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Complex Algebraic Geometry

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Abstract. The conference focused on several topics, classical and modern, in the classification theory of compact algebraic and Kähler varieties, and on several methods, from singularity theory, topology, homological algebra, Geometric Invariant Theory and Moduli theory, char p methods.

Mathematics Subject Classification (2010): 14xx, 18xx, 32xx, 53xx.

Introduction by the Organisers

The workshop *Complex Algebraic Geometry*, organized by Fabrizio Catanese (Bayreuth), Christopher Hacon (Salt Lake City), Yujiro Kawamata (Tokyo) and Bernd Siebert (Hamburg), drew together 52 participants from all over the world.

There were several young PhD students and PostDocs, and a quite remarkable group of established leaders of the fields related to the thematic title of the workshop. It was quite difficult to decide which talks to choose for the program, in view of the variety of very attractive options. Eventually, thanks to the kind offer of some senior participants to decline the offer to deliver a talk, we ended with 21 50 minutes talks, all followed by a lively discussion.

As usual at an Oberwolfach Meeting, the mathematical discussions continued outside the lecture room throughout the day and the night. The Conference fully realized the aim of setting in contact mathematicians with different specializations and non uniform background, of presenting new fashionable topics alongside with new insights on long standing classical open problems.

A central role was played by classification theory of projective and Kähler varieties, their minimal models, vanishing theorems, generic positivity , base point freeness, and the role of singularities. (for instance pertaining to the classification of). There were talks on new results on algebraic surfaces, on irregular varieties, quotients of Abelian varieties, Fano manifolds, and compactifications of the vector group. Some talks were dedicated to the plane Cremona group and to the use of derived categories for rationality questions.

Chow and Hilbert schemes, GIT limits, stability, moduli spaces, were another direction which was present. The action of the absolute Galois group on moduli spaces and on the topology and Hodge structure of varieties was also another theme. Finally, different approaches to moduli spaces of curves with symmetries were presented.

In spite of the title of the conference, also characteristic p methods and problems were exposed.

The variety of striking results and the very interesting and challenging proposals presented in the workshop made the participation highly rewarding. We hope that these abstracts will give a clear and attractive picture, which will be useful to the mathematical community.

Workshop: Complex Algebraic Geometry

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Abstracts

Faithful Actions of Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) and Change of Fundamental Group

Ingrid Bauer
(joint work with F. Catanese, F. Grunewald)

The results presented in this talk are contained in [2]. We begin with the following observation:

Remark 1. *1) $\sigma \in \mathrm{Aut}(\mathbb{C})$ acts on $\mathbb{C}[z_0, \dots z_n]$, by sending*

$$P(z) = \sum_{I=(i_0,\dots,i_n)} a_I z^I \mapsto \sigma(P)(z) := \sum_{I=(i_0,\dots,i_n)} \sigma(a_I) z^I.$$

2) Let X be a projective variety $X \subset \mathbb{P}^n_{\mathbb{C}}$, $X := \{z | f_i(z) = 0 \ \forall i\}$. The action of σ extends coordinatewise to $\mathbb{P}^n_{\mathbb{C}}$, and carries X to the set $\sigma(X)$ which is another variety, denoted by X^σ, and called the conjugate variety*. In fact, since $f_i(z) = 0$ implies $\sigma(f_i)(\sigma(z)) = 0$, one has that $X^\sigma = \{w | \sigma(f_i)(w) = 0 \ \forall i\}$.*
3) Likewise, if $f \colon X \to Y$ is a morphism, its graph Γ_f is a subscheme of $X \times Y$, hence we get a conjugate morphism $f^\sigma \colon X^\sigma \to Y^\sigma$.

In the 60's J. P. Serre showed in [5] that there exists a field automorphism $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety X defined over $\bar{\mathbb{Q}}$ such that X and the Galois conjugate variety X^σ have non isomorphic fundamental groups, in particular they are not homeomorphic.

We prove here a strong sharpening of the phenomenon discovered by Serre: observe in this respect that, if $\mathfrak{c}$ denotes complex conjugation, then X and $X^{\mathfrak{c}}$ are diffeomorphic.

Theorem 2. *If $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is not in the conjugacy class of $\mathfrak{c}$, then there exists a surface isogenous to a product X such that X and the Galois conjugate surface X^σ have non isomorphic topological fundamental groups.*

Remark 3. *Since the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ leaves the algebraic fundamental group invariant, we have that the profinite completions of $\pi_1(X)$ and of $\pi_1(X^\sigma)$ are isomorphic.*

This result is obtained in several steps. To an algebraic number a and $g \geq 3$ we associate the hyperelliptic curve C_a of genus g defined by the equation

$$w^2 = (z-a)(z+2g)\Pi_{i=0}^{2g-1}(z-i).$$

Let $F_a \colon C_a \to \mathbb{P}^1$ be a certain functorial Belyi function and denote by $\psi_a \colon D_a \to \mathbb{P}^1$ the normal closure of C_a .

Remark 4. *1) We denote by G_a the monodromy group of D_a and observe that there is a subgroup $H_a \subset G_a$ acting on D_a such that $D_a/H_a \cong C_a$.*
2) Observe moreover that the degree d of the Belyi function F_a depends not only on the degree of the field etension $[\mathbb{Q}(a) : \mathbb{Q}]$, but much more on the height of the

algebraic number a; one may give an upper bound for the order of the group G_a in terms of these.

The pair (D_a, G_a) that we get is a so called triangle curve, according to the following definition:

Definition 5. *1) A G-* marked variety *is a triple (X, G, α) where $\alpha\colon X \times G \to X$ is an effective action of the group G on X*
Two marked varieties (X, G, α), (X', G, α') are said to be isomorphic *if there is an isomorphism $f\colon X \to X'$ transporting the action $\alpha\colon X \times G \to X$ into the action $\alpha'\colon X' \times G' \to X'$, i.e., such that*

$$f \circ \alpha = \alpha' \circ (f \times \mathrm{id}) \Leftrightarrow \eta' = \mathrm{Ad}(f) \circ \eta, \quad \mathrm{Ad}(f)(\phi) := f\phi f^{-1}.$$

2) A marked curve (D, G, η) consisting of a smooth projective curve of genus g and an effective action of the group G on D is said to be a marked triangle curve of genus g *if $D/G \cong \mathbb{P}^1$ and the quotient morphism $p\colon D \to D/G \cong \mathbb{P}^1$ is branched in three points.*

Without loss of generality we may assume that the three branch points in $\mathbb{P}^1$ are $\{0, 1, \infty\}$ and we may choose a monodromy representation $\mu\colon \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to G_a$ corresponding to the normal ramified covering $\psi_a\colon D_a \to \mathbb{P}^1$. Denote further by $\tau_0,\ \tau_1,\ \tau_\infty$ the images of geometric loops around $0,\ 1,\ \infty$. Then we have that G_a is generated by $\tau_0,\ \tau_1,\ \tau_\infty$ and $\tau_0 \cdot \tau_1 \cdot \tau_\infty = 1$. By Riemann's existence theorem the datum of these three generators of the group G_a determines a *marked triangle curve* (see [3]).

Theorem 6. *To any algebraic number $a \notin \mathbb{Z}$ there corresponds, through a canonical procedure (depending on an integer $g \geq 3$), a marked triangle curve (D_a, G_a).*

This correspondence yields a faithful action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *on the set of isomorphism classes of marked triangle curves.*

Let us recall now the basic definitions underlying our next construction: the theory of surfaces isogenous to a product, introduced in [3] (see also [4]), and which holds more generally for varieties isogenous to a product.

Definition 7. *1) A surface* isogenous to a (higher) product *is a compact complex projective surface S which is a quotient $S = (C_1 \times C_2)/G$ of a product of curves of resp. genera $g_1, g_2 \geq 2$ by the free action of a finite group G. It is said to be* unmixed *if the embedding $i\colon G \to \mathrm{Aut}(C_1 \times C_2)$ takes values in the subgroup (of index at most two) $\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$.*
2) A Beauville surface *is a surface isogenous to a (higher) product which is* rigid, *i.e., it has no nontrivial deformation.*
3) An étale marked surface *is a triple (S', G, η) such that the action of G is fixpoint free. An étale marked surface can also be defined as a quintuple (S, S', G, η, F) where $\eta\colon G \to \mathrm{Aut}(S')$ is an effective free action, and $F\colon S \to S'/G$ is an isomorphism.*

Remark 8. *Consider the coarse moduli space $\mathfrak{M}_{x,y}$ of canonical models of surfaces of general type X with $\chi(\mathcal{O}_X) = x, K_X^2 = y$. We denote by $\mathfrak{M}$ the disjoint union*

$\cup_{x,y\geq 1}\mathfrak{M}_{x,y}$, *and we call it the* moduli space of surfaces of general type. *Fix a finite group G and consider the moduli space $\hat{\mathfrak{M}}^G_{x,y}$ for étale marked surfaces (X, X', G, η, F), where the isomorphism class $[X] \in \mathfrak{M}_{x,y}$. This moduli space $\hat{\mathfrak{M}}^G_{x,y}$ is empty or is a finite étale covering space of $\mathfrak{M}_{x,y}$.*

Recall the following result concerning surfaces isogenous to a product ([3], [4]):

Theorem 9. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product. Then any surface X with the same topological Euler number and the same fundamental group as S is diffeomorphic to S. The corresponding subset of the moduli space $\mathfrak{M}^{top}_S = \mathfrak{M}^{diff}_S$, corresponding to surfaces homeomorphic, resp, diffeomorphic to S, is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.*

If S is a Beauville surface (i.e., S is rigid) this implies: $X \cong S$ or $X \cong \bar{S}$. It follows also that a Beauville surface is defined over $\bar{\mathbb{Q}}$, whence $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the discrete subset of the moduli space $\mathfrak{M}$ of surfaces corresponding to Beauville surfaces. We make the following

Conjecture 10. *The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the discrete subset of the moduli space $\mathfrak{M}$ of surfaces of general type corresponding to Beauville surfaces.*

We can prove the following:

Theorem 11. *The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of connected components of the (coarse) moduli space of étale marked surfaces isogenous to a higher product.*

With a rather elaborate strategy (i.e., showing that the kernel $mathfrakK$ of the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ has to be Abelian, which implies by a result of Fried and Jarden, that $mathfrakK$ is trivial) we can then show the stronger result:

Theorem 12. *The absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of connected components of the (coarse) moduli space of surfaces of general type.*

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A tale of two surfaces

Arnaud Beauville

1. Introduction

The aim of the talk was to point out a link between two surfaces which have appeared recently in the literature: the *surface of cuboids* ([8], [6]) and the surface (actually a family of surfaces) discovered by Schoen [7]. We showed that both surfaces give rise to a surface X with $q = 4$, whose canonical map is 2-to-1 onto a complete intersection of 4 quadrics $\Sigma \subset \mathbb{P}^6$ with 48 nodes. In the first case (§2) X is a quotient $(C \times C')/(\mathbb{Z}/2)^2$, where C and C' are genus 5 curves with a free action of $(\mathbb{Z}/2)^2$. In the second case (§3), X is a double étale cover of the Schoen surface. Despite their similarity the two families of surfaces X constructed are different, in fact they have different homotopy type.

When the canonical map of a surface X of general type has degree > 1 onto a surface, that surface either has $p_g = 0$ or is itself canonically embedded ([1], Th. 3.1). Our surfaces X provide one more example of the latter case, which is rather exceptional (see [4] for a list of the examples known so far).

2. The surface of cuboids and its deformations

In $\mathbb{P}^4$, with coordinates $(x, y; u, v, w)$, we consider the curve C given by

$$u^2 = a(x,y) \quad , \quad v^2 = b(x,y) \quad , \quad w^2 = c(x,y) \tag{1}$$

where a, b, c are quadratic forms in x, y. We assume that the zeros of a, b, c form a set of 6 distinct points. Then C is a smooth curve of genus 5, canonically embedded. It is preserved by the group $\Gamma_+ \cong (\mathbb{Z}/2)^3$ which acts on $\mathbb{P}^4$ by changing the signs of u, v, w. The subgroup $\Gamma \subset \Gamma_+$ (isomorphic to $(\mathbb{Z}/2)^2$) which changes an even number of signs acts freely on C.

Proposition 1. *Let C, C' be two genus 5 curves of type (1), and let X be the quotient of $C \times C'$ by the diagonal action of $\Gamma \cong (\mathbb{Z}/2)^2$.*

1) *X is a minimal surface of general type with $q = 4$, $p_g = 7$, $K^2 = 32$.*

2) *The involution i_X of X defined by the action of $\Gamma_+/\Gamma \cong \mathbb{Z}/2$ has 48 fixed points. The canonical map $\mathrm{can}_X : X \to \mathbb{P}^6$ factors through i_X, and induces an isomorphism of X/i_X onto a complete intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.*

Proof : 1) The computation of the numerical invariants of X is straightforward.

2) Let us denote by $(x', y'; u', v', w')$ the coordinates on C', and by a', b', c' the corresponding quadratic forms. A basis of the canonical space $H^0(X, K_X) = \big(H^0(C, K_C) \otimes H^0(C', K_{C'})\big)^\Gamma$ is given by the elements

$$X = x \otimes x' \quad Y = x \otimes y' \quad Z = y \otimes x' \quad T = y \otimes y' \quad U = u \otimes u' \quad V = v \otimes v' \quad W = w \otimes w'$$

They satisfy the relations

$$XT - YZ = 0 \ , \ U^2 = A(X,Y,Z,T) \ , \ V^2 = B(X,Y,Z,T) \ , \ W^2 = C(X,Y,Z,T) \ ,$$

where A, B, C are quadratic forms satisfying $A(X,Y,Z,T) = a(x,y) \otimes a(x',y')$, and the analogous relations for B and C.

Let Σ be the surface defined by these 4 equations. Since i_X acts trivially on $H^0(X, K_X)$, the canonical map can_X induces a map from X/i_X onto Σ. Since $K_X^2 = 32 = 2\deg(\Sigma)$, this map is one-to-one, hence an isomorphism.

The number ν of fixed points of i_X can be computed directly; it can also be deduced from the formula $\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_\Sigma) - \dfrac{\nu}{4}$, which follows from Riemann-Roch. □

Example. Let us take for C and C' the curve C_0 defined by

$$u^2 = xy \quad , \quad v^2 = x^2 - y^2 \quad , \quad w^2 = x^2 + y^2 \ .$$

We get for Σ the following equations :

$$XT = YZ = U^2 \quad , \quad V^2 = X^2 - Y^2 - Z^2 + T^2 \quad , \quad W^2 = X^2 + Y^2 + Z^2 + T^2 \ ;$$

or, after the linear change of variables $X = \mathsf{x} + \mathsf{t}$, $T = \mathsf{t} - \mathsf{x}$, $Y = \mathsf{y} + \mathsf{iz}$, $Z = \mathsf{y} - \mathsf{iz}$, $U = \mathsf{u}$, $V = 2\mathsf{v}$, $W = 2\mathsf{w}$:

$$\mathsf{t}^2 = \mathsf{x}^2 + \mathsf{y}^2 + \mathsf{z}^2 \quad , \quad \mathsf{u}^2 = \mathsf{y}^2 + \mathsf{z}^2 \quad , \quad \mathsf{v}^2 = \mathsf{x}^2 + \mathsf{z}^2 \qquad \mathsf{w}^2 = \mathsf{x}^2 + \mathsf{y}^2 \ .$$

These are the equations of the *surface of cuboids*, studied in [8], [6]. It encodes the relations in a cuboid (= rectangular box) between the sides $\mathsf{x}, \mathsf{y}, \mathsf{z}$, the diagonals of the faces $\mathsf{u}, \mathsf{v}, \mathsf{w}$, and the big diagonal t. Thus the surface of cuboids belongs to a 6-dimensional family of intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.

The curve C_0 is isomorphic to the modular curve $X(8)$, and the map $C_0 \times C_0 \to \Sigma$ can be described in terms of theta functions [5].

3. The Schoen surface

The Schoen surfaces S have been defined in [7], and studied in [3]. A Schoen surface S is contained in its Albanese variety A; it has the following properties:

a) $K_S^2 = 16$, $p_g = 5$, $q = 4$ (hence $\chi(\mathcal{O}_S) = 2$);

b) The involution (-1_A) induces an involution i_S of S with 40 fixed points. The canonical map $\mathrm{can}_S : S \to \mathbb{P}^4$ factors through i_S, and induces an isomorphism of S/i_S onto the complete intersection of a quadric and a quartic in $\mathbb{P}^4$ with 40 nodes [3].

Let ℓ be a line bundle of order 2 on A; we denote by $\pi : B \to A$ the corresponding étale double cover, and put $X := \pi^{-1}(S)$.

Proposition 2. 1) *X is a minimal surface of general type with $q = 4$, $p_g = 7$, $K_X^2 = 32$.*

2) *For an appropriate choice of ℓ, the involution (-1_B) induces an involution i_X of X with 48 fixed points. The canonical map $\mathrm{can}_X : X \to \mathbb{P}^6$ factors through i_X, and induces an isomorphism of X/i_X onto the complete intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.*

Idea of proof : 1) Since $\pi : X \to S$ is an étale double cover, we have $K_X^2 = 32$ and $\chi(\mathcal{O}_X) = 4$; using Schoen's original construction one finds $q = 4$, hence $p_g = 7$.

2) The surface X has a natural action of $(\mathbb{Z}/2)^2$, given by the involution i_X induced by (-1_B) and the involution τ associated to the double covering $X \to S$. We want to determine how these involutions act on $H^0(X, K_X)$. The decomposition of $H^0(X, K_X)$ into eigenspaces for τ is

$$H^0(X, K_X) \cong H^0(S, K_S) \oplus H^0(S, K_S \otimes \ell) \ .$$

The key point of the proof is the following

Claim. *One can choose ℓ so that i_X acts trivially on $H^0(X, K_X)$.*

Idea of proof : This is equivalent to saying that i_S acts trivially on $H^2(S, \ell)$. One uses the holomorphic Lefschetz formula to translate this into a property of ℓ with respect to the fixed points of i_S, then some coding theory to prove that some line bundles ℓ satisfy this property.

