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Nonlinear Waves and Dispersive Equations

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Abstract. Nonlinear dispersive equations are models for nonlinear waves in a wide range of physical contexts. Mathematically they display an interplay between linear dispersion and nonlinear interactions, which can result in a wide range of outcomes from finite time blow-up to scattering. They are linked to many areas of mathematics and physics, ranging from integrable systems and harmonic analysis to fluid dynamics and general relativity. The conference did focus on the analytic aspects and PDE aspects.

Mathematics Subject Classification (2010): 37L50.

Introduction by the Organisers

This field has experienced a continuous growth during the last two decades. Yet many difficult problems remain open. This attracts many strong mathematicians and is the engine for further growth. Recently there has been a number of very active and successful directions:

(1) The notion of 'minimal blow-up solution', based on seminal work of Kenig-Merle and Tao-Visan became an amazingly successful approach to global well-posedness and scattering for critical focusing dispersive equations below soliton energy.
(2) The study of dynamics near solitons. There is a variety of recent results deepening our insights and tending towards a more global understanding of solutions.
(3) Important models from physics became accessible. There have been several talks on global well-posedness results for water waves with small localized data.

The main focus of the field nowadays seems to be towards understanding large data dynamics for a variety of models, as well as the small data evolution for classes of strongly nonlinear dispersive equations.

The conference brought together people working in dispersive equations coming from various different areas: Applied analysis, harmonic analysis and partial differential equations. Many of the participants and the speakers have been younger mathematicians.

Workshop: Nonlinear Waves and Dispersive Equations

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Abstracts

The initial value problem for the binormal flow with rough data

Valeria Banica
(joint work with Luis Vega)

The talk surveys a series of results on singularity formation for the binormal flow (BF). This equation is the geometric flow of curves $\chi(t)$ in $\mathbb{R}^3$ governed by:

$$\chi_t = \chi_x \wedge \chi_{xx}.$$

Here x stands for the arclength parameter of the curve $\chi(t)$; we denote (T, n, b) the tangent, normal and binormal vector, and (c, τ) the curvature and the torsion. Then using the Frenet system the equation becomes $\chi_t = cb$. It was derived in Da Rios's PhD under the supervision of Levi-Civita in 1906 as a model for the dynamics in a 3D inhomogeneous inviscid Newtonian fluid of a vortex filament located at time t on a $\mathbb{R}^3$ curve $\chi(t, x)$ ([7], see also [1, 13, 14, 15]). The approximation uses a truncation of Biot-Savart's law. It is also known as the Local Induction Approximation (LIA) or vortex filament equation (VFE). We recall here that the binormal flow is also conjectured to model the evolution of quantized vortex filaments in a Bose condensate in the incompressible limit. In a related setting, it was proved in [6] that mean curvature flow governs the dynamics in parabolic Ginzburg-Landau equation.

Although in general part of the complexity of the fluid equations might be lost through this approximation, this model is a simple and very rich one. For instance, it is a completely integrable equation. It is worth pointing out also that the $\mathbb{S}^2$ tangent vector $T(t, x)$ satisfies the Schrödinger map arising in ferromagnetism theory. Moreover, a link with the 1D cubic Schrödinger equation is made at the second order of derivative in space as follows. If one considers the filament function $\psi(t, x) = c(t, x)e^{i\int_0^x \tau(t,s)ds}$, it is easy to check that it satisfies the nonlinear Schrödinger equation (NLSE):

$$i\psi_t + \psi_{xx} + \left(|\psi|^2 - A(t)\right)\psi = 0,$$

where $A(t)$ is in terms of the curvature and torsion $(c, \tau)(t, 0)$. This fundamental remark has been made by Hasimoto in 1972 and it has allowed the transfer of informations from (NLSE) to (BF) ([9], see also [12] for the cases when the curvature vanishes). Conversely, given $A(t)$ and $\psi(t)$ a solution of (NLSE), there exists then a simple way to construct a solution $\chi(t)$ of (BF), but recovering the geometric properties of $\chi(t)$ is not obvious at all.

