-2}(L\cap H_\xi)$$

for any element ξ of the unit sphere S^{2n-1} of $\mathbb{R}^{2n}$, does it follow that

$$\mathrm{HVol}_{2n}(K) \le \mathrm{HVol}_{2n}(L)?$$

Analytic solutions of the Busemann-Petty problem in different settings are based on establishing a connection between a certain distribution and the problem. In the following two propositions we describe the connection between the Busemann-Petty problem in $\mathbb{H}^n_{\mathbb{C}}$ and the distribution $\frac{\|x\|_K^{-2}}{1-\left(|x|\|x\|_K^{-1}\right)^2}$.

Proposition 1. *Let K be an R_θ-invariant star body in $\mathbb{R}^{2n}$, contained in the open unit ball, such that $\frac{\|x\|_K^{-2}}{1-\left(|x|\|x\|_K^{-1}\right)^2}$ is a positive definite distribution on $\mathbb{R}^{2n}$. And let L be an R_θ-invariant star body in $\mathbb{R}^{2n}$, contained in the open unit ball, so that*

$$\mathrm{HVol}_{2n-2}(K\cap H_\xi) \le \mathrm{HVol}_{2n-2}(L\cap H_\xi)$$

for every $\xi \in \mathrm{S}^{2n-1}$. Then

$$\mathrm{HVol}_{2n}(K) \le \mathrm{HVol}_{2n}(L)\,.$$

Proposition 2. *Suppose there is an infinitely smooth complex convex body K in B^n with strictly positive curvature so that $\frac{\|x\|_K^{-2}}{1-\left(|x|\|x\|_K^{-1}\right)^2}$ is not a positive definite distribution on $\mathbb{R}^{2n}$. Then one can perturb the body K to construct another infinitely smooth complex convex body L with strictly positive curvature, so that for every $\xi \in \mathrm{S}^{2n-1}$*

$$\mathrm{HVol}_{2n-2}(L\cap H_\xi) \le \mathrm{HVol}_{2n-2}(K\cap H_\xi)\,,$$

but

$$\mathrm{HVol}_{2n}(L) > \mathrm{HVol}_{2n}(K)\,.$$

Next we construct counterexamples to the Busemann-Petty problem in $\mathbb{H}^n_{\mathbb{C}}$ for $n \ge 4$.

Theorem 1. *There exist R_θ-invariant h-convex bodies K and L in $\mathbb{R}^{2n}$ with $n \geq 4$ satisfying*

$$\mathrm{HVol}_{2n-2}(K \cap H_\xi) < \mathrm{HVol}_{2n-2}(L \cap H_\xi)$$

for every $\xi \in \mathrm{S}^{2n-1}$, but

$$\mathrm{HVol}_{2n}(K) > \mathrm{HVol}_{2n}(L)\,.$$

In view of Theorem 1, we only have to find out what happens in dimensions one, two and three. The case of the complex dimension one is trivial. For the case of complex dimension two we prove:

Lemma 1. *For any R_θ-invariant star body K in $\mathbb{R}^4$, contained in the unit ball, the distribution $\frac{\|x\|_K^{-2}}{1-\left(|x|\|x\|_K^{-1}\right)^2}$ is positive definite.*

And for complex dimension three we show:

Lemma 2. *There is an infinitely smooth R_θ-invariant convex body K in $\mathbb{R}^6$ contained in the unit ball for which the distribution $\frac{\|x\|_K^{-2}}{1-\left(|x|\|x\|_K^{-1}\right)^2}$ is not positive definite.*

Combining the above result we conclude:

Theorem 2. *The answer to the Busemann-Petty problem in the complex hyperbolic n-space, $\mathbb{H}^n_{\mathbb{C}}$, is affirmative for $n \leq 2$ and negative for $n \geq 3$.*

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On the concavity of entropy power in the Brunn-Minkowski theory

Matthieu Fradelizi
(joint work with Arnaud Marsiglietti)

Elaborating on the similarity between the entropy power inequality and the Brunn-Minkowski inequality, Costa and Cover conjectured in [4] the $\frac{1}{n}$-concavity of the (outer) parallel volume of measurable sets as an analogue of the concavity of entropy power. We investigate this conjecture and study its relationship with known geometric inequalities.

Costa and Cover [4] noticed the similarity between the entropy power and the Brunn-Minkowski inequalities: for every independent random vectors X, Y in $\mathbb{R}^n$, with finite entropy and for every compact sets A and B in $\mathbb{R}^n$ one has

$$N(X+Y) \geq N(X) + N(Y) \quad \text{and} \quad |A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}},$$

where

$$N(X) = \frac{1}{2\pi \mathrm{e}} \mathrm{e}^{\frac{2}{n} H(X)}$$

denotes the entropy power of X, $H(X) = -\int f \ln f$ is the entropy of X with density f and where $|\cdot|$ denote the n-dimensional Lebesgue measure. Applying the Brunn-Minkowski inequality to $B = \varepsilon B_2^n$ and letting ε tend to 0 one gets the classical isoperimetric inequality

$$\frac{|\partial A|}{|A|^{\frac{n-1}{n}}} \geq n|B_2^n|^{\frac{1}{n}} = \frac{|\partial B_2^n|}{|B_2^n|^{\frac{n-1}{n}}},$$

where the outer Minkowski surface area is defined by

$$|\partial A| = \lim_{\varepsilon \to 0} \frac{|A + \varepsilon B_2^n| - |A|}{\varepsilon},$$

whenever the limit exists. In the same way, Costa and Cover applied the entropy power inequality to $Y = \sqrt{\varepsilon} G$, where G is a standard Gaussian random vector (the $\sqrt{\varepsilon}$ comes from the homogeneity of entropy power $N(\sqrt{\varepsilon}X) = \varepsilon N(X)$) and by letting ε tend to 0, they obtained the following "isoperimetric entropy inequality"

$$N(X)I(X) \geq n, \quad \text{where} \quad I(X) = \int \frac{|\nabla f|^2}{f}$$

is the Fisher information of X, thanks to de Bruijn's identity

$$\frac{\mathrm{d}}{\mathrm{d}t} H(X + \sqrt{t}G) = \frac{1}{2} I(X + \sqrt{t}G),$$

which states that the Fisher information is the derivative of the entropy along the heat semi-group. Notice that this "isoperimetric entropy inequality" is equivalent to the Log-Sobolev inequality for the Gaussian measure, see [1] chapter 9.

This analogy between the results of the Information theory and the Brunn-Minkowski theory was later extended and further explained and unified through Young's inequality by Dembo [5] and later on by Dembo, Cover and Thomas [6]. Each of these theories deal with a fundamental inequality, the Brunn-Minkowski inequality for the Brunn-Minkowski theory and the entropy power inequality for the Information theory. The objects of each theories are fellows: to the compact sets in the Brunn-Minkowski theory correspond the random vectors in the Information theory, the Gaussian random vectors play the same role as the Euclidean balls, the entropy power N corresponds to the $1/n$ power of the volume $|\cdot|^{1/n}$ and, taking logarithm, the entropy H is the analogue of the logarithm of the volume $\log|\cdot|$. Hence one can conjecture that properties of one theory fit into the other theory.

Thus, Costa and Cover [4], as an analogue of the concavity of entropy power with added Gaussian noise, which states that

$$t \mapsto N(X + \sqrt{t}G)$$

is a concave function (see [3] and [8]), formulated the following conjecture:

Let A be a bounded measurable set in $\mathbb{R}^n$ then the function $t \mapsto |A + tB_2^n|^{\frac{1}{n}}$ is concave on $\mathbb{R}_+$.

They also showed using the Brunn-Minkowski inequality that this conjecture holds true if A is a convex set.

In this talk, we investigate this conjecture and study its relationship with known geometric inequalities. We prove that the conjecture holds true in dimension 1 for all measurable sets and in dimension 2 for connected sets. In dimension $n \geq 3$, we establish that the connectivity hypothesis is not enough and that the conjecture is false in general. We then discuss additional hypotheses which ensure its validity: we conjecture that it holds true for sufficiently large t and we establish it for special sets A.

Notice that Guleryuz, Lutwak, Yang and Zhang [7] also pursued these analogies between the two theories and more recently, Bobkov and Madiman [2] established an analogue in Information theory of the Milman's reverse Brunn-Minkowski inequality.

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Addition

Richard J. Gardner
(joint work with Daniel Hug, Wolfgang Weil, Lukas Parapatits, Franz E. Schuster)

The talk gave an overview of some of the results in three papers, representing joint work with Daniel Hug and Wolfgang Weil of the Karlsruhe Institute of Technology [1], [2] and Lukas Parapatits and Franz Schuster of the Vienna University of Technology [3].

In [1], an investigation is launched into the fundamental characteristics of operations between sets, with a focus on compact convex sets and star sets (compact sets star-shaped with respect to the origin) in $\mathbb{R}^n$. It is proved that if $n \geq 2$, with three trivial exceptions, an operation between origin-symmetric compact convex sets is continuous in the Hausdorff metric, $GL(n)$ covariant, and associative if and only if it is L_p addition for some $1 \leq p \leq \infty$. It is also demonstrated that if $n \geq 2$, an operation $*$ between arbitrary compact convex sets is continuous in the Hausdorff metric, $GL(n)$ covariant, and has the identity property (i.e., $K*\{o\} = K = \{o\}*K$ for all compact convex sets K, where o denotes the origin) if and only if it is Minkowski addition. These results are obtained via characterizations of operations that are projection covariant, meaning that the operation can take place before or after projection onto subspaces, with the same effect. For example, projection covariant operations $*$ between origin-symmetric compact convex sets are precisely those given by the formula

$$h_{K*L}(x) = h_M\left(h_K(x), h_L(x)\right),$$

for all $x \in \mathbb{R}^n$ and some 1-unconditional compact convex set M in $\mathbb{R}^2$. This turns out to be equivalent to

$$K * L = K \oplus_M L,$$

where the operation $\oplus_M$, called M-addition and first introduced by V. Protasov, is defined by

$$K \oplus_M L = \{ax + by : x \in K, y \in L, (a, b) \in M\}.$$

In [3], the goal is to obtain a characterization of Blaschke addition $\sharp$. The main result is that if $n \geq 3$, then an operation $*$ between origin-symmetric convex bodies in $\mathbb{R}^n$ is uniformly continuous in the Lévy-Prokhorov metric (i.e., the usual Lévy-Prokhorov metric between the surface area measures) and $GL(n)$ covariant if and only if $K * L = aK \sharp bL$, for some $a, b \geq 0$ and all origin-symmetric convex bodies K and L. Along the way, it is shown that if $n \geq 3$, then an operation $*$ between origin-symmetric zonoids in $\mathbb{R}^n$ is continuous in the Hausdorff metric and $GL(n)$ covariant if and only if $K * L = aK + bL$, for some $a, b \geq 0$ and all origin-symmetric zonoids K and L.

A full set of examples is provided in [1] and [3] showing that none of the various properties assumed in these results can be omitted.

The Orlicz-Brunn-Minkowski theory, introduced by Lutwak, Yang, and Zhang, is a new extension of the classical Brunn-Minkowski theory. It represents a generalization of the L_p-Brunn-Minkowski theory, analogous to the way that Orlicz spaces generalize L_p spaces. For a convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(1) > 0$, an appropriate way of combining arbitrary sets in $\mathbb{R}^n$ is introduced in [2]. This new operation, called Orlicz addition, has several desirable properties, but is not associative unless $\varphi(t) = t^p$ for some $p \geq 1$. It is shown that Orlicz addition is very closely related to M-addition. Inequalities of the Brunn-Minkowski type are obtained, both for M-addition and Orlicz addition. The new Orlicz-Brunn-Minkowski inequality implies the L_p-Brunn-Minkowski inequality. An Orlicz-Minkowski inequality is also obtained, involving a new Orlicz mixed volume equal to the first variation of volume with respect to Orlicz addition, that generalizes the L_p-Minkowski inequality. This has connections with the conjectured log-Brunn-Minkowski inequality of Lutwak, Yang, and Zhang, and in fact these two inequalities together are shown to split the classical Brunn-Minkowski inequality.

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Geometry of the L_q-centroid bodies of an isotropic log-concave measure

Apostolos Giannopoulos
(joint work with Pantelis Stavrakakis, Antonis Tsolomitis, Beatrice-Helen Vritsiou)

Given a convex body K of volume 1 or a log-concave probability measure μ on $\mathbb{R}^n$, we define the L_q-centroid bodies $Z_q(K)$ or $Z_q(\mu)$, $q \in (0, +\infty)$, through their support function $h_{Z_q(K)}$ or $h_{Z_q(\mu)}$, which is defined as follows: for every $y \in \mathbb{R}^n$,

$$(1) \quad h_{Z_q(K)}(y) := \|\langle \cdot, y\rangle\|_{L_q(K)} = \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q},$$

$$h_{Z_q(\mu)}(y) := \|\langle \cdot, y\rangle\|_{L_q(\mu)} = \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^q d\mu(x)\right)^{1/q}.$$

These bodies then incorporate information about the distribution of linear functionals with respect to the uniform measure on K or with respect to the probability measure μ. The L_q-centroid bodies were introduced, under a different normalization, by Lutwak, Yang and Zhang in [6], while in [7] for the first time, and in [8] later on, Paouris used geometric properties of them to acquire detailed information about the distribution of the Euclidean norm with respect to the uniform measure on isotropic convex bodies. An asymptotic theory for the L_q-centroid bodies has since been developed in the context of isotropic measures and it seems to advance in parallel with all recent developments in the area.

Recall that a convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and if its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that

$$(2) \qquad \int_K \langle x, \theta\rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1}. Similarly, a log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if its barycenter is at the origin and if its inertia matrix is the identity matrix; in that case the isotropic constant of the measure is defined as

$$(3) \qquad L_\mu := \sup_{x\in\mathbb{R}^n} \left(f_\mu(x)\right)^{1/n},$$

where f_μ is the density of μ with respect to the Lebesgue measure. One very well-known open question in the theory of isotropic measures is the hyperplane conjecture, which asks if there exists an absolute constant $C > 0$ such that

$$(4) \qquad L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leq C$$

for all $n \geq 1$. Bourgain proved in [1] that $L_n \leq c\sqrt[4]{n}\log n$, while Klartag [5] obtained the bound $L_n \leq c\sqrt[4]{n}$.

