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Heat Kernels, Stochastic Processes and Functional Inequalities

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ABSTRACT. The general topic of the 2013 workshop *Heat kernels, stochastic processes and functional inequalities* was the study of linear and non-linear diffusions in geometric environments: finite and infinite-dimensional manifolds, metric spaces, fractals and graphs, including random environments. The workshop brought together leading researchers from analysis, probability and geometry and provided a unique opportunity for interaction of established and young scientists from these areas.

Unifying themes were heat kernel analysis, mass transport problems and related functional inequalities such as Poincaré, Sobolev, logarithmic Sobolev, Bakry-Emery, Otto-Villani and Talagrand inequalities. These concepts were at the heart of Perelman's proof of Poincaré's conjecture, as well as of the development of the Otto calculus, and the synthetic Ricci bounds of Lott-Sturm-Villani. The workshop provided participants with an opportunity to discuss how these techniques can be used to approach problems in optimal transport for non-local operators, subelliptic operators in finite and infinite dimensions, analysis on singular spaces, as well as random walks in random media.

Mathematics Subject Classification (2010): 58J65, 58J35, 60J45, 60K37, 60F17, 53C23.

Introduction by the Organisers

The workshop *Heat kernels, stochastic processes and functional inequalities*, organised by Masha Gordina (University of Connecticut), Takashi Kumagai (RIMS, Kyoto University), Laurent Saloff-Coste (Cornell University), and Karl-Theodor Sturm (University of Bonn) was well attended with over 50 participants from

Australia, Austria, Canada, France, Germany, Israel, Italy, Japan, Luxembourg, Poland, Portugal, United Kingdom, and USA. The program consisted of 26 talks and 5 short contributions, leaving sufficient time for discussions. The general topic of the workshop was the study of linear and non-linear diffusions in geometric environments: finite and infinite-dimensional manifolds, metric spaces, fractals and graphs, including random environments. The workshop was successful in bringing together leading experts in three different major fields of mathematics: analysis, stochastics and geometry. It also provided a unique opportunity for interaction of established and young scientists from these areas. One after-dinner session was devoted to short communications by junior participants of the workshop.

One of the topical focuses of the workshop was related to curvature-dimension bounds, optimal transport, and heat flow in metric measure spaces (Matthias Erbar, Nicola Gigli, Arnaud Guillin, Andrea Mondino) as well as in discrete spaces (Jan Maas). Recent developments in understanding degenerate and singular spaces were presented in talks on generalized curvature-dimension conditions in sub-Riemannian geometry both in finite and infinite dimensions. Classical curvature conditions are problematic in these setting, but Poincaré and other functional inequalities are very useful. Several techniques were discussed including the generalized curvature-dimension condition, optimal transport, stability under concentration limits (Fabrice Baudoin, Tai Melcher, Takashi Shioya).

Methods of Dirichlet forms and heat kernel estimates play key roles in a number of topics discussed during the workshop. In particular, they provide a useful tool in proving various functional inequalities in absence of a well-defined geometry. Analysis on fractals, non-local operators, RWRE *etc* are some examples of applications. The workshop had talks in a numbers of these topics: non-local operators (Zhen-Qing Chen, Moritz Kassmann), analysis on fractals (Jun Kigami, Naotaka Kajino, Michael Hinz, Ben Hambly), functional inequalities on various metric measure spaces (Martin Barlow, Richard Bass, Thierry Coulhon, Wolfgang Woess). Currently, major research activity is devoted to invariance principles for random conductance models. This also was a topical focus of the workshop (Chris Burdzy, David Croydon, Sebastian Andres). One talk was on the scaling limit of extreme processes for the two dimensional discrete Gaussian Free Field (Marek Biskup). It is linked to potential theory of the Gaussian Free Field via estimations of Green functions.

Two talks were centered on stochastic differential geometry and heat flow in the case when the underlying manifold changes along a geometric flow such as Ricci flow (Ionel Popescu, Anton Thalmaier). Another example of a stochastic version of a deterministic construction was a talk about a stochastic Euler-Poincaré variational principle for a classical Lagrangian on a general Lie group (Ana Bela Cruzeiro). Its classical analogue is Arnold's picture of the Euler flow as a geodesic on the Lie group of diffeomorphisms.

This diversity of topics and mix of participants stimulated many extensive and fruitful discussions. It also helped initiate new collaborations, in particular for the

young researchers, and strengthen existing ties between researchers in different fields of mathematics.

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Abstracts

Metric measure spaces and Dirichlet forms for lower bounds on the Ricci curvature

NICOLA GIGLI

Motivated by Gromov's precompactness theorem and by the works of Fukaya [9] and Cheeger-Colding [5], [6], [7], [8], in the seminal papers [16] and [19],[20] Lott-Villani on one side and Sturm on the other independently proposed a definition of 'having Ricci curvature bounded from below by K and dimension bounded above by N ' for metric measure spaces, these being called $\text{CD}(K, N)$ spaces (in [16] only the cases $K = 0$ or $N = \infty$ were considered). Here K is a real number and N a real number ≥ 1 , the value $N = \infty$ being also allowed. Their approach is based on the study of the convexity properties of certain entropy functionals w.r.t. the quadratic transportation distance W_2 .

The crucial properties of their definition are the compatibility with the smooth Riemannian case and the stability w.r.t. measured-Gromov-Hausdorff convergence. In particular, the class of $\text{CD}(K, N)$ spaces contains the closure of the set of Riemannian manifolds with Ricci bounded from below by K and dimension bounded from above by N , where closure is intended w.r.t. the measured-Gromov-Hausdorff topology. Furthermore, several natural and expected geometric/analytic properties (like the Bishop-Gromov inequality and Bonnet-Myers theorem) can be established on $\text{CD}(K, N)$ spaces.

Yet, as proved by Cordero-Erasquin, Sturm and Villani (see the last theorem in [21]), the class of $\text{CD}(K, N)$ spaces contains also Finsler structures. For instance, the space $(\mathbb{R}^d, d_{\|\cdot\|}, \mathcal{L}^d)$, where \mathcal{L}^d is the Lebesgue measure and $d_{\|\cdot\|}$ is the distance induced by the norm $\|\cdot\|$, is always a $\text{CD}(0, d)$ space, regardless of the choice of the norm. This is a potential issue, because we know from the work of Cheeger-Colding that Finsler and non-Riemannian manifolds cannot occur as mGH-limits of smooth manifolds with Ricci curvature uniformly bounded from below. Furthermore, the geometry of Finsler manifolds can be radically different from that of Riemannian ones.

As such, the class of $\text{CD}(K, N)$ seems 'too large' to produce the correct intrinsic point of view on lower Ricci bounds and it is natural to ask whether there exists another - more restrictive - synthetic notion of lower Ricci bound which retains the stability properties and rules out Finsler-like geometries.

A proposal in this direction has been made in [2] by myself, Ambrosio and Savaré for the case $N = \infty$, where the class $\text{RCD}(K, \infty)$ has been introduced. The basic idea is to enforce the $\text{CD}(K, \infty)$ condition with the requirement that the heat flow is linear (see also [1]). The root of such idea is in the celebrated paper [15], where Jordan-Kinderlehrer-Otto showed that the heat flow can be seen as gradient flow of the relative entropy w.r.t. the W_2 distance on probability measures. On $\text{CD}(K, \infty)$ spaces, the information that we have, which is in fact the only information available, is that the relative entropy is K -convex w.r.t. the

distance W_2 and is therefore quite natural to study its gradient flow w.r.t. W_2 . This has been done in [10], where it has been shown that such gradient flow is unique. Notice that according to the analysis done by Ohta-Sturm in [18], despite the fact that the normed space $(\mathbb{R}^d, d_{\|\cdot\|}, \mathcal{L}^d)$ is $\text{CD}(0, \infty)$, the distance W_2 never decreases along two heat flows unless the norm comes from a scalar product, in this sense the stated uniqueness result is non-trivial and obtained with a very ad-hoc argument. In [10] it has been also proved that such gradient flow is stable w.r.t. mGH-convergence of compact spaces (see [2] and [13] for generalizations). On the Euclidean space, there is at least one other way of seeing the heat flow as gradient flow: the classical viewpoint of gradient flow in L^2 of the Dirichlet energy. The fact that these two gradient flows produce the same evolution has been generalized in various directions. Among others, one important contribution to the topic has been made by Ohta-Sturm in [17], where they proved that the two approaches produce the same evolution on Finsler manifolds, leading in non-Riemannian manifolds to a non-linear evolution. It is therefore reasonable to ask whether the same sort of identification holds on general $\text{CD}(K, \infty)$ spaces. In such setting, the role of the Dirichlet form is taken by the functional $f \mapsto E(f) := \frac{1}{2} \int |\nabla f|^2 \, \text{d}\mathbf{m}$, where the object $|\nabla f|$ is the 2-minimal weak upper gradient behind the definition of Sobolev functions. Notice that E is in general not a quadratic form, in line with the case of Finsler geometries. Following the strategy proposed in [12] for the case of Alexandrov spaces, in [4] it has been proved that indeed on $\text{CD}(K, \infty)$ spaces the two gradient flows produce the same evolution, which we can therefore undoubtedly call heat flow.

With this understanding of the heat flow, the definition of $\text{RCD}(K, \infty)$ spaces as $\text{CD}(K, \infty)$ spaces where such flow is linear comes out quite naturally: not only it is a stable condition which in the smooth case singles out Riemannian manifolds from Finsler ones, but in the non-smooth world also provides a natural bridge between optimal transport theory and Sobolev calculus. Indeed, to require that the heat flow is linear is equivalent to require that the energy functional E is a quadratic form or, which is the same, that the Sobolev space $W^{1,2}$ built on our metric measure space is Hilbert. Also, the fact that on $\text{RCD}(K, \infty)$ spaces the energy E is a Dirichlet energy allows to make connections with the Bakry-Émery Γ_2 calculus, which furnishes a way to speak about lower Ricci curvature bounds for diffusion operators in the abstract context of Dirichlet forms. It turns out that the two approaches to lower Ricci curvature bounds, via optimal transport and via Γ_2 calculus, are in fact equivalent in high generality ([12], [2], [3]).

Then the appropriate finite dimensional notion of $\text{RCD}(K, N)$ space can be introduced as:

$$\text{RCD}(K, N) := \text{CD}(K, N) \cap \text{RCD}(K, \infty),$$

and in this setting geometric rigidity results like the Abresh-Gromoll inequality and the Cheeger-Colding-Gromoll splitting theorem has been recently established ([11], [14]).

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On the equivalence of Bochner's inequality and the entropic curvature dimension condition on metric measure spaces

MATTHIAS ERBAR

(joint work with Kazumasa Kuwada, Karl-Theodor Sturm)

Bochner's inequality is one of the most fundamental estimates in geometric analysis on Riemannian manifolds. It states that

$$(1) \quad \frac{1}{2} \Delta |\nabla u|^2 - \langle \nabla u, \nabla \Delta u \rangle \geq K \cdot |\nabla u|^2 + \frac{1}{N} \cdot |\Delta u|^2$$

for each smooth function u on a Riemannian manifold (M, g) provided $K \in \mathbb{R}$ is a lower bound for the Ricci curvature on M and $N \in (0, \infty]$ is an upper bound for the dimension of M . In the talk we presented recent results establishing an analogous Bochner inequality on metric measure spaces (X, d, m) with linear heat flow and satisfying the (reduced) curvature-dimension condition. Indeed, we will also prove the converse: if the heat flow on a mms (X, d, m) is linear then an appropriate version of (1) (for the canonical gradient and Laplacian on X) will imply the reduced curvature-dimension condition. Besides that, we also derive new, sharp W_2 -contraction results for the heat flow as well as pointwise gradient estimates and prove that each of them is equivalent to the curvature-dimension condition.

The **curvature-dimension condition** $\text{CD}(K, N)$, introduced by the third named author [3] and in a slightly modified form by Lott & Villani [4], is a sophisticated tightening up of the much simpler $\text{CD}(K, \infty)$ -condition introduced as a synthetic Ricci bound for metric measure spaces. While the latter simply states that the Boltzmann entropy is K -convex along L^2 -Wasserstein geodesics, the latter involves the N -dependent Renyi entropy functional and has no simple interpretation in terms of convexity.

A completely different approach to generalized curvature-dimension bounds is the **energetic curvature-dimension condition** $\text{BE}(K, N)$, set forth in the pioneering work of Bakry and Émery [5]. It applies to the general setting of Dirichlet forms and the associated Markov semigroups and takes the form of (1) where the Laplacian is replaced by the generator of the semigroup and the squared modulus of the gradient by the *carré du champ* operator.

The relation between the two notions of curvature bounds based on optimal transport and Dirichlet forms has been studied in large generality by Ambrosio, Gigli, Savaré and co-workers in a series of recent works, see e.g.[1].

The key tool of their analysis is a powerful calculus on metric measure spaces which allows them to match the two settings. Starting from a metric measure structure they introduce the so called Cheeger energy which takes over the role of the 'standard' Dirichlet energy and is obtained by relaxing the L^2 -norm of the slope of Lipschitz functions. A key result is the identification of the L^2 -gradient flow of the Cheeger energy with the Wasserstein gradient flow of the entropy.

This is the mms equivalent of the famous result by Jordan–Kinderlehrer–Otto and allows one to define unambiguously a heat flow in metric measure spaces.

A mms is called *infinitesimally Hilbertian* if the heat flow is linear. This is equivalently to the Cheeger energy being the associated Dirichlet form. We denote its domain by $W^{1,2}(X, d, m)$. For infinitesimally Hilbertian mms, Ambrosio–Gigli–Savaré prove that $\text{BE}(K, \infty)$ is equivalent to $\text{CD}(K, \infty)$, see [2].

We extend this equivalence taking into account also the dimension bound. Our approach strongly relies on properties and consequences of a new curvature-dimension condition, the so-called **entropic curvature dimension condition** $\text{CD}^e(K, N)$. It simply states that the Boltzmann entropy Ent is (K, N) -convex on the Wasserstein space $\mathcal{P}_2(X, d)$. Here a function u on an interval $I \subset \mathbb{R}$ is called (K, N) -convex if

$$u'' \geq K + \frac{1}{N} \cdot (u')^2.$$

A function U on a geodesic space is called (K, N) -convex if it is (K, N) -convex along each unit speed geodesic. Our first result is the following

Theorem 1. *For a non-branching mms the entropic curvature-dimension condition $\text{CD}^e(K, N)$ is equivalent to the reduced curvature-dimension condition $\text{CD}^*(K, N)$.*

We say that a metric measure space satisfies the **Riemannian curvature-dimension condition** $\text{RCD}(K, N)$ if it is infinitesimally Hilbertian and satisfies $\text{CD}^e(K, N)$. This notion turns out to have the natural stability properties. Namely, the $\text{RCD}(K, N)$ condition is preserved under Gromov–Hausdorff convergence and tensorization of metric measure spaces and holds globally if and only if it holds locally.

The geometric intuition coming from the analysis of (K, N) -convex functions and their gradient flows leads to a new form of the **Evolution Variation Inequality** $\text{EVI}_{K, N}$ on the Wasserstein space. Until now, the notion of $\text{EVI}_{K, N}$ gradient flow was known only without dimension term (i.e. with $N = \infty$). In particular, it turned out that $\text{RCD}(K, \infty)$ spaces can be characterized by the fact that the heat flow is an $\text{EVI}_{K, \infty}$ gradient flow of the entropy. Here we obtain a reinforcement of this result.

Theorem 2. *A mms (X, d, m) satisfies $\text{RCD}(K, N)$ if and only if every $\mu_0 \in \mathcal{P}_2(X, d)$ is the starting point of a curve $(\mu_t)_{t \geq 0}$ in $\mathcal{P}_2(X, d)$ such that for any other $\nu \in \mathcal{P}_2(X, d)$ and a.e. $t > 0$:*

$$(2) \quad \frac{d}{dt} s_{K/N} \left(\frac{1}{2} W_2(\mu_t, \nu) \right)^2 + K \cdot s_{K/N} \left(\frac{1}{2} W_2(\mu_t, \nu) \right)^2 \leq \frac{N}{2} \left(1 - \frac{U_N(\nu)}{U_N(\mu_t)} \right)$$

Here $U_N(\mu) = \exp \left(-\frac{1}{N} \text{Ent}(\mu) \right)$ and $s_{K/N}(r) = \sqrt{\frac{N}{K}} \sin \left(\sqrt{\frac{K}{N}} r \right)$ (provided $K > 0$) and with the usual re-interpretation in the case $K \leq 0$.

This curve is unique and coincides with the heat flow which we denote in the following by $\mu_t = H_t\mu_0$. The Evolution Variation Inequality $\text{EVI}_{K,N}$ as stated above immediately implies new, sharp contraction estimates (or, more precisely, expansion bounds) in Wasserstein metric for the heat flow.

Theorem 3. *Let (X, d, m) be a $\text{RCD}(K, N)$ space. Then for any $\mu, \nu \in \mathcal{P}_2(X, d)$ and $s, t > 0$:*

$$(3) \quad s_{K/N} \left(\frac{1}{2} W_2(H_t\mu, H_s\nu) \right)^2 \leq e^{-K(s+t)} s_{K/N} \left(\frac{1}{2} W_2(\mu, \nu) \right)^2 + \frac{N}{K} \left(1 - e^{-K(s+t)} \right) \frac{(\sqrt{t} - \sqrt{s})^2}{2(s+t)}.$$

Extending previous work of the second named author, we prove that these W_2 -expansion bounds are in intimate correspondence to pointwise gradient estimates of Bakry–Ledoux type.

Theorem 4. *Assume that (X, d, m) is infinitesimally Hilbertian and satisfies W_2 -expansion bound (3). Then for any f of finite Cheeger energy:*

$$(4) \quad |\nabla H_t f|_w^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)} |\Delta H_t f|^2 \leq e^{-2Kt} H_t(|\nabla f|_w^2).$$

Here $|\nabla f|_w$ denotes the weak upper gradient of f introduced in [1]. Differentiating the latter inequality at $t = 0$ leads to the Bochner formula for the canonical gradients and Laplacians on mms.

Theorem 5. *Assume that (X, d, m) is infinitesimally Hilbertian and satisfies the gradient estimate (4). Then for all $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X, d, m)$ and all $g \in D(\Delta)$ bounded and non-negative with $\Delta g \in L^\infty(X, m)$ we have*

$$(5) \quad \frac{1}{2} \int \Delta g |\nabla f|_w^2 dm - \int g \langle \nabla(\Delta f), \nabla f \rangle dm \geq K \int g |\nabla f|_w^2 dm + \frac{1}{N} \int g (\Delta f)^2 dm.$$

Finally we close the circle proving also the converse implication.

Theorem 6. *Assume that (X, d, m) is infinitesimally Hilbertian. Then the Bochner inequality $\text{BE}(K, N)$ (5) implies the entropic curvature-dimension condition $\text{CD}^e(K, N)$.*

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Optimal transport in discrete settings

JAN MAAS

(joint work with Eric Carlen, Matthias Erbar, Nicola Gigli)

In the last decade there has been significant breakthroughs in the analysis and geometry of metric measure spaces based on optimal transport. Among the key insights are the following two connections between the Boltzmann-Shannon entropy and the 2-Wasserstein metric on the space of probability measures:

- (1) Lower bounds on the *Ricci curvature* are equivalent to convexity properties of the entropy along 2-Wasserstein geodesics [17, 5, 18].
- (2) The *heat flow* can be interpreted as the gradient flow of the entropy with respect to the 2-Wasserstein metric [10, 16].

These results have led to a rich theory of Ricci curvature on metric measure spaces, with remarkable geometric and analytic consequences [LV09, 19, 1]. Nevertheless, the theory breaks down in discrete settings, since the 2-Wasserstein space over a discrete space becomes degenerate. In particular, there exist no absolutely continuous curves with respect to W_2 , hence no geodesics and no gradient flows.

In [12] a new metric has been introduced on the space of probability measures, which allows us to formulate discrete analogues of (1) and (2). The framework consist of a Markov kernel K on a finite set \mathcal{X} . We assume that there exists a reversible probability measure π , i.e., the detailed balance conditions $K(x, y)\pi(x) = K(y, x)\pi(y)$ hold for all $x, y \in \mathcal{X}$. In this setting, the discrete Laplacian $\Delta\psi(x) = \sum_{y \in \mathcal{X}} K(x, y)(\psi(y) - \psi(x))$ is the generator of the continuous time Markov semigroup associated with K . For a probability density ρ (with respect to π), we define the relative entropy by $\text{Ent}_\pi(\rho) = \sum_{x \in \mathcal{X}} \rho(x) \log \rho(x) \pi(x)$.

The associated *non-local transportation metric* is then defined using the following discrete analogue of the Benamou–Brenier formula [2]:

$$\mathcal{W}(\rho_0, \rho_1)^2 := \inf_{\rho, \psi} \left\{ \frac{1}{2} \int_0^1 \sum_{x, y \in \mathcal{X}} (\psi_t(x) - \psi_t(y))^2 \hat{\rho}_t(x, y) K(x, y) \pi(x) dt \right\},$$

where the infimum runs over all curves $\rho : [0, 1] \rightarrow \mathcal{P}(\mathcal{X})$ in the space of probability densities, and all functions $\psi : [0, 1] \times \mathcal{X} \rightarrow \mathbb{R}$ satisfying the ‘discrete continuity equation’

$$\frac{d}{dt} \rho_t(x) + \sum_{y \in \mathcal{X}} (\psi_t(x) - \psi_t(y)) \hat{\rho}_t(x, y) K(x, y) = 0$$

with boundary conditions $\rho|_{t=0} = \rho_0$ and $\rho|_{t=1} = \rho_1$. In these formulae, the quantity $\hat{\rho}(x, y)$ denotes the *logarithmic mean* of $\rho(x)$ and $\rho(y)$:

$$\hat{\rho}(x, y) := \int_0^1 \rho(x)^{1-\alpha} \rho(y)^\alpha d\alpha.$$

The metric \mathcal{W} allows us to prove a discrete analogue of (1):

Theorem 1. [4, 12, 14] *The discrete ‘heat flow’ given by the continuous time Markov semigroup $(e^{t\Delta})_{t \geq 0}$ is the gradient flow of the relative entropy Ent_π with respect to the metric \mathcal{W} .*

Since the metric \mathcal{W} takes over the role of the 2-Wasserstein metric in discrete settings, it is possible to formulate the following discrete analogue of (2): a reversible Markov kernel K is said to have Ricci curvature bounded from below by $\kappa \in \mathbb{R}$ if the κ -convexity inequality

$$\text{Ent}_\pi(\rho_t) \leq (1-t)\text{Ent}_\pi(\rho_0) + t\text{Ent}_\pi(\rho_1) - \frac{\kappa}{2}t(1-t)\mathcal{W}(\rho_0, \rho_1)^2$$

holds along all \mathcal{W} -geodesics $(\rho_t)_{t \in [0,1]}$. This notion allows us to extend some well known results involving Ricci curvature to the discrete setting.

Theorem 2. [7] *The following assertions hold for $\kappa > 0$.*

- (1) *(Discrete Bakry–Émery Theorem) If $\text{Ric}(K) \geq \kappa$, then the modified logarithmic Sobolev inequality (MLSI)*

$$\text{Ent}_\pi(\rho) \leq \frac{1}{4\kappa} \sum_{x,y \in \mathcal{X}} (\rho(x) - \rho(y)) (\log \rho(x) - \log \rho(y)) K(x,y) \pi(x)$$

holds for all $\rho \in \mathcal{P}(\mathcal{X})$.

- (2) *(Discrete Otto–Villani Theorem) If K satisfies the MLSI, then the modified Talagrand inequality*

$$\mathcal{W}(\rho, \mathbf{1}) \leq \sqrt{\frac{2}{\kappa} \text{Ent}_\pi(\rho)}$$

holds for all $\rho \in \mathcal{P}(\mathcal{X})$.

Several other results involving this notion of Ricci curvature have been obtained recently. In particular, Mielke [15] showed that for an arbitrary reversible Markov chain on a finite space, the Ricci curvature is bounded from below by a possibly negative constant. Another key result in [15] is a geodesic convexity result for discretisations of one-dimensional Fokker-Planck equations. A tensorisation result for products of Markov chains has been obtained in [7].

Although the definition of \mathcal{W} formally resembles the Benamou–Brenier formulation of the 2-Wasserstein metric, it is not clear a priori how these metrics are related. In a joint work with Gigli [9] we proved that the rescaled non-local transportation metrics \mathcal{W}_N on the d -dimensional discrete torus with mesh size $\frac{1}{N}$ converge, when $N \rightarrow \infty$, to the 2-Wasserstein distance W_2 on the continuous torus. The convergence is in the sense of Gromov–Hausdorff convergence of metric spaces.

Let us finally mention that transportation metrics based on the Benamou–Brenier formula have been recently introduced in several other settings. Gradient flow structures for discrete porous medium equations have been studied in a joint work with Erbar [8]. Erbar also studied non-local transportation metrics on infinite state spaces [6]. In particular, he obtained a gradient flow structure for

fractional heat equation $\partial_t \rho = -(-\Delta)^{\alpha/2} \rho$, as well as geodesic convexity of the entropy in this setting.