Once this is done, one concludes as follows. Choose bases $(x_0, \ldots, x_4)$ and (u, v) of the $(+1)$ and (-1)-eigenspaces in $H^0(X, K_X)$ with respect to τ. The elements u^2, uv, v^2 of $H^0(X, K_X^{\otimes 2})$ are invariant under τ and i_X, therefore they are pull-back of i_S-invariant forms in $H^0(S, K_S^{\otimes 2})$. Such a form comes from an element of $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}}(2))$. Thus we have equations

$$u^2 = a(x) \qquad uv = b(x) \qquad v^2 = c(x) \qquad q(x) = 0$$

where a, b, c, q are quadratic forms in $x_0, \ldots, x_4$, and q is the quadratic form in $\mathbb{P}^4$ vanishing on the image of can_S. Geometric considerations show that the subvariety Σ of $\mathbb{P}^6$ defined by these 4 quadratic forms is a surface. Since i_X acts trivially on $H^0(X, K_X)$, the canonical map $X \to \mathbb{P}^6$ induces a map $X/i_X \to \Sigma$ which has degree 1, hence is an isomorphism.

Finally the number of fixed points of i_X is computed as in §2. ☐

Details can be found in [2].

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Birational stability of the orbifold cotangent bundle

Frédéric Campana
(joint work with Mihai Păun)

The extension of geometric properties from (complex) projective varieties X to (orbifold) pairs (X, Δ) has proven to be unavoidable in birational classification, not only because of the so called LMMP (Log-Minimal Model Program), but also to deal with the multiple fibres of fibrations $f : X \to Y$, encoded in an 'orbifold' base (Y, Δ_f) which permits to a certain extent to express the geometric properties of X in terms of the generic fibre X_y of f, together with the 'geometry' of (Y, Δ_f). The main task being to define appropriately the geometric invariants of such (orbifold) pairs. This extension from the category of varieties to the category of (orbifold) Log-pairs is usually done to the expense of only minor technicalities, while the range of applications is immensely widened. This is once more the case at hand in what follows.

In the sequel, (X, Δ) will be a l.c (for 'Log-canonical') pair, with X a normal and connected complex projective variety of dimension n, and $\Delta := \sum_j a_j.D_j$ an effective divisor with rational coefficients $0 \leq a_j \leq 1$, supported on the (finite) union D of the $D_j's$, prime and distinct Weil divisors on X. These pairs interpolate between the two extreme cases when $\Delta = 0$ (and then $(X, 0) = X$), and the purely logarithmic case where $\Delta = D$ (and then (X, D) is identified with the quasi-projective variety $X - D$). In the general case, (X, Δ) stands for a virtual ramified cover of X ramifying to order $m_j := \frac{1}{1-a_j}$ over D_j.

Theorem 1. *If $K_X + \Delta$ is pseudo-effective, then $\Omega^1(X, \Delta)$ is 'generically positive' ('gsp' for short).*

Recall that a $\mathbb{Q}$-line bundle L is pseudo-effective if $L + a.H$ is $\mathbb{Q}$-effective for any polarization H on X and any positive rational number a. A torsion-free sheaf F is said to be 'gsp' if, for any H and any quotient Q of F, we have: $det(Q).H^{n-1} \geq 0$. This is tested on 'Mehta-Ramanathan curves' C, generic complete intersections of high multiples of H. An essential property of such curves is to be 'free' (i.e.: they can be chosen to avoid any given codimension 2 subset of X). This property will be used crucially in each step of the proof.

When $\Delta = 0$, we recover Miyaoka's generic semi-positivity theorem, because K_X is pseudo-effective ('pseff' for short) if and only if X is not uniruled (i.e.: covered by rational curves). The original proof of Miyaoka however mixed char $p > 0$ and char 0 arguments, and could not be adapted to the present orbifold context. Our proof is in char 0 only.

A second case when the statement of the theorem does not need any new definition is when $(X, \Delta = D)$ is purely logarithmic: $\Omega^1(X, D)$ is then defined to be $\Omega^1_U(Log(D_{|U}))$ on the Zariski open set U with codimension at most two complement consisting of points where X is smooth and D is of normal crossings (or even smooth). One then defines $\Omega^1(X, D)$ $u_*(\Omega^1_U(Log(D_{|U})))$.

In the general case, we define similarly $\Omega^1(X,\Delta)$ over U first and take the extension as above. Assume thus that we are near a point $x\in U$, and have local coordinates $(x_1,...,x_n)$ such that D is supported in the union of the coordinates hyperplanes. For $1\leq j\leq n$, let $a_j=\frac{b_j}{c_j}$ be the coefficient of the coordinate hyperplane $x_j=0$ in Δ, with $0\leq b_j\leq c_j$ coprime integers. When $a_j=0$ (resp. $a_j=1$), we thus have: $b_j=0,c_j=1$ (resp. $b_j=c_j=1$). Otherwise $1\leq b_j<c_j$.

Morally, $\Omega^1(X,\Delta)$ is then, locally, the locally free sheaf of O_X-modules generated by the $\frac{dx_j}{x_j}^{a_j}$. This does not make sense over X but it does by taking $\pi^*(\frac{dx_j}{x_j}^{a_j}=(k.c_j).y^{k_j\cdot(c_j-b_j)}.\frac{dy_j}{y_j}$, if $\pi:Y\to X$ is any local ramified cover defined by: $\pi(y_1,...,y_n)=(y_1^{k_1\cdot c_1},....,y_n^{k_n\cdot c_n})$. In this way, $\pi^*(\Omega^1(X,\Delta))$ can be defined locally, for any choice of positive integers $k_1,....,k_n$.

In order to have a global definition of $\pi^*(\Omega^1(X,\Delta))$, we take a global cyclic ramified cover $\pi:Y\to X$ associated to any reduced section of $k.c.H-D$, for $k>0$ sufficiently large and $c:=lcm(c_j)$. The Galois group is then $\mathbb{Z}_{k.c}$. We say that $\pi^*(\Omega^1(X,\Delta))$ is 'gap' if for any G-invariant quotient sheaf Q of $\pi^*(\Omega^1(X,\Delta))$, and any H, $det(Q).(\pi^*(H))^{n-1}\geq 0$.

Remark 1. This property turns out to be independent of the choices made. This follows from the proof of the theorem. A direct conceptual proof were more interesting.

Corollary 1. *Let (X,Δ) be as above (l.c, thus). Assume that $K_X+\Delta$ is pseff, and that $(K_X+\Delta).H^{n-1}=0$ for some H. Let L a line bundle on X together with an inclusion $\pi^*(L)\to\otimes^m(\pi^*(\Omega^1(X,\Delta)))$, for some $m>0$. Then $L.H^{n-1}\leq 0$.*

When $\Delta=0$, this says for example that the covariant holomorphic tensors on X are 'parallel' if K_X is numerically trivial, a conclusion also obtained via Ricci-flat Kähler metrics and Bochner formula. But the 'orbifold' version above applies to many more situations.

The proof of the above theorem rests on Bogomolov-Mc Quillan algebraicity criterion for foliations and a refinement of Viehweg's weak positivity theorem for direct images of relative pluricanonical sheaves, taking into account the multiple fibres of a fibration (even when $\Delta=0$, these intervene crucially in the proof).

Using the existence of Log-minimal models by Birkar-Cascini-Hacon-Mc Kernan ([BCHM]), we can deduce:

Corollary 2. *Let (X,D) be a purely l.c logarithmic pair. Assume the existence of a big line bundle L on X, together with an injection: $L\to\otimes^m(\Omega^1_X(Log(D)))$. Then K_X+D is big.*

Let us explain the idea when $K:=K_X+D$ is nef: we then have: $L=aH+E$ for some $a>0$ and E effective. From the gsp property we obtain: $a.H.K^{n-1}\leq L.K^{n-1}\leq c.K.K^{n-1}=c.K^n$, where $c=c(m,n)>0$ is such that $det(\Omega^1_X(Log(D)))=c.K$. This is because $K=lim(H_n:=K+\frac{1}{n})$. We now conclude by the Hodge index theorem: $a.(H^n)^{\frac{1}{n}}\leq a.H.K^{n-1}\leq c.K^n$ that $(\frac{a}{c})^n.H^n\leq K^n$. Thus K^n has positive volume and is big.

The case when K is pseff is easily reduced to the preceding case by using [BCHM].

It remains to show that K has to be pseff: if not replace D by $D + t.H$ where $t > 0$ is minimal such that $K + tH$ is pseff. This t is rational, by [BCHM] again. But $\Omega^1(X, D)$ injects naturally into $\Omega^1(X, D + tH)$. Thus $K + tH$ its big, which contradicts the minimality of t. (Strictly speaking, for this step we need to prove the statement above for $(X, D + tH)$ assuming it to be pseff, but the proof is the same while the statement requires replacing L by $\pi^*(L)$, and similarly for $\Omega^1(X, \Delta)$).

Using the existence of the Viehweg-Zuo line bundle (which gives a big line bundle L together with an injection as above in the situation below), we get:

Theorem 2. *Let $f : Z \to B$ be an algebraic proper submersion between connected quasi-projective manifolds. Assume that the fibres of f all have a semi-ample canonical bundle, and that the 'variation' of f is maximal, that is: the rank of the Kodaira-Spencer map is equal to $dim(B)$ at the generic point of B. Then B is of Log-general type (i.e: $K_X + D$ is big, if $B = X - D$, where X is smooth projective, and D a normal crossing divisor on X).*

This statement (sometimes called 'Shafarevich hyperbolicity conjecture', and conjectured by Viehweg-Zuo) was proved when $dim(B) \leq 3$ by Kebekus-Kovács.

On base point freeness in positive characteristic

Paolo Cascini

(joint work with Hiromu Tanaka and Chenyang Xu)

0.1. **Introduction.** Mori's cone theorem and Kawamata-Shokurov base point free theorem represent two of the main tools in the study of the birational geometry of varieties defined over the field of complex number (e.g. [3]). The natural generalisation of these results to varieties defined over an arbitrary algebraically closed field is still open. The purpose of our work is to extend many of the results which, over $\mathbb{C}$, are obtained as applications of these two theorems to varieties defined over a field of positive characteristic.

More specifically, let X be a normal variety defined over an algebraically closed field k of characteristic $p > 0$ and let B be an effective $\mathbb{R}$-divisor such that $K_X + B$ is $\mathbb{R}$-Cartier. Fix a closed point $x \in X$. For any D effective, and for any positive integer e, we consider the trace map

$$Tr^e_X(D)\colon F^e_*(\mathcal{O}_X(-(p^e - 1)K_X - D)) \to \mathcal{O}_X.$$

A pair (X, B) is *strongly F-regular* at x if, for every effective divisor E, there exists a positive integer e such that $Tr^e_X(\ulcorner(p^e - 1)B\urcorner + E)$ is surjective at x. By a result of Hara and Watanabe, if (X, B) is of strongly F-regular type then (X, B) is klt. Moreover, combining together results by Hara and Watanabe and Tagaki, if (X, B) is a pair over $\mathbb{C}$ then (X, B) is klt if and only if its reduction modulo p is strongly F-regular for any sufficiently large prime p.

0.2. **Strictly nef divisors.** Recall that a divisor D on a normal projective variety X is said to be *strictly nef* if $D \cdot C > 0$ for all C curve in X. Note that, even in positive characteristic, strictly nef divisors are not necessarily ample. Our first result is the following:

Main Theorem 1: Let (X, B) be a strongly F-regular pair, where B is an effective $\mathbb{R}$-divisor. Assume that A is an ample $\mathbb{R}$-divisor such that $K_X + A + B$ is strictly nef. Then $K_X + A + B$ is ample.

As a consequence, we obtain the following rationality theorem:

Corollary Let (X, B) be a strongly F-regular pair, where B is an effective $\mathbb{Q}$-divisor. Assume that $K_X + B$ is not nef and A is an ample $\mathbb{Q}$-divisor. Let

$$\lambda := \min\{t > 0 \mid K_X + B + tA \text{ is nef }\}.$$

Then there exists a curve C in X such that $(K_X + \lambda A + B) \cdot C = 0$. In particular, λ is a rational number.

0.3. **Divisors of maximal nef dimension.** A divisor D over a normal projective variety X is of *maximal nef dimension* if $D \cdot C > 0$ for all C movable curve in X. In this case we have:

Main Theorem 2: Let X be a normal projective variety. Assume that A is an ample $\mathbb{R}$-divisor and $B \geq 0$ is a $\mathbb{R}$-divisor such that $K_X + B$ is $\mathbb{R}$-Cartier and $K_X + A + B$ is nef and of maximal nef dimension. Then $K_X + A + B$ is big.[1]

By the main result of [1] we obtain that if X is a normal projective variety defined over an uncountable algebraically closed field k, and L is a nef $\mathbb{R}$-divisor, then there exists an open set $U \subseteq X$ and a proper morphism $\varphi\colon U \to V$, such that L is numerically trivial on a very general fibre F of φ and for a very general point x, we have that $L \cdot C = 0$ if and only if C is contained in the fibre of φ containing x.

Thus, combining the results above we obtain the following:

Theorem: Let X be a normal projective variety. Assume that A is an ample $\mathbb{R}$-divisor, $B \geq 0$ is an $\mathbb{R}$-divisor such that $L = K_X + A + B$ is nef but not big. Then X is covered by rational curves R such that

$$L \cdot R = 0 \qquad \text{and} \qquad -2\dim X \leq (K_X + B) \cdot R < 0.$$

0.4. **Threefolds.** We now consider projective threefolds defined over an algebraically closed field of positive charcteristic. We first obtain the following:

Weak Cone Theorem: Let X be a $\mathbb{Q}$-factorial projective threefolds. Let B be an effective $\mathbb{Q}$-divisor on X whose coefficients are strictly less than one. Assume that $K_X + B$ is not nef. Then there exist an ample $\mathbb{Q}$-divisor A such that $K_X + A + B$ is not nef and finitely many curves $C_1, \cdots, C_r$ on X such that

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+A+B\geq 0} + \sum_{i=1}^{r} \mathbb{R}_{\geq 0}[C_i].$$

[1] The same result was independently obtained by J. M^{c}Kernan using different methods

Thus, by combining together earlier results of Kollár, Keel and Hacon and Xu, we are able to obtain the following weak version of the minimal model program for threefolds:

MMP for 3-folds: Let X be a $\mathbb{Q}$-factorial terminal projective threefold defined over $\overline{\mathbb{F}}_p$ with $p > 5$. Then there exists a K_X-negative birational contraction $f : X \dashrightarrow Y$ to a $\mathbb{Q}$-factorial terminal projective threefold such that one of the following is true:

(1) if K_X is pseudo-effective, then K_Y is nef;
(2) if K_X is not pseudo-effective, then there exist a K_Y-negative extremal ray R of $\overline{NE}(Y)$ and a surjective morphism $g : Y \to Z$ to a normal projective variety such that $g_*\mathcal{O}_Y = \mathcal{O}_Z$ and for every curve C in Y, $g(C)$ is a point if and only if $[C] \in R$.

Finally, we obtain the following:

Weak Base Point Free Theorem: Let (X, B) be a projective three dimensional log canonical pair for some big $\mathbb{Q}$-divisor $B \geq 0$ such that $K_X + B$ is nef. Assume that $p > \frac{2}{a}$ for any coefficient a of B.

(1) If $K_X + B$ is not numerically trivial, then

$$\kappa(X, K_X + B) = \nu(X, K_X + B) = n(X, K_X + B).$$

(2) If $\kappa(X, K_X + B) = 1$ or 2, then $K_X + B$ is semiample.
(3) If $k = \overline{\mathbb{F}}_p$, and all coefficients of B are strictly less than 1, then $K_X + B$ is semiample.

This talk is based on our recent preprint [2].

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Abelian varieties in Brill–Noether loci and irregular surfaces

Ciro Ciliberto

(joint work with Margarida Mendes Lopes, Rita Pardini)

This is a report on joint work in progress with Margarida Mendes Lopes and Rita Pardini.

In [1] the authors posed the problem of studying, and possibly classifying, situations like this:

(*) C is a smooth, projective, complex curve of genus g, Z is an irreducible r–dimensional subvariety of a Brill–Noether locus $W^s_d(C) \subsetneq J^d(C)$, and Z is stable under translations by the elements of an abelian subvariety $A \subsetneq J(C)$ of dimension $a > 0$ (if so, we will say that Z is A*–stable*).

Actually in [1] the variety Z *is* the translate of a positive dimensional proper abelian subvariety of $J(C)$, while the above more general formulation was given in [6].

The motivation for studying (*) resides, among other things, in a theorem of Faltings (see [7]) to the effect that if X is an abelian variety defined over a number field $\mathbb{K}$, and $Z \subsetneq X$ is a subvariety not containing any translate of a positive dimensional abelian subvariety of X, then the number of rational points of Z over $\mathbb{K}$ is finite. The idea in [1] was to apply Faltings' theorem to the d–fold symmetric product $C(d)$ of a curve C defined over a number field $\mathbb{K}$. If C has no positive dimensional linear series of degree d, then $C(d)$ is isomorphic to its Abel–Jacobi image $W_d(C)$ in $J^d(C)$, thus $C(d)$ has finitely many rational points over $\mathbb{K}$ if $W_d(C)$ does not contain any translate of a positive dimensional abelian subvariety of $J(C)$. The suggestion in [1] is that, if, by contrast, $W_d(C)$ contains the translate of a positive dimensional abelian subvariety of $J(C)$, then C should be *quite special*, e.g., it should admit a map to a curve of lower positive genus (curves of this kind clearly are in situation (*)). This idea was tested in [1], where a number of partial results were proven for low values of d.

The problem was taken up in [6] where, among other things, it is proven that if (*) holds, then $r+a+2s \leqslant d$, and, if in addition $d+r \leqslant g-1$, then $r+a+2s = d$ if and only if:

(a) there is a degree 2 morphism $\varphi\colon C \to C'$, with C' a smooth curve of genus a, such that $A = \varphi^*(J(C'))$ and $Z = W_{d-2a-2s}(C) + \varphi^*(J^{a+s}(C'))$.

In [6] there is also the following example with $(d,s) = (g-1,0)$:

(b) there is a degree 2 morphism $\varphi\colon C \to C'$, with C' a smooth curve of genus $g' = r+1$, A is the Prym variety of φ and $Z = \varphi_*^{-1}(K_{C'}) \cong A$.

One more family of examples we discovered is the following:

(c) C is hyperelliptic, there is a degree 2 morphism $\varphi\colon C \to C'$ with C' a smooth curve of genus a such that $g > 2a+1$, $A = \varphi^*(J(C'))$, $0 < s < g-1$ and $Z = \varphi^*(J^a(C')) + W_{d-2s-2a}(C) + W^s_{2s}(C)$ (notice that $W^s_{2s}(C)$ is a point).

The result in [6] goes in the direction indicated in [1]. The unfortunate feature of it is the hypothesis $d+r \leqslant g-1$ which turns out to be *quite strong*. To understand how strong it is, consider the case $(d,s) = (g-1,0)$, which is indeed the crucial one (see [6, Proposition 3.3]) and in which Debarre–Fahaloui's theorem is void.

