Mathematisches Forschungsinstitut Oberwolfach

Report No. 31/2013

DOI: 10.4171/OWR/2013/31

Quadratic Forms and Linear Algebraic Groups

Organised by
Detlev Hoffmann, Dortmund
Alexander Merkurjev, Los Angeles
Jean-Pierre Tignol, Louvain-la-Neuve

16 June – 22 June 2013

Abstract. Topics discussed at the workshop "Quadratic Forms and Linear Algebraic Groups" included besides the algebraic theory of quadratic and Hermitian forms and their Witt groups several aspects of the theory of linear algebraic groups and homogeneous varieties, as well as some arithmetic aspects pertaining to the theory of quadratic forms over function fields or number fields.

Mathematics Subject Classification (2010): 11Exx, 11R20, 12D15, 12Gxx, 14C15, 14C25, 14C35, 14E08, 14G20, 14L15, 14M17, 16K20, 16K50, 19Gxx, 20Gxx.

Introduction by the Organisers

The half-size workshop was organized by Detlev Hoffmann (Dortmund), Alexander Merkurjev (Los Angeles), and Jean-Pierre Tignol (Louvain-la-Neuve), and was attended by 26 participants. Funding from the Leibniz Association within the grant "Oberwolfach Leibniz Graduate Students" (OWLG) provided support toward the participation of one young researcher. Additionally, the "US Junior Oberwolfach Fellows" program of the US National Science Foundation funded travel expenses for one post doc from the USA.

The workshop was the twelfth Oberwolfach meeting on the algebraic theory of quadratic forms and related structures, following a tradition initiated by Manfred Knebusch, Albrecht Pfister, and Winfried Scharlau in 1975. Throughout the years, the theme of quadratic forms has consistently provided a meeting ground where methods from various areas of mathematics successfully cross-breed. Its scope now includes aspects of the theory of linear algebraic groups and their homogeneous spaces over arbitrary fields, but the analysis of quadratic forms over specific fields, such as function fields and fields of characteristic 2, was also the focus of discussions. The talks covered a wide range of topics including, among others, cohomological invariants, local-global principles and patching, field invariants pertaining to quadratic and hermitian forms and to central simple algebras, and a proof of the Grothendieck-Serre conjecture on principal bundles over certain regular semi-local rings.

As the workshop was held simultaneously with another half-size workshop on the related subject of "The Arithmetic of Fields", the organizers of both meetings jointly decided to schedule each morning two plenary talks, one from each group. Ten lectures were thus addressed to the participants of both workshops; they were given by Eva Bayer-Fluckiger, David Grimm, David Leep, Raman Parimala, and Venapally Suresh from the "Quadratic Forms" side, and by Pierre Dèbes, Ido Efrat, Julia Hartmann, Laurent Moret-Bailly, and David Zywina from the "Arithmetic of Fields" side. Concerning details for these latter talks, we refer to the "Arithmetic of Fields" report in this volume. The program also comprised eleven lectures held in parallel with those of the other group. The lectures were of 50 minutes each.

Workshop: Quadratic Forms and Linear Algebraic Groups

Table of Contents

Abstracts

Failure of the local-global principle for isotropy of quadratic forms over surfaces

Asher Auel
(joint work with R. Parimala and V. Suresh)

Let X be an integral scheme, K its function field, Ω the set of rank 1 discrete valuations on K, and K_v the completion of K at $v \in \Omega$. We assume throughout that 2 is invertible on X. Let q be a nondegenerate quadratic form over K and $q_v = q \otimes_K K_v$. The *local-global principle* for isotropy of quadratic forms is the statement: if q_v is isotropic for all $v \in \Omega$ then q is isotropic over K. A natural question is: does the local-global principle hold for a given function field K?

We mention three examples. First, the local-global principle holds if K is a global field by the Hasse–Minkowski theorem. Second, let K be the function field of a smooth proper curve X over an algebraically closed field k. Here, Ω is in bijection with the set of closed points of X. By Tsen's theorem, all quadratic forms of dimension ≥ 3 are isotropic. An anisotropic form q of dimension 2 is similar to the norm form of a separable quadratic field extension L/K, corresponding to a finite flat quadratic cover $Y \to X$ between smooth proper curves. Then q_v is isotropic if and only if the fiber of $Y \to X$ is split over the closed point corresponding to $v \in \Omega$. Hence q_v is isotropic for all $v \in \Omega$ if and only if $Y \to X$ is étale (indeed, k is algebraically closed). The Riemann–Hurwitz formula implies that this is only possible if the genus of X is positive. We conclude that the local-global principle holds over K if and only if $X = \mathbb{P}^1$. Third, there is a similar situation when K is the function field of a smooth proper curve X over a complete discretely valued field k. In this case, the local-global principle holds when $X = \mathbb{P}^1$, fails in general for quadratic forms of dimension 2 over higher genus curves, and holds for forms of dimension ≥ 3, by the results of Colliot-Thélène, Parimala, and Suresh [7] using the patching techniques of Harbater, Hartmann, and Krashen [8].

It is the second example above that we generalize to higher dimension.

Theorem 1. *Let k be an algebraically closed field of characteristic not 2 and K the function field of a surface X over k. Then there are counterexamples to the local-global principle for quadratic forms of dimension 4 over K.*

We remark that K is a C_2-field, hence all quadratic forms of dimension ≥ 5 are isotropic. Earlier, there were known counterexamples to the local-global principle over special classes of surfaces yet the question was still open for rational surfaces.

These counterexamples arise as an application of classification results for quadratic forms of dimension 4. Given a nondegenerate quadratic form q of dimension 4 over a field k (of any characteristic), the even Clifford algebra $C_0(q)$ is a quaternion algebra over the *discriminant extension*, which is an étale quadratic k-algebra l. Similar quadratic forms yield isomorphic even Clifford algebras. Conversely, given a quaternion algebra A over l, which has trivial corestriction to k, there is an associated similarity class $q_{A/l/k}$ of quadratic forms of dimension 4 over k, called the *norm form.* In fact, the even Clifford algebra and norm form define inverse bijections between the set of similarity classes of nondegenerate quadratic forms of dimension 4 with discriminant extension l/k and the set of isomorphism classes of quaternion algebras over l with trivial corestriction to k, see [10, IV.15.B].

This has been generalized to a classification of regular quadratic forms of dimension 4 over affine schemes by Knus, Parimala, and Sridharan [9], and more generally, regular line bundle-valued quadratic forms of dimension 4 by [2, §5.3], in terms of Azumaya quaternion algebras A over étale quadratic covers $Y \to X$. A *line bundle-valued quadratic form* (E, q, L) over a scheme X is the datum of a locally free $\mathscr{O}_X$-module E of finite rank, an invertible sheaf L, and a quadratic form $q : E \to L$. The even Clifford algebra $C_0(E, q, L)$ was defined by Bichsel and Knus [5]. The notion of similarity is replaced by *projective similarity*, which allows for scaling by global units as well as tensoring by invertible modules.

We generalize these classification results to the degenerate context. Let X be an integral scheme with 2 invertible and $D \subset X$ a divisor. A line bundle-valued quadratic form (E, q, L) has *simple degeneration* along D if its restriction to $X \smallsetminus D$ is regular and if for each point x of D, the quadratic form $q \otimes_{\mathscr{O}_X} \mathscr{O}_{X,x}$ has discriminant in $\mathfrak{m}_{X,x} \smallsetminus \mathfrak{m}_{X,x}^2$ and contains a regular subform of codimension 1. If X is regular, then the center of $C_0(E, q, L)$ defines the finite flat quadratic *discriminant cover* $Y \to X$. If (E, q, L) has simple degeneration and even dimension, then $C_0(E, q, L)$ becomes an Azumaya algebra over Y, a result of Kuznetsov [11, Prop. 3.13]. Our main construction is, given an Azumaya quaternion algebra A over Y, a line bundle-valued norm form $q_{A/Y/X}$ of dimension 4 over X.

Theorem 2 ([3]). *Let X be a regular integral scheme of dimension ≤ 2 with 2 invertible and $Y \to X$ a finite flat quadratic cover with regular branch divisor D. Then the even Clifford algebra and norm form define inverse bijections between the set of projective similarity classes of quadratic forms (E, q, L) of dimension 4 with simple degeneration and discriminant cover $Y \to X$ and the set of isomorphism classes of Azumaya quaternion algebras over Y having split norm to X.*

We now review the key ingredients of the proof. The first is a norm (or corestriction) map for Azumaya algebras with respect to finite flat covers of schemes of dimension ≤ 2. Our construction uses Zariski patching techniques of Ojanguren, relying on results of Colliot-Thélène and Sansuc [6, §2]. For the Brauer group, such a norm map was defined in greater generality by Deligne in SGA 4, Exp. 17, §6.2. Second, we prove the smoothness of the nonreductive special orthogonal group scheme $\mathrm{SO}(E, q, L)$ over X associated to a quadratic form with simple degeneration, which allows to extend the exceptional isomorphisms of type ${}^2\mathsf{A}_1 = \mathsf{D}_2$ to this context. Third, we prove the Grothendieck–Serre conjecture for such special orthogonal (and projective) group schemes over discrete valuation rings. The proof then proceeds by patching the classical norm form (for étale quadratic covers) over $X \smallsetminus D$ with suitably chosen quadratic form models having simple degeneration over the local rings of generic points of components of D.

Finally, to construct counterexamples to the local-global principle for the function field K of a smooth proper surface X over an algebraically closed field k, we have two cases. First, if ${}_2\mathrm{Br}(X) \neq 2$, then a given 2-torsion Brauer class α has a quaternion algebra representative by the "period = index" result of Artin [1]. By "purity for division algebras" for schemes of dimension ≤ 2, a result going back to Auslander and Goldman [4], there exists an Azumaya quaternion algebra A on X whose generic fiber is α. Then the reduced norm $\mathrm{Nrd} : A \to \mathscr{O}_X$ is a locally isotropic quadratic form by Tsen's theorem, yet is anisotropic over K. Second, in the case when ${}_2\mathrm{Br}(X) = 0$, we utilize our results. We prove a geometric lemma showing that there always exists a finite flat quadratic cover $Y \to X$ between smooth surfaces, having smooth branch divisor, such that ${}_2\mathrm{Br}(Y) \neq 0$. Then as in the previous case, there exists a nonsplit Azumaya quaternion algebra A over Y, which now has split norm to X by our hypothesis in this case. Since the norm form $q_{A/Y/X}$ of dimension 4 has simple degeneration, it contains a regular subform of rank 3, hence is locally isotropic by Tsen's theorem. Finally, the norm form is anisotropic over K since its even Clifford algebra gives back the nonsplit algebra A over Y, appealing to the fact that a quadratic form of rank 4 is isotropic if and only if its even Clifford algebra is split over the discriminant extension.

Similar considerations can lead to counterexamples to the local-global principle for quadratic forms of dimension 4 over function fields of curves over totally imaginary number fields. An important open question remains: does the local-global principle hold for quadratic forms of dimension ≥ 5 over such fields?

References

[1] M. Artin, *Brauer-Severi varieties*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin, 1982, pp. 194–210.

[2] Asher Auel, *Clifford invariants of line bundle-valued quadratic forms*, MPIM preprint series 2011-33, 2011.

[3] Asher Auel, R. Parimala, and V. Suresh, *Quadric surface bundles over surfaces*, preprint arXiv:1207.4105, 2012.

[4] Maurice Auslander and Oscar Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409.

[5] W. Bichsel and M.-A. Knus, *Quadratic forms with values in line bundles*, Contemp. Math. **155** (1994), 293–306.

[6] J.-L. Colliot-Thélène and J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Ann. **244** (1979), no. 2, 105–134.

[7] Jean-Louis Colliot-Thélène, Raman Parimala, and Venapally Suresh, *Patching and local-global principles for homogeneous spaces over function fields of p-adic curves*, Comment. Math. Helv. **87** (2012), no. 4, 1011–1033.

[8] David Harbater, Julia Hartmann, and Daniel Krashen, *Applications of patching to quadratic forms and central simple algebras*, Invent. Math. **178** (2009), no. 2, 231–263.

[9] M.-A. Knus, R. Parimala, and R. Sridharan, *On rank* 4 *quadratic spaces with given Arf and Witt invariants*, Math. Ann. **274** (1986), no. 2, 181–198.

[10] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, Colloquium Publications, vol. 44, AMS, 1998.

[11] Alexander Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. Math. **218** (2008), no. 5, 1340–1369.

Upper bounds for Euclidean minima of abelian fields

Eva Bayer
(joint work with Piotr Maciak)

Let K be an algebraic number field, and let O_K be its ring of integers. Let $\mathrm{N} : K \to \mathbf{Q}$ be the absolute value of the norm map. The number field K is said to be *Euclidean* (with respect to the norm) if for every $a, b \in O_K$ with $b \neq 0$ there exist $c, d \in O_K$ such that $a = bc + d$ and $\mathrm{N}(d) < \mathrm{N}(b)$. It is easy to check that K is Euclidean if and only if for every $x \in K$ there exists $c \in O_K$ such that $\mathrm{N}(x - c) < 1$. This suggests to look at

$$M(K) = \mathrm{sup}_{x \in K} \mathrm{inf}_{c \in O_K} \mathrm{N}(x - c),$$

called the *Euclidean minimum* of K.

The determination of Euclidean number fields and Euclidean minima is a classical problem – see for instance the survey of Lemmermeyer [L 95], as well as the tables of Cerri [C 07]. Another classical problem is to find *upper bounds* for $M(K)$ in terms of the degree $n = [K : \mathbf{Q}]$ of the number field K, and of the absolute value d_K of its discriminant. Upper bounds valid for arbitrary number fields exist since the early 1950's, due to work of Clarke and Davenport. In [BF 06], it is proved that

$$M(K) \leq 2^{-n} d_K.$$

If K is totally real, then a conjecture attribruted to Minkowski states that

$$M(K) \leq 2^{-n} \sqrt{d_K}.$$

This is known for $n \leq 8$ (cf. [HGRS 11]). One can also try to prove the conjecture for some families of number fields. This is done in [BF 06], [BFN 05] and [BFS 06] for certain cyclotomic fields. It is natural to ask the same question for *abelian* number fields. We have

Theorem. [BFM 13] *Let p be an odd prime number, and let K be an abelian number field of conductor p^r. If $r \geq 2$, then we have*

$$M(K) \leq 2^{-n} \sqrt{d_K}.$$

In particular, Minkowski's conjecture holds for totally real number fields of conductor p^r, when p is an odd prime and $r \geq 2$.