Self-similar solutions of (BF), that is solutions of the type $\chi(t, x) = \sqrt{t}G\left(\frac{x}{\sqrt{t}}\right)$, were studied since the 70's in works on vortex dynamics in superfluids, ferromagnetism, in aortic heart valve leaflet miocardic modeling. They form a family $\{\chi_a\}_{a\in\mathbb{R}^{+*}}$ characterized by the explicit curvature and torsion $(c_a, \tau_a)(t, x) = \left(\frac{a}{\sqrt{t}}, \frac{x}{2t}\right)$. At time $t = 0$ the curve $\chi_a(t)$ becomes the reunion of two half-lines, so a corner-type singularity is generated in finite time ([8]). At the level of vortex filament dynamics in fluids, an analogy can be done between the evolution of χ_a and the "delta wing" vortex appearing when a fluid flows over a triangular obstacle. More precisely, in [11] numerical simulations for self-similar solutions of the binormal flow are shown to be in correlation with the physical experiment.

The question that motivated our series of papers [2, 3, 4, 5] is whether the formation of singularity in finite time for self-similar solutions is stable or not. For instance, if one consider a small perturbation of $\chi_a(1)$ and let it evolve through (BF), does it generate also a singularity in finite time, and if it is the case, which is the geometrical description of the singularity. In [2, 3, 4] we have answered these questions and this allowed us to treat in [5] the issue of considering as initial data for (BF) curves having a corner.

The main theorem in [5] gives, under suitable assumptions, the existence and description of solutions of (BF) generated by curves with a corner, for positive and negative times. Its companion theorem describes the evolution of perturbations of self-similar solutions up to a singularity formation in finite time, and beyond this time.

Concerning the proofs, in order to understand perturbations of χ_a we used Hasimoto's transform followed by a pseudo-conformal transform. Thus our starting point was the large time behavior of small data for:

$$iu_t + u_{xx} + \frac{1}{2t}\left(|u+a|^2 - a^2\right)(u+a) = 0.$$

We noted that the zero Fourier mode of $u(t)$ turns out to grow logarithmically in time for generic data (see Appendix B of [3]). We proved long range scattering results for this equation; the proof required a specific analysis of the linear equation and the introduction of new appropriate spaces. Then we were able to compute asymptotics in space and in time for the tangent and normal vectors of the solution $\chi(t)$ of (BF) constructed from the (NLSE) solution for $t > 0$. These allowed us to get the existence of a trace of the tangent and modulated normal vectors at time $t = 0$. Then by analyzing these traces we displayed how to recover from them the parameter a and the final state of $u(t)$. This yielded a recipe for constructing curves with data having with a corner. Finally this recipe allowed the construction of $\chi(t)$ for negative times.

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Global well-posedness for the Cubic Dirac equation in the critical space.

Ioan Bejenaru
(joint work with Sebastian Herr)

This is a report on the results obtained in [1]. For $M > 0$, the cubic Dirac equation for the spinor field $\psi : \mathbb{R}^4 \to \mathbb{C}^4$ is given by

$$(-i\gamma^\mu \partial_\mu + M)\psi = \langle \gamma^0 \psi, \psi \rangle \psi, \tag{1}$$

where we use the summation convention. Here, $\gamma^\mu \in \mathbb{C}^{4\times 4}$ are the Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices and $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{C}^4$.

Using scaling arguments, it turns out that the problem becomes critical in $H^1(\mathbb{R}^3)$. Local well-posedness was obtained in $H^s(\mathbb{R}^3)$, $s > 1$ (subcritical range) in [3]. Global well-posedness and scattering was proved in [5] for small initial data in $H^s(\mathbb{R}^3)$, $s > 1$ as well as for small initial data in H^1 with some regularity in the angular variable in [4].

The main idea in the above mentioned papers is as follows. The linear part of the Dirac equation is closely related to a half-Klein-Gordon equation. In the subcritical case one can make use of the end-point Strichartz estimate $L_t^2 L_x^\infty$ for the Klein-Gordon equation, while in the critical case certain spherically averaged versions of the same estimate hold true, see [4]. Both of the above strategies reach their limitations when one considers the equation (1) with small but general data in $H^1(\mathbb{R}^3)$, cp. [5, p. 181, l. 1-5].

Our strategy is to provide a replacement for the missing end-point estimate:

$$\|e^{it\langle D\rangle} P_k u_0\|_{L_t^2 L_x^\infty} \lesssim 2^k \|P_k u_0\|_{L_x^2}. \tag{2}$$

where P_k is the standard Littlewood-Paley operator localizing at frequency 2^k and $\langle D\rangle$ is the Fourier multiplier with symbol $\sqrt{|\xi|^2+1}$. This is a problem of independent interest whose applications are expected to go beyond the cubic Dirac equation.