In a non-commutative setting, it has been shown in a joint work with Carlen [3] that the fermionic Fokker-Planck equation is the gradient flow of the von Neumann entropy on the space of density matrices. Gradient flow structures for different quantum evolution equations have been obtained in [13].

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Comparison of quenched and annealed invariance principles for random conductance model

KRZYSZTOF BURDZY

(joint work with Martin Barlow, m Timar)

Let $d \geq 2$ and let E_d be the set of all non oriented edges in the d -dimensional integer lattice, that is, $E_d = \{e = \{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$. Let $\{\mu_e\}_{e \in E_d}$ be a random process with non-negative values, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process $\{\mu_e\}_{e \in E_d}$ represents random conductances. We write $\mu_{xy} = \mu_{yx} = \mu_{\{x, y\}}$ and set $\mu_{xy} = 0$ if $\{x, y\} \notin E_d$. Set

$$\mu_x = \sum_y \mu_{xy}, \quad P(x, y) = \frac{\mu_{xy}}{\mu_x},$$

with the convention that $0/0 = 0$ and $P(x, y) = 0$ if $\{x, y\} \notin E_d$. For a fixed $\omega \in \Omega$, let $X = \{X_t, t \geq 0, P_\omega^x, x \in \mathbb{Z}^d\}$ be the continuous time random walk on \mathbb{Z}^d , with transition probabilities $P(x, y) = P_\omega(x, y)$, and exponential waiting times with mean $1/\mu_x$. The corresponding expectation will be denoted E_ω^x . For a fixed $\omega \in \Omega$, the generator \mathcal{L} of X is given by

$$(1) \quad \mathcal{L}f(x) = \sum_y \mu_{xy}(f(y) - f(x)).$$

In [BD] this is called the *variable speed random walk* (VSRW) among the conductances μ_e . This model, of a reversible (or symmetric) random walk in a random environment, is often called the Random Conductance Model.

We are interested in functional Central Limit Theorems (CLTs) for the process X . Given any process X , for $\varepsilon > 0$, set $X_t^{(\varepsilon)} = \varepsilon X_{t/\varepsilon^2}$, $t \geq 0$. Let $\mathcal{D}_T = D([0, T], \mathbb{R}^d)$ denote the Skorokhod space, and let $\mathcal{D}_\infty = D([0, \infty), \mathbb{R}^d)$. Write d_S for the Skorokhod metric and $\mathcal{B}(\mathcal{D}_T)$ for the σ -field of Borel sets in the corresponding topology. Let X be the canonical process on \mathcal{D}_∞ or \mathcal{D}_T , P_{BM} be Wiener measure on $(\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ and let E_{BM} be the corresponding expectation. We will write W for a standard Brownian motion. It will be convenient to assume that $\{\mu_e\}_{e \in E_d}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that X is defined on $(\Omega, \mathcal{F}) \times (\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ or $(\Omega, \mathcal{F}) \times (\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$. We also define the averaged or annealed measure \mathbf{P} on $(\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ or $(\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$ by

$$\mathbf{P}(G) = \mathbb{E} P_\omega^0(G).$$

Definition 1. For a bounded function F on \mathcal{D}_T and a constant matrix Σ , let $\Psi_\varepsilon^F = E_\omega^0 F(X^{(\varepsilon)})$ and $\Psi_\Sigma^F = E_{\text{BM}} F(\Sigma W)$.

(i) We say that the *Quenched Functional CLT* (QFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on \mathcal{D}_T we have $\Psi_\varepsilon^F \rightarrow \Psi_\Sigma^F$ as $\varepsilon \rightarrow 0$, with \mathbb{P} -probability 1.

(ii) We say that the *Weak Functional CLT* (WFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on \mathcal{D}_T we have $\Psi_\varepsilon^F \rightarrow \Psi_\Sigma^F$ as $\varepsilon \rightarrow 0$, in \mathbb{P} -probability.

(iii) We say that the *Averaged (or Annealed) Functional CLT* (AFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on \mathcal{D}_T we have $\mathbb{E} \Psi_\varepsilon^F \rightarrow \Psi_\Sigma^F$. This is the same as standard weak convergence with respect to the probability measure \mathbf{P} .

If we take Σ to be non-random then since F is bounded, it is immediate that QFCLT \Rightarrow WFCLT \Rightarrow AFCLT. In general for the QFCLT the matrix Σ might depend on the environment $\mu_e(\omega)$. However, if the environment is stationary and ergodic, then Σ is a shift invariant function of the environment, so must be \mathbb{P} -a.s. constant.

In [DFGW] it is proved that if μ_e is a stationary ergodic environment with $\mathbb{E} \mu_e < \infty$ then the WFCLT holds. It is an open question as to whether the QFCLT holds under these hypotheses. For the QFCLT in the case of percolation see [BeB, MP, SS], and for the Random Conductance Model with μ_e i.i.d see [BP, M1, BD, ABDH]. In the i.i.d. case the QFCLT holds (with $\sigma > 0$) for any distribution of μ_e provided $p_0 = \mathbb{P}(\mu_e = 0) < p_c(\mathbb{Z}^d)$.

Definition 2. We say an environment (μ_e) on \mathbb{Z}^d is *symmetric* if the law of (μ_e) is invariant under symmetries of \mathbb{Z}^d .

If (μ_e) is stationary, ergodic and symmetric, and the WFCLT holds with limit ΣW then the limiting covariance matrix $\Sigma^T \Sigma$ must also be invariant under symmetries of \mathbb{Z}^d , so must be a constant $\sigma \geq 0$ times the identity.

Our main result concerns the relation between the weak and quenched FCLT.

Theorem 3. *Let $d = 2$ and $p < 1$. There exists a symmetric stationary ergodic environment $\{\mu_e\}_{e \in E_2}$ with $\mathbb{E}(\mu_e^p \vee \mu_e^{-p}) < \infty$ and a sequence $\varepsilon_n \rightarrow 0$ such that*

(a) *the WFCLT holds for $X^{(\varepsilon_n)}$ with limit W ,*

but

(b) *the QFCLT does not hold for $X^{(\varepsilon_n)}$ with limit ΣW for any Σ .*

Remark 4. (1) Under the weaker condition that $\mathbb{E} \mu_e^p < \infty$ and $\mathbb{E} \mu_e^{-q} < \infty$ with $p < 1$, $q < 1/2$ we have the full WFCLT for $X^{(\varepsilon)}$ as $\varepsilon \rightarrow 0$, i.e., not just along a sequence ε_n . However, the proof of this is very much harder and longer than that of Theorem 3(a). A sketch argument will be posted on the arxiv – see [BBTA]. (Since our environment has $\mathbb{E} \mu_e = \infty$ we cannot use the results of [DFGW].)

(2) Biskup [Bi] has proved that the QFCLT holds with $\sigma > 0$ if $d = 2$ and (μ_e) are symmetric and ergodic with $\mathbb{E}(\mu_e \wedge \mu_e^{-1}) < \infty$.

(3) A forthcoming paper by Andres, Deuschel and Slowik proves that the QFCLT holds (in \mathbb{Z}^d , $d \geq 2$) for stationary symmetric ergodic environments (μ_e) under the conditions $\mathbb{E} \mu_e^p < \infty$, $\mathbb{E} \mu_e^{-q} < \infty$, with $p^{-1} + q^{-1} < 2/d$.

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Quenched invariance principles for random walks and elliptic diffusions in random media with boundary

DAVID A. CROYDON

(joint work with Zhen-Qing Chen, Takashi Kumagai)

The work reported here establishes, via a Dirichlet form extension theorem and making full use of two-sided heat kernel estimates, quenched invariance principles for random walks and random divergence forms in random media with a boundary.

Whilst the current literature provides some powerful techniques for overcoming the technical challenges involved in proving quenched invariance principles for random walks in reversible random environments, such as studying ‘the environment viewed from the particle’ or the ‘harmonic corrector’ for the walk (see [3] for a survey of the recent developments in the area), these are not without their limitations. Most notably, at some point, the arguments applied all depend in a fundamental way on the translation invariance or ergodicity under random walk transitions of the environment. As a consequence, some natural variations of the problem are not covered. Consider, for example, supercritical percolation in a half-space $\mathbb{Z}_+ \times \mathbb{Z}^{d-1}$, or possibly an orthant of \mathbb{Z}^d . Again, there is a unique infinite cluster (Figure 1 shows a simulation of such in the first quadrant of \mathbb{Z}^2), upon which one can define a random walk. Given the known invariance principle for percolation on \mathbb{Z}^d [2, 4, 5], one would reasonably expect that this process would converge, when rescaled diffusively, to a Brownian motion reflected at the boundary. After all, as is illustrated in the figure, the ‘holes’ in the percolation cluster that are in contact with the boundary are only on the same scale as those away from it. However, the presence of a boundary means that the translation invariance/ergodicity properties necessary for applying the existing arguments directly are lacking. (This point was discussed briefly in [2, Section B], see also [3,

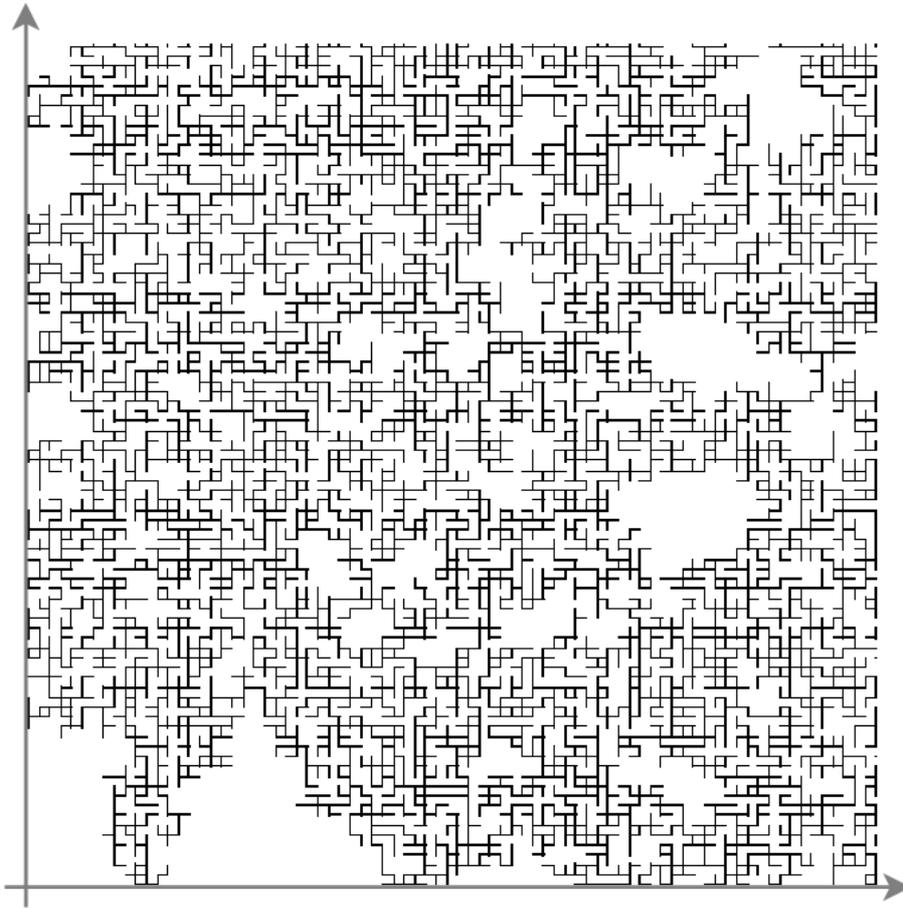


FIGURE 1. A section of the unique infinite cluster for supercritical percolation on \mathbb{Z}_+^2 with parameter $p = 0.52$.

Problem 1.9]; in fact, in both these references, the example of random walk on a percolation cluster/amongst random conductances in a half-space was explicitly listed as an open problem.) Providing an approach for overcoming this issue, and thereby establishing invariance principles in examples such as that just described, is our contribution.

The method we develop for proving quenched invariance principles for random walks in random environments with a boundary encompasses two novel aspects: a Dirichlet form extension argument and the full use of detailed heat kernel estimates. For the first of these, the key observation is that by applying a quenched invariance principle for the full-space case, one can check that the random walk run until it hits the boundary converges to a killed Brownian motion. Noting that the reflected Brownian motion can be seen as the maximal Silverstein's extension of the killed process, a Dirichlet form characterisation then allows one to reconstruct the desired scaling limit for the non-killed processes. The extra technical conditions needed to do this include establishing that the non-killed processes admit a subsequence that converges to a conservative symmetric Hunt process with continuous sample paths. It is in checking this that we make full use of the

heat kernel. In particular, we provide sufficient conditions for the necessary sub-sequential convergence that involve the Hölder continuity of harmonic functions. This continuity property can be verified in examples using relatively straightforward adaptations of existing two-sided heat kernel bounds. We remark that the corrector-type methods for full-space models, such as the approach of [2], often require only upper bounds on the heat kernel.

Rather than introduce the technical framework needed to state the above results precisely, let us instead present here an illustrative application of our main result in the case of the random conductance model. This result verifies the conjecture described above concerning the diffusive behaviour of the random walk on a supercritical percolation cluster on a half-space, quarter-space, etc. Firstly, fix $d_1, d_2 \in \mathbb{Z}_+$ such that $d_1 \geq 1$ and $d := d_1 + d_2 \geq 2$, and define a graph $(\mathbb{L}, E_{\mathbb{L}})$ by setting $\mathbb{L} := \mathbb{Z}_+^{d_1} \times \mathbb{Z}^{d_2}$ and $E_{\mathbb{L}} := \{e = \{x, y\} : x, y \in \mathbb{L}, |x - y| = 1\}$. Next, let $\mu = (\mu_e)_{e \in E_{\mathbb{L}}}$ be a collection of independent and identically distributed random variables on $[0, \infty)$, defined on a probability space Ω with measure \mathbb{P} , such that

$$p_1 := \mathbb{P}(\mu_e > 0) > p_c^{\text{bond}}(\mathbb{Z}^d),$$

where $p_c^{\text{bond}}(\mathbb{Z}^d) \in (0, 1)$ is the critical probability for bond percolation on \mathbb{Z}^d – this condition means that there \mathbb{P} -a.s. exists a unique infinite cluster \mathcal{C}_1 connected by edges with $\mu_e > 0$. We will make the following further assumptions on the distribution of μ :

$$\mathbb{P}(\mu_e \in (0, c)) = 0 \text{ for some } c > 0, \text{ and also } \mathbb{E}(\mu_e) < \infty.$$

In particular, note that this framework includes the special case of supercritical percolation ($\mathbb{P}(\mu_e = 1) = p_1 = 1 - \mathbb{P}(\mu_e = 0)$). Finally, for a given realisation of μ , let $Y = (Y_t)_{t \geq 0}$ be the associated variable speed random walk (VSRW), where we recall that the VSRW on a (weighted) graph is the continuous time Markov process that jumps from a vertex x at a rate equal to the total weight of edges adjacent to x to a neighbour selected at random according to the probability distribution proportional to the corresponding edge weights. (In the percolation setting, similar results to that stated can be obtained for the so-called constant speed random walk (CSRW), which has mean one exponential holding times, or the discrete time random walk.)

Theorem 1. *For almost-every realisation of μ , it holds that the rescaled VSRW $Y^n = (Y_t^n)_{t \geq 0}$, as defined by $Y_t^n := n^{-1}Y_{n^2 t}$, started from $Y_0^n = x_n \in n^{-1}\mathcal{C}_1$, where $x_n \rightarrow x \in \mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$, converges in distribution to $(B_{\sigma t})_{t \geq 0}$, where $\sigma \in (0, \infty)$ is a deterministic constant and $(B_t)_{t \geq 0}$ is standard Brownian motion on $\mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$, started from x and reflected at the boundary.*

Similarly, one could consider compact limiting sets, replacing Y^n in the above theorem by the rescaled version of the variable speed random walk on the largest percolation cluster contained in a box $[-n, n]^d \cap \mathbb{Z}^d$, for example. Our general results allow a quenched invariance principle to be established in this case as well, with the limiting process being Brownian motion in the box $[-1, 1]^d$, reflected at

the boundary. As a corollary, we refine the existing knowledge of the mixing time asymptotics for the sequence of random graphs in question from a tightness result [1] to an almost-sure convergence one.

Although in the percolation setting we only consider relatively simple domains with ‘flat’ boundaries, the principal reason for this is to ensure that deriving the percolation estimates required for our proofs is manageable. In the case when we restrict to uniformly elliptic random conductances, so that controlling the clusters of extreme conductances is no longer an issue, we are able to derive quenched invariance principles in more general domains.

Finally, we note that the same techniques can also be applied in the random divergence form setting.

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Invariance Principle for the Random Conductance Model in a degenerate ergodic Environment

SEBASTIAN ANDRES

(joint work with Jean-Dominique Deuschel, Martin Slowik)

We consider the Euclidean lattice \mathbb{Z}^d with $d \geq 2$. Let E_d be the set of non oriented nearest neighbour bonds: $E_d = \{e = \{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$. The random environment is given by stationary ergodic random variables $(\omega_e, e \in E_d)$ on $[0, \infty)$, defined on a probability space (Ω, \mathbb{P}) . We write $\omega_{xy} = \omega_{\{x, y\}} = \omega_{yx}$, and set $\mu_x = \sum_y \omega_{xy}$. We will study a continuous time random walk $X = (X_t, t \geq 0, P_x^\omega, x \in \mathbb{Z}^d)$ on \mathbb{Z}^d , which jumps according to the transitions $P(x, y) = \omega_{xy}/\mu_x$ and whose generator is given by:

$$\mathcal{L}^\omega f(x) = \sum_{y \sim x} \omega_{xy} (f(y) - f(x)).$$

We denote by $p_t^\omega(x, y)$, $x, y \in \mathbb{Z}^d$, $t \geq 0$, the corresponding transition densities. From now on we assume that $\mathbb{P}[0 < \omega_e < \infty] = 1$.

This model, of a symmetric random walk in a random environment, is known in the literature as the *Random Conductance Model*. We are interested in the \mathbb{P} almost sure or quenched long range behaviour, in particular in obtaining

- i) **Gaussian bounds (GB)** on $p_t^\omega(x, y)$: There exist random variables $V_x(\omega)$ with stretched-exponential tails such that when $t \geq V_x$:

$$p_t^\omega(x, y) \leq c_1 t^{-d/2} e^{-c_2 |x-y|^2/t}, \quad \text{if } t \geq |x - y|,$$

and similar lower bounds.

- ii) **Quenched functional CLT (QFCLT)** for X : Let $X_t^{(n)} = \frac{1}{n} X_{n^2 t}$, and W be a Brownian motion on \mathbb{R}^d . Then for \mathbb{P} -a.a. ω , under P_0^ω ,

$$X^{(n)} \Rightarrow \Sigma \cdot W.$$

Here Σ denotes a (non-degenerate?) diffusivity matrix.

Having both, GB and the QFCLT – following arguments in [BH] – one can obtain a **local limit theorem** for $p_t^\omega(x, y)$, that is

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T} |n^d p_{n^2 t}^\omega(0, \lfloor nx \rfloor) - k_t(x)| = 0, \quad \mathbb{P}\text{-a.s.}, T > 0,$$

($k_t(x)$ denoting the transition kernel of $\Sigma \cdot W$)

In the case of i.i.d. conductances these questions have been studied by a number of different authors under various restrictions on the law of ω_e .

(1) If $\omega_e \in \{0, 1\}$ and $\mathbb{P}[\omega_e > 0] > p_c$, then X becomes a random walk on a supercritical percolation cluster – see [B1] for GB, [SS, BB, MP] for a QFCLT.

(2) In the uniformly elliptic case where $\mathbb{P}(c^{-1} \leq \omega_e \leq c) = 1$ for some $c \geq 1$, GB follow from the results in [Del], and a QFCLT is proved in [SS].

(3) The case with conductances bounded from above $\mathbb{P}(0 < \omega_e \leq 1) = 1 - \mathbb{P}(\omega_e = 0) > p_c$, is treated in [BBHK, BP, Ma1]. A QFCLT is proved in [BP, Ma1], with a strictly positive diagonal diffusivity matrix. Further [BBHK] shows that GB *do not hold* in general in this case for $d \geq 5$. The reason for this lies in the possibility for the random walk to get trapped, which can cause a sub-Gaussian heat kernel decay and thus also a failure of a local limit theorem.

(4) The case when ω_e is bounded from below: $\mathbb{P}(1 \leq \omega_e < \infty) = 1$, is studied in [BD], and GB and a QFCLT are derived.

(5) All approaches have been synthesized together in [ABDH], where a QFCLT with a non-degenerate diffusivity matrix is proven for general nonnegative i.i.d. conductances provided $\mathbb{P}[\omega_e > 0] > p_c$.

To summarize, already in the i.i.d. situation we find ourselves in the somewhat unusual situation when the path distribution satisfies an FCLT even with a non-degenerate limit, but the heat kernel may decay anomalously, so we have a CLT without a local CLT.

Now let us consider general stationary ergodic environments. Here, it has been shown in [Bi] that the QFCLT holds in dimension $d = 1, 2$ provided $\mathbb{E}[\omega_e] < \infty$ and $\mathbb{E}[(\omega_e)^{-1}] < \infty$. Further, an example in $d = 2$ for an ergodic environment has been constructed in [BBT], satisfying $\mathbb{E}[(\omega_e)^p \vee (\omega_e)^{-p}] < \infty$, $p < 1$, for which

the QFCLT fails. Our main result now extends these existing results to higher dimensions.

Theorem 1 (see [ADS]). *Suppose $d \geq 2$. Let $(\omega_e)_{e \in E_d}$ be stationary ergodic and $p, q \in (1, \infty]$ be such that $1/p + 1/q < 2/d$ and assume that*

$$\mathbb{E}[(\omega_e)^p] < \infty \quad \text{and} \quad \mathbb{E}[(\omega_e)^{-q}] < \infty$$

for any $e \in E_d$. Then, QFCLT holds with a deterministic non-degenerate Σ .

In view of the above mentioned results in $d = 2$ it is obvious that the moment conditions in Theorem 1 are not optimal. However, apart from extending the existing results to the case of ergodic environments in higher dimensions, we use a different, quite analytical approach to prove the QFCLT in Theorem 1, which – in contrast to the earlier results in arbitrary dimensions mentioned before – can avoid the explicit use of Gaussian heat kernel estimates. Such an approach can be advantageous especially in the context of the Random Conductance Model, where one can get into the situation when Gaussian bounds fail although a the QFCLT holds.

The basic strategy is to use the 'classical' Kipnis-Varadhan approach and to construct the **corrector** $\chi(\omega, x) : \Omega \times \mathbb{Z}^d \rightarrow \mathbb{R}^d$ such that for \mathbb{P} -a.a. ω the process $M_t = X_t - \chi(\omega, X_t)$ is a P_0^ω -martingale. Then, by standard arguments one proves a FCLT for the martingale part, i.e. $\frac{1}{n}M_{n^2t} \Rightarrow \Sigma \cdot W_t$. The difficult part of the proof is to show that after rescaling the corrector vanishes in the limit in the sense that $\sup_{t \leq T} \frac{1}{n}\chi(\omega, X_{n^2t}) \rightarrow 0$ in P_0^ω -probability. This follows once one has established the **sublinearity** of the corrector, i.e.

$$(1) \quad \lim_{n \rightarrow \infty} \max_{|x| \leq n} \frac{|\chi(\omega, x)|}{n} = 0 \quad \mathbb{P}\text{-a.s.}$$

Until now every proof of (1) required of some kind of Gaussian heat kernel estimates. Our approach starts with the observation that the process (M_t) is a martingale if for \mathbb{P} -a.e. ω , $\chi(\omega, \cdot)$ is a solution of the Poisson equation

$$\mathcal{L}^\omega u = \nabla^* V^\omega,$$

where $V^\omega : E_d \rightarrow \mathbb{R}^d$ is the local drift given by $V^\omega(\{x, y\}) = \omega_{xy}(y - x)$ and ∇^* is the divergence operator associated with the discrete gradient. The idea to prove (1) is now to use the well-established Moser iteration technique to show a maximum inequality in the spirit of the proofs for Harnack inequalities. In a second step one uses the moment conditions and the ergodicity of the environment to control the constants appearing in the iteration by the ergodic theorem.

Finally, we remark that – although we do not use Gaussian bounds in the proof – we show in an upcoming paper elliptic and parabolic Harnack inequalities under some assumptions similar to those of Theorem 1 and as a consequence we obtain some heat kernel bounds and a local limit theorem.

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Curvature bounds and heat kernel methods in subriemannian manifolds

FABRICE BAUDOIN

(joint work with N. Garofalo, M. Bonnefont, B. Kim, I. Munive, J. Wang)

In this talk, I presented some recent developments about the study of functional inequalities for subelliptic diffusion operators. The talk was based on joint works with N. Garofalo, M. Bonnefont, B. Kim, I. Munive, J. Wang.

To introduce the relevant setting we consider a smooth, connected manifold \mathbb{M} endowed with a smooth measure μ and a smooth, locally subelliptic, Markov generator L which is symmetric with respect to μ . As above, we associate with L the following symmetric, first-order, differential bilinear form:

$$(1) \quad \Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf), \quad f, g \in C^\infty(\mathbb{M}).$$

There is a genuine distance d canonically associated with L which is continuous and defines the topology of \mathbb{M} . It is given by

$$(2) \quad d(x, y) = \sup \{ |f(x) - f(y)| \mid f \in C^\infty(\mathbb{M}), \|\Gamma(f)\|_\infty \leq 1 \}, \quad x, y \in \mathbb{M},$$

where for a function g on \mathbb{M} we have let $\|g\|_\infty = \operatorname{ess\,sup}_{\mathbb{M}} |g|$.