A motivation for our work is a recent reduction [4] of the hyperplane conjecture to the study of geometric properties of the L_q-centroid bodies, and in particular to the study of the parameter

$$I_1\big(K, Z_q^\circ(K)\big) := \int_K \|x\|_{Z_q^\circ(K)}(x)dx = \int_K h_{Z_q(K)}(x)dx. \tag{5}$$

The main result of [4] is, in a sense, a continuation of Bourgain's approach to the problem and, roughly speaking, can be formulated as follows: Given $q \geq 2$ and $\frac{1}{2} \leq s \leq 1$, an upper bound of the form $I_1(K, Z_q^\circ(K)) \leq C_1 q^s \sqrt{n} L_K^2$ for all isotropic convex bodies K in $\mathbb{R}^n$ leads to the estimate

$$L_n \leq \frac{C_2 \sqrt[4]{n} \log n}{q^{\frac{1-s}{2}}}. \tag{6}$$

Bourgain's estimate may be recovered by choosing $q = 2$, however, clarifying the behaviour of $I_1(K, Z_q^\circ(K))$ might allow one to use much larger values of q. This behaviour is most naturally related to the geometry of the bodies $Z_q(K)$, and especially how this geometry is affected by or affects the geometry of the body K. This is not yet fully understood and, in view of (6), we believe that its deeper study would be very useful.

In the range $2 \leq q \leq \sqrt{n}$ some basic global parameters of the bodies $Z_q(\mu)$ are completely determined: the volume radius and the mean width of $Z_q(\mu)$ are of the same order $\sqrt{q}$. We provide new information on the local structure of $Z_q(\mu)$, which in turn has some interesting consequences. Our first main result concerns proportional projections of the centroid bodies.

Theorem 1. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Fix $1 \leq \alpha < 2$. For every $0 < \varepsilon < 1$ and any $q \leq \sqrt{\varepsilon n}$ there are $k \geq (1-\varepsilon)n$ and $F \in G_{n,k}$ such that*

$$P_F\big(Z_q(\mu)\big) \supseteq c(2-\alpha)\varepsilon^{\frac{1}{2}+\frac{2}{\alpha}} \sqrt{q}\, B_F, \tag{7}$$

where $c > 0$ is an absolute constant (independent of α, ε, the measure μ, q or n). Moreover, for any $2 \leq q \leq \varepsilon n$ there are $k \geq (1-\varepsilon)n$ and $F \in G_{n,k}$ such that

$$P_F\big(Z_q(\mu)\big) \supseteq \frac{c_1(2-\alpha)\varepsilon^{\frac{1}{2}+\frac{2}{\alpha}}}{L_{\varepsilon n}} \sqrt{q}\, B_F \supseteq \frac{c_2(2-\alpha)\varepsilon^{\frac{1}{4}+\frac{2}{\alpha}}}{\sqrt[4]{n}} \sqrt{q}\, B_F, \tag{8}$$

where $c_1, c_2 > 0$ are absolute constants.

Let us mention that the dual result is a direct consequence of the low M^*-estimate, since the mean width of $Z_q(\mu)$ is known to be of the order of $\sqrt{q}$: if $2 \leq q \leq \sqrt{n}$ and if $\varepsilon \in (0,1)$ and $k = (1-\varepsilon)n$, then a subspace $F \in G_{n,k}$ satisfies

$$P_F(Z_q^\circ(\mu)) \supseteq \frac{c_1\sqrt{\varepsilon}}{\sqrt{q}} B_F \tag{9}$$

with probability greater than $1 - \exp(-c_2 \varepsilon n)$, where $c_1, c_2 > 0$ are absolute constants.

Next. we discuss bounds for the covering numbers of a Euclidean ball by $Z_q(\mu)$. It was proved in [2] and [3] that if μ is an isotropic log-concave measure on $\mathbb{R}^n$ then, for any $1 \le q \le n$ and $t \ge 1$,

$$\log N\big(Z_q(\mu), c_1 t\sqrt{q}B_2^n\big) \le c_2\frac{n}{t^2} + c_3\frac{\sqrt{qn}}{t}, \tag{10}$$

where $c_1, c_2, c_3 > 0$ are absolute constants. Using Theorem 1 we obtain regular entropy estimates for the dual covering numbers.

Theorem 2. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. Assume $1 \le \alpha < 2$. Then, for any $q \le \sqrt{n}$ and any*

$$1 \le t \le \min\big\{\sqrt{q},\ c_1(2-\alpha)^{-1}(n/q^2)^{\frac{\alpha+4}{2\alpha}}\big\}$$

we have

$$\log N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big) \le c(\alpha)\frac{n}{t^{\frac{2\alpha}{\alpha+4}}}\max\left\{\log\frac{2q}{t^2},\ \log\frac{1}{(2-\alpha)t}\right\}, \tag{11}$$

where $c(\alpha) \le C(2-\alpha)^{-2/3}$ and c_1, C are absolute constants. Moreover, for any $2 \le q \le n$ and any

$$1 \le t \le \min\left\{\sqrt{q},\ c_2(2-\alpha)^{-1}L_n\left(\frac{n}{q}\right)^{\frac{\alpha+4}{2\alpha}}\right\}$$

we have

$$\log N\big(\sqrt{q}B_2^n, tZ_q(\mu)\big) \le c(\alpha)L_n^{\frac{2\alpha}{\alpha+4}}\frac{n}{t^{\frac{2\alpha}{\alpha+4}}}\max\left\{\log\frac{2q}{t^2},\ \log\frac{L_n}{(2-\alpha)t}\right\}, \tag{12}$$

where $c(\alpha)$ is as above and c_2 is an absolute constant.

Note that, since $Z_q(\mu) \supseteq B_2^n$, we are interested in bounds for the above covering numbers when t is in the interval $[1, \sqrt{q}]$. An analysis of the restrictions in Theorem 2 shows that, given any $q \le n^{3/7}$, (11) holds true with any t in the "interesting" interval, while the same is true for (12) as long as $q \le \sqrt{L_n}n^{3/4}$. Although all these estimates are most probably not optimal, we can still conclude that $Z_q(\mu)$, with $q \le n^{3/7}$ or $q \le \sqrt{L_n}n^{3/4}$, is a β-regular convex body for some concrete positive value of β. As a consequence of this fact we get an upper bound for the parameter

$$M\big(Z_q(\mu)\big) = \int_{S^{n-1}} \|x\|_{Z_q(\mu)}\, d\sigma(x).$$

Recall that the dual Sudakov inequality of Pajor and Tomczak-Jaegermann provides 2-regular entropy estimates for the numbers $N(B_2^n, tC)$ in terms of $M(C)$, namely it shows that

$$\log N(B_2^n, tC) \le cn\left(\frac{M(C)}{t}\right)^2$$

for every $t \ge 1$. We use in a converse manner the entropy estimates of Theorem 2 to obtain non-trivial upper bounds for $M(Z_q(\mu))$.

Theorem 3. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $1 \leq q \leq n^{3/7}$,*

$$M\big(Z_q(\mu)\big) \leq C\frac{(\log q)^{5/6}}{\sqrt[6]{q}}. \tag{13}$$

Moreover, for every q such that $L_n^2 \log^2 q \leq q \leq \sqrt{L_n}\, n^{3/4}$,

$$M\big(Z_q(\mu)\big) \leq C\frac{\sqrt[3]{L_n}(\log q)^{5/6}}{\sqrt[6]{q}}. \tag{14}$$

Observe now that, if K is an isotropic convex body in $\mathbb{R}^n$ with isotropic constant L_K, then the measure μ_K with density $f_{\mu_K}(x) := L_K^n \mathbf{1}_{K/L_K}(x)$ is isotropic and, for every $q > 0$, it holds that $Z_q(K) = L_K Z_q(\mu_K)$. Using also the fact that $M(K) \leq M(Z_q(K))$ for every symmetric convex body K and every $q > 0$, we can use the above bounds for $M(Z_q(\mu_K))$ to obtain an upper bound for $M(K)$ in the isotropic case.

Theorem 4. *Let K be a symmetric isotropic convex body in $\mathbb{R}^n$. Then,*

$$M(K) \leq C\frac{\sqrt[4]{L_n}(\log n)^{5/6}}{L_K \sqrt[8]{n}}.$$

This is a question that until recently had not attracted much attention. Valettas, using a slightly different approach, has shown that

$$M(K) \leq \frac{C(\log n)^{1/3}}{\sqrt[12]{n} L_K}$$

for every symmetric isotropic convex body K in $\mathbb{R}^n$, where $C > 0$ is an absolute constant. Note that, on the other hand, there are many approaches concerning the corresponding question about the mean width that give the best currently known estimate:

$$w(K) \leq Cn^{3/4} L_K$$

for every isotropic convex body K in $\mathbb{R}^n$. Nevertheless, this problem as well remains open (for a discussion about it, see [3] and the references therein).

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On the roots of the Wills functional

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(joint work with Jesús Yepes Nicolás)

For convex bodies $K, E \in \mathcal{K}^n$ (compact and convex sets in $\mathbb{R}^n$) and a non-negative real number λ, the well-known *Steiner formula* states that the volume of the Minkowski addition $K + \lambda E$ can be expressed as a polynomial of degree (at most) n in the parameter λ,

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i;$$

here the coefficients $\mathrm{W}_i(K; E)$ are called the *relative quermassintegrals* of K with respect to E, and they are a special case of the more general defined *mixed volumes* (see e.g. [5, s. 5.1]). In particular, $\mathrm{W}_0(K; E) = \mathrm{vol}(K)$ and $\mathrm{W}_n(K; E) = \mathrm{vol}(E)$.

In [6] Wills introduced and studied the functional

$$W(\lambda K) = \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{W}_i(K; B_n)}{\kappa_i} \lambda^{n-i}$$

due to a possible relation with the lattice-point enumerator $G(K) = \#(K \cap \mathbb{Z}^n)$, and conjectured that $W(K)$ was an upper bound for $G(K)$. Here B_n denotes the n-dimensional unit ball and $\kappa_n = \mathrm{vol}(B_n)$. Although Hadwiger showed that Wills' conjecture was wrong, the Wills functional turned out to have many interesting properties. For instance, in [1] Hadwiger proved, among others, the following useful integral representation of $W(K)$:

$$W(K) = \int_{\mathbb{R}^n} e^{-\pi d(x,K)^2} dx, \tag{1}$$

where $d(x, K)$ denotes the Euclidean distance between $x \in \mathbb{R}^n$ and K. Recently, Kampf [2] has shown that an analogous formula remains true when the 'distance' $d_E(x, K)$, between $x \in \mathbb{R}^n$ and K, relative to a convex body E with $0 \in \mathrm{int}\, E$, is considered, i.e.

$$\int_{\mathbb{R}^n} e^{-\pi d_E(x,K)^2} dx = \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{W}_i(K; E)}{\kappa_i}.$$

Moreover, this functional can be defined in a more general setting replacing the function $e^{-\pi t^2}$ by another one $G(t)$ properly associated to a measure μ on the nonnegative real line $\mathbb{R}_{\geq 0}$:

$$\int_{\mathbb{R}^n} G\big(d_E(x, K)\big) dx \quad \text{with} \quad G(t) = \mu\big([t, +\infty)\big), \ t \geq 0.$$

Then it can be proved that, if the moments $m_i(\mu) = \int_0^{+\infty} t^i d\mu(t)$ exist and are finite, it holds

$$(2) \qquad W_\mu(K;E) = \int_{K+\operatorname{lin} E} G\big(d_E(x,K)\big)dx = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E) m_i(\mu).$$

Clearly the classical Wills functional (1) is obtained from (2) when $G(t) = e^{-\pi t^2}$ and $E = B_n$. Thus, the following question arises in a natural way: can the Steiner functional $\sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)$ be obtained as a particular case of the generalized Wills functional? We prove the following result:

Theorem 1. *Let $K, E \in \mathcal{K}^n$ with $0 \in \operatorname{relint} E$. Then*

$$\sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E) = \lim_{\sigma\to 0^+} \int_{K+\operatorname{lin} E} \left(\int_{d_E(x,K)}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-1)^2}{2\sigma^2}} dt \right) dx.$$

Moreover, such an expression for the Steiner functional, in which a non-discrete *measure μ on $\mathbb{R}_{\geq 0}$ is considered, is only possible through a 'pass to the limit' process.*

Motivated by the previous works of Henk, Hernández Cifre and Saorín [3, 4] on the roots of the Steiner polynomial, if we take the corresponding generalized Wills polynomial

$$g^{\mu}_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E) m_i(\mu) z^i$$

(cf. (2)) considered as a formal polynomial in a complex variable $z \in \mathbb{C}$, we are interested in studying the structure and properties of its roots; more precisely we ask whether there are *common* properties not depending on the measure μ. To this end, let $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$ be the set of complex numbers with non-negative imaginary part, and for any dimension $n \geq 2$, let

$$(3) \qquad \mathcal{R}^{\mu}_{W}(n) = \big\{ z \in \mathbb{C}^+ : g^{\mu}_{K;E}(z) = 0 \text{ for some } K, E \in \mathcal{K}^n, \dim(K+E) = n \big\}$$

be the set of all roots of all (non-trivial) generalized Wills polynomials in the upper half-plane. Moreover, let $f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E) z^i$ and let

$$\mathcal{R}(n) = \big\{ z \in \mathbb{C}^+ : f_{K;E}(z) = 0 \text{ for some } K, E \in \mathcal{K}^n, \dim(K+E) = n \big\}$$

be the set of all roots of all Steiner polynomials (in the upper half-plane). We prove the following results.

Theorem 2.

i) $\mathcal{R}^{\mu}_{W}(2) \subsetneq \big\{ z \in \mathbb{C}^+ : \operatorname{Re}(z) \leq 0 \big\}$.
ii) $\mathcal{R}^{\mu}_{W}(n)$ *is a convex cone containing the non-positive real axis.*
iii) $\mathcal{R}^{\mu}_{W}(n)$ *is closed.*
iv) $\mathcal{R}^{\mu}_{W}(n) \subsetneq \mathcal{R}^{\mu}_{W}(n+1)$ *for all $n \geq 2$.*
v) $\mathcal{R}^{\mu}_{W}(n) \supsetneq \mathcal{R}(n)$ *for all $n \geq 2$.*

We notice that some of these properties do not hold if discrete or signed measures are allowed in the definition of our functional (2).

Moreover, from the fact that the cone of roots of Steiner polynomials is always contained in the cone of roots of generalized Wills functionals with the same dimension (item (v) in Theorem 2), several additional properties can be obtained from the known results for the Steiner polynomial (see [4]):

Corollary 3.

i) $\mathbb{R}_{\leq 0} \subsetneq \mathcal{R}_W^{\mu}(2)$.

ii) *If* $n \geq 10$ *then* $\{z \in \mathbb{C}^+ : \mathrm{Re}(z) \leq 0\} \subsetneq \mathcal{R}_W^{\mu}(n)$.

iii) *Let* $\gamma \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$. *Then there exists* $n_\gamma \in \mathbb{N}$ *with* $\gamma \in \mathcal{R}_W^{\mu}(n)$ *for all* $n \geq n_\gamma$.