The proof uses packing and covering invariants of number fields, following a method of [BF 06]. A key ingredient is the determinantion of the *trace form* of the ring of integers.

References

[BF 06] E. Bayer–Fluckiger. Upper bounds for Euclidean minima of algebraic number fields, *J. Number Theory* **121** (2006), 305–323.

[BFM 13] E. Bayer–Fluckiger, P. Maciak. Upper bounds for the Euclidean minima of abelian fields of odd prime power conductor, *Math. Ann.* (to appear).

[BFN 05] E. Bayer–Fluckiger, G. Nebe. On the Euclidean minimum of some real number fields, *Journal dethéorie des nombres de Bordeaux* **17** (2005), 437-454.

[BFS 06] E. Bayer–Fluckiger, I. Suarez. Ideal lattices over totally real number fields and Euclidean minima, *Archiv Math.*, **86** (2006), 217-225.

[C 07] J-P. Cerri. Euclidean minima of totally real number fields: algorithmic determination, *Math. Comp.* **76** (2007), 1547-1575.

[HGRS 11] R. J. Hans-Gill, M. Raka and R. Sehmi. On Conjectures of Minkowski and Woods for $n = 8$, *Acta Arith.* **147** (2011), 337-385.

[L 95] F. Lemmermeyer. The Euclidean algorithm in algebraic number fields, *Expo. Math.* **13** (1995), 385-416.

On cohomological invariants of adjoint algebraic groups

Vladimir Chernousov

In my talk we discussed some consequences of Merkurjev's classification of cohomological invariants of adjoint algebraic groups in degree 3.

We first recall what is a cohomological invariant. Let G be a split semisimple algebraic group defined over a field k and M a discrete $\mathrm{Gal}(k_s/k)$-module. Consider two functors from the category **Fields** of field extensions of k into the category**Sets** of sets: the functor $H^1(-,G)$ of isomorphism classes of G-torsors and the functor of the abelian Galois cohomology groups $H^n(-,M)$ with coefficients in M. A cohomological M-invariant (or invariant with coefficients in M) in degree n is a morphism $a : H^1(-,G) \to H^n(-,M)$ of our two functors. An invariant a is called normalized if $a([\xi]) = 0$ where ξ is a trivial cocycle (torsor).

The group of all normalized invariants in degree n with coefficients in M is denoted by $\mathrm{Inv}^n(G,M)_{norm}$. It is an interesting (and difficult) open problem of computing this group even in the simplest case when G is simple and $M = \mathbb{Z}/2$. We refer to [2] and [3] for basic properties and known results on cohomological invariants of algebraic groups.

In degree $n = 3$ the group $\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{norm}$ contains an obvious subgroup $\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2)_{dec}$ consisting of decomposable invariants which are cup product of invariants in degree 2 (all of them come from Tits algebras) with constant invariants in degree 1. This group is canonically isomorphic to $\widehat{C} \otimes k^\times$ where C is the kernel of a simply connected covering $\widetilde{G} \to G$ and hence the problem of computing of $\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{norm}$ is reducing to computing of the quotient group

$$\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{ind} := \mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{norm}/\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2)_{dec}$$

called the group of indecomposable invariants.

In a recent beautiful paper [6] A. Merkurjev constructed an exact sequence consisting of 5 terms involving $\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{norm}$, $\mathrm{CH}^2(BG)_{tors}$, where BG is the classifying space of G, and some other arithmetical data related to G and

C. Using this sequence he showed that if additionally G is simple adjoint then $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{ind} \neq 0$ if and only if G is of type C_n, D_n with n divisible by 4, E_6 and E_7. Furthermore, he also proved that for a split G of type C_n, D_n, E_6 with n divisible by 4 one has

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{ind} \simeq \mathbb{Z}/2$$

and for a split group G of type E_7

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{ind} \simeq \mathbb{Z}/3.$$

It is worth mentioning that in the case $\mathrm{char}(k) \neq 2$ and G is of type C_{4m} a nontrivial indecomposable invariant was constructed in [5]. The sketch of construction such an invariant for type D_{4m} was done in [6] using the same ideas as for C_{4m}. As for a nontrivial indecomposable invariant for type E_7, it was constructed in [4].

In the first part of my talk we gave a uniform construction of nontrivial indecomposable cohomological invariants with coefficients in $\mathbb{Z}/2$ for split adjoint groups of types C_{4m}, D_{4m} and E_6. The main idea is to consider orthogonal representations $\lambda : G \to O(f)$ of the group G in question. Any such representation induces a natural mapping $H^1(F, G) \to H^1(F, O(f))$ where F/k is any field extension. Recall that elements of $H^1(F, O(f))$ are in one-to-one correspondence with isomorphism classes of nondegenerate quadratic forms over F having the same dimension as f. Thus to every $[\xi] \in H^1(F, G)$ we may associate in a functorial way a nondegenerate quadratic form f_ξ.

If now n is a maximal positive integer such that for all field extensions F/k and all cocycles $\xi \in Z^1(F, G)$ the classes of $f_\xi - f$ are contained in $I^n(F)$ then we have a well-defined nontrivial cohomological invariant

$$a_\lambda : H^1(-, G) \longrightarrow I^n/I^{n+1} \simeq H^n(-, \mathbb{Z}/2)$$

in degree n with coefficients in $\mathbb{Z}/2$. Recall that the last isomorphism is due to Voevodsky's theorem.

It easily follows from our construction that $n \geq 2$. Therefore to construct a nontrivial indecomposable cohomological invariant for G in question in degree 3 we need only to produce an orthogonal representation λ of G with the properties:

(1) $f_\xi - f \in I^3(F)$ for all field extensions F/k and all $[\xi] \in H^1(F, G)$;
(2) there exists a field extension F/k and a class $[\xi] \in H^1(F, G)$ such that $f_\xi - f \notin I^4(F)$;
(3) a_λ is indecomposable.

Take λ to be the adjoint representation $ad : G \hookrightarrow \mathrm{GL}(\mathrm{Lie}(G))$ of G. It is well known that $ad(G) \subset O(f)$ where f is a normalized Killing form on $\mathrm{Lie}(G)$. Let ϕ be a canonical mapping $\phi : \mathrm{Spin}(f) \to O(f)$. It induces the following diagram:

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{Spin}(f) \\ \downarrow & & \downarrow \\ G & \longrightarrow & O(f) \end{array}$$

where H is the connected component of the preimage $\phi^{-1}(G)$.

Proposition 1. *If G is an adjoint group of one of the following types C_{4m}, D_{4m} or E_6 then $H \simeq G$.*

According to the proposition any cocycle $\xi \in Z^1(F, G)$ can be viewed as a cocycle with coefficients in $\mathrm{Spin}(f)$. It then follows immediately that the class of the quadratic form $f_\xi - f$ lives in $I^3(F)$ so that property (a) follows.

As for properties (2) and (3) we may assume without loss of generality that k is algebraically closed. Let $T \subset G$ me a maximal split torus and let $c \in \mathrm{Aut}(G)$ be such that $c^2 = 1$ and $c(t) = t^{-1}$ for every $t \in T$. It is known that such an automorphism exists and it is inner for G in question so that $c \in G$. Call A_0 the kernel of "multiplication by 2" on T. Let $A = A_0 \times \{1, c\}$ be the subgroup of G generated by A_0 and by the element c defined above. The group A is isomorphic to $(\pm 1)^{r+1}$.

Take $F = k(t_1, \ldots, t_n, u)$ where n is the rank of G and $t_1, \ldots, t_n$ and u are independent indeterminates. We have

$$H^1(F, A) = H^1(F, \mathbb{Z}/2) \times \ldots \times H^1(F, \mathbb{Z}/2).$$

Identify $H^1(F, \mathbb{Z}/2)$ with $F^\times/(F^\times)^2$ as usual. Then u and the t_i's define elements (u) and (t_i) of $H^1(F, \mathbb{Z}/2)$. Let ξ_A be the element of $H^1(F, A)$ with components $((t_1), \ldots, (t_r), (u))$. Let $\xi = \xi_G$ be the image of ξ_A in $H^1(F, G)$. Using results of [1] one can prove the following proposition which establishes properties (2) and (3).

Proposition 2. *In the notation above the quadratic form $f_\xi - f$ is not hyperbolic and $f_\xi - f \notin I^4(F)$.*

In the second part of the talk with the use of Merkurjev's classification [6] of cohomological invariants of adjoint groups in degree 3 and the above game with Killing forms we constructed new cohomological invariants for groups of type C_n, D_n.

Theorem 3. *Let G be a split adjoint group over a field k of characteristic $\neq 2$ of type C_n or D_n where $n = 4m + i$ and $i = 1, 2, 3$. Then there exists a nontrivial cohomological invariant $c_4 : H^1(-, G) \to H^4(-, \mathbb{Z}/2)$ in degree 4 with coefficients in $\mathbb{Z}/2$.*

References

1. V. Chernousov, J.-P. Serre, *Lower Bounds for Essential Dimensions via Orthogonal Representations*, Journal of Algebra, 305 (2006), 1055-1070.
2. S. Garibaldi, *Cohomological invariants: exceptional groups and spin groups (notes for the Lens mini-cours) with an appendix by Detlev W. Hoffmann*, Memoirs of the American Mathematical Society 200 (2009), no. 937.
3. S. Garibaldi, A. Merkurjev, J.-P. Serre, *Cohomological invariants in Galois cohomology*, American Mathematical Society, Providence, RI, 2003.
4. S. Garibaldi and Ph. Gille, *Algebraic groups with few subgroups*, J. Lond. Math. Soc. (2), 80 (2009), no. 2, 405–430.
5. S. Garibaldi, R. Parimala, J.-P. Tignol, *Discriminant of symplectic involutions*, Pure Appl. Math. Q., 5 (2009), no. 1, 349–374.
6. A. Merkurjev, *Degree three cohomological invariants of semisimple groups*, Preprint (2013).

Pfister involutions in characteristic two

Andrew Dolphin

Over fields of characteristic different from 2 it is well-known that a quadratic form of dimension a 2 power is anisotropic or hyperbolic over any field extension if and only it is similar to a Pfister form. That is, if and only if it is a tensor product of 2 dimensional quadratic forms. Since we can associate to any symmetric bilinear form, and hence to any quadratic form over a field of characteristic different from 2, an orthogonal involution on a split central simple algebra, it is natural to consider whether there are central simple algebras with involution with analogous properties to Pfister forms.

Let (A, σ) be a central simple algebra of degree a 2 power with orthogonal involution over a field F. We denote the central simple algebra (resp. the algebra with involution) obtained by extending scalars over a field extension K/F as A_K (resp. $(A, \sigma)_K$).

Assuming that the characteristic of F is different from 2, in [1] it is asked whether the following are equivalent:

(1) (A, σ) is isomorphic to a product of quaternion algebras with involution.
(2) For all field extensions K/F such that A_K is split, there exists a Pfister form π over K such that $(A, \sigma)_K$ is isomorphic to the adjoint algebra with involution of π.
(3) For any field extension L/F, $(A, \sigma)_L$ is either anisotropic or hyperbolic.

That (1) implies (2) is know as the Pfister Factor Conjecture, and was proven in [2]. That (1) implies (3), and the equivalence of (2) and (3), follows from the Pfister Factor Conjecture and the non-hyperbolic splitting result of [5]. The converse implication, (2) or (3) implies (1), is still an open question in general.

Analogous questions may be asked when we consider fields of characteristic 2. However, just as the theories of quadratic forms and bilinear forms diverge when the characteristic of the base field is 2, we also get two divergent objects when considering algebras with orthogonal involution.

Firstly, we have algebras with quadratic pairs (see [7, Section 5]). These are objects that play an analogous role to quadratic forms as algebras with involution do to symmetric bilinear forms. That is, to any nonsingular quadratic form we may associate an 'adjoint' quadratic pair defined on a split central simple algebra. In this case, one may formulate the above question in a completely analogous way for decomposable quadratic pairs. That is, an algebra with quadratic pair that is isomorphic is a product of quaternion algebras with involution and a quadratic pair on a quaternion algebra. In this case, only (1) implies (2) has been shown in upcoming joint work with K. J. Becher.

Alternatively we may formulate the question in terms of algebras with orthogonal involution and symmetric bilinear forms over a field of characteristic 2. The theory of symmetric bilinear forms in characteristic 2 has several features that mean we must be slightly more careful in our formulation of the analogous question to that posed in [1]. Over fields of characteristic different from 2, all 2-dimensional isotropic symmetric bilinear forms are isometric to the hyperbolic plane. This is not true in characteristic 2, and the wider variety of isotropic 2-dimensional forms means that we must use the weaker property of metabolicity rather than hyperbolicity in the formulation of our problem and be more careful with our statement. For example, there exist metabolic bilinear forms of dimension a 2 power that are not similar to bilinear Pfister forms. That is, they are not isometric to a tensor product of 2-dimensional bilinear forms. Metabolicity for algebras with involution is studied in [3].

Conversely however, the isotropy behaviour of symmetric bilinear forms over quadratic separable extensions is particularly simple. Anisotropic symmetric bilinear forms remain anisotropic over any separable extension (see [6, (10.2.1)]). We can often exploit this property to investigate symmetric bilinear forms over fields of characteristic 2 with much simpler methods than those needed over fields of characteristic different from 2.

We therefore ask the following question. Let (A, σ) be a non-metabolic central simple algebra with orthogonal involution over a field of characteristic 2. Are the following equivalent:

(1) (A, σ) is isomorphic to a product of quaternion algebras with involution.
(2) For all field extensions K/F such that A_K is split, there exists a bilinear Pfister form π over K such that $(A, \sigma)_K$ is isomorphic to the adjoint involution of π.
(3) For any field extension L/F, $(A, \sigma)_L$ is either anisotropic or metabolic.

For this question, that (1) implies (2) is very simply shown by passing to a separable closure of K and using the fact that symmetric bilinear forms do not become anisotropic over a separable extension. As in the case of characteristic different from 2, (3) or (2) implies (1) is still open in general.

