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Algebraic Groups

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Abstract. Linear algebraic groups is an active research area in contemporary mathematics. It has rich connections to algebraic geometry, representation theory, algebraic combinatorics, number theory, algebraic topology, and differential equations. The foundations of this theory were laid by A. Borel, C. Chevalley, T. A. Springer and J. Tits in the second half of the 20th century. The Oberwolfach workshops on algebraic groups, led by Springer and Tits, played an important role in this effort as a forum for researchers, meeting at approximately 3 year intervals since the 1960s. The present workshop continued this tradition, featuring a number of important recent developments in the subject.

Mathematics Subject Classification (2010): 14Lxx, 17Bxx, 20Gxx.

Introduction by the Organisers

Linear algebraic groups originated in the work of E. Picard in the mid-19th century. Picard assigned a "Galois group" to an ordinary differential equation. This construction was developed into what is now known as "differential Galois theory" by J. F. Ritt and E. R. Kolchin in the 1930s and 40s. Their work was a precursor to the modern theory of algebraic groups, founded by A. Borel, C. Chevalley, T. A. Springer and J. Tits in the second half of the 20th century. The Oberwolfach workshops on algebraic groups, organized by Springer and Tits, played an important role in this effort as a forum for researchers, meeting at (approximately) three year intervals since the 1960s.

The present workshop continued this tradition. There were 52 participants from 12 countries (Australia, Canada, China, Denmark, France, Germany, Great

Britain, India, Italy, Russia, Switzerland and the United States). The scientific program consisted of 24 lectures and a problem session. The lectures were selected from over 35 proposals submitted by the participants. There were two mini-series, by Roland Lötscher on *Essential and Canonical Dimension* and Nicolas Ressayre on *The semigroup of branching rules*; each consisted of two expository lectures. The other speakers covered a broad range of topics, including

- spherical varieties,
- geometry and topology of Grassmannians and flag varieties,
- structure of linear algebraic groups over imperfect fields,
- the Cremona group,
- geometric invariant theory,
- W-algebras,
- Soergel bimodules, and
- Kazhdan-Lusztig polynomials.

Recreational activities during the workshop included a Tuesday night classical music performance by one of the participants (Igor Rapinchuk) and a traditional Wednesday afternoon hike.

Workshop: Algebraic Groups

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Abstracts

W-algebras and instanton moduli spaces

Alexander Braverman
(joint work with Michael Finkelberg and Hiraku Nakajima)

This talk is based on a joint work in progress with M. Finkelberg and H. Nakajima. Let G be a simply laced simply connected algebraic group over $\mathbb{C}$. Let $\mathcal{U}_G^d$ denote the Uhlenbeck partial compactification of the moduli space of G-bundles on $\mathbb{P}^2$ endowed with a trivialization at the "infinite" line $\mathbb{P}^1 \subset \mathbb{P}^2$. It has a natural action of the group G (by changing the trivialization) and of the group $GL(2)$. Set

$$M = \bigoplus_{d=0}^{\infty} IH^*_{G\times GL(2)}(\mathcal{U}_G^d)$$

where $IH^*_{G\times GL(2)}$ stands for equivariant intersection cohomology. Let also M_{loc} denote the corresponding localized equivariant intersection cohomology.

Our main result is (roughly speaking) a construction of the action of the W-algebra associated to the affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ on M_{loc} (there is a more precise version which does not require localization) satisfying some natural properties (in particular, it makes M_{loc} into a universal Verma module over the W-algebra). This result was conjectured by physicists (it is a part of the so called "AGT correspondence"). In the case $G = SL(n)$ such an action was constructed earlier by Maulik-Okounkov and Schiffmann-Vasserot.

Curve Neighborhoods of Schubert varieties

Anders S. Buch
(joint work with Leonardo C. Mihalcea)

1. The main result

The title of this talk refers to a recent paper [4] with Mihalcea, but my talk is also closely related to joint work with Chaput, Mihalcea, and Perrin [2].

Let X be a non-singular complex variety, let $\Omega \subset X$ be a closed subvariety, and let $d \in H_2(X) = H_2(X;\mathbb{Z})$ be a degree. The *curve neighborhood* $\Gamma_d(\Omega)$ is defined as the closure of the union of all rational curves in X of degree d that meet Ω. For example, if $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\Omega = \mathbb{P}^1 \times \{0\}$, then $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$, and we have $\Gamma_{(1,0)}(\Omega) = \Omega$ and $\Gamma_{(0,1)}(\Omega) = X$.

I will focus on the case where $X = G/P$ is a generalized flag variety, defined by a semisimple complex Lie group G and a parabolic subgroup P. I also fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. In this case it was proved in [2] that, if Ω is irreducible, then $\Gamma_d(\Omega)$ is irreducible. Notice also that $\Gamma_d(\Omega)$ is B-stable whenever Ω is B-stable. It follows that if Ω is a Schubert variety in X, then $\Gamma_d(\Omega)$ is also a Schubert variety.

It is natural to ask which Schubert variety this is. In other words, if we know the Weyl group element representing Ω, then what is the Weyl group element representing $\Gamma_d(\Omega)$? This question is related to several aspects of the quantum cohomology and quantum K-theory of homogeneous spaces, including two-point Gromov-Witten invariants, the (equivariant) quantum Chevalley formula [6, 7], the minimal powers of the deformation parameter q that appear in quantum products of Schubert classes [6], and a degree distance formula for cominuscule varieties [5] that played an important role in a generalization of the kernel-span technique from [1] and the quantum equals classical theorem from [3].

Let $W = N_G(T)/T$ be the Weyl group of G and let $W_P = N_P(T)/T \subset W$ be the Weyl group of P. We let $W^P \subset W$ denote the subset of minimal length representatives for the cosets in W/W_P. Each element $w \in W$ defines a Schubert variety $X(w) = \overline{Bw.P} \subset X$; if $w \in W^P$ then $\dim X(w) = \ell(w)$. The set of T-fixed points in X is $X^T = \{w.P \mid w \in W^P\}$. We let R be the root system of G, with positive roots R^+ and simple roots $\Delta \subset R^+$.

We describe the curve neighborhood of a Schubert variety in terms of the *Hecke product* of Weyl group elements, which can be defined as follows. For $w \in W$ and $\beta \in \Delta$ we set

$$w \cdot s_\beta = \begin{cases} w\, s_\beta & \text{if } \ell(ws_\beta) > \ell(w); \\ w & \text{if } \ell(ws_\beta) < \ell(w). \end{cases}$$

Given an additional element $w' \in W$ and a reduced expression $w' = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_\ell}$, we then define $w \cdot w' = w \cdot s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_\ell} \in W$, where the simple reflections are Hecke-multiplied to w in left to right order. This defines an associative monoid product on W. The Hecke product is compatible with the Bruhat order on W, for example we have $v \leq v' \Rightarrow u \cdot v \cdot w \leq u \cdot v' \cdot w$ for all $u, v, v', w \in W$.

Given a positive root $\alpha \in R^+$ with $s_\alpha \notin W_P$, let $C_\alpha \subset X$ be the unique T-stable curve that contains the points $1.P$ and $s_\alpha.P$. The main result of [4] is the following theorem, which makes it straightforward to compute the Weyl group element representing the curve neighborhood $\Gamma_d(X(w))$.

Theorem 1. *Assume that* $0 < d \in H_2(X)$, *and let* $\alpha \in R^+$ *be any positive root that is maximal with the property that* $[C_\alpha] \leq d \in H_2(X)$. *Then we have* $\Gamma_d(X(w)) = \Gamma_{d-[C_\alpha]}(X(w \cdot s_\alpha))$.

We remark that the homology group $H_2(X)$ can be identified with the coroot lattice of R modulo the coroots corresponding to P, in such a way that the class $[C_\alpha] \in H_2(X)$ is the image of the coroot $\alpha^\vee$. Theorem 1 therefore makes simultaneous use of the orderings of roots and coroots, which gives rise to interesting combinatorics.

2. Degree distance formula

Theorem 1 can be used to give simple proofs of several well known results concerning the quantum cohomology of generalized flag varieties. Here we will sketch a proof of the degree distance formula for cominuscule varieties due to Chaput, Manivel, and Perrin [5].

Assume that $X = G/P$ where P is a maximal parabolic subgroup of G, and let $\gamma \in \Delta$ be the unique simple root such that $s_\gamma \notin W_P$. Then $H_2(X) = \mathbb{Z}$ has rank one, and the generator $[X(s_\gamma)] \in H_2(X)$ can be identified with $1 \in \mathbb{Z}$. The variety X is called *cominuscule* if, when the highest root $\rho \in R^+$ is expressed as a linear combination of simple roots, the coefficient of γ is one. This implies that $\rho = w_P.\gamma$ where w_P denotes the longest element in W_P. In particular, since $\rho^\vee - \gamma^\vee$ is a linear combination of the coroots of P, we obtain $[C_\rho] = [C_\gamma] = 1 \in H_2(X)$. Given any effective degree $d \in H_2(X)$, it therefore follows from Theorem 1 that

$$\Gamma_d(X(w)) = \Gamma_{d-1}(X(w \cdot s_\gamma)) = \cdots = X(w \cdot s_\gamma \cdot s_\gamma \cdot \ldots \cdot s_\gamma)$$

where s_γ is repeated d times. Since $s_\rho = w_P s_\gamma w_P$, this identity is equivalent to the expression

$$\Gamma_d(X(w)) = X(w \cdot w_P s_\gamma \cdot w_P s_\gamma \cdot \ldots \cdot w_P s_\gamma)\,, \tag{1}$$

with $w_P s_\gamma$ repeated d times.

Given two points $x, y \in X$, let $d(x, y)$ denote the smallest possible degree of a rational curve in X from x to y. This number is determined by the following result from [5].

Corollary (Chaput, Manivel, Perrin)**.** *Let $u \in W^P$. Then $d(1.P, u.P)$ is the number of occurrences of s_γ in any reduced expression for u.*

Proof. For $d \in H_2(X)$, it follows from (1) that $u.P \in \Gamma_d(X(1))$ if and only if u has a reduced expression with at most d occurrences of s_γ. Now set $d = d(1.P, u.P)$ and observe that $u.P \in \Gamma_d(X(1)) \smallsetminus \Gamma_{d-1}(X(1))$. We deduce that u has a reduced expression with exactly d occurrences of of s_γ. The corollary now follows from Stembridge's result [8] that u is fully commutative, i.e. any reduced expression for u can be obtained from any other by interchanging commuting simple reflections. ☐

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On the rationality of some homogeneous spaces

Mathieu Florence
(joint work with Michel Van Garrel)

Let k be a field and let G/k be a linear algebraic group. Let V be a finite-dimensional generically free representation of G over k. We say that Noether's problem for G has a positive answer if the (birational) quotient V/G is a stably rational k-variety. The 'no-name lemma' asserts that this property does not depend on the choice of V.
Assume that Noether's problem for G has a positive answer. Given V as before, we consider the following problem:

$Q(G, V)$: is V/G -rational- over k?

To the author's knowledge, there is no known counterexample to this question.
We now focus on the case where $G = \mathrm{GL}_1(A)$, the group of invertible elements of a finite-dimensional k-algebra A. It is well-known that Noether's problem has a positive answer for such a group.
Note that a positive answer to $Q(\mathrm{GL}_1(A), V)$ for all -étale- k-algebras A and all generically free representations of these is equivalent to Voskresenskii's widely open conjecture, stating that a k-torus which is stably rational should be rational.
The main result of this talk is a generalization of a theorem by Klyachko, and can be stated as follows.

Let A and B be two finite-dimensional k-algebras, of coprime dimensions. Assume that A and B satisfy a mild hypothesis, that we shall not state here, but that is for instance verified if A and B are semi-simple algebras. Denote by G the algebraic group $\mathrm{GL}_1(A \otimes_k B)$ and by H the subgroup of G generated by $\mathrm{GL}_1(A)$ and $\mathrm{GL}_1(B)$; that is, the subgroup generated by pure tensors. Then the quotient G/H is a rational variety.

Klyachko proved this result when A and B are étale k-algebras. Note that it is very difficult to extract from his proof an explicit birational isomorphism between G/H and an affine space in this case. Our proof gives a quite simple, easily computable birational isomorphism.

To finish, we discussed a few other examples where $Q(G, V)$ has a positive answer.

Properties of pseudo-reductive groups and central extensions

Ofer Gabber

The analysis of arguments in the theory of pseudo-reductive groups and in particular the construction from birational groups laws in [1] Ch. 9 and [3] leads to general statements concerning pairs (G, S) where G is a smooth connected affine algebraic group over the gound field k and S is a k-torus in G. As in [1] 3.3, one defines subgroups $H_A(G)$ (denoted $U_A(G)$ if $0 \notin A$) when A is a subsemigroup of $X^*(S)_{\mathbb{Q}}$. For a non-zero α in $X^*(S)_{\mathbb{Q}}$ consider the open ray (α) of positive rational multiples of α and the closed ray $\{0\} \cup (\alpha)$. We say that (G, S) is of Weyl type if (after splitting S) for every non-zero weight α of S on $\mathrm{Lie}(G)$, the normalizer of S in $H_{\mathbb{Q}\alpha}(G)$ is not $Z_G(S)$. We consider in the following definitions only morphisms of pairs $(G', S') \to (G, S)$ where $S' \to S$ is an isogeny. We say the morphism is central if $G' \to G$ is surjective with central kernel. We say that $(G', S') \to (G, S)$ is wpsl if for every open ray λ, $U_\lambda(G') \to U_\lambda(G)$ is injective on k_s-points; we call it psl if it is wpsl and $Z_{G'}(S') \to Z_G(S)$ has finite kernel on k_s-points. A psl morphism with target (G, S) is determined uniquely (up to a unique isomorphism) by giving the morphisms $(Z_{G'}(S'), S') \to (Z_G(S), S)$ and $U_\lambda(G') \to U_\lambda(G)$ for every open ray λ, and such data comes from a psl morphism iff certain conditions involving only the restrictions to sums of two closed rays are satisfied. A system of morphisms with target $U_\lambda(G)$ for all open rays λ comes from a wpsl (equivalently from a psl) morphism of pairs iff the relevant part of the above conditions holds.

Theorem 1. Let (G, S) be a pair of Weyl type, $(\alpha_i)_{1 \le i \le n}$ a basis of radicial rays. Then the category of wpsl morphisms $(G', S') \to (G, S)$ where (G', S') is of Weyl type with the same radicial rays as (G, S), is equivalent to the 2-fibred product of the corresponding categories for $(H_{\mathbb{Q}\alpha_i}(G), S)$ over the corresponding category for $(Z_G(S), S)$.

Theorem 2. Let (G, S) be a pair such that S surjects onto $G/\mathscr{D}(G)$ and $S_1 \to S$ an isogeny of k-tori. Then the category of central morphisms $(G', S') \to (G, S)$ such that $S_1 \to S$ dominates $S' \to S$ has an initial object, which remains initial after any ground field extension.

Theorem 2 implies that a perfect smooth connected group G has a universal central extension and that for G generated by tori there is a universal central extension with unipotent kernel. For the universal central extension $G' \to G$ for perfect G as above one has that G' is perfect and the space of coinvariants $\mathrm{Lie}(G')_{G'}$ is zero, so $G'(k_s)$ is perfect and its image in $G(k_s)$ is the derived group of $G(k_s)$. Also one has that the formal group $\hat{G}'$ is a universal central extension of $\hat{G}$.

For a perfect pseudo-reductive k-group G with a maximal torus S, let $(G', S') \to (G, S)$ be the universal central extension and construct the largest intermediate extension (G'', S') having the property that the automorphisms of G (over k_s)

restricting to the identity on S lift to automorphisms of G'' inducing the identity on $Z_{G''}(S')$. G'' is quasi-reductive and maps to any pseudo-reductive central extension of G. If $\mathrm{char}(k) \neq 2$ or $\mathrm{char}(k) = 2$ and $[k : k^2] \leq 2$, G'' is a product of Weil restrictions of simply connected groups, basic exotic groups and basic non-reduced groups in the sense of [1]. In characteristic 2, G'' generalizes the construction in terms of Clifford algebras in [1] Prop. 9.1.9. If S is split, G'' is k-rational and $G''(k) = G''(k)^+$.

If A is a finite commutative k-algebra then $G = \mathrm{R}_{A/k}(\mathrm{SL}_n)$ $(n \geq 2)$ is perfect and by analogues of [2] Th. 3.4 and 3.7, the kernel of the universal central extension of G is the target $D'(A)$ (resp. $D(A)$) of the universal Steinberg cocycle (resp. bilinear Steinberg cocycle) on $\mathrm{R}_{A/k}(\mathbb{G}_m)$ for $n = 2$ (resp. $n > 2$). When A is generated by one element such Steinberg cocycles vanish. When A is a field, $D(A)$ and $D'(A)$ should be successive extensions of groups of logarithmic forms.

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W-algebras and Whittaker coinvariants

Simon M. Goodwin

(joint work with Tomoyuki Arakawa)

The finite W-algebra $U(\mathfrak{g}, e)$ associated to a reductive Lie algebra $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$ is a quantization of the Slodowy slice through the nilpotent orbit of e. They were introduced to the mathematics literature by Premet in [8] in 2002, and have subsequently attracted a great deal of research interest, see for example Losev's survey article [5]. Under a slightly different guise finite W-algebras and affine W-algebras have been studied in mathematical physics since the 1980s, see for example [3] and the references therein; the different definitions are now known to be equivalent.

The definition of $U(\mathfrak{g}, e)$ involves a nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ and a character χ of $\mathfrak{m}$ constructed from e. Then we define $\mathfrak{m}_\chi = \{x - \chi(x) \mid x \in \mathfrak{m}\}$, which is a Lie subalgebra of $U(\mathfrak{g})$. The W-algebra $U(\mathfrak{g}, e)$ associated to $\mathfrak{g}$ and e is defined by

$$\begin{aligned} U(\mathfrak{g}, e) &= (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi)^{\mathfrak{m}_\chi} = \{u + U(\mathfrak{g})\mathfrak{m}_\chi \mid \mathfrak{m}_\chi u \subseteq U(\mathfrak{g})\mathfrak{m}_\chi\} \\ &\cong {}^{\mathfrak{m}_\chi}(U(\mathfrak{g})/\mathfrak{m}_\chi U(\mathfrak{g})) = \{u + \mathfrak{m}_\chi U(\mathfrak{g}) \mid u\mathfrak{m}_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g})\}. \end{aligned}$$

The isomorphism between the two definitions involves a "shift" as explained in [1, Cor. 2.9].

An important connection between the representation theory of $U(\mathfrak{g}, e)$ and $U(\mathfrak{g})$ stems from the Whittaker invariants functor $H^0(\mathfrak{m}_\chi, ?) : U(\mathfrak{g})\text{-mod} \to U(\mathfrak{g}, e)\text{-mod}$ given by $H^0(\mathfrak{m}_\chi, M) = M^{\mathfrak{m}_\chi}$. It was proved by Skryabin, [10], that this functor is an equivalence of categories when restricted to $\mathfrak{m}_\chi$-Whittaker $U(\mathfrak{g})$-modules, i.e. $U(\mathfrak{g})$-modules on which $\mathfrak{m}_\chi$ acts locally nilpotently. Subsequently this link has been developed in work of Losev, Premet and Ginzburg leading to a bijective correspondence between:

- the isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules up to the action of the component group Γ_e of the centralizer of e; and
- the primitive ideals of $U(\mathfrak{g})$ with associated variety given by the closure of the nilpotent orbit of e.

This correspondence is explained in [5, Theorem 5.4] where further references are also given.