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Almost all shadow boundaries have finite measure

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(joint work with David Larman, Peter Mani)

If C is a convex body in $\mathbb{R}^n$ and X a k-dimensional linear subspace of $\mathbb{R}^n$, we denote by $\mathbf{S}(C, X)$ the shadow boundary of C over X which is defined as the collection of all points which belong to C and to one of its tangent $(n-k)$-flats orthogonal to X.

Definition 1. $\mathbf{S}(C, X) = \{p \in C : (p+X^{\perp}) \cap \mathrm{int}(C) = \emptyset\} = C \cap \pi_X^{-1}[\mathrm{rel\ bd}(\pi_X C)]$ *is the shadow boundary of* C *over* X.

Following up on a research problem of 1957, Klee asked if the boundary of a d-dimensional convex body could contain line segments in all directions [1]. In 1960 McMinn [2] answered this question by showing that:

The set D of directions of line segments lying on the boundary of a 3-dimensional convex body is contained in the union of the ranges of a countable family of Lipschitz functions each on the 1-dimensional closed unit ball B_1 to the surface of the 2-dimensional unit sphere S_2. By virtue of the Lipschitz nature of these functions, they possess total differentials (Lebesgue measure) almost everywhere and their ranges are compact and have finite one dimensional measure.

Besicovitch followed with a simplification of his proof in 1963 [3]. Finally Ewald, Larman and Rogers generalised the result to n dimensions in their 1970 publication [4]. Specifically, they proved:

Theorem (Ewald, Larman and Rogers [4])
If $1 \leq r \leq n-1$ and K a convex body in $\mathbb{E}^n$, the points $\pm G(F)$, corresponding to the r-flats F in $\mathbb{E}^n$ meeting the boundary of K in a set of linear dimension r, form a set on I_r^n of σ-finite $r(n-r-1)$-dimensional Hausdorff measure.

The shadow boundary of a convex body over a subspace X of $\mathbb{R}^n$ is the set of points of its boundary which project onto the boundary of its shadow on X. We call a shadow boundary *sharp* if its projection is injective.
From Ewald, Larman and Rogers' result we know that almost all shadow boundaries are sharp.

We have proved a further property of these sharp shadow boundaries which was first suggested at a workshop in Siegen in 1974 by Peter McMullen. He asked whether *sharp shadow boundaries also have finite "length"*.
This was answered in the affirmative by Peter Steenaerts in 1985 [6] for the cases where X is an l-dimensional subspace of $\mathbb{E}^n$ and $l = 1$ or $n-1$.
Our result generalises this to shadow boundaries over subspaces of any dimension l, where $1 \leq l \leq n$.

For simplicity, we shall only give the proof of this result for polytopes. The proof for general convex bodies requires rectifiability and other geometric measure theory concepts mostly taken from the works of Federer.
Notation:

- $\Gamma(k)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$.
- The measures $\gamma(k)$ on $\Gamma(k)$ is given by:
$$\gamma(k)[M] = \mathcal{O}_n\{r \in O(n) : r[\mathbb{R}^k] \in M\} \text{ where } M \subset \Gamma(k).$$
- $\mathscr{H}^n$ is the n-dimensional Hausdorff measure.
- $W_i(C)$ denotes the i^{th} Minkowski Quermass integral of the convex body C in $\mathbb{R}^n$. Quermass integrals are related to the concept of mixed volumes. See Chapter 4 in R. Schneider [7] for more information.

Definition 2. $\mathbf{P}(C,k) = \{X \in \Gamma(k) : \pi_X|_{\mathbf{S}(C,X)}$ *is injective*$\}$ *is the set of subsets* X *in* $\Gamma(k)$ *for which the shadow boundary* $\mathbf{S}(C,X)$ *is sharp.*

Theorem 1. *For any integer* k *with* $1 \leq k < n$ *there is a number* $\alpha(k,n) > 0$ *such that the equation*
$$\int \mathscr{H}^{k-1}\big(\mathbf{S}(P,X)\big)\, \mathrm{d}\gamma(k)[X] = \alpha(k,n) W_{n-k+1}(P)$$
holds for each n*-polytope* $P \subset \mathbb{R}^n$.

Proof. Consider the set $F(P)$ of all $(k-1)$-dimensional faces of the polytope P. Let $\alpha(P,G)$ be the exterior angle of P at $G \in F(P)$. The incidence function $\varepsilon_P : \Gamma(k) \times F(P) \to \{0,1\}$ is given by

$$\begin{aligned} \varepsilon_P(X,G) &= 1, \qquad \text{if } [\operatorname{aff} G + X] \cap \text{ relint } \big(P\big) = \emptyset \text{ and} \\ \varepsilon_P(X,G) &= 0, \qquad \text{otherwise.} \end{aligned}$$

Using Fubini's Theorem, the definition and properties of $\varepsilon_P(X,G)$ and the relation between the exterior angle $\alpha(P,G)$ and the Quermassintegral $W_{n-k+1}(P)$ [7], we establish

$$\begin{aligned} \int_{\Gamma(k)} \mathscr{H}^{k-1} & [\mathbf{S}(P,X)] \,\mathrm{d}\gamma(k)(X) \\ &= \int_{\mathbf{P}(P,k)} \mathscr{H}^{k-1}[\mathbf{S}(P,X)] \,\mathrm{d}\gamma(k)(X) \\ &= \sum_{G \in F(P)} \mathscr{H}^{k-1}(G) \int_{\Gamma(k)} \varepsilon_P(X,G) \,\mathrm{d}\gamma(k)[X] \\ &= \sum_{G \in F(P)} \mathscr{H}^{k-1}(G)\alpha(P,G) \\ &= a(k,n)W_{n-k+1}(P), \end{aligned}$$

where $a(k,n) > 0$ do not depend on the polytope P. □

For convex bodies: We go on to show:

Theorem 2. *If C is a convex body in $\mathbb{R}^n$ and l an integer with $1 \le l \le n-1$, then*

$$\int \mathscr{H}^{l-1}\big(\mathbf{S}(C,X)\big) \,\mathrm{d}\gamma(l) = \alpha(l,n)W_{n-l+1}(C)$$

holds.

Proof. (vague outline)

- use concept of rectifiability studied at length by Federer for sets and introduce the concept of a rectifiable map.
- We approximate our convex body by a sequence of polytopes.
- Define the maps $\varphi(C,l) : \Gamma(l) \to [0,\infty]$, for $l \in \{1,\dots,n-1\}$, by

$$\varphi(C,l)[X] = \mathscr{H}^{l-1}\big(\mathbf{S}(C,X)\big).$$

- Suffices to show:

$$\varphi(C,l)|_{\mathbf{P}} \text{ is lower semicontinuous and} \tag{1}$$

$$\varphi(C,l)[X] \le \liminf_{j\to\infty} \varphi(Q_j,l)[X], \text{ for every } X \in \mathbf{P}. \tag{2}$$

□

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Stability and separation in volume comparison problems

Alexander Koldobsky

A typical volume comparison problem asks whether inequalities

$$f_K(\xi) \le f_L(\xi), \qquad \forall \xi \in S^{n-1}$$

imply $|K| \le |L|$ for any K, L from a certain class of origin-symmetric convex bodies in $\mathbb{R}^n$, where f_K is a geometric characteristic of K. One can have in mind the section function $f_K(\xi) = |K \cap \xi^\perp|$ or the projection function $f_K(\xi) = |K|\xi^\perp|$, where $|K|$ stands for volume of proper dimension and $\xi^\perp$ is the central hyperplane perpendicular to $\xi \in S^{n-1}$.

In the case where the answer to a volume comparison problem is affirmative, one can ask the following stability question. Suppose that $\epsilon > 0$ and

$$f_K(\xi) \le f_L(\xi) + \epsilon, \qquad \forall \xi \in S^{n-1}. \tag{1}$$

Does there exist a constant c such that for every $\epsilon > 0$

$$|K|^{\frac{n-1}{n}} \le |L|^{\frac{n-1}{n}} + c\epsilon? \tag{2}$$

Stability results are related to hyperplane inequalities as follows. Suppose stability holds for both pairs K, L and L, K with the same constant c. Interchanging K and L in the stability result, one gets a volume difference inequality:

$$\left| |K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} \right| \le c \max_{\xi \in S^{n-1}} |f_K(\xi) - f_L(\xi)|. \tag{3}$$

Suppose now that the function f_L converges to zero uniformly with respect to ξ when $L = \beta B_2^n$ is a multiple of the unit Euclidean ball and $\beta \to 0$. Then, when $L = \beta B_2^n$ and $\beta \to 0$, the inequality (3) turns into a hyperplane inequality:

$$|K|^{\frac{n-1}{n}} \le c \max_{\xi \in S^{n-1}} f_K(\xi). \tag{4}$$

One can also consider a separation problem. Suppose that $\epsilon > 0$ and

$$f_K(\xi) \le f_L(\xi) - \epsilon, \qquad \forall \xi \in S^{n-1}. \tag{5}$$

Does there exist a constant c such that for every $\epsilon > 0$

$$|K|^{\frac{n-1}{n}} \le |L|^{\frac{n-1}{n}} - c\epsilon? \tag{6}$$

In the case where the answer is affirmative we get another kind of a volume difference inequality:

$$|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} \ge c \min_{\xi \in S^{n-1}} \left(f_L(\xi) - f_K(\xi)\right). \tag{7}$$

Again, if $f_{\beta B_2^n}$ converges to zero uniformly in ξ when $\beta \to 0$, we get the following version of a hyperplane inequality:

$$|L|^{\frac{n-1}{n}} \ge c \min_{\xi \in S^{n-1}} f_L(\xi). \tag{8}$$

This strategy was applied in [1] to several functions f_K including the section and projection functions. In [2] similar inequalities were proved for arbitrary measure with continuous density in place of volume. Sections of lower dimensions were considered in [5], and stability and hyperplane inequalities for complex convex bodies were proved in [3, 6]. For more results of this kind, along with a survey, see [4].

For example, the following stability result was proved in [2]. Let f be an even non-negative continuous function on $\mathbb{R}^n$, let μ be the measure with density f, let K and L be origin-symmetric star bodies in $\mathbb{R}^n$, and let $\epsilon > 0$. Suppose that K is an intersection body and that for every $\xi \in S^{n-1}$,

$$\mu(K \cap \xi^\perp) \le \mu(L \cap \xi^\perp) + \epsilon.$$

Then

$$\mu(K) \le \mu(L) + \frac{n}{n-1} c_n |K|^{1/n} \epsilon,$$

where $c_n = |B_2^n|^{\frac{n-1}{n}} / |B_2^{n-1}| < 1$. The corresponding hyperplane inequality is as follows. If K is an intersection body in $\mathbb{R}^n$, then

$$\mu(K) \le \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ |K|^{1/n}.$$

The constant in the latter inequality is sharp.

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A short proof of Paouris' theorem

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(joint work with R. Adamczak, O. Guédon, R. Latała, K. Oleszkiewicz, A. Pajor, N. Tomczak-Jaegermann)

A random vector in $\mathbb{R}^n$ is called log-concave if it has a log-concave distribution. It is called isotropic if it is centered and its covariance matrix is the identity.

In [3] G. Paouris proved the following theorem.

Theorem 1. *For any log-concave isotropic random vector X in $\mathbb{R}^n$ and any $t \geq 1$*

$$\mathbb{P}\left(|X| \geq Ct\sqrt{n}\right) \leq \exp(-t\sqrt{n}),$$

where $|X|$ denotes the Euclidean length of X.

In fact, he obtained a more general result. Denote the weak L_p-norm by

$$\sigma_p(X) = \sup_{|z|=1} (\mathbb{E}|\langle z, X\rangle|^p)^{1/p}.$$

Theorem 2. *For any log-concave random vector X in $\mathbb{R}^n$ and any $p \geq 1$*

$$(\mathbb{E}|X|^p)^{1/p} \leq C\left(\mathbb{E}|X| + \sigma_p(X)\right).$$

Note that the second theorem implies the first one by the Chebyshev inequality.

In our talk we provide a short proof of the Theorem 2. This is done in 5 steps.

Step 1. We pass to the expectation of a norm of the standard Gaussian vector $G = (g_1, g_2, ..., g_n)$ in $\mathbb{R}^n$, by noticing that

$$\mathbb{E}_X|X|^p = \alpha_p^{-p}\ \mathbb{E}_X\mathbb{E}_G|\langle G, X\rangle|^p = \alpha_p^{-p}\ \mathbb{E}_G|||G|||^p,$$

where $|||z||| = (\mathbb{E}|\langle z, X\rangle|^p)^{1/p}$ and $\alpha_p = (\mathbb{E}|g_1|^p)^{1/p} \approx \sqrt{p/e}$ as $p \to \infty$.

Step 2. Using the concentration of the Gaussian process $|||G|||$, we obtain

$$(\mathbb{E}_X|X|^p)^{1/p} \leq \alpha_p^{-1}\ \left(\mathbb{E}_G|||G||| + \alpha_p\sigma_p(X)\right).$$

Step 3. Applying Gordon's minimax theorem we observe

$$(\mathbb{E}|X|^p)^{1/p} \leq \alpha_p^{-1}\ \left(\mathbb{E}_\Gamma \min_{|z|=1} |||\Gamma z||| + (\alpha_p + \sqrt{d})\sigma_p(X)\right),$$

where $\Gamma = \{g_{i,j}\}$ is $n \times d$ Gaussian matrix (later we choose $d \approx p$).

Step 4. On this step we prove the following proposition.

Proposition 3. *Let Y be a log-concave symmetric d-dimensional random vector. Then*

$$\min_{|z|=1} \left(\mathbb{E}_Y|\langle z, Y\rangle|^d\right)^{1/d} \leq C\,\mathbb{E}|Y|.$$

Step 5. Denote by d the integer satisfying $p \le d < p+1$. Applying Proposition 3,

$$\min_{|z|=1} |||\Gamma z||| = \min_{|z|=1} (\mathbb{E}_X |\langle \Gamma z, X\rangle|^p)^{1/p} = \min_{|z|=1} (\mathbb{E}_X |\langle z, \Gamma^* X\rangle|^p)^{1/p} \le C\, \mathbb{E}_X |\Gamma^* X|.$$

Taking expectation with respect to Γ, we observe

$$\mathbb{E}_\Gamma \min_{|z|=1} |||\Gamma z||| \le C\, \mathbb{E}_\Gamma \mathbb{E}_X |\Gamma^* X| \le C\sqrt{d}\, \mathbb{E}_X |X|.$$

Combining with the previous inequalities, we obtain

$$(\mathbb{E}|X|^p)^{1/p} \le C\alpha_p^{-1} \left(\sqrt{d}\, \mathbb{E}_X|X| + (\alpha_p + \sqrt{d})\sigma_p(X)\right),$$

which implies the result for symmetric random vectors (since $\alpha_p \sim \sqrt{p} \sim \sqrt{d}$). The general case follows by the symmetrization (passing to vector $X - X'$, where X' is an independent copy of X).