We always assume that this distance is finite and that the metric space (\mathbb{M}, d) is complete.

We also suppose, and this is the main new ingredient, that there exists on \mathbb{M} a symmetric, first-order differential bilinear form $\Gamma^Z : C^\infty(\mathbb{M}) \times C^\infty(\mathbb{M}) \rightarrow \mathbb{R}$, satisfying

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h).$$

The following hypothesis on Γ^Z plays a pervasive role in the results that I will describe.

Assumption 1.

$$(3) \quad \Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

Before we proceed with the discussion, we pause to stress that, in the generality in which we work the bilinear differential form Γ^Z , unlike Γ , is not a priori canonical. Whereas Γ is determined once L is assigned, the form Γ^Z in general is not intrinsically associated with L . However, in several geometric examples the choice of Γ^Z will be natural and even canonical, up to a constant. This is the case, for instance, for one of the important geometric examples covered by our analysis: The CR Sasakian manifolds. Roughly speaking, we can think of Γ^Z as an orthogonal complement of Γ : the bilinear form Γ represents the square of the length of the gradient in the horizontal directions, whereas Γ^Z represents the square of the length of the gradient along the vertical directions.

Given the sub-Laplacian L and the first-order bilinear forms Γ and Γ^Z on \mathbb{M} , we now introduce the following second-order differential forms:

$$(4) \quad \Gamma_2(f, g) = \frac{1}{2} [L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)],$$

$$(5) \quad \Gamma_2^Z(f, g) = \frac{1}{2} [L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)].$$

Observe that if $\Gamma^Z \equiv 0$, then $\Gamma_2^Z \equiv 0$ as well. As for Γ and Γ^Z , we will use the notations $\Gamma_2(f) = \Gamma_2(f, f)$, $\Gamma_2^Z(f) = \Gamma_2^Z(f, f)$.

Definition 2 (Generalized Curvature Dimension Inequality). We say that the Markov generator L satisfies the *generalized curvature-dimension inequality* $CD(\rho_1, \rho_2, \kappa, d)$ if there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$, and $0 < d \leq +\infty$ such that the inequality

$$(6) \quad \Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f)$$

hold for every $f \in C^\infty(\mathbb{M})$ and every $\nu > 0$.

Under the above assumptions we explained the following results:

- 1) **Stochastic completeness:** Assume that L satisfy $CD(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \in \mathbb{R}$. Then, the heat semigroup $P_t = e^{tL}$ is stochastically complete, i.e., $P_t 1 = 1$. In particular, the Markov process with generator L is non explosive.

- 2) **Li-Yau type inequalities:** If L satisfies $\text{CD}(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \in \mathbb{R}$, then for any bounded $f \in C^\infty(\mathbb{M})$, such that $f, \sqrt{\Gamma(f)}, \sqrt{\Gamma^Z(f)} \in L^2_\mu(\mathbb{M})$, $f \geq 0$, $f \neq 0$, the following inequality holds for $t > 0$:

$$\Gamma(\ln P_t f) + \frac{2\rho_2}{3} t \Gamma^Z(\ln P_t f) \leq \left(1 + \frac{3\kappa}{2\rho_2} - \frac{2\rho_1}{3} t\right) \frac{L P_t f}{P_t f} + \frac{d\rho_1^2}{6} t - \frac{d\rho_1}{2} \left(1 + \frac{3\kappa}{2\rho_2}\right) + \frac{d \left(1 + \frac{3\kappa}{2\rho_2}\right)^2}{2t}.$$

- 3) **Scale-invariant parabolic Harnack inequality:** If L satisfies $\text{CD}(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \geq 0$, then for every $(x, s), (y, t) \in \mathbb{M} \times (0, \infty)$ with $s < t$ one has

$$u(x, s) \leq u(y, t) \left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{D}{d} \frac{d(x, y)^2}{4(t-s)}\right),$$

with $u(x, t) = P_t f(x)$, and $f \in C^\infty(\mathbb{M})$ such that $f \geq 0$ and bounded. Here $D = \left(1 + \frac{3\kappa}{2\rho_2}\right) d$.

- 4) **Off-diagonal Gaussian lower and upper bounds for the heat kernel:** If L satisfies $\text{CD}(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \geq 0$, then for any $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon) = C(d, \kappa, \rho_2, \varepsilon) > 0$, which tends to ∞ as $\varepsilon \rightarrow 0^+$, such that for every $x, y \in \mathbb{M}$ and $t > 0$ one has

$$\frac{C(\varepsilon)^{-1}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{Dd(x, y)^2}{d(4-\varepsilon)t}\right) \leq p(x, y, t) \leq \frac{C(\varepsilon)}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{(4+\varepsilon)t}\right).$$

- 5) **Liouville type theorem:** If L satisfies $\text{CD}(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \geq 0$, then there exists no positive solution of $Lf = 0$.
- 6) **Bonnet-Myers type theorem:** Suppose that L satisfy $\text{CD}(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 > 0$. Then, the metric space (\mathbb{M}, d) is compact in the metric topology and we have

$$(7) \quad \text{diam } \mathbb{M} \leq 2\sqrt{3}\pi \sqrt{\frac{\kappa + \rho_2}{\rho_1 \rho_2} \left(1 + \frac{3\kappa}{2\rho_2}\right) d}.$$

Smoothness properties of heat kernel measures on infinite-dimensional nilpotent groups

TAI MELCHER

(joint work with Fabrice Baudoin, Daniel Dobbs, Maria Gordina)

Quasi-invariance under translations and other smoothness properties are fundamental topics of study for measures in infinite dimensions. This report presents results for heat kernel measures on certain infinite-dimensional nilpotent Lie groups, in both elliptic and hypoelliptic settings. Standard techniques for proving these types of results rely on certain functional analytic inequalities which often involve

a lower bound on the Ricci curvature. Such bounds are unavailable in the hypoelliptic (or sub-Riemannian) setting. Nilpotent Lie groups serve as an important first example in the study of sub-Riemannian geometries, and thus positive results in these settings are necessary in the development of a general sub-Riemannian theory.

The focus of the present report is a specific construction of infinite-dimensional Lie groups, the simplest of which may be thought of as infinite-dimensional analogues of Heisenberg groups, in the sense that they are step 2 stratified. In particular, let (W, H, μ) denote an abstract Wiener space and \mathbf{C} a finite-dimensional Hilbert space. Then $\mathfrak{g} = W \oplus \mathbf{C}$ endowed with a continuous, step 2 nilpotent Lie bracket satisfying $[\mathfrak{g}, \mathfrak{g}] = \mathbf{C}$ is called an *infinite-dimensional Heisenberg-like algebra*. The nilpotence allows an explicit construction of a group operation on $G = W \times \mathbf{C}$ via the Baker-Campbell-Hausdorff-Dynkin formula. These groups provide a natural first framework for studying hypoelliptic heat kernel measures in infinite dimensions.

Elliptic heat kernel measures in the infinite-dimensional Heisenberg setting were studied in [4]. Brownian motion $\{\xi_t\}_{t \geq 0}$ may be constructed on these groups in the standard way via the “rolling map”

$$(1) \quad d\xi_t = L_{\xi_t^*} \circ dB_t \text{ with } \xi_0 = e,$$

where B_t is Brownian motion on \mathfrak{g} , and the *elliptic heat kernel measure* is the end point distribution $\mu_T = \text{Law}(\xi_T)$ for ξ the solution to (1). In [4], Driver and Gordina proved quasi-invariance for μ_T under translations by elements of the *Cameron-Martin subgroup* $G_{CM} = H \times \mathbf{C}$. In fact, they gave an explicit quasi-invariance formula for the “Wiener measure” $\mu = \text{Law}(\xi_\cdot)$ on the path space of G under translations by absolutely continuous paths taking values in G_{CM} and a subsequent first-order integration by parts formula.

Building on these results, we prove in [3] that elliptic path space and heat kernel measures on infinite-dimensional Heisenberg-like groups satisfy a strong definition of smoothness. We give a direct proof that one may perform arbitrary integration by parts for elliptic heat kernel measures in this setting, and the logarithmic derivatives of these measures are integrable to all orders. This method of direct proof generates explicit formulae for these derivatives, analogous to the Hermite polynomials. Typically, it is not possible to verify smoothness in this sense and weaker interpretations must be made.

To consider the hypoelliptic regime, note that one may think of the requirement that $[W, W] = \mathbf{C}$ as satisfaction of Hörmander’s condition, and thus one might expect that, for η the solution to (1) considered now as a stochastic differential equation being driven by a Brownian motion B_t on W , the end point distribution $\nu_T = \text{Law}(\eta_T)$ would also satisfy smoothness properties. We will call ν_T the *hypoelliptic heat kernel measure* on G .

In [2], we prove a Cameron-Martin type quasi-invariance theorem for hypoelliptic heat kernel measures on infinite-dimensional Heisenberg-like groups. These constitute the first results of their type for hypoelliptic heat kernel measures in infinite dimensions. Again, typical proofs of quasi-invariance rely on uniform lower

bounds on the Ricci curvature of appropriate finite-dimensional approximation groups, and such bounds are unavailable in the hypoelliptic setting. However, a lower bound on the Ricci curvature is equivalent to the curvature-dimension inequality, and the proofs in [2] substitute uniform bounds from a generalized curvature-dimension inequality (as first introduced in [1]) on the finite-dimensional approximation groups. These bounds allow the proof of the requisite functional analytic inequalities to prove quasi-invariance for the limiting measure. The strong Feller property is also proved in [2] for the associated semi-group; this is an important property in probabilistic potential theory in infinite dimensions. In particular, it plays a key role in the ergodicity of a diffusion.

Nilpotent Lie groups serve as a first example in the study of sub-Riemannian geometries, and thus general nilpotent models beyond the stratified step 2 Heisenberg construction and subsequent results in these settings are of importance. Of course, before addressing any issues of smoothness for hypoelliptic diffusions, one should first resolve these questions in the elliptic case. Some progress has been made in this direction in [5], in which we replace the finite-dimensional Hilbert space \mathbf{C} in the Heisenberg construction with a finite-dimensional nilpotent Lie algebra \mathfrak{v} . Constructing a Lie bracket on $\mathfrak{g} = W \times \mathfrak{v}$ is equivalent to constructing an extension of (the abelian Lie algebra) W over \mathfrak{v} . This may be done in such a way that \mathfrak{g} is nilpotent, and again this allows the explicit description of $W \times \mathfrak{v}$ as a nilpotent Lie group. We call this construction a *semi-infinite Lie group*, and in [5] we show that elliptic heat kernel measures on semi-infinite Lie groups of arbitrary step are quasi-invariant under translations by the Cameron-Martin subgroup. We also prove a log Sobolev inequality holds in this setting.

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Stochastic Target Approach to Ricci Flow on Surfaces

IONEL POPESCU

(joint work with Robert W. Neel)

This report is around the joint work with Robert W. Neel on the stochastic approach of Ricci flow on surfaces.

The first step in the program starts with the observation that metric evolving under the Ricci flow on surfaces preserves the conformal class of the metric, thus the Ricci flow equation can for the conformal factor is a second order non-linear equation. This allows us to find a stochastic target representation of the conformal factor. We define here a notion of a reachable set which in is just the set of points. One can interpret this to a certain extent as a weak solution to the Ricci flow.

The first result in this direction is that if the Ricci flow has a smooth solution then the reachable set is the same as the evolving manifold under the Ricci flow. This is particularly useful because on one hand can be turned as a representation result for the conformal factor of the Ricci flow and on the other hand can be used to prove the uniqueness of the Ricci flow.

All the results we pointed out for the Ricci flow can be extended also for the normalized Ricci flow under which the total area of the surface stays constant. It is known from the Ricci flow literature (Cao an also Hamilton) that the normalized Ricci flow has long time existence. The real result in the Ricci flow literature is that the normalized Ricci flow converges to a constant curvature metric as the time goes to infinity. We are able to reprove this result in the case of the surfaces of non-positive Euler characteristic.

It is desirable to see this result from an ergodic perspective, namely that the curvature gets averaged in long run and this explains why the metric converges to a metric of constant curvature. We bring this picture to light via coupling.

The main tools in doing this is the coupling of various process, particularly time changed Brownian motions. There are several steps into this. The first step is proving that (against a metric of constant curvature) the conformal factor converges to a constant. This can be regarded as a C^0 convergence. We do this by using a comparison of the distance function with a Bessel process. The second step is to prove convergence in C^1 and this means that we want to show that the gradient of the conformal factor converges to 0. This is again done using coupling of two particles combined with a functional inequality satisfied by the supremum of the gradient at time t .

The real difficult part is to show that the Hessian of the conformal factor converges to 0. To carry this through we use a coupling of three particles. This is highly nontrivial because the equation on the first hand is nonlinear and the space we work with is not flat. This is based on a hint in one of the papers of M. Cranston and involves the construction of three processes X_t, Y_t, Z_t such that Z_t is always on the geodesic between X_t and Y_t (at least for small time t) and such that it satisfies several crucial symmetry properties. These symmetries play the main role in a key cancellation which allows us to get the Hessian estimates.

The estimates on higher derivatives follow by induction.

Nonlocal Energies, Operators and Anomalies

MORITZ KASSMANN

In the talk we study functions $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ or time-independent versions $u : \mathbb{R}^d \rightarrow \mathbb{R}$, satisfying a certain nonlocal equation in a domain $\Omega \subset \mathbb{R}^d$ which might be bounded.

1. SET-UP

Let $(\mu(x, dy))_{x \in \mathbb{R}^d}$ be a family of measures on \mathbb{R}^d satisfying

$$\sup_{x \in \mathbb{R}^d} \int \min(|x - y|^2, 1) \mu(x, dy) < +\infty.$$

We assume that $\mu(x, dy)dx$ is a symmetric measure on $\mathbb{R}^d \times \mathbb{R}^d$. This means, for every set $A \times B \subset \mathbb{R}^d \times \mathbb{R}^d$ and every $t \geq 0$

$$(1) \quad \int_A \int_B \mu(x, dy) dx = \int_B \int_A \mu(x, dy) dx.$$

Let us assume that for some $\alpha \in (0, 2)$ and $\Lambda \geq 1$, for every $x_0 \in \mathbb{R}^d$, $\rho > 0$ and $v \in C^{0,1}(B_\rho(x_0))$

$$(K_1) \quad \rho^{-2} \int_{|x_0 - y| \leq \rho} |x_0 - y|^2 \mu(x_0, dy) + \int_{|x_0 - y| > \rho} \mu(x_0, dy) \leq \Lambda \rho^{-\alpha},$$

and for $B = B_\rho(x_0)$

$$(K_2) \quad \begin{aligned} & \Lambda^{-1} \iint_{B \times B} [v(x) - v(y)]^2 \mu(x, dy) dx \\ & \leq (2 - \alpha) \iint_{B \times B} \frac{[v(x) - v(y)]^2}{|x - y|^{d+\alpha}} dx dy \leq \Lambda \iint_{B \times B} [v(x) - v(y)]^2 \mu(x, dy) dx. \end{aligned}$$

We assume these conditions to hold for all $\rho > 0$ because it simplifies our presentation. It would be sufficient to assume (K_1) , (K_2) for small ρ , say for all $\rho \in (0, 1]$.

Before we mention examples of measures, let us draw the attention to Section 2 of [4] where related results are discussed in detail.

1. A standard example satisfying the above assumptions is given by $\mu(x, dy) = |x - y|^{-d-\alpha} dy$ for some $\alpha \in (0, 2)$. In this case the constant Λ would depend on α and blow up for $\alpha \rightarrow 2-$. This would affect our results because the constants in our theorems depend (only) on d , a lower bound for α and Λ .

2. If, instead, one chooses $\mu(x, dy) = (2 - \alpha)|x - y|^{-d-\alpha} dy$, then assumptions (K_1) , (K_2) are satisfied for one fixed Λ and all for all α satisfying $\alpha_0 \leq \alpha < 2$. In this sense our results are robust, i.e. the constants stay bounded for $\alpha \rightarrow 2-$.

This is a natural assumption, because for this choice the operators L studied below converge to a $c(d)(-\Delta)u$ as $\alpha \rightarrow 2-$, where $c(d)$ is a dimensional constant.

Let us not worry about $\alpha \rightarrow 2-$ for the remaining examples. A nice example is:

$$(2) \quad \mu_{axes}(x, dy) = \sum_{i=1}^d \left[|x_i - y_i|^{-1-\alpha} dy_i \prod_{j \neq i} \delta_{x_j}(dy_j) \right],$$

4. Fix $\gamma \in (0, 2)$ and $0 < s \leq 1$ and set

$$(3) \quad \mu_{cusp}(x, dy) = \chi_{\{|z_2| > |z_1|^s \vee |z_1| > |z_2|^s\}}(x - y) |x - y|^{-d-\alpha} dy.$$

Then the effective “order” α - used in (K_1) , (K_2) - will be less than γ .

2. NONLOCAL OPERATORS IN DIVERGENCE FORM

Let us first look at what we call nonlocal operators in divergence form. One example is

$$(4) \quad (Lu)(t, x) = \text{p. v.} \int_{\mathbb{R}^d} (u(t, y) - u(t, x)) a(t, x, y) \mu(x, dy)$$

where $a(t, x, y) \in [1, 2]$.

Note that in general, even if u is smooth, the expression $(Lu)(t, x)$ is not defined due to the possible singularity at $x = y$. In order to circumvent this problem, one defines the solution property with the help of test functions and a bilinear form $\mathcal{E}(u, v)$ which, if the measures are smooth, satisfies

$$\mathcal{E}(u, v) = \int (-Lu)(x) v(x) dx.$$

This observation explains the use of the terminology “divergence form”. The form \mathcal{E} is symmetric because, for t fixed, the measure $a(t, x, y) \mu(x, dy) dx$ is a symmetric measure on $\mathbb{R}^d \times \mathbb{R}^d$.

Theorem [2], [4] : *Assume (K_1) and (K_2) hold true for some $\Lambda \geq 1$ and $\alpha \in (\alpha_0, 2)$. There is a constant $C = C(d, \alpha_0, \Lambda)$ such that for every supersolution u of $\partial_t u - Lu = f$ on $Q = (-1, 1) \times B_2(0)$ which is nonnegative in $(-1, 1) \times \mathbb{R}^d$ the following inequality holds:*

$$(HI) \quad \|u\|_{L^1(U_\ominus)} \leq C \left(\inf_{U_\oplus} u + \|f\|_{L^\infty(Q)} \right)$$

where $U_\oplus = (1 - (\frac{1}{2})^\alpha, 1) \times B_{1/2}(0)$, $U_\ominus = (-1, -1 + (\frac{1}{2})^\alpha) \times B_{1/2}(0)$.

The proof of this result follows the strategy of [5]. We need several algebraic inequalities and a new version of the weighted Poincaré inequality proved in [1]. As a corollary to this result, one obtains important Hölder apriori estimates. Note that the measures enter the result (i.e. the constants in the statements) only through Λ, α_0 but not through α ! One interesting phenomenon (anomaly) is that a strong Harnack inequality does not hold instead of (HI) when considering solutions

instead of subsolutions. In this respect nonlocal operators are different from local differential operators of second order.

3. FULLY NONLINEAR PROBLEMS

Let us define nonlinear operators with the help of linear ones of the form:

$$(5) \quad Lv(x) = \frac{1}{2} \int_{\mathbb{R}^d} [v(x+h) - 2v(x) + v(x-h)] K(h) dh,$$

where $K : \mathbb{R}^d \rightarrow [0, \infty)$ is appropriate. We fix numbers $\alpha_0, \delta, \lambda, \Lambda$ with $\delta \in (0, 1)$, $\alpha_0 \in (0, 2)$, $0 < \lambda \leq \Lambda$. For $\alpha \in (0, 2)$ we define $\mathcal{L}_0 = \mathcal{L}_0(\alpha, \delta, \lambda, \Lambda, n)$ to be the class of all operators L of the form (5) with corresponding symmetric kernels K satisfying

$$(6) \quad (2 - \alpha)1_{V(h)} \frac{\lambda}{|h|^{n+\alpha}} \leq K(h) \leq (2 - \alpha) \frac{\Lambda}{|h|^{n+\alpha}}, \quad h \in \mathbb{R}^n \setminus \{0\}$$

for some conical set $V = V_\xi \subset \mathbb{R}^n$ of the form $V_\xi = \{z \in \mathbb{R}^n \mid |\langle z/|z|, \xi \rangle| \geq \delta\}$ with $\xi \in \mathbb{S}^{n-1}$. Note that ξ may depend on K whereas δ is fixed. In Caffarelli/Silvestre ('09) the authors consider the case $\delta = 0$.

We prove regularity estimates with constants which depend on $\alpha \in (\alpha_0, 2)$ only through α_0 . In particular they stay bounded for $\alpha \rightarrow 2-$. Let us introduce the maximal and the minimal operator with respect to \mathcal{L}_0 by

$$M_{\mathcal{L}_0}^+ u(x) = \sup_{L \in \mathcal{L}_0} Lu(x) \quad \text{and} \quad M_{\mathcal{L}_0}^- u(x) = \inf_{L \in \mathcal{L}_0} Lu(x).$$

If u is sufficiently regular, then the extremal operators can be represented more explicitly.

Our main result is the following a-priori Hölder regularity estimate.

Theorem [3]: *Let $\alpha \in (\alpha_0, 2)$ and u be a bounded function in \mathbb{R}^n such that $M_{\mathcal{L}_0}^+ u \geq -C'$ in $B_1(0)$ and $M_{\mathcal{L}_0}^- u \leq C'$ in $B_1(0)$ in the viscosity sense for some constant $C' > 0$. There are constants $\beta \in (0, 1)$ and $C \geq 1$ depending only on $n, \lambda, \Lambda, \alpha_0$ and δ such that $u \in C^\beta(\overline{B_{1/2}(0)})$ and*

$$(7) \quad \|u\|_{C^\beta(\overline{B_{1/2}(0)})} \leq C \left(\sup_{\mathbb{R}^n} |u| + C' \right).$$

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Non-local Perturbations and Heat Kernel Estimates

ZHEN-QING CHEN

(joint work with Jieming Wang)

Quite a lot of progress has been made during the last ten years in establishing DeGiorgi-Nash-Moser-Aronson type theory for symmetric discontinuous Markov processes, or equivalently, for symmetric non-local operators

$$\mathcal{L}f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(y) - f(x))J(x, y)dy$$

in the distributional sense. Here $J(x, y) \geq 0$ is a symmetric measurable kernel. For example, it is shown in [1] that if $J(x, y) \asymp |x - y|^{-(d+\alpha)}$ for some $0 < \alpha < 2$, then there is a Feller process X having \mathcal{L} as its infinitesimal generator. Moreover, X has a jointly continuous transition density function $p(t, x, y)$ with respect to the Lebesgue measure on \mathbb{R}^d and

$$(1) \quad p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}$$

for every $t > 0$ and $x, y \in \mathbb{R}^d$. Here for two non-negative functions f and g , the notation $f \asymp g$ means that there is a constant $c \geq 1$ so that $c^{-1}f \leq g \leq cf$ on their common domain of definitions. For real numbers $a, c \in \mathbb{R}$, we use $a \vee c$ and $a \wedge c$ to denote $\max\{a, c\}$ and $\min\{a, c\}$, respectively. When $J(x, y) \asymp |x - y|^{-(d+\alpha)} + |x - y|^{-(d+\beta)}$ for some $0 < \beta < \alpha < 2$, it is established in [2] that there is a Feller process X having \mathcal{L} as its infinitesimal generator. Moreover, X has a jointly continuous transition density function $p(t, x, y)$ with respect to the Lebesgue measure on \mathbb{R}^d and

$$(2) \quad p(t, x, y) \asymp \left(t^{-d/\alpha} \wedge t^{-d/\beta} \right) \wedge \left(\frac{t}{|x - y|^{d+\alpha}} + \frac{t}{|x - y|^{d+\beta}} \right)$$

for every $t > 0$ and $x, y \in \mathbb{R}^d$.

The purpose of this talk is to present some recent progress in the study of DeGiorgi-Nash-Moser-Aronson type theory for non-symmetric non-local operators

$$(3) \quad \begin{aligned} \mathcal{L}f(x) &= \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x))j(x, z)dz \\ &= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z \mathbf{1}_{\{|z| \leq 1\}} \rangle) j(x, z) dz, \end{aligned}$$

where $j(x, z)$ is a bounded kernel that is symmetric in z : $j(x, z) = j(x, -z)$.