We show that (1) implies (3) using the splitting results of [4] and the refined Witt decomposition shown in [8].

References

[1] E. Bayer-Fluckieger, R. Parimala, and A. Quéguiner-Mathieu. Pfister involutions. *Proc. Indian Acad. Sci. Math. Sci.*, 113:365–377, 2003.

[2] K. J. Becher. A proof of the Pfister Factor Conjecture. *Invent. Math.*, 173(1):1–6, 2008.

[3] A. Dolphin. Metabolic involutions. *Journal of Algebra*, 336(1):286–300, 2011.

[4] A. Dolphin. Decomposition of algebras with involution in characteristic 2. *Journal of Pure and Applied Algebra*, 217(9):1620–1633, 2013.

[5] N. A. Karpenko. Hyperbolicity of orthogonal involutions. *Doc. Math.*, Extra volume: Andrei A. Suslin sixtieth birthday:371–392, 2010.

[6] M. Knebusch. *Grothendieck und Wittringe von nichtausgearteten symmetrischen Bilinearformen.* Math-natw. Klasse 3. Sitzb. Heidelberg Akademie Wiss., 1969/1970.

[7] M.-A. Knus, A.S. Merkurjev, M. Rost, and J.-P. Tignol. *The Book of Involutions*, volume 44 of *Colloq. Publ., Am. Math. Soc.* Am. Math. Soc., 1998.

[8] A. Laghribi and P. Mammone. Hyper-isotropy of bilinear forms in characteristic 2. *Contemporary Mathematics*, 493:249–269, 2009.

Gabber's compactifications of algebraic groups and homogeneous spaces

Philippe Gille

Introduction

Let k be a field, k_s a separable closure and let $\overline{k}$ be an algebraic closure of k.

If G/k is an algebraic group (or more generally an homogeneous space $X = G/H$), it is of interest to compactify it in an equivariant way with respect to the left action. More precisely, by a G–equivariant compactification of X, we mean a G-open and dense embedding $X \hookrightarrow X^c$ where X^c is a projective k–scheme.

In geometric invariant theory, we require more properties for compactifications: smoothness, nice boundary, "modular understanding" of the functor of points, ... Let us mention here for example the de Concini-Procesi "wonderful" compactification of semisimple adjoint groups by [CP] and two different compactifications of the linear groups (Kausz [K], Huruguen [Hu, §2.2]).

The "wonderful compactification" is actually equivariant for $G \rtimes \mathrm{Aut}(G)$; this is also the case when compactifiying the one dimensional split torus $\mathbb{G}_m$ by the projective line $\mathbf{P}^1_k$. One cannot require that in the general case as we will see for higher tori. We have $\mathrm{Aut}(\mathbb{G}_m^2) = \mathrm{GL}_2(\mathbb{Z})$ and we claim that there is no projective compactification of $T = \mathbb{G}_m^2$ which is $\mathbb{G}_m^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$-equivariant. Assume that T^c is such a compactification; the toric surface T^c can be desingularized in a canonical way so that we can assume that T^c is a smooth toric surface. But the automorphism group of T^c is in this case algebraic by a result of Harbourne [Ha, cor. 1.4], this is a contradiction. However for the split tori $\mathbb{G}_m^n$, the theory of fans permits to contruct equivariant compactifications under $\mathbb{G}_m^n \rtimes \Gamma$ for an arbitrary finite subgroup Γ of $\mathrm{GL}_n(\mathbb{Z})$ (Brylinski, Künnemann, see [CTHS]); by Galois descent it provides nice compactications of arbitrary k–tori.

Our purpose is mainly arithmetic so that we are not interested in the geometry of the compactifications but wish to control the rational points of the boundary. To be clearer, let us start with the following fact taken from the proof of lemma 12 in [CTS].

Lemma 1. *Let T/k be an anisotropic torus. Then there exists a T–compactification T^c such that $T(k) = T^c(k)$.*

Proof. Let K/k be a finite Galois extension which splits T. Then there exists an embedding $i : T \to \big(R_{K/k}(\mathbb{G}_{m,K})\big)^n$ where $R_{K/k}(\mathbb{G}_{m,K})$ denotes the Weil restriction torus. We denote by $R^1_{K/k}(\mathbb{G}_{m,K}) = \ker\big(R_{K/k}(\mathbb{G}_{m,K}) \to \mathbb{G}_m\big)$ the kernel of the norm map $N_{K/k}$. Since T anisotropic, T embeds in $\big(R^1_{K/k}(\mathbb{G}_{m,K})\big)^n$ and the statement boils down to the case of the norm one torus $H = R^1_{K/k}(\mathbb{G}_{m,K})$. The norm one torus H embeds in an equivariant way to the projective hypersurface of equation $x^{[K:k]} = N_{K/k}(y)$ inside $\mathbf{P}(k \oplus K)$. □

This fact admits the following generalization.

Theorem 2. *(Borel-Tits,* [BT, th. 8.2]*). Assume that k is perfect and that G/k is affine and smooth. Then the following are equivalent:*

(1) G is k-wound, i.e. G admits no k–subgroup isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$;

(2) G admits a G–equivariant compactification G^c such that $G(k) = G^c(k)$.

Note that (2) $\Longrightarrow$ (1) is trivial since a k–embedding $f : \mathbb{G}_a \to G$ extends to a k–map $\widetilde{f} : \mathbf{P}^1_k \to G^c$. Since G is affine, f cannot extend to a k–morphism $\mathbf{P}^1_k \to G$, so that $\widetilde{f}(\infty) \in G^c(k) \setminus G(k)$. Similarly a k–embedding for $\mathbb{G}_m \to G$ provides at least one point of $G^c(k) \setminus G(k)$.

Remark 3. Still assuming that k is perfect, a general smooth connected k-group G fits in an exact sequence $1 \to L \to G \to A \to 1$ where A is an abelian variety and L is a smooth affine k-group (Chevalley). Note that G is k-wound iff L is k-wound, so that (2) $\Longrightarrow$ (1) holds. For (1) $\Longrightarrow$ (2), the trick is to compactify fiberwise with respect to $G \to A$, that is by taking the contracted product $G^c = G \wedge^L L^c$ over A. For a non-connected smooth G, we repeat the same trick with the exact sequence $1 \to G^0 \to G \to F \to 1$ where F is finite étale over k and G^0 the neutral component of G [DG, II.5.1].

Gabber's compactifications

From now one, we assume that k is of characteristic $p > 0$. Gabber proved a similar statement than Theorem 2.

Theorem 4. [G, Theorem B] *Let G/k be an algebraic group. Then the following are equivalent:*

(1) G is k-wound, i.e. G admits no k–subgroup isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$;

(2) G admits a G–equivariant compactification G^c such that $G(k) = G^c(k)$.

Furthermore, if G is affine, we can assume that G^c is equipped with an ample line bundle which is G–linearized.

Again the theorem is the implication (1) $\Longrightarrow$ (2), namely the construction of compactifications.

Remarks 5. (a) Note that G is not assumed to be smooth.

(b) The affine case is the main one. Assuming it holds, one deals firstly with the case of G not affine and connected. It fits then in an exact sequence $1 \to L \to G \to A \to 1$ where L is affine and A an abelian variety [SGA3, 12.5.(5)]. The same trick than in Remark 3 works and provides the desired compactification of G from a compactification of L and idem to pass to the non-connected case.

(c) The theorem was previously known in the case of a commutative k–group as a result by Bosch-Lütkebohmert-Raynaud, [BLR, §10.2, th. 7]).

(e) Let F/k be a field extension such that G_F is F-wound. If G^c is a Gabber's compactification of G/k, it can be shown than G^c_F is is a Gabber's compactification of G_F/F.

From now on, we deal then with an affine algebraic k-group G. The talk discussed mainly the reduction of the theorem to the smooth case. The k-group G admits a maximal closed k–subgroup denoted by $G^\dagger$ [CGP, C.4.1]. It is characterized by the fact that $G^\dagger(k_s) = G(k_s)$ or equivalently that $(G/G^\dagger)(k_s) = \{x_0\}$. For reducing to the smooth case, the main new step is to provide a nice compactification of the homogeneous space $X = G/G^\dagger$.

Theorem 6. *(Gabber, 2011, see* [GGMB]*). The homogeneous space X admits a G–compactification X^c equipped with an ample G–linearized line bundle such that $X^c(k_s) = \{x_0\}$.*

In the proof we gave the full proof of this last result which was used in the talk by Laurent Moret-Bailly. From this fact, the reduction to the smooth case goes by a dévissage argument from the $G^\dagger$-torsor $G \to G/G^\dagger$ of the same vein than lemma 15 of [BLR, §10.2].

References

[BT] A. Borel, J. Tits, *Groupes réductifs*, Publ. Math. IHES **27**(1965), 55–151.

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete **21** (1990), Springer-Verlag.

[CTHS] J.-L. Colliot-Thélène, D. Harari, A. N. Skorobogatov, *Compactification équivariante d'un tore*, Expositiones math. **23** (2005), 589-606.

[CTS] J.-L. Colliot-Thélène and J.-J. Sansuc, *La R-équivalence sur les tores*, Annales de l'École Normale Supérieure **10** (1977), 175–229.

[CP] C. de Concini, C. Procesi, *Complete symmetric varieties*, Invariant theory (Montecani, 1982), Lecture Note in Math. **996** (1983), 589-606.

[CGP] B. Conrad, O. Gabber, G. Prasad, *Pseudo-reductive groups*, Cambridge University Press (2010).

[DG] M. Demazure, P. Gabriel, *Groupes algébriques*, North-Holland (1970).

[G] O. Gabber, *On pseudo-reductive groups and compactifications theorems*, Oberwolfach report (2011).

[GGMB] O. Gabber, P. Gille, L. Moret-Bailly, *Torseurs topologiques sur les corps henséliens*, en préparation.

[Ha] B. Harbourne, *Rational surfaces with infinite automorphism group and no antipluricanonical curve*, Proc. AMS **99** (1987), 409-414.

[Hu] M. Huruguen, *Compactification d'espaces homogènes sphériques sur un corps quelconque*, thèse (Grenoble, 2011), author's URL.

[K] I. Kausz, *A modular compactification of the general linear group*, Doc. Math. **5** (2000), 553-594.

[SGA3] M. Demazure, A. Grothendieck, *Schémas en groupes* I, II, III, Lecture Notes in Math **151, 152, 153**, Springer-Verlag, New York (1970) et réédition : Documents Mathématiques **7, 8**, Société Mathématique de France (2011).

Permutation modules and motives of geometrically rational surfaces

Stefan Gille

This is a report on the work in progress [4]. Let F be a perfect field with algebraic closure $\bar{F}$ and absolute Galois group $G_F = \mathrm{Gal}(\bar{F}/F)$. We set $\bar{X} := \bar{F} \times_F X$ for an F-scheme X. Let S be a geometrically rational F-surface, by which we understand a smooth projective and geometrically integral F-scheme of dimension 2, such that $\bar{S} := \bar{F} \times_F S$ is birational isomorphic to the 2-dimensional projective space over $\bar{F}$.

Assume that S has an F-rational point. Then S decomposes

$$S \simeq \mathrm{Spec}F \oplus M \oplus \underline{\mathbb{Z}}(2)$$

in the category of (effective) Chow motives with integral coefficients $\mathfrak{Chow}(F)$. (Here $\underline{\mathbb{Z}}(1)$ denotes the Tate motive and we have set $\underline{\mathbb{Z}}(i) := \underline{\mathbb{Z}}(1)^{\otimes i}$ for $i \geq 0$.)

To understand the middle part M of this decomposition one observes first that the natural homomorphisms

$$(1) \qquad \mathrm{Hom}_{\mathfrak{Chow}(F)}((\mathrm{Spec}E) \otimes \underline{\mathbb{Z}}(1), M) \longrightarrow \mathrm{Hom}_{G_F}(\mathrm{CH}_0(\bar{E}), \mathrm{CH}_1(\bar{S})),$$

$$\alpha \longmapsto (\bar{F} \times_F \alpha)_*$$

and

$$(2) \qquad \mathrm{Hom}_{\mathfrak{Chow}(F)}(M, (\mathrm{Spec}E) \otimes \underline{\mathbb{Z}}(1)) \longrightarrow \mathrm{Hom}_{G_F}(\mathrm{CH}_1(\bar{S}), \mathrm{CH}_0(\bar{E})),$$

$$\beta \longmapsto (\bar{F} \times_F \beta)_*$$

are isomorphisms for any étale F-algebra E. (The second isomorphism uses the fact that since S is a geometrically rational surface the intersection pairing $\mathrm{Pic}\bar{S} \times \mathrm{Pic}\bar{S} \longrightarrow \mathbb{Z}$ is a regular symmetric bilinear form which is G_F-invariant.)

As well known the G_F-module $\mathrm{CH}_0(\bar{E})$ is a permutation module for all étale algebras E and every G_F-permutation module is isomorphic to $\mathrm{CH}_0(\bar{E})$ for some étale F-algebra E. Using this fact and the Rost nilpotence theorem for geometrically rational surfaces in $\mathfrak{Chow}(F)$, see [2, 3], one deduces from the above isomorphisms (1) and (2) the following result:

Theorem. *Let S be a geometrically rational F-surface, such that $S(F) \neq \emptyset$. Then the motive of S in $\mathfrak{Chow}(F)$ is 0-dimensional (equivalently, a direct summand of $\underline{\mathbb{Z}} \oplus [(\mathrm{Spec}E) \otimes \underline{\mathbb{Z}}(1)] \oplus \underline{\mathbb{Z}}(2)$ for some étale F-algebra E) if and only if* $\mathrm{Pic}\bar{S}$ *is a direct summand of a G_F-permutation module.*

The proof of this theorem is constructive and can be used to compute the motive of some geometrically rational surfaces. For instance, if S is a Del Pezzo surface of degree 6 with rational point then $\mathrm{Pic}\bar{S}$ is always a direct summand of a G_F-permutation module and it turns out that the middle part of the motive of S is a direct summand of $(\mathrm{Spec}E)(1)$, where E is the product of a cubic and a quadratic étale algebra over F.

Remarks.

(i) In a letter [1] to the author Colliot-Thélène showed that $\mathrm{Pic}\bar{S}$ is a direct summand of a G_F-permutation module if and only if $\mathrm{CH}_0(L \times_F S)$ is torsion free for all field extensions $L \supseteq F$.