The Whittaker coinvariants functor $H_0(\mathfrak{m}_\chi, ?) : U(\mathfrak{g})\text{-mod} \to U(\mathfrak{g}, e)\text{-mod}$ given by $H_0(\mathfrak{m}_\chi, M) = M/\mathfrak{m}_\chi M$ was the focus of this talk. When restricted to modules that are finitely generated over $\mathfrak{m}_\chi$ this functor is exact and the images are finite dimensional. In joint work (in preparation) with Arakawa we have considered the image under $H_0(\mathfrak{m}_\chi, ?)$ of irreducible highest weight modules $L(\lambda)$ for $U(\mathfrak{g})$. We have shown that $H_0(\mathfrak{m}_\chi, L(\lambda))$ is a direct sum over Γ_e-conjugates, of $L(\Lambda)$ with possible multiplicities, where $L(\Lambda)$ is the irreducible highest weight module for $U(\mathfrak{g}, e)$ "corresponding to λ"; these irreducible modules are defined in [1, Section 4]. Moreover, the multiplicity of each summand is equal and we give a reduction to calculating this multiplicity to the case where e is distinguished; in particular proving that it is equal to 1 in case e is of standard Levi type. In case $\mathfrak{g}$ is of type A this is all known from the work of Brundan and Kleshchev, see [2, Corollary 8.24]. Central to our approach is the relationship between Harish-Chandra bimodules for $U(\mathfrak{g})$ and $U(\mathfrak{g}, e)$ as developed by Losev in [6], which builds on work of Ginzburg in [4]. Our methods are also applicable for the case of affine W-algebras, so our results have analogues there. They also lead to a theoretical method of calculating characters of finite dimensional irreducible modules for W-algebras lying in the image of the Whittaker coinvariants functor.

Recently, Losev has introduced generalized Soergel functors, which give a significant generalization of the Whittaker coinvariants functor, see [7]. For e even the Whittaker coinvariants functor is a special case of a generalized Soergel functor. Losev has proved analogues of our result and deeper theorems about these generalized Soergel functors.

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Quiver varieties of type D

Anthony Henderson
(joint work with Anthony Licata)

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and G its adjoint group. It is well known that G acts with finitely many orbits on the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. Given $e \in \mathcal{N}$, there is a standard construction of a transverse slice $\mathcal{S}_e$ to the orbit $G \cdot e$ in $\mathcal{N}$, known as the *Slodowy variety*. Namely, one chooses $f \in \mathcal{N}$ so that $\{e, [e, f], f\}$ is an $\mathfrak{sl}_2$-triple, and defines $\mathcal{S}_e$ as $\{x \in \mathcal{N} \,|\, [x - e, f] = 0\}$.

Let $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ denote the Springer resolution of $\mathcal{N}$, defined by $\widetilde{\mathcal{N}} = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \,|\, x \in \mathfrak{b}\}$ where $\mathcal{B}$ is the variety of Borel subalgebras of $\mathfrak{g}$. Here $\mu(x, \mathfrak{b}) = x$. Slodowy proved in [9] that the restriction of μ to $\widetilde{\mathcal{S}}_e = \mu^{-1}(\mathcal{S}_e)$ is a resolution of $\mathcal{S}_e$. Note that the Springer fibre $\mathcal{B}_e = \mu^{-1}(e)$ is a subvariety of $\widetilde{\mathcal{S}}_e$.

In the classical types, the nilpotent orbits are labelled by partitions λ. We write $\mathcal{S}_\lambda^{\mathrm{X}_\ell}$, $\widetilde{\mathcal{S}}_\lambda^{\mathrm{X}_\ell}$, $\mathcal{B}_\lambda^{\mathrm{X}_\ell}$ to mean $\mathcal{S}_e$, $\widetilde{\mathcal{S}}_e$, $\mathcal{B}_e$ where $\mathfrak{g}$ is of type X_ℓ (X being either A, B, C, or D) and e is in the nilpotent orbit labelled by λ.

There are many similarities between the variety $\widetilde{\mathcal{S}}_e$ with its subvariety $\mathcal{B}_e$ and the *quiver variety* $\mathfrak{M}$ with its subvariety $\mathfrak{L}$, introduced by Nakajima in [7, 8]. To define these quiver varieties one needs to specify a finite graph Γ, a dominant weight λ for the Kac–Moody algebra corresponding to the graph, and a weight μ of the integrable highest-weight representation with highest weight λ. For what follows, it suffices to consider the case where Γ is a simply-laced Dynkin diagram of type X_ℓ, the dominant weight λ belongs to the root lattice, and μ is the zero weight. In this case we will write the quiver varieties as $\mathfrak{M}_\lambda^{\mathrm{X}_\ell}$ and $\mathfrak{L}_\lambda^{\mathrm{X}_\ell}$.

In type A, the connection between Slodowy and quiver varieties is as close as could be. Explicitly, suppose $\mathfrak{g} = \mathfrak{sl}_n$ and λ is a partition of n, viewed as a dominant weight for $\mathfrak{g}$ in the usual way. Then as a special case of a theorem of Maffei [5], there is an isomorphism of varieties

$$\widetilde{\mathcal{S}}_{\lambda^{\mathsf{t}}}^{\mathrm{A}_{n-1}} \cong \mathfrak{M}_\lambda^{\mathrm{A}_{n-1}} \tag{1}$$

under which $\mathcal{B}_{\lambda^{\mathsf{t}}}^{\mathrm{A}_{n-1}}$ corresponds to $\mathfrak{L}_\lambda^{\mathrm{A}_{n-1}}$. Here λ^{t} is the partition dual to λ.

In other types, one cannot expect such a general statement. As suggested by Mirković and Vybornov [6], the situation is reminiscent of the connection between orbit closures in the nilpotent cone and in the affine Grassmannian. When $\mathfrak{g} = \mathfrak{sl}_n$, the analogue of (1) is the fact, discovered by Lusztig in [3], that the nilpotent orbit closure $\overline{\mathcal{O}_\lambda}$ is isomorphic to an open subvariety of the Schubert variety $\overline{\mathsf{Gr}_\lambda}$ in the affine Grassmannian Gr. In other types, such isomorphisms hold only for certain 'small' nilpotent orbits, described in [1]; these include, in classical types, the orbits labelled by partitions with two columns. Correspondingly, one might hope to find quiver varieties isomorphic to the varieties $\widetilde{\mathcal{S}}_e$ where e belongs to a 'big' nilpotent orbit, e.g., in classical types, one labelled by a partition with two rows. Part of the motivation for this hope is the idea of a *symplectic duality* between quiver varieties and slices in the affine Grassmannian [2].

The first nontrivial case is when e belongs to the subregular orbit. Here a result of Brieskorn shows that $\widetilde{\mathcal{S}}_e$ is the minimal resolution of a Kleinian singularity [9], which is known to be isomorphic to a quiver variety [7]. For example, for $n \geq 3$ we have an isomorphism

$$\widetilde{\mathcal{S}}^{\mathrm{C}_n}_{(2n-2,2)} \cong \mathfrak{M}^{\mathrm{D}_{n+1}}_{\varpi_2} \tag{2}$$

under which $\mathcal{B}^{\mathrm{C}_n}_{(2n-2,2)}$ corresponds to $\mathfrak{L}^{\mathrm{D}_{n+1}}_{\varpi_2}$. Here the numbering of nodes of the Dynkin diagram of type D_{n+1} is such that the nodes interchanged by the diagram involution are n and $n+1$, and the fundamental weight ϖ_2 is the highest root.

We have found the following generalization of (2).

Theorem 1. *When $\mathfrak{g} = \mathfrak{sp}_{2n}$ (for $n \geq 3$) and $(2n-k,k)$ is a partition labelling a nilpotent orbit in $\mathfrak{g}$, we have an isomorphism*

$$\widetilde{\mathcal{S}}^{\mathrm{C}_n}_{(2n-k,k)} \cong \begin{cases} \mathfrak{M}^{\mathrm{D}_{n+1}}_{\varpi_k}, & \text{if } k < n \ (k \text{ must then be even}), \\ \mathfrak{M}^{\mathrm{D}_{n+1}}_{\varpi_n+\varpi_{n+1}}, & \text{if } k = n \text{ is even}, \\ \mathfrak{M}^{\mathrm{D}_{n+1}}_{2\varpi_n} (\cong \mathfrak{M}^{\mathrm{D}_{n+1}}_{2\varpi_{n+1}}), & \text{if } k = n \text{ is odd}, \end{cases}$$

under which $\mathcal{B}^{\mathrm{C}_n}_{(2n-k,k)}$ corresponds to $\mathfrak{L}^{\mathrm{D}_{n+1}}_{\varpi_k}$ etc. as appropriate.

Our proof of this statement begins by observing that $\widetilde{\mathcal{S}}^{\mathrm{C}_n}_{(2n-k,k)}$ can be regarded as the fixed-point subvariety of $\widetilde{\mathcal{S}}^{\mathrm{A}_{2n-1}}_{(2n-k,k)}$ for the involution induced by the diagram involution of $\mathfrak{sl}_{2n}$. Crucially, the partition dual to $(2n-k,k)$, regarded as the dominant weight $\varpi_k + \varpi_{2n-k}$ for $\mathfrak{sl}_{2n}$, is stable under this diagram involution (which would not be true outside the two-row case). Hence we can define a diagram involution of the quiver variety $\mathfrak{M}^{\mathrm{A}_{2n-1}}_{\varpi_k+\varpi_{2n-k}}$. Analysing Maffei's proof of the isomorphism (1), we show that the involutions on both sides correspond. We then pass to the fixed-point subvarieties to obtain the theorem.

The last step involves an interesting general fact, that quiver varieties of type D_{n+1} arise as (connected components of) the fixed-point subvarieties of diagram involutions of quiver varieties of type A_{2n-1}. Roughly speaking, the branching comes about because the vector spaces in the 'middle' are split into (+1)- and (−1)-eigenspaces. This respects the subvarieties $\mathfrak{L}$, as may be seen from Lusztig's description of those subvarieties in [4].

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Spherical varieties in arbitrary characteristic

Friedrich Knop

Let G be a connected reductive group defined over an algebraically closed ground field k of arbitrary characteristic p. A normal G-variety X is called *spherical* if a Borel subgroup B of G has an open orbit in X. For $p = 0$ there exists by now an extensive body of research on spherical varieties culminating in a complete classification in terms of certain combinatorial structures (Luna-Vust [11], Luna [10], Losev [8], Cupit-Foutou [4], Bravi-Pezzini [2]).

In the talk, we described our progress towards extending the structural theory of spherical varieties to ground fields of arbitrary characteristic.

A rational function $f \neq 0$ on X is called B-semiinvariant if there is a character χ_f of B such that $b \cdot f = \chi_f(b) f$ for all $b \in B$. The set of characters χ_f arising this way is denoted by $\Xi(X)$. It is a subgroup of $\Xi(B)$, the character group of B. Its rank is called the *rank of* X. Put $\Xi_{\mathbb{Q}}(X) := \Xi(X) \otimes \mathbb{Q}$, $\Xi_p(X) := \Xi(X) \otimes \mathbb{Z}[p^{-1}]$, and $N_{\mathbb{Q}}(X) := \mathrm{Hom}(\Xi(X), \mathbb{Q})$.

Let $\mathcal{V}(X)$ be the set of G-invariant valuations $v : k(X)^* \to \mathbb{Q}$. Then $\mathcal{V}(X)$ can be embedded into $N_{\mathbb{Q}}(X)$ via the map $v \mapsto (\chi_f \mapsto v(f))$ and is thereby identified with a finitely generated rational convex cone with non-empty interior (see [5]).

Therefore, $\mathcal{V}(X)$ can be described by a set of linear inequalities

$$\sigma_1 \leq 0, \ldots, \sigma_s \leq 0$$

The set $\Sigma(X) = \{\sigma_1, \ldots, \sigma_s\}$ can be made unique by two requirements: *a)* none of the σ_i is redundant and *b)* each σ_i is primitive element in the lattice $\mathbb{Z}S \cap \Xi_p(X)$ where $\mathbb{Z}S$ is the root lattice of G.

Definition. *The elements of* $\Sigma(X)$ *are then called the* spherical roots of X.

Using an auxiliary Weyl group invariant scalar product $(\cdot, \cdot)$ on $\Xi(B)$ one can define the reflection s_σ of $\Xi_{\mathbb{Q}}(X)$ about any non-zero $\sigma \in \Xi_{\mathbb{Q}}(X)$ by the usual formula $s_\sigma(\chi) = \chi - 2\frac{(\chi,\sigma)}{(\sigma,\sigma)}\sigma$.

Definition. *The reflections* s_σ *about spherical roots* σ *of* X *generate a subgroup* $W_X \subseteq \mathrm{GL}(\Xi_{\mathbb{Q}}(X))$ *called the* little Weyl group of X.

Our first main result is:

Theorem 1 ([6, Thm. 4.3])**.** *Let X be a spherical variety. Then:*

(1) The little Weyl group W_X is finite.
(2) The groups $\Xi_p(X)$ and $\mathbb{Z}S \cap \Xi_p(X)$ are W_X-invariant.
(3) The set $R_X := W_X\Sigma(X)$ is a (finite) root system with Weyl group W_X.

Since the little Weyl group is generated by reflections about codimension-1-faces of the valuation cone, we get:

Corollary. *The valuation cone $\mathcal{V}(X)$ of a spherical variety X is a union of Weyl chambers for the little Weyl group W_X.*

For the proof of Theorem 1, we first classify all abstract spherical roots, i.e., all elements of $\Xi(B)$ which are spherical root for some spherical variety. Using "localization at Σ" (see below), this amounts to listing all spherical varieties of rank 1. Thereby we extended a classification by Akhiezer [1] in characteristic zero. The rest of the proof is then a case-by-case argument.

Schalke, [12], has shown that the valuation cone can indeed be the union of more than one Weyl chamber. Her example is $X = \mathrm{SL}(3)/\mathrm{SO}(3)$ where k is of characteristic two. On the other hand, a celebrated theorem of Brion, [3, Thm. 3.5], asserts that $\mathcal{V}(X)$ consists of a single Weyl chamber when $p = 0$. It turns out that $p = 2$ is the only bad prime. This is our second main result:

Theorem 2 ([6, Thm. 4.6])**.** *Let X be a spherical variety which is defined over a field of characteristic $p \neq 2$. Then $\Sigma(X)$ is a system of simple roots for the root system $R_X = W_X\Sigma(X)$.*

Corollary. *Let $p \neq 2$. Then the valuation cone $\mathcal{V}(X)$ of a spherical variety X is a Weyl chamber for the little Weyl group W_X.*

For the proof of Theorem 2 is suffices to show that $(\sigma, \tau) \leq 0$ for any two distinct spherical roots of X. Using the classification of rank-1-spherical varieties we show first that there are very few pairs σ, τ of abstract spherical roots which have $(\sigma, \tau) > 0$ and which satisfy a certain compatibility condition. Note that σ and τ are spherical roots but possibly not of the same spherical variety. So, the exceptional cases are then ruled out by proving that the exceptional pairs σ, τ are never spherical roots of the same variety X. For this we use the geometrical arguments on the interplay between spherical roots and colors in the companion paper [7].

In that paper, we extend mostly results of Luna, [9], from characteristic zero to arbitrary characteristic. These are, in particular, two reduction techniques which apply to so-called complete toroidal spherical varieties:

- *Localization at S.* Here S is the set of simple roots of S. For any subset $S' \subseteq S$ this technique produces a spherical $G^{S'}$-variety $X^{S'}$ (where $G^{S'} \subseteq G$ is the Levi subgroup with simple roots S') such that $\Sigma(X^{S'})$ consists of those spherical roots of X which are supported in S'. Geometrically, this technique is equivalent to studying the open Białynicki-Birula cell for a 1-parameter subgroup of G which is adapted to S'.

- *Localization at* Σ. This technique is similar but here we start with a subset $\Sigma' \subseteq \Sigma(X)$ (satisfying some technical condition in case $p = 2$). The result is a spherical G-variety X' with $\Sigma(X') = \Sigma'$. Geometrically, X' is (the normalization of) a particular orbit closure in X.

A novel feature of [7] are certain p-powers which are attached to any spherical variety in positive characteristic. Using them, we made an attempt to define the combinatorial structure of a *spherical system* for spherical varieties over fields of positive characteristic.

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Positivity in T-Equivariant K-theory of flag varieties associated to Kac-Moody groups

Shrawan Kumar

Let G be any symmetrizable Kac-Moody group over $\mathbb{C}$ completed along the negative roots and $G^{\min} \subset G$ be the 'minimal' Kac-Moody group. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus and W the Weyl group. Let $\bar{X} = G/B$ be the 'thick' flag variety (introduced by Kashiwara) which contains the standard KM flag ind-variety $X = G^{\min}/B$. Let T be the quotient torus $H/Z(G^{\min})$, where $Z(G^{\min})$ is the center of $G^{\min}$. Then, the action of H on $\bar{X}$ (and X) descends to an action of T. We denote the representation ring of T by $R(T)$. For any $w \in W$, we have the Schubert cell $C_w := BwB/B \subset X$, the Schubert variety $X_w := \overline{C_w} \subset X$, the opposite Schubert cell $C^w := B^- wB/B \subset \bar{X}$, and the opposite Schubert variety $X^w := \overline{C^w} \subset \bar{X}$.

Let $K_T^{\mathrm{top}}(X)$ be the T-equivariant topological K-group of the ind-variety X. Let $\{\psi^w\}_{w\in W}$ be the 'basis' of $K_T^{\mathrm{top}}(X)$ given by Kostant-Kumar. (Actually our ψ^w is a slight twist of their basis.)

Express the product in topological K-theory $K_T^{\mathrm{top}}(X)$:

$$\psi^u \cdot \psi^v = \sum_w p_{u,v}^w \psi^w, \quad \text{for } p_{u,v}^w \in R(T).$$

Then, the following result is our main theorem. This generalizes one of the main results of Anderson-Griffeth-Miller (which was conjectured earlier by Graham-Kumar) from the finite to any symmetrizable Kac-Moody case.

Theorem 1. *For any $u, v, w \in W$,*

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\, p_{u,v}^w \in \mathbb{Z}_+[(e^{-\alpha_1} - 1), \ldots, (e^{-\alpha_r} - 1)],$$

where $\{\alpha_1, \ldots, \alpha_r\}$ are the simple roots.

As an immediate consequence of the above theorem (by evaluating at 1), we obtain the following result which generalizes one of the main results of Brion (conjectured by A.S. Buch) from the finite to any symmetrizable Kac-Moody case.

Corollary 2. *For any $u, v, w \in W$,*

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\, a_{u,v}^w \in \mathbb{Z}_+,$$

where $a_{u,v}^w$ are the structure constants of the product in $K^{\mathrm{top}}(X)$ under the basis $\psi_o^w := 1 \otimes \psi^w$.

The proof relies heavily on algebro-geometric techniques. We realize the structure constants $p_{u,v}^w$ as the coproduct structure constants in the structure sheaf basis $\{\mathscr{O}_{X_w}\}_{w\in W}$ of the T-equivariant K-group $K_o^T(X)$ of finitely supported T-equivariant coherent sheaves on X. Let $K_T^0(\bar{X})$ denote the Grothendieck group of T-equivariant coherent $\mathscr{O}_{\bar{X}}$-modules $\mathcal{S}$. Then, there is a 'natural' pairing

$$\langle\, ,\, \rangle : K_T^0(\bar{X}) \otimes K_0^T(X) \to R(T),$$

coming from the T-equivariant Euler-Poincaré characteristic. Define the T-equivariant coherent sheaf $\xi^u := e^{-\rho}\mathcal{L}(\rho)\omega_{X^u}$ on $\bar{X}$, where

$$\omega_{X^u} := \mathscr{E}xt^{\ell(u)}_{\mathscr{O}_{\bar{X}}}\big(\mathscr{O}_{X^u}, \mathscr{O}_{\bar{X}}\big) \otimes \mathcal{L}(-2\rho)$$

is the dualizing sheaf of X^u. For any character e^λ of H, let $\mathcal{L}(\lambda)$ be the G-equivariant line bundle on $\bar{X}$ associated to the character $e^{-\lambda}$ of H. We show that the basis $\{[\xi^w]\}$ is dual to the basis $\{[\mathscr{O}_{X_w}]\}_{w\in W}$ under the above pairing.