Our first result is the full classification of the cases in which (*) happens and $d = r+a+2s$. We then prove that, with no further assumption, either (a) or (b) or (c) occurs.

The idea of the proof is not so different, in principle, from the one proposed in [6] in the restricted situation considered there. Indeed, one uses the A–stability of Z and its maximal dimension to produce linear series on C which are not birational, in fact composed with a degree 2 irrational involution. The main tool in [6], inspired by [1], is a Castelnuovo's type of analysis for the growth of the dimension of certain linear series.

Our approach also consists in producing a non birational linear series on C, but it is in a sense more direct. We consider (*) with $(d,s)=(g-1,0)$ and $a+r=g-1$, i.e., the basic case, in which Z is contained in $W_{g-1}(C)$, which is a translate of the theta divisor $\Theta\subset J(C)$. This immediately produces, using the Gauss map of Θ restricted to Z, a base point free sublinear series L of dimension r of the canonical series of C. It turns out that Z is birational to an irreducible component of the variety $C(g-1,L)\subset C(g-1)$ consisting of all divisors of degree $g-1$ contained in some divisor of L. The A–stability of Z implies that $C(g-1,L)$ has some other component besides the one birational to Z, and this forces L to be non–birational. Once one knows this, a (rather subtle) analysis of the map determined by L and of its image leads to the conclusion.

The motivation for considering this problem is for us quite different from the one of [1, 6]. It is in fact related to the study of irregular surfaces S of general type, where situation (*) presents itself in a rather natural way. For example, let $C\subset S$ be a smooth, irreducible curve, and assume C corresponds to the general point of an irreducible component $\mathcal{C}$ of the Hilbert scheme of curves on S which dominates $\mathrm{Pic}^0(S)$. There is also only one irreducible component $\mathcal{K}$ of the Hilbert scheme of curves homologous to canonical curves on S which dominates $\mathrm{Pic}^0(S)$ (this is called the *main paracanonical system*). The curves in $\mathcal{C}$ cut out on C divisors which are residual, with respect to $|K_C|$, of divisors cut out by curves in $\mathcal{K}$. Consider now the one of the two systems $\mathcal{C}$ and $\mathcal{K}$ whose curves cut on C divisors of minimal degree $d=\min\{C^2,K_S\cdot C\}$, and denote by s the dimension of the general fibre of this system over $\mathrm{Pic}^0(S)$. Then we have a natural restriction map $\mathrm{Pic}^0(S)\dashrightarrow W^s_d(C)\subset J^d(C)$, whose image is a q–dimensional abelian variety contained in $W^s_d(C)$, which is what happens in (*). Thus understanding (*) would provide us with the understanding of (most) curves on irregular surfaces.

Our aforementioned result on (*), even if restricted to the very special case in which Z has maximal dimension, turns out to be useful in surface theory. For example, if S is a minimal, irregular surface of general type, then $K_S^2\geqslant 2p_g$ (see [5]) and we are able to classify surfaces for which $K_S^2=2p_g$. Precisely we prove that minimal, irregular surface of general type with $K_S^2=2p_g$ are of one of the following types:

(i) $q=1$, an infinite family with any $p_g\geqslant 1$, which are suitable double covers of elliptic scrolls (classified in [8], see also [3]);

(ii) $p_g=q=2$, $K_S^2=4$, double covers of principally polarised abelian surfaces (A,Θ) branched along a divisor in $|2\Theta|$;

(iii) $p_g=q=3$, $K_S^2=6$, symmetric products of smooth curves of genus 3 (classified in [4]);

(iv) $p_g = q = 4$, $K_S^2 = 8$, products of genus 2 curves (classified in [2]).
The idea of the proof is as follows. Results in [5] yield $q \leqslant 4$ with equality only in case (iv). The case $q = 1$ was treated in [8]. The case $\chi = 1$, $q = 3$, was treated in [4]. The case $\chi = 1$, $q = 2$ leads to (ii), thus solving a conjecture which has been open for some years. The case $\chi \geqslant 2$, $2 \leqslant q \leqslant 3$, is excluded in the following way. One checks that, if such a surface S exists, then $\mathrm{Albdim}(S) = 2$. So by Severi's inequality $2p_g = K_S^2 \geqslant 4\chi$. This leads to the only numerical possibility $q = 3, p_g = 4, K_S^2 = 8$, which is ruled out using an analysis of the canonical map.

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Non-simplicity of the planar Cremona group (after S. Cantat and S. Lamy)

Igor Dolgachev

The Cremona group $\mathrm{Cr}_F(n)$ in dimension n over a field F is the group of birational transformation of the projective space $\mathbb{P}_F^n$. In purely algebraic terms it is the group of automorphisms $\mathrm{Aut}_F(F(t_1, \ldots, t_n))$ of the field of rational functions with coefficients in F. In the case $n = 1$, the group is isomorphic to the simple algebraic group $\mathrm{PGL}_2(F)$, but in the case $n > 1$ it does not admit any sensible algebraic structure. However, one can define the corresponding non-representable functor on the category of F-algebras [4]. In particular, the Lie algebra of the Cremona group makes sense, and it is isomorphic to the infinite-dimensional Lie algebra $\mathrm{Der}_F(F(t_1, \ldots, t_n))$ of derivations of $F(t_1, \ldots, t_n)$ over F.

In 1895 F. Enriques asked whether the group $\mathrm{Cr}_{\mathbb{C}}(2)$ is simple as an abstract group. Apparently not being aware of Enriques's question, Yu. Manin in Moscow in the sixties and D. Mumford in the early seventies posed the same question. Here are the arguments pro and contra of the simplicity statement.

Pro:
1) The Lie algebra $\mathrm{Der}_F(F(t_1,\dots,t_n))$ is a simple Lie algebra (L. Makar-Limanov).
2) By Noether's Factorization Theorem (see [5]), the group $\mathrm{Cr}_{\mathbb{C}}(2)$ is generated by its subgroup of projective transformations $\mathrm{PGL}_3(\mathbb{C})$ and a single transformation $(t_1,t_2)\mapsto(t_1^{-1},t_2^{-1})$, called the standard quadratic transformation. By a remark of M. Gizatullin, any proper normal subgroup H of $\mathrm{Cr}_{\mathbb{C}}(2)$ does not contain a non-identical projective transformation.

3) Gizatullin proved that the normal subgroup containing a Cremona transformation given by homogeneous polynomials of degree ≤ 7 coincides with the whole group [7].
4) Previous attempts to find a non-injective homomorphism of $\mathrm{Cr}_{\mathbb{C}}(2)$ some other group of Cremona transformations had failed.

Conra:V. Danilov proved that the subgroup of $\mathrm{Aut}_F(F[t_1,\dots,t_n])$ that consists of automorphisms of the polynomial algebra $F[t_1,\dots,t_n]$ with the jacobian equal to the identity is not simple [3].

In my talk I explain a recent remarkable result of Serge Cantat and Stéphane Lamy that gives the negative answer to the question of Enriques. The goal of my talk was to make the community of algebraic geometers to be aware of this result and hint on the methods of its proof coming from a different area of mathematics.

Theorem 1. *For any algebraically closed field F, the group $\mathrm{Cr}_{\mathbb{F}}(2)$ contains proper normal subgroups.*

In fact, one can construct an explicit birational transformations g of the plane such that the smallest normal subgroup containing some power g^n is proper.

The proof of the theorem is based on a known representation of the group $\mathrm{Cr}_{\mathbb{F}}(2)$ in the group of isometries of the infinite-dimensional hyperbolic space associated with the Néron-Severi space of Manin's bubble space of a smooth projective surface S obtained by blowing up all points in the plane including infinitely near points [8]. Its element are pairs $(D,\sum m_ix_i)$, where D is divisor class on S and $\sum m_ix_i$ is an element of the free abelian group generated by the set of closed points and infinitely points on S. The intersection product is defined by $\langle(D,\sum m_ix_i),(D',\sum m_i'x_i)\rangle = D\cdot D'-\sum m_im_i'$. This space $\mathcal{Z}(S)$, the Néron-Severi space of the bubble space, equipped with this pairing becomes a hyperbolic space of infinite dimension. The Cremona group $\mathrm{Cr}_F(2)$ has a natural faithful action by isometries of $\mathcal{Z}(\mathbb{P}^2)$.

An example of a subgroup of $\mathrm{Cr}_F(2)$ is a subgroup H of the group of biregular automorphisms of a rational surface S admitting a birational morphism $\pi: S\to \mathrm{P}_F^2$. In this case the group H acts on the Néron-Severi group $\mathrm{NS}(S)$ equipped with the intersection of divisor classes pairing. The space $\mathrm{NS}(S)$ can be viewed as the orthogonal space of $\mathcal{Z}(S)$ in $\mathcal{Z}(\mathbb{P}^2)$. It is a free module of some some rank ρ and in a natural hyperbolic orthonormal basis $(e_0,\dots,e_n)$ of $\mathrm{NS}(S)$ formed by $e_0=\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and the classes e_i of exceptional curves E_i of π, the action g_* of $g\in H$ is defined by the classical characteristic matrix of a Cremona transformation (see [5]).

We will restrict ourselves with trying to find a proper subgroup of the Cremona group among subgroups of a rational surface, although the paper deals with a more general case. Let $\mathbb{H}$ be the real hyperbolic space associated with the real vector space $V = \mathrm{NS}(S)_{\mathbb{R}}$, i.e. the connected component of the image of the light cone $\{x \in \mathbb{H} : x^2 > 0\}$ in $\mathbb{H} \setminus \{0\}/\mathbb{R}_+$ containing the ample cone. Its boundary consists of positive rays of isotropic vectors.

The group of isometries of a real hyperbolic space is a connected component of the projectivized orthogonal group of V. An isometry σ is called *hyperbolic* if it has two different fixed points on the boundary. It corresponding to two real eigenvalues $\lambda(\sigma) > 1$ and $\lambda(\sigma)^{-1}$ in V, all other eigenvalues are complex numbers of absolute value 1. There is a unique geodesic line $\mathrm{Ax}(\sigma)$ and σ preserves this line and acts on it by translating a point $x \in \mathrm{Ax}(\sigma)$ to a point $\sigma(x)$ with distance $\mathrm{dist}(x, \sigma(x))$ equal to $L(\sigma) = \log \lambda(\sigma)$.

Let ϵ, B be two positive numbers. A subset A of $\mathbb{H}$ is called (ϵ, B)-*rigid* if $\mathrm{diam}(A \cap_\epsilon \sigma(A)) \geq B$ for some isometry σ implies $\sigma(A) = A$. Here $A \cap_\epsilon \sigma(A)$ is the set of points whose distance to A and $\sigma(A)$ is less than or equal to ϵ. It is called ϵ-rigid if it is (ϵ, B)-rigid for some B. It is clear that, if σ is ϵ-rigid, then it is ϵ'-rigid for $\epsilon' < \epsilon$. In fact, the converse is true:if $\epsilon > 2\theta = 16 \log 3$ and $\mathrm{Ax}(\sigma)$ is 2θ-rigid, then it is also ϵ-rigid.

A hyperbolic isometry σ in a group G of isometries is called *tight* if $\mathrm{Ax}(\sigma)$ is 2θ-rigid, and, for all $\tau \in G$, $\tau(\mathrm{Ax}(\sigma) = \mathrm{Ax}(\sigma)$ implies $\tau \circ \sigma \circ \tau^{-1} = \sigma$ or σ^{-1}.

The main result from hyperbolic which is used in the proof of the theorem is the following.

Theorem 2 (Normal subgroup theorem)**.** *Let G be a group of isometries of a hyperbolic space $\mathbb{H}$. Suppose that $\sigma \in G$ is tight and satisfies $\frac{1}{20}L(g) > 60\theta + 2B$ for some $B > 0$. Then any element $\tau \neq 1$ in the smallest normal subgroup $\langle\langle\sigma\rangle\rangle \subset G$ containing σ satisfies the following alternative:either τ is conjugate to σ, or $L(\tau) > L(g)$.*

In particular, since $L(\sigma^2) > L(\sigma)$ and σ^2 is not conjugate to σ (they have different eigenvalues larger than 1), we obtain that $\langle\langle g^2\rangle\rangle$ is a proper normal subgroup of G.

To apply this theorem to the Cremona group we need a geometric condition that g_* satisfies the assumption of the Normal subgroup theorem.

Theorem 3. *Let S be a rational surface over F such that $\mathrm{Aut}(S)$ acts faithfully on $\mathrm{NS}(S)$ and let $g \in \mathrm{Aut}(S)$ such that $\sigma = g_*$ is hyperbolic. Let $V_g \subset \mathrm{NS}(S)_{\mathbb{R}}$ be the plane spanned by two isotropic eigenvectors of g_*. Assume that g_* acts identically on $V_g^\perp \cap \mathrm{NS}(S)$. Then $\mathrm{Ax}(g_*)$ is rigid, any $h \in \mathrm{Bir}(S) \cong \mathrm{Cr}_F(2)$ which preserves $\mathrm{Ax}(g_*)$ is an automorphism of S, and g_* is a tight element of $\mathrm{Bir}(S)$. In particular, for sufficiently large n, the group $\langle\langle g^n\rangle\rangle$ is a proper normal subgroup of $\mathrm{Cr}_F(2)$.*

Example 1. Let S be a general Coble surface, i.e. a rational surface obtained by blowing up the ten nodes $x_1, \ldots, x_{10}$ of a general rational plane curve of of degree 6. It is known that $\mathrm{Aut}(S)$ acts faithfully on $\mathrm{NS}(S)$ and its image in the subgroup of isometries of the orthogonal complement $K_S^\perp$ of the canonical class in $\mathrm{NS}(S)$ is equal to the 2-congruence subgroup $W(2)$, i.e. the group of isometries such that, for any $x \in K_S^\perp$, $\frac{1}{2}(\sigma(x) - x) \in K_X^\perp$ (see [1]).

Let $e_0, e_1, \dots, e_{10}$ be the canonical basis of $\mathrm{NS}(S)$. Consider the following divisors:

$$D_1 = 6e_0 - 2\sum_{i=1}^{8} e_i - e_9 - e_{10}, \quad D_2 = 6e_0 - 2\sum_{i=1}^{8} e_i - e_7 - e_8.$$

We have $D_1^2 = D_2^2 = 2$ and $D_1 \cdot D_2 = 4$. The plane L spanned by the divisor classes of D_1 and D_2 in $\mathrm{NS}(S)_\mathbb{R}$ contains two isotropic vectors. Consider the isometry ϕ of L defined by $D_1 \mapsto 4D_1 - D_2$ and $D_2 \mapsto D_1$. Then ϕ^2 maps D_1 to $15D_1 - 4D_2$ and D_2 to $4D_1 - D_2$. Let σ be defined as an isometry that coincides with ϕ^2 on L and with the identity on $L^\perp \cap \mathrm{NS}(S)$. Note that $K_X = -3e_0 + \sum_{i=1}^{10} e_i \in L^\perp$, hence σ acts on $K_X^\perp$ and, obviously, belongs to the 2-level congruence subgroup. Thus there exists $g \in \mathrm{Aut}(S)$ such that $g_* = \sigma$. Applying the previous proposition, we obtain that some power of g normally generates a proper subgroup of the Cremona group. A direct computation shows that

$$\sigma(e_0) = 73e_0 - 24(e_1 + \cdots + e_6) - 30(e_7 + e_8) - 6(e_9 + e_{10}).$$

As the corresponding Cremona transformation, g is given by the linear system of curves of degree 73 with base point of multiplicity 24 at $x_1, \dots, x_6$, of multiplicity 30 at x_7, x_8 and multiplicity 6 at x_9, x_{10}.

Example 2. Let E be an elliptic curve with complex multiplication by $i = \sqrt{-1}$ and let $X = E \times E/(\tau)$, where τ acts diagonally by multiplication by i. The surface X is a rational surface, it is the quotient of the Kummer surface $\mathrm{Kum}(E \times E)$ by a non-symplectic involution. The group $\mathrm{PGL}_2(\mathbb{Z}[i])$ acts on X in an obvious manner. Cantat and Lamy prove that any matrix $M \in \mathrm{SL}(\mathbb{Z})$ such that it is congruent to the identity matrix modulo 2 with $|\mathrm{tr}(M)| \geq 3$ defines an element g of $\mathrm{Aut}(X)$ such some power of g normally generates a proper normal subgroup of the Cremona group.

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Characterization of varieties of Fano type via singularities of Cox rings

Yoshinori Gongyo
(joint work with A. Sannai, S. Okawa, S. Takagi)

The notion of Cox rings was defined in [HK], generalizing Cox's homogeneous coordinate ring [Co] of projective toric varieties.

Let X be a normal ($\mathbb{Q}$-factorial) projective variety over an algebraically closed field k. Suppose that the divisor class group $\mathrm{Cl}(X)$ is finitely generated and free, and let $D_1, \cdots, D_r$ be Weil divisors on X which form a basis of $\mathrm{Cl}(X)$. Then the ring

$$\bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^n} H^0(X, \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r)) \subseteq k(X)[t_1^{\pm}, \cdots, t_r^{\pm}]$$

is called the Cox ring of X. If the Cox ring of a variety X is finitely generated over k, X is called a ($\mathbb{Q}$-factorial) Mori dream space. This definition is equivalent to the geometric one given in the original definition of MDS ([HK, Proposition 2.9]). Projective toric varieties are Mori dream spaces and their Cox rings are isomorphic to polynomial rings [Co]. The converse also holds [HK], characterizing toric varieties via properties of Cox rings.

We say that X is of *Fano type* if there exists an effective $\mathbb{Q}$-divisor Δ on X such that $-(K_X + \Delta)$ is ample and (X, Δ) is klt. It is known by [BCHM] that $\mathbb{Q}$-factorial varieties of Fano type are Mori dream spaces. Since projective toric varieties are of Fano type, this result generalizes the fact that projective toric varieties are Mori dream spaces. Therefore, in view of the characterization of toric varieties mentioned above, it is natural to expect a similar result for varieties of Fano type. The purpose of this paper is to give a characterization of varieties of Fano type in terms of the singularities of their Cox rings.

Main Theorem [with A. Sannai, S. Okawa, S. Takagi]: Let X be a $\mathbb{Q}$-factorial normal projective variety over an algebraically closed field of characteristic zero. Then X is of Fano type if and only if its Cox ring is finitely generated and has only log terminal singularities.

Our proof of Main Theorem 1 is based on the notion of global F-regularity, which is defined for projective varieties over a field of positive characteristic via splitting of Frobenius morphisms. A projective variety over a field of characteristic zero is said to be of globally F-regular type if its modulo p reduction is globally F-regular for almost all p. Schwede–Smith [SS] proved that varieties of Fano type are of globally F-regular type, and they asked whether the converse is true. We give an affirmative answer to their question in the case of Mori dream spaces.