We consider the non-local operators \mathcal{L}^b where $j(x, z)$ is of the following type:

$$(4) \quad j^b(x, z) = \frac{\mathcal{A}(d, -\alpha)}{|z|^{d+\alpha}} + b(x, z) \frac{\mathcal{A}(d, -\beta)}{|z|^{d+\beta}}.$$

Here $0 < \beta < \alpha < 2$ and $\mathcal{A}(d, -\alpha)$ is a normalizing constant so that the corresponding operator \mathcal{L}^b is $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ when $b \equiv 0$. In other words, we

consider

$$(5) \quad \mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b,$$

where

$$\mathcal{S}^b f(x) := \mathcal{A}(d, -\beta) \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}}) \frac{b(x, z)}{|z|^{d+\beta}} dz$$

Clearly, when $b(x, z) \equiv a \geq 0$, $\mathcal{L}^b = \Delta^{\alpha/2} + a\Delta^{\beta/2}$, which is the infinitesimal generator of a Lévy process that is the independent sum of a symmetric α -stable process and a symmetric β -stable process with weight $a^{1/\beta}$. It follows from (2) and a scaling argument that $\Delta^{\alpha/2} + a\Delta^{\beta/2}$ has a fundamental solution $p_a(t, x, y)$ that enjoys the following two-sided estimates:

$$(6) \quad p_a(t, x, y) \asymp \left(t^{-d/\alpha} \wedge (at)^{-d/\beta} \right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right)$$

for every $t > 0$ and $x, y \in \mathbb{R}^d$.

Theorem 1. *For every bounded function b on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition $b(x, z) = b(x, -z)$, there is a unique continuous function $q^b(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ that satisfies the Duhamel's formula*

$$q^b(t, x, y) = p_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q^b(t-s, x, z) \mathcal{S}_z^b p_0(s, z, y) dz ds$$

$(0, \varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$ with $|q^b(t, x, y)| \leq cp_1(t, x, y)$ on $(0, \varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$ for some $\varepsilon, c > 0$, and that

$$\int_{\mathbb{R}^d} q^b(t, x, y) q^b(s, y, z) dy = q^b(t+s, x, z) \quad \text{for every } t, s > 0 \text{ and } x, z \in \mathbb{R}^d.$$

Moreover, the following holds.

- (i) for each $t > 0$ and $x \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} q^b(t, x, y) dy = 1$.
- (ii) For every $f \in C_b^2(\mathbb{R}^d)$,

$$T_t^b f(x) - f(x) = \int_0^t T_s^b \mathcal{L}^b f(x) ds,$$

where $T_t^b f(x) = \int_{\mathbb{R}^d} q^b(t, x, y) f(y) dy$.

The last property says that q^b is the fundamental solution of \mathcal{L}^b in the distributional sense. Unlike the gradient perturbation for $\Delta^{\alpha/2}$, in general the kernel $q^b(t, x, y)$ in Theorem 1 can take negative values. For example, this is the case when $b \equiv -1$, that is, when $\mathcal{L}^b = \Delta^{\alpha/2} - \Delta^{\beta/2}$, according to the next theorem.

Theorem 2 *Suppose that $x \mapsto b(x, z)$ is continuous for a.e. $z \in \mathbb{R}^d$. Then $q^b(t, x, y) \geq 0$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ if and only if for each $x \in \mathbb{R}^d$, $j^b(x, z) \geq 0$ for a.e. $z \in \mathbb{R}^d$*

Note that when $q^b(t, x, y)$ is non-negative, it follows from Theorem 1 that it generates a strong Markov process X^b . For each bounded function $b(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ and $\lambda > 0$, define

$$m_{b,\lambda} = \operatorname{ess\,inf}_{x,z \in \mathbb{R}^d, |z| > \lambda} |b(x, z)| \quad \text{and} \quad M_{b,\lambda} = \operatorname{ess\,sup}_{x,z \in \mathbb{R}^d, |z| > \lambda} |b(x, z)|.$$

Theorem 3. For every $A > 0$, and $\lambda > 0$, there is a positive constant $C_1 = C_1(d, \alpha, \beta, A, \lambda)$ such that for any bounded b satisfying $j^b(x, z) \geq 0$ with $\|b\|_\infty \leq A$,

$$(7) \quad 0 < q^b(t, x, y) \leq C_1 p_{M_{b+,\lambda}}(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d.$$

Moreover, for every $\varepsilon > 0$, there is a positive constant $C_2 = C_2(d, \alpha, \beta, A, \lambda, \varepsilon)$ such that for any b with $\|b\|_\infty \leq A$ so that

$$(8) \quad j^b(x, z) \geq \varepsilon |z|^{-(d+\alpha)} \quad \text{for a.e. } x, z \in \mathbb{R}^d$$

we have

$$(9) \quad C_2 p_{m_{b+,\lambda}}(t, x, y) \leq q^b(t, x, y) \leq C_1 p_{M_{b+,\lambda}}(t, x, y)$$

for $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$. The kernel $q^b(t, x, y)$ uniquely determines a Feller process $X^b = (X_t^b, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d)$. The Feller process X^b is conservative and has a Lévy system $(J^b(x, y)dy, t)$, where $J^b(x, y) = j^b(x, y-x)$. Moreover, for each $x \in \mathbb{R}^d$, (X^b, \mathbb{P}_x) is the unique solution to the martingale problem $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ with initial value x . Here $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on \mathbb{R}^d .

Note that when (8) holds, there is $R_0 > 0$ so that $b(x, z) \geq 0$ for all $|z| \geq R_0$. The estimate (9) provides quantitative information on how the heat kernel estimates depending on b in a continuous way.

Theorem 4. Let $A \geq 0$ and $\varepsilon > 0$. There is a constant $C = C(d, \alpha, \beta, A, \varepsilon) \geq 1$ so that for any bounded b with $\|b\|_\infty \leq A$ and

$$j^b(x, z) \geq \varepsilon \left(\frac{1}{|z|^{d+\alpha}} + \frac{1}{|z|^{d+\beta}} \right) \quad \text{for a.e. } x, z \in \mathbb{R}^d,$$

we have

$$C^{-1} p_1(t, x, y) \leq q^b(t, x, y) \leq C p_1(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d.$$

We refer the reader to [3] for the proof of these results as well as further results, including Hölder continuity of parabolic functions of \mathcal{L}^b and parabolic Harnack inequality.

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Periodic and non-periodic aspects of heat kernel asymptotics on Sierpiński carpets

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The purpose of this talk was to present the author's recent results in [5, 6] on various short time asymptotics of the canonical heat kernel on generalized Sierpiński carpets, which are among the most typical examples of *infinitely ramified* self-similar fractals and have been intensively studied e.g. in [1, 2, 7, 3, 4].

1. GENERALIZED SIERPIŃSKI CARPETS & THEIR CANONICAL DIRICHLET FORM

We first collect basic facts concerning generalized Sierpiński carpets and their canonical self-similar Dirichlet form, following [5, Section 5] and [6, Section 5].

We fix the following setting throughout this abstract.

Framework 1. Let $d, l \in \mathbb{N}$, $d \geq 2$, $l \geq 2$ and set $Q_0 := [0, 1]^d$. Let $S \subset \{0, 1, \dots, l-1\}^d$ be non-empty, and for each $i \in S$ define $f_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $f_i(x) := l^{-1}i + l^{-1}x$. Set $Q_1 := \bigcup_{i \in S} f_i(Q_0)$, which satisfies $Q_1 \subset Q_0$. Let K be the *self-similar set associated with* $\{f_i\}_{i \in S}$, i.e. the unique non-empty compact subset of \mathbb{R}^d such that $K = \bigcup_{i \in S} f_i(K)$, and set $F_i := f_i|_K$ for $i \in S$. Also let $\rho : K \times K \rightarrow [0, \infty)$ be the Euclidean metric on K given by $\rho(x, y) := |x - y|_{\mathbb{R}^d}$, set $d_f := \log_l \#S$, which is the Hausdorff dimension of (K, ρ) , and set $\mu := \mathcal{H}_\rho^{d_f}(K)^{-1} \mathcal{H}_\rho^{d_f}$, where $\mathcal{H}_\rho^{d_f}$ denotes the d_f -dimensional Hausdorff measure on (K, ρ) .

Note that μ is the unique Borel measure on K such that $\mu(K_w) = (\#S)^{-m}$ for any $m \in \mathbb{N}$ and any $w = w_1 \dots w_m \in S^m$, where $F_w := F_{w_1} \circ \dots \circ F_{w_m}$ and $K_w := F_w(K)$.

The following definition is essentially due to M. T. Barlow and R. F. Bass [1].

Definition 2 (Generalized Sierpiński carpets). $\text{GSC}(d, l, S) := (K, S, \{F_i\}_{i \in S})$ is called a *generalized Sierpiński carpet* if and only if S satisfies the following conditions:

- (GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of \mathbb{R}^d with $f(Q_0) = Q_0$.
- (GSC2) (Connectedness) Q_1 is connected.
- (GSC3) (Non-diagonality) $\text{int}_{\mathbb{R}^d}(Q_1 \cap \prod_{k=1}^d [(i_k - \varepsilon_k)l^{-1}, (i_k + 1)l^{-1}])$ is either empty or connected for any $(i_k)_{k=1}^d \in \mathbb{Z}^d$ and any $(\varepsilon_k)_{k=1}^d \in \{0, 1\}^d$.
- (GSC4) (Borders included) $\{(x_1, 0, \dots, 0) \in \mathbb{R}^d \mid x_1 \in [0, 1]\} \subset Q_1$.

As special cases of Definition 2, $\text{GSC}(2, 3, S_{\text{SC}})$ and $\text{GSC}(3, 3, S_{\text{MS}})$ are called the *Sierpiński carpet* and the *Menger sponge*, respectively, where $S_{\text{SC}} := \{0, 1, 2\}^2 \setminus \{(1, 1)\}$ and $S_{\text{MS}} := \{(i_1, i_2, i_3) \in \{0, 1, 2\}^3 \mid \sum_{k=1}^3 \mathbf{1}_{\{1\}}(i_k) \leq 1\}$ (see Figure 1).

In the rest of this abstract, we assume that $\text{GSC}(d, l, S) = (K, S, \{F_i\}_{i \in S})$ is a generalized Sierpiński carpet. We set $V_0 := K \setminus (0, 1)^d$, which is regarded as the “*boundary of K*” on account of the following fact: $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $m \in \mathbb{N}$ and any $w, v \in S^m$ with $w \neq v$.

Thanks to the results of Barlow and Bass [1], Kusuoka and Zhou [7] and Barlow, Bass, Kumagai and Teplyaev [2], it is known that there exists a unique self-similar Laplacian on K in the sense of Theorem 4 below.

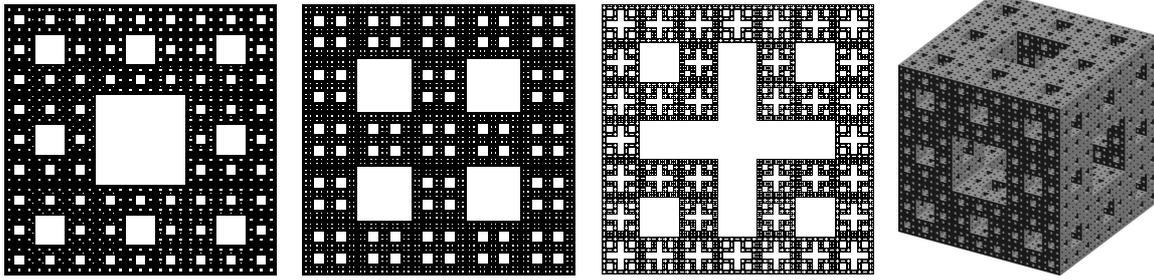


FIGURE 1. Sierpiński carpet, some other generalized Sierpiński carpets with $d = 2$ and Menger sponge

Definition 3. We define

$$(1) \quad \mathcal{G}_0 := \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^d \text{ with } f(Q_0) = Q_0\},$$

which forms a subgroup of the group of homeomorphisms of K by virtue of (GSC1).

Theorem 4 ([2, Theorems 1.2 and 4.32], cf. [4, Proposition 5.1], [5, Proposition 5.9]). *There exists a unique (up to constant multiples of \mathcal{E}) non-zero conservative symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ with the following properties:*

(GSCDF1) *If $u \in \mathcal{F} \cap C(K)$ and $g \in \mathcal{G}_0$ then $u \circ g \in \mathcal{F}$ and $\mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u)$.*

(GSCDF2) $\mathcal{F} \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{F} \text{ for any } i \in S\}$.

(GSCDF3) *There exists $r \in (0, \infty)$ such that for any $u \in \mathcal{F} \cap C(K)$,*

$$(2) \quad \mathcal{E}(u, u) = \sum_{i \in S} \frac{1}{r} \mathcal{E}(u \circ F_i, u \circ F_i).$$

Moreover, $r \in (0, \infty)$ for which (GSCDF3) holds is unique and satisfies $r \leq l^{-2} \#S$.

Set $\tau := \#S/r$ and $d_w := \log_l \tau$, so that $d_w \geq 2$ (in fact, $d_w > 2$ if $\#S < l^d$; see e.g. [1, Remarks 5.4-1.]). Then we have the following heat kernel estimate.

Theorem 5 ([1, Theorem 1.3], [2, Theorem 4.30 and Remark 4.33]). $(K, \mu, \mathcal{E}, \mathcal{F})$ admits a (unique) continuous heat kernel $p = p_t(x, y) : (0, \infty) \times K \times K \rightarrow \mathbb{R}$, and there exist $c_1, c_2 \in (0, \infty)$ such that for any $(t, x, y) \in (0, 1] \times K \times K$,

$$(3) \quad \frac{c_1}{t^{d_f/d_w}} \exp\left(-\left(\frac{\rho(x, y)^{d_w}}{c_1 t}\right)^{\frac{1}{d_w-1}}\right) \leq p_t(x, y) \leq \frac{c_2}{t^{d_f/d_w}} \exp\left(-\left(\frac{\rho(x, y)^{d_w}}{c_2 t}\right)^{\frac{1}{d_w-1}}\right).$$

2. MAIN RESULTS

Now we are in the stage of stating the main results of [5, 6]. The first theorem concerns *non-log-periodic* oscillatory behavior of $p_t(x, x)$ as $t \downarrow 0$ for “generic” $x \in K$. Recall that ν is called a *self-similar measure on K* if and only if ν is a Borel probability measure on K satisfying $\nu \circ F_i = \nu_i \nu$ for any $i \in S$ for some $(\nu_i)_{i \in S} \in (0, \infty)^S$. Recall also that $f : (0, \infty) \rightarrow (0, \infty)$ is said to *vary regularly at 0* if and only if the limit $\lim_{t \downarrow 0} f(\alpha t)/f(t)$ exists in $(0, \infty)$ for any $\alpha \in (0, \infty)$.

Theorem 6 ([5, Theorem 5.11]). *If $\#S < l^d$, then there exist $c_{\text{NP}} \in (0, \infty)$ and a Borel subset N of K satisfying $\nu(N) = 0$ for any self-similar measure ν on K , such that for any $x \in K \setminus N$,*

(NRV) $p_{(\cdot)}(x, x)$ does **not** vary regularly at 0,

(NP) $\limsup_{t \downarrow 0} \left| t^{d_f/d_w} p_t(x, x) - G(-\log t) \right| \geq c_{\text{NP}}$ for **any** periodic $G : \mathbb{R} \rightarrow \mathbb{R}$.

The proof of Theorem 6 makes heavy use of (2), (3) and the following fact: *there exist $y, z \in K \setminus V_0$ such that $\liminf_{t \downarrow 0} p_t(y, y)/p_t(z, z) > 1$.*

On the other hand, the (spectral) partition function $\mathcal{Z}(t) := \int_K p_t(x, x) d\mu(x)$ is expected to exhibit log-periodic behavior, for $\mathcal{Z}(t)$ can be written as $\mathcal{Z}(t) = \sum_{n \in \mathbb{N}} e^{-\lambda_n t}$ by using the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ of the associated Laplacian on K and $\{\lambda_n\}_{n \in \mathbb{N}}$ should strongly reflect the self-similarity of the space. Indeed, we have the following log-periodic asymptotics of $\mathcal{Z}(t)$, improving [3, Theorem 4.1].

Theorem 7 ([6, Theorem 5.11]). *Set $d_k := \log_l \#(S \cap (\mathbb{Z}^{d-k} \times \{0\}^k))$ for $k \in \{0, 1, \dots, d\}$ ($d_0 = d_f$). Then there exist $c_3 \in (0, \infty)$ and continuous log τ -periodic functions $G_k : \mathbb{R} \rightarrow \mathbb{R}$, $k \in \{0, \dots, d\}$, with G_0, G_1 being $(0, \infty)$ -valued, such that*

$$(4) \quad \mathcal{Z}(t) = \sum_{k=0}^d t^{-d_k/d_w} G_k(-\log t) + O\left(\exp\left(-c_3 t^{-\frac{1}{d_w-1}}\right)\right) \quad \text{as } t \downarrow 0.$$

Note that $d_{d-1} = 1$ and $d_d = 0$ by (GSC4) and that for each $k \in \{0, 1, \dots, d\}$, d_k is the box-counting and Hausdorff dimensions of $K \cap (\mathbb{R}^{d-k} \times \{0\}^k)$ with respect to ρ . Note also that $d_k > d_{k+1}$ for any $k \in \{0, \dots, d-1\}$ by (GSC4) and (GSC1).

Remark 8. (1) *The author has no idea whether any one of the periodic functions G_k , $k \in \{0, \dots, d\}$ in Theorem 7 is non-constant when $\#S < l^d$.*

(2) *The author has no idea which signs G_k takes for $k \in \{2, \dots, d\}$ when $\#S < l^d$.*

(3) (4) is valid also for the partition function of the Dirichlet Laplacian on $K \setminus V_0$, with G_0 unchanged and G_1 multiplied by a strictly negative constant.

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Optimal transport approach to Skorokhod embedding

MARTIN HUESMANN

(joint work with Mathias Beiglboeck, Walter Schachermayer)

Given a centered probability measure λ on \mathbb{R} with finite second moment, the Skorokhod problem consists in finding a stopping time τ such that the stopped Brownian motion B_τ has law λ . There are many different solutions to this problem (e.g. see [1] for a survey). Some of these solutions enjoy certain optimality properties. From an optimal transport point of view, these properties can be stated as being solutions to the following minimization problem

$$\inf_{\tau: B_\tau \sim \lambda} \mathbb{E}[c(B_{t \leq \tau}, \tau)].$$

For instance, the “cost function” $c(B_{t \leq \tau}, \tau) = \tau^2$ gives rise to the Root embedding. In this particular case, the optimal stopping time is given by a hitting time of a barrier. We want to think about the stopping time as being a coupling between the Wiener measure \mathbb{W} and the measure λ . Therefore, we interpret a stopping time τ as a measure

$$\mu(d\omega, dt) := \delta_{\tau(\omega)}(dt)\mathbb{W}(d\omega)$$

on $C(\mathbb{R}_+) \times \mathbb{R}_+$. This corresponds to a Monge solution of the minimization problem. Then it is natural to also consider *randomized stopping times* μ corresponding to Kantorovich solutions. They are defined to be measures on $C(\mathbb{R}_+) \times \mathbb{R}_+$ admitting a disintegration $(\mu_\omega)_\omega$ with respect to the Wiener measure such that

- (1) $\mu_\omega \in \mathcal{P}(\mathbb{R}_+)$ almost surely and
- (2) $\omega \mapsto \mu_\omega(A)$ is \mathcal{F}_t measurable for all $A \in \mathcal{B}([0, t])$ and $t \geq 0$.

The advantage is that the set of all randomized stopping times embedding λ is compact. If the functional

$$\mu \mapsto \mathbb{E}_\mu[c(B_{s \leq t}, t)] = \int c(\omega_{s \leq t}, t) \mu(d\omega, dt)$$

is lower semicontinuous the compactness ensures the existence of an optimal solution μ_o . To get hands on the optimizer we derive a dual problem and a variant of cyclical monotonicity which allows us to get geometrical information on the support of the optimizer. In the special case of the above mentioned cost function $c(B_{s \leq t}, t) = t^2$ the monotonicity ensures that there no two paths f, g stopped at time s respectively $t > s$, such that there is a time $s < u < t$ where g crosses the level of $f(s)$, i.e. $g(u) = f(s)$. The reason is the convexity of t^2 . If two such paths existed it would be cheaper to stop g at time u and “transfer” the piece $g_{\lfloor u, t \rfloor}$ to the stopped stub $f_{\lfloor 0, s \rfloor}$ to get a path $\tilde{f} : [0, s + t - u] \rightarrow \mathbb{R}$. This allows us to define a barrier

$$\mathcal{R} := \{(x, t) : \exists (f, s) \in \text{supp}(\mu_o), f(s) = x, t \geq s\}.$$

Using the monotonicity again one sees that μ_o coincides with the hitting time of \mathcal{R} . As being the hitting time of a barrier is a non-convex property this directly yields uniqueness of the solution. Hence, this could be interpreted as a Brenier result for the Skorokhod problem. By a similar reasoning we recover the Rost

and Azema Yor solutions to the Skorokhod embedding problem. Moreover, this approach works without any changes in any dimension, thereby extending the Root and Rost embedding to arbitrary finite dimensions.

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Sobolev Inequality and Quenched invariance principle for Diffusions in Periodic Potential

MOUSTAPHA BA

(joint work with Pierre Mathieu)

We prove here, using stochastic analysis methods; the invariance principle for a \mathbb{R}^d -diffusions $d \geq 2$; involving in periodic potential beyond uniform boundedness assumptions. The potential is not assumed to have any regularity. So the stochastic calculus theory for processes associated to Dirichlet forms used to justify the existence and uniqueness of this process starting for almost all $x \in \mathbb{R}^d$; allows us to show an invariance principle for almost all starting point (individual invariance principle). For that, we show then one Sobolev inequality to bound the probability of transition associated to the time changed diffusion for times large enough and deduce the existence of one bounded density. This property allows us to prove the tightness of the sequence of processes in the uniform topology; the proof of the convergence in finite dimensional distribution is very standard: construction of corrector and central limit theorem for martingale with continuous time. The approach used here is the same as in Mathieu 2008: the notion of time changed process by an additive functional.

Upper escape rate of random walks

XUEPING HUANG

(joint work with Yuichi Shiozawa)

The study of escape rate of Markov processes originated from the celebrated Khinchin's law. For the Brownian motion on a Riemannian manifold, an effective upper escape rate function has been obtained in the work of Grigor'yan-Hsu, and Hsu-Qin in terms of volume growth.

We prove an analogous upper escape rate function for the continuous time random walk on a locally finite weighted graph. The discrete setting causes serious technical obstructions in adapting the a priori estimate methods. Inspired by the recent work of Folz on stochastic completeness, we overcome this difficulty by modifying the original graph to one that is more amenable for estimates and then transfer the result back. A large deviation type result is proven along the way to relate the upper rate function of the modified graph to that of the original one.

Parabolic Harnack inequality for the random conductance model in a degenerate environment

MARTIN SLOWIK

(joint work with Sebastian Andres, Jean-Dominique Deuschel)

Consider the d -dimensional Euclidean lattice (\mathbb{Z}^d, E^d) for $d \geq 2$. We study the nearest-neighbor (random) conductance model, that is a reversible continuous-time Markov process $\{X_t : t \geq 0\}$ on \mathbb{Z}^d with generator, L , which acts on bounded functions $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ as

$$(L^\pi f)(x) = \sum_{y \sim x} \frac{\omega(\{x, y\})}{\pi(x)} (f(y) - f(x)),$$

where $\omega = \{\omega(e) \in (0, \infty) : e \in E^d\}$ is a family of positive edge weights and $\pi : \mathbb{Z}^d \rightarrow (0, \infty)$ is a speed measure. In this talk, we give a characterization of the parabolic Harnack inequality by summability conditions of ω and π on large balls. The proof is based on an application of Moser's iteration method. Parabolic Harnack inequalities for reversible Markov chains on graphs have been established so far under the assumption of uniform ellipticity of the weights. In this respect, we generalize the results obtained in [2].

Of particular interest is the case that the conductances ω are itself random. We assume that the law of the conductances in such a situation is ergodic with respect to space shifts. One application of the Harnack inequality is to prove the Hölder regularity of the transition density. Based on arguments given in [1], the Hölder regularity is a key ingredient to establish a quenched local limit theorem.

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Martingales on manifolds with time-dependent connection

ROBERT PHILIPOWSKI

(joint work with Hongxin Guo, Anton Thalmaier)

Let M be a d -dimensional differentiable manifold, $\pi : \mathcal{F}(M) \rightarrow M$ its frame bundle and $(\nabla(t))_{t \geq 0}$ a family of linear connections on M depending smoothly on t . Let $X = (X_t)_{t \geq 0}$ be an M -valued continuous semimartingale. We say that an $\mathcal{F}(M)$ -valued semimartingale $U = (U_t)_{t \geq 0}$ is a $(\nabla(t))_{t \geq 0}$ -horizontal lift of X if $\pi X = U$

and there is an \mathbb{R}^d -valued semimartingale $Z = (Z_t)_{t \geq 0}$ such that

$$dU_t = \sum_{i=1}^d H_i^{\nabla(t)}(U_t) \circ dZ_t^i,$$

here $H_1^{\nabla(t)}, \dots, H_d^{\nabla(t)}$ are the standard horizontal vector fields with respect to the connection $\nabla(t)$. U and Z are uniquely determined by X and the choice of an initial frame $U_0 \in \mathcal{F}(M)$, and we call Z the $(\nabla(t))_{t \geq 0}$ -antidevelopment of X . As in the case of a fixed connection we have the following result:

Theorem 1. *The following conditions are equivalent:*

- (1) *The $(\nabla(t))_{t \geq 0}$ -antidevelopment of X is an \mathbb{R}^d -valued local martingale.*
- (2) *For all smooth functions $f : M \rightarrow \mathbb{R}$ the process*

$$f(X_t) - f(X_0) - \frac{1}{2} \int_0^t \text{Hess}^{\nabla(s)} f(dX_s, dX_s)$$

is a real-valued local martingale.