Proposition 3. *For any $u, w \in W$,*

$$\mathscr{E}xt^j_{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w}) = 0, \quad \textit{for all } j \neq \ell(u).$$

Thus,

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\xi^u, \mathscr{O}_{X_w}) = 0, \quad \textit{for all } j > 0.$$

We also prove the following local Tor vanishing result, which is a certain cohomological analogue of the proper intersection property of X^u with X_w.

Lemma 4. *For any* $u, w \in W$*,*

$$\mathscr{T}or_j^{\mathscr{O}_{\bar{X}}}(\mathscr{O}_{X^u}, \mathscr{O}_{X_w}) = 0, \qquad \textit{for all } j > 0.$$

We show that the Richardson varieties $X_w^v := X_w \cap X^v \subset \bar{X}$ are irreducible, normal and Cohen-Macaulay, for short CM. Then, we construct a desingularization Z_w^v of X_w^v.

Many ideas are taken from very interesting papers of Brion and Anderson-Griffeth-Miller. However, there are several technical difficulties to deal with arising from the infinite dimensional setup, which has required various different formulations and more involved proofs. Some of the major differences are:

(1) In the finite case one just works with the opposite Schubert varieties X^u and their very explicit BSDH desingularizations which allows an easy calculation of the dualizing sheaf of the analogue of $\tilde{\mathscr{Z}}$. In our general symmetrizable Kac-Moody set up, we need to consider the Richardson varieties X_w^u and their desingularizations Z_w^u. Our desingularization Z_w^u is not as explicit as the BSDH desingularization. Then, we need to draw upon the result due to Kumar-Schwede that X_w^u have Kawamata log terminal singularities and use this result to calculate the dualizing sheaf of $\tilde{\mathscr{Z}}$.

(2) Instead of considering just one flag variety in the finite case, we need to consider the 'thick' flag variety and the standard ind flag variety and the pairing between them. Moreover, the identification of the basis of $K_T^0(\bar{X})$ dual to the basis of $K_0^T(X)$ given by the structure sheaf of the Schubert varieties X_w is more delicate.

(3) In the finite case one uses Kleiman's transversality result for the flag variety X. In our infinite case, to circumvent the absence of Kleiman's transversality result, we needed to prove various local Ext and Tor vanishing results.

On degenerate flag varieties

Peter Littelmann

(joint work with Evgeny Feigin, Misha Finkelberg, Ghislain Fourier, Giovanni Cerulli Irelli, and Markus Reineke)

1. Definition and construction

For simplicity let us assume that k is an algebraically closed field of characteristic zero. For a construction in arbitrary characteristic using Chevalley groups and the Kostant lattice see [FFL3].

We fix a semisimple connected algebraic group G, let $B \subset G$ be a Borel subgroup and $T \subset B$ a maximal torus. Denote by B^- the opposite Borel subgroup such that $B \cap B^- = T$ and let $N^- \subset B^-$ be its unipotent radical. The Lie algebras of these groups are denoted by $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{b} = \mathrm{Lie}\, B$, $\mathfrak{t} = \mathrm{Lie}\, T$, $\mathfrak{b}^- = \mathrm{Lie}\, B^-$ and $\mathfrak{n}^- = \mathrm{Lie}\, N^-$.

Let Φ be the root system of G and let Φ^+ be the set of positive roots corresponding to the choice of B. The group N^- has a natural filtration by the normal subgroups $N_s^- = \prod_{\substack{\beta\in\Phi^+ \\ ht(\beta)\geq s}} U_{-\beta}$, where ht denotes the height function on Φ^+, and the group $N^{-,a} := \prod_{s\geq 1} N_s^-/N_{s+1}^-$ is a connected abelian unipotent group, endowed with a natural T-action such that $\mathfrak{n}^{-,a} = \mathrm{Lie}\, N^{-,a}$ is isomorphic to $\mathfrak{n}^-$ as a T-module. By identifying $\mathfrak{n}^-$ with $\mathfrak{g}/\mathfrak{b}$, we get a natural B-action on $\mathfrak{n}^{-,a}$ ($= \mathfrak{n}^-$ by the above mentioned isomorphism of vector spaces), this action can be lifted to a B-action on $N^{-,a}$.

Definition 1.1. $G^a = B \ltimes N^{-,a}$.

Recall the PBW-filtration on the enveloping algebra $U(\mathfrak{n}^-)$ of $\mathfrak{n}^-$:

$$U(\mathfrak{n}^-)_s = \langle 1, X_1 \cdots X_t \mid 0 \leq t \leq s, X_1, \ldots, X_t \in \mathfrak{n}^- \rangle.$$

The associated graded algebra is $S(\mathfrak{n}^-)$, the symmetric algebra over the vector space $\mathfrak{n}^-$. For a dominant weight λ let $V(\lambda)$ be the associated irreducible finite dimensional representation of G. Fix a highest weight vector v_λ, we define the subspace $V(\lambda)_s \subseteq V(\lambda)$ by $V(\lambda)_s = U(\mathfrak{n}^-)_s v_\lambda$ and we set $(V(\lambda)_{-1} = 0)$: $V(\lambda)^a = \bigoplus_{s\geq 0} V(\lambda)_s/V(\lambda)_{s-1}$. It can be easily proved that

Lemma 1.2. *$V(\lambda)^a$ is a G^a-module.*

By the support $\sup \lambda$ of λ we mean the subset of fundamental weights such that the coefficient a_ϖ in $\lambda = \sum_\varpi a_\varpi \varpi$ is nonzero.

Let P_λ be the normalizer in G of the line kv_λ, then P_λ is a standard parabolic subgroup of G. The *generalized flag variety* $\mathcal{F}_\lambda$ is the orbit

$$\mathcal{F}_\lambda = G.[v_\lambda] \simeq G/P_\lambda.$$

In analogy to this classical construction we define:

Definition 1.3. The *degenerate flag variety* $\mathcal{F}_\lambda^a$ is the closure of the highest weight orbit in $\mathbb{P}(V^a(\lambda))$:

$$\mathcal{F}_\lambda^a := \overline{G^a.[v_\lambda]} \subseteq \mathbb{P}(V^a(\lambda)).$$

Remark 1.4. $G^a.[v_\lambda] = N^{-,a}.[v_\lambda]$ is the orbit of a unipotent group and hence an affine variety, so one has to take the closure to get a projective variety.

In the following we compare some properties of the classical flag variety and the degenerate flag variety:

classical flag variety	**degenerate flag variety**
$\mathcal{F}_\lambda$ depends only on $\sup\lambda$	open question (∗) except for G of type A and C
$\mathcal{F}_\lambda$ is smooth	not true in general, $\mathcal{F}^a_\lambda$ is normal, has rational singularities and is Gorenstein for G of type A and C
$\mathcal{F}_\lambda$ is a homogeneous G-variety and has a finite number of B-orbits	G^a has in general an infinite number of orbits in $\mathcal{F}^a_\lambda$ Example: $G = SL_4$, $\lambda = \varpi_1 + \varpi_2$
$\mathcal{F}_\lambda$ has the Bott-Samelson variety as a natural desingularization for the embedded Schubert varieties	a desingularization for $\mathcal{F}^a_\lambda$ is not known except for G of type A and C

Example 1.5. If $\lambda = \varpi$ is a cominuscule fundamental weight, then the unipotent radical of P_ϖ is commutative. Let $N^-_{P_\varpi}$ be the unipotent radical of the opposite parabolic subgroup. It is then easy to see that

$$G^a[v_\varpi] = N^{-,a}[v_\varpi] = N^-_{P_\varpi}[v_\varpi].$$

Since $N^-_{P_\varpi}$ is already abelian, the $N^-_{P_\varpi}$-module structure of $V^a(\lambda)$ and $V(\lambda)$ do not differ and hence

$$\mathcal{F}^a_\varpi = \overline{G^a[v_\varpi]} = \overline{N^{-,a}[v_\varpi]} = \overline{N^-_{P_\varpi}[v_\varpi]} = G/P_\varpi.$$

Example 1.6. Let $w_1, \dots, w_{2n}$ be a basis of a $2n$-dimensional vector space W. We fix a non-degenerate symplectic form $\langle\cdot,\cdot\rangle$ defined by the conditions $\langle w_i, w_{2n+1-i}\rangle = 1$ for $1 \le i \le n$ and $\langle w_i, w_j\rangle = 0$ for all $1 \le i, j \le 2n$, $j \ne 2n+1-i$. We realize the symplectic group Sp_{2n} as the group of automorphisms of W leaving the form invariant. For $k = 1, \dots, n$ we write $W = W_{k,1} \oplus W_{k,2} \oplus W_{k,3}$, where

$$W_{k,1} = \mathrm{span}(w_1, \dots, w_k),\ W_{k,3} = \mathrm{span}(w_{2n-k+1}, \dots, w_{2n})$$
$$W_{k,2} = \mathrm{span}(w_{k+1}, \dots, w_{2n-k}).$$

Denote by $p_{1,3}$ the projection $p_{1,3} : W \to W_{k,1} \oplus W_{k,3}$, and let $\varpi_1, \dots, \varpi_n$ be the fundamental weights (enumeration as in Bourbaki). Recall that

$$\mathcal{F}_{\varpi_k} = \{U \in Gr_k(W) \mid U \text{ is isotropic}\}.$$

For the degenerate case we get the following description [FFiL]:

$$\mathcal{F}^a_{\varpi_k} = \{U \in Gr_k(W) \mid pr_{1,3}(U) \text{ is isotropic}\}.$$

2. A basis for $V(\lambda)^a$, and how to prove (∗)

In the following we consider only the simple Lie algebra $\mathfrak{sl}_{n+1}$. For more details see [FFL1], for the construction in the case of the symplectic group see [FFL2].

Let R^+ be the set of positive roots of $\mathfrak{sl}_{n+1}$. Let α_i, ω_i $i = 1, \dots, n$ be the simple roots and the fundamental weights. All positive roots of $\mathfrak{sl}_{n+1}$ are of the form $\alpha_p + \alpha_{p+1} + \dots + \alpha_q$ for some $1 \le p \le q \le n$. In the following we denote such a root by $\alpha_{p,q}$, for example $\alpha_p = \alpha_{p,p}$.

We think of the roots as an indexing system for a diagram of empty boxes. Such a diagram can be filled in with natural numbers $\mathbf{s} = (s_{i,j})_{1\le i\le j\le n}$. We think of such a filled in diagram as a multi-exponent and we associate to this element the monomial $F^{\mathbf{s}} = \prod_{1\le i\le j\le n} F_{\alpha_{i,j}}^{n_{i,j}} \in S(\mathfrak{n}^-)$.

Definition 2.1. A *Dyck path* (or simply *a path*) is a sequence

$$\mathbf{p} = (\beta(0), \beta(1), \ldots, \beta(k)),\ \ k \ge 0$$

of positive roots satisfying the following conditions:

i) the first and last elements are simple roots. More precisely, $\beta(0) = \alpha_i$ and $\beta(k) = \alpha_j$ for some $1 \le i \le j \le n$;

ii) the elements in between obey the following recursion rule: If $\beta(s) = \alpha_{p,q}$ then the next element in the sequence is of the form either $\beta(s+1) = \alpha_{p,q+1}$ or $\beta(s+1) = \alpha_{p+1,q}$.

Definition 2.2. For an integral dominant $\mathfrak{sl}_{n+1}$-weight $\lambda = \sum_{i=1}^n m_i\omega_i$ let $S(\lambda)$ be the set of all multi-exponents $\mathbf{s} = (s_\beta)_{\beta\in R^+} \in \mathbb{N}^{R^+}$ such that for all Dyck paths $\mathbf{p} = (\beta(0), \ldots, \beta(k))$

$$s_{\beta(0)} + s_{\beta(1)} + \cdots + s_{\beta(k)} \le m_i + m_{i+1} + \cdots + m_j, \tag{1}$$

where $\beta(0) = \alpha_i$ and $\beta(k) = \alpha_j$.

Theorem 2.3. [FFL1, Fe1]

i) $\{F^{\mathbf{s}}v_\lambda \mid \mathbf{s} \in S(\lambda)\}$ *is a basis for the deformed representation* $V(\lambda)^a$.

ii) $V(\lambda+\mu)^a$ *is isomorphic to the* G^a*-submodule generated by* $v_\lambda \otimes v_\mu$ *in* $V(\lambda)^a \otimes V(\mu)^a$*, so we have a natural injection*

$$V(\lambda+\mu)^a \hookrightarrow V(\lambda)^a \otimes V(\mu)^a.$$

iii) $\mathcal{F}^a_\lambda$ *depends only on the support* $\sup\lambda$ *of* λ.

iv) $\mathcal{F}^a_\lambda$ *can be realized as a subvariety of a product of Graßmann varieties. More precisely, for simplicity assume* λ *is regular, then*

$$\mathcal{F}^a_\lambda = \{(U_1, \ldots, U_n) \in Gr_1(k^{n+1}) \times \ldots \times Gr_n(k^{n+1}) \mid p_2(U_1) \subseteq U_2, p_3(U_2) \subseteq U_3, \ldots, p_{n-1}(U_{n-2}) \subseteq U_{n-1}\}$$

where $Gr_j(k^{n+1})$ *denotes the Graßmann variety of* j*-dimensional subspaces of* k^{n+1} *and* p_i *is the projection along the line* ke_i.

Theorem 2.4. [Fe1]

i) The degenerate flag variety $\mathcal{F}^a_\lambda$ *and the flag variety* $\mathcal{F}_\lambda$ *can be connected by a flat family* $\mathcal{F}^t$, $t \in k$*, of projective varieties such that* $\mathcal{F}_\lambda \simeq \mathcal{F}^t$ *for all* $t \ne 0$ *and* $\mathcal{F}^a_\lambda \simeq \mathcal{F}^0$.

ii) For simplicity assume λ *is regular. For the embedding*

$$\mathcal{F}^a_\lambda \hookrightarrow Gr_1(k^{n+1}) \times \ldots \times Gr_n(k^{n+1}) \subseteq \prod_{j=1,\ldots,n} \mathbb{P}(\Lambda^j k^{n+1})$$

let R be the multi homogeneous coordinate ring of $\mathcal{F}^a_\lambda$. Then, as a G^a-module,

$$R = \bigoplus_{\lambda \text{ dominant weight}} (V(\lambda)^a)^*.$$

The desingularization of the degenerate flag varieties can be explicitly described in type A and C, see [FF, FFiL]. We present in the following the construction only in the case of type A. For simplicity we assume λ to be a regular dominant weight, and since $\mathcal{F}^a_\lambda$ depends only on the support, we often omit the index λ.

Theorem 2.5. *Let $\widehat{\mathcal{F}^a} \subseteq \prod_{i=1}^n Gr_i(k^{n+1})^{n+1-i}$ be the collection of subspaces*

- $V_{i,j}$, $1 \leq i \leq j \leq n$, $V_{i,j} \in Gr_i(k^{n+1})$;
- $V_{i,j} \subseteq span(e_1, e_2, \ldots, e_{i-1}, e_i, e_{j+1}, e_{j+2}, \ldots, e_n, e_{n+1})$;
- $V_{i,j} \subseteq V_{i+1,j}$, *and* $pr_{j+1}(V_{i,j}) \subseteq V_{i,j+1}$.

Then $\widehat{\mathcal{F}^a}$ is a smooth projective variety, and the natural map

$$\widehat{\mathcal{F}^a} \to \mathcal{F}^a, \quad ((V_{i,j}))_{i,j} \mapsto (V_{i,i})_i,$$

is a desingularization. The varieties $\widehat{\mathcal{F}^a}$ and $\mathcal{F}^a$ are Frobenius split, $\mathcal{F}^a$ is a normal variety, locally a complete intersection, and has rational singularities. Further

$$H^0(\mathcal{F}^a, i^*_\lambda(\mathcal{O}(1)))^* \simeq V(\lambda)^a, \quad H^j(\mathcal{F}^a, i^*_\lambda(\mathcal{O}(1))) = 0 \;\forall\, j \geq 1.$$

Remark 2.6. The desingularization has one feature in common with the classical Bott-Samelson construction: using projections, one can view $\widehat{\mathcal{F}^a}$ as a tower of $\mathbb{P}^1$-fibrations. The construction in the symplectic case is similar, see [FFiL].

3. Cellular decompositions for $\mathcal{F}^a_\lambda$ in type A

In the classical case the flag variety has a nice cellular decomposition given by the Schubert cells, but since the degenerate flag variety is not anymore smooth, such a nice structure can in general not be expected. But in the case of type A some of these nice properties can be generalized to the degenerate case.

It has been observed in [CFR1] that the degenerate flag varieties can be identified with certain quiver Grassmannians of the equioriented quiver of type A_n. More precisely, $\mathcal{F}^a$ is isomorphic to the quiver Grassmannian $Gr_{\dim_A}(A \oplus A^*)$, where A and A^* are the path algebra of the equioriented A_n-quiver, resp. its dual. This observation can be used to get a deeper understanding of the geometry and combinatorics of the degenerate flag varieties. In [CFR1, CFR2] they study the varieties $\mathcal{F}^a$ using the techniques from the theory of quiver Grassmannians. For example, the fact that $\mathcal{F}^a$ can be realized as a quiver Grassmannian has as a consequence that a larger group $\mathcal{A}$ acts on $\mathcal{F}^a$.

Denote by h_n the normalized median Genocchi number. The first numbers are as follows: 1, 2, 7, 38, 295, 3098.

Theorem 3.1. [Fe2, CFR1] *Let $G = SL_{n+1}$. The group $\mathcal{A}$ acts on $\mathcal{F}^a$ with a finite number of orbits, the number is equal to h_{n+1}. Each orbit is an affine cell, containing exactly one T-fixed point. The orbits are labeled by elements of DC_{n+1}.*

It is also possible to define a function ℓ on the set DC_n and obtain in this way a formula for the Poincaré polynomial similar to the classical case:

$$P_{\mathcal{F}^a} = \sum_{D \in DC_{n+1}} t^{2\ell(D)}.$$

These methods can be refined to prove that the number of 1-dimensional T-orbits is finite and get in addition an explicit description of the moment graph of $\mathcal{F}^a$. In [CFR2] they describe explicitly the smooth and singular loci of the degenerate flag varieties and they give a combinatorial formula for the Poincaré polynomial of the smooth locus.

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Essential dimension and canonical dimension of algebraic groups

Roland Lötscher

These two talks provided a survey on two integer-valued invariants of algebraic groups. Both of them concern the structure and complexity of torsors of an algebraic group and are linked to the study of various algebraic objects like quadratic forms, central simple algebras, octonion algebras etc.

The first invariant is called "essential dimension" and was introduced around 1995 by J. Buhler and Z. Reichstein, see [2, 17]. The essential dimension $\mathrm{ed}(G)$ of an algebraic group G over a field F is defined as the least integer n such that every G-torsor over a field extension K/F is isomorphic to a G-torsor obtained by scalar extension from an intermediate field of transcendence degree at most n over

F. This is roughly the number of independent parameters needed to write down G-torsors. For example a O_n-torsor corresponds to a non-degenerate quadratic form of dimension n and can be written down in diagonalized form (assuming $\mathrm{char}F \neq 2$) with n parameters, so that $\mathrm{ed}(O_n) \leq n$.

The second invariant, called "canonical dimension", addresses the question on how far G-torsors are from being trivial. It was introduced by G. Berhuy and Z. Reichstein in [1] around 2004. The value $\mathrm{cdim}(G)$ is defined as the least integer n such that every splitting field of a G-torsor over some field extension K/F contains a subfield of transcendence degree at most n over K that also splits the torsor. We always have $\mathrm{cdim}(G) \leq \dim G$.

By a result of A. Merkurjev [14, Proposition 4.4] $\mathrm{ed}(G)$ is zero if and only if G is *special*, i.e., all G-torsors over field extensions are trivial. Special algebraic groups were introduced by J.-P. Serre and classified over algebraically closed fields by A. Grothendieck in the 1950's, see [6]. If G is defined over a perfect field, $\mathrm{cdim}(G) = 0$ if and only if G^0 is special (in case of algebraically closed field F this was proven in [1]; it is not hard to prove it in general). Except of this coincidence the values of $\mathrm{ed}(G)$ and $\mathrm{cdim}(G)$ are quite independent.