Classical Strichartz estimates are usually derived from global estimates for the full kernel or for the kernel localized at a dyadic frequency. Our approach differs in that we seek to fully understand the precise nature of the decay in various regions and then provide a microlocal approach to the Strichartz estimates. By localizing the kernel beyond the dyadic localization we are able to identify adapted frames in which the $L^2 L^\infty$ estimates are true. Essentially we prove an estimate of type

$$\sum_{\omega\in\Omega(k)} \|e^{\pm it\langle D\rangle} P_{k,\omega} u_0\|_{L^2_{t_{\lambda,\omega}} L^\infty_{t_{\lambda,\omega}}} \lesssim 2^k \|P_k u_0\|_{L^2}. \tag{3}$$

where $\Omega(k)$ is a discrete set depending on k, $P_{k,\omega}$ localizes at frequency 2^k and in some angular region described by ω and $\lambda = \lambda(k)$ depends only on k.

(3) needs to be complemented with energy estimates in similar frames, that is to provide estimates of type $\|e^{\pm it\langle D\rangle} P u_0\|_{L^\infty_{t_{\lambda,\omega}} L^2_{t_{\lambda,\omega}}}$ where P is an operator which localizes in a certain region on the Fourier side. These estimates depend on the region where P localizes and on the frame $(t_{\lambda,\omega}, x_{\lambda,\omega})$; in particular there are null frames in which the estimate fails, but this is bypassed by using standard frames (t,x). This part of the theory is in the spirit to the work of Tataru for Wave Maps, see [6].

Combining the Strichartz estimates (3) with the energy estimates requires some additional structure in the nonlinearity since otherwise logarithmic divergences would occur. Fortunately we are able to take advantage of a null structure exhibited by the nonlinearity in the spirit of the work on the Dirac-Klein-Gordon equations in [2].

By employing some additional structures, such as $X^{s,b}$ type spaces, we are able to establish global well-posedness and scattering for the cubic Dirac equation (1) with small initial data in the critical space H^1.

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Gibbs measure evolution and probabilistic global well-posedness for radial nonlinear Schrödinger and wave equations on the unit ball

Aynur Bulut
(joint work with Jean Bourgain)

In a series of recent works, we establish global well-posedness for Gibbs measure evolutions of the radial nonlinear Schrödinger and wave equations on the unit balls of $\mathbb{R}^2$ and $\mathbb{R}^3$. The relevant initial value problems are

$$\text{(NLS)}\qquad \begin{cases} iu_t + \Delta u - |u|^p u &= 0, \quad (t,x) \in [0,T) \times B_d, \quad d = 2,3, \\ u|_{t=0} &= u_0, \\ u|_{x\in\partial B_d} &= 0, \end{cases}$$

with $u : [0,T) \times B_d \to \mathbb{C}$, and

$$\text{(NLW)}\qquad \begin{cases} w_{tt} - \Delta w + |w|^p w &= 0, \quad (t,x) \in [0,T) \times B_3, \\ (w,w_t)|_{t=0} &= (w_0, w_1), \\ (w,w_t)|_{\partial B_d} &= 0, \end{cases}$$

with $w : [0,T) \times B_3 \to \mathbb{R}$.

Fix $d \in \{2,3\}$, and let (e_n) denote the sequence of radial eigenfunctions of $-\Delta$ on B_d (with vanishing boundary conditions), with associated eigenvalues $(\lambda_n)^2$ arranged in increasing order. Writing $v = w_0 + i(\sqrt{-\Delta})^{-1}w_1$, the NLW equation becomes

$$iv_t - (\sqrt{-\Delta})v + (\sqrt{-\Delta})^{-1}|\mathrm{Re}\, u|^p \mathrm{Re}\, u = 0.$$

Both NLS and NLW thus describe Hamiltonian evolutions. Letting P_N be the truncation operator defined by $P_N(\sum_{n\in\mathbb{N}} \alpha_n e_n) = \sum_{n\le N} \alpha_n e_n$, the truncated initial value problems

$$\text{(NLS)}_N\qquad \begin{cases} iu_t + \Delta u - P_N(|u|^p u) &= 0, \\ u|_{t=0} &= P_N u_0, \\ u|_{x\in\partial B_d} &= 0, \end{cases}$$

$$\text{(NLW)}_N\qquad \begin{cases} iv_t - (\sqrt{-\Delta})v - P_N\Big[(\sqrt{-\Delta})^{-1}(|\mathrm{Re}\, u|^p \mathrm{Re}\, u)\Big] &= 0, \\ v|_{t=0} &= P_N v_0, \\ v|_{x\in\partial B_d} &= 0, \end{cases}$$

are therefore equipped with associated Gibbs measures of the form

$$\mu_G^{(N)}(d\phi) = e^{-\frac{1}{p+2}\|P_N\phi\|_{L_x^{p+2}}^{p+2}}\mu_F^{(N)}(d\phi),$$

which is invariant under the finite-dimensional evolution (the measure $d\mu_F^{(N)} = e^{\frac{1}{2}\|\nabla\phi\|_{L_x^2}^2}\prod_{n=1}^N d^2\phi$ in this expression denotes the free probability measure associated to the linear evolution).