(ii) Using the same method one can show the following (which seems to be well known):

Theorem. *Let X be a smooth projective F-scheme, such that the motive of X is geometrically split in the category of Chow motives with rational coefficients $\mathfrak{Chow}(F,\mathbb{Q})$, i.e. the motive of $\bar{X}$ is isomorphic to a (finite) direct sum of twists of Tate motives in $\mathfrak{Chow}(\bar{F},\mathbb{Q})$. Then the motive of X is zero dimensional in $\mathfrak{Chow}(F,\mathbb{Q})$.*

References

[1] J.-L. Colliot-Thélène, *Brief an Author*, vom 8.Mai 2013.

[2] S. Gille, *The Rost nilpotence theorem for geometrically rational surfaces*, Invent. Math. **181** (2010), 1–19.

[3] S. Gille, *On Chow motives of surfaces*, J. reine angew. Math. (to appear).

[4] S. Gille, *Permutation modules and motives of geometrically rational surfaces*, in preparation.

Valuations on real function fields and lower bounds for the pythagoras number

David Grimm

The pythagoras number $p_2(F)$ of a field F is by definition the smallest $n \in \mathbb{N}$ such that every sum of squares in F is equal to a sum of n squares in F, or ∞ if no such n exists. When F is the function field of a variety V over $\mathbb{R}$ (or any real closed field), A. Pfister showed that $p_2(F) \leq 2^d$. Finding the exact value of $p_2(F)$ (or just good lower bounds) is an open problem.

W. Kucharz showed in [K1] for real function fields $F/\mathbb{R}$ in d variables that $p_2(F) \geq d+1$, and he obtains the same lower bound more generally for real closed base fields in [K2]. He derives this bound from a more general result on minimal sets of generators for certain finitely generated ideals in the so called *real holomorphy ring* of $F/\mathbb{R}$ as defined in [B, p. 148]. The latter relies on Hironaka's resolution of singularities and of points of indeterminacy of rational functions. Furthermore, computations of Chern classes of vector bundles are used, and this part of the proof does not seem to generalize to the situation of varieties V over arbitrary formally real base fields K with formally real function field $F = K(V)$ (unless V contains a smooth K-rational point, or a closed point of odd degree).

I presented a more elementary proof for the lower bound $p_2(F) \geq d+1$ that does not need to assume that the base field K of F/K is real closed. Furthermore, Hironaka's resolution results or computations of Chern classes of vector bundles are not needed.

The case $d \geq 3$ is easily dealt with. We use the fact that if $F = K(V)$ is formally real, then V contains a smooth closed point P with formally real residue field $K(P)$. The generic point of the exceptional fiber of the blowing-up of V along P then yields a discrete valuation with real residue field $K(P)(X_1,\ldots,X_{d-1})$. Simple valuation theoretic considerations show that $p_2(F) \geq p_2(K(P)(X_1,\ldots,X_{d-1}))$, and for real rational function fields in at least two variables we have the better lower bound $p_2(K(P)(X_1,\ldots,X_{d-1})) \geq (d-1)+2 = d+1$ due to an iteration argument based on the *Cassels-Pfister theorem* and the fact that the bound holds when $d-1=2$ due to [CEP].

If $d=2$, the same argument yields that $p_2(F) \geq p_2(K(P)(X))$. However, it is known that $p_2(K(P)(X)) < 3$ can occur even when K is not real closed (e.g. for $K=\mathbb{R}((t))$). So we need a different argument when $d=2$. The key is the observation that it is sufficient to find a discrete valuation on F with nonreal residue field in which -1 is not a square (a well chosen lift of a nontrivial representation of zero as a sum of three squares then exhibits the lower bound $p_2(F) \geq 3$). In geometric terms, it is sufficient to find a geometrically irreducible curve C on the surface V (which we can assume to be projective and normal) that does not contain points with formally real residue field. The generic point of C in V will then yield a valuation with residue field $K(C)$. The way to obtain the existence of such a curve is by considering hyperplane sections of V with respect to some well chosen embedding in projective space. After enlarging the embedding dimension via a Veronese map if necessary, we have V embedded in a larger variety W that is defined over $\mathbb{Q}$ inside projective space while finding at the same time a hyperplane H defined over $\mathbb{Q}$ that has no common $\mathbb{R}$-points with W (and hence in particular with V). The completeness of the first order theory of real closed fields together with Bertini's theorem for generic hyperplane sections shows that after some (rational) small ϵ-variation of the coefficients of H, we have that $C = H \cap V$ is a smooth geometrically connected curve over K (and hence in particular geometrically irreducible) that contains no point with formally real residue field.

In the case $d=1$, we have in the rational case obviously that $p_2(K(X)) > 1$, as the pythagorean closure of a non-pythagorean field is always an infinite field extension as was shown by Diller and Dress [B, Theorem 3.8].

Kucharz' result on finitely generated ideals of the real holomorphy ring of a function field $F/\mathbb{R}$ does not only yield the lower bound $p_2(F) \geq d+1$ for the pythagoras number of a function field $F/\mathbb{R}$ in d variables, but in fact for all higher even pythagoras numbers $p_{2m}(F)$ as well (which is by definition the smallest $n \in \mathbb{N}$ such that every sum of $2m$-th powers is a sum of n such powers). In fact, my more elementary approach generalizes also to the $2m$-th pythagoras number. However, since the proof for dimension $d \geq 3$ is a mere reduction to the case of a rational function field $p_{2m}(F) \geq p_{2m}(K(P)(X_1,\ldots,X_{d-1}))$, it remains the task to find good lower bounds for the $2m$-th pythagoras number of the latter kind. One such lower bound (also for base fields that are not real closed) can be obtained by adapting Kucharz' proof to the situation of varieties over formally real fields that contain a smooth rational point. The resulting lower bound $p_{2m}(K(P)(X_1,\ldots,X_{d-1})) \geq (d-1)+1 = d$ in the rational case is slightly too bad to prove Kucharz' bound $p_{2m}(F) \geq d+1$ for arbitrary real d-dimensional function field F over general formally real fields. Summarized we obtain:

Theorem. *Let F/K a real function field in d variables and let $m \in \mathbb{N}$. Then $p_{2m}(F) \geq d+1$ when $m=1$, or $d \leq 2$, or when the embedding $K \hookrightarrow F$ admits a section $F \to K \cup \{\infty\}$. In the remaining cases we have $p_{2m}(F) \geq d$.*

Idealy, I would like to show $p_{2m}(F) \geq d+1$ unconditionally, which with my method of proof would require a better lower bound for rational function fields. Note that the *Cassels-Pfister theorem* for quadratic forms does not generalize to higher degree forms, so it is not evident how one can obtain better lower bounds for the $2m$-th pythagoras number of higher dimensional rational function fields from a good lower bound in small dimensions when $m \geq 2$.

References

[B] E. Becker. *The real holomorphy ring and sums of 2nth powers*, Real algebraic geometry and quadratic forms (Rennes, 1981), Lecture Notes in Math. 959, (1982), 139–181

[CEP] J.W.S. Cassels, W.J. Ellison, A. Pfister. *On sums of squares and on elliptic curves over function fields*, Journal of Number Theory 3, (1971), 125–149

[K1] W. Kucharz. *Invertible ideals in real holomorphy rings*, J. Reine Angew. Math. 395 (1989), 171–185

[K2] W. Kucharz. *Generating ideals in real holomorphy rings*, J. Algebra 144 (1991), 1–7

Hermitian forms over quaternion algebras

Nikita A. Karpenko

(joint work with Alexander Merkurjev)

We study a hermitian form h over a quaternion division algebra Q over a field. If the characteristic of the field is 2, we additionally assume that h is *alternating* (as defined in [4, §4.A]). For *generic* h and Q (defined as in the beginning of [3, §6]), for any integer $i \in [1,\ n/2]$, where $n := \dim_Q h$, we show in our Main Theorem ([3, Theorem 10.1]) that the variety of i-dimensional (over Q) totally isotropic right subspaces of h is 2-*incompressible* (see [1] for definition). The proof is based on a computation of the Chow ring for the classifying space of a certain parabolic subgroup in a split simple adjoint affine algebraic group of type C_n made in [3, §5]. As an application, we determine the smallest value of Alexander Vishik's J-invariant of a non-degenerate quadratic form divisible by a 2-fold Pfister form ([3, Corollary 11.3]); we also determine the biggest values of the canonical dimensions of the orthogonal Grassmannians associated to such quadratic forms (see [3, Corollary 11.2]).

The general outline of the paper follow the pattern of [2], where hermitian forms over quadratic extension fields (in place of quaternion algebras) and quadratic forms divisible by 1-fold Pfister forms have been treated.

Acknowledgements. We thank Burt Totaro for useful advices. The speaker gratefully acknowledges hospitality of the Fields Institute for Research in Mathematical Sciences (Toronto, Ontario), the Department of Mathematics of University of Toronto, and Thematic Program on Torsors, Nonassociative Algebras and Cohomological Invariants (January–June 2013).

References

[1] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.
[2] Karpenko, N. A. Unitary Grassmannians. *J. Pure Appl. Algebra 216*, 12 (2012), 2586–2600.
[3] Karpenko, N. A., and Merkurjev, A. S. Hermitian forms over quaternion algebras. Linear Algebraic Groups and Related Structures (preprint server) 498 (2013, May 13), 22 pages.
[4] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

Some field invariants in characteristic two related to the u-invariant

Ahmed Laghribi

Let F be a field of characteristic 2, and ϕ a quadratic form over F. We denote by $\mathrm{ql}(\phi)$ (*resp.* $\dim\phi$) the quasilinear part of ϕ (*resp.* the dimension of ϕ). The form ϕ is called nonsingular if $\dim \mathrm{ql}(\phi) = 0$, singular if $\dim \mathrm{ql}(\phi) > 0$, and totally singular if $\dim\phi = \dim \mathrm{ql}(\phi)$. The integer $\dim\phi - \dim \mathrm{ql}(\phi)$ is called the regular dimension of ϕ, we denoted it by $\mathrm{rdim}\phi$.

Let $Q(F)$ (*resp.* $T(F)$) denote the set of F-quadratic forms (*resp.* the set of totally singular F-quadratic forms) up to isometry. Because of the distinction between singular forms and nonsingular forms, Baeza introduced the two invariants:

$$u(F) = \sup\{\dim\phi \mid \phi \text{ is nonsingular and anisotropic}\},$$

$$\hat{u}(F) = \sup\{\dim\phi \mid \phi \in Q(F) \text{ is anisotropic}\}.$$

Obviously, $u(F) \leq \hat{u}(F)$, and the invariant $u(F)$ takes only even values because a nonsingular form is of even dimension.

An important question consists in giving the possible values of the $\hat{u}$-invariant (*resp.* u-invariant) of a field of characteristic 2. In [4] Mammone, Tignol and Wadsworth proved the following:

(A) There exists a field F such that $u(F) = \hat{u}(F) = 6$.
(B) For any integer $n \geq 2$, there exists a field F such that $u(F) = 2n$ and $\hat{u}(F) = \infty$.
(C) For any integers n, m such that $2^m \geq 2n \geq 4$ and $m \geq n-1$, there exists a field F such that $u(F) = 2n$ and $\hat{u}(F) = 2^m$.

Also Mammone, Moresi and Wadsworth proved that the $\hat{u}$-invariant can not take the integers $2^n - 1$ ($n \geq 2$) and 5 [3]. As we may verify in [4], the fields F in (B) and (C) satisfy the condition $\hat{u}(F) = [F : F^2]$, or equivalently, $\hat{u}(F) = \sup\{\dim\phi \mid \phi \in T(F) \text{ is anisotropic}\}$. Also, the claim that the $\hat{u}$-invariant can not take the integers $2^n - 1$ ($n \geq 2$) is based on the double inequality $[F : F^2] \leq \hat{u}(F) \leq 2[F : F^2]$ [3]. So, many information on the values of the $\hat{u}$-invariant are controlled by totally singular forms. Moreover, concerning the field F in (C), we do not know if there exists an anisotropic, not totally singular, form ϕ such that $\hat{u}(F) = \dim\phi$. This motivates the idea to get information on the supremum of the dimensions of anisotropic not totally singular forms, independently of the forms in $T(F)$. To this end, we consider the set $Q'(F) = Q(F) \setminus T(F)$, and we introduce the following invariant:

$$\tilde{u}(F) = \sup\{\dim \phi \mid \phi \in Q'(F) \text{ is anisotropic}\}.$$

Also, for $r, s \geq 1$ integers, we consider the following invariants:

$$u_r(F) = \sup\{\dim \phi \mid \phi \in Q'(F) \text{ is anisotropic and } \mathrm{rdim}\phi = 2r\},$$

$$\tilde{u}_s(F) = \sup\{\dim \phi \mid \phi \in Q'(F) \text{ is anisotropic and } \dim \mathrm{ql}(\phi) = s\}.$$

In what follows we give some elementary relations between the invariants u, $\hat{u}$, $\tilde{u}$, u_r and $\tilde{u}_s$:

- $u(F) \leq \tilde{u}(F) \leq \hat{u}(F)$.
- For any integers $r, s \geq 1$, we have:

$$\begin{cases} u_r(F) \leq \tilde{u}(F) \geq \tilde{u}_s(F) \\ u_{r+1}(F) \leq u_r(F) + 1 \\ \tilde{u}_{s+1}(F) \leq \tilde{u}_s(F) + 1. \end{cases} \qquad (\star)$$

- $u_1(F) \leq \tilde{u}(F) \leq 2u_1(F) - 2$.
- If $u(F) = 2n$, then $u(F) \leq u_{n-k}(F) + k$ for $0 \leq k \leq n-1$.
- If $u_n(F) = 2n$, then $u(F) = u_n(F)$.
- If $\tilde{u}(F) \neq \hat{u}(F)$, then $\hat{u}(F) = [F : F^2]$.

Our aim is to treat the following question:

Question 1. What are the possible values of the invariants $\tilde{u}$, u_r and $\tilde{u}_s$?