Definition 2. We call X a $(\nabla(t))_{t \geq 0}$ -martingale if these equivalent conditions are satisfied.

The importance of this notion of martingale lies in its relation with the non-linear heat equation:

Theorem 3. *Let N be another differentiable manifold and $T_1 < T_2$. Let $(g(t))_{T_1 \leq t \leq T_2}$ be a smooth family of Riemannian metrics on M and $(\tilde{\nabla}(t))_{T_1 \leq t \leq T_2}$ a smooth family of connections on N . Then a smooth function $u : [T_1, T_2] \times M \rightarrow N$ is a solution of the non-linear heat equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta^{g(t), \tilde{\nabla}(t)} u$$

if and only if for any M -valued $(g(T_2 - t))_{0 \leq t \leq T_2 - T_1}$ -valued Brownian motion $(X_t)_{0 \leq t \leq T_2 - T_1}$ the N -valued process $(u(T_2 - t, X_t))_{0 \leq t \leq T_2 - T_1}$ is a $(\tilde{\nabla}(t))_{0 \leq t \leq T_2 - T_1}$ -martingale.

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Dimensional contraction via Markov transportation distance

ARNAUD GUILLIN

(joint work with François Bolley, Ivan Gentil)

Contraction properties of (Markovian) semigroups have always been an important probabilistic or analyst tool, as it enables to study for the example the existence of invariant probability measures, and long time behaviour of solutions of various linear (Fokker-Planck, kinetic Fokker-Planck, ...) or non linear (McKean-Vlasov, porous medium...) partial differential equations. One important aspect is of course the distance in which we measure this contraction. If (weighted) total variation has often been the first choice of probabilists, L^2 , Sobolev or Fourier norms were usually picked by PDE specialist ([CT07]). However, recent progress has shown that the Kantorovitch-Rubinstein-Wasserstein distance (Wasserstein distance in short) is a particular relevant choice (see for example [Ott01, CMV06, BGG12] and reference book [Vil09]). The Wasserstein distance between two Borel probability measures ν and μ on a Polish metric space (E, d) is defined by $W_2^2(\mu, \nu) = \inf \int d^2(x, y) d\pi(x, y)$, where the infimum runs over all probability measures π in $E \times E$ with marginals μ and ν . The space of probability measures with finite second moment, equipped with the Wasserstein distance, has been the subject and the starting point of many works. We refer to the books [AGS08, Vil03, Vil09] for the main background on this issue and its interplay with the optimal transportation problem.

Geometric properties of metric spaces are an important and vast issue with many different problems, and a particularly relevant one is the notion of curvature which has recently attracted a lot of attention [Vil09, Stu06b, Stu06a, AGS08]. The link between the Wasserstein distance and the curvature can be (very roughly) described as follows. Let $(P_t)_{t \geq 0}$ denote the heat semigroup on a smooth and complete (and connex) Riemannian manifold (M, g) : it solves the heat equation $\partial_t u = \Delta_g u$ where Δ_g is the Laplace-Beltrami operator on M . Let also μ be the Riemannian measure on (M, g) and d the associated Riemannian distance. Then a fundamental result, due to K.-T. Sturm and M. von Renesse in [SvR05], says that the Ricci curvature of the manifold is bounded from below by a constant $R \in \mathbb{R}$ if and only if

$$(1) \quad W_2(P_t f \mu, P_t g \mu) \leq e^{-Rt} W_2(f \mu, g \mu),$$

for any $t \geq 0$ and any probability density f, g with respect to μ . This result is important since it gives a bridge between curvature and transportation distance.

A crucial challenging problem consists in understanding the role of the dimension in the contraction property in Wasserstein distance. This has been recently performed in the following two remarkable results by finding an upper bound of the distance $W_2(P_t f \mu, P_s g \mu)$ with two *different* times $s, t > 0$.

- The first result is due to K. Kuwada in [Kuw13] : the Ricci curvature of the n -dimensional manifold M is bounded from below by a constant $R \in \mathbb{R}$ is and

only if

$$(2) \quad W_2^2(P_t f \mu, P_s g \mu) \leq A(s, t, R) W_2^2(f \mu, g \mu) + B(s, t, n, R),$$

for any $s, t > 0$ and any probability density f, g with respect to the measure μ , for appropriate functions $A, B \geq 0$.

• The second result is due to M. Erbar, K. Kuwada and K.-T. Sturm [EKS13] : the Ricci curvature of the n -dimensional manifold M is bounded from below by a constant $R \in \mathbb{R}$ if and only if

$$s_{\frac{R}{n}} \left(\frac{1}{2} W_2(P_t f \mu, P_s g \mu) \right)^2 \leq e^{-R(t+s)} s_{\frac{R}{n}} \left(\frac{1}{2} W_2(f \mu, g \mu) \right)^2 + \frac{n}{R} (1 - e^{-R(s+t)}) \frac{(\sqrt{t} - \sqrt{s})^2}{2(t+s)}$$

for any $s, t > 0$ and any probability density f, g with respect to the measure μ ; here $s_r(x) = \sin(\sqrt{r}x)/\sqrt{r}$ if $r > 0$ and $s_r(t) = \sinh(\sqrt{-r}x)/\sqrt{-r}$ if $r < 0$.

In these two results the dimension n appears only when the Wasserstein distance is applied to different times s and t . When $t = s$ in (2) then the positive additional terms vanish and we only recover the classical contraction inequality (1).

The aim of this paper is to take the dimension into account and to improve inequality (1) for solutions considered at the *same* time. For instance we prove that the usual Laplacian operator in \mathbb{R}^n satisfies

$$(3) \quad W_2^2(P_T f dx, P_T g dx) \leq W_2^2(f dx, g dx) - \frac{2}{n} \int_0^T (\text{Ent}_{dx}(P_t f) - \text{Ent}_{dx}(P_t g))^2 dt,$$

for any $T \geq 0$ and any probability density f, g with respect to the Lebesgue measure dx ; here $\text{Ent}_{\mu}(g) = \int g \log g d\mu$ is the entropy. This inequality is stronger than (1) since \mathbb{R}^n associated with the Euclidean distance has a null Ricci curvature and then satisfies (1) with $R = 0$.

An other important step in understanding curvature is the link between the synthetic curvature notion of Lott-Sturm-Villani [LV09, Stu06b, Stu06a], contraction property, gradient/commutation type property and Bakry-Emery curvature condition (see the recent [AGS11, AGS12, EKS13]). Indeed, as is well known, the condition on the lower bound of the Ricci curvature is related to the so-called curvature-dimension criterion proposed by D. Bakry and M. Émery in [BÉ85]. In particular the curvature-dimension $CD(R, n)$ for $R \in \mathbb{R}$ and $n \geq 1$ is satisfied for the Laplace-Beltrami operator on a Riemannian manifold if and only if the Ricci curvature of the manifold is uniformly bounded from below by R and the dimension of the manifold is larger than n . The curvature-dimension criterion reads as follows $\Gamma_2(f) \geq R\Gamma(f) + \frac{1}{n}(Lf)^2$.

To reach a dimensional contraction inequality we thus have some large choice of curvature notions. We will work with a new distance based on the evolution equation, and adapted to it. This new distance, called *Markov transportation distance*, will be more adapted to the Bakry-Emery curvature-dimension condition formulation and to the Γ_2 operator. Basically, it is defined with the dynamical approach proposed by J.-D. Benamou and Y. Brenier in [BB00] : for any probability

density f and g with respect to the Lebesgue measure in \mathbb{R}^n ,

$$W_2^2(fdx, gdx) = \inf \int_0^1 \int \frac{|w_s|^2}{\rho_s} dx ds,$$

where the infimum runs over all path $(\rho_s)_{s \in [0,1]}$ such that $\partial_s \rho_s + \nabla \cdot w_s = 0$ ($\nabla \cdot$ stand for the usual divergence operator) for a vector field $(w_s)_{s \in [0,1]}$ and satisfying $\rho_0 = f$ and $\rho_1 = g$. This dynamical approaches is the starting point of the generalization to other distances in [DNS09]. Following [BB00], we define the Markov transportation distance by

$$T_2^2(f\mu, g\mu) = \inf \int_0^1 \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds,$$

where the infimum runs over all path $(\rho_s)_{s \in [0,1]}$ such that $\partial_s \rho_s + Lh_s = 0$, satisfying $\rho_0 = f$ and $\rho_1 = g$. In this abstract formulation, the discrete or non-local operator can be studied in a similar way. A fundamental instance is that of $L = \Delta - \nabla V \cdot \nabla$ on \mathbb{R}^n , with carré du champ $\Gamma(f) = |\nabla f|^2$ and invariant measure $\mu = e^{-V}$: in this case $W_2(f\mu, g\mu) \leq T_2(f\mu, g\mu)$ since the infimum defining the distances runs over a smaller set for T_2 than for W_2 .

One of the main goal is to show in a simple way how to reach dimension dependent contraction property in the Wasserstein distance for the specific heat semigroup on \mathbb{R}^n (inequality (3)). We prove that under the curvature-dimension condition $CD(R, n)$, for any $T > 0$,

$$T_2^2(P_T f\mu, P_T g\mu) \leq e^{-2RT} T_2^2(f\mu, g\mu) - \frac{2}{n} \int_0^T e^{-2R(T-t)} (\text{Ent}_\mu(P_t g) - \text{Ent}_\mu(P_t f))^2 dt,$$

for any probability density f, g with respect to the measure μ .

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Some new relations between $\mathbf{CD}^*(K, N)$, the dimensional Bakry-Emery condition and the EVI property of gradient flows

ANDREA MONDINO

(joint work with Luigi Ambrosio, Giuseppe Savaré)

Given a smooth Riemannian manifold (M, g) , the celebrated Bochner identity states that for every $f \in C^\infty(M)$ one has

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathit{Hess} f|^2 + \mathit{Ric}(\nabla f, \nabla f) + g(\nabla\Delta f, \nabla f),$$

where Δ is the Laplace-Beltrami operator and Ric is the Ricci tensor. If $\dim(M) \leq N$ and $\mathit{Ric} \geq Kg$ then the identity above give the so called *dimensional Bochner inequality*, also referred to as *dimensional Bakry-Émery condition* $BE(K, N)$

$$(1) \quad \frac{1}{2}\Delta|\nabla f|^2 \geq \frac{1}{N}|\Delta f|^2 + K|\nabla f|^2 + g(\nabla\Delta f, \nabla f).$$

Recall that the dimensional Bochner inequality is a fundamental tool in the study of Riemannian manifolds with Ricci lower bounds (Splitting theorem of Cheeger-Gromoll, Lichnerowicz bound on spectral gap, bounds on the topological complexity by Gallot-Gromov and Anderson, etc.).

The goal of this work is to establish a dimensional Bochner inequality in a non smooth setting; a special role is played by some non linear gradient flow. Let us introduce the framework.

$(X, \mathbf{d}, \mathbf{m})$ indicates a metric measure space, m.m.s. for short; i.e., (X, \mathbf{d}) is a complete and separable metric space (possibly non compact) and \mathbf{m} is a probability measure on it (in the setting of smooth Riemannian manifolds \mathbf{m} corresponds to the volume measure multiplied by a suitable Gaussian and \mathbf{d} is the usual Riemannian distance).

In this framework, using tools of optimal transportation, Lott-Villani [13] and Sturm [15]-[16] detected the class of the so called $\text{CD}(K, N)$ -spaces having Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1, \infty]$; this notion is compatible with the classical one in the smooth setting (i.e., a Riemannian manifold has dimension less or equal to N and Ricci curvature greater or equal to K if and only if it is a $\text{CD}(K, N)$ -space), it is stable under measured Gromov-Hausdorff convergence, and it implies fundamental properties as the Bishop-Gromov volume growth, Bonnet-Myers diameter bound, the Lichnerowicz spectral gap, the Brunn-Minkowski inequality, etc.

On the other hand, some basic properties like the local-to-global and the tensorization are not clear for the $\text{CD}(K, N)$ condition. In order to remedy to this inconvenient, Bacher-Sturm [7] introduced a (a priori) weaker notion of curvature called *reduced curvature condition*, and denoted with $\text{CD}^*(K, N)$, which satisfies the aforementioned missing properties and share the same nice geometric features of $\text{CD}(K, N)$ (but some of the inequalities may not have the optimal constant; for more details the relations between $\text{CD}(K, N)$ and $\text{CD}^*(K, N)$ see [9] and [8]).

As a matter of facts, both the $\text{CD}(K, N)$ and $\text{CD}^*(K, N)$ conditions include Finsler geometries [14]-[17]. In order to isolate the Riemannian-like structures, Ambrosio-Gigli-Savaré [4] (see also [1] for a simplification of the axiomatization and the extension to σ -finite measures) introduced the class of $\text{RCD}(K, \infty)$ -spaces. Such notion strengthens the $\text{CD}(K, \infty)$ condition with the linearity of the heat flow (for the definition of heat flow in the non smooth setting see below; notice that on a smooth Finsler manifold, the linearity of the Heat flow is equivalent to say that the manifold is, in fact, Riemannian); as proved in [12], the $\text{RCD}(K, \infty)$ condition is also stable under measured Gromov-Hausdorff convergence.

In order to define the Heat flow, first of all recall that on a m.m.s. (X, d, \mathbf{m}) we cannot speak of differential (or gradient) of a function f but at least the modulus of the differential is \mathbf{m} -a.e. well defined, it is called weak upper differential and it is denoted with $|Df|_w$ (see [3]). With this object one defines the Cheeger energy of a measurable function $f : X \rightarrow \mathbb{R}$ as

$$\text{Ch}(f) := \frac{1}{2} \int_X |Df|_w^2 \, d\mathbf{m}$$

if $|Df|_w \in L^2(X, \mathbf{m})$, and $+\infty$ otherwise. Since Ch is convex and lowersemi-continuous on $L^2(X, \mathbf{m})$, one can apply the classical theory of gradient flows of convex functionals in Hilbert spaces [2] and define the heat flow \mathbf{H}_t as the unique L^2 -gradient flow of Ch . The infinitesimal generator of this semigroup is called Laplacian and it is denoted with Δ . Let us remark that in general Δ is *not* a linear operator, and it is linear if and only if the Heat flow \mathbf{H}_t is linear.

In order to keep track of all the three conditions above (lower bound on the Ricci curvature, upper bound on the dimension and Riemannian-like behaviour), we introduce the class $\text{RCD}^*(K, N)$. Such class consists of those m.m. spaces which satisfy the $\text{CD}^*(K, N)$ condition and have linear Heat flow. Also the $\text{RCD}^*(K, N)$ condition is stable under measured Gromov-Hausdorff convergence, so that limit

spaces of Riemannian manifolds with Ricci curvature bounded below by K and dimension bounded above by N are $\text{RCD}^*(K, N)$ -spaces.

One of the main achievements of the work [6] we present at the workshop is that the $\text{RCD}^*(K, N)$ condition is equivalent to the dimensional Bochner inequality (1) (properly understood in a weak sense).

More precisely we prove that, under the assumption of linearity of the Heat flow (also called infinitesimal Hilbertianity, cf. [11]), the space $(X, \mathbf{d}, \mathbf{m})$ is $\text{CD}^*(K, N)$ if only if the *nonlinear diffusion equation*

$$\partial_t \rho_t = \Delta P(\rho)$$

solves a suitable Evolution Variational Inequality (adapted to take into account the effect of the curvature and of the nonlinearity) for the gradient flow of the corresponding entropy functional $\int_X U(\rho) \mathbf{d}\mathbf{m}$, in the metric space of the probability measures on X endowed with the L^2 -Kantorovich-Wasserstein transportation distance. Here, $U \in C^1(0, \infty)$ and $P \in C^1[0, \infty)$ are related by

$$P(\rho) = \rho U'(\rho) - U(\rho)$$

and P satisfies the McCann displacement convexity condition

$$\rho P'(\rho) - P(\rho) \geq -\frac{1}{N} P(\rho) \quad .$$

By relaxing a bit the regularity assumptions on U and P (this is possible, for instance, when $\mathbf{m}(X) < \infty$) this class of entropy functionals includes in particular the N -Renyi entropy $-N\rho^{1-1/N}$ corresponding to the fast-diffusion equation $\partial_t \rho = \Delta \rho^{1-1/N}$. Finally, we prove that this characterization of the $\text{CD}^*(K, N)$ condition in terms of the Evolution Variational Inequality is equivalent to the dimensional Bochner inequality (1) (always under the infinitesimal Hilbertianity assumption).

Let us mention that some of the above results have been obtained independently (and slightly before us) with different techniques by Erbar-Kuwada-Sturm [10].

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Stochastic variational principles and FBSde’s on Lie groups

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(joint work with Marc Arnaudon, Xin Chen)

INTRODUCTION

There are two possible methods of generalizing Geometric Analysis to a stochastic framework: one, inspired by early work of J. M. Bismut, consists in considering systems with randomly perturbed Lagrangians. The other, more inspired in Feynman’s path integral approach, starts from classical Lagrangians but computed on stochastic paths. In particular here the position is random but the velocity, taken as an average over all possible stochastic paths, is deterministic. We develop the second approach in the context of Lie groups.

1. SEMI-MARTINGALES IN A LIE GROUP

Let G be a Lie group endowed with a left (*right*) invariant metric and a left (*right*) invariant connection, e its identity element and $\mathcal{G} \simeq T_e(G)$ the corresponding Lie algebra. For a vector field $u(\cdot) \in C([0, T]; \mathcal{G})$ we consider the semi-martingale

$$dg(t) = T_e L_{g(t)} \left(\sum_k H_k \circ dW_t^k - \frac{1}{2} \sum_i \nabla_{H_k} H_k dt + u(t) dt \right), \quad g(0) = e$$

where $H_k \in \mathcal{G}$ (not necessarily orthogonal or even a basis), $T_a L_{g(t)} : T_a G \rightarrow T_{g(t)a} G$ is the differential of the left translation $L_{g(t)}(x) := g(t)x$, $\forall x \in G$ at the point $x = a \in G$, W_t^k i.i.d. \mathbb{R} valued Brownian motions.

The Itô contraction term is given by $\frac{1}{2} \sum_k \nabla_{H_k} H_k$.

The derivative in time of a G -valued semi-martingale ξ , $\xi(0) = x$, is defined in the following way: if $\eta(t) := \int_0^t t_{0 \leftarrow s}^\nabla \circ d\xi(s)$, where $t_{0 \leftarrow s}^\nabla$ denotes Itô parallel transport associated with the connection ∇ , we define

$$D_t^\nabla \xi(t) := t_{t \leftarrow 0}^\nabla D_t \eta$$

with

$$D_t \eta = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} E_t[\eta_{t+\epsilon} - \eta_t]$$

In particular we have $D_t^\nabla g(t) = T_e L_{g(t)} u(t)$. This notion of derivative allows to consider the action functional

$$S^{\nabla, \langle \rangle}(\xi(\cdot)) = \frac{1}{2} E \left[\int_0^T L(\xi(\cdot), T_{\xi(t)} L_{\xi(t)^{-1}} D_t^\nabla \xi(t)) dt \right]$$

For the variational principle we consider the right (*left*) derivatives,

$$\partial_R S^{\nabla, \langle \rangle}(\xi(\cdot)) = \frac{d}{d\epsilon} \Big|_{\epsilon=0} S^{\nabla, \langle \rangle}(\xi_{\epsilon, v}(\cdot))$$

with $\xi_{\epsilon, v}(t) = \xi(t)e_{\epsilon, v}(t)$, $e_{\epsilon, v}(t) = e + \epsilon \int_0^t T_e L_{e_{\epsilon, v}(s)} \dot{v}(s) ds$ for $v \in C^1([0, T]; \mathcal{G})$ such that $v(0) = v(T) = 0$.

2. STOCHASTIC EULER-POINCARÉ REDUCTION THEOREM

We have the following result, that generalizes to the stochastic framework the well known Euler-Poincaré reduction theorem of Geometric Mechanics (case where $H_k = 0$).

Theorem 1. *The G -valued semi-martingale $g(\cdot)$ defined above is a critical point of $S^{\nabla, \langle \rangle}$ if and only if the non-random $u(\cdot)$ satisfies the equation,*

$$\frac{d}{dt} u(t) = ad_{\tilde{u}(t)}^* u(t) + K(u(t)),$$

where

$$\tilde{u}(t) := u(t) - \frac{1}{2} \sum_k \nabla_{H_k} H_k,$$

for $u \in \mathcal{G}$, $ad_u^* : \mathcal{G} \rightarrow \mathcal{G}$ is the adjoint of ad_u with respect to the metric $\langle \rangle$, and the operator $K : \mathcal{G} \rightarrow \mathcal{G}$ is defined as

$$\langle K(u), v \rangle = - \langle u, \frac{1}{2} \sum_k (\nabla_{ad_v H_k} H_k + \nabla_{H_k} (ad_v H_k)) \rangle, \quad \forall u, v \in \mathcal{G}$$

(in the case of right-invariant metric, the signs in the r.h.s. of the equation for u are changed).

An important particular case of the theorem above is when ∇ is the (right invariant) Levi-Civita connection with respect to $\langle \rangle$ and $\nabla_{H_k} H_k = 0$ for each k . Then K is de Rham-Hodge Laplacian,

$$K(u) = -\frac{1}{2} \sum_k (\nabla_{H_k} \nabla_{H_k} u + R(u, H_k) H_k) \quad \forall u \in \mathcal{G}$$

3. APPLICATIONS

We focus in the application to the Navier-Stokes equations on the two dimensional torus \mathbb{T}^2 . To obtain these we consider the (right invariant) Lie group to be $G = G_V^s := \{g := \mathbb{T}^2 \rightarrow \mathbb{T}^2 \text{ is a volume preserving bijection map, } g, g^{-1} \in H^s\}$, where H^s is the s -th order Sobolev space ($s > 2$). (Note that this is not quite a Lie group since left translation is not smooth but the results still hold).

The Lie algebra is

$$g_V^s = \{X : H^s(\mathbb{T}^2; T\mathbb{T}^2), \pi(X) = e, \operatorname{div} X = 0\}$$

the inner product

$$\langle X, Y \rangle^0 := \int_{\mathbb{T}^2} \langle X(x), Y(x) \rangle_x dx, \quad X, Y \in g_V^s$$

and the (right invariant) Levi-Civita connection $\nabla_X^0 Y = P_e(\nabla_X Y)$ where P_e denotes the orthogonal (L^2) projection onto the divergence free part in the Hodge decomposition of $H^s(T\mathbb{T}^2)$. We consider the vector fields

$$A_k(\theta) = \lambda(|k|)(k_2, -k_1)\cos(k \cdot \theta) \quad B_k(\theta) = \lambda(|k|)(k_2, -k_1)\sin(k \cdot \theta)$$

where $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$, $k = (k_1, k_2) \in \mathbb{Z}^2$, $k \cdot \theta = k_1\theta_1 + k_2\theta_2$ and $\lambda(|k|)$ verifies certain growth conditions ensuring existence of the corresponding Brownian motion g . We have $\nabla_{A_k}^0 A_k = 0$, $\nabla_{B_k}^0 B_k = 0 \forall k$, and the s.d.e. becomes

$$dg(t, \theta) = \sum_k (A_k(g(t, \theta)) \circ dW_t^{k,1} + B_k(g(t, \theta)) \circ dW_t^{k,2}) + u(t, g(t, \theta))dt, \quad g(0, \theta) = \theta$$

The stochastic Euler-Poincaré theorem then reads:

The semi-martingale g is a critical point of the action functional $S^{\nabla^0, \langle \cdot \rangle^0}$ if and only if u satisfies the Navier-Stokes equation,

$$\frac{\partial u}{\partial t} = -u \cdot \nabla u + \frac{\nu}{2} \Delta u - \nabla p(t), \quad \operatorname{div} u = 0.$$

4. ASSOCIATED BACKWARD SDES

Consider the case where $\{H_k\}$ is an o.n. basis of the Lie algebra, ∇ Levi-Civita connection, $\nabla_{H_k} H_k = 0$.