Accordingly, we consider randomly chosen initial data

$$u_0 = \phi_\omega := \sum_{n\in\mathbb{N}} \frac{g_n(\omega)}{\lambda_n} e_n, \quad (w_0, w_1) = \left(\sum_{n\in\mathbb{N}} \frac{h_n(\omega)}{\lambda_n} e_n, \sum_{n\in\mathbb{N}} k_n(\omega) e_n\right),$$

where ω belongs to a probability space Ω, (h_n) and (k_n) are sequences of real-valued Gaussian random variables on Ω, and (g_n) is a sequence of complex-valued Gaussian random variables. This data belongs almost surely to the space $H_x^s(B_d)$ (resp., $H_x^s(B_d) \times H_x^s(B_d)$) for $s < \frac{1}{2}$, while the $H_x^{\frac{1}{2}}$ (resp, $H_x^{\frac{1}{2}} \times H_x^{-\frac{1}{2}}$) norm is a.s. infinite. Since the critical regularity for the local well-posedness theory is $s_c = \frac{d}{2} - \frac{2}{p}$, the initial value problems fall a.s. into the ill-posed regime when $p \geq 2$ in the three-dimensional setting and when $p \geq 4$ in the two-dimensional setting.

We now state our main results (see also the announcement [1]). In each of the following theorems, we have fixed $0 < s < \frac{1}{2}$, $0 < T < \infty$, and convergence is taken with respect to the space $C_t([0,T); H_x^s(B_d))$.

Theorem 0.1 (3D NLW, [2])**.** *Fix $p < 4$. Then the sequence of solutions to the truncated NLW on B_3 is almost surely convergent.*

Note that the case $p < 3$ was treated in [6] using fixed point techniques (see also [5] for related results showing how enhanced bounds arising from the randomization can overcome the ill-posedness in many settings).

Theorem 0.2 (2D NLS, [3])**.** *Fix $p \in 2\mathbb{N}$. Then the sequence of solutions to the truncated NLS on B_2 is a.s. convergent.*

The restriction to even integers is not essential, and more general nonlinearities can also be treated. Note that the case $p < 4$ was covered in [7].

Theorem 0.3 (3D NLS, [4])**.** *Fix $p = 2$. Then the sequence of solutions to the truncated NLS on B_3 is a.s. convergent.*

This proof of this theorem is the most delicate of the three results, and requires careful choice of function spaces balanced against estimates of the nonlinearity involving a number of delicate probabilistic bounds.

A common feature in each of these results is that for p sufficiently large it is not possible to close the estimates necessary for a fixed-point iteration. We therefore pursue an alternative approach based on a direct proof of convergence of the sequence through a careful analysis of frequency interactions and bootstrap-type estimates. A key ingredient is the use of the invariance of the Gibbs measure even at the local level, which allows for the consideration of short-time intervals in completing the bootstrap argument.

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Soliton resolution for equivariant wave maps to the sphere

Raphaël Côte

We consider finite energy corotationnal wave maps $\psi : \mathbb{R}_t \times [0,+\infty)_r \to \mathbb{R}$ solution to

$$\text{(WM)} \qquad \begin{cases} \partial_{tt}\psi - \partial_{rr}\psi - \dfrac{1}{r}\partial_r\psi + \dfrac{f(\psi)}{r^2} = 0 \\ (\psi, \partial_t\psi)|_{t=0} = (\psi_0, \psi_1) \end{cases} \qquad \text{with } f = gg',$$

where $g : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^3$ function. For a function $\vec{\phi} = (\phi_0, \phi_1)$, define the energy

$$E(\vec{\phi}) := \int_0^\infty \left(|\phi_1(r)|^2 + |\partial_r\phi_0(r)|^2 + \frac{|g(\phi_0(r))|^2}{r^2} \right) rdr.$$