Constant finite groups of $\mathrm{ed}(G) \leq 1$ have been classified in [11] and those of $\mathrm{ed}(G) \leq 2$, when F is algebraically closed of characteristic 0, in [4]. In contrast all finite algebraic groups have canonical dimension 0.

Some of the highlights in the field of essential dimension are the exact computation of $\mathrm{ed}(G)$ for finite p-groups (in case F contains a primitive p-th root of unity) in [10], for algebraic tori (split by a Galois extension of p-power degree) in [13] and for Spin groups (assuming characteristic 0) in [3] and [14]. Opposed to this there are seemingly simple problems in essential dimension like computing $\mathrm{ed}(\mathbb{Z}/11\mathbb{Z})$ over the field $F = \mathbb{Q}$ for which every attempt at a solution has failed so far.

Some of the most intriguing problems in the surveyed field are the computation of the essential dimension of symmetric groups S_n and projective linear groups PGL_d. These problems are strongly related to the problem of simplifying a general degree n polynomial equation by means of Tschirnhaus transformations and to the structure description of central simple algebras of degree d, respectively. The exact values for $\mathrm{ed}(S_n)$ are known up to $n = 6$ by [2] (assuming $\mathrm{char}(F) \neq 2$ in case $n = 6$) and, when F has characteristic 0, for $n = 7$ by [5]. A strong lower bound on $\mathrm{ed}(\mathrm{PGL}_n)$ (for $n = p^r$ and $\mathrm{char}(F) \neq p$) has been established in [15].

The strongest results on the values $\mathrm{cdim}(G)$ are known for split semisimple algebraic groups G. There is a variant of canonical dimension of G with respect to a prime p, denoted $\mathrm{cdim}_p(G)$. The precise value of $\mathrm{cdim}_p(G)$ has been computed for every split simple algebraic group G in [9, 19]. In some cases $\mathrm{cdim}(G) = \mathrm{cdim}_p(G)$ when G has only one torsion prime p. This happens e.g. for $G = \mathrm{PGL}_{p^r}$ and for $G = \mathrm{SO}_n$ (where $p = 2$).

In a very different direction the canonical dimension of (non-split) algebraic tori has been studied in [12, 8]. These two papers give partial affirmative answers on the following question:

Open question: Is $\mathrm{cdim}(T) = \dim T$ for every anisotropic algebraic torus T split by a Galois extension of prime power degree?

For further reading: Excellent and detailed surveys on essential dimension and to a lesser extent on canonical dimension are provided by [18, 16]. A very good survey on canonical dimension of projective homogeneous varieties, which is related to canonical dimension of split semisimple algebraic groups, can be found in [7].

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Quantum K-theory of homogeneous spaces

Nicolas Perrin

(joint work with Anders S. Buch, Pierre-Emmanuel Chaput, and Leonardo C. Mihalcea)

1. Introduction

The quantum cohomology of a projective rational homogeneous space X is now better understood. However, its quantum K-theory has only been studied since very recently. In this note, we describe some of the properties of quantum cohomology that one would like to extend to quantum K-theory and explain that many of these results would follow from rational connectedness results on subvarieties of the moduli space of stable maps to X.

This note is based on a joint project with A. Buch, P.-E. Chaput and L. Mihalcea. For more details, we refer the reader to the papers [3, 8, 4, 5, 6].

2. Quantum cohomology and K-theory

2.1. **Stable maps.** We recall here some notation. Let $X = G/P$ be a homogeneous space, where G is any semisimple complex linear algebraic group and P a maximal parabolic subgroup. Given an effective class $d \in H_2(X;\mathbb{Z})$ and an integer $n \geq 0$, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(X,d)$ parametrizes the set of all n-pointed stable genus-zero maps $f : C \to X$ with $f_*[C] = d$, and is equipped with a total evaluation map $\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_n) : \overline{\mathcal{M}}_{0,n}(X,d) \to X^n := X \times \cdots \times X$. A detailed construction of this space can be found in the survey [9].

Let $\mathbf{d} = (d_0, d_1, \ldots, d_r)$ be a sequence of effective classes $d_i \in H_2(X;\mathbb{Z})$, let $\mathbf{e} = (e_0, \ldots, e_r) \in \mathbb{N}^{r+1}$, and set $|\mathbf{d}| = \sum d_i$ and $|\mathbf{e}| = \sum e_i$. We consider stable maps $f : C \to X$ in $\overline{\mathcal{M}}_{0,|\mathbf{e}|}(X, |\mathbf{d}|)$ defined on a chain C of $r+1$ projective lines, such that the i-th projective line contains e_i marked points (numbered from $1 + \sum_{j<i} e_i$ to $\sum_{j\leq i} e_i$) and the restriction of f to this component has degree d_i. To ensure that such a map is indeed stable, we demand that $e_i \geq 1 + \delta_{i,0} + \delta_{i,r}$ whenever $d_i = 0$. Let $M_{\mathbf{d},\mathbf{e}} \subset \overline{\mathcal{M}}_{0,|\mathbf{e}|}(X, |\mathbf{d}|)$ be the closure of the locus of all such stable maps.

Given subvarieties $\Omega_1, \ldots, \Omega_m$ of X with $m \leq |\mathbf{e}|$, define a boundary Gromov-Witten variety by $M_{\mathbf{d},\mathbf{e}}(\Omega_1, \ldots, \Omega_m) = \bigcap_{i=1}^m \mathrm{ev}_i^{-1}(\Omega_i) \subset M_{\mathbf{d},\mathbf{e}}$. We also define the varieties $\Gamma_{\mathbf{d},\mathbf{e}}(\Omega_1, \ldots, \Omega_m) = \mathrm{ev}_{|\mathbf{e}|}(M_{\mathbf{d},\mathbf{e}}(\Omega_1, \ldots, \Omega_m)) \subset X$.

If no sequence $\mathbf{e}$ is specified, we will use $\mathbf{e} = (3)$ when $r = 0$ and $\mathbf{e} = (2, 0, \ldots, 0, 1)$ when $r > 0$; this convention will only be used when $d_i \neq 0$ for $i > 0$.

2.2. **Definition of the product.** The quantum cohomology and K-theory rings of X are algebras over $\Lambda = \mathbb{Z}[[H_2(X;\mathbb{Z})]]$, which as a Λ-modules are given by $\mathrm{QH}(X) = H(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda$ and $\mathrm{QK}(X) = K(X) \otimes_{\mathbb{Z}} \Lambda$. The multiplicative structure is defined, for classes $\alpha, \beta \in H^*(X;\mathbb{Z})$ or classes $A, B \in K(X)$, by

$$\alpha \star \beta = \sum_d (\alpha \star \beta)_d q^d \text{ and } A \star B = \sum_d (A \star B)_d q^d$$

with $(\alpha \star \beta)_d = \mathrm{ev}_{3*}(\mathrm{ev}_1^* \alpha \cdot \mathrm{ev}_2^* \beta)$ and

$$(A \star B)_d = \sum_{\mathbf{d}=(d_0,\dots,d_r)} (-1)^r \, \mathrm{ev}_{3*}(\mathrm{ev}_1^* \alpha \cdot \mathrm{ev}_2^* \beta \cdot \mathcal{O}_{M_{\mathbf{d}}}),$$

where the second sum runs over all sequences $\mathbf{d} = (d_0, \dots, d_r)$ with $|\mathbf{d}| = d$ and $d_i > 0$ for $i > 0$. Note that in the above we consider push-forward as a map between Chow groups which does not change the definition for rational homogeneous spaces. General results (see [1, 2, 10]) imply that these products are commutative and associative.

We recall some results on $\mathrm{QH}(X)$ for X a projective rational homogeneous space. There are natural basis $(\sigma^u)_{u \in W_X}$ of $H^*(X;\mathbb{Z})$ and $(\mathcal{O}^u)_{u \in W_X}$ of $K^*(X)$ where W_X is a subset of the Weyl group W of $\mathrm{Aut}(X)$. One defines the structure constants $c_{u,v}^w(d)$ and $\kappa_{u,v}^w(d)$ of the product as follows. For $u, v \in W_X$ we write

$$\sigma^u \star \sigma^v = \sum_{d,w} c_{u,v}^w(d) q^d \sigma^w \text{ and } \mathcal{O}^u \star \mathcal{O}^v = \sum_{d,w} \kappa_{u,v}^w(d) q^d \mathcal{O}^w.$$

In the following we denote by $\mathrm{QH}^*(X)_{loc}$ the localisation of $\mathrm{QH}^*(X)$ obtained by inverting the quantum parameters.

Theorem 1. *Let $\alpha, \beta \in H^*(X;\mathbb{Z})$.*

(ı) The product $\alpha \star \beta$ is finite i.e. $(\alpha \star \beta)_d = 0$ *for d large.*

(ıı) We have $c_{u,v}^w(d) \geq 0$.

(ııı) There is an explicit representation $\rho : \pi_1(Aut(X)) \to \mathrm{QH}^*(X)_{loc}$ given by a map $w : \pi_1(Aut(X)) \to W_X$ such that

$$\rho(\epsilon) \cdot \sigma^u = \sigma^{w(\epsilon)} \star \sigma^u = q^{d_\epsilon(u)} \sigma^{w(\epsilon)u}.$$

The first result follows from the definition and the fact that rational homogeneous spaces are Fano. The second result can be found in [9] and the last result was proved in [7]. One would like to extend the above results to $\mathrm{QK}(X)$. One of the main problems in computing quantum K-theoretic products is the computation of push-forwards. General results of Kollár [11] give tools to compute such push-forwards under rational connectedness assumptions.

Definition 1. *Let X_u and X^v be two opposite Schubert varieties. We say that X satisfies the property $RC(\mathbf{d}, X_u, X^v)$ if the general fiber of the morphism*

$$\mathrm{ev}_{|\mathbf{e}|} : M_{\mathbf{d},\mathbf{e}}(X_u, X^v) \to \Gamma_{\mathbf{d},\mathbf{e}}(X_u, X^v)$$

is rationaly connected.

Proving the property $RC(\mathbf{d}, X_u, X^v)$ even in very special situation leads to interesting results. In particular, we obtain the following (see [5]).

Theorem 2. *Assume that $Pic(X) = \mathbb{Z}$ and that $RC(\mathbf{d}, \text{point}, \text{point})$ holds for $\mathbf{d} = (d)$ with d large. Then for all classes $A, B \in K(X)$ and d large, we have $(A \star B)_d = 0$.*

Several results on rational connectedness were obtained in [8]. In [4], we considered the special case where X is a cominuscule homogeneous space and proved the above finiteness of the quantum product in K-theory; more precisely the vanishing $(\alpha \star \beta)_d = 0$ for $d > d_X(2)$ where $d_X(2)$ is the minimal degree of rational curves connecting any two point in X. The bound is sharp is this case.

The cominuscule case has a simpler behaviour. In particular, we prove the following (see [6]).

Theorem 3. *Assume that X is cominuscule and that $RC(\mathbf{d}, X_u, X^v)$ holds for all $\mathbf{d}$, X_u and X^v. We have*

$$(-1)^{deg(\sigma^u)+deg(\sigma^v)+deg(\sigma^w)+\int_d c_1(X)} \kappa_{u,v}^w(d) \geq 0.$$

Using this result and several geometric constructions we also prove (see [6]).

Theorem 4. *There is an explicit representation $\rho : \pi_1(Aut(X)) \to \mathrm{QK}^*(X)_{loc}$ given by a map $w : \pi_1(Aut(X)) \to W_X$ such that*

$$\rho(\epsilon) \cdot \mathcal{O}^u = \mathcal{O}^{w(\epsilon)} \star \mathcal{O}^u = q^{d_\epsilon(u)} \mathcal{O}^{w(\epsilon)u}.$$

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Algebraic groups and the Cremona group

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Let X be an irreducible algebraic variety. The group $\operatorname{Bir} X$ of its birational automorphisms is endowed with the Zariski topology by means of the notion of "algebraic family" in $\operatorname{Bir} X$, [9], [4] (the idea of "algebraic families", in the frame of subsets of $\operatorname{Aut}X$, was first embodied in [8]). If G is an algebraic group and a homomorphism $\varphi : G \to \operatorname{Bir} X$ is an algebraic family, then $\varphi(G)$ is called an *algebraic subgroup of* $\operatorname{Bir} X$, [6]. In this case $\operatorname{Ker}\varphi$ is closed in G, the natural map $G/\operatorname{Ker}\varphi \hookrightarrow \varphi(G)$ is a homeomorphism, and $\dim \varphi(G)$ (defined in [8]) is $\dim G/\operatorname{Ker}\varphi$. In particular, if G is a (linear) algebraic torus, then $\varphi(G)$ is called an *algebraic torus of* $\operatorname{Bir} X$.

$\operatorname{Bir} \mathbf{A}^n$ is the *Cremona group* Cr_n *of rank* n. We distinguish in it the subgroups $\operatorname{Aut}^* \mathbf{A}^n \subset \operatorname{Aut} \mathbf{A}^n \supset \mathrm{Aff}_n \supset \mathrm{GL}_n \supset \mathrm{SL}_n \supset D_n \supset D_n^*$, where $\operatorname{Aut}^* \mathbf{A}^n := \{g \in \operatorname{Aut} \mathbf{A}^n \mid g$ leaves fixed the standard volume form on $\mathbf{A}^n\}$, D_n is the maximal diagonal torus in GL_n, and $D_n^* := D_n \cap \mathrm{SL}_n$. The direct limit Cr_∞ of the tower of natural embeddings $\mathrm{Cr}_1 \hookrightarrow \mathrm{Cr}_2 \hookrightarrow \mathrm{Cr}_3 \hookrightarrow \cdots$ is called *the Cremona group of infinite rank*, [6], [7].

Below all algebraic varieties are taken over an algebraically closed field k of characteristic zero.

Conjugacy. We first discuss conjugacy of some algebraic subgroups of Cr_n.

Theorem 1 (Tori in Cr_n)**.**

(i) ([1, Cor. 2]) Cr_n *contains no tori of dimension* $> n$ *and every* r*-dimensional torus in* Cr_n *for* $r = n, n-1, n-2$ *is conjugate to* D_r.

(ii) ([7, Cor. 2.5(a)]) Cr_n *for* $n \geqslant 5$ *contains* $(n-3)$*-dimensional tori that are not conjugate to subtori of* D_n.

(iii) ([7, Thm. 2.6]) *Every* r*-dimensional torus in* Cr_n *is conjugate in* Cr_{n+r} *to* D_r.

(iv) ([6, Thm. 2(v)]) *Every* r*-dimensional torus in* Cr_∞ *is conjugate to* D_r.

Theorem 1 implies that every n-dimensional torus in Cr_n is maximal, Cr_n contains no maximal $(n-1)$- and $(n-2)$-dimensional tori, but if $n \geqslant 5$, then Cr_n contains maximal $(n-3)$-dimensional tori.

The following generalizes the results of [1], [2] to the case of disconnected groups.

Theorem 2 ([6, Thm. 8])**.** *Let* G *be an algebraic subgroup of* $\operatorname{Aut} \mathbf{A}^n$ *such that* G^0 *is a torus.*

(i) *If* $\dim G = n$ *or* $n-1$, *then there is an element* $g \in \operatorname{Aut} \mathbf{A}^n$ *such that* $gGg^{-1} \subset \mathrm{GL}_n$ *and* $gG^0g^{-1} \subset D_n$.

(ii) *If* $G \subset \operatorname{Aut}^* \mathbf{A}^n$ *and* $\dim G = n-1$, *then there is an element* $g \in \operatorname{Aut}^* \mathbf{A}^n$ *such that* $gGg^{-1} \subset \mathrm{SL}_n$ *and* $gG^0g^{-1} = D_n \cap \mathrm{SL}_n$.

Theorem 3 (Classification of diagonalizable subgroups of Aff_n up to conjugacy in Cr_n, [6, Thm. 1])**.**

(i) *Diagonalizable subgroups of* Aff_n *are conjugate in* Cr_n *if and only if they are isomorphic.*

(ii) *Any diagonalizable subgroup G of* Aff_n *is conjugate in* Cr_n *to a unique closed subgroup of* D_n *of the form*

$$\ker \varepsilon_{r+1}^{d_1} \cap \ldots \cap \ker \varepsilon_{r+s}^{d_s} \cap \ker \varepsilon_{r+s+1} \cap \ldots \cap \ker \varepsilon_n$$

where $r = \dim G$*,* $d_1 | \ldots | d_s$ *are all the invariant factors of* G/G^0*, and* ε_i *is the* i*-th coordinate character of* D_n*.*

Corollary.

(i) *Every torus T of* Aff_n *is conjugate in* Cr_n *to* D_r*,* $r = \dim T$*.*
(ii) *Isomorphic finite Abelian subgroups of* Aff_n *are conjugate in* Cr_n*.*
(iii) ([3, Thm. 1]) *Elements of* Aff_n *of the same finite order are conjugate in* Cr_n*.*

Theorems 2 and 3 are applied in [6] for obtaining the following classifications of algebraic subgroups of a group A up to conjugacy in a group B:

- $A = B = \mathrm{Aut}\,\mathbf{A}^n$:
  (i) diagonalizable subgroups of dimension $\geqslant n-1$,
  (ii) maximal n-dimensional subgroups G such that G^0 is a torus;
- $A = B = \mathrm{Aut}^*\mathbf{A}^n$:
  (a) diagonalizable subgroups of dimension $n-1$,
  (b) maximal $(n-1)$-dimensional subgroups G such that G^0 is a torus;
- $A = B = \mathrm{Aut}\,\mathbf{A}^3$:
  (i) 1-dimensional tori;
- $A = \mathrm{Aut}\,\mathbf{A}^n$, $B = \mathrm{Cr}_n$:
  (i) diagonalizable subgroups of dimension $\geqslant n-1$.

Normalizers. Given a group G and its subgroup H, denote by $N_H(G)$ the normalizer of H in G.

Theorem 4 ([6, Thm. 6])**.** *Let G be a diagonalizable subgroup of* $\mathrm{Aut}\,\mathbf{A}^n$ *and let* $\dim G \geqslant n-1$*. Then* $N_{\mathrm{Aut}\,\mathbf{A}^n}(G)$ *is an algebraic subgroup of* $\mathrm{Aut}\,\mathbf{A}^n$*.*

In fact, if $k[\mathbf{A}^n]^G \neq k$ for G from Theorem 4, then $N_{\mathrm{Aut}\,\mathbf{A}^n}(G)$ is explicitly described in [6, Thm. 6]. In particular, $N_{\mathrm{Aut}\,\mathbf{A}^n}(D_n) = N_{\mathrm{GL}_n}(D_n)$ and $N_{\mathrm{Aut}^*\mathbf{A}^n}(D_n^*) = N_{\mathrm{SL}_n}(D_n^*)$. This solves Problem 1.2 in [5].

Jordan decomposition in $\mathrm{Bir}\,X$*.* Let X be an algebraic variety. Call an element $g \in \mathrm{Bir}\,X$ *algebraic* if there is an algebraic subgroup G of $\mathrm{Bir}\,X$ containing g. If G is affine, the Jordan decomposition $g = g_s g_u$ in G is defined. By [6], g_s and g_n depend only of g, not on G. Hence, calling $g = g_s g_u$ the *Jordan decomposition of g in* $\mathrm{Bir}\,X$, we get the well-defined notion. In Cr_n every algebraic element admits the Jordan decomposition since every algebraic subgroup of Cr_n is affine.

Torsion primes for $\mathrm{Bir}\,X$*.* Let G be a connected reductive algebraic group and let p be a prime integer. Recall that p is called *torsion prime for G* if there is a finite Abelian p-subgroup of G not contained in any torus of G. The set $\mathrm{Tors}(G)$ of all torsion primes for G is finite. It is explicitly described for every simple G.