Theorem A: Let X be a $\mathbb{Q}$-factorial Mori dream space over a field of characteristic zero. Then X is of Fano type if and only if it is of globally F-regular type.

Theorem A is a key to the proof of Main Theorem 1, so we outline its proof here. The only if part was already proved by [SS, Theorem 5.1], so we explain the if part. Since X is a $\mathbb{Q}$-factorial Mori dream space, we can run a $(-K_X)$-MMP which terminates in finitely many steps. A $(-K_X)$-MMP $X_i \dashrightarrow X_{i+1}$ usually makes the singularities of X_i worse as i increases, but in our setting, we can check that each X_i is also of globally F-regular type. This means that each X_i has only log terminal singularities, so that a $(-K_X)$-minimal model becomes of Fano type. Finally we trace back the $(-K_X)$-MMP above and show that in each step the property of being of Fano type is preserved, concluding the proof.

In order to prove Main Theorem 1, we also show that if X is a $\mathbb{Q}$-factorial Mori dream space of globally F-regular type, then modulo p reduction of a multi-section ring of X is the multi-section ring of modulo p reduction X_p of X for almost all p. The proof is based on the finiteness of contracting rational maps from a fixed Mori dream space, vanishing theorems for globally F-regular varieties and cohomology-and-base-change arguments. This result enables us to apply the theory of F-singularities to a Cox ring of X and, as a consequence, we see that that a $\mathbb{Q}$-factorial Mori dream space over a field of characteristic zero is of globally F-regular type if and only if its Cox ring has only log terminal singularities. Thus, Main Theorem 1 follows from Theorem A.

I also report the following theorem. Remark that in the following theorem we do not assume the MDS-ness.

Theorem B (with S. Takagi): Let S be a normal projective surface over an algebraically closed field of characteristic zero. If S is of dense globally F-split type (resp. globally F-regular type), then it is of Calabi–Yau type (resp. Fano type).

One of the key ingredients in the proof is to show that taking the Zariski decomposition of the anti-canonical divisor of a surface of dense globally F-split type commutes with reduction modulo p. The globally F-regular case of Theorem B immediately follows from this fact.

The proof of the globally F-split case is much more involved. First, by taking the minimal resolution, we may assume that S is smooth. If S is not rational, then the problem can be reduced to whether the projective bundle of a rank 2 vector bundle of degree zero over an elliptic curve is globally F-split. This question was already answered by Mehta and Srinivas [MS], so we suppose that S is rational. Using the Zariski decomposition of $-K_S$ and a result of Laface and Testa [LT] on rational surfaces, we can reduce to the case where $-K_S$ is nef and there exists an effective divisor D linearly equivalent to $-K_S$. We can assume in addition that the modulo p reduction S_p of S is a minimal elliptic surface and the reduction D_p of D is an indecomposable curve of canonical type. We then make use of the classification of singular fibers (Kodaira's table) to see that if D_p is not of type I_n, then (S_p, D_p) has to be globally F-split for infinitely many p. Finally, since a fiber of type I_n is a normal crossing divisor and global F-splitting implies log canonicity (see [HW, Theorem 3.9]), we conclude that (S, D) is log canonical, that is, S is of Calabi-Yau type.

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Étale fundamental groups of klt spaces, flat sheaves, and quotients of Abelian varieties

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(joint work with Stefan Kebekus and Thomas Peternell)

Working with a singular complex algebraic variety X, one is often interested in comparing the (étale) fundmental group or the set of finite étale covers of X with that of its smooth locus X_{reg}. For example, Lefschetz theorems for singular varieties only hold for the smooth locus of X and for the smooth locus of a general hyperplane section. More precisely, one may ask the following.

What are the obstructions to extend finite étale covers of X_{reg} to X? How do the étale fundamental groups of X and of its smooth locus differ?

We answer these questions for projective varieties X with Kawamata log terminal (klt) singularities, a class of varieties that is important in the Minimal Model Program. The main result of our upcoming paper [GKP13], see Theorem 1 below, asserts that there are no infinite towers of finite Galois morphisms over a klt base variety where all morphisms are étale in codimension one, but branched over a small set. In a certain sense, this result can be seen as saying that the difference between the sets of étale covers of X and of X_{reg} is small in case X is klt.

1. Main results

While the main technical result of [GKP13] is quite general, and its formulation is therefore somewhat involved, for many applications the following special case suffices.

Theorem 1. *Let X be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is Kawamata log terminal (klt). Assume we are given a sequence of finite, surjective morphisms that are étale in codimension one,*

$$X = Y_0 \xleftarrow{\gamma_1} Y_1 \xleftarrow{\gamma_2} Y_2 \xleftarrow{\gamma_3} Y_3 \xleftarrow{\gamma_4} \cdots .$$

If the composed morphisms $\gamma_1 \circ \cdots \circ \gamma_i : Y_i \to X$ are Galois for every $i \in \mathbb{N}^+$, then all but finitely many of the morphisms γ_i are étale.

Here, a finite, surjective morphism $\gamma : Y \to X$ is called *Galois*, if it is the quotient morphism for the action of a finite group G acting algebraically on Y. The statement does not continue to hold if one drops the "Galois" assumption on the composed morphisms $\gamma_1 \circ \cdots \circ \gamma_i$ (in my talk I discussed singular Kummer surfaces $S = A/\pm 1$, where A is an abelian surface, and towers of endomorphisms of S induced by isogenies of A). The proof of Theorem 1 is by induction using the singularity stratification of X. The base of the induction is secured by a recent result of Chenyang Xu on the finiteness of local algebraic fundamental groups of klt singularities, see [Xu12].

Before stating an almost direct consequence of Theorem 1, let us recall that if Y is a complex algebraic variety, the *étale fundamental group* $\widehat{\pi}_1(Y)$ is isomorphic to the profinite completion of the topological fundamental group of the associated complex space Y^{an}, cf. [Mil80, § 5].

Theorem 2 (Extension of étale covers from the smooth locus of klt spaces)**.** *Let X be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is klt. Then, there exists a normal variety $\widetilde{X}$ and a finite, surjective Galois morphism $\gamma : \widetilde{X} \to X$, étale in codimension one, such that the following, equivalent conditions hold.*

1. *(1) Any finite, étale cover of $\widetilde{X}_{\mathrm{reg}}$ extends to a finite, étale cover of $\widetilde{X}$.*
2. *(2) The natural map $\widehat{\iota}_* : \widehat{\pi}_1(\widetilde{X}_{\mathrm{reg}}) \to \widehat{\pi}_1(\widetilde{X})$ of étale fundamental groups induced by the inclusion of the smooth locus, $\iota : \widetilde{X}_{\mathrm{reg}} \to \widetilde{X}$, is an isomorphism.*

2. Applications to flat sheaves and to quotients of Abelian varieties

Flat sheaves. Let us recall that if Y is a complex algebraic variety and $\mathscr{G}$ is a holomorphic vector bundle on the underlying complex space Y^{an}, we call $\mathscr{G}$ *flat* if it is defined by a representation of the topological fundamental group $\pi_1(Y^{an})$. An algebraic vector bundle or locally free sheaf is called *flat* if and only the the associated complex vector bundle is flat. With this terminology, we have the following consequence of Theorem 2.

Theorem 3 (Extension and algebraicity theorem for flat sheaves)**.** *Let X be a normal, complex, projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor Δ such that (X, Δ) is klt. Then, there exists a normal variety $\widetilde{X}$ and a finite,* *surjective Galois morphism* $\gamma : \widetilde{X} \to X$*, étale in codimension one, such that the following holds: if* $\widetilde{\mathscr{F}}^\circ$ *is any flat holomorphic vector on* $\widetilde{X}^{an}_{\mathrm{reg}}$*, then there exists a flat algebraic vector bundle* $\widetilde{\mathscr{F}}$ *on* $\widetilde{X}$ *such that* $\widetilde{\mathscr{F}}|_{\widetilde{X}^{an}_{reg}} \cong \widetilde{\mathscr{F}}^\circ$*.*

Quotients of abelian varieties. Consider a Ricci-flat compact Kähler manifold whose second Chern class vanishes. As a classical consequence of Yau's results on the existence of a Kähler-Einstein metric, X is then covered by a complex torus, cf. [Kob87, Ch. IV, Cor. 4.15]. Building on our results discussed so far, we are able to generalise this to the singular case, when X is assumed to have singularities as they appear in the Minimal Model Program.

Theorem 4 (Characterisation of torus quotients)**.** *Let* X *be a normal, complex, projective variety of dimension* n *with at worst canonical singularities. Assume that* X *is smooth in codimension two and that the canonical divisor is numerically trivial,* $K_X \equiv 0$*. Further, assume that there exists an ample divisor* H *on* X *and a desingularisation* $\pi : \widetilde{X} \to X$ *such that*

$$c_2(\mathscr{T}_{\widetilde{X}}) \cdot \pi^*(H)^{n-2} = 0.$$

Then, there exists an Abelian variety A *and a finite, surjective, Galois morphism* $A \to X$ *that is étale in codimension two.*

A more general statement for threefolds was proven in [SBW94].

3. Sketch of the proof of Theorem 4

Suppose we are given a projective variety X as in Theorem 4. As all assumptions on X are invariant under taking finite, surjective Galois morphisms that are étale in codimension one, an application of Theorem 3 allows us to assume that any flat holomorphic vector bundle on X^{an}_{reg} extends to a flat algebraic vector bundle on X. In order to show the claim, it then suffices to show that under these assumptions the variety X is actually smooth, cf. the introduction to Section 2.

As $K_X \equiv 0$, Miyaoka's Generic Nefness Theorem implies that $\mathscr{T}_X$ is semistable with respect to H, see for example [GKP11]. Consequently, the theorem of Mehta-Ramanathan implies that for any $m \gg 0$, and any general complete intersection surface S for $|mH|$, the restriction $\mathscr{T}_X|_S$ is an $H|_S$-semistable vector bundle on the smooth projective surface S (here we use the assumption on the codimension of the singular set of X) with vanishing first and second Chern class. It then follows (essentially from the Kobayashi-Hitchin correspondence) that $\mathscr{T}_X|_S$ is flat, see for example [Sim92]; i.e, $\mathscr{T}_X|_S$ is given by a representation of $\pi_1(S)$. Furthermore, in the situation under discussion Hamm's version [BS95, Thm. 2.3.1] of Lefschetz' Theorem states that the natural morphism $\pi_1(S) \to \pi_1(X_{\mathrm{reg}})$ is an isomorphism, and hence can be used to define a representation of $\pi_1(X_{\mathrm{reg}})$, and therefore a flat holomorphic vector bundle $\mathscr{G}^\circ$ on X_{reg}, which by our WLOG assumption extends to a flat algebraic vector bundle $\mathscr{G}$ on the whole of X. As $\mathscr{G}|_S$ is isomorphic to $\mathscr{T}_X|_S$, the tangent sheaf $\mathscr{T}_X$ is therefore a flat vector bundle. In particular, $\mathscr{T}_X$ is locally free. The Lipman-Zariski-Conjecture for klt spaces, as proven for example in [GKKP11], hence implies that X is smooth.

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Equivariant compactifications of the vector group

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(joint work with Baohua Fu)

Let $G = \mathbf{C}^n$ be the complex vector group of dimension n. An *equivariant compactification* of G is a G-action $A : G \times X \to X$ on a projective variety X of dimension n with an open orbit $O \subset X$. In particular, the orbit O is G-equivariantly biregular to G. Given a projective variety X, such an action A is called an *EC-structure* on X, in abbreviation of 'Equivariant Compactification-structure'. Let $A_1 : G \times X_1 \to X_1$ and $A_2 : G \times X_2 \to X_2$ be EC-structures on two projective varieties X_1 and X_2. We say that A_1 and A_2 are *isomorphic* if there exist a linear automorphism $F : G \to G$ and a biregular morphism $\iota : X_1 \to X_2$ with the commuting diagram

$$\begin{array}{ccc} G \times X_1 & \xrightarrow{A_1} & X_1 \\ (F,\iota)\downarrow & & \downarrow \iota \\ G \times X_2 & \xrightarrow{A_2} & X_2. \end{array}$$

In [3], Hassett and Tschinkel studied EC-structures on projective space $X = \mathbf{P}^n$. They discovered that there are many distinct isomorphism classes of EC-structures on $\mathbf{P}^n$ if $n \geq 2$ and infinitely many of them if $n \geq 6$. They posed the question whether a similar phenomenon occurs when X is a smooth quadric hypersurface. This was answered negatively in [6], using arguments along the line of Hassett-Tschinkel's approach. A further study was made in [1] where the authors raised the corresponding question when X is a Grassmannian. Even for simplest examples like the Grassmannian of lines on $\mathbf{P}^4$, a direct generalization of the arguments in [3] or [6] seems hard.

Our main result gives a uniform conceptual answer to these questions, as follows.

Theorem *Let X be a Fano manifold of dimension n with Picard number 1, different from $\mathbf{P}^n$. Assume that X has a family of minimal rational curves whose VMRT $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ at a general point $x \in X$ is smooth. Then all EC-structures on X are isomorphic.*

Recall that $\mathcal{C}_x$ is the union of tangent directions to rational curves of minimal degree through x. Theorem has the following consequence.

Corollary *Let $X \subset \mathbf{P}^N$ be a projective submanifold of Picard number 1 such that for a general point $x \in X$, there exists a line of $\mathbf{P}^N$ passing through x and lying on X. If X is different from the projective space, then all EC-structures on X are isomorphic.*

It is well-known that when X has a projective embedding satisfying the assumption of Corollary, some family of lines lying on X gives a family of minimal rational curves, for which the VMRT $\mathcal{C}_x$ at a general point $x \in X$ is smooth (e.g. by Proposition 1.5 of [4]). Thus Corollary follows from Theorem. Corollary answers Arzhantsev-Sharoyko's question on Grassmannians and also gives a more conceptual answer to Hassett-Tschinkel's question on a smooth quadric hypersurface, as a Grassmannian or a smooth hyperquadric can be embedded into projective space with the required property. In fact, all known examples of Fano manifolds of Picard number 1, which admit EC-structures, can be embedded into projective space with the property described in Corollary. These include all irreducible Hermitian symmetric spaces and some non-homogeneous examples coming from Proposition 6.14 of [2].

The proof of Theorem is a simple consequence of the Cartan-Fubini type extension theorem in [5].

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MJ-discrepancy and Shokurov's conjectures

Shihoko Ishii

If X is a normal $\mathbb{Q}$-Gorenstein variety, by taking an appropriate resolution $f: Y \to X$, we can define the discrepancy divisor $K_{Y/X}$ (call it the "usual discrepancy divisor"). By using this discrepancy, we can define canonical (resp. log canonical) singularities and also multiplier ideals. If X is not normal or $\mathbb{Q}$-Gorenstein, the usual discrepancy is not defined. We think of another kind of discrepancy which works also for more general settings. Let X be an equidimensional reduced scheme of finite type over algebraically closed field k of characteristic zero.

Let $\mathfrak{a}$ be a non zero ideal on X and let $f\colon Y \to X$ be a log resolution for the product of $\mathfrak{a}$ with the Jacobian ideal $\mathcal{J}_X$ of X. Such a resolution has the property that the image of the canonical map $f^*(\Omega_X^d) \to \Omega_Y^d$ (where $d = \dim(X)$) can be written as $\mathcal{O}_Y(-\widehat{K}_{Y/X}) \cdot \Omega_Y^d$, for some effective divisor $\widehat{K}_{Y/X}$ on Y. This is the *Mather discrepancy divisor*, and if we write $\mathcal{J}_X \cdot \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$, then the difference $\widehat{K}_{Y/X} - J_{Y/X}$ is called the *Mather-Jacobian discrepancy divisor.* By using this discrepancy as an replacement of the usual discrepancy, we can define MJ-canonical singularities, log MJ-canonical singularities, MJ-minimal log discrepancy and MJ-multiplier ideal ([2]) for an arbitrary equidimensional reduced scheme X of finite type over k. The Mather-Jacobian discrepancy has good properties, sometimes better properties than the usual discrepancy. The most distinguished property is the following Inversion of Adjunction ([1],[3]):

Let X be embedded to a non-singular variety A as a closed subscheme of codimension c, and W a proper closed subset of X. Let $\widetilde{\mathfrak{a}} \subseteq \mathcal{O}_A$ be an ideal sheaf such that $\mathfrak{a} := \widetilde{\mathfrak{a}}\mathcal{O}_X$ is a non-zero ideal sheaf of $\mathcal{O}_X$, and let $I_X \subseteq \mathcal{O}_A$ be the ideal sheaf defining X. Then we obtain the formula on MJ-minimal log discrepancies:

$$\widehat{\mathrm{mld}}(W; X, \mathfrak{a}\mathcal{J}_X) = \widehat{\mathrm{mld}}(W; A, \widetilde{\mathfrak{a}} I_X^c).$$

By making use of this formula, we obtain the following:

(1) The answer to the Mather-Jacobian version of Shokurov's conjectures about minimal log discrepancy ([4])
(2) The fact that small deformations of log MJ-canonical singularities (resp. MJ-canonical singularities) are log MJ-canonical singularities (resp. MJ-canonical singularities).
(3) The complete list of MJ-canonical singularities and log MJ-canonical singularities of dimension ≤ 2.

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Dynamical Systems and categories

L. Katzarkov

Over the last several years, a variety of new categorical structures have been discovered by physicists. Furthermore, it has become transparently evident that the higher categorical language is beautifully suited to describing cornerstone concepts in modern theoretical physics.

The goal of my talk is to describe the connection between Dynamical Systems and Categories.

Recent work of Cantat-Lamy on the Cremona group and Blanc-Cantat on dynamical spectra suggests that there is a deep parallel between the study of groups of birational automorphisms on one hand, and mapping class groups on the other. Under this parallel, the dynamical degree of a birational map plays the role of the entropy of a pseudo-Anosov map. We consider these developments from the perspective of derived categories and their groups of autoequivalences. In a striking series of papers Gaiotto-Moore-Neitzke and Bridgeland-Smith have established a connection between Teichmüller theory and the theory of stability conditions on triangulated categories. An analogy between the Teichmüller geodesic flow and the wall crossing on the space of stability conditions had been noticed previously in the works of Kontsevich and Soibelman.