The main results on this question are as follows:

(1) For the $\tilde{u}$-invariant:

Theorem 2. *(a) $\tilde{u}(F) \neq 3, 5, 7$.*
(b) For any integers $n \geq 2$, $m \geq 2n-2$ and $k \geq m+2$, there exists a field F such that $u(F) = 2n$, $\tilde{u}(F) = u_1(F) = 2 + 2^m$ and $\hat{u}(F) = 2^k$.
(c) For any integer $n \geq 2$, there exists a field F such that $u(F) = 2n$ and $\tilde{u}(F) = u_1(F) = \infty$.

The fields given in statements (b) and (c) of Theorem 2 have the property that the invariants $\tilde{u}$ and u_1 coincide. In general, these two invariants are different, for example, we proved that the field F given in (A) satisfies $u_1(F) = 4$ and $\tilde{u}(F) = 6$.

(2) For the u_r-invariant:

Theorem 3. *(a) For any integer $n \geq 1$, there exists a field F such that $u_1(F) = 2^n + 1$.*
(b) For any integers $m, r \geq 1$, there exists a field F such that $u_r(F) = 2r + 2^m$.
(c) For any integer $r \geq 2$, there exists a field F such that $u_r(F) = 2r$ and $u_{r-1}(F) = 2r - 1$. In particular, $u_r(F) = u_{r-1}(F) + 1$.

In view of the inequality $u_{r+1}(F) \leq u_r(F) + 1$ given in ($\star$), and statement (c) of Theorem 3, we proved that for $r = 2^n$ ($n \geq 1$), there exists a field F such that $u_r(F) < u_{r-1}(F) + 1$.

Moreover, it follows from ($\star$) that if $u_r(F)$ if finite, then $u_k(F)$ is also finite for any $k \geq r+1$, but the converse is not true in general. Indeed, for any integer $r \geq 1$, we proved the existence of a field F such that $u_r(F) = \infty$ and $u_{r+1}(F) = 2(r+1)$, in particular, $u_k(F) < \infty$ for any $k \geq r + 1$.

(3) For the $\tilde{u}_s$-invariant:

Proposition 4. *For any integers $l, r, s \geq 1$ such that $2^{l-1} < s \leq 2^l$ et $r \geq \frac{2^l - s}{2}$, there exists a field F such that $\tilde{u}_s(F) = 2r + s$.*

The proofs of the results above are based on the following:

(1) Let $D = [b_1, a_1) \otimes_F \cdots \otimes_F [b_r, a_r)$ be a tensor product of n quaternion F-algebras, and $K = F(\sqrt{c_1}, \cdots, \sqrt{c_m})$ such that $[K : F] = 2^m$. If $D \otimes_F K$ is a division algebra, then $u(F) \geq 2(r + 1)$, $\hat{u}(F) \geq 2^{r+m}$, and $\phi := a_1[1, b_1] \perp \cdots \perp a_r[1, b_r] \perp \langle\langle c_1, \cdots, c_m \rangle\rangle$ is anisotropic, in particular, $\tilde{u}(F) \geq 2r + 2^m$.

(2) Let D be a division F-algebra of degree 2^n, and ϕ an anisotropic F-quadratic form such that $\mathrm{rdim}\phi = 2r$ ($r \geq 1$) and $\dim \mathrm{ql}(\phi) = s$. Then, $D \otimes_F F(\phi)$ is a division algebra in the following cases:

- $\dim \phi > 2 + \dim_F D$.
- $r = n$ and $s \geq 2$.
- $r > n$ and $s \geq 1$.

The point (2) is based on the index reduction theorems [4, Th. 3, Th. 4]. The proof of statement (a) of Theorem 3 uses the following result: *If ϕ and ψ are anisotropic F-quadratic forms such that* $\dim \phi = 2^n + 1$ *($n \geq 1$),* $\dim \psi > \dim \phi$ *and* $\mathrm{rdim}\phi = \mathrm{rdim}\psi = 2$*, then ϕ is anisotropic over $F(\psi)$.* This result, which is a consequence of some results in [5], was proved by another argument using the fact that any anisotropic not totally singular F-quadratic form of dimension $2^n + 1$ becomes an anisotropic Pfister neighbor over a suitable extension of F.

In additional to statement (b) of Theorem 2, the $\tilde{u}$-invariant can take any power of 2. Indeed, it was proved in [3] that there exists a field F such that $u(F) = \hat{u}(F) = 2^m$ ($m \geq 0$), in particular, this field satisfies $\tilde{u}(F) = 2^m$ because $u(F) \leq \tilde{u}(F) \leq \hat{u}(F)$. Still for even integers, we do not know if the $\tilde{u}$-invariant takes the integers $4 + 2^m$ for $m \geq 3$. For odd integers, and in view of statement (a) of Theorem 2, we ask the following question:

Question 5. Does there exist a field F such that $\tilde{u}(F) = 9$?

Moreover, using the double inequality $u_1(F) \leq \tilde{u}(F) \leq 2u_1(F) - 2$, we conclude that $\tilde{u}(F) \neq 2^n - 1$ ($n \geq 2$) when $u_1(F)$ is a power of 2.

We also discussed the following classical question:

Question 6. (Universality) Suppose that $u(F) < \infty$. Is any anisotropic nonsingular F-quadratic form of dimension $u(F)$ universal?

This question has a positive answer if $u(F) = 2$ or 4 (due to Baeza [1]). We proved that the answer is also positive if $u(F) = 6$. If $u(F) = 8$, we have the following partial result: *If ϕ is an anisotropic nonsingular F-form of dimension* 8 *and $c \in F^*$ such that* $\operatorname{ind} C(c\phi) \leq 2$, *then $\phi \perp \langle c \rangle$ is isotropic.*

References

[1] R. Baeza, *Comparing u-invariants of fields of characteristic* 2*,* Bol. Soc. Bras. Mat. **13 (1)**, 105–114 (1982).

[2] D. W. Hoffmann, A. Laghribi, *Quadratic forms and Pfister neighbors in characteristic* 2*,* Trans. Amer. Math. Soc. **356** (2004), 4019–4053.

[3] P. Mammone, R. Moresi, A. Wadsworth, *u-invariant of fields of characteristic* 2*,* Math. Z. **208** (1991), 335–347.

[4] P. Mammone, J.-P. Tignol, A. Wadsworth, *Fields of characteristic* 2 *with prescribed u-invariant,* Math. Ann. **290**, 109–128 (1991).

[5] B. Totaro, *Birational geometry of quadrics in characteristic* 2*,* J. Algebraic Geom. **17** (2008), no. 3, 577–597.

The u-invariant of a rational function field

David Leep

The (classical) u-invariant of a field F, written $u(F)$, is the maximum dimension of an anisotropic quadratic form defined over F. We set $u(F) = \infty$ if no such maximum exists. The main question of this report is the problem of computing $u(k(t))$ where k is a field and $k(t)$ is the rational function field over k. Throughout this report, k denotes a field with $\operatorname{char} k \neq 2$.

Proposition 1. $2u(k) \leq 2\sup\{u(E) \mid [E : k] < \infty\} \leq u(k(t))$.

Proof. The first inequality is trivial and the second inequality is proved using standard valuation theory. □

Proposition 2. *Let $[E : k] = r$. Then $u(E) \leq \frac{r+1}{2}u(k)$.*

Proof. See [L], Theorem 2.10. □

Proposition 2, currently the best known upper bound for $u(E)$, is not strong enough to even suggest the finiteness of $u(k(t))$ in Proposition 1. We now pursue a second approach.

Let $u_k(r, m)$ denote the smallest integer such that every system of r quadratic forms defined over k in more than $u_k(r, m)$ variables vanishes on an m-dimensional affine linear space defined over k. Set $u_k(r, m) = \infty$ if no such integer exists. Note that $u_k(1, 1) = u(k)$.

Proposition 3. $2u(k) \leq u_k(2, 1) \leq u(k(t))$.

Proof. Let q_1 and q_2 be two quadratic forms defined over k. Let $q_1 + tq_2$ denote the polynomial sum over $k(t)$. The Amer-Brumer theorem (see [A] and [B]) states that q_1 and q_2 have a nontrivial common zero over k if and only $q_1 + tq_2$

is isotropic over $k(t)$. This immediately implies that $u_k(2,1) \leq u(k(t))$. The inequality $2u(k) \leq u_k(2,1)$ comes from considering two anisotropic forms q_1 and q_2 in disjoint variables. □

Proposition 4. *Let Q be a regular quadratic form defined over $k(t)$. There exist quadratic forms q_1, q_2 defined over k and an integer $l \geq 0$ such that $q_1 + tq_2 \simeq_{k(t)} l\mathbb{H} \perp Q$.*

Proposition 5 (Amer's Theorem, [A])**.** *Let q_1 and q_2 be two quadratic forms defined over k. Then q_1 and q_2 vanish on a common m-dimensional affine linear space over k if and only if $q_1 + tq_2$ vanishes on an m-dimensional affine linear space over $k(t)$.*

Lemma 6.

(1) $2u(k) \leq u_k(2,1) \leq 3u(k)$.
(2) $u_k(2,m) + 2 \leq u_k(2,m+1) \leq u_k(2,m) + 3$ *for all* $m \geq 1$.
(3) $u_k(2,1) + 2(m-1) \leq u_k(2,m) \leq u_k(2,1) + 3(m-1)$ *for all* $m \geq 1$.

Proof. Proofs of these inequalities can be found in [L]. □

Theorem 7. $u(k(t)) = \sup_{m\geq 1}\{u_k(2,m) - 2(m-1)\}$.

Proof. By Lemma 6, we can assume that $u(k)$ is finite and thus $u_k(2,m)$ is finite for all $m \geq 1$.

For arbitrary $m \geq 1$, let q_1, q_2 be two quadratic forms defined over k in $u_k(2,m)$ variables such that q_1, q_2 do not vanish on a common m-dimensional vector space defined over k. By Proposition 5, $q_1 + tq_2$ doesn't vanish on an m-dimensional vector space over $k(t)$. Thus we have $q_1 + tq_2 \simeq_{k(t)} Q \perp l\mathbb{H} \perp \mathrm{rad}(q_1 + tq_2)$, where Q is anisotropic over $k(t)$ and $l + \dim(\mathrm{rad}(q_1 + tq_2)) \leq m - 1$. Then

$$u(k(t)) \geq \dim Q = u_k(2,m) - 2l - \dim(\mathrm{rad}(q_1 + tq_2)) \geq u_k(2,m) - 2(m-1).$$

Thus, $u(k(t)) \geq \sup_{m\geq 1}\{u_k(2,m) - 2(m-1)\}$.

Let Q be an anisotropic quadratic form defined over $k(t)$. By Proposition 4, there exist quadratic forms q_1 and q_2 defined over k such that $q_1 + tq_2 \simeq_{k(t)} Q \perp (m-1)\mathbb{H}$ for some integer $m \geq 1$. We have $\dim(q_1 + tq_2) \leq u_k(2,m)$, otherwise q_1 and q_2 would vanish on a common m-dimensional vector space over k and thus $q_1 + tq_2$ would also vanish on an m-dimensional vector space over $k(t)$ by the trivial implication of Proposition 5. Thus

$$\dim(Q) \leq u_k(2,m) - 2(m-1) \leq \sup_{m\geq 1}\{u_k(2,m) - 2(m-1)\}.$$

Therefore, $u(k(t)) \leq \sup_{m\geq 1}\{u_k(2,m) - 2(m-1)\}$. □

The second inequality in Proposition 3 is contained in Theorem 7 when $m = 1$.

Corollary 8. $u(k(t)) \leq N$ *if and only if* $u_k(2,m) \leq 2(m-1) + N$ *for all* $m \geq 1$.

The estimate in Lemma 6 (3) implies that

$$u_k(2,1) \leq u_k(2,m) - 2(m-1) \leq u_k(2,1) + (m-1)$$

for all $m \geq 1$. By Corollary 8, these estimates are not strong enough to conclude the finiteness of $u(k(t))$.

I have recently improved the estimates in Lemma 6 to obtain the following result.

Theorem 9. *$u_k(2,m) \leq M + \frac{5}{2}(m-1)$ for some positive constant M and for all $m \geq 1$.*

This improvement of Lemma 6, the first improvement since [L], is still not strong enough to prove the finiteness of $u(k(t))$, but there is hope that additional improvements will still be possible.

References

[A] M. Amer, *Quadratische Formen über Funktionenkörpern*, unpublished dissertation, Johannes Gutenberg-Universität, Mainz (1976).

[B] A. Brumer, *Remarques sur les couples de formes quadratiques*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 16, A679–A681.

[L] D. Leep, *Systems of quadratic forms*, J. reine angew. Math. **350**, (1984), 109-116.

A proof of conjecture of Serre and Grothendieck on principal bundles over regular local rings containing infinite fields

Ivan Panin

(joint work with Roman Fedorov)

Assume that U is a regular scheme, $\mathbf{G}$ is a reductive U-group scheme, and $\mathcal{G}$ is a principal $\mathbf{G}$-bundle. It is well known that such a bundle is trivial locally in étale topology (see e.g. [Gro3, Section 6]) but in general not in Zariski topology. Grothendieck and Serre conjectured that $\mathcal{G}$ is trivial locally in Zariski topology, if it is trivial at all the generic points. More precisely

Conjecture (Grothendieck-Serre)**.** *Let R be a regular local ring, let K be its field of fractions. Let $\mathbf{G}$ be a reductive group-scheme over $U := \operatorname{spec} R$, let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle. If $\mathcal{G}$ is trivial over $\operatorname{spec} K$, then it is trivial. Equivalently, the map*

$$H^1_{ét}(R, \mathbf{G}) \to H^1_{ét}(K, \mathbf{G}),$$

induced by the inclusion of R into K, has a trivial kernel.

The main result of this talk is Theorem 1. It asserts that this conjecture holds for regular local rings R, containing infinite fields. Our proof was inspired by the theory of affine Grassmannians. It is also based significantly on the geometric part of the paper [PSV1] of the second author with A. Stavrova and N. Vavilov.

Here is a list of known results in the same vein, corroborating the Grothendieck–Serre conjecture.

• The case where the group scheme $\mathbf{G}$ comes from the ground field k is completely solved by J.-L. Colliot-Thélène, M. Ojanguren, M. S. Raghunatan and O. Gabber: in [CTO] and [Rag1], [Rag2] when k is infinite; O. Gabber [Gab] announced a proof for an arbitrary ground field k.