Denote $\bar{u}(t, \cdot) = -u(T - t, \cdot)$;

$$\frac{d}{dt} \bar{u}(t) = ad_{\bar{u}(t)}^* \bar{u}(t) - K(\bar{u}(t))$$

Denoting \mathcal{D}_t the stochastic covariant derivative along g associated with ∇ and writting $\Gamma(u, v) = T_e L_g(\nabla_{T_g L_{g^{-1}} u} T_g L_{g^{-1}} v)$, $u, v \in T_g G$, we have the following characterization of the equations of motion as solutions of a forward-backward stochastic differential equation:

Theorem 2. *If a vector field $\bar{u} \in C^1([0, T]; T_e G)$ solves the equation $\frac{d}{dt} \bar{u}(t) = ad_{\bar{u}(t)}^* \bar{u}(t) - K(\bar{u}(t))$ with final condition $\bar{u}(T) = u_T$ then*

$$(g(t), X(t)) := T_e L_{g(t)} \bar{u}(t), \quad Z_k(t) := T_e L_{g(t)} \nabla_{H_k} \bar{u}(t)$$

solves the FB stochastic system:

$$\begin{aligned} dg(t) &= \sum_k T_e L_{g(t)} H_k \circ dW_t^k \\ \mathcal{D}_t X(t) &= \sum_k Z_k(t) dW_t^k - \frac{1}{2} Ric(X(t)) dt \\ &\quad + \sum_k [(\langle X(t), Z_k(t) + \Gamma(X(t), \tilde{H}_k(g(t))) \rangle)] \tilde{H}_k(g(t)) dt \\ g(0) &= e, \quad X(T) = T_e L_{g(T)} u_T \end{aligned}$$

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Extreme points of the two dimensional discrete Gaussian Free Field

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(joint work with Oren Louidor)

The Discrete Gaussian Free Field (DGFF) is a family of zero-mean Gaussian random variables $\{h_x : x \in V_N\}$ indexed by the vertices of a box V_N of side N in, say, the integer lattice \mathbb{Z}^d . The covariance $\text{Cov}(h_x, h_y)$ is given by the Green function $G_N(x, y)$ of the simple symmetric random walk on \mathbb{Z}^d that is killed upon exiting from V_N . The two-dimensional case, $d = 2$, is particularly interesting because there the Green function admits the scaling relations

$$(1) \quad G_N(x, x) = g \log N + O(1) \quad \text{and} \quad G_N(x, y) = g \log \frac{N}{|x - y|} + O(1)$$

for all $x, y \in V_N$ sufficiently away from V_N^c with $|x - y| \gg 1$. Here $g := 2/\pi$ (although $g := (2\pi)^{-1}$ has been used as well). As a consequence, the two-dimensional DGFF exhibits numerous scale-invariant properties.

In the work reported on here, we are concerned with the behavior of extreme points of the two-dimensional DGFF. We will focus on the extreme local maxima — i.e., the tips of large “peaks” — in typical field configurations $\{h_x : x \in V_N\}$. These are captured by the empirical point measure

$$(2) \quad \eta_{N,r}(A \times B) := \sum_{x \in V_N} 1_{\{x/N \in A\}} 1_{\{h_x - m_N \in B\}} 1_{\{h_x = \max_{z \in \Lambda_r(x)} h_z\}},$$

on $[0, 1]^2 \times \mathbb{R}$, for a suitable centering sequence m_N . The restriction to “high peaks” comes in the form of the restriction to points that maximize the configuration in

an r -neighborhood thereof. Let $\text{PPP}(\lambda)$ stand for the Poisson point process with intensity measure λ (which may itself be random). For the sequence

$$(3) \quad m_N := 2\sqrt{g} \log N - \frac{3}{4}\sqrt{g} \log \log N,$$

we then state and prove:

Theorem 1. *There is a random measure $Z(dx)$ on $[0, 1]^2$ with $Z([0, 1]^2) < \infty$ a.s. and $Z(A) > 0$ a.s. for any open set $A \subset [0, 1]^2$ such that for any r_N with $r_N \rightarrow \infty$ and $r_N/N \rightarrow 0$,*

$$(4) \quad \eta_{N,r_N} \xrightarrow[N \rightarrow \infty]{} \text{PPP}(Z(dx) \otimes e^{-\alpha h} dh),$$

where $\alpha := 2/\sqrt{g}$ — which in present normalization reads $\alpha = \sqrt{2\pi}$.

The proof splits into two rather disjoint steps: First, since we know from earlier work (specifically, Ding and Zeitouni [3]) that $\{\eta_{N,r_N} : N \geq 1\}$ is tight, we extract a subsequential limit η and prove the following identity

$$(5) \quad E(e^{-\langle \eta, f \rangle}) = E(e^{-\langle \eta, f_t \rangle})$$

for any $f: [0, 1]^2 \times \mathbb{R} \rightarrow \mathbb{R}$ that is continuous with compact support, where

$$(6) \quad f_t(x, h) = -\log E^0(e^{-f(x, h + W_t - \frac{\alpha}{2}t)}),$$

with E^0 denoting expectation with respect to the standard Brownian motion W_t . The relation (5) can be interpreted as invariance of the law of η under “Dysonization” of the sample points by independent copies of the process $W_t - \frac{\alpha}{2}t$. Liggett [4] then implies that η takes the form of a Poisson process in (4).

The second step of the proof is an argument for uniqueness of the limit process, and thus uniqueness of distribution of the random measure $Z(dx)$. This is achieved by enhancing, only slightly, the main results of Bramson, Ding and Zeitouni [2]. To illustrate this step, note that (4) yields

$$(7) \quad P\left(\max_{x \in V_N} h_x - m_N \leq t\right) \xrightarrow[N \rightarrow \infty]{} E(e^{-\alpha^{-1} Z([0,1]^2) e^{-\alpha t}}).$$

So convergence in law for the (centered) absolute maximum of the field (which is what [2] provides for us) determines the Laplace transform of the total mass of the Z -measure, and thus its law as well.

The random Z -measure exhibits hosts of interesting properties including a version of conformal invariance and interpretations via multiplicative Gaussian cascades. We also show that Z is a.s. non-atomic and has a form associated with the so called derivative martingale for the Continuum GFF. The extreme points of DGFF thus exhibit a structure similar to that found in the context of Branching Brownian Motion and Branching Random Walk. We refer to recent preprint [1] for more details and full discussion of earlier and related work.

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Sharp Isoperimetric Inequalities and Sobolev Inequalities under Curvature-Dimension-Diameter Condition

EMANUEL MILMAN

Let (M^n, g) denote an n -dimensional ($n \geq 2$) complete oriented smooth Riemannian manifold, and let μ denote a probability measure on M having density Ψ with respect to the Riemannian volume form vol_g .

Definition (Generalized Ricci Tensor). Given $q \in [0, \infty]$ and assuming that $\Psi > 0$ and $\log(\Psi) \in C^2$, we denote by $Ric_{g, \Psi, q}$ the following generalized Ricci tensor:

$$(1) \quad Ric_{g, \Psi, q} := Ric_g - \nabla_g^2 \log(\Psi) - \frac{1}{q} \nabla_g \log(\Psi) \otimes \nabla_g \log(\Psi) = Ric_g - q \frac{\nabla_g^2 \Psi^{1/q}}{\Psi^{1/q}} .$$

Note that $Ric_{g, \Psi, \infty} = Ric_g - \nabla_g^2 \log(\Psi)$ and that $Ric_{g, \Psi, 0} = Ric_g$ when Ψ is constant. Here as usual Ric_g denotes the Ricci curvature tensor and ∇_g denotes the Levi-Civita covariant derivative.

Definition (Curvature-Dimension-Diameter Condition). (M^n, g, μ) is said to satisfy the Curvature-Dimension-Diameter Condition $CDD(\rho, n + q, D)$ ($\rho \in \mathbb{R}$, $q \in [0, \infty]$, $D \in (0, \infty]$), if μ is supported on the closure of a geodesically convex domain $\Omega \subset M$ of diameter at most D , having (possibly empty) C^2 boundary, $\mu = \Psi \cdot vol_g|_{\Omega}$ with $\Psi > 0$ on $\overline{\Omega}$ and $\log(\Psi) \in C^2(\overline{\Omega})$, and as 2-tensor fields:

$$Ric_{g, \Psi, q} \geq \rho g \quad \text{on } \Omega .$$

When $\Omega = M$ and $D = +\infty$, the latter definition coincides with the celebrated Bakry–Émery Curvature-Dimension condition $CD(\rho, n + q)$. Indeed, the generalized Ricci tensor incorporates information on curvature and dimension from both the geometry of (M, g) and the measure μ , and so ρ may be thought of as a generalized-curvature lower bound, and $n + q$ as a generalized-dimension upper bound.

Let (Ω, d) denote a separable metric space, and let μ denote a Borel probability measure on (Ω, d) . The Minkowski (exterior) boundary measure $\mu^+(A)$ of a Borel set $A \subset \Omega$ is defined as $\mu^+(A) := \liminf_{\varepsilon \rightarrow 0} \frac{\mu(A_\varepsilon^d) - \mu(A)}{\varepsilon}$, where $A_\varepsilon^d := \{x \in \Omega; \exists y \in A \ d(x, y) < \varepsilon\}$. The isoperimetric profile $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) : [0, 1] \rightarrow$

$\mathbb{R}_+ \cup \{+\infty\}$ is defined as $\mathcal{I}(v) := \inf \{\mu^+(A); \mu(A) = v\}$. In our manifold-with-density setting, we will always assume that the metric d is given by the induced geodesic distance on (M, g) , and write $\mathcal{I} = \mathcal{I}(M, g, \mu)$. When $(\Omega, d) = (\mathbb{R}, |\cdot|)$, we also define $\mathcal{I}^\flat = \mathcal{I}^\flat(\mathbb{R}, |\cdot|, \mu) : [0, 1] \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ by $\mathcal{I}^\flat(v) := \inf \{\mu^+(A); \mu(A) = v, A = (-\infty, \xi) \text{ or } A = (\xi, \infty)\}$.

When $\rho > 0$, sharp isoperimetric inequalities under the $CD(\rho, n + q)$ condition are known and well understood, thanks to the existence of comparison model spaces on which equality is attained. The first such result was obtained by M. Gromov, who identified the n -Sphere as the extremal model space in the constant density case ($q = 0$), thereby extending P. Lévy's isoperimetric inequality on the sphere. The case when $q = +\infty$ was treated by Bakry and Ledoux (and later Morgan), who showed that the corresponding model space is the Real line equipped with a Gaussian density. An extension of these results to $q \in (0, \infty)$ was subsequently obtained by Bayle.

However, in all other cases, none of previously known results (by Croke, Bérard, Besson, Gallot and others) yield *sharp* isoperimetric inequalities for all $v \in (0, 1)$. The purpose of this work is to fill this gap, providing a sharp isoperimetric inequality under the $CDD(\rho, n + q, D)$ condition in the entire range $\rho \in \mathbb{R}$, $q \in [0, \infty]$, $D \in (0, \infty]$ and $v \in (0, 1)$, in a single unified framework. In particular, for each choice of parameters, we identify the *model spaces* which are extremal for the isoperimetric problem. Our results seem new even in the classical constant-density case ($q = 0$) when $\rho \leq 0$ and $D < \infty$ or when $\rho > 0$ and $D < \pi\sqrt{(n-1)/\rho}$.

1. RESULTS

Given $\delta \in \mathbb{R}$, set as usual:

$$s_\delta(t) := \begin{cases} \sin(\sqrt{\delta}t)/\sqrt{\delta} & \delta > 0 \\ t & \delta = 0 \\ \sinh(\sqrt{-\delta}t)/\sqrt{-\delta} & \delta < 0 \end{cases}, \quad c_\delta(t) := \begin{cases} \cos(\sqrt{\delta}t) & \delta > 0 \\ 1 & \delta = 0 \\ \cosh(\sqrt{-\delta}t) & \delta < 0 \end{cases}.$$

Given a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(0) \geq 0$, we denote by $f_+ : \mathbb{R} \rightarrow \mathbb{R}_+$ the function coinciding with f between its first non-positive and first positive roots, and vanishing everywhere else.

Definition. Given $H, \rho \in \mathbb{R}$ and $m \in (0, \infty]$, set $\delta := \rho/m$ and define:

$$J_{H, \rho, m}(t) := \begin{cases} (c_\delta(t) + \frac{H}{m}s_\delta(t))_+^m & m \in (0, \infty) \\ \exp(Ht - \frac{\rho}{2}t^2) & m = \infty \end{cases}.$$

Remark. Observe that $J_{H, \rho, m}$ coincides with the solution J to the following second order ODE, on the maximal interval containing the origin where such a solution exists:

$$-(\log J)'' - \frac{1}{m}((\log J)')^2 = -m \frac{(J^{1/m})''}{J^{1/m}} = \rho, \quad J(0) = 1, \quad J'(0) = H.$$

Lastly, given a non-negative integrable function f on a closed interval $L \subset \mathbb{R}$, we denote for short $\mathcal{I}(f, L) := \mathcal{I}(\mathbb{R}, |\cdot|, \mu_{f,L})$, where $\mu_{f,L}$ is the probability measure supported in L with density proportional to f there. Similarly, we set $\mathcal{I}^b(f, L) := \mathcal{I}^b(\mathbb{R}, |\cdot|, \mu_{f,L})$. When $\int_L f(x)dx = 0$ we set $\mathcal{I}^b(f, L) = \mathcal{I}(f, L) \equiv +\infty$, and when $\int_L f(x)dx = +\infty$ we set $\mathcal{I}^b(f, L) = \mathcal{I}(f, L) \equiv 0$.

Theorem 1. Let (M^n, g, μ) satisfy the $CDD(\rho, n+q, D)$ condition with $\rho \in \mathbb{R}$, $q \in [0, \infty]$ and $D \in (0, +\infty]$. Then:

$$(2) \quad \mathcal{I}(M, g, \mu) \geq \inf_{H \in \mathbb{R}, a, b \geq 0, a+b \leq D} \mathcal{I}^b(J_{H, \rho, n+q-1}, [-a, b]) ,$$

where the infimum is interpreted pointwise on $[0, 1]$. In fact, the infimum above is always attained (when $D = \infty$ at $a = b = \infty$), one can always use $b = D - a$, and the \mathcal{I}^b may be replaced by \mathcal{I} , leading to the same lower bound.

The bound (2) was deliberately formulated to cover the entire range of values for ρ , n , q and D simultaneously, indicating its universal character, but it may be easily simplified as follows:

Corollary 2. Under the same assumptions and notation as in Theorem 1, and setting $\delta := \frac{\rho}{n+q-1}$:

Case 1 - $q < \infty$, $\rho > 0$, $D < \pi/\sqrt{\delta}$:

$$\mathcal{I}(M^n, g, \mu) \geq \min_{\xi \in [0, \pi/\sqrt{\delta}-D]} \mathcal{I}^b(\sin(\sqrt{\delta}t)^{n+q-1}, [\xi, \xi + D]) .$$

Case 2 - $q < \infty$, $\rho > 0$, $D \geq \pi/\sqrt{\delta}$:

$$\mathcal{I}(M^n, g, \mu) \geq \mathcal{I}^b(\sin(\sqrt{\delta}t)^{n+q-1}, [0, \pi/\sqrt{\delta}]) .$$

Case 3 - $q < \infty$, $\rho = 0$, $D < \infty$:

$$\mathcal{I}(M^n, g, \mu) \geq \min(\inf_{\xi \geq 0} \mathcal{I}^b(t^{n+q-1}, [\xi, \xi + D]), \mathcal{I}^b(1, [0, D])) .$$

Case 4 - $q < \infty$, $\rho < 0$, $D < \infty$:

$$\mathcal{I}(M^n, g, \mu) \geq \min \left\{ \begin{array}{l} \inf_{\xi \geq 0} \mathcal{I}^b(\sinh(\sqrt{-\delta}t)^{n+q-1}, [\xi, \xi + D]) , \\ \mathcal{I}^b(\exp(\sqrt{-\delta}(n+q-1)t), [0, D]) , \\ \inf_{\xi \in \mathbb{R}} \mathcal{I}^b(\cosh(\sqrt{-\delta}t)^{n+q-1}, [\xi, \xi + D]) \end{array} \right\} .$$

Case 5 - $q = \infty$, $\rho \neq 0$, $D < \infty$:

$$\mathcal{I}(M^n, g, \mu) \geq \min_{\xi \in \mathbb{R}} \mathcal{I}^b(\exp(-\frac{\rho}{2}t^2), [\xi, \xi + D]) .$$

Case 6 - $q = \infty$, $\rho > 0$, $D = \infty$: $\mathcal{I}(M^n, g, \mu) \geq \mathcal{I}^b(\exp(-\frac{\rho}{2}t^2), \mathbb{R})$.

Case 7 - $q = \infty$, $\rho = 0$, $D < \infty$: $\mathcal{I}(M^n, g, \mu) \geq \min_{H \geq 0} \mathcal{I}^b(\exp(Ht), [0, D])$.

In all the remaining cases, we have the trivial bound $\mathcal{I}(M^n, g, \mu) \geq 0$.

Note that when q is an integer, $\mathcal{I}^\flat(\sin(\sqrt{\delta}t)^{n+q-1}, [0, \pi/\sqrt{\delta}])$ coincides (by testing spherical caps) with the isoperimetric profile of the $(n+q)$ -Sphere having Ricci curvature equal to ρ , and so Case 2 with $q = 0$ recovers the Gromov–Lévy isoperimetric inequality; for general $q < \infty$, Case 2 was obtained by Bayle. Case 6 recovers the Bakry–Ledoux isoperimetric inequality. To the best of our knowledge, all remaining cases are new. To illuminate the transition between Cases 1 and 2, note that if (M^n, g, μ) satisfies the $CD(\rho, n+q)$ condition with $\rho > 0$, the diameter of M is bounded above by $\pi/\sqrt{\delta}$: when $q = 0$ this is the classical Bonnet–Myers theorem, which was extended to $q > 0$ by Bakry–Ledoux and Qian. As for the sharpness, we have:

Theorem 3. For any $n \geq 2$, $\rho \in \mathbb{R}$, $q \in [0, \infty]$ and $D \in (0, \infty]$, the lower bounds provided in Corollary 2 (or equivalently, the one provided in Theorem 1) on the isoperimetric profile of (M^n, g, μ) satisfying the $CDD(\rho, n+q, D)$ condition, are sharp, in the sense that they cannot be pointwise improved.

We conclude that with the exception of Cases 2 and 6 above, there is no *single* model space to compare to, and that a simultaneous comparison to a natural *one parameter family* of model spaces is required, nevertheless yielding a sharp comparison result.

2. SAMPLE APPLICATION

Using the well-known relation between isoperimetric inequalities and Sobolev-type inequalities, we can obtain as a corollary of our main result improved estimates on the Sobolev constants of spaces satisfying the $CDD(\rho, n+q, D)$ condition. As an example, we employ an observation of Ledoux, refined by Beckner, stating that an metric-measure-space satisfying a Gaussian isoperimetric inequality, also satisfies a log-Sobolev inequality, to deduce:

Corollary 4. Let (M^n, g, μ) satisfy the $CDD(\rho, \infty, D)$ condition. Then:

$$\int |\nabla f|^2 d\mu \geq \lambda_{\rho, D}^2 \int f^2 \log f^2 d\mu, \quad \lambda_{\rho, D} := \frac{c}{\int_0^{CD} \exp(-\frac{\rho}{2}t^2) dt},$$

for any locally-Lipschitz function $f : M^n \rightarrow \mathbb{R}$ with $\int f^2 d\mu = 1$, where $c, C > 0$ are numeric constants.

This result improves, in certain regimes of the pair $(\rho, \rho D^2)$, the previously best known estimates on the log-Sobolev constant in the above setting, due to Wang and Bakry–Ledoux–Qian.

Energy of cutoff functions and heat kernel upper bounds

MARTIN T. BARLOW

(joint work with Sebastian Andres)

Let (\mathcal{X}, d, μ) be a metric measure space with a strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$. Write $B(x, r)$ for the ball centre x and radius r , and $V(x, r) = \mu(B(x, r))$. We will assume that \mathcal{X} is unbounded in the metric d , and satisfies *volume doubling*: that is, there exists a constant C_V such that

$$V(x, 2r) \leq C_V V(x, r), \quad x \in \mathcal{X}, r > 0.$$

Let $p_t(x, y)$ be the heat kernel associated with $(\mathcal{X}, \mathcal{E})$. Define a space-time scaling function $t = \Psi(r)$ by

$$(1) \quad \Psi(r) = \Psi_{\beta_L, \beta}(r) = \begin{cases} r^{\beta_L} & \text{if } 0 \leq r \leq 1, \\ r^\beta & \text{if } r > 1. \end{cases}$$

Set

$$(2) \quad \Phi(R, t) = \begin{cases} \left(\frac{R^{\beta_L}}{t}\right)^{1/(\beta_L-1)}, & \text{if } t \leq R, \\ \left(\frac{R^\beta}{t}\right)^{1/(\beta-1)}, & \text{if } t \geq R. \end{cases}$$

We consider heat kernel upper bounds associated with Ψ , of the form

$$(3) \quad p_t(x, y) \leq \frac{\exp(-\Phi(c_2 d(x, y), t))}{V(x, \Psi^{-1}(c_1 t))}, \quad t > 0, x, y \in \mathcal{X}.$$

If this holds we say $\text{UHK}(\Psi)$ holds for $(\mathcal{X}, \mathcal{E})$, and if further a lower bound of the same form (but with different constants c_i) also holds then we say that $\text{HK}(\Psi)$ holds. Heat kernel bounds of this type hold for various regular fractals – for examples see [2]. In these cases one has $\beta_L = \beta$. Situations when $\beta_L \neq \beta$ arise for examples such as the ‘cable system’ of the Sierpinski gasket graph.

We are interested in finding characterisations of $\text{UHK}(\Psi)$, and in particular ones which are *stable*. A stable characterisation is one such that, if it holds for \mathcal{E} , and if $(\mathcal{E}', \mathcal{F})$ is another Dirichlet form on $L^2(\mathcal{X})$, satisfying

$$C^{-1} \mathcal{E}(f, f) \leq \mathcal{E}'(f, f) \leq C \mathcal{E}(f, f) \text{ for all } f \in \mathcal{F},$$

then it holds for \mathcal{E}' .

In the ‘classical’ case $\Psi(r) = \Psi_{2,2}(r) = r^2$ such a stable characterisation is known. We say $(\mathcal{X}, \mathcal{E})$ satisfies the Faber-Krahn inequality $\text{FK}(\Psi)$ if there exists a constant C_F and $\nu > 0$ such that for any ball $B = B(x, r)$ and open set $D \subset B$, the smallest Dirichlet eigenvalue $\lambda_1(D)$ satisfies

$$(4) \quad \lambda_1(D) \geq \frac{C_F}{\Psi(r)} \left(\frac{\mu(B)}{\mu(D)}\right)^\nu.$$

In view of the variational characterisation of $\lambda_1(D)$, this condition is stable.

Theorem 1. (See [7, 8].) *Suppose that (\mathcal{X}, d, μ) satisfies volume doubling. The following are equivalent:*

- (a) $(\mathcal{X}, \mathcal{E})$ satisfies $FK(\Psi_{2,2})$,
- (b) $(\mathcal{X}, \mathcal{E})$ satisfies $UHK(\Psi_{2,2})$.

It is easy to see that this cannot extend to general Ψ : the condition $FK(\Psi)$ is monotone in Ψ , while this is not the case in general for $UHK(\Psi)$. In more fundamental terms, the condition $FK(\Psi)$ implies that the exit time of the Hunt process X associated with \mathcal{E} from a ball $B(x, r)$ is at most of order $\Psi(r)$, but does not preclude the possibility it might be much quicker. An additional condition giving a lower bound on such exit times is needed. Such a condition, denoted $CS(\beta)$, was found in the context of the two-sided bounds $HK(\Psi)$ in [3, 4]. In this talk we introduce a similar condition; it is very likely that it can replace $CS(\beta)$ in [3, 4].

We call a function φ a *cutoff function* for open sets $D_1 \subset D_2$ if $\varphi = 1$ on D_1 and $\varphi = 0$ on D_2 . We say that $CSA(\Psi)$ holds if there exists a constant C_S such that for every $x \in \mathcal{X}$, $R > 0$, $r > 0$, there exists a cutoff function φ for $B(x, R) \subset B(x, R + r)$ such that, writing $U = B(x, R + r) - B(x, R)$, for all $f : U \rightarrow \mathbb{R}$,

$$(5) \quad \int_U f^2 d\Gamma(\varphi, \varphi) \leq \frac{1}{8} \int_U \varphi^2 d\Gamma(f, f) + C_S \Psi(r)^{-1} \int_U f^2 dm.$$

Here CS stands for ‘cutoff-Sobolev’, and A for ‘annulus’.

Theorem 2. (See [1].) *Suppose that (\mathcal{X}, d, μ) satisfies volume doubling. The following are equivalent:*

- (a) $(\mathcal{X}, \mathcal{E})$ satisfies $FK(\Psi)$ and $CSA(\Psi)$,
- (b) $(\mathcal{X}, \mathcal{E})$ satisfies $UHK(\Psi)$.

The proof uses the approach of Davis [6] to prove a general version of the Davis-Gaffney inequality. Using CSA one can construct an exponentially increasing function ψ which mimics the role of $\exp(\lambda d(x_0, x))$ in Davis’ paper. This gives

Proposition 1. Suppose that $CSA(\Psi)$ holds. Then if $d(x_1, x_2) = R$ and f_i have support in $B(x_i, R/4)$,

$$\int f_1(y_1) f_2(y_2) p_t(y_1, y_2) \mu(dy_1) \mu(dy_2) \leq \|f_1\|_2 \|f_2\|_2 c \exp(-\Psi(cR, t)).$$

This estimate does not in general yield pointwise bounds. However, if combined with a mean value inequality, as in [5, 7], it does imply $UHK(\Psi)$. This mean value inequality follows from $CSA(\Psi)$ and $FK(\Psi)$. For further details see [1].