We take all these discoveries further. First, we define and study entropy in the context of triangulated and A_∞-categories. More specifically we construct and study a categorical version of the notion of *dynamical entropy*. Dynamical entropy typically arises as a measure of the complexity of a dynamical system. This notion exists in a variety of flavors, e.g. the Kolmogorov-Sinai measure-theoretic entropy, the topological entropy of Thurston and Gromov, algebraic entropy, etc. In analogy with these notions, we define the entropy of an exact endofunctor of a triangulated category with a generator.

In the case of saturated (smooth and proper) A_∞-categories we prove the following foundational results:

Theorem 1. *In the saturated case, the entropy of an endofunctor may be computed as a limit of Poincaré polynomials of Ext-groups.*

This result is connected to classical dynamical systems:

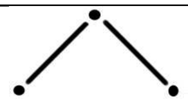
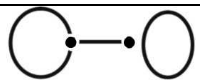
Theorem 2. *In the saturated case (under a certain generic technical condition), there is a lower bound on the entropy given by the logarithm of the spectral radius of the induced action on Hochschild homology.*

We develop further the parallel with dynamical systems. We build on the following basic correspondences:

1) geodesics $\leftrightarrow$ stable objects.

2) compactifications of Teichmüller spaces ↔ stability conditions.
3) classical entropy of pseudo-Anosov transformations ↔ categorical entropy.
4) categories ↔ differential equations.

We record our findings in the following table:

Category	Stable objects	Stab. cond.	Density of phases	Diff. eq.
A_n		$e^{P(z)}(dz)^2$	NO	$\left((\frac{d}{dz})^2 + e^{P(z)}\right)f = 0$
$\hat{A_n}$		$q(z)(dz)^2$	YES	Schrödinger eq.
$Tot\left(\begin{array}{c}A_2\\ \downarrow\\ C\end{array}\right)$	Spectral networks	$H^0(K^2) \oplus H^0(K^3)$	YES	Lax pair

We develop further the connection between categories and differential equations. Figure 1 suggests that one can study the WKB approximation of flat connections via harmonic maps to buildings.

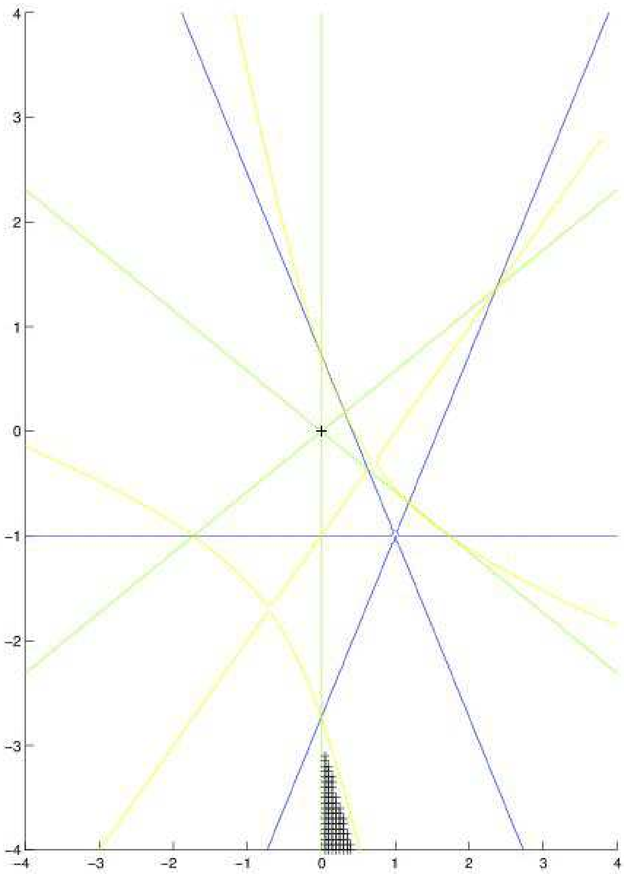


FIGURE 1. Harmonic maps to Buildings and WKB.

The corresponding categories are given by singularities of the harmonic map - the so called spectral networks. This project makes a connection between categories and differential equations. We show that there is an interpretation of the higher dimensional Cremona group and certain groups of autoequivalences.

In such a way we approach a classical question in Algebraic Geometry posed by Enriques in 1886 - show that the higher dimensional Cremona group is not simple.

Steenbrink vanishing extended

Sándor J Kovács

The importance of rational singularities has been demonstrated for decades through various applications. Log terminal singularities (of all stripes) are rational and this single fact has far reaching consequences in the minimal model program. Unfortunately, not all singularities that appear in the minimal model program are rational. In particular, the class of log canonical singularities which emerges as the most important class in many applications, for instance in moduli theory, is not necessarily rational.

The class of Du Bois singularities is an enlargement of the class of rational singularities. Even though this notion was introduced several decades ago [DB81, Ste83], it has remained relatively obscure for a long time. It was recently proved that log canonical singularities are Du Bois [KK10] and this fact has started a flurry of activities and Du Bois singularities are becoming central in the minimal model program and related areas.

An important application of Du Bois singularities appeared in [GKKP11] and in some other articles that grew out of it [Dru13, Gra13]. The way Du Bois singularities were used in these articles is through a vanishing theorem that can be considered a generalization of a vanishing theorem due to Steenbrink [Ste85].

The notion of Du Bois singularities was recently extended for pairs in [Kov11] and in this talk I reported on an extension of the vanishing theorem used in [GKKP11] to Du Bois pairs:

Theorem [Kov13] *Let X be a normal variety and $\pi : Y \to X$ a resolution of singularities. Let $\Sigma \subseteq X$ be a subvariety and E the reduced exceptional divisor of π and $\Gamma = E \cup (\pi^{-1}\Sigma)_{red}$. Assume that (X, Σ) is a Du Bois pair. Then for all p,*

$$R^{\dim X-1}\pi_*\big(\Omega^p_Y(\log\Gamma)(-\Gamma)\big) = 0.$$

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Covering semigroups

Vik. S. Kulikov

Let $f : E \to F$ be a finite morphism between complex non-singular irreducible projective curves. Let us fix a point $q \in F$ that is not a branch point of f and order the points of E lying over q. We call the morphism f with a fixed ordering of the points of $f^{-1}(q)$ a *marked covering.*

Consider the fundamental group $\pi_1(F \setminus B, q)$ of the complement of the branch set $B \subset F$ of a marked covering f of degree $d = \deg f$. Then, the ordering of the points of $f^{-1}(q)$ defines a homomorphism $f_* : \pi_1(F \setminus B, q) \to \mathcal{S}_d$ of $\pi_1(F \setminus B, q)$ to the symmetric group $\mathcal{S}_d$. Due to irreducibility of E, the image $\mathrm{im} f_* = G \subset \mathcal{S}_d$ acts transitively on $f^{-1}(q)$. We fix the embedding $G \hookrightarrow \mathcal{S}_d$.

The movement along a standard simple loops γ around branch points $b \in B$ defines the *local monodromy* $f_*(\gamma) \in G$ of f at b. The homotopy class of this standard loop, and hence the local monodromy, are defined by b uniquely only up to conjugation, in G. We denote by $O \subset G$ the union of the conjugacy classes of all the local monodromies of f and call the pair (G, O) the *equipped group.* The collection $\tau = (\tau_1 C_1, \dots, \tau_m C_m)$, where $C_1, \dots, C_m$ list all the conjugacy classes included in O and τ_i counts the number of branch points of f with the local monodromies belonging to C_i, is called the *monodromy type* of f. Below, we will assume that the elements of O generate G.

The degree d marked coverings of F with monodromy group G having n branch points with local monodromies from O form a so called Hurwitz space $\mathrm{HUR}_{(G,O),n}(F, q)$. The same coverings, but with fixed monodromy type τ and $\sum \tau_i = n$ form its subspace $\mathrm{HUR}_{d,G,\tau}(F)$ which consists of some its connected components and this space is called the *Hurwitz space of degree d coverings of F having ramification type τ.*

In the case $F = \mathbb{P}^1$, $G = \mathcal{S}_d$, and O is the set of transpositions, the famous Clebsch – Hurwitz Theorem states that $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(\mathbb{P}^1)$ consists of a single irreducible component if $\tau = (nO)$ with even $n \geq 2(d-1)$ and it is empty otherwise. Generalizations of Clebsch – Hurwitz Theorem were obtained in [11], [2], and [6] – [8]. In particular, Clebsch – Hurwitz Theorem was extended to the following cases: in [11], if all but two local monodromies are transpositions; and in [6], if there are at least $3(d-1)$ transpositions among the local monodromies. In [7], it is proved that for an equipped group $(\mathcal{S}_d, O)$ such that the first conjugacy class C_1 of O contains an odd permutation leaving fixed at least two elements, the Hurwitz space $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(\mathbb{P}^1)$ is irreducible if τ_1 is big enough. On the other hand, the example in [11] shows that $\mathrm{HUR}_{8,\mathcal{S}_8,\tau}(\mathbb{P}^1)$ consists of at least two irreducible components if $\tau = (1C_1, 1C_2, 1C_3)$, where C_1 is the conjugacy class of the permutation $(1,2)(3,4,5)$, C_2 is the conjugacy class of $(1,2,3)(4,5,6,7)$, and C_3 is the conjugacy class of $(1,2,3,4,5,6,7)$. Articles [2] and [8] are devoted to partial generalizations of Clebsch – Hurwitz Theorem to the case of arbitrary group G. In particular, in [8], it was proved that for a fixed equipped finite group (G, O) the number of irreducible components of $\mathrm{HUR}_{d,G,\tau}(\mathbb{P}^1)$ (if it is non-empty) does not depend on τ if all τ_i are big enough.

For higher genus, the irreducibility of $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(F)$ is proved in [3] under hypothesis that $n \geq 2d$ and all local monodromies are transpositions. After that, this result was improved, first, in [4] where the hypothesis $n \geq 2d$ was replaced by $n \geq 2d-2$, and next, in [9], where the second hypothesis was replaced by assumption that all but one local monodromies are transpositions. In addition, the result of [6], mentioned above, was generalized in [10] to the coverings of curves of arbitrary genus.

The aim of my talk is to extend results of [6] – [8] from $F = \mathbb{P}^1$ to the case of F of arbitrary genus. The approach used there for counting the number of irreducible components of $\mathrm{HUR}_{d,G,\tau}(\mathbb{P}^1)$ is based on a systematic work with semigroups over groups; in particular, factorization semigroups $S(G, O)$ with factors belonging to O play the crucial role in this study, especially since subsets of elements of type τ of subsemigroup $S(G,O)_{\mathbf{1}}^G \subset S(G, O)$ are in a canonical bijection with the sets of irreducible components of the Hurwitz space $\mathrm{HUR}_{d,G,\tau}(\mathbb{P}^1)$.

To treat the coverings of projective curves of arbitrary genus we generalize the notion of factorization semigroups to that of semigroups of marked coverings. One can consider different levels of the equivalence relations of coverings and so we introduce, respectively, different species of semigroups of marked coverings. The equivalence relation of the level that is most appropriate to construction of Hurwitz spaces is based essentially on moving of branch points, while that the level most appropriate to topological classification of coverings includes, in addition, the action on the base of coverings by the whole mapping class group. In particular, considering the coverings up to moving of branch points we introduce a semigroup $G\mathbb{S}_d(G, O)$ of marked degree d coverings with monodromy group G and set of local monodromies $O \subset G$. If we consider the same coverings up to the action of the modular group, then we obtain another semigroup, which we denote by $GW\mathbb{S}_d(G, O)$. They are related by a natural epimorphism $\Phi : G\mathbb{S}_d(G, O) \to GW\mathbb{S}_d(G, O)$ of semigroups. Certain specific subsemigroups of these two semigroups are in a canonical bijection with the set of irreducible components of the Hurwitz space $\mathrm{HUR}_{d,G}(F)$ and, respectively, the set of topological classes of marked degree d coverings of F with monodromy groups G.

By definition, the *monodromy type* of an element $s = (f : E \to F)$ belonging to one of these semigroups is the collection $\tau(s) = (\tau_1 C_1, \dots, \tau_m C_m)$ of local monodromies of f. The monodromy type behaves additively and gives a homomorphism from semigroups of coverings to the semigroup $\mathbb{Z}^m_{\geq 0}$. Therefore, for any constant $T \in \mathbb{N}$, there appear well defined subsemigroups

$$G\mathbb{S}_{d,T}(G,O) = \{s \in G\mathbb{S}_d(G,O) \mid \tau_i(s) \geq T \text{ for } i = 1, \dots, m\}$$

and

$$GW\mathbb{S}_{d,T}(G,O) = \{s \in GW\mathbb{S}_d(G,O) \mid \tau_i(s) \geq T \text{ for } i = 1, \dots, m\}.$$

The main results are as follows[1].

Theorem 1. *For any equipped finite group (G,O) such that the elements of O generate the group G, there is a constant $T \in \mathbb{N}$ such that the restriction of Φ to $G\mathbb{S}_{d,T}(G,O)$ is an isomorphism of $G\mathbb{S}_{d,T}(G,O)$ and $GW\mathbb{S}_{d,T}(G,O)$.*

In [8], an *ambiguity index* $a_{(G,O)}$ was defined for each equipped finite group (G,O).

Theorem 2. *For each equipped finite group (G,O), $O = C_1 \sqcup \dots \sqcup C_m$, such that the elements of O generate the group G, there is a constant T such that for any projective irreducible non-singular curve F the number of irreducible components of each non-empty Hurwitz space $HUR_{d,G,\tau}(F)$ is equal to $a_{(G,O)}$ if $\tau_i \geq T$ for all $i = 1, \dots, m$.*

Theorem 3. *Let C be the conjugacy class of an odd permutation $\sigma \in \mathcal{S}_d$ such that σ leaves fixed at least two elements. Then there is a constant N_C such that for any projective irreducible non-singular curve F the Hurwitz space $\mathrm{HUR}_{d,\mathcal{S}_d,\tau}(F)$ is irreducible if C enters in τ with a factor $\geq N_C$.*

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From surfaces of general type to stable (log) surfaces

Wenfei Liu
(joint work with Sönke Rollenske)

Stable surfaces come to help compactify the moduli space of surfaces of general type. They have semi-log-canonical singularities and ample canonical divisors. One asks if the facts about the canonical models of surfaces of general type still hold for the more general stable surfaces.

Together with Sönke Rollenske I investigated the pluri-log-canonical maps and the geography of stable log surfaces ([LR12, LR13]). Here the notion of stable log surface is slightly more general than that of stable surfaces, in that, a reduced boundary is allowed in the log case.

1. Pluri-log-canonical maps

Theorem 1. *Let* (X, Δ) *be a stable log surface and* I *its (global) index. Then* $mI(K_X + \Delta)$ *is base point free for* $m \geq 4$ *and* $mI(K_X + \Delta)$ *is very ample for* $m \geq 8$.

Remark 1. We can do better if the singularities are assumed to be mild. For example, $5K_X$ is very ample if X has only semi-canonical singularities.

We prove the base point freeness by applying a Reider-type result of Kawachi's on the normalisation combined with a detailed analysis of the non-normal locus.

Our result on pluri-log-canonical embeddings are somewhat more involved. We follow an approach due to Catanese, Franciosi, Hulek, and Reid: for every subscheme of length two find a pluri-log-canonical curve containing it and then prove that this curve is embedded by $|mI(K_X + \Delta)|$ for $m \geq 8$.

As further quests in this topic it would be interesting to address the follow problems:

(1) Let (X, Δ) be a stable log surface and U the (open) Gorenstein locus of (X, Δ). What is the optimal number r such that rK_U induces a birational map or an embedding?
(2) Is there a stable log surface (X, Δ) such that $5I(K_X + \Delta)$ is not very ample? Such a surface (if exists) tends to be Gorenstein, i.e., its index I is 1.

2. Geography

The geography problem of stable log surfaces asks: for which $(a,b) \in \mathbb{Q}_{>0}\times\mathbb{Z}$, do there exist stable log surfaces (X,Δ) such that $(K_X+\Delta)^2 = a$ and $\chi(\omega_X(\Delta)) = b$? At the moment we are only able to answer the question for Gorenstein stable log surfaces.

Theorem 2. *Let (X,Δ) be a Gorenstein stable log surface. Then*

(i) $(K_X+\Delta)^2 \geq \chi(\omega_X(\Delta)) - 2$ (stable Noether inequality)*;*

(ii) $(K_X+\Delta)^2 \geq -\chi(\omega_X(\Delta))$ (P_2-inequality)*.*

Theorem2, (i) was proved by Sakai ([S80]) when X is normal. The problem in the nonnormal case is that, X could have arbitrarily many irreducible components, so that one can not apply Sakai's result to the normalisation $\bar{X}$ directly. We solve this issue by showing that enough log canonical sections in $H^0(\bar{X}, K_{\bar{X}}+\bar{D}+\bar{\Delta})$ get lost in glueing $\bar{X}$ back to X, where $\bar{D}$ (resp. $\bar{\Delta}$) is the conductor divisor (resp. the strict transform of Δ) in $\bar{X}$.

Theorem2, (ii) follows simply from

$$\chi(\omega_X(\Delta)^{\otimes 2}) = h^0(X, 2(K_X+\Delta)) \geq 0.$$

Surprisingly the P_2-inequality is almost sharp, as shown by an example. In particular $\chi(\omega_X(\Delta))$ could be arbitrarily negative for Gorenstein stable log surfaces.

For a general stable log surface (X,Δ), we make the following speculation:

$$h^0(X, K_X+\Delta) \leq \ulcorner (K_X+\Delta)^2 \urcorner + 2.$$

The normal case has been treated by Tsunoda and Zhang ([TZ92]).

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Enriques surfaces as neighbors of rational surfaces (and vice versa)

Shigeru Mukai

Enriques surfaces are similar to K3 surfaces. Both are of Kodaira dimension one and can be studied lattice theoretically by virtue of Torelli type theorem. But Enriques surfaces, with the same birational invariants $q = p_g = 0$, are similar to rational surfaces too. They mildly degenerate to rational surfaces with quotient singularities of type (1,1)/4, and change into rational elliptic surfaces by logarithmic transformation. This similarity and intimacy is very useful in studying Enriques and rational surfaces. In my talk I discussed Theorems A–C on automorphism groups from this view point.

Let R_5 be a *quintic del Pezzo surface*, the blow-up of the projective plane $\mathbb{P}^2$ at four points in general position, say, at $(x_0 : x_1 : x_2) = (1 : \pm 1 : \pm 1)$. It has ten *lines*, that is, smooth rational curves of anti-canonical degree one, and the dual graph of their configuration is the Petersen graph. There are 15 intersection points among the ten lines in total. Let R_{-10} be the blow-up of R_5 at the 15 intersection points. The Cremona transformation $(x_0 : x_1 : x_2) \mapsto (1/x_0 : 1/x_1 : 1/x_2)$ induces an involution of R_{-10}, which we denote by σ. Taking conjugate by the action of $\operatorname{Aut} R_5 \simeq \mathfrak{S}_5$, we obtain five involutions $\sigma = \sigma_1, \ldots, \sigma_5$.