We would also like to mention that the log-concavity was used only in Proposition 3. Thus our proof extends to the class of random vectors satisfying the following $H(p,\lambda)$ assumption.

Definition. *Let $p > 0$, $m = \lceil p \rceil$, and $\lambda \ge 1$. We say that a random vector X in E satisfies $H(p,\lambda)$ assumption if for every linear mapping $A : E \to \mathbb{R}^m$ such that $Y = AX$ is non-degenerate in $\mathbb{R}^m$ there exists a gauge $\|\cdot\|$ on $\mathbb{R}^m$ such that $\mathbb{E}\|Y\| < \infty$ and*

$$(\mathbb{E}\|Y\|^p)^{1/p} \le \lambda\, \mathbb{E}\|Y\|.$$

More precisely we have.

Theorem 4. *Let $p > 0$ and $\lambda \ge 1$. If a random vector X in a finite dimensional Euclidean space satisfies $H(p,\lambda)$, then*

$$(\mathbb{E}|X|^p)^{1/p} \le C(\lambda\, \mathbb{E}|X| + \sigma_p(X)).$$

Finally we prove that the class of so-called κ-concave probability measures satisfies this assumption if $-1 < \kappa \le 0 < p < -1/\kappa$.

Recall here that the notion of a κ-concave measure was introduced by C. Borell as follows: a measure μ on $\mathbb{R}^n$ is called κ-concave if for every measurable A, B and every $\theta \in [0,1]$,

$$\mu(\theta A + (1-\theta)B) \ge (\theta\mu(A)^\kappa + (1-\theta)\mu(B)^\kappa)^{1/\kappa}.$$

The case $\kappa = 0$ corresponds to a log-concave measure.

The talk is based on the papers [1, 2].

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Sharp Isoperimetric Inequalities and Model Spaces for Curvature-Dimension-Diameter Condition

Emanuel Milman

Let (M^n, g) denote an n-dimensional ($n \geq 2$) complete oriented smooth Riemannian manifold, and let μ denote a probability measure on M having density Ψ with respect to the Riemannian volume form vol_g.

Definition (Generalized Ricci Tensor)**.** *Given $q \in [0, \infty]$ and assuming that $\Psi > 0$ and $\log(\Psi) \in C^2$, we denote by $Ric_{g,\Psi,q}$ the following generalized Ricci tensor:*

$$(1) \quad Ric_{g,\Psi,q} := Ric_g - \nabla_g^2 \log(\Psi) - \frac{1}{q} \nabla_g \log(\Psi) \otimes \nabla_g \log(\Psi) = Ric_g - q \frac{\nabla_g^2 \Psi^{1/q}}{\Psi^{1/q}} \ .$$

Note that $Ric_{g,\Psi,\infty} = Ric_g - \nabla_g^2 \log(\Psi)$ and that $Ric_{g,\Psi,0} = Ric_g$ when Ψ is constant. Here as usual Ric_g denotes the Ricci curvature tensor and ∇_g denotes the Levi-Civita covariant derivative.

Definition (Curvature-Dimension-Diameter Condition)**.** *(M^n, g, μ) is said to satisfy the Curvature-Dimension-Diameter Condition $CDD(\rho, n+q, D)$ ($\rho \in \mathbb{R}$, $q \in [0, \infty]$, $D \in (0, \infty]$), if μ is supported on the closure of a geodesically convex domain $\Omega \subset M$ of diameter at most D, having (possibly empty) C^2 boundary, $\mu = \Psi \cdot vol_g|_\Omega$ with $\Psi > 0$ on $\overline{\Omega}$ and $\log(\Psi) \in C^2(\overline{\Omega})$, and as 2-tensor fields:*

$$Ric_{g,\Psi,q} \geq \rho g \quad \text{on } \Omega \ .$$

When $\Omega = M$ and $D = +\infty$, the latter definition coincides with the celebrated Bakry–Émery Curvature-Dimension condition $CD(\rho, n+q)$. Indeed, the generalized Ricci tensor incorporates information on curvature and dimension from both the geometry of (M, g) and the measure μ, and so ρ may be thought of as a generalized-curvature lower bound, and $n+q$ as a generalized-dimension upper bound.

Let (Ω, d) denote a separable metric space, and let μ denote a Borel probability measure on (Ω, d). The Minkowski (exterior) boundary measure $\mu^+(A)$ of a Borel set $A \subset \Omega$ is defined as $\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon}$, where $A^d_\varepsilon := \{x \in \Omega; \exists y \in A \ \ d(x,y) < \varepsilon\}$. The isoperimetric profile $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) : [0,1] \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as $\mathcal{I}(v) := \inf \{\mu^+(A); \mu(A) = v\}$. In our manifold-with-density setting, we will always assume that the metric d is given by the induced geodesic distance on (M, g), and write $\mathcal{I} = \mathcal{I}(M, g, \mu)$. When $(\Omega, d) =$

$(\mathbb{R}, |\cdot|)$, we also define $\mathcal{I}^\flat = \mathcal{I}^\flat(\mathbb{R}, |\cdot|, \mu) : [0,1] \to \mathbb{R}_+ \cup \{+\infty\}$ by $\mathcal{I}^\flat(v) := \inf\{\mu^+(A); \mu(A) = v \ , \ A = (-\infty, \xi) \text{ or } A = (\xi, \infty)\}$.

When $\rho > 0$, sharp isoperimetric inequalities under the $CD(\rho, n+q)$ condition are known and well understood, thanks to the existence of comparison model spaces on which equality is attained. The first such result was obtained by M. Gromov, who identified the n-Sphere as the extremal model space in the constant density case ($q = 0$), thereby extending P. Lévy's isoperimetric inequality on the sphere. The case when $q = +\infty$ was treated by Bakry and Ledoux (and later Morgan), who showed that the corresponding model space is the Real line equipped with a Gaussian density. An extension of these results to $q \in (0, \infty)$ was subsequently obtained by Bayle.

However, in all other cases, none of previously known results (by Croke, Bérard, Besson, Gallot and others) yield *sharp* isoperimetric inequalities for all $v \in (0,1)$. The purpose of this work is to fill this gap, providing a sharp isoperimetric inequality under the $CDD(\rho, n+q, D)$ condition in the entire range $\rho \in \mathbb{R}$, $q \in [0, \infty]$, $D \in (0, \infty]$ and $v \in (0,1)$, in a single unified framework. In particular, for each choice of parameters, we identify the *model spaces* which are extremal for the isoperimetric problem. Our results seem new even in the classical constant-density case ($q = 0$) when $\rho \leq 0$ and $D < \infty$ or when $\rho > 0$ and $D < \pi\sqrt{(n-1)/\rho}$.

1. Results

Given $\delta \in \mathbb{R}$, set as usual:

$$s_\delta(t) := \begin{cases} \sin(\sqrt{\delta} t)/\sqrt{\delta} & \delta > 0 \\ t & \delta = 0 \\ \sinh(\sqrt{-\delta} t)/\sqrt{-\delta} & \delta < 0 \end{cases} \quad , \quad c_\delta(t) := \begin{cases} \cos(\sqrt{\delta} t) & \delta > 0 \\ 1 & \delta = 0 \\ \cosh(\sqrt{-\delta} t) & \delta < 0 \end{cases} .$$

Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) \geq 0$, we denote by $f_+ : \mathbb{R} \to \mathbb{R}_+$ the function coinciding with f between its first non-positive and first positive roots, and vanishing everywhere else.

Definition. *Given $H, \rho \in \mathbb{R}$ and $m \in (0, \infty]$, set $\delta := \rho/m$ and define:*

$$J_{H,\rho,m}(t) := \begin{cases} \left(c_\delta(t) + \frac{H}{m} s_\delta(t)\right)_+^m & m \in (0, \infty) \\ \exp(Ht - \frac{\rho}{2} t^2) & m = \infty \end{cases} .$$

Remark. *Observe that $J_{H,\rho,m}$ coincides with the solution J to the following second order ODE, on the maximal interval containing the origin where such a solution exists:*

$$-(\log J)'' - \frac{1}{m}((\log J)')^2 = -m \frac{(J^{1/m})''}{J^{1/m}} = \rho \ , \ J(0) = 1 \ , \ J'(0) = H \ .$$

Lastly, given a non-negative integrable function f on a closed interval $L \subset \mathbb{R}$, we denote for short $\mathcal{I}(f, L) := \mathcal{I}(\mathbb{R}, |\cdot|, \mu_{f,L})$, where $\mu_{f,L}$ is the probability measure supported in L with density proportional to f there. Similarly, we set $\mathcal{I}^\flat(f, L) :=$

$\mathcal{I}^\flat(\mathbb{R}, |\cdot|, \mu_{f,L})$. When $\int_L f(x)dx = 0$ we set $\mathcal{I}^\flat(f,L) = \mathcal{I}(f,L) \equiv +\infty$, and when $\int_L f(x)dx = +\infty$ we set $\mathcal{I}^\flat(f,L) = \mathcal{I}(f,L) \equiv 0$.

Theorem 1.1. *Let (M^n, g, μ) satisfy the $CDD(\rho, n+q, D)$ condition with $\rho \in \mathbb{R}$, $q \in [0,\infty]$ and $D \in (0, +\infty]$. Then:*

$$\mathcal{I}(M, g, \mu) \geq \inf_{H\in\mathbb{R}, a,b\geq 0, a+b\leq D} \mathcal{I}^\flat \left(J_{H,\rho,n+q-1}, [-a,b]\right) \ , \tag{2}$$

where the infimum is interpreted pointwise on $[0,1]$. In fact, the infimum above is always attained (when $D = \infty$ at $a = b = \infty$), one can always use $b = D - a$, and the $\mathcal{I}^\flat$ may be replaced by $\mathcal{I}$, leading to the same lower bound.

The bound (2) was deliberately formulated to cover the entire range of values for ρ, n, q and D simultaneously, indicating its universal character, but it may be easily simplified as follows:

Corollary 1.2. *Under the same assumptions and notation as in Theorem 1.1, and setting $\delta := \frac{\rho}{n+q-1}$:*

Case 1 - $q < \infty$**,** $\rho > 0$**,** $D < \pi/\sqrt{\delta}$**:**

$$\mathcal{I}(M^n, g, \mu) \geq \inf_{\xi\in[0,\pi/\sqrt{\delta}-D]} \mathcal{I}^\flat \left(\sin(\sqrt{\delta}t)^{n+q-1}, [\xi, \xi+D]\right).$$

Case 2 - $q < \infty$**,** $\rho > 0$**,** $D \geq \pi/\sqrt{\delta}$**:**

$$\mathcal{I}(M^n, g, \mu) \geq \mathcal{I}^\flat \left(\sin(\sqrt{\delta}t)^{n+q-1}, [0, \pi/\sqrt{\delta}]\right).$$

Case 3 - $q < \infty$**,** $\rho = 0$**,** $D < \infty$**:**

$$\mathcal{I}(M^n, g, \mu) \geq \min(\inf_{\xi\geq 0} \mathcal{I}^\flat(t^{n+q-1}, [\xi, \xi+D]), \mathcal{I}^\flat(1, [0,D])).$$

Case 4 - $q < \infty$**,** $\rho < 0$**,** $D < \infty$**:**

$$\mathcal{I}(M^n, g, \mu) \geq \min \left\{ \begin{array}{c} \inf_{\xi\geq 0} \mathcal{I}^\flat(\sinh(\sqrt{-\delta}t)^{n+q-1}, [\xi, \xi+D]) \ , \\ \mathcal{I}^\flat(\exp(\sqrt{-\delta}(n+q-1)t), [0,D]) \ , \\ \inf_{\xi\in\mathbb{R}} \mathcal{I}^\flat(\cosh(\sqrt{-\delta}t)^{n+q-1}, [\xi, \xi+D]) \end{array} \right\}.$$

Case 5 - $q = \infty$**,** $\rho \neq 0$**,** $D < \infty$**:**

$$\mathcal{I}(M^n, g, \mu) \geq \inf_{\xi\in\mathbb{R}} \mathcal{I}^\flat(\exp(-\frac{\rho}{2}t^2), [\xi, \xi+D]).$$

Case 6 - $q = \infty$**,** $\rho > 0$**,** $D = \infty$**:** $\mathcal{I}(M^n, g, \mu) \geq \mathcal{I}^\flat(\exp(-\frac{\rho}{2}t^2), \mathbb{R})$.

Case 7 - $q = \infty$**,** $\rho = 0$**,** $D < \infty$**:** $\mathcal{I}(M^n, g, \mu) \geq \inf_{H\geq 0} \mathcal{I}^\flat(\exp(Ht), [0,D])$.

In all the remaining cases, we have the trivial bound $\mathcal{I}(M^n, g, \mu) \geq 0$.

Note that when q is an integer, $\mathcal{I}^\flat(\sin(\sqrt{\delta}t)^{n+q-1}, [0, \pi/\sqrt{\delta}])$ coincides (by testing spherical caps) with the isoperimetric profile of the $(n+q)$-Sphere having Ricci curvature equal to ρ, and so Case 2 with $q = 0$ recovers the Gromov–Lévy isoperimetric inequality; for general $q < \infty$, Case 2 was obtained by Bayle. Case 6 recovers the Bakry–Ledoux isoperimetric inequality. To the best of our knowledge, all remaining cases are new. To illuminate the transition between Cases 1 and 2, note that if (M^n, g, μ) satisfies the $CD(\rho, n+q)$ condition with $\rho > 0$, the diameter of M is bounded above by $\pi/\sqrt{\delta}$: when $q = 0$ this is the classical Bonnet-Myers theorem, which was extended to $q > 0$ by Bakry–Ledoux and Qian. As for the sharpness, we have:

Theorem 1.3. *For any $n \geq 2$, $\rho \in \mathbb{R}$, $q \in [0, \infty]$ and $D \in (0, \infty]$, the lower bounds provided in Corollary 1.2 (or equivalently, the one provided in Theorem 1.1) on the isoperimetric profile of (M^n, g, μ) satisfying the $CDD(\rho, n + q, D)$ condition, are sharp, in the sense that they cannot be pointwise improved.*

We conclude that with the exception of Cases 2 and 6 above, there is no *single* model space to compare to, and that a simultaneous comparison to a natural *one parameter family* of model spaces is required, nevertheless yielding a sharp comparison result.