• The case of an arbitrary reductive group scheme over a discrete valuation ring is completely solved by Y. Nisnevich in [Nis].

• The case where $\mathbf{G}$ is an arbitrary torus over a regular local ring was settled by J.-L. Colliot-Thélène and J.-J. Sansuc in [CTS].

• For some simple group schemes of classical series the conjecture is solved in works of the second author, A. Suslin, M. Ojanguren and K. Zainoulline [PS], [OP], [Zai], [OPZ].

• Under an isotropy condition on $\mathbf{G}$ the conjecture is proven in a short series of preprints [PSV1], [PSV2], [Pan].

• The case of strongly inner simple adjoint group schemes of type E_6 and E_7 is done by the second author, V. Petrov, and A. Stavrova in [PPS]. No isotropy condition is imposed there.

• The case when $\mathbf{G}$ is of the type F_4 with trivial g_3-invariant is settled by V. Chernousov in [Che].

Main results

Recall that an R-group scheme $\mathbf{G}$ is called *reductive*, if it is affine and smooth as an R-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $s : R \to \Omega$ the scalar extension $\mathbf{G}_\Omega$ is a connected reductive algebraic group over Ω. This definition of a reductive R-group scheme coincides with [DG, Exp. XIX, Defn. 2.7]. A well-known conjecture due to J.-P. Serre and A. Grothendieck [Ser, Remarque, p.31], [Gro1, Remarque 3, p.26-27], and [Gro2, Remarque 1.11.a] asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme $\mathbf{G}$ over R the map

$$H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(K, \mathbf{G}),$$

induced by the inclusion of R into K, has a trivial kernel. The following theorem, which is the main result of the present paper, asserts that this conjecture is valid, provided that R contains an infinite field.

Theorem 1. *Let R be a regular semi-local domain containing an infinite field, and let K be its field of fractions. Let $\mathbf{G}$ be a reductive group scheme over R. Then the map*

$$H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(K, \mathbf{G}),$$

induced by the inclusion of R into K, has a trivial kernel.

In other words, under the above assumptions on R and $\mathbf{G}$, each principal $\mathbf{G}$-bundle over R having a K-rational point is trivial.

Theorem 1 has the following

Corollary. *Under the hypothesis of Theorem 1, the map*

$$H^1_{\text{ét}}(R, \mathbf{G}) \to H^1_{\text{ét}}(K, \mathbf{G}),$$

induced by the inclusion of R into K, is injective.

Theorem 2. *Let R be the semi-local ring of finitely many closed points on an irreducible smooth affine variety over an infinite field k, set $U = \operatorname{spec} R$. Let $\mathbf{G}$ be a simple simply-connected group scheme over U (see* [DG, Exp. XXIV, Sect. 5.3] *for the definition).*

Let $Z \subset \mathbb{P}^1_U$ be a closed subset, quasi-finite and surjective over U. Let $\mathcal{G}$ be a principal $\mathbf{G}$-bundle on $\mathbb{P}^1_U$ such that its restriction to $\mathbb{P}^1_U - Z$ is trivial. Then there exists a closed subscheme $Y \subset \mathbb{P}^1_U$ étale over U such that $Y \cap Z = \emptyset$ and the restriction of $\mathcal{G}$ to $\mathbb{P}^1_U - Y$ is a trivial $\mathbf{G}$-bundle.

The main idea is to choose Y (finite and étale over U such that $\mathbf{G}$ has a parabolic group subscheme defined over Y and modify the bundle $\mathcal{G}$ in a neighborhood of Y). We use the technique of henselization. An essentially equivalent proof is based on formal loops.

Acknowledgements. The authors collaborated during Summer schools "Contemporary Mathematics", organized by Moscow Center for Continuous Mathematical Education. They would like to thank the organizers of the school for the perfect working atmosphere. The first author benefited from talks with D. Arinkin and A. Braverman. A part of the work was done, while he was a member of Max Planck Institute for Mathematics in Bonn. The second author thanks very much A. Stavrova and N. Vavilov for the very stimulating interest to the topic. He also thanks the RFBR-grant 13–01–00429–a for the support.

References

[Che] Vladimir Chernousov. *Variations on a theme of groups splitting by a quadratic extension and Grothendieck-Serre conjecture for group schemes F_4 with trivial g_3 invariant.* Doc. Math., (Extra volume: Andrei A. Suslin sixtieth birthday):147–169, 2010.

[CTO] Jean-Louis Colliot-Thélène and Manuel Ojanguren. *Espaces principaux homogènes localement triviaux.* Inst. Hautes Études Sci. Publ. Math., (75):97–122, 1992.

[CTS] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. *Principal homogeneous spaces under flasque tori: applications.* J. Algebra, 106(1):148–205, 1987.

[DG] M. Demazure and A. Grothendieck. *Schémas en groupes. III: Structure des schémas en groupes réductifs.* Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153. Springer-Verlag, Berlin, 1970.

[Gab] Ofer Gabber. Announced and still unpublished.

[Gro1] Alexander Grothendieck. *Torsion homologique et sections rationnelles.* Anneaux de Chow et applications, Séminaire Claude Chevalley, number 3. Paris, 1958.

[Gro2] Alexander Grothendieck. *Le groupe de Brauer. II. Théorie cohomologique.* Dix Exposés sur la Cohomologie des Schémas, pages 67–87. North-Holland, Amsterdam, 1968.

[Gro3] Alexander Grothendieck. *Technique de descente et théorèmes d'existence en géometrie algébrique. I. Généralités. Descente par morphismes fidèlement plats.* Séminaire Bourbaki, Vol. 5, Exp. No. 190., pages 299–327. Soc. Math. France, Paris, 1995.

[Nis] Yevsey Nisnevich. *Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional regular local rings.* C. R. Acad. Sci. Paris Sér. I Math., 309(10):651–655, 1989.

[OP] Manuel Ojanguren and Ivan Panin. *Rationally trivial Hermitian spaces are locally trivial.* Math. Z., 237(1):181–198, 2001.

[OPZ] M. Ojanguren, I. Panin, and K. Zainoulline. *On the norm principle for quadratic forms.* J. Ramanujan Math. Soc., 19(4):289–300, 2004.

[Pan] I. Panin. *On Grothendieck–Serre's conjecture concerning principal G-bundles over reductive group schemes:II.* ArXiv e-prints, 0905.1423v3, April 2013.

[PPS] I. Panin, V. Petrov, and A. Stavrova. *Grothendieck-Serre conjecture for adjoint groups of types E_6 and E_7 and for certain classical groups.* ArXiv e-prints, 0905.1427, December 2009.

[PS] I. A. Panin and A. A. Suslin. *On a conjecture of Grothendieck concerning Azumaya algebras.* St.Petersburg Math. J., 9(4):851–858, 1998.

[PSV1] I. Panin, A. Stavrova, and N. Vavilov. *On Grothendieck–Serre's conjecture concerning principal G-bundles over reductive group schemes:I.* ArXiv e-prints, 0905.1418v3, April 2013.

[PSV2] I. Panin, A. Stavrova, and N. Vavilov. *Grothendieck-Serre Conjecture I: Appendix.* ArXiv e-prints, 0910.5465, October 2009.

[Rag1] M. S. Raghunathan. *Principal bundles admitting a rational section.* Invent. Math., 116(1-3):409–423, 1994.

[Rag2] M. S. Raghunathan. *Erratum: "Principal bundles admitting a rational section" [Invent. Math.* **116** *(1994), no. 1-3, 409–423; MR1253199 (95f:14093)].* Invent. Math., 121(1):223, 1995.

[Ser] Jean-Pierre Serre. *Espaces fibrés algébrique.* Anneaux de Chow et applications, Séminaire Claude Chevalley, number 3. Paris, 1958.

[Zai] Kirill Zainoulline. *On Grothendieck conjecture about principal homogeneous spaces for some classical algebraic groups.* St.Petersburg Math. J., 12(1):117–143, 2001.

Period-index and u-invariant questions for fields

R. Parimala

(joint work with V. Suresh)

Let F be a field of characteristic not 2. The u-invariant $u(F)$ is defined to be the maximum dimension of anisotropic quadratic forms over F. The behavior of the u-invariant under rational function field extensions is very little understood.

For any field F, the Brauer p-dimension $Br_p dim(F)$ of F is defined as the least positive integer d such that for any central simple algebra A defined over any finite extension of F of exponent a power of p, the index of A divides the dth power of the exponent. The Brauer dimension of F is the maximum of the Brauer p-dimensions of F as p varies over all primes. The behavior of the Brauer dimension of a field again is very little understood under rational function field extensions.

There is a class of fields where there is way to understand the u-invariant and the Brauer dimension under rational function field extensions. Let K be a complete discrete valued field of characteristic zero with residue field κ. Let F be the function field in one variable over K. Suppose $char(\kappa) = p$. Let l be a prime not equal to p. Harbater, Hartmann and Krashen prove that for a prime l not equal to p, if $Br_l dim(\kappa') \leq d$ for every finite extension κ' of κ and if $br_l dim(E) \leq d+1$ for every function field E in one variable over κ, then $Br_l dim(F) \leq d+2$. This result for K a p-adic field is due to Saltman. It remained open whether $Br_p dim(F)$ is finite for function fields of p-adic curves.

Let κ be a field of characteristic $p > 0$. The p-rank of κ is d if $[\kappa : \kappa^p] = p^d$. We prove that if K is a complete discrete valued field of characteristic zero with residue field κ of characteristic $p > 0$ with p-rank of κ equal to d, then, for a function field F in one variable over K, $Br_p dim(F) \leq 2d + 2$. We also prove that if the residue field κ is a perfect field of characteristic 2, $u(F) \leq 8$. For function fields of p-adic curves, it follows that the Brauer dimension is 2. Further the u-invariant of function fields of dyadic curves is 8, a result due to Heath-Brown and Leep.

The main ingredients in the proof are Kato's filtration of the p-part of the Brauer group of a complete discrete valued field of characteristic zero with residue field of characteristic p and the patching theorems of Harbater-Hartmann-Krashen.

Skew hermitian forms over quaternion algebras

Anne Quéguiner-Mathieu
(joint work with Jean-Pierre Tignol)

The dimension, the discriminant and the Clifford invariant are classical invariants of quadratic forms that do extend to the setting of central simple algebras with orthogonal involutions. Under some additional assumptions on the algebra, namely that it has index dividing half its degree, one can also define an analogue of the Arason invariant, which is the degree 3 invariant occuring in Milnor's conjecture. If the underlying algebra has degree 12, this invariant need not be represented by a cohomology class of order 2, as opposed to what happens for quadratic forms, and also for involutions in smaller degree (see [4]).

In a joint paper in preparation with Jean-Pierre Tignol, we study this invariant in degree 12 and index 2. In particular, we prove the following, which gives a very explicit description of the algebras with involution we are interested in:

Theorem 1. *Let (A, σ) be a degree 12 algebra with orthogonal involution. We assume A is Brauer equivalent to a quaternion algebra Q and σ has trivial discriminant and trivial Clifford invariant. Then, there exists quaternion algebras Q_i and H_i, for $1 \leq i \leq 3$ such that $H_1H_2H_3$ is split, Q_iH_i is Brauer equivalent to Q for all i, and*

$$(A, \sigma) \in \boxplus_{1\leq i\leq 3}(Q_i, {}^-) \otimes (H_i, {}^-),$$

meaning that (A, σ) is a direct sum of those three degree 4 products.

Note that such a direct sum is not uniquely defined by the data of the summands (see [2]).

Given an additive decomposition of (A,σ) as in the theorem, we may consider the finite subgroup U of $\mathrm{B}r_2(F)$ generated by the Brauer classes of the quaternion algebras Q_i, H_i. It contains at most 8 elements, all of index at most 2. It appears that the Arason invariant of σ, which takes values in $H^3(F,\mathbb{Q}/\mathbb{Z})/F^\times\cdot[Q]$ is closely related to the homology of the complex in Galois cohomology which was introduced and studied by Peyre in [3], namely

$$F^\times\cdot U\to H^3(F,\mathbb{Q}/\mathbb{Z})\to H^3(F(X),\mathbb{Q}/\mathbb{Z}),$$

where $F^\times\cdot U$ denotes the subgroup of $H^3(F,\mathbb{Q}/\mathbb{Z})$ which consists of cup products $(\lambda)\cdot\alpha$, where $\lambda\in F^\times$ and $\alpha\in U$, and X is the product of the Severi-Brauer varieties of the quaternion algebras of U. One of the main results of our paper asserts

Theorem 2. *The homology of Peyre's complex is generated by the class of the Arason invariant of* (A,σ).

Note that the algebras with involution (A,σ) we are studying here are always split by a quadratic extension of the base field (since they are of index 2), and also hyperbolic over a quadratic extension of F (see [1]). Nevertheless, they are not always split and hyperbolic over a quadratic extension of F. Using the results mentioned above, we give a necessary and sufficient condition under which such a quadratic splitting field does exist. In particular, we provide an example of an (A,σ) for which the Arason invariant is represented by a cohomology class of order 2, and yet, (A,σ) does not admit any quadratic splitting field.

References

[1] S. Garibaldi, A. Quéguiner-Mathieu, Pfister's theorem for orthogonal involutions of degree 12, Proceedings of the AMS **137** (2009), 1215-1222.

[2] I. Dejaiffe, Somme orthogonale d'algèbres à involution et algèbre de Clifford, Comm. Algebra **26(5)** (1995), 1589-1612.

[3] E. Peyre, Products of Severi-Brauer varieties and Galois cohomology, in K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), 369-401, Proc. Sympos. Pure Math., 58, Part 2 Amer. Math. Soc., Providence, RI.

[4] A. Quéguiner-Mathieu, J.-P. Tignol, Cohomological invariants for orthogonal involutions on degree 8 algebras, J. K-Theory **9(2)** (2012), 333-358.