Let X be an irreducible algebraic variety. Since the notion of tori in $\mathrm{Bir}\,X$ is well-defined, replacing G in the above definition by $\mathrm{Bir}\,X$ or its subgroup yields the well-defined notion. In particular, we consider what is obtained when G is replaced by Cr_n, $\mathrm{Aut}\,\mathbf{A}^n$, or $\mathrm{Aut}^*\mathbf{A}^n$ as the definitions of torsion primes for these groups. In [6] is proved that

$$\mathrm{Tors}(\mathrm{Cr}_1) = \{2\}, \quad \mathrm{Tors}(\mathrm{Cr}_2) = \{2,3,5\}, \quad \mathrm{Tors}(\mathrm{Cr}_n) = \{2,3,\ldots\} \text{ for any } n \geqslant 2,$$
$$\mathrm{Tors}(\mathrm{Aut}\,\mathbf{A}^n) = \mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n) = \{\varnothing\} \text{ for } n \leqslant 2$$

(in particular, the torsion primes for Cr_2 are the same as that for the exceptional simple algebraic group E_8).

Questions. One can prove that if finite subgroups of Aff_n are isomorphic, then they are conjugate in Cr_{2n}.

Question 1. *Are there finite isomorphic subgroups of* Aff_n *that are not conjugate in* Cr_n?

Question 2. *Are the sets* $\mathrm{Tors}(\mathrm{Cr}_n)$, $\mathrm{Tors}(\mathrm{Aut}\,\mathbf{A}^n)$, *and* $\mathrm{Tors}(\mathrm{Aut}^*\mathbf{A}^n)$ *finite*?

Question 3. *What is the minimal n such that 7 lies in one of them*?

As is known, Jordan decompositions in algebraic groups have the properties:
— every semisimple element of a connected group lies in its torus;
— the set of all unipotent elements is closed;
— the conjugacy class of every semisimple element of a connected reductive group is closed;
— the closure of the conjugacy class of every element g of a connected reductive group contains g_s.

Question 4. *Are there analogues or modifications of these properties for the groups* Cr_n, $\mathrm{Aut}\,\mathbf{A}^n$, *and* $\mathrm{Aut}^*\mathbf{A}^n$?

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Derived subalgebras of nilpotent centralisers and completely prime primitive ideals

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(joint work with Lewis Topley)

Let $\Bbbk$ be an algebraically closed field of characteristic 0 and let G be a simple algebraic group of adjoint type over $\Bbbk$. Given an element x in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ we write G_x for the stabiliser of x in G and denote by $\mathfrak{g}_x$ the Lie algebra of G_x.

Let $U(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$ and denote by $\mathcal{X}$ the set of all primitive ideals of $U(\mathfrak{g})$. The graded algebra associated with the canonical filtration of $U(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$ which we identify with $S(\mathfrak{g}^*)$ by using the Killing form on $\mathfrak{g}$. Using Commutative Algebra we then attach to $I \in \mathcal{X}$ two important invariants: the associated variety $\mathrm{VA}(I)$ and the associated cycle $\mathrm{AC}(I)$. The variety $\mathrm{VA}(I)$ is the zero locus in $\mathfrak{g}$ of the G-stable ideal $\mathrm{gr}(I)$ of $S(\mathfrak{g}^*)$ and $\mathrm{AC}(I)$ is a formal linear combination $\sum_{i=1}^{l} m_i \mathcal{V}(\mathfrak{p}_i)$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ are the minimal primes of $S(\mathfrak{g}^*)$ over $\mathrm{Ann}_{S(\mathfrak{g}^*)}\,\mathrm{gr}(U(\mathfrak{g})/I)$ and $m_1, \ldots, m_l$ are their multiplicities. Since the variety $\mathrm{VA}(I)$ coincides with the Zariski closure of a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ by Joseph's theorem, we have that $\mathrm{AC}(I) = m_I \overline{\mathcal{O}}$ for some $m_I \in \mathbb{N}$. The positive integer m_I is often referred to as the *multiplicity* of $\mathcal{O}$ in $U(\mathfrak{g})/I$ and abbreviated as $\mathrm{mult}_{\mathcal{O}}(U(\mathfrak{g})/I)$.

From now on we fix a nonzero nilpotent element $e \in \mathfrak{g}$ and include it into an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$. Let $U(\mathfrak{g}, e)$ be the finite W-algebra associated with the pair $(\mathfrak{g}, e)$, a non-commutative filtered deformation of the coordinate algebra $\Bbbk[e + \mathfrak{g}_f]$ on the Slodowy slice $e + \mathfrak{g}_f$ regarded with its Slodowy grading. Recall that $U(\mathfrak{g}, e) = \big(\mathrm{End}_{\mathfrak{g}}\, Q_e\big)^{\mathrm{op}}$ where Q_e stands for the generalised Gelfand–Graev $\mathfrak{g}$-module associated with e; see [13], [4] for more detail. By a result of Skryabin, proved in the appendix to [13], the right $U(\mathfrak{g}, e)$-module Q_e is free and for any irreducible $U(\mathfrak{g}, e)$-module V the $\mathfrak{g}$-module $Q_e \otimes_{U(\mathfrak{g},e)} V$ is irreducible. As a consequence, the annihilator $I_V := \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_e \otimes_{U(\mathfrak{g},e)} V\big)$ is a primitive ideal of $U(\mathfrak{g})$.

Let $\mathcal{O}$ be the adjoint G-orbit of e and define $\mathcal{X}_{\mathcal{O}} := \{I \in \mathcal{X} \mid \mathrm{VA}(I) = \overline{\mathcal{O}}\}$. By [14], $I_V \in \mathcal{X}_{\mathcal{O}}$ for any finite dimensional irreducible $U(\mathfrak{g}, e)$-module V, whilst [7], [15] and [5] show that any $I \in \mathcal{X}_{\mathcal{O}}$ has the form I_W for some finite dimensional irreducible $U(\mathfrak{g}, e)$-module W. Furthermore, there is a natural action of the component group $\Gamma = G_e/G_e^{\circ}$ on the set $\mathrm{Irr}\, U(\mathfrak{g}, e)$ of all isoclasses of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules and the primitive ideal I_W depends only on the class $[W] \in \mathrm{Irr}\, U(\mathfrak{g}, e)$. In [9] Losev showed that

$$\mathrm{mult}_{\mathcal{O}}(U(\mathfrak{g})/I_W) = [\Gamma : \Gamma_W] \cdot (\dim W)^2 \qquad (1)$$

where Γ_W denotes the stabiliser of the isoclass $[W]$ in Γ and the equality $I_W = I_{W'}$ holds if and only if $[W'] = {}^{\gamma}[W]$ for some $\gamma \in \Gamma$. By Goldie's theory, for any $I \in \mathcal{X}$ the prime Noetherian ring $U(\mathfrak{g})/I$ embeds into a full ring of fractions which is isomorphic to the matrix algebra $\mathrm{Mat}_n(\mathcal{D}_I)$ over a skew-field $\mathcal{D}_I$ (called the

Goldie field of $U(\mathfrak{g})/I$). The positive integer $n = n_I$ coincides with the Goldie rank of $U(\mathfrak{g})/I$ which is often abbreviated as $\mathrm{rk}(U(\mathfrak{g})/I)$.

Recall that a primitive ideal I is called *completely prime* if $U(\mathfrak{g})/I$ is a domain. It is well known that this happens if and only if $\mathrm{rk}(U(\mathfrak{g})/I) = 1$. Classifying the completely prime primitive ideals of $U(\mathfrak{g})$ is an old-standing classical problem of Lie Theory which remains unresolved outside type A. If $I = I_V \in \mathcal{X}_{\mathcal{O}}$, where $[V] \in \operatorname{Irr} U(\mathfrak{g}, e)$, then the main result of [16] states that the number

$$q_I := \frac{\dim V}{\mathrm{rk}(U(\mathfrak{g})/I)}$$

is a positive integer and, moreover, $q_I = 1$ provided that the Goldie field $\mathcal{D}_I$ is isomorphic to the skew-field of fractions of a Weyl algebra. The integrality of q_I implies that I_V is completely prime whenever $\dim V = 1$ (this fact also follows from results of Mœglin [12] and Losev [7]). We mention for completeness that outside type A there are examples of completely prime primitive ideals $I \in \mathcal{X}_{\mathcal{O}}$ for which $q_I > |\Gamma|$ (see [16, Remark 4.3]), but it is proved in [10] for $\mathfrak{g}$ classical (and conjectured for $\mathfrak{g}$ exceptional) that $q_I = 1$ whenever the central character of I is integral.

We are interested in those $I \in \mathcal{X}_{\mathcal{O}}$ for which $\mathrm{mult}_{\mathcal{O}}(U(\mathfrak{g})/I) = 1$; we call such primitive ideals *multiplcity free*. Any multiplicity free primitive ideal is completely prime, but the converse is not always true outside type A. Let $\mathcal{S}_1, \ldots, \mathcal{S}_t$ be all sheets of $\mathfrak{g}$ containing $\mathcal{O}$. For $1 \le i \le t$, set $r_i = \dim \mathcal{S}_i - \dim \mathcal{O}$, the rank of $\mathcal{S}_i$, and define $r(e) := \max_{1 \le i \le t} r_i$. Let $\mathfrak{c}_e = \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$. The adjoint action of G_e on $\mathfrak{g}_e$ induces the trivial action of the connected group G_e° on $\mathfrak{c}_e$ and hence gives rise to a natural action of Γ on $\mathfrak{c}_e$. We denote by $\mathfrak{c}_e^\Gamma$ the subspace of all $x \in \mathfrak{c}_e$ such that $\gamma(x) = x$ for all $\gamma \in \Gamma$. Put $c(e) := \dim(\mathfrak{c}_e)$ and $c_\Gamma(e) := \dim(\mathfrak{c}_e^\Gamma)$.

Let $U(\mathfrak{g}, e)^{\mathrm{ab}} = U(\mathfrak{g}, e)/I_c$ where I_c is the two-sided ideal of $U(\mathfrak{g}, e)$ generated by all commutators $u \cdot v - v \cdot u$ with $u, v \in U(\mathfrak{g}, e)$ and denote by $\mathcal{E}$ the maximal spectrum of $U(\mathfrak{g}, e)^{\mathrm{ab}}$. This affine variety parametrises the 1-dimensional representations of $U(\mathfrak{g}, e)$ and is acted upon by the the component group Γ (it is known that Γ acts on $U(\mathfrak{g}, e)^{\mathrm{ab}}$ by algebra automorphisms). We denote by $\mathcal{E}^\Gamma$ the set of all $\eta \in \mathcal{E}$ such that $\gamma(\eta) = \eta$ for all $\gamma \in \Gamma$. Let I_Γ be the ideal of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ generated by all $\phi - \phi^\gamma$ with $\phi \in U(\mathfrak{g}, e)^{\mathrm{ab}}$ and $\gamma \in \Gamma$. Then $\mathcal{E}^\Gamma$ coincides with the zero locus of I_Γ in $\mathcal{E}$. Define $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma := U(\mathfrak{g}, e)^{\mathrm{ab}}/I_\Gamma$.

It follows from (1) that $I = I_V$ is multiplicity free if and only if $\dim V = 1$ and $\Gamma_V = \Gamma$. Thus, in order to classify the multiplicity-free primitive ideals in $\mathcal{X}_{\mathcal{O}}$ we need to determine the variety $\mathcal{E}^\Gamma$. Thanks to [15, Theorem 1.2] we know that $\dim \mathcal{E} = r(e)$ and the number of irreducible components of $\mathcal{E}$ is greater than or equal to t. Thus, the variety $\mathcal{E}$ is irreducible only if e lies in a unique sheet of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}_n$, this condition is satisfied for any nilpotent element e and [15, Corollary 3.2] states that $U(\mathfrak{sl}_n, e)^{\mathrm{ab}}$ is a polynomial algebra in $r(e)$ variables. Our first main result is a generalisation of that to all Lie algebras of classical types. We call an element $a \in \mathfrak{g}$ *non-singular* if it lies in a unique sheet of $\mathfrak{g}$.

Theorem 1. *If e is a nilpotent element in a classical Lie algebra $\mathfrak{g}$, then the following are equivalent:*

(i) *e is non-singular;*
(ii) *$c(e) = r(e)$;*
(iii) *$U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra in $r(e)$ variables.*

Although the polynomiality of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ occurs rather infrequently outside type A, our next result shows that the algebras $U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$ exhibit a very uniform behaviour:

Theorem 2. *If e is any nilpotent element in a classical Lie algebra $\mathfrak{g}$, then $U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$ is isomorphic to a polynomial algebra in $c_{\Gamma}(e)$ variables. In particular $\mathcal{E}^{\Gamma}$ is a single point if and only if $c_{\Gamma}(e) = 0$.*

As an obvious corollary of Theorem 2 we deduce that the variety $\mathcal{E}^{\Gamma}$ is isomorphic to an affine space for any nilpotent element in a classical Lie algebra and hence is irreducible.

In order to prove Theorems 1 and 2 we have to look very closely at the centralisers of nilpotent elements in classical Lie algebras. Suppose $\mathfrak{g}$ is one of $\mathfrak{so}_N$ or $\mathfrak{sp}_N$. It is well known that to any nilpotent element $e \in \mathfrak{g}$ one can attach a partition $\lambda \in \mathcal{P}_{\epsilon}(N)$ where $\epsilon = 1$ if $\mathfrak{g} = \mathfrak{so}_N$ and $\epsilon = -1$ if $\mathfrak{g} = \mathfrak{sp}_N$. Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of N with $\lambda_1 \geq \cdots \geq \lambda_n \geq 1$ is in $\mathcal{P}_{\epsilon}(N)$ if there is a map $i \mapsto i'$ on the set of indices $\{1, \ldots, n\}$ satisfying $i' \in \{i-1, i, i+1\}$ and $i'' = i$ and such that $\lambda_{i'} = \lambda_i$ and $i' = i$ if and only if $\epsilon(-1)^{\lambda_i} = -1$ for all i. We call a pair of indices $(i, i+1)$ with $1 \leq i < n$ a *2-step* of λ if $i' = i$, $(i+1)' = i+1$ and $\lambda_{i-1} \neq \lambda_i \geq \lambda_{i+1} \neq \lambda_{i+2}$ where our convention is that $\lambda_i = 0$ for $i \in \{0, n+1\}$. We denote by $\Delta(\lambda)$ the set of all 2-steps of λ and set

$$s(\lambda) := \textstyle\sum_{i=1}^{n} [(\lambda_i - \lambda_{i+1})/2].$$

We call a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n}, \lambda_{2n+1}\, \lambda_{2n+2}) \in \mathcal{P}_1(N)$ *exceptional* if $N = 4m+2$ for some $m \in \mathbb{N}$ and there exists a $k \leq n$ such that the parts $\lambda_{2k+1}, \lambda_{2k+2}$ are odd and the parts λ_i with $i \notin \{2k+1, 2k+2\}$ are all even.

It should be mentioned that for any $(i, i+1) \in \Delta(\lambda)$ the integers λ_i and λ_{i+1} have the same parity. If $(i, i+1) \in \Delta(\lambda)$ and $i > 1$ (resp. $i = 1$), then we call λ_{i-1} and λ_{i+2} (resp. λ_3) the *boundary* of $(i, i+1)$. We say that a 2-step $(i, i+1)$ is *good* if its boundary and λ_i have the opposite parity.

Theorem 3. *Let $\mathfrak{g}$ be one of $\mathfrak{so}_N$ or $\mathfrak{sp}_N$, where $N \geq 2$, and let e be a nilpotent element of $\mathfrak{g}$ associated with a partition $\lambda \in \mathcal{P}_{\epsilon}(N)$. Then the following hold:*

(i) *$c(e) = s(\lambda) + |\Delta(\lambda)|$;*
(ii) *$c_{\Gamma}(e) = s(\lambda)$ unless $\mathfrak{g} = \mathfrak{so}_N$ with $N = 4m+2$ and $\lambda \in \mathcal{P}_1(N)$ is exceptional, in which case $c_{\Gamma}(e) = s(\lambda) + 1$;*
(iii) *e is non-singular if and only if all 2-steps of λ are good.*

If $\lambda \in \mathcal{P}_1(4m+2)$ is exceptional, then it is immediate from the definitions that $|\Delta(\lambda)| = 1$ and the only 2-step of λ is good. Therefore, any nilpotent element e

of $\mathfrak{g} = \mathfrak{so}_{4m+2}$ associated with λ is non-singular. It is also straightforward to see that any such e is a Richardson element of $\mathfrak{g}$.

Now suppose that $\mathfrak{g}$ is an exceptional Lie algebra. In this case our results are less complete because we have to exclude the following seven induced orbits:

TABLE 0. UNRESOLVED CASES.

F_4	E_6	E_7	E_8	E_8	E_8	E_8
$\mathsf{C}_3(\mathsf{a}_1)$	$\mathsf{A}_3+\mathsf{A}_1$	$\mathsf{D}_6(\mathsf{a}_2)$	$\mathsf{E}_6(\mathsf{a}_3)+\mathsf{A}_1$	$\mathsf{D}_6(\mathsf{a}_2)$	$\mathsf{E}_7(\mathsf{a}_2)$	$\mathsf{E}_7(\mathsf{a}_5)$

It is worth mentioning that all orbits listed in Table 0 are non-special in the sense of Lusztig.

Theorem 4. *Let $\mathfrak{g}$ be an exceptional Lie algebra and suppose that e is an induced nilpotent element of $\mathfrak{g}$. Then the following hold:*

(i) $\mathcal{E}^\Gamma \neq \emptyset$.

(ii) *If e is not listed in the first six columns of Table 0 and lies in a single sheet of $\mathfrak{g}$, then $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra in $c(e)$ variables.*

(iii) *If e is not listed in Table 0, then $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is isomorphic to a polynomial algebra in $c_\Gamma(e)$ variables.*

The traditional way of classifying the completely prime ideals $I \in \mathcal{X}$ parallels Borho's classification of the sheets of $\mathfrak{g}$; see [1]. Here one aims to show that if the orbit $\mathcal{O}$ is induced from a rigid orbit $\mathcal{O}_0$ in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, then the majority of $I \in \mathcal{X}_\mathcal{O}$ can be obtained as the annihilators in $U(\mathfrak{g})$ of induced $\mathfrak{g}$-modules $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{p}}(E) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$ where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{n}$ and E is an irreducible $\mathfrak{p}$-module with the property that $\mathfrak{n} \cdot E = 0$ and $I_0 := \mathrm{Ann}_{U(\mathfrak{l})} E$ is a completely prime primitive ideal of $U(\mathfrak{l})$ such that $\mathrm{VA}(I_0) = \overline{\mathcal{O}}_0$.

Theorem 5. *Let $I \in \mathcal{X}_\mathcal{O}$ be a multiplicity-free primitive ideal associated with an induced nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$. If $\mathfrak{g}$ is exceptional assume further that $\mathcal{O}$ is not listed in Table 0. Then there exists a proper parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with a Levi subalgebra $\mathfrak{l}$ and a rigid nilpotent orbit $\mathcal{O}_0$ in $\mathfrak{l}$ such that $\mathcal{O}$ is induced from $\mathcal{O}_0$ and $I = I(\mathfrak{p}, E)$ where E is an irreducible $U(\mathfrak{p})$-module with the trivial action of the nilradical of $\mathfrak{p}$. Moreover, the primitive ideal $I_0 = \mathrm{Ann}_{U(\mathfrak{l})} E$ is completely prime and $\mathrm{VA}(I_0) = \overline{\mathcal{O}}_0$.*

Theorem 5 can be regarded as a generalisation of Mœglin's theorem [11] on completely prime primitive ideals of $U(\mathfrak{sl}_n)$. It is quite possible that it holds for all induced orbits in $\mathfrak{g}$.