Vertical versus horizontal Poincaré inequalities on the Heisenberg group

Assaf Naor
(joint work with Vincent Lafforgue)

The discrete Heisenberg group, denoted $\mathbb{H}$, is the group generated by two elements $a, b \in \mathbb{H}$, with the relations asserting that the commutator $[a, b] = aba^{-1}b^{-1}$ is in the center of $\mathbb{H}$. Thus $\mathbb{H}$ is given by the presentation

$$\mathbb{H} = \langle a, b \,|\, a[a, b] = [a, b]a \ \wedge\ b[a, b] = [a, b]b \rangle \,.$$

Write $c = [a, b]$ and let $e_{\mathbb{H}}$ denote the identity element of $\mathbb{H}$. The left-invariant word metric on $\mathbb{H}$ induced by the symmetric generating set $\{a, b, a^{-1}, b^{-1}\}$ is denoted $d_W(\cdot, \cdot)$. For $n \in \mathbb{N}$ let $B_n = \{x \in \mathbb{H} : \ d_W(x, e_{\mathbb{H}}) \leq n\}$ denote the corresponding closed ball of radius n.

A Banach space $(X, \|\cdot\|_X)$ is said to be uniformly convex if for every $\varepsilon \in (0, 1)$ there exists $\delta \in (0, 1)$ such that every $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$ and $\|x - y\|_X \geq \varepsilon$ satisfy $\|x + y\|_X \leq 2(1 - \delta)$. The supremum over those $\delta \in (0, 1)$ for which this holds true is denoted $\delta_{(X,\|\cdot\|_X)}(\varepsilon)$, and is called the modulus of uniform convexity of $(X, \|\cdot\|_X)$. An important theorem of Pisier asserts that every uniformly convex Banach space $(X, \|\cdot\|_X)$ admits an equivalent norm $\|\cdot\|$ for which there exist $q \in [2, \infty)$ and $\eta \in (0, 1)$ such that $\delta_{(X,\|\cdot\|)}(\varepsilon) \geq (\eta\varepsilon)^q$ for all $\varepsilon \in (0, 1)$. For concreteness we recall that if $p \in (1, \infty)$ then ℓ_p satisfies such an estimate with $q = \max\{p, 2\}$.

Theorem 1 (Vertical versus horizontal Poincaré inequality)**.** *For every $\eta \in (0, 1)$ and $q \in [2, \infty)$ there exists $K = K(\eta, q) \in (0, \infty)$ with the following property. Suppose that $(X, \|\cdot\|_X)$ is a Banach space satisfying $\delta_{(X,\|\cdot\|_X)}(\varepsilon) \geq (\eta\varepsilon)^q$ for every*

$\varepsilon \in (0,1)$. *Then for every* $n \in \mathbb{N}$ *and every* $f : \mathbb{H} \to X$ *we have*

$$(1)\quad \sum_{k=1}^{n^2} \sum_{x\in B_n} \frac{\|f(xc^k)-f(x)\|_X^q}{k^{1+q/2}} \le K \sum_{x\in B_{21n}} \Big(\|f(xa)-f(x)\|_X^q + \|f(xb)-f(x)\|_X^q\Big).$$

The constant 21 appearing in the range of the summation on the right hand side of (1) is an artifact of our proof and is not claimed to be sharp. The important point here is that the summation on the right hand side of (1) is over $x \in B_{\lambda n}$ for some universal constant $\lambda \in \mathbb{N}$. One can clearly make the same statement for word metrics induced by other finite symmetric generating sets of $\mathbb{H}$: the choice of generating set will only affect the value of λ.

An inspection of our proof of Theorem 1 reveals that $K^{1/q} \lesssim 1/\eta$, but we will not explicitly track the value of such constants in the ensuing discussion. Here, and in what follows, we use $A \lesssim B$ and $B \gtrsim A$ to denote the estimate $A \leqslant CB$ for some absolute constant $C \in (0,\infty)$. If we need C to depend on parameters, we indicate this by subscripts, thus e.g. $A \lesssim_\alpha B$ means that $A \leqslant C_\alpha B$ for some $C_\alpha \in (0,\infty)$ depending only on α. We shall also use the notation $A \asymp B$ for $A \lesssim B \ \wedge\ B \lesssim A$, and similarly $A \asymp_\alpha B$ stands for $A \lesssim_\alpha B \ \wedge\ B \lesssim_\alpha A$.

We call (1) a "vertical versus horizontal Poincaré inequality" for the following reason. The right hand side of (1) is the ℓ_q norm of the discrete horizontal gradient of f: it measures the "local" variation of f along the edges of the Cayley graph of $\mathbb{H}$ (a.k.a. the horizontal edges in $\mathbb{H}$). The left hand side of (1) measures the "global" variation of f along the center of $\mathbb{H}$ (a.k.a. the vertical direction in $\mathbb{H}$). Theorem 1 asserts that the global vertical variation of f is always bounded by its local horizontal variation. Thus, if the right hand side of (1) is small then f must collapse distances along the center of $\mathbb{H}$.

The (bi-Lipschitz) distortion of a finite metric space (M, d_M) in a Banach space $(X, \|\cdot\|_X)$, denoted $c_X(M,d_M) \in [1,\infty)$, is the infimum over those $D \in [1,\infty)$ for which there exists an embedding $f : M \to X$ that satisfies $d_M(x,y) \le \|f(x) - f(y)\|_X \le D d_M(x,y)$ for all $x,y \in M$. When $X = \ell_p$ for some $p \in [1,\infty)$ it is customary to write $c_{\ell_p}(M,d_M) = c_p(M,d_M)$. The quantity $c_2(M,d_M)$ is known as the Euclidean distortion of (M,d_M). Suppose that $(X,\|\cdot\|_X)$ satisfies the assumption of Theorem 1 and that $f : \mathbb{H} \to X$ satisfies $d_W(x,y) \le \|f(x) - f(y)\|_X \le D d_W(x,y)$ for all $x,y \in B_{22n}$. Since $d_W(c^k, e_{\mathbb{H}}) \asymp \sqrt{k}$ for every $k \in \mathbb{N}$ and $|B_m| \asymp m^4$ for every $m \in \mathbb{N}$, Theorem 1 applied to f yields the following estimate.

$$(2)\qquad n^4 \log n \lesssim \sum_{k=1}^{n^2} n^4 \frac{k^{q/2}}{k^{1+q/2}} \lesssim_X n^4 D^q.$$

We therefore obtain the following corollary of Theorem 1.

Corollary 2 (Sharp nonembeddabilty of balls in $\mathbb{H}$)**.** *Fix* $\eta \in (0,1)$ *and* $q \in [2,\infty)$. *Suppose that* $(X,\|\cdot\|_X)$ *is a Banach space satisfying* $\delta_{(X,\|\cdot\|_X)}(\varepsilon) \ge (\eta\varepsilon)^q$ *for every*

$\varepsilon \in (0,1)$. *Then for every* $n \in \mathbb{N}$ *we have*

$$c_X\left(B_n, d_W\right) \gtrsim_\eta (\log n)^{1/q}.$$

Corollary 2 yields an estimate on $c_X(B_n, d_W)$ in terms of the modulus of uniform convexity of $(X, \|\cdot\|_X)$ which is asymptotically best possible, up to constant factors that are independent of n. The following corollary states this explicitly for the case of special interest $X = \ell_p$, though one could equally well state such results for a variety of concrete spaces for which the modulus of uniform convexity has been computed (e.g., the same conclusion holds true with ℓ_p replaced by the Schatten class S_p).

Corollary 3. *For every integer* $n \geq 2$ *we have*

$$p \in (1,2] \implies c_p(B_n, d_W) \asymp_p \sqrt{\log n},$$

and

$$p \in [2,\infty) \implies c_p(B_n, d_W) \asymp_p (\log n)^{1/p}.$$

Random points in a convex body and the Log-Concave Ensemble of random matrices

Alain Pajor

In a paper on the algorithmic complexity of computing volume in high dimensions, Kannan, Lovász and Simonovits asked for the following question (1996):

Question (KLS [3]) *Let* K *be a convex body in* $\mathbb{R}^n$. *Given* $\varepsilon > 0$, *how many independent points* X_i *uniformly distributed on* K *are needed for the empirical covariance matrix to approximate the covariance matrix up to* ε *with overwhelming probability?*

In other words, let $X \in \mathbb{R}^n$ be a random vector with the identity as covariance matrix – such a vector is called *isotropic*– we would like that with high probability,

$$(1-\varepsilon)|\theta|^2 \leq \frac{1}{N}\sum_{1}^{N} |\langle X_i, \theta\rangle|^2 \leq (1+\varepsilon)|\theta|^2, \quad \text{for all } \theta \in \mathbb{R}^n.$$

In this talk, we will survey recent results of R. Adamczak, A. E. Litvak, A. Pajor and N. Tomczak-Jaegermann [1] and [2]. We discuss in particular the following result which answers completely KLS question.

Let $X_1, \ldots, X_N \in \mathbb{R}^n$ be independent centered random vectors with log-concave distribution and with the identity as covariance matrix. We show that with overwhelming probability one has

$$\sup_{x\in S^{n-1}} \left| \frac{1}{N}\sum_{i=1}^{N} \left(|\langle X_i, x\rangle|^2 - \mathbb{E}|\langle X_i, x\rangle|^2\right) \right| \leq C\sqrt{\frac{n}{N}},$$

where C is an absolute positive constant. This result is valid in a more general framework when the linear forms $(\langle X_i, x\rangle)_{i\leq N, x\in S^{n-1}}$ and the Euclidean norms

$(|X_i|/\sqrt{n})_{i\le N}$ exhibit uniformly a sub-exponential decay. As a consequence, if A denotes the random matrix with columns (X_i), then with overwhelming probability, the extremal singular values $\lambda_{\min}$ and $\lambda_{\max}$ of $AA^\top$ satisfy the inequalities $1 - C\sqrt{\frac{n}{N}} \le \frac{\lambda_{\min}}{N} \le \frac{\lambda_{\max}}{N} \le 1 + C\sqrt{\frac{n}{N}}$ which is a quantitative version of Bai-Yin theorem known for random matrices with i.i.d. entries.

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The Centro-Affine Hadwiger Theorem

Lukas Parapatits

(joint work with Christoph Haberl)

Let $\mathcal{K}^n$ denote the set of convex bodies, i.e. nonempty compact convex subsets of $\mathbb{R}^n$. A valuation μ is a map from $\mathcal{K}^n$ to $\mathbb{R}$ that satisfies

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L) \tag{1}$$

whenever $K \cup L$ is convex. The best known theorem in the theory of valuations is Hadwiger's [2] classification of all rigid motion invariant continuous valuations.

Theorem 1. *A map $\mu\colon \mathcal{K}^n \to \mathbb{R}$ is an $\mathrm{SO}(n)$ invariant translation invariant continuous valuation if and only if there are constants $c_0, c_1, \dots, c_n \in \mathbb{R}$ such that*

$$\mu = c_0V_0 + c_1V_1 + \ldots + c_nV_n.$$

Here, $V_i,\ i = 0, \dots, n$ are the intrinsic volumes. In particular, V_n is the volume and V_0 is the Euler characteristic.

We can also consider valuations which are defined on subsets of $\mathcal{K}^n$ and satisfy (1). Let $\mathcal{K}^n_{oo}$ denote the set of convex bodies which contain the origin in their interiors. The following theorem is a new result [1] which strengthens a previous classification of homogeneous valuations by Ludwig [3].

Theorem 2. *Let $n \ge 2$. A map $\mu\colon \mathcal{K}^n_{oo} \to \mathbb{R}$ is an $\mathrm{SL}(n)$ invariant continuous valuation if and only if there exist constants $c_{-n}, c_0, c_n \in \mathbb{R}$ such that*

$$\mu(K) = c_{-n}V_n(K^*) + c_0V_0 + c_nV_n(K)$$

for all $K \in \mathcal{K}^n_{oo}$.

Here, $K^* := \{y \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } x \in K\}$ denotes the polar body of K.

Another interesting type of valuations are Minkowski valuations. These are maps from (a subset of) $\mathcal{K}^n$ to $\mathcal{K}^n$ which satisfy (1) where $+$ is Minkowski addition, i.e. $K + L := \{x + y : x \in K, y \in L\}$. A Minkowski valuation is called $\mathrm{SL}(n)$ covariant if $\mu(\phi K) = \phi\mu(K)$ for all $\phi \in \mathrm{SL}(n)$. Ludwig [4] already classified all homogeneous $\mathrm{SL}(n)$ covariant continuous Minkowski valuations. With the techniques used in the proof of the above theorem on real valued valuations, we hope to proof the following result.

Conjecture 3. *Let $n \ge 3$. A map $\mu : \mathcal{K}^n_{oo} \to \mathcal{K}^n$ is an $\mathrm{SL}(n)$ covariant continuous Minkowski valuation, if and only if there exist constants $c_1, \dots, c_4 \ge 0$ and $c_5 \in \mathbb{R}$ such that*

$$\mu(K) = c_1 K + c_2(-K) + c_3\,\Pi(K^*) + c_4\,\mathrm{M}(K) + c_5\,\mathrm{m}(K)$$

for all $K \in \mathcal{K}^n_{oo}$.

Here, Π, M and m are the projection body operator, the moment body operator and the moment vector operator, respectively.

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A central limit theorem for projections of the cube

Peter Pivovarov

(joint work with Grigoris Paouris and Joel Zinn)

In this talk, I discussed a central limit theorem for the volume of projections of the cube $B^N_\infty = [-1, 1]^N$ onto a random subspace of dimension n, when n is fixed and $N \to \infty$. To fix the notation, let $n \ge 1$ be an integer and for $N \ge n$, let $G_{N,n}$ denote the Grassmannian manifold of all n-dimensional linear subspaces of $\mathbb{R}^N$. Equip $G_{N,n}$ with the Haar probability measure $\nu_{N,n}$, which is invariant under the action of the orthogonal group. Suppose that $(E(N))_{N \ge n}$ is a sequence of random subspaces with $E(N)$ distributed according to $\nu_{N,n}$. Consider the random variables

$$Z_N := |P_{E(N)} B^N_\infty|, \tag{1}$$

where $P_{E(N)}$ denotes the orthogonal projection onto $E(N)$ and $|\cdot|$ is n-dimensional volume. Then Z_N satisfies the following central limit theorem.

Theorem 1.

$$\frac{Z_N - \mathbb{E}Z_N}{\sqrt{\mathrm{var}(Z_N)}} \xrightarrow{d} \mathcal{N}(0,1) \ \text{as} \ N \to \infty. \tag{2}$$

Here $\xrightarrow{d}$ denotes convergence in distribution and $\mathcal{N}(0,1)$ a standard Gaussian random variable with mean 0 and variance 1.