Theorem A. *The automorphism group of R_{-10} is generated by $\operatorname{Aut} R_5$ and σ. Moreover, it is isomorphic to the semi-direct product of the amalgam of five involutions $\langle \sigma_1 \rangle * \cdots * \langle \sigma_5 \rangle$ by $\mathfrak{S}_5$.*

The double cover of R_5 with branch the union of ten lines is a K3 surface with 15 nodes. The minimal resolution X_4 is a double cover of R_{-10}. (The suffix "4" denotes the discriminant of rank 2 transcendental lattice of the K3 surface.) Hence R_{-10}, with the strict transform of ten lines, is a Coble-Enriques surface in the following sense.

Definition A smooth algebraic surface S with a boundary divisor $B = \sqcup_1^m B_i$ is a *Coble-Enriques surface* of index m if S is the a quotient of a smooth K3 surface X by an involution ε whose fixed point locus is the disjoint union of m smooth rational curves and if B is the branch divisor.

The boundary components B_i's are all smooth rational curves with self intersection number $(B_i^2) = -4$. When index $m = 0$, S is nothing but an Enriques surface. Those with positive index are classified by Dolgachev-Zhang [1]. The maximum index is $m = 10$ and the rational surface R_{-10} above is the unique Coble-Enriques surface of maximum index.

Example (1) Let $\bar{B} \subset \mathbb{P}^2$ be an (irreducible) plane sextic with ten nodes and R_{-1} the blow-up of $\mathbb{P}^2$ at the ten nodes. Then R_{-1} with the strict transform B of $\bar{B}$ is a Coble-Enriques surface of index one.
(2) Let $\bar{B}$ be the union of six lines in $\mathbb{P}^2$. Then the blow-up R_{-6} of $\mathbb{P}^2$ at the 15 intersection points, with the strict transform B of $\bar{B}$, is a Coble-Enriques surface of index six.

Returning to Theorem A, let $L \simeq \mathbb{Z}^{10}$ be the orthogonal complement of the boundary components $B_1, \cdots, B_{10}$ in the Picard lattice $\operatorname{Pic} R_{-10} \simeq \mathbb{Z}^{20}$. The 15 exceptional curves of the blowing up $R_{-10} \to R_5$ define 15 roots, that is, divisor classes of self intersection number -2, in L. The five involutions $\sigma_1, \ldots, \sigma_5$ also define five roots in L. These 20 roots have the same graph as the Enriques surface S of type VII in Kondo [3]. Theorem A is proved in an analogous way to his proof of $\operatorname{Aut} S \simeq \mathfrak{S}_5$.

Remark Since the Picard lattice of the covering K3 surface X_4 is 2-elementary, $\operatorname{Aut} X_4$ is the central extension of $\operatorname{Aut} R_{-10}$ by the covering involution. Hence the latter half of Theorem A also follows from Vinberg [7]. Our proof is the one which eliminates the K3 surface X_4 and Torelli type theorem from his.

Let S be an Enriques surface which has semi-symplectic action of both the alternating group $\mathfrak{A}_6$ of degree six and the group 3^2D_8 of order 72 and are found in [4] and [6]. (An automorphism of an Emriques surface S is *semi-symplectic* if it acts trivially on $H^0(\mathcal{O}_S(2K_S)) \simeq \mathbb{C}$.) The covering K3 surface of S is the one found by Keum-Oguiso-Zhang [2] using the Leech lattice and Leech roots.

Theorem B. *S is isomorphic to the logarithmic transform of the Hesse elliptic surface*

$$R_0 := \mathrm{Bl}_9\, \mathbb{P}^2 \cdots \to \mathbb{P}^1, \quad (x_0 : x_1 : x_2) \mapsto (x_0^3 + x_1^3 + x_2^3 : 3x_0x_1x_2)$$

at the two fibers over $(1 \pm \sqrt{3} : 1)$ *(with multiplicity two).*

By a similar argument with the proof of Theorem A, we have

Theorem C. *The semi-symplectic automorphism group of S is isomorphic to the amalgam* $(3^2D_8) * \mathfrak{A}_6$ *over* 3^2C_4.

The Enriques surface S has 40 roots of $\mathbb{P}^1$'s and involutions. It is interesting to observe that the graph of these 40 roots are the same as that of Example (2). When the six lines tangent to the same conic, the Coble-Eniques surface R_{-6} of index six is the projection of a Kummer quartic surface from one of 16 nodes, say n_0. The boundary $B = \sum_1^6 B_i$ is the image of six tropes passing through n_0. In this case, the 40 roots of R_{-6} consists of 15 $\mathbb{P}^1$'s over the remaining 15 nodes $n_1, \ldots, n_{15}$, 15 involutions of Hutchinson-Göpel type ([5]) and the images of 10 tropes which does not pass through n_0.

Remark The action of $\mathfrak{A}_6$ on the Enriques surface S extends to that of M_{10}, the 2-point stabilizer group of the Mathieu group M_{12} (as permutation group of degree 12). The group M_{10} contains $\mathfrak{A}_6$ as subgroup of index two and the full automorphism group $\mathrm{Aut}\, S$ is the amalgam $(3^2D_8) * M_{10}$ over 3^2C_4.

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On the components of moduli spaces of curves with symmetry

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(joint work with Fabrizio Catanese, Michael Lönne)

I report on a research project aimed at classifying the connected components of the moduli space $S_g(G)$ of smooth curves of genus g with a G-action via numerical and homological invariants of the G-action. G is a finite group and we work over $\mathbb{C}$.

Let us recall that a G-marked curve is a pair (C, a), where C is an algebraic curve and $a: G \to \mathrm{Aut}(C)$ is an injective group homomorphism, hence yielding an effective G-action on C ([2]). An isomorphism $(C_1, a_1) \to (C_2, a_2)$ is a G-equivariant isomorphism of curves $f: C_1 \to C_2$. Families of G-marked curves are defined in the usual way, (C, a) is smooth if C is so and the genus of (C, a) is that of C. Under certain conditions that ensure the stability of (C, a) (e.g. $g \geq 2$), one can prove that the set of smooth G-marked curves of genus g modulo isomorphisms has a structure of quasi-projective variety. The group $\mathrm{Aut}(G)$ acts naturally on it and we denote by $S_g(G)$ the quotient variety.

The first invariant we can use to address our problem of classification comes from the Galois cover $p: C \to C' := C/G$ associated to a given G-marked curve (C, a). The genus g' of C' and the number d of branch points $y_1, \ldots, y_d \in C'$ are numerical invariants (under deformations) of (C, a). Then a first simplification of the problem consists in considering families where the genus g' of C' and the number d of branch points is fixed. In this way one obtains a stratification of $S_g(G)$ and then one asks which strata is irreducible. These strata are related to the so-called Hurwitz spaces [16] and the archetypal result is the theorem of Lüroth-Clebsch [9] and Hurwitz [18] saying that simple coverings of the projective line (C') form an irreducible variety (cf. also [1]). This result has been extended in several ways, see e.g. [17], [19], [20], [22] and the references therein.

A further numerical invariant is provided by the function ν that to each conjugacy class $\mathcal{C} \subset G$ associates the number of branch points with local monodromy lying in $\mathcal{C}$ modulo $\mathrm{Aut}(G)$. Notice that ν determines d and, together with the genus g of C, we deduce g' via Hurwitz's formula:

$$2g - 2 = |G| \left[2g' - 2 + \sum_{i=1}^{d} \left(1 - \frac{1}{m_i} \right) \right],$$

where m_i is the order of the local monodromy of p around y_i. Therefore ν provides a finer stratification of $S_g(G)$ and again one asks whether these strata are irreducible. In the case where G is cyclic, the answer is affirmative: [11] for free actions; [10] when $|G|$ is prime; [3] in the general case.

For finite abelian groups G, ν is not enough to distinguish the components of $S_g(G)$. A further topological invariant of (C, a) is obtained as follows. Let $H \leq G$ be the subgroup generated by the local monodromies of p around the branch points

$y_1, \dots, y_d$, then p factorizes as follows:

$$\begin{array}{ccc} C & \longrightarrow & C'' = C/H \\ & \overset{p}{\searrow} & \downarrow q \\ & & C' \end{array}$$

where $q\colon C'' \to C'$ is an étale Galois covering with group G/H. Let $Bq\colon C' \to B(G/H)$ be a classifying map for q, then $Bq_*([C']) \in H_2(G/H, \mathbb{Z})$ is a topological invariant of (C, a). Combining the results of [13] and [4], one can prove that the locus of $S_g(G)$ consisting of curves with a G-action having the same numerical type ν and the same class in $H_2(G/H, \mathbb{Z})$ is irreducible.

In general, for any finite group G, one defines the *topological type* of (G, a) as the group homomorphism

$$\rho\colon G \to \mathrm{Out}^+(\pi_1(C, x_0))$$

induced by $g \mapsto (a(g)_*\colon \pi_1(C, x_0) \to \pi_1(C, a(g)\cdot x_0))$, modulo the action of $\mathrm{Aut}(G)$ by pre-composition and the adjoint action by the mapping class group Map_g. By a lemma of Lefschetz ρ is injective. Identifying $\mathrm{Out}^+(\pi_1(C, x_0))$ with Map_g, we obtain a finite subgroup $\rho(G) \le \mathrm{Map}_g$. Now, using a result of [4] (cf. also [14] and [3]), we deduce that the locus $S_g(G, \rho) \subset S_g(G)$ of curves with G-action of topological type ρ is irreducible. In particular $\pi_0(S_g(G))$ is in bijection with the set of possible topological types.

Now we reduce our problem to a combinatorial one. A *Hurwitz generating system* is an element $v = (c_1, \dots, c_d, a_1, b_1, \dots, a_{g'}, b_{g'}) \in G^{d+2g'}$ such that the following conditions hold:

i) G is generated by the entries of v;
ii) $c_i \neq 1$, $\forall i$;
iii) $\prod_{i=1}^d c_i \prod_{j=1}^{g'} [a_j, b_j] = 1$.

Denote by $HS(G; g', d)$ the set of such v's. The group $\mathrm{Aut}(G)$ acts on $HS(G; g', d)$ diagonally and the mapping class group $\mathrm{Map}(g', d)$ acts on $HS(G; g', d)/\mathrm{Aut}(G)$. Once we choose a geometric basis for $\pi_1(C' \setminus \{y_1, \dots, y_d\}, y_0)$, we obtain a bijection between the set of topological types of curves C with G-action, for which the genus of C/G is g' and $C \to C/G$ has d branch points, and

$$\left(\frac{HS(G; g', d)}{\mathrm{Aut}(G)}\right) \Big/ \mathrm{Map}_{(g',d)}\,.$$

Our main contributions are the following. We first introduce [6, 7] a new homological invariant, the *ϵ-invariant.* In order to do that, let us fix a free presentation of G, $G = \frac{F}{R}$, where $F = \langle \hat{g} | g \in G \setminus \{1\} \rangle$ is the free group generated by $\hat{g}$, $g \in G \setminus \{1\}$. Then, for any conjugation-invariant subset $\Gamma \subset G$, $\Gamma \neq \{1\}$, set

$$R_\Gamma := \langle\langle [F, R], \hat{a}\hat{b}\hat{c}^{-1}\hat{b}^{-1} | a \in \Gamma\,,\ ab = bc \in G \rangle\rangle \trianglelefteq F$$

and K_Γ the kernel of $\frac{F}{R_\Gamma} \to G$ induced by $\hat{g} \mapsto g$. Finally, we define

$$\mathcal{K} := \left(\coprod_\Gamma K_\gamma \right) /_{\mathrm{Aut}(G)} .$$

One can prove that the map $HS(G; g', d) \to \mathcal{K}$,

$$v \mapsto \prod_{i=1}^{d} \widehat{c_i} \prod_{j=1}^{g'} [\widehat{a_j}, \widehat{b_j}] \in K_{\Gamma_v}, \quad \Gamma_v = \cup_i \{\text{conjugacy class of } c_i\}$$

descends to the map

$$\epsilon\colon \left(\frac{HS(G; g', d)}{\mathrm{Aut}(G)} \right) /_{\mathrm{Map}(g',d)} \to \mathcal{K} .$$

Since $\epsilon([v])$ can be interpreted as an element in a certain quotient of $H_2(BG, BG^1)$, we refer to ϵ as a homological invariant of G-marked curves.

Notice that the ϵ-invariant extends both the numerical type ν and the class $Bq_*([C']) \in H_2(G/H, \mathbb{Z})$ mentioned before. For example, the natural morphism $K_\Gamma \to K_\Gamma^{\mathrm{ab}}$ induces a map $A\colon \mathcal{K} \to (\oplus_{\mathcal{C}} \mathbb{Z}\langle \mathcal{C} \rangle) /_{\mathrm{Aut}(G)}$ (where $\mathcal{C}$ varies over the conjugacy classes of G) such that $\nu = A \circ \epsilon$. We say that $[(n_{\mathcal{C}})_{\mathcal{C}}] \, (\oplus_{\mathcal{C}} \mathbb{Z}\langle \mathcal{C} \rangle) /_{\mathrm{Aut}(G)}$ is *admissible* if

$$\sum_{\mathcal{C}} n_{\mathcal{C}} [\mathcal{C}] = 0 \quad \text{in the abelianized group } G^{\mathrm{ab}} \text{ of } G.$$

Then we prove the following results:

Theorem 1. ([5, 6])
Let $G = D_n$ be the dihedral group of order $2n$. Then the following holds:

i) ϵ is injective for any g' and d;
ii) $\mathrm{Im}(\epsilon)$ is the preimage under A of the admissible elements.

Theorem 2.(Genus stabilization [7])
For any G, g' and d, there exists an integer $s = s(d)$ such that:

i) ϵ is injective $\forall g' > s$;
ii) $\mathrm{Im}(\epsilon)$ is the preimage under A of the admissible elements, if $g' > s$.

Theorem 3.(Branch stabilization [8])
For any G, $\Gamma = \{g_1, \ldots, g_r\} \subset G$ any conjugation-invariant subset $\neq \{1\}$. Assume that $\langle \Gamma \rangle = G$ and set

$$u_\Gamma = (\underbrace{g_1, \ldots, g_1}_{\mathrm{ord}_{g_1}}, \ldots, \underbrace{g_r, \ldots, g_r}_{\mathrm{ord}_{g_r}}) .$$

Then there exists $m \in \mathbb{N}$ such that, $\forall v, w \in HS(G; g', d)$ with $\nu(v) \geq \nu(\underbrace{u_\Gamma, \ldots, u_\Gamma}_{m-\text{times}})$, the following holds:

$$\epsilon([v]) = \epsilon([w]) \quad \Rightarrow \quad [v] = [w] \in \left(\frac{HS(G; g', d)}{\mathrm{Aut}(G)} \right) /_{\mathrm{Map}(g',d)} .$$

Notice that Thm. 2 generalizes [12, Thm. 6.20] and reduces to it when $d = 0$ (étale case), while Thm. 3 extends a result of Conway-Parker (cf. [15]) which holds for $g' = 0$ and when $H_2(G, \mathbb{Z})$ is generated by commutators. A result similar to Thm. 3 has been obtained in [21], with different techniques; it would be interesting to compare the two approaches.

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Minimal models for Kähler threefolds

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(joint work with Andreas Höring)

The minimal model program (MMP) is one of the cornerstones in the classification theory of complex projective varieties. It is fully developped in dimension 3, due to Kawamata, Kollár, Mori, Reid and Shokurov, despite tremendous recent progress in higher dimensions, in particular by Birkar, Cascini, Hacon and McKernan [1]. In the Kähler situation the basic methods from the MMP, such as the base point free theorem, fail. Nevertheless it is expected that the main results should be true also in this more general context.

The goal of this report is to discuss and develop the minimal model program for Kähler threefolds X whose canonical bundle K_X is pseudoeffective, which is joint recent work with A.Höring [10]. To be more specific, we obtain the following result:

Theorem 1. *Let X be a (non-algebraic) normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that K_X is pseudo-effective. Then X has a minimal model, i.e., there exists a MMP*

$$X \dashrightarrow X'$$

such that $K_{X'}$ is nef.

In our context a variety X is said to be $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier *and* a multiple $(K_X^{\otimes m})^{**}$ of the canonical sheaf K_X is locally free. The bimeromorphic map $X \dashrightarrow X'$ exhibiting the minimal model X' decomposes into a finite sequence of divisorial contractions and flips, given by extremal rays in the dual of the Kähler cone. For previous partial results, we refer to [5], [14], [15] and [12].

As to further notations, recall that an irreducible and reduced complex space X is Kähler if there exists a Kähler form ω, i.e. a positive closed real $(1,1)$-form on the smooth part X_{reg} of X, such that the following holds: for every point $x \in X_{\mathrm{sing}}$ there exists an open neighbourhood $x \in U \subset X$ and a closed embedding $i_U : U \subset V$ into an open set $V \subset \mathbb{C}^N$, and a strictly plurisubharmonic C^∞-function $f : V \to \mathbb{C}$ with $\omega|_{U \cap X_{\mathrm{reg}}} = (i\partial\overline{\partial} f)|_{U \cap X_{\mathrm{reg}}}$. In the same way, differential forms of type (p,q) are defined. Dually, the notion of a (positive closed) current on a singular space is defined.

A line bundle L is *pseudo-effective* if $c_1(L)$ is represented by a positive closed current. In case X is projective, this is equivalent to saying that, given an ample line bundle A, for all large m, some power of the bundle $L^m \otimes A$ is effective. L is *nef*, if $c_1(L)$ is in the closure of the Kähler cone, the cone generated by the classes of the Kähler forms. The classes are taken in $N^1(X)$, the space of $d-$closed real $(1,1)-$forms modulo $\partial\overline{\partial}$ of real functions.

A remarkable theorem of Brunella [3] says that a smooth non-algebraic compact Kähler threefold is uniruled if and only its canonical bundle K_X is not pseudo-effective. Thus Theorem 1 states that a non-uniruled Kähler threefold has a minimal model.