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Partition functions and indices of transversally elliptic operators

Claudio Procesi
(joint work with Corrado De Concini and Michèle Vergne)

In 1968 appears the fundamental work of Atiyah and Singer on the index theorem of elliptic operators, a theorem formulated in successive steps of generality. One general and useful setting is for operators on a manifold M which satisfy a symmetry with respect to a compact Lie group G and are elliptic in directions transverse to the G-orbits. The values of the index are generalized functions on G.

In his Lecture Notes describing joint work with I.M. Singer, Atiyah explains how to reduce general computations to the case in which G is a torus, and the manifold M is a complex linear representation $M_X = \oplus_{a \in X} L_a$, where $X \subset \hat{G}$ is a finite list of characters and L_a the one dimensional complex line where G acts by the character $a \in X$. He then computes explicitly in several cases and ends his introduction saying

" ... for a circle (with any action) the results are also quite explicit. However for the general case we give only a reduction process and one might hope for something explicit. This probably requires the development of an appropriate algebraic machinery, involving cohomology but going beyond it."

In the seminar I explained how to provide this algebraic machinery, which turns out to be a spinoff of the theory of splines, and complete this computation.

On the conjecture of Borel and Tits for abstract homomorphisms of algebraic groups

Igor Rapinchuk

The general philosophy in the study of abstract homomorphisms between the groups of rational points of algebraic groups is as follows. Suppose G and G' are algebraic groups that are defined over infinite fields K and K', respectively. Let

$$\varphi\colon G(K) \to G'(K')$$

be an abstract homomorphism between their groups of rational points. Then, under appropriate assumptions, one expects to be able to write φ essentially as a composition $\varphi = \beta \circ \alpha$, where $\alpha\colon G(K) \to {}_{K'}G(K')$ is induced by a *field homomorphism* $\tilde{\alpha}\colon K \to K'$ (and ${}_{K'}G$ is the group obtained from G by base change via $\tilde{\alpha}$), and $\beta\colon {}_{K'}G(K') \to G'(K')$ arises from a K'-defined *morphism of algebraic groups* ${}_{K'}G \to G'$. Whenever φ admits such a decomposition, one generally says that it has a *standard description.*

The main motivation for my work has been the following conjecture of Borel and Tits (see [2], 8.19). Recall that for an algebraic G defined over a field k, one denotes by G^+ the subgroup of $G(k)$ generated by the k-points of split (smooth) connected unipotent k-subgroups.

> Let G and G' be algebraic groups defined over infinite fields k and k', respectively. If $\rho\colon G(k) \to G'(k')$ is any abstract homomorphism such that $\rho(G^+)$ is Zariski-dense in $G'(k')$, then *there exists a commutative finite-dimensional k'-algebra B and a ring homomorphism $f\colon k \to B$ such that*

(BT) $$\rho|_{G^+} = \sigma \circ r_{B/k'} \circ F$$

> *where $F\colon G(k) \to_B G(B)$ is induced by f (${}_BG$ is the group obtained by change of scalars), $r_{B/k'}\colon {}_BG(B) \to R_{B/k'}({}_BG)(k')$ is the canonical isomorphism (here $R_{B/k'}$ denotes the functor of restriction of scalars), and σ is a rational k'-morphism of $R_{B/k'}({}_BG)$ to G'.*

In their fundamental paper [2], Borel and Tits proved the conjecture for G an absolutely almost simple k-isotropic group and G' a reductive group. Shortly after the conjecture was formulated, Tits [10] sketched a proof of (BT) in the case that $k = k' = \mathbb{R}$. Prior to our work, the only other available result was due to L. Lifschitz and A.S. Rapinchuk [4], where the conjecture was essentially proved in the case where k and k' are fields of characteristic 0, G is a universal Chevalley group, and G' is an algebraic group with commutative unipotent radical.

While the above results only deal with abstract homomorphisms of groups of points over *fields*, it should be pointed out that there has also been considerable interest and activity in analyzing abstract homomorphisms of higher rank arithmetic groups and lattices (e.g. the work of Bass, Milnor, and Serre [1] on the congruence subgroup problem and Margulis's Superrigidity Theorem [5, Chap. VII]). However, relatively little was previously known about abstract homomorphisms of groups of points over *general commutative rings*, which has been the primary focus of my work in this area.

To state our results, we first need to fix some notations. Let Φ be a reduced irreducible root system of rank ≥ 2 and G be the corresponding universal Chevalley-Demazure group scheme over $\mathbb{Z}$. For any commutative ring R, we denote by $G(R)^+$ the subgroup of $G(R)$ generated by the R-points of the canonical one-parameter root subgroups (usually called the *elementary subgroup*). Our first result is a rigidity statement for abstract representations

$$\rho\colon G(R)^+ \to GL_n(K),$$

where K is an algebraically closed field. In the statement below, for a finite-dimensional commutative K-algebra B, we view the group of rational points $G(B)$ as an algebraic group over K using the functor of restriction of scalars. Furthermore, given a commutative ring R, we will say that (Φ, R) is a *nice pair* if $2 \in R^\times$ whenever Φ contains a subsystem of type B_2 and $2, 3 \in R^\times$ if Φ is of type G_2.

Theorem 1. [7, Main Theorem] *Let Φ be a reduced irreducible root system of rank ≥ 2, R a commutative ring such that (Φ, R) is a nice pair, and K an algebraically closed field. Assume that R is noetherian if* $\operatorname{char} K > 0$. *Furthermore let G be the universal Chevalley-Demazure group scheme of type Φ and let $\rho\colon G(R)^+ \to GL_n(K)$ be a finite-dimensional linear representation over K of the elementary subgroup $G(R)^+ \subset G(R)$. Set $H = \overline{\rho(G(R)^+)}$ (Zariski closure), and let H° denote the connected component of the identity of H. Then in each of the following situations*

(1) *H° is reductive;*

(2) $\operatorname{char} K = 0$ *and R is semilocal;*

(3) $\operatorname{char} K = 0$ *and the unipotent radical U of H° is commutative,*

there exists a commutative finite-dimensional K-algebra B, a ring homomorphism $f\colon R \to B$ with Zariski-dense image, and a morphism $\sigma\colon G(B) \to H$ of algebraic K-groups such that for a suitable subgroup $\Delta \subset G(R)^+$ of finite index, we have

$$\rho|_\Delta = (\sigma \circ F)|_\Delta,$$

where $F\colon G(R)^+ \to G(B)^+$ is the group homomorphism induced by f.

Thus if $R = k$ is a field of characteristic $\neq 2$ or 3, then R is automatically semilocal and (Φ, R) is a nice pair, so Theorem 1 provides a proof of (BT) in the case that G is split and K is an algebraically closed field of characteristic zero.

Let us now describe some applications of Theorem 1 to the study of character varieties of elementary subgroups of Chevalley groups. Let K be an algebraically closed field of characteristic 0 and R be a finitely generated commutative ring. As above, we suppose that Φ is a reduced irreducible root system of rank ≥ 2 and let G be the corresponding universal Chevalley-Demazure group scheme. Then the elementary subgroup $G(R)^+$ has Kazhdan's property (T) (see [3]), hence is in particular a finitely generated group, and therefore, for any integer $n \geq 1$, one can consider the character variety $X_n(\Gamma)$. Our first result is as follows.

Theorem 2. [8] *Let Φ be a reduced irreducible root system of rank ≥ 2, R a finitely generated commutative ring such that (Φ, R) is a nice pair, and G the universal Chevalley-Demazure group scheme of type Φ. Denote by Γ the elementary subgroup $G(R)^+$ of $G(R)$ and consider n-th character variety $X_n(\Gamma)$ of Γ over an algebraically closed field K of characteristic 0. Then there exists a constant $c = c(R)$ (depending only on R) such that $\kappa_n(\Gamma) := \dim X_n(\Gamma)$ satisfies*

$$\kappa_n(\Gamma) \leq c \cdot n$$

for all $n \geq 1$.

The proof of Theorem 2 exploits extensively our standard descriptions for representations with non-reductive image.

Another application of Theorem 1 has to do with the problem of realizing complex affine varieties as character varieties of suitable finitely generated groups. This question was previously considered by M. Kapovich and J. Millson [6], who showed that any affine variety S defined over $\mathbb{Q}$ is birationally isomorphic to an appropriate character variety of some Artin group Γ. Using Theorem 1, we are able to prove the following result (we now take $K = \mathbb{C}$).

Theorem 3. [9] *Let S be an affine algebraic variety defined over $\mathbb{Q}$. There exist a finitely generated group Γ having Kazhdan's property (T) and an integer $n \geq 1$ such that there is a biregular isomorphism of complex algebraic varieties*

$$S(\mathbb{C}) \to X_n(\Gamma) \setminus \{[\rho_0]\},$$

where ρ_0 is the trivial representation.

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The semigroup of branching rules

Nicolas Ressayre

1. Definition

Let $G \subset \hat{G}$ be two complex connected reductive groups. Fix maximal tori T, $\hat{T}$ and Borel subgroups B and $\hat{B}$. Let $X(T)$ (resp. $X(T)^+$) denote the set of weights (resp. dominant weights) of T. For $\lambda \in X(T)^+$, $V_G(\lambda)$ denotes the irreducible G-module of highest weight λ. Similar notations are used for the group $\hat{G}$. We consider the branching problem of understanding irreducible $\hat{G}$-modules $V_{\hat{G}}(\hat{\lambda})$ as G-modules. More precisely, consider the set

$$\Lambda(G, \hat{G}) = \{(\lambda, \hat{\lambda}) \in X(T)^+ \times X(\hat{T})^+ \ : \ \begin{array}{l} V_G(\lambda)^* \text{ embeds in } V_{\hat{G}}(\hat{\lambda}) \\ \text{as a } G\text{−module} \end{array} \}.$$

This lecture surveys recent results on the question of describing $\Lambda(G, \hat{G})$. The first qualitative result is:

Theorem 1 (Brion-Knop)**.** *As a subset of $X(T) \times X(\hat{T})$, $\Lambda(G, \hat{G})$ is stable by addition. More precisely, it is a finitely generated semigroup.*

This theorem allows to decompose the question of describing $\Lambda(G, \hat{G})$ in three ones:

(1) describe the group $\mathbb{Z}\Lambda(G, \hat{G})$ generated by $\Lambda(G, \hat{G})$;
(2) describe the cone $\mathbb{Q}_{\geq 0}\Lambda(G, \hat{G})$ generated by $\Lambda(G, \hat{G})$;
(3) compare the set $\mathbb{Q}_{\geq 0}\Lambda(G, \hat{G}) \cap \mathbb{Z}\Lambda(G, \hat{G})$ of integral points in the cone with $\Lambda(G, \hat{G})$. This is named the saturation problem.

The group $\mathbb{Z}\Lambda(G, \hat{G})$ is easy to describe. For simplicity assume that no ideal of lg (the Lie algebra of G) is an ideal of $\hat{\text{lg}}$. We have:

Proposition 1. *Let $Z := \{g \in G \ : \ \forall \hat{g} \in \hat{G} \quad g\hat{g} = \hat{g}g\}$ be the intersection of G and the center of $\hat{G}$. Then*

$$\mathbb{Z}\Lambda(G, \hat{G}) = \{(\lambda, \hat{\lambda}) \ : \ \forall z \in Z \quad \lambda(z)^{-1} = \hat{\lambda}(z)\}.$$

2. Inequalities of the convex cone

Relation with Geometric Invariant Theory. Consider the complete flag variety $X = G/B \times \hat{G}/\hat{B}$ of the group $G \times \hat{G}$. Given $(\lambda, \hat{\lambda}) \in X(T)^+ \times X(\hat{T})^+$, we construct a $G \times \hat{G}$-linearized line bundle $\mathcal{L}(\lambda, \hat{\lambda})$ on X such that $H^0(X, \mathcal{L}(\lambda, \hat{\lambda})) = V_G(\lambda) \otimes V_{\hat{G}}(\hat{\lambda})$. In particular, $(\lambda, \hat{\lambda})$ belongs to $\mathbb{Q}_{\geq 0}\Lambda(G, \hat{G})$ if and only if $\mathcal{L}(\lambda, \hat{\lambda})^{\otimes k}$ admits nonzero G-invariant sections, for some positive integer k. This interpretation of the cone $\mathbb{Q}_{\geq 0}\Lambda(G, \hat{G})$ in terms of GIT, allows one in terms of moment map. Let us illustrate this principle by an example.

The Horn cone. If $G = \mathrm{GL}_n(\mathbb{C})$ diagonally embedded in $\hat{G} = G \times G$, the closure in $X(T)^3 \otimes \mathbb{R}$ of the cone $\mathbb{Q}_{\geq 0}\Lambda(G, \hat{G})$ identifies with the set

$$\mathrm{Horn}(n) = \{(\lambda(A), \lambda(B), \lambda(C) \ : \ \begin{array}{l} A,\, B,\, C \text{ are 3 Hermitian matrices} \\ \text{s.t. } A + B + C = 0 \end{array} \}.$$

Here $\lambda(A) \in \mathbb{R}^n$ denotes the spectrum of the Hermitian $n \times n$ matrix A. In 1962, A. Horn purposed the following conjectural description of $\mathrm{Horn}(n)$.

Conjecture 1. *Let λ, μ and ν be three nonincreasing sequences in $\mathbb{R}^n$. Then $(\lambda, \mu, \nu) \in \mathrm{Horn}(n)$ if and only if*

$$\sum_i \lambda_i + \sum_j \mu_j + \sum_k \nu_k = 0,$$

and

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \leq 0,$$

for any subsets I, J and K of $\{1, \dots, n\}$ of same cardinality $1 \leq r \leq n$ such that

$$(\tau^I, \tau^J, \tau^K - (n-r)^r) \in \mathrm{Horn}(r). \tag{1}$$

Here $\tau^I = (i_r - r \geq \dots \geq i_1 - 1)$, if $I = \{i_1 < \dots < i_r\}$.

The Belkale-Kumar product. Consider any projective G-homogeneous space G/P and its cohomology ring $H^*(G/P, \mathbb{Z})$. The Schubert bases is denoted by $(\sigma_w)_{w \in W^P}$. In 2006, Belkale-Kumar defined a new associative product $\odot$ on $H^*(G/P, \mathbb{Z})$. More precisely, $H^*(G/P, \mathbb{Z})$ admits a (multi)-filtration with $\odot$ as associated graded product.

To any 1-PS θ of G is associated a parabolic subgroup $P(\theta)$. Since θ is also a 1-PS of $\hat{G}$ we have a parabolic subgroup $\hat{P}(\theta)$ and an embedding $\phi_\theta : G/P(\theta) \longrightarrow \hat{G}/\hat{P}(\theta)$. The comorphism ϕ^* in cohomology respects the filtration and hence induces a morphism $\phi_\theta^\odot : (H^*(\hat{G}/\hat{P}(\theta), \mathbb{Z}), \odot) \longrightarrow (H^*(G/P(\theta), \mathbb{Z}), \odot)$.

A description of $\mathbb{Q}_{\geq 0}\Lambda(G, \hat{G})$. An indivisible dominant 1-PS θ of T is said to be *special* if the set of weights χ of T acting on $\hat{\mathrm{lg}}/\mathrm{lg}$ such that $\langle \chi, \theta \rangle = 0$ spans an hyperplane of $X(T) \otimes \mathbb{Q}$.

Theorem 2. *Assume that no ideal of* lg *is an ideal of* $\hat{\mathrm{lg}}$. *Let $(\lambda, \hat{\lambda}) \in X(T)^+ \times X(\hat{T})^+$. Then $(\lambda, \hat{\lambda})$ belongs to $\mathbb{Q}_{\geq 0}\Lambda(G, \hat{G})$ iff for any special 1-PS θ, we have*

$$\langle w\theta, \lambda \rangle + \langle \hat{w}\theta, \hat{\lambda} \rangle \leq 0$$

for any $w \in W^{P(\theta)}$ and $\hat{w} \in \hat{W}^{\hat{P}(\theta)}$ such that

$$\phi_\theta^\odot(\sigma_{\hat{w}}) \odot \sigma_w = [pt].$$

Moreover, this system of inequalities is irredundant.

Using GIT, we proved some parts of this statement. We also explained the contributions of Klyachko, Kapovich-Leeb-Milson, Belkale, Berenstein-Sjamaar, Belkale-Kumar and R. to this result. For example, Klyachko's result can be stated as follows:

Theorem 3 (Klyachko 1998)**.** *Conjecture 1 holds if one replaces condition* (1) *by*

$$\sigma_I.\sigma_J.\sigma_K = d[pt] \in H^*(\mathrm{Gr}(r,n),\mathbb{Z}), \tag{2}$$

for some positive d. Here σ_I, σ_J and σ_K are Schubert classes of the Grassmannian $\mathrm{Gr}(r,n)$*.*

3. The question of saturation

In this last section, we consider the case of G diagonally embedded in $\hat{G} = G\times G$. Set $\Lambda(G) := \Lambda(G, G\times G)$.

Observe that the integer d in condition (2) is the Littlewood-Richardson coefficient $c_{\tau_I\,\tau_J}^{\tau_K{}^\vee}$. Then, one can prove that conditions (1) and (2) are equivalent if and only if

$$c_{k\tau_I\,k\tau_J}^{k\tau_K{}^\vee} > 0 \quad \text{for some positive } k \qquad \Rightarrow \qquad c_{\tau_I\,\tau_J}^{\tau_K{}^\vee} > 0. \tag{3}$$

Statement (3) is equivalent to

$$\mathbb{Q}_{\geq 0}\Lambda(G) \cap \mathbb{Z}\Lambda(G) = \Lambda(G), \tag{4}$$

for $G = \mathrm{GL}_n$. In 2000, Knutson-Tao proved this statement ending the proof of the Horn conjecture. Other proofs were given by Derksen-Weyman, Buch, Belkale, Kapovich-Leeb-Milson.

In general the equality (4) does not hold, and we are looking for saturation constants k such that for any $(\lambda,\mu,\nu) \in \mathbb{Z}\Lambda(G)$

$$(\lambda,\mu,\nu) \in \mathbb{Q}_{\geq 0}\Lambda(G) \quad \Rightarrow \quad (k\lambda, k\mu, k\nu) \in \Lambda(G).$$

Theorem 4 (Kapovich-Leeb-Milson)**.** *Assume G simple and simply connected. Let k be the lcd of the coefficients of the longest root expressed in the bases of simple roots.*

Then k^2 is a saturation constant.

For classical G, this result was improved by Belkale-Kumar and Sam:

Theorem 5. *For classical G, 2 is a saturation constant.*

We ended these lectures by two conjectures.

Conjecture 2. *If G is simple and simply laced then* 1 *is a saturation constant.*

Conjecture 3. *If λ, μ and ν are regular then* 1 *is a saturation constant.*

Perverse sheaves on affine Grassmannians, universal Verma modules, and differential operators on the basic affine space

Simon Riche
(joint work with Victor Ginzburg)

This note gives an overview of the main constructions and results of [GR].

Let G be a complex connected reductive algebraic group, $B \subset G$ a Borel subgroup, $T \subset B$ a maximal torus, and let W be the Weyl group of (G, T). Let $\mathfrak{t}$ be the Lie algebra of T, and consider the graded algebra $\mathrm{S}_\hbar := \mathrm{S}(\mathfrak{t})[\hbar]$ (where all natural generators are in degree 2). In [GR] we construct and study three families of graded modules over $\mathrm{S}_\hbar$, endowed with symmetries parametrized by W. These families arise in three very different contexts: the first family is defined in terms of geometry (more precisely asymptotic differential operators on the basic affine space), the second family is defined in terms of algebra (more precisely intertwiners for asymptotic universal Verma modules), and the third family is defined in terms of topology (more precisely equivariant cohomology of spherical perverse sheaves on the affine Grassmannian of the dual group). We prove that these three families are related by natural isomorphisms compatible with the Weyl group symmetries.