Gaussian random matrices play a central role in the proof of Theorem 1, as is often the case with results about projections onto random subspaces $E \in G_{N,n}$. Specifically, let G be an $n \times N$ random matrix with independent columns $g_1, \ldots, g_N$ distributed according to standard Gaussian measure γ_n on $\mathbb{R}^n$, i.e.,

$$d\gamma_n(x) = (2\pi)^{-n/2} e^{-\|x\|_2^2/2} dx.$$

View G as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$. If $C \subset \mathbb{R}^N$ is any convex body, then

$$|GC| = \det(GG^*)^{\frac{1}{2}} |P_E C|, \tag{3}$$

where $E = \mathrm{Range}(G^*)$ is distributed uniformly on $G_{N,n}$; moreover, $\det(GG^*)^{1/2}$ and $|P_E C|$ are independent. The latter fact underlies the Gaussian representation of intrinsic volumes, as proved by B. Tsirelson in [2] (see also [4]). Our proof involves two main steps, first analyzing asymptotic normality of $|GB_\infty^N|$ and then dealing with the quotient $|GB_\infty^N| / \det(GG^*)^{1/2}$.

Setting $X_N := |GB_\infty^N|$ and applying the well-known zonotope volume formula, we have

$$X_N = 2^n \sum_{1 \le i_1 < \ldots < i_n \le N} |\det[g_{i_1} \cdots g_{i_n}]|, \tag{4}$$

where $\det[g_{i_1} \cdots g_{i_n}]$ is the determinant of the matrix with columns $g_{i_1}, \ldots, g_{i_n}$. Similarly, we set $Y_N := \det(GG^*)^{\frac{1}{2}}$ and apply the Cauchy-Binet formula:

$$Y_N = \left(\sum_{1 \le i_1 < \ldots < i_n \le N} |\det[g_{i_1} \cdots g_{i_n}]|^2 \right)^{\frac{1}{2}}. \tag{5}$$

Then $X_N = Y_N Z_N$, where Y_N and Z_N are independent.

The proof employs several ingredients including W. Hoeffding's central limit theorem for U-statistics [5], as well as its application to mixed volumes of random convex sets by R. A. Vitale [6]. The latter implies that X_N is asymptotically normal. On the other hand, it is well-known that Y_N^2 satisfies a central limit theorem (using, e.g., [5]). The proof then rests on careful analysis showing that asymptotic normality of Z_N arises from two contributing terms X_N and Y_N^2. Randomization inequalities for U-statistics from [1] play an important role in the proof. A key proposition is the following:

Proposition 2. *Let X_N, Y_N and Z_N be as defined above. Then*

$$\frac{Z_N - \mathbb{E}Z_N}{N^{\frac{n-1}{2}}} = \alpha_{N,n} \frac{X_N - \mathbb{E}X_N}{N^{n-\frac{1}{2}}} - \beta_{N,n} \frac{Y_N^2 - \mathbb{E}Y_N^2}{N^{n-\frac{1}{2}}} - \delta_{N,n}, \tag{6}$$

where $(\alpha_{N,n})$, $(\beta_{N,n})$ *and* $(\delta_{N,n})$ *are sequences of random variables that converge almost surely to constants, i.e.,*

(i) $\alpha_{N,n} \overset{a.s.}{\to} 1$ *as* $N \to \infty$;
(ii) $\beta_{N,n} \overset{a.s.}{\to} \beta_n > 0$ *as* $N \to \infty$;
(iii) $\delta_{N,n} \overset{a.s.}{\to} 0$ *as* $N \to \infty$.

The fact that $N^{n-\frac{1}{2}}$ appears in both of the denominators on the right-hand side of (6) indicates that both X_N and Y_N^2 must be accounted for in order to capture the asymptotic normality of Z_N. We refer the reader to [3] for the details.

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Simplicial polytopes that maximize the isotropic constant are highly symmetric

Luis Rademacher

The slicing constant L_K is an affine-invariant measure of the spread of a convex body K. For a d-dimensional convex body K, L_K can be defined by $L_K^{2d} = \det(A(K))/(\mathrm{vol}(K))^2$, where $A(K)$ is the covariance matrix of the uniform distribution on K. It is an outstanding open problem to find a tight asymptotic upper bound of the slicing constant as a function of the dimension. It has been conjectured that there is a universal constant upper bound. The conjecture is know to be true for several families of bodies, in particular, highly symmetric bodies such as bodies having an unconditional basis. It is also know that maximizers cannot be smooth. In this work we show progress towards reducing to a highly symmetric case among non-smooth bodies. More precisely, we show that if a simplicial d-polytope K is a maximizer of the slicing constant among d-dimensional convex bodies, then when K is put in isotropic position it must be isohedral, that is, its symmetry group acts transitively upon facets. In particular, all facets are congruent.

The Disk Graph: Recent Developments

Matthias Reitzner

Let η_t be a Poisson point process in $\mathbb{R}^d$ of constant intensity $t > 0$. This Poisson point process consists of infinitly many random points, $\eta_t = \{x_1, x_2, \dots\}$, with the property that the number of points $|\eta_t \cap W|$ in a convex body W is Poisson distributed with parameter $tV_d(W)$.

To define the disk graph $G(\eta_t, \delta_t)$ we take all points of η_t to be the vertices of $G(\eta_t, \delta_t)$ and connect two points $x, y \in \eta_t$ by an edge if

$$\|x - y\| \le \delta_t.$$

The resulting graph $G(\eta_t, \delta_t)$ is a random geometric graph, called disk graph, or sometimes Gilbert graph, interval graph (for $d = 1$) or distance graph. The disk graph is the maybe most natural construction of a random geometric graph, see e.g. the book by Penrose [2].

We are interested in the local behaviour of the disk graph within a convex body W, when $t \to \infty$ and $\delta_t \to 0$. Define the number of edges of $G(\xi, \delta)$ in the window W given by

$$N_t = N(\eta_t, \delta_t) = \frac{1}{2} \sum_{(x,y) \in (\eta_t \cap W)^2_{\neq}} \mathbf{1}(\|x - y\| \le \delta_t)$$

and the total edge length by

$$L_t = L(\eta_t, \delta_t) = \frac{1}{2} \sum_{(x,y) \in (\eta_t \cap W)^2_{\neq}} \mathbf{1}(\|x - y\| \le \delta_t) \|x - y\|.$$

Classical results are the expectations of these quantities which are proved using the Slivnyak-Mecke theorem,

$$\mathbb{E} N_t = \frac{t^2}{2} \big(V(W)\, \kappa_d\, \delta_t^d + O(\delta_t^{d+1}) \big)$$

and

$$\mathbb{E} L_t = \frac{t^2}{2} \Big(V(W)\, \frac{d\, \kappa_d}{d+1}\, \delta_t^{d+1} + O(\delta_t^{d+2}) \Big)$$

for $\delta_t \to 0$. More recently Schulte [4] investigated the asymptotic covariance structure of the suitable rescaled number of edges and total edge length,

$$\tilde{N}_t = N_t / \sqrt{t^2 \delta_t^d \max\{t\delta_t^d, 1\}},\ \ \tilde{L}_t = L_t / \sqrt{t^2 \delta_t^{d+2} \max\{t\delta_t^d, 1\}}$$

and proved

$$\lim_{t \to \infty} \begin{pmatrix} \mathbb{V} \tilde{N}_t & \mathbb{C}(\tilde{N}_t, \tilde{L}_t) \\ \mathbb{C}(\tilde{N}_t, \tilde{L}_t) & \mathbb{V} \tilde{L}_t \end{pmatrix} = \begin{cases} \Sigma_2, & \lim_{t\to\infty} t\, \delta_t^d = 0 \\ c\, \Sigma_1 + \Sigma_2, & \lim_{t\to\infty} t\, \delta_t^d = c \in (0, 1] \\ \Sigma_1 + \frac{1}{c}\, \Sigma_2, & \lim_{t\to\infty} t\, \delta_t^d = c \in (1, \infty) \\ \Sigma_1, & \lim_{t\to\infty} t\, \delta_t^d = \infty \end{cases}$$

with covariance matrices

$$\Sigma_1 = V(W) \begin{pmatrix} \kappa_d^2 & d\,\kappa_d^2/(d+1) \\ d\,\kappa_d^2/(d+1) & (d\,\kappa_d/(d+1))^2 \end{pmatrix},$$

$$\Sigma_2 = \frac{V(W)}{2} \begin{pmatrix} \kappa_d & d\,\kappa_d/(d+1) \\ d\,\kappa_d/(d+1) & d\,\kappa_d/(d+2) \end{pmatrix}.$$

This allows to deduce limit theorems. It was proved in [3] [4] that N_t and L_t satisfy a central limit theorem.

A deeper understanding of the behaviour of the disk graph can be obtained by ordering all distances between two points in the convex body W. This yields the point set

$$\xi_t = \{\|x_1 - x_2\| : \ (x_1, x_2) \in (\eta_t \cap W)^2_{\neq}\}$$

on the positive real line. It was proved by Schulte and Thäle [5] that

(a) $t^{2/d}\xi_t$ converges in distribution to a Poisson point process on $\mathbb{R}_+$ with intensity measure

$$\mu(B) = \frac{\kappa_d}{2} V(W) d \int_B u^{d-1}\, du, B \subset \mathbb{R}_+;$$

(b) and that for $t \geq 1$ the shortest distance $G_t^{(1)}$ satisfies

$$\left| P(t^{2/d} G_t^{(1)} > x) - e^{-\frac{\kappa_d}{2} V(W) x^d} \right| \leq C(x) t^{-\min\{\frac{2}{d}, \frac{1}{2}\}}.$$

As an application we present a connection to a question concerning empty triangles. Given a finite point set X in the plane, the degree of a pair $\{x, y\} \subset X$ is the number of *empty triangles* $t = \operatorname{conv}\{x, y, z\}$, where empty means $t \cap X = \{x, y, z\}$. Define $\deg X$ as the maximal degree of a pair in X.

Here we take X to be the intersection of a Poisson point process with the convex body W. Observe that for any pair $(x, y) \in (\eta_t \cap W)^2_{\neq}$ the degree is clearly bounded by the number of points $\eta_t \cap W$ which has expectation $tV_d(W)$. It turns out that the degree of X is close to this trivial upper bound. It is proved in Bárány, Marckert and Reitzner [1] that for $X = \eta_t \cap W$ there is a constant $c > 0$ such that

$$\mathbb{E}(\deg X) \geq \frac{c}{\ln t}\, t.$$

The proof uses essentially a large deviation inequality for the length of the disk graph.

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On approximation by projections of polytopes with few facets

Mark Rudelson
(joint work with Alexander Litvak, Nicole Tomczak-Jaegermann)

This is a report on paper [4].

One of the standard ways to describe a convex body in computational geometry is the *membership oracle.* The membership oracle of a body $K \subset \mathbb{R}^n$ is an algorithm, which, given a point $x \in \mathbb{R}^n$, outputs whether $x \in K$, or $x \notin K$. If such oracle is constructed, and if the body K has a relatively well-conditioned position, meaning that $rB_2^n \subset K \subset RB_2^n$ with $R/r \le n^C$, then one can construct efficient probabilistic algorithms for estimating the volume of K, its inertia ellipsoid, and other geometric characteristics (see e.g. [3] and [8]). Yet, constructing an efficient membership oracle for a given convex body may be a hard problem [2]. Because of this, it is important to know whether a convex body can be approximated by another body, for which the membership oracle can be efficiently constructed. One natural class of convex bodies for which the construction of the membership oracle is efficient is the projections of a polytope with a few faces. Such polytopes can be realized as projections of sections of a simplex in a dimension comparable to n. This construction is discussed in details in [2]. In particular, the following problem was posed (Problem 4.7.2 in [2]).

Problem. *Let $K \subset \mathbb{R}^n$ be a symmetric convex body and let $P \subset \mathbb{R}^n$ be a projection of a polytope with N facets, which approximates K within a factor of 2. Is it true that in the worst case the number N should be at least exponential in d: $N \ge e^{cd}$ for some absolute constant $c > 1$?*

If $K = B_p^n$ is the unit ball of ℓ_p^n, then this approximation requires only proportional dimension. Moreover, even the existence of an n-dimensional convex body, which cannot be approximated by a projection of a section of a simplex Δ_N with N proportional to n has been an open problem.

Our result provides an affirmative solution to the Barvinok problem above. Furthermore, we prove a lower estimate for the minimal Banach–Mazur distance between a certain convex symmetric body and a projection of a polytope with N facets. This estimate is optimal for all $N > n$ up to logarithmic terms.

Theorem. *Let $n \le N$. There exists an n-dimensional convex symmetric body B, such that for every n-dimensional convex body K obtained as a projection of a section of an N-dimensional simplex one has*

$$d(B, K) \ge c \sqrt{\frac{n}{\ln \frac{2N \ln(2N)}{n}}},$$

where c is an absolute positive constant.

Any projection of a section of a simplex can be realized as a section of a projection of a simplex. Thus, the Theorem above holds for bodies K obtained as a section of a projection of a simplex as well.

To see that the estimate of the Theorem is close to optimal, recall that Barvinok proved in [1] that for every $N \geq 8n$ and every symmetric convex body B in $\mathbb{R}^n$ there exists a section K of an N-dimensional simplex such that

$$d(B,K) \leq C \max\left\{1, \sqrt{\frac{n}{\ln N} \cdot \ln \frac{n}{\ln N}}\right\}.$$

Comparison of these two bounds shows that working with projections of sections of a simplex, as opposed to using sections alone, does not significantly improve the approximation. This is in stark contrast with the situation described in the Quotient of a Subspace Theorem. Recall that the Quotient of a Subspace Theorem of Milman ([5], see also [6] and [7] for the non-symmetric case) states that *given* $\theta \in (0,1)$ *and an* n*-dimensional convex body* K *there exists a projection of a section of* K *whose dimension is greater than* θn *and whose Banach-Mazur distance to the Euclidean ball of the corresponding dimension does not exceed* $C(\theta)$ *(moreover,* $C(\theta)$ *can be chosen such that* $C(\theta) \to 1$ *as* $\theta \to 0^+$*).* On the other hand, it is well-known by a volumetric argument that any n-dimensional section of the N-dimensional cube (or simplex) is at the distance at least $c\sqrt{n/\ln(2N/n)}$ from the n-dimensional Euclidean ball. Thus, in the case of the cube (or simplex) and proportional subspaces/projections, taking just sections leads to $c\sqrt{n}$ distance to the Euclidean ball, while adding one more operation – taking a projection – yields the distance bounded by an absolute constant.

Our result also shows that Quotient of a Subspace Theorem cannot be extended much beyond the Euclidean setting. Even if we start with the simplest (in terms of complexity) convex body – simplex – we cannot obtain an arbitrary convex set by taking a projection of a section.