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Spectral properties and heat kernels for some random fractals

BEN HAMBLY

(joint work with Uta Freiberg, John Hutchinson)

The study of diffusion in disordered media by physicists in the 1970s inspired mathematical work in the 1980s on the development of a theory of diffusion on fractals. The initial approach was to work on deterministic fractals such as the Sierpinski gasket and construct the Laplace operator on the set as the limit of discrete Laplace operators on graph approximations to the set. The introduction of Dirichlet form techniques allowed a nice analytical route into the topic and led to results for a general class of finitely ramified fractals, p.c.f. self-similar sets. These results include Weyl type theorems on the asymptotics of the eigenvalue counting function and sub-Gaussian heat kernel estimates. More challenging fractals, such as the Sierpinski carpet have also been treated with similar results obtained but these are restricted to sets with a high degree of symmetry. One strand of research in the area has been to look at random fractals as the fractal structures that arise from physical systems when they are at or near a phase transition are typically random.

In this work we focus on a class of random fractals based on the Sierpinski gasket. In earlier work two types of randomization were considered. Firstly a randomization in the scaling which preserves some spatial homogeneity with detailed results obtained in [1]. The other is the case of random recursive fractals in which the construction of the fractals puts independent randomness within each subfractal. This was treated in [4, 5]. Our aim here is to look at a class of V -variable fractals introduced by [3] which interpolates between these two cases. They show some combination of scale and spatial randomness and essentially have the property that at each stage of construction the possible random subfractals that can be chosen come from a finite set. A crucial property to exploit in our analysis is that they have ‘necks’. As the subfractals are randomly chosen from a finite set it is possible that all will be the same, and this occurs with positive probability ensuring that such necks happen almost surely. At these levels each subfractal is the

same which allows a nice decomposition of the fractal into more easily controlled pieces.

In the construction we have an integer parameter V which determines the number of possible subfractals at each stage. For $V = 1$ this is exactly the set up for the scale irregular fractals in [1]. As $V \rightarrow \infty$ we allow more and more randomness and the V -variable fractals become random recursive in the limit. Our aim was to see what the range of possible behaviours for the spectral asymptotics and the heat kernel were. Do the fractals behave more like the scale irregular case, with for instance large fluctuations in the spectral asymptotics rather than the random recursive case where the Weyl limit exists?

To be more precise let $V \in \mathbb{N}$. The random V -variable fractals are generated from a possibly uncountable family \mathbf{F} of IFSs where each IFS $F = \{\psi_i^F, i = 1, \dots, N^F\} \in \mathbf{F}$ generates an affine nested fractal. We assume that $3 \leq N^F < \infty$ and that the parameters $r^F = \{r_i^F, i = 1, \dots, N^F\}$, the resistances of each cell, and a set of weights $w^F = \{w_i^F; i = 1, \dots, N^F\}$ are suitably bounded. We write P_V, E_V for the natural probability measure and expectation for these fractals. As the fractals are naturally described by trees we write \mathbf{i} for the address of the branch corresponding to a particular subfractal $K_{\mathbf{i}}$ and $r_{\mathbf{i}}, w_{\mathbf{i}}$ for the product of the randomly chosen resistance and weight parameters associated with the branch.

We write $n(k)$ for the level of construction corresponding to the k -th neck. For \mathbf{i} with length $|\mathbf{i}| = n(k)$, so it is in the k -th neck level, we can define a measure as

$$\mu_{\mathbf{i}} = \mu(K_{\mathbf{i}}) = \mu([\mathbf{i}]) = \frac{w_{\mathbf{i}}}{\sum_{|j|=|\mathbf{i}|} w_j}.$$

The corresponding unit mass measure μ can then be extended to K . Of particular interest are weights of the form $w_i^F = (r_i^F)^\alpha$ for all $F \in \mathbf{F}$ and some fixed $\alpha > 0$. We will call this the flat measure on K .

For the eigenvalue counting function $\mathcal{N}(\lambda) := |\{\lambda_i : \lambda_i \leq \lambda, \lambda_i \text{ an eigenvalue}\}|$ it is possible to determine its growth exponent as the zero of a pressure function.

Theorem 1. The spectral exponent is given by β_0 where $\gamma(\beta_0) = 0$ with

$$\gamma(\beta) = E_V \log \sum_{|\mathbf{i}|=n(1)} (r_{\mathbf{i}} \mu_{\mathbf{i}})^{\beta/2}.$$

That is

$$\frac{d_s(\mu)}{2} := \lim_{\lambda \rightarrow \infty} \frac{\log \mathcal{N}(\lambda)}{\log \lambda} = \frac{\beta_0}{2}, \quad P_V \text{ a.s.}$$

Theorem 2. The spectral exponent for the flat measure ν is given P_V a.s. by

$$\frac{d_s(\nu)}{2} = \frac{d_f}{d_f + 1}.$$

where d_f is the Hausdorff dimension in the resistance metric

$$E_V \log \sum_{|\mathbf{i}|=n(1)} r_{\mathbf{i}}^{d_f} = 0.$$

The spectral dimension d_s is the maximum spectral exponent $d_s(\mu)$ over all measures μ . Equality holds if and only if μ is the flat measure with respect to the resistance metric.

For more refined results we consider the flat measure. There are constants c_1, \dots, c_4 such that the counting function satisfies

$$c_1\phi(\lambda)^{-c_2} \leq \frac{\mathcal{N}(\lambda)}{\lambda^{d_s/2}} \leq c_3\phi(\lambda)^{c_4} \quad P_V \text{ a.s.}$$

where $\phi(\lambda) = \exp(\sqrt{\log \lambda \log \log \lambda})$. In fact the fluctuations can be shown to be of this size, which are the same as those observed in the scale irregular case, and there is no Weyl limit.

For the heat kernel we note that none of the more generic techniques used for say the determination of sub-Gaussian heat kernels hold as we do not have volume doubling of our measure. Thus we employ more bare hands techniques developed for fractals over the years. These need to be further developed to handle this class of fractals. It is possible to get local estimates on the heat kernel for any measure which show that, as in the case of multifractal measures considered by [2], the local behaviour of the spectral dimension is different to the global one in general. For the case of the flat measure they are the same and we just state the result

Theorem 3. If ν is the flat measure in the resistance metric we have constants such that P_V a.s. for ν -almost all $x \in K$

$$c_1\phi(1/t)^{-c_2}t^{-d_s/2} \leq p_t(x, x) \leq c_3\phi(1/t)^{c_4}t^{-d_s/2}, \quad 0 < t < c_5.$$

where $\phi(t) = \exp(\sqrt{\log t \log \log t})$.

Once again there are fluctuations of this size and these agree with those observed in the scale irregular case.

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1-forms, vector fields, and topologically one-dimensional fractals

MICHAEL HINZ

(joint work with Alexander Teplyaev)

We consider compact connected topologically one-dimensional metric spaces that carry a strongly local regular Dirichlet form, cf. [4]. Following the approach to differential 1-forms and derivations associated with Dirichlet forms as proposed by Cipriani and Sauvageot [1, 2] (further elaborated in [6] and in implicit or related form touched in several earlier references), we introduce new notions of *local exactness* and *local harmonicity*. They allow to determine whether a 'space of harmonic 1-forms' in a related Hodge decomposition is trivial or not.

We first survey the construction of 1-forms and derivations as described in [1]. For this basic setup compactness, connectedness or one-dimensionality are not needed. Let X be a locally compact separable metric space, m a nonnegative Radon measure on X with $\text{supp } m = X$, and consider a regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L_2(X, m)$. We write $\mathcal{B} := \mathcal{F} \cap L_\infty(X, m)$ for the algebra of (m -equivalence classes of) energy finite essentially bounded functions. For any $f, g \in \mathcal{B}$ there is a finite signed Radon measure $\Gamma(f, g)$ such that

$$\int_X \varphi d\Gamma(f, g) = \mathcal{E}(\varphi f, g) + \mathcal{E}(f, \varphi g) - \mathcal{E}(fg, \varphi), \quad \varphi \in \mathcal{F} \cap C_c(X),$$

usually referred to as the *mutual energy measure* of f and g , [4]. We consider $\mathcal{B} \otimes \mathcal{B}$, endowed with the nonnegative definite symmetric bilinear form

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} = \int_X bd \, d\Gamma(a, c).$$

Factoring out zero seminorm elements and completing yields a Hilbert space \mathcal{H} , the *space of differential 1-forms associated with $(\mathcal{E}, \mathcal{F})$* , [1]. A left action on \mathcal{H} of the algebra \mathcal{B} and a right action on \mathcal{H} of the algebra $\mathcal{B}_b(X)$ of bounded Borel functions can be defined as the continuous and uniformly bounded linear extensions of

$$c(a \otimes b) := (ac) \otimes b - c \otimes (ab) \quad \text{and} \quad (a \otimes b)d := a \otimes (bd),$$

$c \in \mathcal{B}$, $d \in \mathcal{B}_b(X)$. A linear and bounded *derivation operator* $\partial : \mathcal{B} \rightarrow \mathcal{H}$ is given by $\partial f := f \otimes \mathbf{1}$, and if $\mathcal{E}(\mathbf{1}) = 0$ then $\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f)$, $f \in \mathcal{B}$. The operator ∂ extends to a closed unbounded linear operator $\partial : L_2(X, m) \rightarrow \mathcal{H}$. Its formal adjoint (up to sign convention) $\partial^* : \mathcal{H} \rightarrow L_2(X, m)$, determined by the integration by parts identity

$$\langle u, \partial^* v \rangle_{L_2(X, m)} = -\langle \partial u, v \rangle_{\mathcal{H}}, \quad u \in \mathcal{B}, v \in \text{dom } \partial^*,$$

is an unbounded linear operator with dense domain $\text{dom } \partial^* \subset \mathcal{H}$.

Examples 1. If M is a compact Riemannian manifold, then the closure of

$$\mathcal{E}(f) = \int_M \|df\|_{T^*M}^2 \, d\text{vol}, \quad f \in C^\infty(M),$$

in $L_2(M, dvol)$ yields a local regular symmetric Dirichlet form $(\mathcal{E}, H^1(M))$. Here d denotes the exterior derivative. The derivation operator ∂ coincides with d , seen as an unbounded closed linear operator from $L_2(M, dvol)$ into $\mathcal{H} = L_2(M, dvol, T^*M)$.

From now on assume that X is a compact connected topologically one-dimensional metric space, m is a finite nonnegative Radon measure on X with $\text{supp } m = X$, and $(\mathcal{E}, \mathcal{F})$ is a strongly local regular symmetric Dirichlet form on $L_2(X, m)$ that admits a spectral gap and such that the associated m -symmetric Hunt process satisfies the absolute continuity condition, [4, Sections 4.1 and 4.2].

The spectral gap implies that the image $Im \partial$ of ∂ is a closed linear subspace of \mathcal{H} , we refer to it as *the space of exact 1-forms*. Its complement is denoted by \mathcal{H}^1 , i.e.

$$\mathcal{H} = Im \partial \oplus \mathcal{H}^1.$$

Examples 2. If in the situation of Example 1 we consider the unit circle $M = S^1$ then the preceding line rewrites

$$L_2(S^1, \mu, T^*M) = Im d \oplus \mathbb{H},$$

where μ is the image measure of the one-dimensional Lebesgue measure on $[0, 2\pi)$ under the map $t \mapsto e^{it}$. The space \mathbb{H} of harmonic 1-forms is one-dimensional. See for instance [7] or [8].

We investigate whether \mathcal{H}^1 is trivial or not. To this purpose, we introduce the following notions, cf. [5].

Definition 1. A 1-form $\omega \in \mathcal{H}$ is called *locally exact* if there exist a finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of X and functions $f_\alpha \in \mathcal{B}$, $\alpha \in J$, such that

$$\omega \mathbf{1}_{U_\alpha} = \partial f_\alpha \mathbf{1}_{U_\alpha}, \quad \alpha \in J.$$

A 1-form $\omega \in \mathcal{H}$ is called *locally harmonic* if there exist a finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of X and functions $h_\alpha \in \mathcal{B}$, $\alpha \in J$, such that each h_α is harmonic on U_α (in the Dirichlet form sense) and

$$\omega \mathbf{1}_{U_\alpha} = \partial h_\alpha \mathbf{1}_{U_\alpha}, \quad \alpha \in J.$$

Consider the linear subspaces of \mathcal{H} given by

$$\mathcal{H}_{loc} := \text{clos span } \{\omega \in \mathcal{H} : \omega \text{ locally exact}\}$$

and

$$\mathcal{H}_{loc}^1 := \text{clos span } \{\omega \in \mathcal{H} : \omega \text{ locally harmonic}\}.$$

Recall that any finite open cover of a topologically one-dimensional compact metric space has a finite refinement consisting of open sets with topologically zero-dimensional and therefore totally disconnected boundaries. Decompositions of the domain \mathcal{F} of \mathcal{E} related to these totally disconnected boundaries allow to prove the following results.

Theorem 2.

- (i) We have $\mathcal{H}_{loc} = \mathcal{H}$, i.e. the locally exact 1-forms are dense in \mathcal{H} .
- (ii) We have $\mathcal{H}_{loc}^1 = \mathcal{H}^1$, i.e. the locally harmonic 1-forms are dense in \mathcal{H}^1 .

Details can be found in [5]. Using Definition 1 and some principles behind Theorem 2 we can formulate at least some qualitative statements connecting the space \mathcal{H}^1 and the topology of X . Let $\check{H}^1(X)$ denote the first Čech cohomology of X , [3, 8]. For a topologically one-dimensional space X the dimension of $\check{H}^1(X)$ 'gives the number of independent cycles' of X . In particular, the space $\check{H}^1(X)$ is trivial if and only if X is a (topological) tree.

For the sake of brevity statement (ii) in the following theorem is a simplified special case of the original result in [5].

Theorem 3.

- (i) If \mathcal{H}^1 is nontrivial then $\check{H}^1(X)$ is nontrivial.
- (ii) Assume that points have positive capacity. Then, if $\check{H}^1(X)$ is nontrivial, also \mathcal{H}^1 is nontrivial.

If A is the $L_2(X, m)$ -generator of $(\mathcal{E}, \mathcal{F})$ then we observe $\partial^* \partial f = Af$ (in a weak sense). If we agree to define a related Hodge Laplacian on 1-forms by

$$\Delta_1 := \partial \partial^*$$

(implicitly assuming there are no nonzero 2-forms) and call $\ker \Delta_1$ the space of harmonic 1-forms, then we obtain the following as a consequence of Theorem 2.

Theorem 4.

- (i) The Hodge Laplacian Δ_1 may be seen as an unbounded self-adjoint operator on \mathcal{H} .
- (ii) We have $\ker \Delta_1 = \mathcal{H}^1$, and every 1-form can be written as the orthogonal sum of an exact and a harmonic part.

Our results are applicable to fractals such as Sierpinski gaskets, Sierpinski carpets or Barlow-Evans-Laaksø spaces. They do not depend on the Hausdorff dimension of the space, only the topological dimension matters. Note that the points of these spaces do not have contractible neighborhoods.

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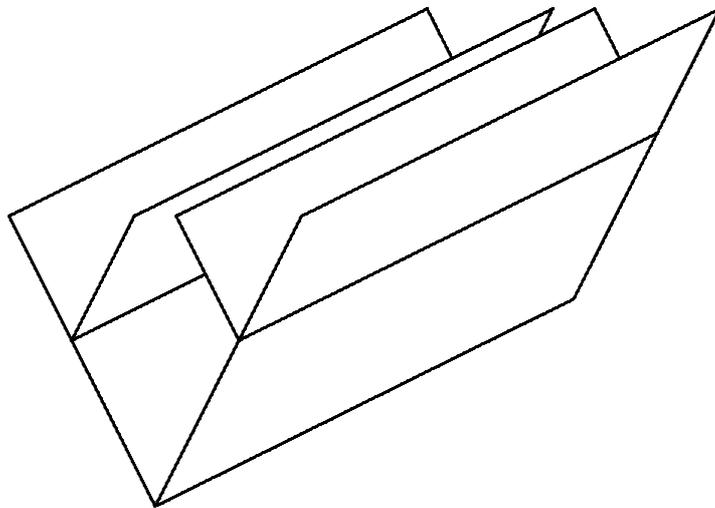
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Brownian motion on treebolic space

WOLFGANG WOESS

(joint work with Alexander Bendikov, Laurent Saloff-Coste, Maura Salvatori)

Treebolic space $\text{HT}(\mathfrak{q}, \mathfrak{p})$ is a 2-dimensional Riemannian complex which is obtained by glueing together horizontal strips of hyperbolic upper half plane \mathbb{H} in a tree-like fashion. The parameter $\mathfrak{q} > 1$ determines the hyperbolic width $\log \mathfrak{q}$ of the strips, while the integer $\mathfrak{p} \geq 2$ is the branching number of the underlying regular tree \mathbb{T} . The latter is understood as a metric tree where each edge is a copy of the unit interval.



The figure shows a compact piece of $\text{HT}(\mathfrak{q}, 2)$. (Each strip extends to infinity in both horizontal directions. Below each strip, another one is attached. Above each strip, $\mathfrak{p} = 2$ strips are attached.)

Treebolic space can also be described as the *horocyclic product* of \mathbb{H} and \mathbb{T} : Let $\mathfrak{h} : \mathbb{T} \rightarrow \mathbb{R}$ denote the Busemann function (height) with respect to a reference end of the metric tree (in our picture, that end sits at the bottom). Then

$$\text{HT}(\mathfrak{q}, \mathfrak{p}) = \{(z, w) \in \mathbb{H} \times \mathbb{T} : \mathfrak{h}(w) = \log_{\mathfrak{q}}(\text{Im } z)\}.$$

We consider a two-parameter family of Laplace operators $\Delta_{\alpha, \beta}$ on $\text{HT}(\mathfrak{q}, \mathfrak{p})$, where $\alpha \in \mathbb{R}$ and $\beta > 0$. Briefly spoken, “nice” functions in the domain of $\Delta_{\alpha, \beta}$ must be such that at any point on one of the bifurcation lines, β times the sum of the vertical derivatives on each of the strips \mathfrak{p} above the point must coincide with the vertical derivative at that point on the unique strip below. In the interior of each strip, $\Delta_{\alpha, \beta}$ is hyperbolic Laplacian with vertical drift parameter α .

Treebolic spaces are key examples that illustrate the theory of strip complexes developed in [1], and in that reference, it is proved (among other) that $\Delta_{\alpha, \beta}$ is essentially self-adjoint. We consider Brownian motion, that is, the Markov process generated by $\Delta_{\alpha, \beta}$. It is transient for any choice of the parameters. Regarding its behaviour in space, we compute the rate of escape and prove a central limit theorem for the distance from the starting point. The rate of escape coincides

with the absolute value of the drift of the vertical projection of the process on the real line. When that drift is non-zero, the central limit theorem also exhibits the same behaviour as the vertical projection. The zero-drift case is more subtle.

Also, $\text{HT}(\mathbf{q}, \mathbf{p})$ has a natural geometric compactification and boundary at infinity. We show that Brownian motion converges almost surely in the topology of the compactification to a boundary-valued limit random variable. That limit variable lives only on part of the boundary, which depends on the sign of the drift of the vertical projection. In case of zero drift, the a.s. limit is a single deterministic point of the boundary.

Those results are contained in [2]. In the forthcoming paper [3], the potential theory, and in particular, the positive harmonic functions of $\Delta_{\alpha, \beta}$ are studied. Every positive harmonic function h decomposes as $h(z, w) = h_1(z) + h_2(w)$, where h_1 and h_2 are non-negative harmonic functions for the projected Laplacians on \mathbb{H} and \mathbb{T} , respectively. As to be expected, the Liouville property (constantness of all bounded harmonic functions) holds precisely when the vertical drift is 0.

Treebolic space has two “sister structures”. On one hand, instead of a horocyclic product of a tree with hyperbolic plane, one may take two regular trees, with possibly different degrees $\mathbf{p}, \mathbf{q} \geq 2$ (integer). The resulting structures are the *Diestel-Leader graphs* $\text{DL}(\mathbf{p}, \mathbf{q})$, and if their degrees coincide, one gets Cayley graphs of *lamplighter groups*. For random walks on these structures, the understanding of their geometry has led to very satisfactory results, see [4], [6] and the references therein. On the other hand, one may take two (again oppositely oriented) hyperbolic planes, possibly with different real parameters $\mathbf{p}, \mathbf{q} > 0$ for their respective curvatures $-\mathbf{p}^2$ and $-\mathbf{q}^2$. Then one obtains the Sol-groups, resp. manifolds (in case of standard parameters, this is one of Thurston’s eight model geometries). Brownian motion on $\text{Sol}(\mathbf{p}, \mathbf{q})$ is studied in detail in [5].

Of those three, treebolic space is hardest to handle because of the singularities which it has at the bifurcation lines. What is common is that the understanding of their geometries as horocyclic products leads to the obtained probabilistic and analytic results.

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Brownian motion, moving metrics and Perelman's entropy formula

ANTON THALMAIER

(joint work with Hongxin Guo, Robert Philipowski)

We discussed notions of stochastic differential geometry in the framework that the underlying manifold evolves along a geometric flow, cf. [1] for a discussion of various flows. Special interest was given to stochastic versions of entropy formulas for positive solutions of the heat equation (or conjugate heat equation) under forward/backward Ricci flow.

I. As a typical example consider backward heat equation under backward Ricci flow on a (not necessarily compact) complete Riemannian manifold M :

$$\begin{cases} \frac{\partial}{\partial t} g = 2 \operatorname{Ric}, \\ \frac{\partial}{\partial t} u + \Delta u = 0. \end{cases}$$

Let $X_t(x)$ be a $g(t)$ -Brownian motion on M starting from x , and let $p(t, x, y)$ be the density of $X_t(x)$ with respect to $\operatorname{vol}_{g(t)}$. Taking the heat kernel measure

$$p(t, x, y) \operatorname{vol}_{g(t)}(dy) = \mathbb{P} \{X_t(x) \in dy\}$$

as reference measure, we note that the integral

$$\int_M u(t, y) p(t, x, y) \operatorname{vol}_{g(t)}(dy) = \mathbb{E}[u(t, X_t(x))]$$

stays constant under the flow, under the technical assumption that the local martingale $u(t, X_t(x))$ is a true martingale.

Proposition (see [2]). *The Boltzmann-Shannon entropy of the measure*

$$\mu_t := u(t, \cdot) p(t, x, y) \operatorname{vol}_{g(t)}(dy) \equiv u(t, X_t(x)) d\mathbb{P}$$

calculates as

$$\mathcal{E}(t) = \int_M (u \log u)(t, y) p(t, x, y) \operatorname{vol}_{g(t)}(dy) = \mathbb{E}[(u \log u)(t, X_t(x))].$$

The first two derivatives of $\mathcal{E}(t)$ are given by

$$\begin{aligned} \mathcal{E}'(t) &= \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t(x)) \right] \equiv \mathbb{E} [(|\nabla \log u|^2 u)(t, X_t(x))] \\ \mathcal{E}''(t) &= 2 \mathbb{E} \left[\left(u |\operatorname{Hess} \log u|^2 \right)(t, X_t(x)) \right]. \end{aligned}$$

Upon time reversal, solutions to the above equations, defined for all $t \geq 0$, correspond to ancient solutions of the heat equation (starting at $t = -\infty$) under forward Ricci flow. Our Proposition allows to classify such solutions according to their entropy.

Corollary. *Assume that $\frac{\partial g}{\partial t} = 2\operatorname{Ric}$ (or more generally $\frac{\partial g}{\partial t} \leq 2\operatorname{Ric}$) and let u be a positive solution of the backward heat equation $\frac{\partial}{\partial t} u + \Delta u = 0$. Let*

$$\theta := \lim_{t \rightarrow \infty} \mathcal{E}'(t) \in [0, +\infty].$$

- Then u is constant if and only if $\theta = 0$.
- If the entropy $\mathcal{E}(t)$ grows sublinearly, i.e. $\lim_{t \rightarrow \infty} \mathcal{E}(t)/t = 0$, then $\theta = 0$, and hence u is constant.
- If $2\text{Ric} - \frac{\partial g}{\partial t}$ is strictly positive everywhere, then nonconstant solutions to the backward heat equation cannot have linear entropy.

II. Similar formulas as in the Proposition have been presented for

$$\begin{cases} \frac{\partial}{\partial t} g = -2 \text{Ric}, \\ \frac{\partial}{\partial t} u + \Delta u = Ru. \end{cases}$$

Now one takes

$$\mathbb{P}_t(x) := \exp \left(- \int_0^t R(s, X_s(x)) ds \right) d\mathbb{P}$$

as reference measure, where $X_t(x)$ is a $g(t)$ -Brownian motion starting from x , and considers the entropy of the measure

$$\mu_t(x) := u(t, X_t(x)) d\mathbb{P}_t(x).$$

This allows to define probabilistic versions of Perelman's \mathcal{F} - and \mathcal{W} -functional which due to the fast decay of the heat kernel do not require compactness of the underlying manifold, cf. [3]. The resulting entropy formulas seem to be appropriate to deal with "No breather theorems" on non-compact manifolds.