Let U be the unipotent radical of B, and consider the basic affine space $\mathscr{X} := G/U$. Let $\mathcal{D}_{\hbar,\mathscr{X}}$ be the sheaf of algebras of asymptotic differential operators on $\mathscr{X}$ (i.e. the Rees algebra of the usual algebra of differential operators on $\mathscr{X}$, filtered by the degree) and let $\mathcal{D}_\hbar(\mathscr{X}) := \Gamma(\mathscr{X}, \mathcal{D}_{\hbar,\mathscr{X}})$. This graded algebra has a G-module structure (induced by the G-action on $\mathscr{X}$ given by multiplication on the left), a T-module structure (induced by the T-action on $\mathscr{X}$ given by multiplication *on the right*), and a structure of $\mathrm{S}_\hbar$-algebra (obtained by differentiating the T-action on $\mathscr{X}$). For V in the category $\mathsf{Rep}(G)$ of finite dimensional algebraic G-modules and $\lambda \in X^*(T)$ we set

$$\mathcal{M}^{\mathrm{geom}}_{V,\lambda} := \big(V \otimes_{\mathbb{C}} {}^{(\lambda)}\mathcal{D}_\hbar(\mathscr{X})\big)^G \langle \lambda(2\check{\rho})\rangle.$$

(Here ${}^{(\lambda)}$ is a twist of the $\mathrm{S}_\hbar$-action by λ, and $\langle\cdot\rangle$ is the shift of the grading.) Using a W-action on $\mathcal{D}_\hbar(\mathscr{X})$ induced by partial Fourier transforms (see [BBP]) we construct for any $w \in W$ and V, λ as above an isomorphism of graded $\mathrm{S}_\hbar$-modules

$$\Phi^{V,\lambda}_w : \mathcal{M}^{\mathrm{geom}}_{V,\lambda} \xrightarrow{\sim} {}^{w}\mathcal{M}^{\mathrm{geom}}_{V,w\lambda}$$

(where ^{w} is a twist of the $\mathrm{S}_\hbar$-action by w). This construction depends on a choice of a root vector in the Lie algebra $\mathfrak{g}$ of G associated with any simple root.

Let now $U_\hbar(\mathfrak{g})$ be the asymptotic universal enveloping algebra of $\mathfrak{g}$ (i.e. the Rees algebra of the usual enveloping algebra of $\mathfrak{g}$, equipped with the Poincaré–Birkhoff–Witt filtration). For $\lambda \in X^*(T)$ we consider the (asymptotic) universal Verma module

$$\mathbf{M}(\lambda) := \mathrm{S}_\hbar\langle \lambda + \rho\rangle \otimes_{U_\hbar(\mathfrak{b})} U_\hbar(\mathfrak{g}).$$

Here $\mathfrak{b}$ is the Lie algebra of B, ρ is the half sum of positive roots, and $\mathrm{S}_\hbar\langle\lambda+\rho\rangle$ is the right $U_\hbar(\mathfrak{b})$-module isomorphic to $\mathrm{S}_\hbar$ as a graded $\mathbb{C}[\hbar]$-module, where the nilpotent radical of $\mathfrak{b}$ acts trivially and any $x \in \mathfrak{t}$ acts by multiplication by $x + \hbar(\lambda+\rho)(x)$.

We will consider $\mathbf{M}(\lambda)$ as a graded $(\mathrm{S}_\hbar, U_\hbar(\mathfrak{g}))$-bimodule. If V is in $\mathsf{Rep}(G)$ there exists a natural $(\mathrm{S}_\hbar, U_\hbar(\mathfrak{g}))$-bimodule structure on $V \otimes_{\mathbb{C}} \mathbf{M}(\lambda)$, and we define

$$\mathcal{M}^{\mathrm{alg}}_{V,\lambda} := \mathrm{Hom}_{(\mathrm{S}_\hbar, U_\hbar(\mathfrak{g}))}\big(\mathbf{M}(0), V \otimes_{\mathbb{C}} \mathbf{M}(\lambda)\big).$$

Adapting a construction appearing in the construction of the dynamical Weyl group (see [TV]) we define for any $w \in W$ and V, λ as above an isomorphism of graded $\mathrm{S}_\hbar$-modules

$$\Theta^{V,\lambda}_w : \mathcal{M}^{\mathrm{alg}}_{V,\lambda} \xrightarrow{\sim} {}^w\mathcal{M}^{\mathrm{alg}}_{V,w\lambda}.$$

Again, this construction depends on a choice of a root vector in $\mathfrak{g}$ associated with any simple root.

For the third construction we let $\check{G}$ be another complex connected reductive algebraic group. We choose a Borel subgroup $\check{B} \subset \check{G}$ and a maximal torus $\check{T} \subset \check{B}$. Let $\mathsf{Gr}_{\check{G}} := \check{G}(\mathscr{K})/\check{G}(\mathscr{O})$ be the associated affine Grassmannian, an ind-projective variety endowed with an action of $\check{G}(\mathscr{O})$. (Here $\mathscr{K} := \mathbb{C}((t))$ and $\mathscr{O} := \mathbb{C}[[t]]$.) We consider the category $\mathsf{Perv}_{\check{G}(\mathscr{O})}(\mathsf{Gr}_{\check{G}})$ of $\check{G}(\mathscr{O})$-equivariant perverse sheaves with complex coefficients on $\mathsf{Gr}_{\check{G}}$ (which by definition are assumed to be supported on a finite union of $\check{G}(\mathscr{O})$-orbits). Then the geometric Satake equivalence (see [MV] and references therein) provides a complex connected reductive algebraic group G (which is dual to $\check{G}$ in the sense of Langlands), a Borel subgroup $B \subset G$, a maximal torus $T \subset B$ and an equivalence of categories

$$\mathbb{S} : \mathsf{Perv}_{\check{G}(\mathscr{O})}(\mathsf{Gr}_{\check{G}}) \xrightarrow{\sim} \mathsf{Rep}(G).$$

Any $\lambda \in X_*(\check{T})$ defines in a natural way a point in $\mathsf{Gr}_{\check{G}}$, and we denote by i_λ the inclusion of this point. The group $\mathbb{C}^\times$ acts on $\mathsf{Gr}_{\check{G}}$ by loop rotation, and the image of i_λ is $\check{T} \times \mathbb{C}^\times$-stable. Moreover, any object $\mathcal{F}$ of $\mathsf{Perv}_{\check{G}(\mathscr{O})}(\mathsf{Gr}_{\check{G}})$ has a canonical $\check{G}(\mathscr{O}) \ltimes \mathbb{C}^\times$-equivariant structure. Hence it makes sense to define the graded module

$$\mathcal{M}^{\mathrm{top}}_{\mathcal{F},\lambda} := \mathsf{H}^\bullet_{\check{T}\times\mathbb{C}^\times}(i_\lambda^! \mathcal{F})$$

over the graded algebra $\mathrm{S}(\check{\mathfrak{t}}^*)[\hbar]$. (Here $\check{\mathfrak{t}}$ is the Lie algebra of $\check{T}$.) Moreover, the action of $N_{\check{G}}(\check{T}) \subset \check{G} \subset \check{G}(\mathscr{O})$ allows to define canonical isomorphisms

$$\Xi^{\mathcal{F},\lambda}_w : \mathcal{M}^{\mathrm{top}}_{\mathcal{F},\lambda} \xrightarrow{\sim} {}^w\mathcal{M}^{\mathrm{top}}_{\mathcal{F},w\lambda}$$

for all w in the Weyl group of $(\check{G}, \check{T})$ and $\mathcal{F}, \lambda$ as above.

The main result of [GR] can be stated as follows.

Theorem. *Assume that the datum of $(T \subset B \subset G)$ is obtained from $(\check{T} \subset \check{B} \subset \check{G})$ via the geometric Satake equivalence. (In particular, this implies that $X_*(\check{T}) = X^*(T)$, and the Weyl groups of (G,T) and $(\check{G},\check{T})$ are canonically identified.)*

1. *For any $\mathcal{F}$ in $\mathsf{Perv}_{\check{G}(\mathscr{O})}(\mathsf{Gr})$ and $\lambda \in X_*(\check{T}) = X^*(T)$ there exist canonical isomorphisms of graded $\mathrm{S}_\hbar$-modules*

$$\mathcal{M}^{\mathrm{geom}}_{\mathbb{S}(\mathcal{F}),\lambda} \cong \mathcal{M}^{\mathrm{alg}}_{\mathbb{S}(\mathcal{F}),\lambda} \cong \mathcal{M}^{\mathrm{top}}_{\mathcal{F},\lambda}.$$

(2) Assume the simple root vectors in $\mathfrak{g}$ are those provided by the geometric Satake equivalence. Then for any $w \in W$ and $\mathcal{F}, \lambda$ as above the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{M}^{\mathrm{geom}}_{\mathbb{S}(\mathcal{F}),\lambda} & \xrightarrow{\sim} & \mathcal{M}^{\mathrm{alg}}_{\mathbb{S}(\mathcal{F}),\lambda} & \xrightarrow{\sim} & \mathcal{M}^{\mathrm{top}}_{\mathcal{F},\lambda} \\
\downarrow{\scriptstyle \Phi_w^{\mathbb{S}(\mathcal{F}),\lambda}} & & \downarrow{\scriptstyle \Theta_w^{\mathbb{S}(\mathcal{F}),\lambda}} & & \downarrow{\scriptstyle \Xi_w^{\mathcal{F},\lambda}} \\
{}^w\mathcal{M}^{\mathrm{geom}}_{\mathbb{S}(\mathcal{F}),w\lambda} & \xrightarrow{\sim} & {}^w\mathcal{M}^{\mathrm{alg}}_{\mathbb{S}(\mathcal{F}),w\lambda} & \xrightarrow{\sim} & {}^w\mathcal{M}^{\mathrm{top}}_{\mathcal{F},w\lambda}
\end{array}$$

In [GR] we also prove a "classical analogue" of this result (corresponding to $\hbar = 0$). In this case there is no "algebraic" family, and the "geometric" family is defined in terms of line bundles on the Grothendieck–Springer resolution $\widetilde{\mathfrak{g}}$. The "symmetries" can be described in terms of the W-action on the regular part of $\widetilde{\mathfrak{g}}$. The "topological" family (and its symmetries) are obtained by forgetting the $\mathbb{C}^\times$-action.

We also observe that the main theorem allows to give new proofs of a result of Ginzburg on a geometric realization of the Brylinski–Kostant filtration (see [Gi]), and of a result of Braverman–Finkelberg on a geometric description of certain dynamical Weyl group operators (see [BF]). This application was our main motivation for introducing the isomorphisms $\Theta_w^{V,\lambda}$.

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Infinite rank classical groups and specialization

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(joint work with Andrew Snowden and Jerzy Weyman)

Throughout we work over the field of complex numbers **C**.

The polynomial representation theory of the general linear group $\mathbf{GL}_n$ has a well-behaved limit for $n \to \infty$. On the level of character rings, this is the passage from the ring of symmetric polynomials in n variables to the ring of symmetric functions in infinitely many variables. One possible model for the $n \to \infty$ limit is the category **Pol** of polynomial functors from the category of vector spaces to itself. The category **Pol** is a semisimple Abelian category, it has a tensor product, and its simple objects are the Schur functors $\mathbf{S}_\lambda$, where λ ranges over all integer partitions. See [5, Chapter 1] for an exposition. **Pol** is characterized by a universal property: for any Abelian symmetric monoidal $\mathbf{C}$-linear category $\mathcal{A}$, a symmetric monoidal tensor functor $\mathbf{Pol} \to \mathcal{A}$ is equivalent to the choice of an object $A \in \mathcal{A}$. This universal property gives a specialization functor $\mathbf{Pol} \to \mathrm{Rep}(\mathbf{GL}_n)$ which simply evaluates a polynomial functor on the vector representation $\mathbf{C}^n$ of $\mathbf{GL}_n$. This specialization functor is exact, and is fundamental for many stabilization properties of the polynomial representation theory of $\mathbf{GL}_n$.

We remark that the category **Pol** has a natural grading via $\deg(\mathbf{S}_\lambda) = |\lambda|$ and many examples of algebra objects (twisted commutative algebras) in the category of graded-finite polynomial functors arise in classical algebraic geometry. This perspective is pursued in [9, 10].

In some cases, considering only polynomial representations is too restrictive (for example, when studying the adjoint representation of $\mathbf{GL}_n$), and also it is natural to ask about analogues of the category **Pol** for the orthogonal and symplectic groups. Such a category was introduced and studied by several authors [1, 6, 7, 8] and can be described as follows. Let $\mathbf{V} = \bigcup_n V_n$ be the union of the defining representations of either the series $\mathbf{GL}_n$, $\mathbf{O}_n$, or $\mathbf{Sp}_{2n}$, and also let $\mathbf{V}_* = \bigcup_n V_n^*$ be the union of the dual of the defining representations. Then $\mathrm{Rep}(\mathbf{O})$ and $\mathrm{Rep}(\mathbf{Sp})$ are defined as the category of representations of either $\mathbf{O}_\infty = \bigcup_n \mathbf{O}_n$ or $\mathbf{Sp}_\infty = \bigcup_n \mathbf{Sp}_{2n}$ which are subquotients of finite direct sums of the tensor spaces $\mathbf{V}^{\otimes N}$. To define $\mathrm{Rep}(\mathbf{GL})$ we use the mixed tensor spaces $\mathbf{V}^{\otimes N} \otimes \mathbf{V}_*^{\otimes M}$.

These categories are no longer semisimple. For example, the map $\mathbf{V} \otimes \mathbf{V}_* \to \mathbf{C}$ in $\mathrm{Rep}(\mathbf{GL})$ does not split. However, they still have a tensor structure. The simple objects V_λ are indexed by partitions in the orthogonal and symplectic case, and by pairs of partitions in the general linear case. The injective envelope of V_λ in the first two cases is $\mathbf{S}_\lambda(\mathbf{V})$ and the injective envelope of $V_{\lambda,\mu}$ is $\mathbf{S}_\lambda(\mathbf{V}) \otimes \mathbf{S}_\mu(\mathbf{V}_*)$ in the third case.

In [11], we showed that $\mathrm{Rep}(G)$ is equivalent to the category of finite length modules over certain twisted commutative algebras. For $G = \mathbf{O}$ and $G = \mathbf{Sp}$, the algebras are $\mathrm{Sym}(\mathrm{Sym}^2)$ and $\mathrm{Sym}(\bigwedge^2)$, which can be thought of as the coordinate rings of the space of infinite size symmetric and skew-symmetric matrices, respectively. For $G = \mathbf{GL}$, there is a similar description in terms of a bivariate twisted commutative algebra which can be thought of as the coordinate ring of the space of infinite size generic matrices. The utility of this perspective is that one can now use techniques from commutative algebra to study these categories.

Also in [11], we characterized these categories in terms of universal properties. Let G be one of $\{\mathbf{O}, \mathbf{Sp}, \mathbf{GL}\}$ and let $\mathcal{A}$ be an Abelian symmetric monoidal $\mathbf{C}$-linear category. Then a left-exact symmetric monoidal tensor functor $\mathrm{Rep}(G) \to \mathcal{A}$ is equivalent to the choice of:

- ($G = \mathbf{O}$): a pair (A, ω) with $A \in \mathcal{A}$ and $\omega \colon \mathrm{Sym}^2 A \to \mathbf{C}$.
- ($G = \mathbf{Sp}$): a pair (A, ω) with $A \in \mathcal{A}$ and $\omega \colon \bigwedge^2 A \to \mathbf{C}$.
- ($G = \mathbf{GL}$): a triple (A, A', ω) with $A, A' \in \mathcal{A}$ and $\omega \colon A \otimes A' \to \mathbf{C}$.

Using these universal properties, we can define specialization functors

$$\Gamma_n\colon \mathrm{Rep}(\mathbf{O}) \to \mathrm{Rep}(\mathbf{O}_n),$$
$$\Gamma_{2n}\colon \mathrm{Rep}(\mathbf{Sp}) \to \mathrm{Rep}(\mathbf{Sp}_{2n}),$$
$$\Gamma_n\colon \mathrm{Rep}(\mathbf{GL}) \to \mathrm{Rep}(\mathbf{GL}_n).$$

They are guaranteed to be left-exact, but will not be right-exact. For example, in $\mathrm{Rep}(\mathbf{Sp})$, we have the injective resolution

$$0 \to V_{1,1,1,1,1,1} \to \bigwedge^6 \mathbf{V} \to \bigwedge^4 \mathbf{V} \to 0,$$

and applying Γ_4, we get a complex concentrated in cohomological degree 1, so $\mathrm{R}^1\Gamma_4(V_{1,1,1,1,1,1}) = \mathbf{C}$.

In fact, in [12], we showed that the derived specialization functors of a simple object are nonzero in at most 1 cohomological degree. The rule to calculate the derived functors can be obtained from a Weyl group action and is analogous to the Borel–Weil–Bott theorem. We explain this now for $\mathrm{Rep}(\mathbf{Sp})$. Let W be the Weyl group of type BC_∞ which acts on integer sequences $(a_1, a_2, \dots)$ with generators $s_i(\dots, a_i, a_{i+1}, \dots) = (\dots, a_{i+1}, a_i, \dots)$ $(i \ge 1)$ and $s_0(a_1, a_2, \dots) = (-a_1, a_2, \dots)$. For $w \in W$, set $\ell(w)$ to be the minimal number of s_i needed to generate w. Given a partition λ, let $\lambda^\dagger$ be its transposed partition. Finally, set $\rho^{2n} = (-(n+1), -(n+2), \dots)$. Then one of two possibilities occurs:

- There exists a non-identity $w \in W$ such that $w(\lambda^\dagger + \rho^{2n}) - \rho^{2n} = \lambda^\dagger$. In this case, $\mathrm{R}^i\Gamma_{2n}(V_\lambda) = 0$ for all $i \ge 0$.
- There exists a unique $w \in W$ such that $w(\lambda^\dagger + \rho^{2n}) - \rho^{2n} = \mu^\dagger$ where μ is a partition with at most n parts. In this case, $\mathrm{R}^{\ell(w)}\Gamma_{2n}(V_\lambda) = V_\mu$ (where μ is interpreted as a dominant weight for $\mathbf{Sp}_{2n}$) and $\mathrm{R}^i\Gamma_{2n}(V_\lambda) = 0$ for $i \neq \ell(w)$.

There are similar rules for the orthogonal case and general linear case, in which case one uses the Weyl groups of type D_∞ and A_∞, respectively. We refer the reader to [12] for more details.

We close by mentioning that the character rings of the categories $\mathrm{Rep}(G)$ and the ring homomorphisms induced from the specialization functors were studied in [2, 3, 4].

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Modular Koszul Duality

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(joint work with Simon Riche and Geordie Williamson)

Given a $\mathbb{Z}$-graded ring E let us consider the abelian category $\mathsf{Mod}^{\mathbb{Z}}{-}E$ of all $\mathbb{Z}$-graded right E-modules. There are two ways to "forget a part of the grading" on its derived category. That is, there are two triangulated functors:

$$\mathcal{D}(\mathsf{Mod}{-}E) \xleftarrow{v} \mathcal{D}(\mathsf{Mod}^{\mathbb{Z}}{-}E) \xrightarrow{\overline{v}} \mathsf{dgDer}{-}E.$$

On the left is simply the derived functor v of forgetting the grading $\mathsf{Mod}^{\mathbb{Z}}{-}E \to \mathsf{Mod}{-}E$. On the right is the derived category of the differential graded ring $(E, d = 0)$, which is obtained as the localisation of homotopy category of differential graded right modules at quasi-isomorphisms (see for example [BL94] for a thorough discussion). The right-hand functor $\overline{v}$ sends a complex of graded modules, thought of as a bigraded abelian group $(M^{i,j})$ (with $(\cdot a) : M^{i,j} \to M^{i,j+|a|}$ for all $a \in E$ homogeneous of degree $|a|$ and differential $d : M^{i,j} \to M^{i+1,j}$) to the differential graded E-module $\overline{v}M$ with $(\overline{v}M)^n := \bigoplus_{i+j=n} M^{i,j}$ and obvious differential. It is this construction which provides the basic homological scaffolding of Koszul duality.