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Non-uniqueness of convex bodies with prescribed volumes of sections and projections

Dmitry Ryabogin
(joint work with Fedor Nazarov, Artem Zvavitch)

As usual, a *convex body* $K \subset \mathbb{R}^d$ is a compact convex subset of $\mathbb{R}^d$ with non-empty interior. We assume that $0 \in K$. We consider the *central section function* A_K:

$$A_K(u) = \mathrm{vol}_{d-1}(K \cap u^{\perp}), \qquad u \in \mathbb{S}^{d-1}, \tag{1}$$

the *maximal section function* M_K:

$$M_K(u) = \max_{t\in\mathbb{R}} \mathrm{vol}_{d-1}(K \cap (u^{\perp} + tu)), \qquad u \in \mathbb{S}^{d-1}, \tag{2}$$

and the *projection function* P_K:

$$P_K(u) = \mathrm{vol}_{d-1}(K|u^{\perp}), \qquad u \in \mathbb{S}^{d-1}. \tag{3}$$

Here $u^{\perp}$ stands for the hyperplane passing through the origin and orthogonal to the unit vector u, $K \cap (u^{\perp} + tu)$ is the section of K by the affine hyperplane $u^{\perp} + tu$, and $K|u^{\perp}$ is the projection of K to $u^{\perp}$. Observe that $A_K \le M_K \le P_K$. It is well known, [Ga], that for origin-symmetric bodies *each* of the functions $M_K = A_K$ and P_K determines the convex body $K \subset \mathbb{R}^d$ uniquely. More precisely, either of the conditions

$$M_{K_1}(u) = M_{K_2}(u) \qquad \forall u \in \mathbb{S}^{d-1},$$

and

$$P_{K_1}(u) = P_{K_2}(u) \qquad \forall u \in \mathbb{S}^{d-1},$$

implies $K_1 = K_2$, provided K_1, K_2 are origin-symmetric and convex.

In this talk, we address the (im)possibility of analogous results for not necessarily symmetric convex bodies.

It is well known, [BF], that on the plane there are convex bodies K that are *not* Euclidean discs, but nevertheless satisfy $M_K(u) = P_K(u) = 1$ for all $u \in \mathbb{S}^1$. These are the *bodies of constant width* 1.

In 1929 T. Bonnesen asked whether *every* convex body $K \subset \mathbb{R}^3$ is uniquely defined by P_K and M_K, (see [BF], page 51). We note that in any dimension $d \ge 3$, it is not even known whether the conditions $M_K \equiv c_1$, $P_K \equiv c_2$ are incompatible for $c_1 < c_2$.

In 1969 V. Klee asked whether the condition $M_{K_1} \equiv M_{K_2}$ implies $K_1 = K_2$, or, at least, whether the condition $M_K \equiv c$ implies that K is a Euclidean ball, see [Kl1].

Recently, R. Gardner and V. Yaskin, together with the first and the third named authors constructed two bodies of revolution K_1, K_2 such that K_1 is origin-symmetric, K_2 is not origin-symmetric, but $M_{K_1} \equiv M_{K_2}$, thus answering the first version of Klee's question but not the second one (see [GRYZ]).

The main results we will present in this talk are the following.

Theorem 1. *If $d = 4$, there exists a convex body of revolution $K \subset \mathbb{R}^d$ satisfying $M_K \equiv const$ that is not a Euclidean ball.*

Theorem 2. *If $d = 4$, there exist two essentially different convex bodies of revolution K_1, $K_2 \subset \mathbb{R}^d$ such that $A_{K_1} \equiv A_{K_2}$, $M_{K_1} \equiv M_{K_2}$, and $P_{K_1} \equiv P_{K_2}$.*

Theorem 1 answers the question of Klee in $\mathbb{R}^4$, and Theorem 2 answers the analogue of the question of Bonnesen in $\mathbb{R}^4$.

Remark 1. *Theorem 1 is actually true in all dimensions, but the construction for $d \neq 4$ is long and rather technical,* [NRZ1]*. Theorem 2 is true, provided $d \geq 4$ is even,* [NRZ2]*. The case of odd dimensions remains open.*

The general idea of the construction of the bodies K_1 and K_2 in Theorem 2 is borrowed from [RY], which attributes it to [GV] and [GSW]. It can be easily understood from the following illustration.

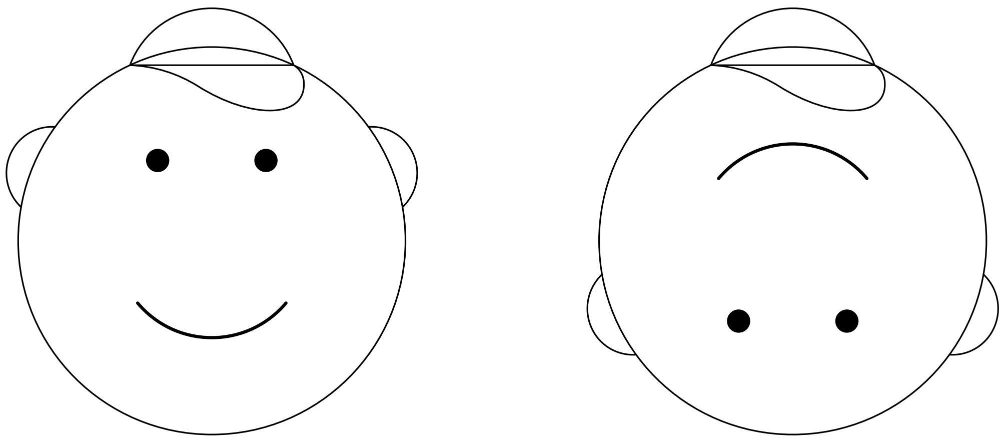

FIGURE 1. Two small-eared round faces in a cap

Here the "ears" and the "cap" will be made very small in order not to destroy the convexity of the bodies.

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Decomposition of polytopes through their inner parallel bodies

Eugenia Saorín Gómez

(joint work with Eva Linke)

Let $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq b_i, 1 \leq i \leq m\} \subset \mathbb{R}^n$ be a polytope with unit outer normal vectors u_i, $1 \leq i \leq m$. The inner parallel body of P at distance $|\lambda|$ is the polytope $P_\lambda = \{x : \langle x, u_i \rangle \leq b_i - |\lambda|, 1 \leq i \leq m\}$. In order P_λ not to be empty, it is necessary to ask for $0 \leq |\lambda| \leq \mathrm{r}(P)$, where $\mathrm{r}(P)$ is the classical inradius of P, i.e., the radius of one of the largest balls fitting inside P.

P_λ is a summand of P if and only if it does exist another polytope Q, such that $P = P_\lambda + Q$, where the sum here is the usual Minkowski sum. In 1969, Shephard proved that for given polytopes P, Q, in order Q to be a summand of P, the following two conditions are necessary and sufficient: $i)$ for every unit vector u and the corresponding faces of P, $F(P, u)$, and Q, $F(Q, u)$, built as the intersection with a supporting hyperplane with outer normal vector u, $\mathrm{dimF(P, u)} \geq \mathrm{dimF(Q, u)}$; $ii)$ for every edge of P, $F(P, u)$, $\mathrm{vol}_1(F(P, u)) \geq \mathrm{vol}_1(F(Q, u))$.

In this talk we address the following question: When is P_λ a summand of P, for $0 \leq |\lambda| \leq \mathrm{r}(P)$? More in particular, we are interested in the polytopes P for which all their inner parallel bodies P_λ, $0 \leq |\lambda| \leq \mathrm{r}(P)$ are summands of them.

In 1978, Sangwine-Yager provided necessary conditions for a general convex body to have all its inner parallel bodies as summands, however no sufficient conditions are known. Using these results together with the mentioned Shephard's criterion, we prove that for polytopes something else can be said.

In dimension 2 the situation is clear: it is not difficult to prove that for any convex polygon P, P_λ is a summand of P for all $0 \leq |\lambda| \leq \mathrm{r}(P)$. In higher dimension, the situation changes drastically. We provide examples of polytopes none of whose inner parallel bodies are summands, as well as, only, some of them.

For a polytope P, we say that P_λ is a nested summand of P, if P_λ is not just a summand of P but also a summand of all P_μ with $|\lambda| < |\mu| \leq 0$. In this talk, we provide a characterization of the polytopes all whose inner parallel bodies are nested summands of them. Furthermore, for such a polytope P we provide a complete description of the summand(s) which complete the decomposition of P. Indeed, it turns out that this is the case for all polygons in $\mathbb{R}^2$. In this particular case we prove also that for any convex body $K \subseteq \mathbb{R}^2$, all its inner parallel bodies are nested summands of it.

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On the distribution of the ψ_2-norm of linear functionals and mean width in isotropic position

Petros Valettas

(joint work with A. Giannopoulos, G. Paouris)

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered (i.e. it has its center of mass at the origin), and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 \, dx = L_K^2$$

for every $\theta \in S^{n-1}$. It is known (see [10]) that for every convex body K in $\mathbb{R}^n$ there exists an invertible affine transformation T such that $T(K)$ is isotropic. Moreover, this isotropic position of K is uniquely determined up to orthogonal transformations; this implies that the isotropic constant L_K is an affine invariant of K.

We say that $\theta \in S^{n-1}$ is a subgaussian direction for K with constant $r > 0$ if $\|\langle \cdot, \theta\rangle\|_{\psi_2} \leqslant r \|\langle \cdot, \theta\rangle\|_1$, where

$$\|f\|_{\psi_2} = \inf\left\{ t > 0 : \int_K \exp\left((|f(x)|/t)^2 \right) \, dx \leqslant 2 \right\}.$$

V. Milman asked if every centered convex body K has at least one "subgaussian" direction (with constant $r = O(1)$). By the formulation of the problem, it is clear that one can work within the class of isotropic convex bodies. Affirmative answers have been given in some special cases. Bobkov and Nazarov (see [1] and [2]) proved that if K is an isotropic 1–unconditional convex body, then $\|\langle \cdot, \theta\rangle\|_{\psi_2} \leqslant c\sqrt{n}\|\theta\|_\infty$ for every $\theta \in S^{n-1}$; a direct consequence is that the diagonal direction is a subgaussian direction with constant $O(1)$. In [11] it is proved that every zonoid has a subgaussian direction with a uniformly bounded constant. Another partial result was obtained in [12]: if K is isotropic and $K \subseteq (\gamma\sqrt{n}L_K)B_2^n$ for some $\gamma > 0$, then

$$\sigma\big(\theta \in S^{n-1} : \|\langle \cdot, \theta\rangle\|_{\psi_2} \geqslant c_1 \gamma t L_K\big) \leqslant \exp(-c_2\sqrt{n}t^2/\gamma)$$

for every $t \geqslant 1$, where σ is the rotationally invariant probability measure on S^{n-1} and $c_1, c_2 > 0$ are absolute constants.

The first general answer to the question was given by Klartag who proved in [8] that every isotropic convex body K in $\mathbb{R}^n$ has a "subgaussian" direction with a constant which is logarithmic in the dimension. An alternative proof with a slightly better estimate was given in [4]. The best known estimate, which appears in [5], follows from an upper bound for the volume of the body $\Psi_2(K)$ with support function

$$h_{\Psi_2(K)}(\theta) := \sup_{2\leqslant q\leqslant n} \frac{\|\langle \cdot, \theta\rangle\|_q}{\sqrt{q}}.$$

It is known that $\|\langle \cdot, \theta\rangle\|_{\psi_2} \simeq \sup_{2\leqslant q\leqslant n} \frac{\|\langle \cdot, \theta\rangle\|_q}{\sqrt{q}}$, and hence, $h_{\Psi_2(K)}(\theta) \simeq \|\langle \cdot, \theta\rangle\|_{\psi_2}$. The main result in [5] states that

$$(*) \qquad \frac{c_1}{\sqrt{n}} L_K \leqslant |\Psi_2(K)|^{1/n} \leqslant \frac{c_2\sqrt{\log n}}{\sqrt{n}} L_K,$$

where $c_1, c_2 > 0$ are absolute constants. A direct consequence of the right hand side inequality in $(*)$ is the existence of subgaussian directions for K with constant $r = O(\sqrt{\log n})$.

The approach in [8], [4] and [5] does not provide estimates on the measure of directions for which an isotropic convex body satisfies a ψ_2-estimate with a given constant r. Klartag obtains some information on this question, but for a different position of K. In [6] we pose the problem of the distribution of the ψ_2-norm of linear functionals on isotropic convex bodies and we provide some first measure estimates. To this end, we introduce the function

$$\psi_K(t) := \sigma\left(\{\theta \in S^{n-1} : h_{\Psi_2(K)}(\theta) \leqslant ct\sqrt{\log n} L_K\}\right).$$

The problem is to give lower bounds for $\psi_K(t)$, $t \geqslant 1$. A general estimate is presented in the next theorem:

Theorem 1. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t \geqslant 1$ we have*

$$(1.9) \qquad \psi_K(t) \geqslant \exp(-cn/t^2),$$

where $c > 0$ is an absolute constant.

For the proof of Theorem 1 we first obtain, for every $\log^2 n \leqslant k \leqslant n$, some information on the ψ_2-behavior of directions in an arbitrary k-dimensional subspace of $\mathbb{R}^n$. Then, this is combined with a simple argument which is based on the fact that the ψ_2-norm is Lipschitz with constant $O(\sqrt{n}L_K)$, and hence, it is stable on a spherical cap of the appropriate radius.

It is known (for example, see [9]) that every isotropic convex body K is contained in $[(n+1)L_K]B_2^n$, and hence, we have $\psi_K(t) = 1$ if $t \geqslant c\sqrt{n/\log n}$. Therefore, the bound of Theorem 1 is of some interest only when $1 \leqslant t \leqslant c\sqrt{n/\log n}$. Actually, if $t \geqslant c\sqrt[4]{n}$ then we have much better information:

Proposition 2. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every*

$$t \geqslant c_1\sqrt[4]{n}/\sqrt{\log n}$$

one has

$$\psi_K(t) \geqslant 1 - e^{-c_2 t^2 \log n},$$

where $c_1, c_2 > 0$ are absolute constants.

The problem of bounding from above the mean width in isotropic position is central in the asymptotic theory of convex bodies. There are several approaches that lead to the best (presently) estimate: for every isotropic convex body K in $\mathbb{R}^n$ one has

$$w(K) \leqslant cn^{3/4} L_K,$$

where $c > 0$ is an absolute constant. The first one appeared in the PhD Thesis of M. Hartzoulaki [7] and was based on a result from [3] regarding the mean width of a convex body under assumptions on the regularity of its covering numbers. The second one is more recent and is due to P. Pivovarov [14]; it relates the question to the geometry of random polytopes with vertices independently and uniformly distributed in K and makes use of the concentration inequality of [13]. A third – very direct – proof of this bound can be based on the "theory of L_q–centroid bodies" which was developed by the second named author (see [6]). In [6] we propose one more approach, which can exploit our knowledge on $\psi_K(t)$ and provide a dichotomy argument in terms of some global parameter $\rho_* \equiv \rho_*(K)$ of the body K. In particular we have the following:

Theorem 3. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, one has*

$$w(K) \leqslant C\sqrt{n} \min\{\sqrt{\rho_*}, \sqrt{n \log n/\rho_*}\} L_K,$$

where $C > 0$ is an absolute constant.