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Completion of the Sierpinski gasket minus the unit interval

JUN KIGAMI

Let $p_1 = \frac{1}{2} + \frac{\sqrt{-3}}{2}$, $p_2 = 0$ and $p_3 = 1$ and let $V_0 = \{p_1, p_2, p_3\}$. Note that V_0 is the set of vertices of an equilateral triangle in \mathbb{C} . Define $f_i : \mathbb{C} \rightarrow \mathbb{C}$ for $i = 1, 2, 3$ by $f_i(z) = (z - p_i)/2 + p_i$. The Sierpinski gasket is the invariant set of $\{f_1, f_2, f_3\}$, i.e. the unique nonempty compact set K which satisfies

$$K = f_1(K) \cup f_2(K) \cup f_3(K).$$

Define $I = \{x | x \in [0, 1]\}$. Our first problem is the following:

What is the completion of $K \setminus I$?

An immediate answer can be K itself, if we associate the Euclidean metric to $K \setminus I$. However, if we consider the shortest path metric d_* on $K \setminus I$, which is defined by

$$d_*(x, y) = \inf \{ \text{length of } \gamma \mid \gamma : [0, 1] \rightarrow K \setminus I \text{ is a rectifiable curve} \\ \text{with } \gamma(0) = x \text{ and } \gamma(1) = y. \},$$

then the completion is not K but $K_\Sigma = (K \setminus I) \cup \Sigma$, where Σ is the ternary Cantor set $\{0, 1\}^\mathbb{N}$. Let π be the natural map from Σ to I defined by $\pi(i_1 i_2 \dots) = \sum_{n \geq 1} \frac{i_n}{2^n}$ for $i_1 i_2 \dots \in \Sigma$ with $i_1, i_2, \dots \in \{0, 1\}$. Define $\tilde{\pi} : K_\Sigma \rightarrow K$ by

$$\tilde{\pi}(x) = \begin{cases} x & \text{if } x \in K \setminus I \\ \pi(x) & \text{if } x \in \Sigma. \end{cases}$$

Then $\tilde{\pi}$ is a continuous. Note that $\tilde{\pi}(\Sigma) = I$.

This story of two different completions sounds like a simple topological twist, but in fact it helps us to understand the trace of the Brownian motion on K onto I . Now let us review the construction of the Brownian motion on the Sierpinski gasket from the view point of a Dirichlet form. Define sequence of nondirected graphs $G_m = (V_m, E_m)$, where V_m is the set of vertices and E_m is the set of edges, inductively by $V_{m+1} = f_1(V_m) \cup f_2(V_m) \cup f_3(V_m)$, $E_0 = \{p_1 p_2, p_2 p_3, p_3 p_1\}$ and $E_{m+1} = \{f_1(e), f_2(e), f_3(e) \mid e \in E_m\}$. Then $\{V_m\}_{m \geq 0}$ is increasing and the closure of $V_* = \cup_{m \geq 0} V_m$ under the Euclidean metric is K . Define $\mathcal{E}_m(u, v)$ for $u, v \in \ell(V_m)$, where $\ell(V_m) = \{u \mid u : V_m \rightarrow \mathbb{R}\}$, by

$$\mathcal{E}_m(u, v) = \frac{1}{2} \left(\frac{5}{3} \right)^m \sum_{p, q \in V_m, pq \in E} (u(p) - u(q))(v(p) - v(q)).$$

Then $\mathcal{E}_m(u|_{V_m}, u|_{V_m}) \leq \mathcal{E}_{m+1}(u|_{V_{m+1}}, u|_{V_{m+1}})$ for any $u : K \rightarrow \mathbb{R}$. Let

$$\mathcal{F} = \{u \mid u : K \rightarrow \mathbb{R}, \lim_{m \rightarrow \infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < +\infty\}$$

and, for $u, v \in \mathcal{F}$, define

$$\mathcal{E}(u, v) = \lim_{m \rightarrow \infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m}).$$

It is known that $(\mathcal{E}, \mathcal{F})$ is a resistance form on K . In particular, we may define the resistance metric $R(x, y)$ on K by

$$R(x, y) = \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u, u)} \mid u \in \mathcal{F}, \mathcal{E}(u, u) > 0 \right\}.$$

We see that

$$R(x, y) \asymp |x - y|^{\alpha-1},$$

where $\alpha = \frac{\log 10/3}{\log 2}$. Let μ be the self-similar measure with weight $(1/3, 1/3, 1/3)$ on K . Then $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(K, \mu)$. The associated diffusion process on K is called the Brownian motion on the Sierpinski gasket.

Next we define the trace of $(\mathcal{E}, \mathcal{F})$ on the unit interval I . Let $\mathcal{F}_I = \{u|_I : u \in \mathcal{F}\}$. For $\varphi, \psi \in \mathcal{F}_I$, define

$$\mathcal{E}_I(\varphi, \psi) = \min\{\mathcal{E}(u, v) | u, v \in \mathcal{F}, u|_I = \varphi, v|_I = \psi\}.$$

$(\mathcal{E}_I, \mathcal{F}_I)$ is called the trace of $(\mathcal{E}, \mathcal{F})$ onto I . $(\mathcal{E}_I, \mathcal{F}_I)$ is a regular Dirichlet form on $L^2(I, dx)$. We are interested in an asymptotic behavior of the jump process associated with the Dirichlet form $(\mathcal{E}_I, \mathcal{F}_I)$. The domain \mathcal{F}_I is identified with

$$\mathcal{F}^{(\alpha)}(I) = \left\{ u \mid u \in L^2(I, dx), \int_I \int_I \frac{(u(x) - u(y))^2}{|x - y|^{\alpha+1}} dx dy < +\infty \right\}$$

A characterization of \mathcal{F}_I as a Besov type space has given by [2]. Let $p_I(t, x, y)$ be the transition density of the jump process associated with $(\mathcal{E}_I, \mathcal{F}_I)$. In [4], one has

$$p_I(t, x, x) \asymp \frac{1}{t^\alpha}.$$

To understand more detailed behavior of $p_I(t, x, y)$, we make use of the space K_Σ . There exists a natural lift $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ to K_Σ , where $\tilde{\mathcal{F}}$ consists of continuous functions on K_Σ , $\{u \circ \tilde{\pi} | u \in \mathcal{F}\} \subseteq \tilde{\mathcal{F}}$, $\mathcal{E}(u, u) = \tilde{\mathcal{E}}(u \circ \tilde{\pi}, u \circ \tilde{\pi})$ for any $u \in \mathcal{F}$ and

$$\mathcal{F} = \{u | u \in \tilde{\mathcal{F}}, u(x) = u(y) \text{ whenever } \tilde{\pi}(x) = \tilde{\pi}(y)\}.$$

Moreover, define $\mathcal{F}_\Sigma = \{f|_\Sigma : f \in \tilde{\mathcal{F}}\}$ and, for $f \in \mathcal{F}_\Sigma$,

$$\mathcal{E}_\Sigma(f, f) = \min\{\tilde{\mathcal{E}}(u, u) | u \in \tilde{\mathcal{F}}, u|_\Sigma = f\}$$

Then we have $u \circ \pi \in \mathcal{F}_\Sigma$ for any $u \in \mathcal{F}_I$, $\mathcal{E}_I(u, u) = \mathcal{E}_\Sigma(u \circ \pi, u \circ \pi)$ and

$$\mathcal{F}_I = \{f | f \in \mathcal{F}_\Sigma, f(x) = f(y) \text{ whenever } \pi(x) = \pi(y)\}.$$

To analyse $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$, we utilize the correspondenc between the process on $K \setminus I$ associated with $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ and a random walk on the infinite binary tree. See [5] for the theory of random walks and their Martin boundaries. Let T be the infinite binary tree, i.e. $T = \cup_{n \geq 0} \{0, 1\}^n$ is the vertices and the collective of the edges is $\{(w, w0), (w, w1) | w \in T\}$. If $w \in \{0, 1\}^n$, we set $|w| = n$. For an edge (w, wi) with $w \in T$ and $i = 0, 1$, attach a conductance $C(w, wi) = (\frac{5}{3})^{|w|}$. Let $(\mathcal{Q}, \mathcal{D})$ be the associated energy and its domain defined by

$$\mathcal{Q}(u, u) = \sum_{w \in T} \sum_{i=0,1} C(w, wi) (u(w) - u(wi))^2$$

and $\mathcal{D} = \{u | \mathcal{Q}(u, u) < +\infty\}$. There is a natural map Φ from \mathcal{D} to $\tilde{\mathcal{F}}$ such that $\tilde{\mathcal{E}}(\Phi(u), \Phi(u)) = \mathcal{Q}(u, u)$. Through Φ , we may identify the Brownian motion on $K \setminus I$ with the random walk on T associated with the conductances $\{C(w, wi)\}_{w \in T, i=0,1}$, which is given by $p(w, wi) = 5/13$ and $p(wi, w) = 2/13$. This random walk is transient. The Cantor set Σ is the Martin boundary. Using the result in [3], if $(\mathcal{Q}_\Sigma, \mathcal{D}_\Sigma)$ be the trace of $(\mathcal{Q}, \mathcal{D})$ onto Σ , then

$$\mathcal{Q}_\Sigma(u, u) = \int_\Sigma \int_\Sigma \frac{|u(x) - u(y)|^2}{d_\Sigma(x, y)^{\alpha+1}} \nu(dx) \nu(dy),$$

where $d_\Sigma(x, y) = 2^{s(x, y)}$ with $s(x, y) = \min\{n | x_n \neq y_n\}$ for $x = x_1x_2\dots, y = y_1y_2\dots \in \Sigma$ and ν is the self-similar measure on Σ with weight $(1/2, 1/2)$. Note that $d_\Sigma(\cdot, \cdot)$ is a metric on Σ . The important fact is that $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma) = (\mathcal{Q}_\Sigma, \mathcal{D}_\Sigma)$. Through π we have an expression of $(\mathcal{E}_I, \mathcal{F}_I)$ as

$$\mathcal{E}_I(\varphi, \varphi) = \int_I \int_I \frac{|\varphi(x) - \varphi(y)|^2}{d_\Sigma(\pi^{-1}(x), \pi^{-1}(y))^{\alpha+1}} dx dy.$$

Note that d_Σ is a metric on Σ . Hence if we write $J(x, y) = d_\Sigma(\pi^{-1}(x), \pi^{-1}(y))^{-(\alpha+1)}$, then $J(x, y)$ is essentially discontinuous. In spite of such a complicated structure of $J(x, y)$, since $J(x, y) \leq c|x - y|^{-(\alpha+1)}$, by [1] we have an upper off-diagonal estimate of transition kernel $p_I(t, x, y)$

$$p_I(t, x, y) \leq c \min \left\{ \frac{1}{t^{1/\alpha}}, \frac{t}{|x - y|^{\alpha+1}} \right\}$$

However, this is not best possible. Moreover off-diagonal lower estimate is much harder and left to further study.

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Concentration, Ricci curvature, and eigenvalues of Laplacian

TAKASHI SHIOYA

In this talk, we consider the convergence of a sequence of spaces:

$$X_n \rightarrow X \quad \text{as } n \rightarrow \infty,$$

where the spaces X and X_n , $n = 1, 2, \dots$, are Riemannian manifolds, or more generally metric measure spaces. In the most studies done before, every X_n is assumed to have a lower bound of curvature and the convergence is with respect to the (measured) Gromov-Hausdorff topology. If every X_n has a lower bound of sectional curvature, then the limit X becomes an Alexandrov space. The study of convergence in this case was applied in Perelman's proof of the geometrization conjecture. If every X_n has a lower bound of Ricci curvature, then Cheeger and Colding studied the structure of the limit space X deeply. However, in all such studies, the uniform boundedness of dimensions is necessary. This talk is about convergence in the case where

$$\dim X_n \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

for which, the (measured) Gromov-Hausdorff topology is no longer useful and instead, the concentration topology introduced by Gromov [2] is suitable.

An mm-space is a complete separable metric space (X, d_X) with a Borel probability measure μ_X . Gromov defined the *observable distance*, say $d_{\text{conc}}(X, Y)$, between two mm-spaces X and Y by the difference between 1-Lipschitz functions on X and those on Y . We say that a sequence of mm-spaces X_n , $n = 1, 2, \dots$, *concentrates to* an mm-space X if $d_{\text{conc}}(X_n, X) \rightarrow 0$ as $n \rightarrow \infty$. We call the topology induced from d_{conc} the *concentration topology*. In the special case where the limit is a one-point space, the concentration $X_n \rightarrow \{p\}$ is equivalent to that for any 1-Lipschitz function $f_n : X_n \rightarrow \mathbb{R}$ there is a constant c_n such that

$$\lim_{n \rightarrow \infty} \mu_{X_n} (|f_n - c_n| > \varepsilon) = 0$$

for any $\varepsilon > 0$, which means that any 1-Lipschitz function on X_n is close to a constant for large n . This is so called ‘concentration of measure phenomenon’, which was first discovered by P. Lévy and developed by V. Milman. The sequence of unit spheres S^n , $n = 1, 2, \dots$, concentrates to a one-point space, which was proved by Lévy. As another example, the product space $S^n \times X$ concentrates to X as $n \rightarrow \infty$ for an mm-space X . One of our main theorems is stated as follows.

Main Theorem A ([1]). *Let X and X_n , $n = 1, 2, \dots$, be mm-spaces such that X is proper. If each X_n satisfies the curvature-dimension condition $\text{CD}(K, N)$ for two real numbers K and $N \geq 1$ and if X_n concentrates to X , then X also satisfies $\text{CD}(K, N)$.*

Here, the curvature-dimension condition $\text{CD}(K, N)$ is a generalization of the condition that the Ricci curvature is bounded below by K and the dimension is bounded above by N . This is defined by Lott-Villani-Sturm [3, 5, 6] by using the theory of optimal mass-transport. In the case where N is finite, Main Theorem A is implied by the result for measured Gromov-Hausdorff convergence [3], because we have a uniform doubling constant, which leads to the precompactness of the sequence $\{X_n\}$ with respect to the Gromov-Hausdorff topology.

The proof of Main Theorem A is much more delicate than that of the result for measured Gromov-Hausdorff convergence [3]. Under a concentration $X_n \rightarrow X$, we have 1-Lipschitz maps $f_n : X_n \rightarrow X$ which is something like a fibration. In the example $S^n \times X \rightarrow X$, the map $f_n : S^n \times X \rightarrow X$ is just taken to be the projection. A difficult point in the proof is that each fiber of f_n may not be small. In fact, in the example $S^n \times X \rightarrow X$, the fiber is S^n , i.e., a constant size. This is different from measured Gromov-Hausdorff convergence.

As an application of Main Theorem A, we have the following.

Main Theorem B ([1]). *For any number k , there is a constant $C_k > 0$ depending only on k such that if M is a connected and closed Riemannian manifold of nonnegative Ricci curvature, then*

$$\lambda_k(M) \leq C_k \lambda_1(M).$$

In Main Theorem B, the independence of the dimension on C_k is a new feature. All known techniques only give an estimate depending on the dimension.

Let us move on to another topic. Gromov [2] introduced a specific natural compactification, say Π , of the space of mm-spaces with the concentration topology. For $a_1, a_2, \dots > 0$, we have an infinite-dimensional Gaussian measure

$$\gamma_{\{a_i^2\}}^\infty = \prod_{i=1}^{\infty} \gamma_{a_i^2}^1$$

on \mathbb{R}^∞ with variance a_1^2, a_2^2, \dots . We call $\Gamma^\infty\{a_i^2\} := (\mathbb{R}^\infty, \|\cdot\|_2, \gamma_{\{a_i^2\}}^\infty)$ an *infinite-dimensional Gaussian space*. In general, $\Gamma^\infty\{a_i^2\}$ is not an mm-space. Nevertheless, we have a corresponding element of the compactification Π , say a *virtual infinite-dimensional Gaussian space*. We have the following theorem.

Main Theorem C ([4]). *Let E^n be an n -dimensional ellipsoid in \mathbb{R}^{n+1} equipped with the Euclidean distance function and the normalized Riemannian volume measure, and let d_n be the diameter of E^n , where $n = 1, 2, \dots$*

- (1) $d_n/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$ if and only if E^n concentrates to a one-point space as $n \rightarrow \infty$.
- (2) $d_n/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$ if and only if E^n concentrates to the largest element of Π as $n \rightarrow \infty$.
- (3) Assume that the i -th semi-axis of E^n is monotone non-increasing in $i \in \{1, 2, \dots, n+1\}$. If the i -th semi-axis of E^n divided by \sqrt{n} converges to a number a_i as $n \rightarrow \infty$ for each $i = 1, 2, \dots$, then E^n concentrates to a virtual infinite-dimensional Gaussian space with variance a_1^2, a_2^2, \dots as $n \rightarrow \infty$.

Note that every former example of concentration is of product type, and that this example is the first discovered one of non-product type. Since Main Theorem C also holds for any subsequence of $\{E^n\}$, we are able to know all the limits of ellipsoids. A similar result for complex projective spaces is also obtained.

A key to the proof of Main Theorem C is the following theorem.

Main Theorem D ([4]). *There exists a metric ρ on the compactification Π such that*

$$\rho \leq d_{\text{conc}}.$$

We cannot expect $\rho \geq C d_{\text{conc}}$, as is seen in the following.

Proposition. There exist two sequences of mm-spaces X_n and Y_n , $n = 1, 2, \dots$, such that

$$\lim_{n \rightarrow \infty} X_n = \lim_{n \rightarrow \infty} Y_n \quad \text{and} \quad \liminf_{n \rightarrow \infty} d_{\text{conc}}(X_n, Y_n) > 0.$$

This proposition is obtained by applying Main Theorem C.

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A new approach to Gaussian heat kernel upper bounds on doubling metric measure spaces

THIERRY COULHON

(joint work with Salahaddine Boutayeb, Adam Sikora)

Let (M, μ) be a measure space and L a non-negative self-adjoint operator on $L^2(M, \mu)$, so that $(e^{-tL})_{t>0}$ is a semigroup of contractions of $L^2(M, \mu)$. Denote by $\mathcal{E}(f)$ the quadratic form $\langle Lf, f \rangle = \|L^{1/2}f\|_2^2$ with domain \mathcal{F} . The semigroup e^{-tL} may (or may not) have a measurable kernel $p_t(x, y)$:

$$e^{-tL}f(x) = \int_M p_t(x, y)f(y) d\mu(y), \quad f \in L^2(M, \mu), \quad \text{a.e. } x \in M.$$

The question is to relate upper estimates of $p_t(x, y)$, especially as $t \rightarrow +\infty$ (especially if μ is infinite), to geometric properties of (M, μ, \mathcal{E}) or (M, d, μ, \mathcal{E}) , where d is a metric on M connected in some way to \mathcal{E} . Some functional analytic methods have been developed in order to estimate from above and below

$$\sup_{x, y \in M} p_t(x, y) = \|e^{-tL}\|_{1 \rightarrow \infty}$$

as a function of $t > 0$ in terms of the so-called L^2 isoperimetric profile of (M, μ, \mathcal{E}) (see for instance the survey [5]). The problem is that to do analysis on (M, μ, \mathcal{E}) or (M, d, μ, \mathcal{E}) , pointwise estimates of $p_t(x, y)$ are needed rather than estimates of $\sup_{x, y \in M} p_t(x, y)$, and this may be a different problem. For instance, if M is a non-negative Ricci curvature manifold and L the Laplace-Beltrami operator on M , $p_t(x, x) \simeq \frac{1}{V(x, \sqrt{t})}$, where $V(x, r) := \mu(B(x, r))$ is the measure of the ball $B(x, r)$ of center $x \in M$ and radius $r > 0$. Even though $V(x, r) \simeq V(y, r)$ for $d(x, y) \simeq r$, $V(x, r)$ may vary drastically with x . Assume the volume doubling property: $V(x, 2r) \leq CV(x, r)$, $\forall x \in M, r > 0$. It has been understood during the last decade that, in the case where M is a Riemannian manifold and L the non-negative Laplace-Beltrami operator on M , suitable upper estimates, lower estimates, and gradient estimates of $p_t(x, y)$ are the key to the understanding of the L^p boundedness of the Riesz transform on M (see [6], [2], [1]). The relationship

between these various estimates is now well understood (see [7], [8], [9], [3]). The present talk was about [4], where we give a purely functional analytic approach to the most basic estimate, namely the on-diagonal upper estimate

$$(DUE) \quad p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})},$$

and more generally

$$(DUE^v) \quad p_t(x, x) \leq \frac{C}{v(x, \sqrt{t})},$$

where v is a general doubling function, which may differ from the volume function V . It had been shown previously in [7] that if L satisfies the so-called Davies-Gaffney estimate, the estimate (DUE^v) self-improves into a full Gaussian upper estimate. We now show that, if \mathcal{E} is a Dirichlet form, if M and v are doubling, and if L again satisfies the Davies-Gaffney estimate (which is equivalent to the finite speed propagation of the associated wave equation), then (DUE^v) is equivalent to

$$(N^v) \quad \|f\|_2^2 \leq C(\|fv_r^{-1/2}\|_1^2 + r^2\mathcal{E}(f)), \quad \forall r > 0, \quad \forall f \in \mathcal{F}.$$

and

$$(GN_q^v), \quad \|fv_r^{\frac{1}{2}-\frac{1}{q}}\|_q^2 \leq C(\|f\|_2^2 + r^2\mathcal{E}(f)), \quad \forall r > 0, \quad \forall f \in \mathcal{F},$$

for $q > 2$ small enough. Here $v_r(x) := v(x, r)$. This is a generalization of the case where $v(x, r) \simeq r^D$, and this also allows one to recover the previously known characterisations of (DUE^v) due to Saloff-Coste and Grigor'yan in the case $v = V$, but without having to go through Moser iteration. One also recovers results from [10].

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The stability of the elliptic Harnack inequality

RICHARD F. BASS

A justly famous theorem of Moser [3] says that if \mathcal{L} is the uniformly elliptic operator in divergence form given by

$$\mathcal{L}f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(\cdot) \frac{\partial f}{\partial x_j}(\cdot) \right) (x)$$

acting on functions on \mathbb{R}^d , where the a_{ij} are also bounded and measurable, then an elliptic Harnack inequality (EHI) holds for functions that are non-negative and harmonic with respect to \mathcal{L} in a domain. This is one of the more important theorems in the study of elliptic and parabolic partial differential equations, and is used, for example, in deriving *a priori* regularity results for harmonic functions and for heat kernels.

The operator \mathcal{L} is associated with the Dirichlet form

$$\mathcal{E}_{\mathcal{L}}(f, f) = \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) dx.$$

If the a_{ij} are bounded and the matrices $a(x) = (a_{ij}(x))$ are uniformly positive definite, then $\mathcal{E}_{\mathcal{L}}$ is comparable to \mathcal{E}_{Δ} , where

$$\mathcal{E}_{\Delta}(f, f) = \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx,$$

which is the Dirichlet form corresponding to the Laplacian. Thus one could rephrase Moser's theorem as saying that whenever the Dirichlet form corresponding to an operator \mathcal{L} is comparable to the Dirichlet form corresponding to the Laplacian, then the EHI holds for non-negative functions that are harmonic with respect to \mathcal{L} in a domain.

We can view Moser's theorem as a stability theorem for the EHI. In this talk we will discuss generalizing this stability property to very general state spaces. We show that provided some mild regularity holds, then whenever two Dirichlet forms \mathcal{E}_1 and \mathcal{E}_2 are comparable with corresponding operators \mathcal{L}_1 and \mathcal{L}_2 , the EHI holds for non-negative harmonic functions with respect to \mathcal{L}_1 if and only if the EHI holds for non-negative harmonic functions with respect to \mathcal{L}_2 .

We also provide a characterization of the EHI. Provided the regularity holds, this characterization can be considered as necessary and sufficient conditions for the EHI.

We do require some mild regularity. For example, one of our assumptions is that volume doubling holds. Whereas the parabolic Harnack inequality implies volume doubling, the example of Delmotte [2] shows that the EHI can hold even though volume doubling does not. Since every known approach to proving an EHI uses volume doubling in an essential way, the problem of finding necessary and sufficient conditions for the EHI to hold without assuming any regularity looks very hard.

Let $V(x, r)$ be the volume of $B(x, r)$ with respect to the measure $\mu(A) = \sum_{x \in A} \mu_x$. Let $C(x, r)$ be the capacity of $B(x, r)$ (a definition is given in the next section). Finally define $E(x, r) = V(x, r)/C(x, r)$. It will turn out that $E(x, r)$ is comparable to the expected time that the process spends in $B(x, r)$ when started at x .

The novel feature of this paper is to introduce the adjusted Poincaré inequality (API):

$$\sum_{y \in B(x, r)} |f(y) - f_{B(x, r)}|^2 \mu_y \leq cE(x, r)\mathcal{E}_{B(x, c'r)}(f, f).$$

Here $c' > 1$, f_A is the average value of f on the set A with respect to the measure μ , and \mathcal{E}_A is the Dirichlet form restricted to the set A . Note that in the usual Poincaré inequality, $E(x, r)$ is replaced by r^β for β equal to some constant, most often, $\beta = 2$.

We also use another inequality, which we call the cut-off inequality (COI). This is closely related to the cut-off Sobolev inequality of [1].

Our first main theorem is that if transience and regularity hold, then the EHI holds if and only if both the COI and API hold. This immediately implies our second theorem, the stability result, which says that if transience and the regularity hold and the EHI holds for a weighted graph, then the EHI holds for every equivalent weighted graph.

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