In order to obtain Koszul duality in the sense of [BGS96] we equip the above picture with some finiteness conditions. Let $\Bbbk$ be a field and let E be a finite dimensional $\mathbb{Z}$-graded $\Bbbk$-algebra of finite global dimension. Consider the categories $\mathsf{Modf}{-}E$ (resp. $\mathsf{Modf}^{\mathbb{Z}}{-}E$) of finite dimensional (resp. finite dimensional $\mathbb{Z}$-graded) right E-modules. Let $\mathsf{dgDerf}{-}E \subset \mathsf{dgDer}{-}E$ denote the full triangulated subcategory with objects finite dimensional differential graded right E-modules. In this setting Koszul duality for category $\mathcal{O}$ means the existence of a finite dimensional

$\mathbb{Z}$-graded $\mathbb{C}$-algebra E together with vertical equivalences of categories:

$$\begin{array}{ccccc} \mathcal{D}^{\mathrm{b}}(\mathsf{Modf}{-}E) & \xleftarrow{v} & \mathcal{D}^{\mathrm{b}}(\mathsf{Modf}^{\mathbb{Z}} - E) & \xrightarrow{\overline{v}} & \mathsf{dgDerf}{-}E \\ \wr\uparrow & & & & \wr\uparrow \\ \mathcal{D}^{\mathrm{b}}(\mathcal{O}_0) & & & & \mathcal{D}^{\mathrm{b}}_{(B)}(G/B,\mathbb{C}) \end{array} \tag{1}$$

Here G denotes a complex semi-simple group with Borel subgroup $B \subset G$ and $\mathcal{D}^{\mathrm{b}}_{(B)}(G/B,\mathbb{C})$ denotes the full subcategory of the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on the flag variety G/B whose cohomology sheaves are constructible with respect to the stratification by Bruhat cells. Such complexes will be called "Bruhat constructible" from now on. On the left $\mathcal{O}_0$ denotes the principal block of category $\mathcal{O}$ for the Langlands dual group $G^\vee$.

We can choose our $\mathbb{Z}$-graded $\mathbb{C}$-algebra E and our vertical equivalences of categories so that the left-hand equivalence preserves the t-structures (and hence is induced from an equivalence of abelian $\mathbb{C}$-categories $\mathsf{Modf}{-}E \xrightarrow{\sim} \mathcal{O}_0$) and such that for all x in the Weyl group W there exists a finite dimensional $\mathbb{Z}$-graded E-module $\widetilde{M}_x$, which specializes on the left-hand side to the Verma module with highest weight $x \cdot 0$, and on the right-hand side to the derived direct image of the constant perverse sheaf on the Bruhat cell BxB/B under the embedding in G/B. We can depict the situation as follows:

$$M(x\cdot 0) \quad \leftarrow\!\shortmid \quad \widetilde{M}_x \quad \mapsto \quad i_{x*}\underline{\mathbb{C}}_{BxB/B}[\ell(x)].$$

Here $x \cdot 0$ means as usual $x\rho - \rho$ where ρ is half the sum of the positive roots, that is those roots whose weight spaces act locally nilpotently on all objects in $\mathcal{O}$.

The goal of our work is to establish analogous statements in the modular setting. To do this, let us choose a field $\mathbb{F}$ of characteristic $\ell > 0$ and consider the full subcategory

$$\mathcal{D}^{\mathrm{b}}_{(B)}(G/B,\mathbb{F}) \subset \mathcal{D}^{\mathrm{b}}(G/B,\mathbb{F})$$

of Bruhat constructible complexes of sheaves on the complex flag variety as above, the only difference being that now we consider sheaves with coefficients in $\mathbb{F}$. On the other side we consider $\mathcal{O}_0(\mathbb{F})$, the "subquotient around the Steinberg weight" as defined in [Soe00]. This is a subquotient of the category of finite dimensional rational representations of the group $G^\vee_{\mathbb{F}}$ over $\mathbb{F}$. In order for this to make sense we need to restrict to the case where the characteristic is bigger than the Coxeter number. This ensures that we can find a dominant weight in the interior of an alcove obtained by stretching an alcove of the affine Weyl group by ℓ.

Under more restrictive assumptions on the characterstic of our coefficients we prove modular analogues of the previous statements:

Theorem 1 ("Modular Koszul duality"). *Suppose that $\ell > |R| + 1$, where R is the root system of G. Then there exists a graded finite dimensional $\mathbb{F}$-algebra E of* *finite global dimension together with vertical equivalences of categories:*

$$\begin{array}{ccccc} \mathcal{D}^{\mathrm{b}}(\mathsf{Modf}-E) & \xleftarrow{v} & \mathcal{D}^{\mathrm{b}}(\mathsf{Modf}^{\mathbb{Z}}-E) & \xrightarrow{\overline{v}} & \mathsf{dgDerf}-E \\ \wr\uparrow & & & & \wr\uparrow \\ \mathcal{D}^{\mathrm{b}}(\mathcal{O}_0(\mathbb{F})) & & & & \mathcal{D}^{\mathrm{b}}_{(B)}(G/B,\mathbb{F}) \end{array} \tag{2}$$

Moreover, for all $x \in W$ there exists a finite dimension graded E-module $\widetilde{M}_x$ which specialises to the standard object M_x in $\mathcal{O}_0(\mathbb{F})$ on the left and to $i_{x*}\underline{\mathbb{F}}_{BxB/B}[\ell(x)]$ on the right. In formulas:

$$M_x \quad \leftarrow\!\shortmid \quad \widetilde{M}_x \quad \mapsto \quad i_{x*}\underline{\mathbb{F}}_{BxB/B}[\ell(x)].$$

Let us emphasize, however, the difference to the non-modular case: in the modular case it is not clear whether the $\mathbb{Z}$-graded algebra E can be chosen such that it vanishes in negative degrees and is semi-simple in degree 0. Hence we are in no position to discuss the Koszulity of E in general.

Part of the proof of this modular Koszul duality can already be found in [Soe00]. More precisely, one of the main results of [Soe00] is an equivalence of categories

$$\mathsf{Modf}-E \xrightarrow{\sim} \mathcal{O}_0(\mathbb{F})$$

where $E = \mathrm{Ext}^\bullet(\mathcal{E},\mathcal{E})$ is the algebra of self-extensions of the direct sum of all parity sheaves $\mathcal{E} := \bigoplus_{y\in W}\mathcal{E}_y$ in the terminology of [JMW09]. Hence all that remains in order to construct a diagram as in the theorem is the construction of an equivalence of triangulated $\mathbb{F}$-categories

$$\mathsf{dgDerf}-E \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}_{(B)}(G/B,\mathbb{F}).$$

Now by general homological algebra if we choose a bounded below injective resolution $\mathcal{E}_y^\bullet$ of each parity sheaf $\mathcal{E}_y$ and set $\mathcal{E}^\bullet := \bigoplus_{y\in W}\mathcal{E}_y^\bullet$ and consider the endomorphism dg-ring $E^\bullet := \mathrm{End}^\bullet(\mathcal{E}^\bullet)$ we will get an equivalence of triangulated categories

$$\mathsf{dgDerf}-E^\bullet \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}_{(B)}(G/B,\mathbb{F}).$$

with the notation on the left meaning objects with finite dimensional homology. By definition the cohomology of the dg-ring $E^\bullet$ is the graded ring $E = \mathrm{Ext}^\bullet(\mathcal{E},\mathcal{E})$ from above. Suppose that we can find another dg-ring D and quasi-isomorphisms $E \xleftarrow{\sim} D \xrightarrow{\sim} E^\bullet$ then we get equivalences

$$\mathsf{dgDerf}-E \xrightarrow{\sim} \mathsf{dgDerf}-D \xrightarrow{\sim} \mathsf{dgDerf}-E^\bullet$$

and the theorem is established.

In order to find our dg-ring D and the desired quasi-isomorphisms we adapt the techniques of [DGMS75] to the context of modular étale sheaves. Here the Frobenius action plays the part which Hodge theory plays in [DGMS75]. In particular, we do not actually work on a complex flag variety, but rather on the flag variety over a finite field $\mathbb{F}_p$ of characteristic different from the characteristic ℓ of $\mathbb{F}$. Also, in order to construct our dg-ring $E^\bullet$ with compatible Frobenius action we instead work with perverse sheaves (and projective resolutions). More details can be found in [RSW12], from which this survey is largely taken.

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Invariant Hilbert schemes and resolutions of categorical quotients

Ronan Terpereau

Let W be an affine variety equipped with an action of a reductive algebraic group G. The *invariant Hilbert scheme* is a quasi-projective scheme that classifies the G-stable closed subschemes Z of W such that the affine algebra of Z is the direct sum of simple G-modules with prescribed finite multiplicities (see [1, 2]). Denoting $h:\ \mathrm{Irr}(G) \to \mathbb{N}$ a function from the set of the irreducible representations of G to the set of the nonnegative integers, we have the following set-theoretical description of the invariant Hilbert scheme:

$$\mathrm{Hilb}_h^G(W) = \left\{ \begin{array}{l} G\text{-stable ideal } I \subset k[W] \text{ such that} \\ k[W]/I \cong \bigoplus_{M \in \mathrm{Irr}(G)} M^{\oplus h(M)} \text{ as a } G\text{-module.} \end{array} \right\}$$

A strong motivation for the study of the invariant Hilbert scheme is given by the classification of varieties with reductive group action (see for instance [4, 5, 6]). Another motivation (studied in my thesis for the classical groups) is to construct canonical resolutions of the quotient variety $W//G = \mathrm{Spec}(k[W]^G)$. When G is finite, this construction is closely related to the much studied MacKay correspondance (see [3]).

In [7, 8], we considered the following cases:

- $G = GL(V)$ acting naturally on $W = V^{\oplus n_1} \oplus V^{*\oplus n_2}$;
- $G = SL(V)$, $O(V)$, $SO(V)$, $Sp(V)$ acting naturally on $W = V^{\oplus n'}$.

Here V stands for the defining representation, and V^* for the dual representation. We denote h_0 the Hilbert function of the general fibers of the quotient map $q:\ W \to W//G$, and $\mathcal{H}_W := \mathrm{Hilb}_{h_0}^G(W)$ the associated invariant Hilbert scheme. There exists a projective morphism

$$\gamma:\ \mathcal{H}_W \to W//G,$$

called the *Hilbert-Chow morphism*, which induces an isomorphism over the largest open subset $U \subset W/\!/G$ over which q is flat. The *main component* of $\mathcal{H}_W$ is the variety defined by

$$\mathcal{H}_W^{\mathrm{prin}} := \overline{\gamma^{-1}(U)}.$$

Then the restriction $\gamma : \mathcal{H}_W^{\mathrm{prin}} \to W/\!/G$ is a proper birational morphism; it is thus a canonical candidate for a resolution of $W/\!/G$. We showed in [7, 8] that γ is indeed a resolution if:

- $G = SL(V)$, and arbitrary $\dim(V)$;
- $G = GL(V)$ or $O(V)$, and $\dim(V) \leq 2$;
- $G = Sp(V)$, and $\dim(V) \leq 4$.

Actually, $\mathcal{H}_W = \mathcal{H}_W^{\mathrm{prin}}$ in these cases. We also showed in [7] that $\mathcal{H}_W$ is singular if $G = GL(V),\ O(V)$ or $SO(V)$ with $\dim(V) = 3$, but the smoothness of $\mathcal{H}_W^{\mathrm{prin}}$ is still an open problem in these cases.

To determine the previous examples, we first obtained a *reduction principle* that allows us to assume that the number of copies of standard representations considered is equal to the dimension of the representations. For instance, to treat the case of $GL(V)$ acting on $V^{\oplus n_1} \oplus V^{*\oplus n_2}$, we just have to consider $V^{\oplus \dim(V)} \oplus V^{*\oplus \dim(V)}$. Another key ingredient is provided by a group action on the invariant Hilbert scheme. More precisely, let

$$G' \subset \mathrm{Aut}^G(W)$$

be a reductive algebraic subgroup. It is known that G' acts on $W/\!/G$ and on $\mathcal{H}_W$, and that the Hilbert-Chow morphism γ is G'-equivariant. Besides, we show that each closed G'-orbit of $\mathcal{H}_W$ contains a finite number of fixed point for the action of a Borel subgroup of G'. To determine these fixed points, we significantly use representation theory of G'. The fixed points give us a lot of information concerning the geometry of $\mathcal{H}_W$. For instance, if there is only one fixed point, then $\mathcal{H}_W$ has to be connected. The next step is to compute (with the help of the software *Macaulay2*) the dimension of the tangent space at each fixed point. If all these dimensions coincide with the dimension of $\mathcal{H}_W^{\mathrm{prin}}$, then $\mathcal{H}_W = \mathcal{H}_W^{\mathrm{prin}}$ is a smooth variety that we can describe explicitly. Else, we get that $\mathcal{H}_W$ is singular.

In [7, 9], we considered the invariant Hilbert scheme related to the symplectic reduction of W by G, when W is a symplectic vector space. More precisely, we take $G \subset GL(V)$ a classical group acting on $W := V^{\oplus d} \oplus V^{*\oplus d}$. Then W is a symplectic representation, and thus there exists a G-equivariant *moment map* $\mu : W \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. The quotient $\mu^{-1}(0)/\!/G$ is called the *symplectic reduction* of W by G. In the examples we consider, $\mu^{-1}(0)/\!/G$ is always a symplectic variety, and we showed in [9] that the restriction $\gamma : \mathcal{H}_{\mu^{-1}(0)}^{\mathrm{prin}} \to \mu^{-1}(0)/\!/G$ is a symplectic resolution if and only if:

- $G = GL(V)$, and $d \leq 1 + \dim(V)$ with d even;
- $G = O(V)$, and $d \leq \frac{1}{2}(1 + \dim(V))$;
- $G = Sp(V)$, and $d \leq 1 + \frac{1}{2}\dim(V)$ with d even.

Moreover, γ is still a resolution of $\mu^{-1}(0)//G$ if $\dim(V) \leq 2$. The case $\dim(V) \geq 3$ remains open. Let us also mention that $\mathcal{H}_{\mu^{-1}(0)} \neq \mathcal{H}^{\text{prin}}_{\mu^{-1}(0)}$ in general.

We now go back to the "classical" setting. In all the examples where $\mathcal{H}_W$ is singular, it would be interesting to determine if it is reduced, irreducible,... and if the variety $\mathcal{H}^{\text{prin}}_W$ is smooth. It would also be interesting to determine if $\mathcal{H}_W$ stays singular for $\dim(V) >> 0$. To do that, the methods used in [7] are unsuitable (too computational) and new approaches are required. We can of course ask the same kind of questions in the "symplectic" setting, i.e. after replacing $\mathcal{H}_W$ by $\mathcal{H}_{\mu^{-1}(0)}$.

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The Hecke algebra and Soergel bimodules

Geordie Williamson

(joint work with Ben Elias)

Let (W, S) be a Coxeter system and $\mathcal{H} = \bigoplus_{x\in W} \mathbb{Z}[q^{\pm 1/2}]T_x$ its Hecke algebra. In 1970 Kazhdan and Lusztig [7] defined a remarkable basis $C'_x = q^{-\ell(x)/2}(\sum_{y\leq x} P_{y,x}T_y)$ for $\mathcal{H}$ and conjectured:

(1) $P_{y,x} \in \mathbb{N}[q]$;

(2) if $C'_x C'_y = \sum \mu^z_{xy} C'_z$ then $\mu^z_{xy} \in \mathbb{N}[q^{\pm 1/2}]$.

This conjecture became known as the *Kazhdan-Lusztig positivity conjecture*. It was solved by Kazhdan and Lusztig [8] for finite and affine Weyl groups by interpreting $P_{y,x}$ as Poincaré polynomials of local intersection cohomology of Schubert varieties. Their proof was generalized using Kac-Moody flag varieties to crystallographic Coxeter groups (groups in which all orders of products of two simple reflections belong to $\{2, 3, 4, 6, \infty\}$) by a number of authors. Kazhdan-Lusztig polynomials have gone on to occupy a central place in Lie theory and representation theory, starting with the famous Kazhdan-Lusztig conjectures on the characters of simple highest weight modules over complex semi-simple Lie algebras. The Kazhdan-Lusztig conjecture was solved in 1981 by Beilinson and Bernstein [1] and Brylinski and Kashiwara [3].

In 1990 Soergel [10] defined certain modules over the coinvariant algebra and used them to give another proof of the Kazhdan-Lusztig conjecture. Like the previous proof of Beilinson-Bernstein and Brylinski-Kashiwara, Soergel's proof relies on results from geometry. However Soergel reduced the reliance to one result, the famous "decomposition theorem" of Beilinson, Bernstein, Deligne and Gabber [2]. Moreover the decomposition theorem has a simple algebraic translation: one hopes that a certain module over the coinvariant algebra decomposes as a direct sum in a way dictated by the Hecke algebra. Though easily stated, this algebraic statement has resisted an algebraic proof for the last twenty years.

In a sequel [11] Soergel introduced equivariant analogues of his modules which have come to be known as Soergel bimodules. More recently [12] Soergel has shown that his bimodules always provided a categorification of the Hecke algebra, and conjectured that a basis given by indecomposable Soergel bimodules categorifies the Kazhdan-Lusztig basis. He explained why his conjecture implies the Kazhdan-Lusztig positivity conjectures.

Since its proof by Beilinson, Bernstein, Deligne and Gabber the decomposition theorem has had two other proofs. The first, by Saito, uses his technology of mixed Hodge modules. He replaces the reliance of the earlier proof on Frobenius weights with weights in the sense of mixed Hodge theory. More recently, de Cataldo and Migliorini [4, 5] have given a proof of the decomposition theorem using classical Hodge theory. Using several beautiful arguments they translate the geometric statement of the decomposition theorem to questions about forms and filtrations on the cohomology of smooth projective varieties, and use Hodge theory (more precisely the weak Lefschetz theorem and the Hodge-Riemann bilinear relations) to prove that these statements hold. In the process they uncover several remarkable signature properties of forms involved in the decomposition theorem, leading to what they dub the "decomposition theorem with signs".

In [6] Ben Elias and the author establish these Hodge theoretic properties for Soergel bimodules in a purely algebraic/combinatorial manner. By adapting the arguments of de Cataldo and Migliorini we are able to deduce Soergel's conjecture. Hence the Kazhdan-Lusztig positivity conjecture holds. The evidence is mounting that Soergel bimodules provide the right setting in which to study Kazhdan-Lusztig theory for arbitrary Coxeter groups.

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Formal group laws, Hecke algebras and algebraic oriented cohomology theories

Kirill Zainoulline
(joint work with Baptiste Calmès, Alex Hoffnung, José Malagón-Lopez, Victor Petrov, Alistair Savage, and Changlong Zhong)

Let G be a split semisimple linear algebraic group over a field and let T be its split maximal torus. We construct a generalized characteristic map relating the, so called, formal group algebra of the character group of T with the algebraic oriented cohomology of the variety of Borel subgroups of G and we show that the kernel of this map is generated by W-invariant elements, where W is the Weyl group of G, hence generalizing the respective results by Demazure [Invariants symétriques entiers des groupes de Weyl et torsion] for Chow groups and Grothendieck's K_0. Using the language of formal group algebras we generalize the construction of the nil Hecke ring of Kostant-Kumar [Nil Hecke ring and Cohomology of G/P for a Kac-Moody group G^*] to the context of an arbitrary algebraic oriented cohomology theory. The resulting object, which we call a formal (affine) Demazure algebra, is parameterized by a one-dimensional commutative formal group law and has the following important property: specialization to the additive and multiplicative periodic formal group laws yields completions of the nil Hecke and the 0-Hecke rings respectively. We also generalize the coproduct structure on nil Hecke rings introduced and studied by Kostant-Kumar [T-equivariant K-theory of generalized flag variety]. As an application we construct an algebraic model of the T-equivariant oriented cohomology of the variety of complete flags.

The talk is based on several joint projects with B. Calmès, A. Hoffnung, J. Malagón-Lopez, V. Petrov, A. Savage and C. Zhong:

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