On the splitting of quasilinear quadratic forms

Stephen Scully

Let (V,ϕ) be a quadratic space over a field F. Among the basic invariants of ϕ are its *anisotropic part* ϕ_{an} and *total index* $i_{\mathrm{t}}(\phi)$, the latter being defined as the largest dimension of a totally isotropic subspace of V. Via a construction due to Knebusch ([Kne76]), these invariants can be extended to an important collection of higher invariants of ϕ. Explicitly, one may define an integer $h(\phi)$

and a sequence $\big(F_r, \phi_r, i_r(\phi)\big)_{0\leq r\leq h(\phi)}$, where F_r is an extension of F, ϕ_r is an anisotropic quadratic form over F_r and $i_r(\phi)$ is a non-negative integer, as follows:

- Set $F_0 = F$, $\phi_0 = \phi_{\mathrm{an}}$ and $i_0(\phi) = i_{\mathrm{t}}(\phi)$.
- Suppose that F_r and ϕ_r are defined. If dim $\phi_r \leq 1$, then $r = h(\phi)$. Otherwise, set F_{r+1} to be the function field $F_r(\phi_r)$ of the integral projective quadric over F_r defined by the vanishing of ϕ_r, and set $\phi_{r+1} = (\phi_{F_{r+1}})_{\mathrm{an}}$ and $i_{r+1}(\phi) = i_{\mathrm{t}}\big((\phi_r)_{F_{r+1}}\big)$.

The sequence $\mathbf{i}(\phi) = \big(i_r(\phi)\big)_{0\leq r\leq h(\phi)}$ is called the *standard splitting pattern of* ϕ. This invariant provides a useful means by which to pre-classify quadratic forms according to what one may term their 'algebraic complexity', and has been the focus of intense study since its introduction to the subject. In recent decades, the advent of powerful 'motivic' methods has led to remarkable progress in this area of research. Among the highlights of this progress, we can mention the following celebrated result of Vishik.

Theorem 1 (A. Vishik, [Vis11]). *Let ϕ be an anisotropic quadratic form of dimension ≥ 2 over a field F of characteristic $\neq 2$, and write* dim $\phi - i_1(\phi) = 2^{r_1} - 2^{r_2} + \ldots + (-1)^{s-1}2^{r_s}$, *where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geq 1$. Let $2 \leq l \leq s$, and let $d_l = \sum_{i=1}^{l-1}(-1)^{i-1}2^{r_i-1} + \epsilon(l)\cdot\sum_{j=l}^{s}(-1)^{j-1}2^{r_j}$, where $\epsilon(l) = 1$ (0) if l is even (odd). Suppose that L is an extension of F such that $i_{\mathrm{t}}(\phi_L) > d_l$. Then $i_{\mathrm{t}}(\phi_L) \geq d_l + i_1(\phi)$.*

Vishik's theorem (based on the existence of so-called *excellent connections* in the motives of smooth quadrics) is a major achievement for the theory of quadratic forms which subsumes a number of earlier breakthroughs, including Karpenko's solution of Hoffmann's conjecture on the possible values of the invariant i_1 ([Kar03]). At present, many results of this kind are limited to the case of fields of characteristic different from 2. For *non-singular* forms, this limitation is largely accounted for by the prominent use of Steenrod operations on mod-2 Chow groups which are not yet constructed in characteristic 2. The characteristic-2 theory of quadratic forms is, however, more complex than its 'good-characteristic' counterpart due to the existence of *singular* anisotropic forms in this setting. If one allows for the case of singular forms, then non-traditional phenomena manifest themselves. Coupled to the fact that contemporary geometric methods are less readily available in the presence of singularities, this has resulted in a comparatively modest progression of the singular theory. The goal of this talk is to present some new results on the standard splitting of quadratic forms for the extreme, yet fundamental, class of *totally singular* (or *quasilinear*) forms.

Assume henceforth that char $F = 2$. We say that ϕ is *quasilinear* if $\phi(v + w) = \phi(v) + \phi(w)$ for all $v, w \in V$. It is not difficult to see in this case that $h(\phi) \leq \log_2(\dim \phi)$, which already highlights the distinction to be drawn between the non-singular and quasilinear settings. Nevertheless, in analogy with the characteristic-not-2 theory, the 'simplest' type of splitting is exhibited here by the diagonal parts of bilinear Pfister forms (or *quasi-Pfister forms*): an anisotropic quasilinear quadratic form ϕ is similar to an n-fold quasi-Pfister form if and only if $\mathbf{i}(\phi) =$

$(2^{n-1}, 2^{n-2}, \dots, 2, 1)$. It is therefore natural to introduce the *quasi-Pfister height of* ϕ as the smallest $r \geq 0$ such that ϕ_r is similar to a quasi-Pfister form (such an r exists). Laghribi ([Lag04]) has described the anisotropic forms of quasi-Pfister height ≤ 1: they are precisely the anisotropic *quasi-Pfister neighbours* (i.e., forms similar to subforms of codimension $< 2^{n-1}$ in n-fold quasi-Pfister forms). The following proposition indicates that the study of this invariant is key for understanding the standard splitting pattern as a whole (for example, an explicit description of the forms of quasi-Pfister height 2 would already yield a full set of restrictions on the possible values of the invariant i_1).

Proposition 2. *Let* ϕ *be an anisotropic quasilinear quadratic form over* F*, and let* $1 \leq s < h_{\mathrm{qp}}(\phi)$*. Then there exists an extension* $\widetilde{F}$ *of* F *such that* $h_{\mathrm{qp}}(\phi_{\widetilde{F}}) \leq s$ *and* $i_r(\phi_{\widetilde{F}}) = i_r(\phi)$ *for all* $0 \leq r < s$*.*

While we cannot address the problem of classifying forms of small quasi-Pfister height (≥ 2) at present, some progress has been made towards understanding the structure of the standard splitting pattern in the quasilinear case. One of the main results achieved in this direction is the following theorem, which is directly analogous to the $l = 2$ case of Vishik's Theorem 1.

Theorem 3. *Let* ϕ *be an anisotropic quasilinear quadratic form of dimension* ≥ 2 *over* F*, and write* $\dim \phi = 2^n + m$*, where* $n \geq 0$ *and* $1 \leq m \leq 2^n$*. Suppose that* L *is an extension of* F *such that* $i_{\mathrm{t}}(\phi_L) < m$*. Then* $i_{\mathrm{t}}(\phi_L) \leq m - i_1(\phi)$*.*

As an immediate corollary of Theorem 3, we obtain the following partial result towards an analogue of Karpenko's theorem on the values of i_1.

Theorem 4. *Let* ϕ *be an anisotropic quasilinear quadratic form of dimension* ≥ 2 *over* F*, and write* $\dim \phi = 2^n + m$*, where* $n \geq 0$ *and* $1 \leq m \leq 2^n$*. Then either* $i_1(\phi) = m$ *or* $i_1(\phi) \leq m/2$*.*

Theorem 4 extends earlier work of Hoffmann and Laghribi ([HL06]), who established the upper bound $i_1(\phi) \leq m$ (which is itself used in the proof of Theorem 4). It may also be deduced from the following result on the general shape of the standard splitting pattern for quasilinear forms.

Theorem 5. *Let* ϕ *be an anisotropic quasilinear quadratic form of dimension* ≥ 2 *over* F *such that* $h_{\mathrm{qp}}(\phi) > 0$*. Then* $i_1(\phi) \leq i_2(\phi) \leq \ldots \leq i_{h_{\mathrm{qp}}(\phi)}$*.*

Theorem 5 shows that the standard splitting pattern admits a hill-type structure in this setting: it is monotonic increasing up to the $h_{\mathrm{qp}}(\phi)^{th}$ entry, and consists of descending powers of 2 thereafter. This novel feature of the quasilinear theory has important further implications. For example, it may be used in conjunction with the above results to establish the following theorem, directly analogous to a conjecture of Hoffmann (cf. [Hof95, §4]) concerning non-singular quadratic forms with so-called *maximal splitting* (which remains wide open, even in characteristic different from 2).

Theorem 6. *Let ϕ be an anisotropic quasilinear quadratic form over F such that $2^n + 2^{n-2} < \dim \phi \leq 2^{n+1}$ for some $n \geq 2$. If $i_1(\phi) = \dim \phi - 2^n$, then ϕ is a quasi-Pfister neighbour.*

As indicated above, much of the geometric approach which has been highly succesful in the characteristic-not-2 theory is not applicable here. A key tool in our approach to the above results is the following 'functorial' property of the standard splitting pattern in the quasilinear setting. The $r = 1$ case of this result, due to Totaro ([Tot08]), is directly analogous to a celebrated result of Karpenko and Merkurjev which establishes a similar functorial property of the canonical dimension of smooth quadrics ([KM03]).

Theorem 7. *Let ϕ and ψ be anisotropic quasilinear quadratic forms of dimension ≥ 2 over F such that $\phi_{F(\psi)}$ is isotropic. Then $h(\psi) \leq h(\phi)$ and:*

(1) $\dim \psi_r \leq \dim \phi_r$ *for all* $1 \leq r \leq h(\psi)$.
(2) $\dim \psi_r = \dim \phi_r$ *for all* $1 \leq r \leq h(\psi)$ *if and only if* $\psi_{F(\phi)}$ *is isotropic.*

References

[HL06] D. W. Hoffmann and A. Laghribi, *Isotropy of quadratic forms over the function field of a quadric in characteristic 2*, J. Algebra **295** (2006), no. 2, 362–386.

[Hof95] D. W. Hoffmann, *Isotropy of quadratic forms over the function field of a quadric*, Math. Z. **220** (1995), no. 3, 461–476.

[Kar03] N.A. Karpenko, *On the first Witt index of quadratic forms*, Invent. Math. **153** (2003), no. 2, 455–462.

[KM03] N. Karpenko and A. Merkurjev, *Essential dimension of quadrics*, Invent. Math. **153** (2003), no. 2, 361–372.

[Kne76] M. Knebusch, *Generic splitting of quadratic forms. I.*, Proc. London Math. Soc. (3) **33** (1976), no. 1, 65–93.

[Lag04] A. Laghribi, *On splitting of totally singular quadratic forms*, Rend. Circ. Mat. Palermo (2) **53** (2004), no. 3, 325–336.

[Tot08] B. Totaro, *Birational geometry of quadrics in characteristic 2*, J. Alg. Geom. **17** (2008), 577–597.

[Vis11] A. Vishik, *Excellent connections in the motives of quadrics*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 1, 183–195.

Higher reciprocity laws and rational points

V. Suresh

(joint work with J.-L. Colliot-Thélène and R. Parimala)

Let K be a number field and Ω_K be the set of places of K. For $v \in \Omega_K$, let K_v denote the completion of K at v. A classical theorem of Hasse and Minkowski asserts that a quadratic form q over K is isotropic if it is isotropic over K_v for all $v \in \Omega_K$.

One has more general local-global principles for homogeneous spaces under connected linear algebraic groups. Let X be a projective homogeneous space under a connected linear algebraic group defined over a number field K. A theorem of Harder asserts that if $X(K_v) \neq \varnothing$, $\forall v \in \Omega_K$, then $X(K) \neq \varnothing$. For principle homogeneous spaces under a semisimple simply connected linear algebraic groups, a similar local-global result holds (Kneser, Harder, Chernousov). For a adjoint, quasi-split or K-rational connected linear algebraic groups over K, a similar local-global principle is a theorem of Sansuc.

Let K be a complete discrete valued field with residue field κ algebraically closed. Let X be a smooth projective curve over K and $F = K(X)$. Let Ω_F be the set of all discrete valuations of F. For $\nu \in \Omega_F$, let F_ν denote the completion of F at ν. Let G be a connected linear algebraic group over F and

$$Ш^1(F,G) = ker(H^1(F,G) \to \prod_{\nu\in\Omega} H^1(F_\nu, G)).$$

The set $Ш^1(F,G)$ classifies all principal homogeneous spaces which have rational points over F_ν for all $\nu \in \Omega_F$. A theorem of Harbater-Hartmann-Krashen asserts that if G is a connected linear algebraic group over F which is F-rational, then $Ш^1(F,G) = \{1\}$.

In this talk we construct an example of a torus T over $F = \mathbb{C}((t))(x)$ with $Ш^1(F,T) \neq \{1\}$, thereby showing that the theorem of Harbater-Hartmann-Krashen need not hold if G is not F-rational.

To construct our example we introduce an obstruction using a Bloch-Ogus complex.

Signatures of hermitian forms and applications

Thomas Unger
(joint work with Vincent Astier)

In [1], [2] and [3] we started developing the theory of signatures of hermitian forms, defined over central simple algebras with involution (with respect to orderings on the base field), inspired by [4]. In contrast to classical signatures of quadratic forms, signatures of hermitian forms should be considered as relative invariants. Below we present a summary of our work thus far.

Let F be a formally real field with space of orderings X_F and Witt ring $W(F)$. Let (A,σ) be an F-algebra with involution, i.e. a pair consisting of a finite-dimensional F-algebra A, whose centre $Z(A)$ satisfies $[Z(A) : F] \leq 2$, and which is assumed to be either simple (if $Z(A)$ is a field) or a direct product of two simple algebras (if $Z(A) = F \times F$), and an F-linear involution $\sigma : A \to A$. For $\varepsilon \in \{-1, 1\}$ let $W_\varepsilon(A,\sigma)$ be the Witt group of Witt equivalence classes of ε-hermitian forms defined on finitely generated right A-modules. This is a $W(F)$-module. All forms are assumed to be non-singular and are identified with their classes in $W_\varepsilon(A,\sigma)$.

Let A be Brauer equivalent to an F-division algebra D and let ϑ be an involution on D of the same kind as σ. Then (A,σ) and (D,ϑ) are Morita equivalent and we obtain a (non-canonical) isomorphism of $W(F)$-modules $W_\varepsilon(A,\sigma) \simeq W_{\varepsilon\mu}(D,\vartheta)$ with $\mu \in \{-1, 1\}$. For the purpose of the study of signatures we may assume that $\varepsilon = \mu = 1$, cf. [1, 2.1].