Note that from this Theorem we recover the best known estimate, up to some logarithmic in dimension term.

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On the inverse Klain map

Thomas Wannerer
(joint work with Lukas Parapatits)

The Klain map is a useful tool in the theory of even valuations. While the Klain map is continuous and monotone, we show that its inverse is in general neither continuous nor monotone. More precisely, in a joint work with Parapatits [1] we characterize the class of those centrally symmetric convex bodies at which every i-homogeneous valuation depends continuously on its Klain function. A similar characterization regarding the monotonicity of the inverse Klain map is also presented. We explain how this result can be used to show that McMullen's decomposition is not possible in the class of translation-invariant, continuous, positive valuations. This implies that there exists no McMullen decomposition for translation-invariant, continuous Minkowski valuations, which solves a problem first posed by Schneider and Schuster [2].

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On the volume of polars of convex bodies

Manuel Weberndorfer

The Blaschke-Santaló inequality is one of the most famous isoperimetric inequalities. It states that the volume of the polar of a convex body is bounded from above by the reciprocal of the volume of the convex body K itself. More precisely,

$$V(K^*) \leq w_n^2 \frac{1}{V(K)},$$

where K^* denotes the the polar body with respect to the Santaló point, and w_n denotes the volume of the Euclidean unit n-ball.

The dual problem to find a corresponding sharp lower bound for $V(K^*)$ is now open for around 80 years. This is not due to a lack of interest and, in fact, substantial inroads have been made recently (see e.g. [4, 5, 7, 10, 15]). For instance, sharp lower bounds for the volume of polars of convex bodies in terms of the volume of the John ellipsoid JK, and the dual Legendre ellipsoid L_2K, have been obtained: Barthe [2] has established that

$$V(K^*) \geq \frac{w_n(n+1)^{(n+1)/2}}{n^{n/2}n!}\frac{1}{V(JK)}, \tag{1}$$

and Lutwak, Yang, and Zhang [13], have proved the inequality

$$V(K^o) \geq \frac{w_n(n+1)^{(n+1)/2}}{n^{n/2}n!}\frac{1}{V(L_2K)}, \tag{2}$$

where K^o denotes the polar body with respect to the origin. Moreover, Reisner has obtained the sharp lower bound for $V(K^*)$ in terms of the volume of K under the additional assumption that K is a zonoid [8, 16]. In this talk we present two theorems: The first of these theorems has inequalities (1) and (2) as special cases, and the second theorem extends Reisner's result for zonotopes.

First, we consider Wulff shapes

$$W_{\nu,f} = \{x \in \mathbb{R}^n : \ x \cdot u \leq f(u) \text{ for all } u \in \operatorname{supp}\nu\}$$

generated by positive continuous functions f on the sphere and isotropic f-centered measures ν on the sphere. Here, a measure ν is called isotropic if

$$\int_{S^{n-1}} |x \cdot u|^2 \, d\nu(u) = |x|^2 \qquad \text{for every } x \in \mathbb{R}^n,$$

and f-centered if the centroid of $f d\nu$ lies at the origin. In a joint work with F. E. Schuster, we have obtained a sharp lower bound for the volume of polars of Wulff shapes that immediately implies inequalities (1) and (2). The proof uses ideas of Ball [1], Barthe [3], and Lutwak, Yang, and Zhang [13]. We remark that the corresponding result in the origin-symmetric setting has been obtained by Lutwak, Yang, and Zhang [12].

Theorem 1 (F. E. Schuster and W. [17]). *Suppose f is a positive continuous function on S^{n-1} and ν is an isotropic f-centered measure. Then*

$$V(W^o_{\nu,f}) \geq \frac{(n+1)^{(n+1)/2}}{n!}\|f\|^{-n}_{L_2(\nu)}$$

with equality if and only if $\operatorname{conv}\operatorname{supp}\nu$ *is a regular simplex inscribed in S^{n-1} and f is constant on* $\operatorname{supp}\nu$.

Next we move on to generalizations of Reisner's result for zonotopes. Recently, Campi and Gronchi [6] have considered the volume of polars of L_p zonotopes $Z_p\Lambda$

associated with finite spanning sets $\Lambda \subseteq \mathbb{R}^n$. For $p \geq 1$, these are defined by their support function

$$h(Z_p\Lambda, u) = \sqrt[p]{\sum_{v\in\Lambda} |v \cdot u|^p}.$$

Campi and Gronchi have obtained the inequality

$$V(Z_p^*\Lambda) \geq c_{p,n} \frac{1}{V(Z_1\Lambda)},$$

where $c_{p,n} = V(Z_p^*\Lambda_\perp)V(Z_1\Lambda_\perp)$ for the canonical basis $\Lambda_\perp$. To extend their result in the spirit of the asymmetric L_p Brunn-Minkowski theory (see [9, 11]), we consider *asymmetric* L_p zonotopes $Z_p^+\Lambda$ defined by their support function

$$h(Z_p^+\Lambda, u) = \sqrt[p]{\sum_{v\in\Lambda} \max\{v \cdot u, 0\}^p}.$$

Using ideas and results of Campi, Gronchi, Meyer, and Reisner [6, 14], we have established a sharp lower bound for the polars of these asymmetric L_p zonotopes in terms of $V(Z_1^+\Lambda)$, along with its equality conditions. Note that it contains Reisner's result for zonotopes as a special case.

Theorem 2 (W. [18]). *Suppose $p \geq 1$ and $\Lambda \subseteq \mathbb{R}^n \setminus \{o\}$ is finite and spanning. Then*

$$V(Z_p^{+,*}\Lambda) \geq c_{p,n}^+ \frac{1}{V(Z_1^+\Lambda)},$$

where $c_{p,n}^+ = V(Z_p^{+,*}\Lambda_\perp)V(Z_1^+\Lambda_\perp)$. Equality holds for $p > 1$ if and only if Λ is a linear image of the canonical basis. If $p = 1$, then equality holds if and only if Z_1^+ is a parallelepiped.

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Mean Section Bodies and Surface Area Measures

Wolfgang Weil

(joint work with Paul Goodey)

For $2 \leq k \leq d-1$, the k-th mean section body, $M_k(K)$, of a convex body K in $\mathbb{R}^d$, was introduced in [2] as the Minkowski sum of all sections of K by k-dimensional flats. Here, we show that the characterization of these mean section bodies is equivalent to the solution of the general Minkowski problem, namely that of giving the characteristic properties of those measures on the unit sphere which arise as surface area measures (of arbitrary degree) of convex bodies. This equivalence arises from an analysis of Berg's [1] solution of the Christoffel problem. Using the functions introduced by Berg, we give an integral representation of the support function of $M_k(K)$ in terms of the $(d+1-k)$-th surface area measure of K. Our results are based on Fourier transform techniques from [3] which also yield a stability version of the fact that $M_k(K)$ determines K uniquely.

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Divergence inequalities for log concave functions

Elisabeth Werner
(joint work with Umut Caglar, Carsten Schütt)

There is a general approach to extend invariants and inequalities of convex bodies to the corresponding invariants and inequalities for functions. Among the best known affine isoperimetric inequalities is the Blaschke Santaló inequality ([4, 7, 8]. The corresponding inequalities for log concave functions were proved by Ball [3] and Artstein, Klartag and Milman [1] (see also [5, 6]). A stronger inequality than the Blaschke Santaló inequality is the affine isoperimetric inequality for convex bodies. It involves the affine surface area $as_1(K) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x)$. Here, κ is the Gauss curvature and μ the usual surface measure on the boundary ∂K. Then

$$\frac{as_1(K)}{as(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-1}{n+1}},$$

with equality if and only if K is an ellipsoid.

The equivalent of this inequality for log concave functions was established in [2]: For every log-concave function $\varphi : \mathbb{R}^n \to [0,\infty)$ with enough smoothness and integrability properties,

$$\int_{\text{supp}(\varphi)} \varphi \, \ln\Big(\det\left(\text{Hess}\left(-\ln\varphi\right)\right)\Big) dx \leq 2\,\text{Ent}(\varphi) + \|\varphi\|_{L^1} \ln(2\pi e)^n, \tag{1}$$

where, for $\varphi \in L^1(\mathbb{R}^n, dx)$, the Lebesgue integrable functions on $\mathbb{R}^n$, $\|\varphi\|_{L^1} = \int_{\mathbb{R}^n} \varphi dx$, $\text{Hess}(\varphi) = \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n}$ is the Hessian of φ, and

$$\text{Ent}(\varphi) = \int_{\text{supp}(\varphi)} \varphi \ln \varphi dx - \|\varphi\|_{L^1} \ln \|\varphi\|_{L^1} \tag{2}$$

is the entropy of φ.

Thus, the affine isoperimetric inequality corresponds to a reverse log Sobolev inequality for entropy.

The characterization of equality in inequality (1) remained an open problem. Here, we not only settle the equality characterization, but also strengthen and generalize inequality (1) and prove the following entropy inequality for log concave functions.

Theorem 1. *Let $f : (0,\infty) \to \mathbb{R}$ be a convex function. Let $\varphi : \mathbb{R}^n \to [0,\infty)$ be an upper semi-continuous log-concave function with $0 \in int\,(supp(\varphi))$ which belongs to $C^2(supp(\varphi)) \cap L^1(\mathbb{R}^n, dx)$ and is such that φ° and* $\varphi f\left(\frac{e^{\frac{\langle \nabla \varphi, x\rangle}{\varphi}}}{\varphi^2} det\left(Hess\left(-\ln\varphi\right)\right)\right)$

are in $L^1(supp(\varphi), dx)$. *Then*

$$\int_{supp(\varphi)} \varphi\; f\left(e^{\langle \frac{\nabla\varphi}{\varphi}, x\rangle} \varphi^{-2} \left(\det\left(Hess\left(-\ln\varphi\right)\right)\right)\right) dx$$

$$\geq\; f\left(\frac{\int \varphi^\circ dx}{\int \varphi dx}\right) \left(\int_{supp(\varphi)} \varphi dx\right). \tag{3}$$

If f *is concave, the inequality is reversed. If* f *is linear, equality holds in (3).*

Moreover, if f *is strictly convex resp. concave, equality holds in (3) iff* $\varphi(x) = Ce^{-\langle Ax, x\rangle}$*, where* C *is a positive constant and* A *is an* $n \times n$ *symmetric positive definite matrix.*

Here, $\nabla\varphi$ denotes the gradient of φ and $\varphi^\circ = \inf_{y\in\mathbb{R}^n}\left[\frac{e^{-\langle x,y\rangle}}{\varphi(y)}\right]$ [1] is the dual function of φ.

The characterization of the equality case is obtained as a consequence of the uniqueness of the solution of a certain Monge Ampere differential equation.

If we let $f(t) = \ln t$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let* $\varphi : \mathbb{R}^n \to [0, \infty)$ *be an upper semi-continuous log-concave function with* $0 \in int(supp(\varphi))$ *which belongs to* $C^2(supp(\varphi)) \cap L^1(\mathbb{R}^n, dx)$ *and is such that* φ° *and* $\varphi \ln\left(\det\left(Hess\left(-\ln\varphi\right)\right)\right) \in L^1(supp(\varphi), dx)$. *Then*

$$\int_{supp(\varphi)} \varphi\; \ln\left(\det\left(Hess\left(-\ln\varphi\right)\right)\right) dx$$

$$\leq 2\,\mathrm{Ent}(\varphi) + \|\varphi\|_{L^1} \ln\left[e^n \left(\int\varphi\right)\left(\int\varphi^\circ\right)\right], \tag{4}$$

with equality iff $\varphi(x) = Ce^{-\langle Ax, x\rangle}$*, where* C *is a positive constant and* A *is an* $n \times n$ *symmetric positive definite matrix.*

Inequality (4) is stronger than (1): inequality (1) follows from inequality (4) with the functional form of the Blaschke Santaló inequality [1, 3, 5, 6].

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Detecting symmetry in star bodies

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(joint work with D. Ryabogin)

Let K be a convex body in $\mathbb{R}^n$, i.e. a compact convex set with a non-empty interior. We say that K is *origin-symmetric* if $K = -K$. The presence of origin-symmetry is an essential assumption in various problems. Many results that hold for origin-symmetric convex bodies fail in the absence of the symmetry condition. For example, origin-symmetric convex bodies are uniquely determined by the volumes of their projections or central sections, while this is not true for general convex bodies; see [1]. Thus, detecting symmetry in convex bodies is one of the fundamental questions in convex geometry and geometric tomography.

In [4] we suggest a new method of detecting symmetry. Let K be a star body and let $C(\xi, z)$ be the cone $\{x \in \mathbb{R}^n : x \cdot \xi = |x|z\}$, where $\xi \in S^{n-1}$, $z \in (-1, 1)$, and $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n$ is the usual inner product in $\mathbb{R}^n$. In this notation, z is the cosine of the angle between x and ξ. For $z \in (-1, 1)$, we define the *conical section function* $C_{K,\xi}(z)$ by

$$C_{K,\xi}(z) = \mathrm{vol}_{n-1}(K \cap C(\xi, z)).$$

Clearly, if K is an origin-symmetric star body, then for each ξ the function $C_{K,\xi}(z)$ is an even function of z, and therefore has a critical point at $z = 0$. In [4] we show that the converse statement is also true.

Theorem 1. *Let K be a C^1 star body in $\mathbb{R}^n$. Assume that for each $\xi \in S^{n-1}$ the function $C_{K,\xi}(z)$ has a critical point at $z = 0$. Then the body K is origin-symmetric.*

This theorem is an analog of the result by Makai, Martini and Ódor [3], which can be stated as follows.

Theorem 2. *Let K be a C^1 star body in $\mathbb{R}^n$. If for every $\xi \in S^{n-1}$ the function $A_{K,\xi}(t)$ has a critical point at $t = 0$, then K is origin-symmetric.*

Here, $A_{K,\xi}(t)$ is the *parallel section function* defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap (\xi^\perp + t\xi)), \quad t \in \mathbb{R},$$

and $\xi^\perp = \{x \in \mathbb{R}^n : x \cdot \xi = 0\}$ is the hyperplane passing through the origin and orthogonal to the vector ξ.

Makai, Martini and Ódor proved Theorem 2 in the class of convex bodies, in which case the C^1-smoothness assumption can be dropped. Using the same reasoning, it can be shown that Theorem 1 also holds true for convex bodies without the smoothness assumption.

The techniques that we use to prove our results were developed by Koldobsky (see [2]) and are based on the Fourier transform of distributions. Using these methods we also give a new proof of Theorem 2.

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