We now explain the methods and main steps for proving Theorem 1. Restricting therefore from now on to varieties X ($\mathbb{Q}-$factorial, terminal singularities) with K_X pseudoeffective, we consider the divisorial Zariski decomposition [2]

$$K_X = \sum_{j=1}^{r} \lambda_j S_j + N(K_X).$$

Here the S_j are irreducible surfaces, λ_j are positive real numbers and $N(K_X)$ is an $\mathbb{R}-$line bundle which is "nef in codimension one", in particular $N(K_X)$ is pseudo-effective on every surface. If $K_X|_{S_j}$ is not pseudoeffective, one shows that the surface S_j is uniruled. It follows that K_X is not nef (in the sense of [8]) if and only if there exists a curve $C \subset X$ such that $K_X \cdot C < 0$. We then show how deformation theory on the threefold X and the (possibly singular) surfaces S_j can be used to establish an analogue of Mori's bend and break technique. As a consequence we derive the cone theorem for the Mori cone $\overline{NE}(X)$.

It is important to note that the Mori cone $\overline{NE}(X)$ is not the correct object to consider in the non-algebraic setting: even if we find a bimeromorphic morphism $X \to Y$ contracting exactly the curves lying on some K_X-negative extremal ray in $\overline{NE}(X)$, it is not clear that Y is a Kähler space. The Mori cone is simply to small in the non-algebraic context. However it had been observed in [14] that the Kähler condition is preserved if we contract extremal rays

$$R \subset \overline{NA}(X),$$

the cone generated by positive closed currents of bidimension $(1,1)$. In general, $\overline{NE}(X)$ is a proper subcone of $\overline{NA}(X)$, even if X is projective. Based on the description of $\overline{NA}(X)$ by Demailly and Păun [7], we derive from the cone theorem for $\overline{NE}(X)$ the following cone theorem:

Theorem 2. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Then there exists a countable family $(\Gamma_i)_{i\in I}$ of rational curves on X such that*

$$0 < -K_X \cdot \Gamma_i \leq 4$$

and

$$\overline{NA}(X) = \overline{NA}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].$$

We now fix a K_X-negative extremal ray $R = \mathbb{R}^+[\Gamma_i] \subset \overline{NA}(X)$ and prove the existence of the contraction of R. In other words, we are going to construct a morphism

$$\varphi = \varphi_R : X \to Y$$

to a normal *Kähler* space Y contracting exactly those curve lying in R. If the curves $C \subset X$ with class $[C] \in R$ cover a divisor S, we can use generalisations of Grauert's criterion [9] to contract S.

If the curves in the extremal ray cover only a 1-dimensional set C (i.e. the contraction, if it exists, is small), the problem is more subtle. By Grauert's criterion it is sufficient and necessary to find an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is ample and has support on C. In practice it is very difficult to compute the conormal sheaf, even for the reduced curve C. However since the curves in C belong to an extremal ray there exists a nef and big cohomology class α which is zero exactly on the curves in R; the class α being the analogon of the nef supporting divisor in the projective case. Considering once again the divisorial Zariski decomposition $K_X = \sum_{j=1}^r \lambda_j S_j + N(K_X)$, we now make a case distinction. If there exists a surface S_j such that $S_j \cdot C < 0$, this gives one direction where the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is ample. Moreover we prove that $\alpha|_{S_j}$ is nef and big, so an application of the Hodge index theorem yields another direction where $\mathcal{I}/\mathcal{I}^2$ is ample.

Thus we are left with the case where $N(K_X) \cdot C < 0$. If X is projective, Nakayama [13, III, 4.b] gives a very short argument: if H is an ample divisor, some multiple of the class $N(K_X) + \varepsilon H$ with $0 < \varepsilon \ll 1$ gives a linear system without fixed component, so C is contained in a local complete intersection curve having ample conormal bundle along C, so we conclude as in the first case. In the non-algebraic case we use again the deep results by Demailly-Păun [7] and Boucksom [2] to prove that there exists a modification $\mu : \tilde{X} \to X$ and a Kähler form $\tilde{\alpha}$ such that $\mu_* \tilde{\alpha} = \alpha$. Analysing the positivity of the μ-exceptional divisor we construct an ideal sheaf $\mathcal{I}$ having the required properties. In summary we have proven the contraction theorem:

Theorem 3. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Let $\mathbb{R}^+[\Gamma_i]$ be a K_X-negative extremal ray in $\overline{NA}(X)$. Then the contraction of $\mathbb{R}^+[\Gamma_i]$ exists in the Kähler category.*

Since Mori's theorem [11] assures the existence of flips also in the analytic category, termination of the process being elementary, we can now run the MMP and obtain Theorem 1.

By [6], Theorem 0.3, this also implies that the non-vanishing conjecture holds for compact Kähler threefolds:

Corollary 1. *Let X be a normal $\mathbb{Q}$-factorial compact (non-projective) Kähler threefold with at most terminal singularities. Then X is uniruled if and only if $\kappa(X) = -\infty$.*

Actually one can obtain a little more, using [15]:

Corollary 2. *Let X be a normal $\mathbb{Q}$-factorial compact (non-projective) Kähler threefold with at most terminal singularities. Suppose that K_X is nef. Then mK_X is spanned for some positive m, unless (possibly) there is no positive-dimensional* *subvariety through the very general point of* X *and* X *is not bimeromorphic to* T/G *where* T *is a torus and* G *a finite group acting on* T.

The remaining problem to solve abundance for Kähler threefolds completely is to prove the following well-known

Conjecture *Let* X *be a smooth compact Kähler threefold or a normal* $\mathbb{Q}$*-factorial compact Kähler threefold with at most terminal singularities. Assume there is no positive-dimensional subvariety through the very general point of* X*. Then* X *is bimeromorphic to* T/G *with* T *a torus and* G *a finite group acting on* T.

Recently, Campana, Demailly and Verbitsky [4] obtained some results towards this conjecture. Using the results of [10], these results can be generalized as follows.

Theorem 4. *Let* X *be a* $\mathbb{Q}-$*factorial Kähler threefold with only terminal singularities without divisors. Suppose furthermore that there is no positive-dimensional subvariety through a very general point of* X*. Then there exists a finite morphism* $\tilde{X} \to X$ *étale outside a finite set, the singular locus of* X*, such that* $\tilde{X}$ *is a torus. If* X *is even smooth, then* X *is itself a torus.*

Proof. By Corollary 1, $\kappa(X) \geq 0$. Since X does not contain any divisor, there exists a number m such that $mK_X \simeq \mathcal{O}_X$. Then, following [4], we take a finite cover $\tilde{X} \to X$, étale in codimension 1, such that $K_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}$. Now Riemann-Roch for Gorenstein threefolds gives $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. Since $\tilde{X}$ is not algebraic, with only rational singularities, $h^2(\mathcal{O}_{\tilde{X}}) \neq 0$, hence $\tilde{X}$ has positive irregularity $q(\tilde{X}) = h^1(\mathcal{O}_{\tilde{X}})$. Thus we obtain a non-trivial Albanese map $\tilde{X} \to \tilde{A}$, again using the fact that X has only rational singularities. It is now obvious to conclude that α is an isomorphism.
If X is actually smooth, then right away $\chi(X, \mathcal{O}_X) = 0$ and we conclude as before that X is a torus (see [4], Lemma 1.4).

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Multiplicative Hodge structures of conjugate varieties

Stefan Schreieder

In my talk I reported on the results in [6]. For a smooth complex projective variety X and a field automorphism σ of the complex numbers, the conjugate variety X^σ is defined via the fiber product diagram

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\sigma^*} & \mathrm{Spec}(\mathbb{C}). \end{array}$$

To put it another way, X^σ is the smooth variety whose defining equations in some projective space are given by applying σ to the coefficients of the equations of X. As abstract schemes – but in general not as schemes over Spec($\mathbb{C}$) – X and X^σ are isomorphic.

The aim of this talk is to study to which extent cohomological and Hodge theoretic data on X and X^σ coincides. Let me first state some previously known results.

(1) Pull-back of forms induces a σ-linear isomorphism between the algebraic de Rham complexes of X and X^σ. This induces an isomorphism of complex Hodge structures

$$H^*(X,\mathbb{C}) \otimes_\sigma \mathbb{C} \xrightarrow{\sim} H^*(X^\sigma,\mathbb{C}),$$

where $\otimes_\sigma\mathbb{C}$ means that the tensor product is taken where $\mathbb{C}$ maps to $\mathbb{C}$ via σ, see [3]. In particular, Hodge and Betti numbers of conjugate varieties coincide.

(2) The singular cohomology with $\mathbb{Q}_\ell$-coefficients coincides on smooth complex projective varieties with ℓ-adic étale cohomology. Since étale cohomology does not depend on the structure morphism to Spec($\mathbb{C}$), we obtain isomorphisms of graded $\mathbb{Q}_\ell$-, resp. $\mathbb{C}$-algebras,

$$H^*(X,\mathbb{Q}_\ell) \xrightarrow{\sim} H^*(X^\sigma,\mathbb{Q}_\ell) \quad \text{and} \quad H^*(X,\mathbb{C}) \xrightarrow{\sim} H^*(X^\sigma,\mathbb{C}),$$

where the latter depends on an embedding $\mathbb{Q}_\ell \subseteq \mathbb{C}$.

(3) There are conjugate varieties which are not homeomorphic. The first such examples were found 1964 by Serre [7], who showed that the topological fundamental groups of X and X^σ may in fact be non-isomorphic. More recently, Bauer–Catanese–Grunewald showed in [1] that this actually happens for any nontrivial element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is not contained in the conjugacy class of complex conjugation; see also Ingrid Bauer's talk on this work.

(4) In 2009, Charles constructed in [2] conjugate varieties X and X^σ with

$$\pi_1(X) \cong \pi_1(X^\sigma) \cong \mathbb{Z}^8 \quad \text{and} \quad H^*(X,\mathbb{R}) \not\cong H^*(X,\mathbb{R}).$$

Given the above results, this work is motivated by two questions. The first one is of topological nature and can for instance be found in Reed's Oxford thesis [5].

Question 1. Do there exist simply connected conjugate varieties X, X^σ which are non-homeomorphic?

The second question will be motivated by the Hodge conjecture. In order to state it, we define for any subfield $K \subseteq \mathbb{C}$, the space of K-rational (p,p)-classes on X by

$$H^{p,p}(X,K) := H^{p,p}(X) \cap H^{2p}(X,K);$$

the corresponding graded K-algebra is denoted by $H^{*,*}(X,K)$. The Hodge conjecture predicts that $H^{*,*}(X,\mathbb{Q})$ is generated by algebraic cycles. Since each algebraic cycle $Z \subseteq X$ induces a canonical cycle $Z^\sigma \subseteq X^\sigma$ and vice versa, the Hodge conjecture implies the following weaker conjecture:

$$H^{*,*}(X,\mathbb{Q}) \cong H^{*,*}(X^\sigma,\mathbb{Q}). \tag{1}$$

Apart from the (few) cases where the Hodge conjecture is known, and apart from Deligne's result [4] which settles (1) for abelian varieties, this conjecture is wide open, see [3]. The above consequence of the Hodge conjecture motivates our second question:

Question 2. For which subfields $K \subseteq \mathbb{C}$ is it true that

$$H^{*,*}(X,K) \cong H^{*,*}(X^\sigma,K) \tag{2}$$

holds for all conjugate varieties X, X^σ?

If $K = \mathbb{Q}(iw)$ with $w^2 \in \mathbb{N}$ is an imaginary quadratic number field, then the real part, as well as $1/w$ times the imaginary part of a $\mathbb{Q}(iw)$-rational (p,p)-class is $\mathbb{Q}$-rational. Hence,

$$H^{*,*}(-,\mathbb{Q}(iw)) \cong H^{*,*}(-,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(iw).$$

The Hodge conjecture therefore predicts (2) for imaginary quadratic number fields K. In [6], I am able to settle all remaining cases:

Theorem 1. *Let $K \subseteq \mathbb{C}$ be a subfield, not contained in an imaginary quadratic number field. Then there exist conjugate smooth complex projective varieties whose graded algebras of K-rational (p,p)-classes are not isomorphic.*

By Theorem 1, there are conjugate smooth complex projective varieties X, X^σ with

$$H^{*,*}(X,\mathbb{C}) \ncong H^{*,*}(X^\sigma,\mathbb{C}).$$

This shows the following:

Corollary 1. *The complex Hodge structure on the complex cohomology algebra of smooth complex projective varieties is not invariant under the* $\mathrm{Aut}(\mathbb{C})$*-action on varieties.*

Building upon some examples I construct in the proof of Theorem 1, I can extend the above mentioned result of Charles substantially:

Theorem 2. *Any birational equivalence class of complex projective varieties in dimension* ≥ 10 *contains conjugate smooth complex projective varieties whose real cohomology algebras are non-isomorphic.*

Since the fundamental group of smooth complex projective varieties is a birational invariant, Theorem 2 answers Question 1:

Corollary 2. *Let G be the fundamental group of a smooth complex projective variety. Then there exist conjugate smooth complex projective varieties with fundamental group G, but non-isomorphic real cohomology algebras.*

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Derived categories of some surfaces of general type and rationality questions

Pawel Sosna

(joint work with Christian Böhning, Hans-Christian Graf von Bothmer)

It has become commonplace to study the geometry of a smooth complex projective variety Z through its bounded derived category of coherent sheaves $\mathsf{D}^\mathsf{b}(Z)$. Since $\mathsf{D}^\mathsf{b}(Z)$ is usually fairly complicated, one can hope that sometimes it can be "decomposed" into hopefully simpler pieces:

Definition. A *semiorthogonal decomposition* (s.d.) of $\mathsf{D}^{\mathsf{b}}(Z)$ as above is a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_m$ satisfying:

(1) $\mathcal{A}_j \subset \mathcal{A}_i^{\perp} := \{T \mid \mathrm{Hom}(A_i, T) = 0 \; \forall A_i \in \mathcal{A}_i\}$ for all $i > j$.

(2) For all $D \in \mathsf{D}^{\mathsf{b}}(Z)$ there exists a sequence of maps

$$0 = D_m \longrightarrow D_{m-1} \longrightarrow \dots \longrightarrow D_1 \longrightarrow D_0 = D$$

such that the cone of the map $D_i \to D_{i-1}$ is contained in $\mathcal{A}_i$ for all $i = 1, \dots, m$.

We will write a semiorthogonal decomposition as $\mathsf{D}^{\mathsf{b}}(Z) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$.

As an example, if $E \in \mathsf{D}^{\mathsf{b}}(Z)$ is *exceptional*, that is, $\mathrm{Hom}(E, E[k]) = \mathbb{C}$ for $k = 0$ and 0 otherwise, then $\mathsf{D}^{\mathsf{b}}(Z) = \langle E^{\perp}, E \rangle$, where we write E for the triangulated category generated by E (this category is just $\mathsf{D}^{\mathsf{b}}(\mathrm{Spec}(\mathbb{C}))$). For instance, any line bundle on a Fano variety is exceptional. The same holds for line bundles on surfaces with $p_g = q = 0$. If $\mathsf{D}^{\mathsf{b}}(Z) = \langle \mathcal{A}, E_1, \dots, E_m \rangle$ is an s.d. and all E_i are exceptional, we call $(E_1, \dots, E_m)$ an *exceptional collection.* Note that if the E_i are line bundles, condition (1) boils down to $H^k(Z, E_j \otimes E_i^{-1}) = 0$ for all k and all $i > j$.

Concerning the interplay between semiorthogonal decompositions and geometry, consider the following example. If V is a cubic threefold, we have $\mathsf{D}^{\mathsf{b}}(V) = \langle \mathcal{A}_V, \mathcal{O}, \mathcal{O}(1) \rangle$. It was shown in [2] that two cubic threefolds V and V' are isomorphic if and only if the categories $\mathcal{A}_V$ and $\mathcal{A}_{V'}$ are equivalent. The proof relies on reconstructing the intermediate Jacobian from $\mathcal{A}_V$ using Bridgeland stability conditions.

One dimension higher, Kuznetsov proved in [7] that

$$\mathsf{D}^{\mathsf{b}}(W) = \langle \mathcal{A}_W, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$$

if W is a smooth cubic fourfold. Furthermore, for the fourfolds W which are known to be rational, the category $\mathcal{A}_W$ is equivalent to $\mathsf{D}^{\mathsf{b}}(S)$ for a smooth projective complex K3 surface S . Kuznetsov then conjectured that a cubic fourfold is rational if and only if $\mathcal{A}_W$ is the bounded derived category of a smooth projective K3 surface.

In this approach to the rationality question one wants to define a Clemens–Griffiths component of the bounded derived category which is invariant under birational transformations. To be able to do this, it would be useful to know whether any s.d. can be extended to a maximal one, i.e. one whose components do not admit any semiorthogonal decompositions. A natural approach to establish this, is to use invariants which are additive on semiorthogonal decompositions, such as Hochschild homology $\mathsf{HH}_{\bullet}$ or the Grothendieck group K_0. More precisely, folklore conjectures state that if $\mathcal{A} \neq 0$ is a component in a s.d., then $\mathsf{HH}_{\bullet}(\mathcal{A}) \neq 0$, and similarly for K_0. Another ingredient in this categorical approach to rationality of cubic fourfolds is a conjectural Jordan–Hölder type property for semiorthogonal decompositions. Note that this property was known to fail for general triangulated categories but it was unknown whether it does hold for categories of the form $\mathsf{D}^{\mathsf{b}}(Z)$.

The purpose of the talk was to give an outline of the proofs of the theorems below. To formulate them, consider the Fermat quintic $Y = \{x_1^5 + \ldots + x_4^5 = 0\}$ in $\mathbb{P}^3_{\mathbb{C}}$ and the action of $G = \mathbb{Z}/5 = \langle \xi \rangle$ on $\mathbb{P}^3$ given by $x_i \mapsto \xi^i x_i$. The classical Godeaux surface is defined as the quotient $X = Y/G$. It is a surface of general type with $p_g = q = 0$ whose canonical bundle K_X is ample. Furthermore, $K_X^2 = 1$, $\mathrm{Pic}(X) = \mathbb{Z}^9 \oplus \mathbb{Z}/5$, and, since the Bloch conjecture holds for X, one can check that $K_0(X) = \mathbb{Z}^{11} \oplus \mathbb{Z}/5$.

Theorem 1 ([3]). *There exists a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(X) = \langle \mathcal{A}, \mathcal{L}_1, \ldots, \mathcal{L}_{11} \rangle,$$

where $\mathcal{L}_i \in \mathrm{Pic}(X)$ and the category $\mathcal{A}$ is non-trivial with $\mathsf{HH}_\bullet(\mathcal{A}) = 0$, $K_0(\mathcal{A}) = \mathbb{Z}/5$.