Let $P \in X_F$, let F_P denote a real closure of F at P and consider

$$(\star) \qquad W(A,\sigma) \longrightarrow W(A\otimes_F F_P, \sigma\otimes \mathrm{id}) \xrightarrow{\mathscr{M}_P} W_\varepsilon(D_P,\vartheta_P) \xrightarrow{\mathrm{sign}_P} \mathbb{Z},$$

where the first map is induced by scalar extension, the second map is an isomorphism of $W(F_P)$-modules induced by Morita equivalence and sign_P is either the classical signature isomorphism if $\varepsilon = 1$ and $(D_P,\vartheta_P) \in \{(F_P,\mathrm{id}), (F_P(\sqrt{-1}), -), ((-1,-1)_{F_P}, -)\}$ (where $-$ denotes conjugation and quaternion conjugation, respectively), or $\varepsilon = -1$ and $\mathrm{sign}_P \equiv 0$ if $(D_P,\vartheta_P) \in \{(F_P,\mathrm{id}), ((-1,-1)_{F_P}, -), (F_P\times F_P, \widehat{\ })\}$ (where $\widehat{\ }$ denotes the exchange involution), and where in each case the indicated involutions are obtained after a further application of Morita equivalence. We call $\mathrm{Nil}[A,\sigma] := \{P \in X_F \mid \mathrm{sign}_P \equiv 0\}$ the set of *nil-orderings* of (A,σ). It depends only on the Brauer class of A and the type of σ. In addition it is clopen in X_F [1, 6.5]. We write $\widetilde{X}_F := X_F \setminus \mathrm{Nil}[A,\sigma]$.

Definition 1. Let $h \in W(A,\sigma)$, $P \in X_F$ and $\mathscr{M}_P$ as in $(\star)$. The *M-signature* of h at $(P,\mathscr{M}_P)$ is defined by $\mathrm{sign}_P^{\mathscr{M}_P} h := \mathrm{sign}_P(\mathscr{M}_P(h\otimes F_P))$ and is independent of the choice of F_P.

If we choose a different Morita map $\mathscr{M}'_P$ in $(\star)$, then $\mathrm{sign}_P^{\mathscr{M}'_P} h = \pm\,\mathrm{sign}_P^{\mathscr{M}_P} h$, cf. [1, 3.4], which prompts the question if there is a way to make the M-signature independent of the choice of Morita equivalence. It follows from [1, 6.4] and [2, 3.2] that:

Theorem 2. *There exists $H \in W(A,\sigma)$ such that $\mathrm{sign}_P^{\mathscr{M}_P} H \neq 0$ for all $P \in \widetilde{X}_F$.*

Definition 3. Let $P \in \widetilde{X}_F$, let $\mathscr{M}_P$ be any Morita map as in $(\star)$, let H be as in (2) and let $\delta \in \{-1,1\}$ be the sign of $\mathrm{sign}_P^{\mathscr{M}_P} H$. Let $h \in W(A,\sigma)$. The *H-signature of h at P* is defined by $\mathrm{sign}_P^H h := \delta\,\mathrm{sign}_P^{\mathscr{M}_P} h$. If $P \in \mathrm{Nil}[A,\sigma]$, we set $\mathrm{sign}_P^H h := 0$.

The H-signature at P is independent of the choice of Morita equivalence $\mathscr{M}_P$ and is a refinement of the definition of signature in [4], the latter not being defined when σ becomes hyperbolic over $A\otimes_F F_P$, cf. [1, 3.11]. The H-signature has many pleasing properties, cf. [5, 4.1] for *(iv)* and [1, 3.6, 8.1] for the other statements:

Theorem 4.

(i) *Let h be a hyperbolic form over (A,σ), then $\mathrm{sign}_P^H h = 0$.*

(ii) *Let $h_1, h_2 \in W(A,\sigma)$, then $\mathrm{sign}_P^H(h_1 \perp h_2) = \mathrm{sign}_P^H h_1 + \mathrm{sign}_P^H h_2$.*

(iii) *Let $h \in W(A,\sigma)$ and $q \in W(F)$, then $\mathrm{sign}_P^H(q\cdot h) = \mathrm{sign}_P q \cdot \mathrm{sign}_P^H h$.*

(iv) *(Pfister's local-global principle) Let $h \in W(A,\sigma)$. Then h is a torsion form if and only if $\mathrm{sign}_P^H h = 0$ for all $P \in X_F$.*

(v) *(Going-up) Let $h \in W(A,\sigma)$ and let L/F be an algebraic extension of ordered fields. Then $\mathrm{sign}_Q^{H\otimes L}(h\otimes L) = \mathrm{sign}_{Q\cap F}^H h$ for all $Q \in X_L$.*

(vi) *(Going-down: Knebusch trace formula) Let L/F be a finite extension of ordered fields and let $h \in W(A\otimes_F L, \sigma\otimes\mathrm{id})$. Then $\mathrm{sign}_P^H(\mathrm{Tr}^*_{A\otimes_F L} h) = \sum_{P\subseteq Q\in X_L} \mathrm{sign}_Q^{H\otimes L} h$ for all $P \in X_F$, where $\mathrm{Tr}^*_{A\otimes_F L} h$ denotes the Scharlau transfer induced by the A-linear homomorphism $\mathrm{id}_A \otimes \mathrm{Tr}_{L/F} : A\otimes_F L \to A$.*

The pair $(\ker \operatorname{sign}_P, \ker \operatorname{sign}_P^H)$ is a prime m-ideal of the $W(F)$-module $W(A, \sigma)$ whenever $P \in \widetilde{X}_F$ in the following sense, cf. [2, 4.1]:

Definition 5. Let R be a commutative ring and let M be an R-module. An *m-ideal* of M is a pair (I, N) where I is an ideal of R, N is a submodule of M, and such that $I \cdot M \subseteq N$.

An m-ideal (I, N) of M is *prime* if I is a prime ideal of R (we assume that all prime ideals are proper), N is a proper submodule of M, and for every $r \in R$ and $m \in M$, $r \cdot m \in N$ implies that $r \in I$ or $m \in N$.

We obtain a classification à la Harrison and Lorenz-Leicht, cf. [2, 5.5, 5.7]:

Theorem 6. *Let (I, N) be a prime m-ideal of the $W(F)$-module $W(A, \sigma)$.*

(*a*) *If $2 \notin I$, then one of the following holds:*
   (*i*) *There exists $P \in X_F$ such that $(I, N) = (\ker \operatorname{sign}_P, \ker \operatorname{sign}_P^H)$.*
   (*ii*) *There exist $P \in X_F$ and a prime $p > 2$ such that $(I, N) = \big(\ker(\pi_p \circ \operatorname{sign}_P), \ker(\pi \circ \operatorname{sign}_P^H)\big)$, where $\pi_p : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ and $\pi : \operatorname{Im} \operatorname{sign}_P^H \to \operatorname{Im} \operatorname{sign}_P^H / (p \cdot \operatorname{Im} \operatorname{sign}_P^H)$ are the canonical projections.*

(*b*) *If $2 \in I$, then $I = I(F)$, the fundamental ideal of $W(F)$. Furthermore, a pair $(I(F), N)$ is a prime m-ideal of $W(A, \sigma)$ if and only if N is a proper submodule of $W(A, \sigma)$ with $I(F) \cdot W(A, \sigma) \subseteq N$.*

When $2 \in I$, N is not uniquely determined by I (in contrast to the $2 \notin I$ case), since there are in general several proper submodules N of $W(A, \sigma)$ containing $I(F) \cdot W(A, \sigma)$, such as $I(F) \cdot W(A, \sigma)$ itself and $I(A, \sigma)$, the submodule of $W(A, \sigma)$ consisting of all classes of forms of even rank. In general $I(F) \cdot W(A, \sigma) \neq I(A, \sigma)$, cf. [2, 5.8]. Also, $I(A, \sigma)$ can be singled out by a natural property, cf. [2, 5.10].

The following result is [1, 7.2]:

Theorem 7. *Let $h \in W(A, \sigma)$. The total H-signature* $\operatorname{sign}^H h : X_F \to \mathbb{Z}, P \mapsto \operatorname{sign}_P^H h$ *is continuous* (*with respect to the Harrison topology on X_F and the discrete topology on $\mathbb{Z}$*).

Finally, we present some results from [3]. Let $C(X_F, \mathbb{Z})$ denote the ring of continuous functions from X_F to $\mathbb{Z}$ and consider the group homomorphism $\operatorname{sign}^H : W(A, \sigma) \to C(X_F, \mathbb{Z}), h \mapsto \operatorname{sign}^H h$.

Theorem 8. *For every $f \in C(X_F, \mathbb{Z})$ there exists $n \in \mathbb{N}$ such that $2^n f \in \operatorname{Im} \operatorname{sign}^H$. In other words, the cokernel of* sign^H *is a 2-primary torsion group.*

Definition 9. The *stability index* of (A, σ) is the smallest $k \in \mathbb{N}$ such that $2^k C(X_F, \mathbb{Z}) \subseteq \operatorname{Im} \operatorname{sign}^H$ if such a k exists and ∞ otherwise. It is independent of the choice of H. The group $\operatorname{coker} \operatorname{sign}^H$ is up to isomorphism independent of the choice of H. We denote it by $S_H(A, \sigma)$ and call it the *stability group* of (A, σ).

It follows from Theorems 4(*iv*) and 8 and from [6, 6.1] that

Theorem 10. *Let $W_t(A, \sigma)$ denote the torsion subgroup of $W(A, \sigma)$. The sequence*

$$0 \longrightarrow W_t(A, \sigma) \longrightarrow W(A, \sigma) \xrightarrow{\operatorname{sign}^H} C(X_F, \mathbb{Z}) \longrightarrow S_H(A, \sigma) \longrightarrow 0$$

is exact. The groups $W_t(A,\sigma)$ and $S_H(A,\sigma)$ are 2-primary torsion groups.

References

[1] V. Astier and T. Unger, Signatures of hermitian forms and the Knebusch Trace Formula (2012), `http://arxiv.org/abs/1003.0956`

[2] V. Astier and T. Unger, Signatures of hermitian forms and "prime ideals" of Witt groups (2013), `http://arxiv.org/abs/1303.3494`

[3] V. Astier and T. Unger, Stability index of algebras with involution (2013), in preparation.

[4] E. Bayer-Fluckiger and R. Parimala, Classical groups and the Hasse principle, *Ann. of Math. (2)* **147** (1998), no. 3, 651–693.

[5] D.W. Lewis and T. Unger, A local-global principle for algebras with involution and hermitian forms, *Math. Z.* **244** (2003), 469–477.

[6] W. Scharlau, Induction theorems and the structure of the Witt group, *Invent. Math.* **11** (1970), 37–44.

Reporter: David Grimm

Participants

Prof. Dr. Asher Auel
Courant Institute for Math. Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
UNITED STATES

Prof. Dr. Sanghoon Baek
KAIST
Dept. of Mathematical Sciences
291 Daehak-ro, Yuseong-gu
305-701 Daejeon
KOREA, REPUBLIC OF

Prof. Dr. Eva Bayer-Fluckiger
EPFL
SB IMB CSAG (Batiment MA)
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Karim Johannes Becher
Dept. Wiskunde & Informatica
Universiteit Antwerpen
M.G. 224
Middelheimlaan 1
2020 Antwerpen
BELGIUM

Dr. Baptiste Calmes
Université d'Artois
Faculté Jean Perrin
Rue Jean Souvraz - SP 18
62307 Lens Cedex
FRANCE

Prof. Dr. Vladimir Chernousov
Dept. of Mathematics & Statistics
University of Alberta
632 Central Academic Bldg.
Edmonton AB T6G 2G1
CANADA

Dr. Andrew Dolphin
Université Catholique de Louvain
SST/ICT/INMA - Euler
Avenue Georges Lemaitre, 4-6
1348 Louvain-la-Neuve
BELGIUM

Dr. Philippe Gille
ENS Paris
45, rue d'Ulm
75005 Paris
FRANCE

Prof. Dr. Stefan Gille
Dept. of Mathematics & Statistics
University of Alberta
632 Central Academic Bldg.
Edmonton AB T6G 2G1
CANADA

Dr. David Maximilian Grimm
EPFL
SB IMB CSAG (Batiment MA)
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Detlev Hoffmann
Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund
GERMANY

Dr. Nikita Karpenko
Département Mathématiques
Université P. et M. Curie
4, place Jussieu
75252 Paris Cedex 05
FRANCE

Dr. Ahmed Laghribi
Laboratoire de Mathématiques
Faculté Jean Perrin
Rue Jean Souvraz - SP 18
62307 Lens Cedex
FRANCE

Prof. Dr. David B. Leep
Department of Mathematics
University of Kentucky
715 Patterson Office Tower
Lexington, KY 40506-0027
UNITED STATES

Prof. Dr. Alexander S. Merkurjev
Department of Mathematics
University of California at
Los Angeles
405 Hilgard Avenue
Los Angeles, CA 90095-1555
UNITED STATES

Prof. Dr. Ivan A. Panin
Laboratory of Algebra at St. Petersburg
Department of V.A. Steklov Institute of
Mathematics (P.O.M.I.)
Russian Academy of Sciences
27, Fontanka
191 023 St. Petersburg
RUSSIAN FEDERATION

Prof. Dr. Raman Parimala
Emory University
Dept. of Mathematics & Computer
Science
400 Dowman Drive
Atlanta GA 30307
UNITED STATES

Prof. Dr. Anne
Queguiner-Mathieu
Mathématiques
LAGA - CNRS: UMR 7539
Université Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
93430 Villetaneuse
FRANCE

Dr. Melanie Raczek
Institut de Mathématique
Université Catholique de Louvain
Chemin du Cyclotron, 2
1348 Louvain-la-Neuve
BELGIUM

Stephen Scully
Department of Mathematics
The University of Nottingham
Nottingham NG7 2RD
UNITED KINGDOM

Dr. Anastasia Stavrova
Dept. of Mathematics & Mechanics
St. Petersburg State University
Starys Petergof
Universitetsky Pt., 28
198 504 St. Petersburg
RUSSIAN FEDERATION

Prof. Dr. Venapally Suresh
Dept. of Mathematics & Computer
Science
Emory University
400, Dowman Dr.
Atlanta, GA 30322
UNITED STATES

Prof. Dr. Jean-Pierre Tignol
ICTEAM Institute
Université Catholique de Louvain
Ave. Georges Lemaitre 4-6; Bte L 4.05.1
1348 Louvain-la-Neuve
BELGIUM

Dr. Thomas Unger
School of Mathematical Sciences
University College Dublin
Belfield
Dublin 4
IRELAND

Dr. Alexander Vishik
Department of Mathematics
The University of Nottingham
University Park
Nottingham NG7 2RD
UNITED KINGDOM

Prof. Dr. Kirill Zainoulline
Dept. of Mathematics & Statistics
University of Ottawa
585 King Edward Avenue
Ottawa, Ont. K1N 6N5
CANADA