Theorem 2 ([5]). *There exists a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(X) = \langle \mathcal{B}, \mathcal{M}_1, \ldots, \mathcal{M}_9 \rangle,$$

where $\mathcal{M}_j \in \mathrm{Pic}(X)$ and the category $\mathcal{B}$ contains no exceptional object. In particular, $\mathsf{D}^{\mathsf{b}}(X)$ does not satisfy the Jordan–Hölder property for semiorthogonal desompositions.

The proof of the first result proceeds in several steps. First one works on the level of $N(X)$, the Picard group modulo torsion, and constructs a sequence of (classes of) line bundles L_i satisfying $\chi(L_i, L_j) = 0$ for $i > j$. One then has to make sure that one can find line bundles $\mathcal{L}_i$ having this numerical behaviour and satisfying

$$H^0(X, \mathcal{L}_j \otimes \mathcal{L}_i^{-1}) = H^2(X, \mathcal{L}_j \otimes \mathcal{L}_i^{-1}) = H^0(X, K_X \otimes \mathcal{L}_i \otimes \mathcal{L}_j^{-1})^* = 0 \text{ for } i > j.$$

One of the main ingredients in this step is a classification of effective degree 1 divisors on X. Lastly, one can use the torsion in $\mathrm{Pic}(X)$ to twist away unwanted sections, giving the exceptional sequence $(\mathcal{L}_1, \ldots, \mathcal{L}_{11})$. By the additivity of $\mathsf{HH}_\bullet$ and K_0 on semiorthogonal decompositions, the stated properties of $\mathcal{A}$, called a *quasi-phantom category*, are immediate.

For the second result one starts by observing that $N(X) \cong \mathrm{Pic}(S)$, where S is a del Pezzo surface of degree 1. On the other hand, effectiveness of line bundles is very different on both sides: Roughly speaking, one can find line bundles $\mathcal{L}$ on S and $\overline{\mathcal{L}}$ on X corresponding to each other via the above isomorphism of lattices, and satisfying $\chi(\mathcal{L}) = 0 = \chi(\overline{\mathcal{L}})$. However, $R\Gamma^\bullet(\mathcal{L}) \neq 0$, while $R\Gamma^\bullet(\overline{\mathcal{L}})$ can be zero. The idea is then to find a collection of line bundles whose Euler pairing χ vanishes and where such a bundle occurs as a difference, meaning that this collection can never be lifted to an exceptional collection on S, while a lifting will indeed be possible on X. The shape of this particular sequence consisting of nine elements readily implies that $\mathcal{B}$ cannot have any exceptional object, concluding the proof.

Remark. Quasi-phantoms were shown to exist on other "fake del Pezzo surfaces" as well, see, for instance, [1], where the Grothendieck group of the quasi-phantom is $(\mathbb{Z}/2)^6$. There also exist *phantom categories*, that is, categories $\mathcal{A}$ appearing in a semiorthogonal decomposition and satisfying $\mathsf{HH}_\bullet(\mathcal{A}) = 0 = K_0(\mathcal{A})$. This was proved in [4] for the generic determinantal Barlow surface and in [6] for products of surfaces of general type admitting quasi-phantoms whose Grothendieck groups have coprime order.

Finally note that the existence of (quasi-)phantoms is not restricted to surfaces of general type, since, for example, one can blow up $\mathbb{P}^5$ in the Godeaux surface.

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Toward the Chow ring of the moduli space of genus 6 curves

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(joint work with Nikola Penev)

This is a report on work in progress. We believe the arguments to be complete, but this is only certain once full details of the arguments are written up. We will work throughout with Chow rings with $\mathbb{Q}$-coefficients.

The modern algebro-geometric study of the moduli space of curves was initiated by Faber's papers [F1, F2] on the Chow ring of $\mathscr{M}_3$ and $\mathscr{M}_4$, which made clear that much could be understood about Chow rings of $\mathscr{M}_g$, and Faber's conjecture [F3], which suggested the existence of an incredibly rich structure in the "tautological" part of the Chow ring. (Looijenga's seminal paper [L] must be mentioned in this context.) Earlier work of Mumford [Mum], and Witten's conjecture [W] (with its many remarkable proofs) provided the backbone for these papers.

Mumford earlier described the Chow ring of $\mathscr{M}_2$ in his landmark paper [Mum], and Izadi later determined the Chow ring of $\mathscr{M}_5$, [I]. In genus up to 5 the Chow ring is all tautological, and of a particular form, $\mathbb{Q}[\kappa_1]/(\kappa_1^{g-1})$. Simpler proofs of these facts were given in [FL], by describing perfect stratifications.

In genus 6, the tautological ring is more complicated; it was determined by Faber [F3] (and is given in the right side of (1)). At first this looks like an ugly ring, but it is not. Instead, you should consider Faber's conjecture as suggesting that the tautological ring of $\mathscr{M}_g$ should be of a particularly beautiful form, with simple generators, certain (beautifully combinatorially) defined top intersections, and "Poincaré duality" forcing the structure of the ring. Faber showed in [F3] and in later work that the tautological ring is indeed of this form for $g \leq 23$ (and in particular, in our case $g = 6$).

Our main result says that in fact the full Chow ring has this structure.

Main Theorem. *The Chow ring of $\mathscr{M}_6$ (with $\mathbb{Q}$-coefficients) is all tautological, and thus is given by:*

$$A^*(\mathscr{M}_6) = \mathbb{Q}[\kappa_1, \kappa_2]/(127\kappa_1^3 - 2304\kappa_1\kappa_2, 113\kappa_1^4 - 36864\kappa_2^2). \tag{1}$$

In particular, in $H^*(\mathscr{M}_6)$, all "algebraic cohomology" is tautological, and the natural map $A^*(\mathscr{M}_6) \to H^{2*}(\mathscr{M}_6)$ is an injection.

We briefly describe the new points of view which make the Main Theorem possible.

(i) We take advantage of the fact that Faber has already described the tautological ring completely; we will show that all classes in a certain generating set are tautological, and do not worry about describing relations.

(ii) We cut up $\mathscr{M}_6$ into locally closed strata as is traditional, but we do not worry about whether the strata have nontrivial Chow classes. Instead, we choose strata which are group quotients, and use a theorem of Vistoli (from [V]) to show that the Chow rings are generated by Chern classes of some natural vector bundle. We then show that the fundamental class of the stratum is tautological, and also relate the Chern classes of the vector bundle to the Hodge bundle to show that they too are tautological.

(iii) The case of trigonal curves requires some work and some new ideas.

(iv) For a large open subset $\mathscr{M}_6^M$ of $\mathscr{M}_6$ (those curves which have finitely many g^1_4's, or which are bi-elliptic), we use Mukai's fundamental work [Muk] describing each of the corresponding curves as a complete intersection in $G(2,5)$, and in particular we construct a rank 5 vector bundle V on the open subset $\mathscr{M}_6^M$ (the "Mukai-general curves"), relativizing Mukai's construction. We reduce to showing that the Chern classes of V are tautological on $\mathscr{M}_6^M$.

(v) We then show the Mukai bundle V on $\mathscr{M}_6^M$ is a subbundle of the rank 6 bundle of quadric relations on the canonical curve. We use this to show that the Chern classes of V are all tautological.

The fact that we can determine $A^*(\mathscr{M}_6)$ requires a number of fortunate coincidences. But we hope that aspects of our methods will be useful in other circumstances. As a first example, it seems plausible that such methods can show that $A^*(\mathscr{M}_g)$ is finitely generated for $g = 7, 8, 9$, using Mukai's description of large open subsets of $\mathscr{M}_g$ in this genus range. (There seems no complelling reason to believe that $A^*(\mathscr{M}_g)$ is finitely generated in general.)

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Nonexistence of asymptotically chow semistable limit

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(joint work with Xiaowei Wang)

Given a smooth curve of genus at least 2, Mumford showed that it is asymptotically chow stable. For canonically polarized smooth surface, the asymptotical chow stability was proved by Gieseker and for general dimensional case, it was showed by Donaldson. As GIT stability automatically yields a compact moduli space, then it is natural to ask the following question

Question. Does asymptotically chow semi stability yields a compact moduli space?

Our work [2] gives a negative answer to this question, which has been expected by people for long time, via an indirect way of comparing different notions of stability.

KSBA stability is another notion which was invented to compactify the moduli space of canonically polarized manifolds via minimal model theory. In fact, for a family of canonically polarized manifolds, its KSBA stable limit is just the fiber of the canonically model of its semi stable reduction, which is precisely the same construction as in Deligne-Mumford. Such canonically model exists because of the finite generation of the canonical ring as proved in [1].

For a family of n-dimensional polarized projective varieties $(\mathcal{X}, \mathcal{L})$ over a complete smooth curve C, we can define the Donaldson-Futaki invariant to be

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}/X) = (n+1)(L_t^n)(\mathcal{L}^n \cdot K_{\mathcal{X}}) - n(\mathcal{L}^{n+1})(L_t^{n-1} \cdot K_{X_t}).$$

where (X_t, L_t) is the general fiber of $(\mathcal{X}, \mathcal{L})$. When the general fibers are canonically polarized manifolds, i.e. $L_t = rK_{X_t}$, this formula is simplified to

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}/X) = C((n+1)(\mathcal{L}^n \cdot K_{\mathcal{X}}) - \frac{n}{r}\mathcal{L}^{n+1}),$$

for $C = L_t^n > 0$.

We first show that if $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}^s, \omega^{[r]})$ are two polarized families over C, which over an open set $C^0 \subset C$, they parametrize the isomorphic family of canonically polarized manifolds, and the latter one is a KSBA family, then

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \mathrm{DF}(\mathcal{X}^s, \omega^{[r]}).$$

In other words, the KSBA compactification minimizes the Donaldson-Futaki invariants among all compactifications. Moreover, if $\mathcal{X}$ is normal, then the equality of Donaldson-Futaki invariants will imply $(\mathcal{X}, \mathcal{L}) \cong (\mathcal{X}^s, \omega^{[r]})$.

On the other hand, for a family of polarized variety $(\mathcal{X}, \mathcal{L})/C$, we can also define its geometric height $h(\mathcal{X}, \mathcal{L})$ by considering the degree of the chow line bundle restricting on the induced section in the relative Hilbert scheme. And there is the equality

$$h(\mathcal{X}, \mathcal{L}^{\otimes k}) = C \cdot \mathrm{DF}(\mathcal{X}, \mathcal{L}^{\otimes k})k^{2n} + O(k^{2n-1}),$$

where C is a positive constant.

The last observation we made is that if we are provided a family of polarized varieties whose general fibers are chow semistable over an open smooth curve, then the compactification whose every fiber is chow semistable minimizes the geometric height. This is obtained by comparing the zeros of invariant sections on the Hilbert scheme and use the definition of semi-stable point.

So now let us assume we have asymptotically chow semistable compactification $(\mathcal{X}, \mathcal{L}^{\otimes k})$ of a family of canonically polarized manifolds. Then for every k, among all compactifications, the geometric height $h(\mathcal{X}, \mathcal{L}^{\otimes k})$ is minimal. In particular we know the leading term, which is the Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L}^{\otimes k})$ also should be minimal. Since $\mathcal{X}$ is normal, we conclude that indeed $(\mathcal{X}, \mathcal{L})$ will be the KSBA compactification.

However, examples of a family of canonically polarized manifolds whose KSBA limit is not asymptotically chow semistable has been known for a long time. One explicit example was given by

$$\mathcal{X}/C = (w^{m-6}(xyz^4 + y^6) + w^{m-10}z^{10} + t^{30}w^m + x^m + y^m + z^m = 0) \in \mathbb{P}(x,y,z,w) \times \mathbb{C}[t],$$

for $m \geq 30$. When $t \neq 0$, this is a family of smooth canonically polarized surfaces. But we can explicitly calculate the KSBA limit which has a singularity with multiplicity larger than 8. In particular, it is not asymptotically chow semistable due to Mumford's calculation. Therefore, we verify this family provides an example which answers the Question negatively.

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Bhargava's formula and the Hilbert scheme of points

Takehiko Yasuda
(joint work with Melanie Machett Wood)

In this talk, we tried to relate two similar formulas, Bhargava's formula counting extensions of a local field and a formula concerning the Hilbert scheme of points.

Let K be a local field with the residue field having q elements. Bhargava [1] proved that for a positive integer n,

$$\sum_{[E:K]=n} \frac{1}{\sharp \mathrm{Aut}(E)} q^{-v_K(d_{E/K})} = \sum_{i=0}^{n-1} P(n, n-i) q^{-i}. \tag{1}$$

Here E runs over étale K-algebras of degree n modulo isomorphism and $P(n, n-i)$ denotes the number of partitions of n into exactly $n-i$ parts. Let G_K be the absolute Galois group of K and Γ a finite group. Following [4] and [5], we put

$$M(K, \Gamma, c) := \frac{1}{\sharp\Gamma} \sum_{\rho: G_K \to \Gamma} q^{-c(\rho)},$$

where ρ runs over continuous homomorphisms and c is some real-valued function in ρ. Kedlaya [4] reduced Bhargava's formula (1) to the form,

$$M(K, S_n, a) = \sum_{i=0}^{n-1} P(n, n-i) q^{-i}, \tag{2}$$

with a the Artin conductor induced by the standard representation of S_n.

To see the other formula, consider the Hilbert-Chow morphism

$$\mathrm{Hilb}^n(\mathbb{A}_k^2) \to S^n \mathbb{A}_k^2$$

from the Hilbert scheme of n points on the affine plane to the nth symmetric product with k a base field. It is known that the morphism is a crepant resolution (see [2]). Let $E \subset \mathrm{Hilb}^n(\mathbb{A}_k^2)$ be the preimage of the origin of $S^n\mathbb{A}_k^2$. If $k = \mathbb{F}_q$, then we see

$$\sharp E(\mathbb{F}_q) = \sum_{i=0}^{n-1} P(n, n-i) q^{i}, \tag{3}$$

using a cell decomposition by Ellingsrud and Stømme [3]. We find an obvious similarity between (1)=(2) and (3).

The relation between them is explained to some extent in terms of the wild McKay correspondence. Let X be the quotient scheme associated to a faithful $\mathcal{O}_K$-linear Γ-action on $\mathbb{A}^n_{\mathcal{O}_K}$ without peudo-reflection. Suppose that there exists a crepan resolution $Y \to X$. We put $E \subset Y$ to be the preimage of the origin of $X(\kappa)$ with κ the residue field of K. Then the following equality is conjectured in [6] as a variant of a conjecture in [8].

Conjecture (The wild McKay correspondence). We have

$$\sharp E(\kappa) = M(K, \Gamma, -w).$$

Here w is the weight function coming from a study of motivic integration over wild Deligne-Mumford stacks [8].

The conjecture holds when K is a power series field of characteristic p, Γ is the cyclic group of order p and the Γ-action on $\mathbb{A}^n_{\mathcal{O}_K}$ is already defined over the coefficient field [7].

Our main result is the following.

Theorem 1. *The above conjecture holds when* $\Gamma = S_n$, $X = S^n \mathbb{A}^2_{\mathcal{O}_K}$ *and* $Y = \mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})$*. In particular, we have*

$$M(K, S_n, -w) = \sum_{i=0}^{n-1} P(n, n-i) q^i.$$

Thus the right hand sides of formulas (2) and (3) become the same if we replace the Artin conductor a with the weight w. The proof of the theorem is based on Bhargava's more precise formula in [1] and a comparison of a and w.

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Semi stable Higgs bundles and representations of fundamental groups over positive and mixed characteristic

Kang Zuo

Let k be an algebraic closure of finite fields with odd characteristic p and X a smooth projective scheme over the Witt ring $W(k)$. To an object $(M, Fil^\bullet, \nabla, \Phi)$ in Fontaine-Faltings category $MF^{\nabla}_{[0,n]}(X)$, where M is a vector bundle over X with an integrable connection ∇,

$$\{0\} = Fil^{n+1} \subset Fil^n \subset Fil^{n-1} \subset \cdots \subset Fil^0 = M$$

is a filtration of $\mathcal{O}_X-$module satisfying Griffiths-transversality and Φ is a relative Frobenius acting on M satisfying the strongly $p-$divisible property, one associates a crystalline representation of the fundamental group of the generic fibre X^0 of X under the so-called arithmetic Riemann-Hilbert correspondence developed by Fontaine and Faltings ([Fa1], [Fo]). On the other hand by taking the grading of (M, ∇) with respect to the filtration $Fil^\bullet \subset M$ one obtains a Higgs bundle in the form $(\bigoplus_{s+t=n} E^{s,t}, \theta)$ over X (called system of Hodge bundles), which is semi-stable and with trivial Chern classes. In fact, one obtains immediately a purely algebraic proof for the semistablity of Higgs bundles arising from geometry over $\mathbb{Z}/p^2$. Faltings conjectures that semi-stable Higgs bundles with trivial Chern classes over X corresponds to representations of $\pi_1(X^0 \times_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p)$ ([Fa2]). In our project we intend to study this conjecture by finding the analogue of the Simpson's correspondence over the complex numbers [S]. The main discovery in our projects is introducing intermediate notions *strongly semistable Higgs bundles* and *(quasi)periodic Higgs bundles* connecting semistable Higgs bundles and objects in $MF^{\nabla}_{[0,n]}(X)$([SZ], [LSZ]) In char. p the both notions rely on the Cartier's inverse constructed by Ogus and Vologodsky in their work on char. p nonabelian Hodge theory ([OV]). A lifting of Cartier's inverse to mixed characteristic is constructed in our project, which is used for the notion (quasi)periodicity in the mixed characteristic. Using the strong Higgs semistablity we define a self map on the moduli space of semistable Higgs bundles on X with trivial Chern classes. The periodic points in the moduli space under this map is a flavor of Analysis, looks like solutions of the Higgs-Yang-Mills equation on Higgs bundles over $\mathbb{C}$. We get the following results:

1) There is one to one correspondence between the category of periodic Higgs bundles and Fontaine-Faltings category. Hence via the arithmetic Riemann-Hilbert correspondence there is one to one correspondence between the category of periodic Higgs bundles and the category of crystalline representations of $\pi_1(X^0)$. The statement over char. p generalizes a theorem du to H. Lange and U. Stuhler ([LS]).

2)A Higgs bundle with trivial Chern classes is strongly semistbale if and only it is quasiperiodic.

3) A semistable Higgs bundle with trivial Chern classes of rank ≤ 3 is strongly semistable.

Conjecture Any semistable Higgs bundle with trivial Chern classes is always strongly semistable.

Hence by **2)** any semistable Higgs bundle with trivial Chern classes is always quasiperiodic.

At the moment we are trying to find relations between quasiperiodic Higgs bundles and representations of $\pi_1(X^0 \times_{\mathbb{Q}_p} K)$, where K is a ramified field extension of $\mathbb{Q}_p$.

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