At least formally, a wave map $\vec{\psi} = (\psi, \partial_t\psi)$ preserves the energy. If $\vec{\phi}$ has finite energy, then ϕ continuous and bounded, and has well defined limits at 0 and $+\infty$, which cancel g: we denote them $\phi(0)$ and $\phi(\infty)$. If $\vec{\phi}$ is a wave map, these limits do not depend on time. This motivates the introduction of the set where g vanishes

$$\mathcal{V} := \{\ell \in \mathbb{R} \mid g(\ell) = 0\}.$$

Also, let

$$G(x) := \int_0^x |g(y)|dy.$$

Our goal in this paper is to obtain a similar classification for wave maps of arbitrarily large energy, that is to relax the bound on the energy, inspired by the works of Duyckaerts Kenig and Merle [7, 8, 9, 10] in the context of the 3D $\dot{H}^1$-critical wave equation. It extends previous works [14, 2, 4, 5].

We provide a description of a wave map into decoupled profiles, a so called soliton resolution. It turns out that these profiles are harmonic maps and linear scattering terms. Recall that a harmonic map is a solution Q of finite energy of

$$\partial_{rr} Q + \frac{1}{r}\partial_r Q = \frac{f(Q)}{r^2}.$$

(Hence $(Q,0)$ is a finite energy stationary wave map).

On the other hand, given $\ell \in \mathcal{V}$, we define the linearized wave map flow around ℓ:

$$(\mathrm{LW}_\ell) \qquad \partial_{tt}\phi - \partial_{rr}\phi - \frac{1}{r}\partial_r\phi + \frac{g'(\ell)^2}{r^2}\phi = 0.$$

We make the following assumptions on the metric g:

- (A1) $G(x) \to \pm\infty$ as $x \to \pm\infty$.
- (A2) $\mathcal{V}$ is discrete,
- (A3) For all $\ell \in \mathcal{V}$, $g'(\ell) \in \{-1, 1\}$.

(A1) and (A2) are natural non degeneracy assumptions. (A3) captures the case of $g = \sin$ which correspond to wave maps with target manifold $\mathbb{S}^2$. We also have a weaker result under the relaxed assumption

- (A3') For all $\ell \in \mathcal{V}$, $g'(\ell) \in \{-2, -1, 1, 2\}$,

which encompasses the 4D radial Yang-Mills equation as well.

We work in the functional space $H \times L^2$ defined by the norm

$$\|\phi_0\|_H^2 := \int_0^\infty \left(|\partial_r \phi_0(r)|^2 + \frac{|\phi_0(r)|^2}{r^2} \right) r dr.$$

Theorem 1. *We make assumptions (A1)-(A2)-(A3').*

Let $\vec{\psi}(t)$ be a finite energy wave map. Then there exist a sequence of time $t_n \uparrow T^+(\vec{\psi})$, an integer $J \geq 0$, J sequences of scales $\lambda_{J,n} \ll \cdots \ll \lambda_{2,n} \ll \lambda_{1,n}$ and J harmonic maps $Q_1, \ldots, Q_J$

$$(1) \qquad \vec{\psi}(t_n) = \sum_{j=1}^{J} \left(Q_j\left(\cdot/\lambda_{j,n}\right) - Q_j(\infty), 0\right) + \vec{\phi}_n + \vec{b}_n.$$

where $\vec{b}_n$ vanishes in the following sense: for any sequence $\lambda_n > 0$, and $A > 0$

$$\|b_{n,0}\|_{H(t_n/A \leq r \leq A\lambda_n)} \to 0,$$

(so that $\|b_{n,0}\|_{L^\infty} \to 0$) and $\|b_{n,1}\|_{L^2} \to 0$; and ϕ_n is as follows:

1. *If $T^+(\vec{\psi}) < +\infty$, then $J \geq 1$, $\lambda_{1,n} \ll T^+(\vec{\psi}) - t_n$ and there exists a fixed function $\vec{\phi}$ of finite energy such that*
$$\vec{\phi}_n = \vec{\phi}.$$
2. *If $T^+(\vec{\psi}) = +\infty$, denote $\ell = \psi(\infty)$, there exists a solution $\vec{\phi}_L(t) \in \mathcal{C}(\mathbb{R}, H \times L^2)$ to the linear wave equation (LW_ℓ) such that*
$$\vec{\phi}_n = (\ell, 0) + \vec{\phi}_L(t_n).$$

If we furthermore assume (A3), then (in both cases)

$$\|\vec{b}_n\|_{H\times L^2} \to 0 \quad as \quad n \to +\infty.$$

The first step in the proof is to choose a sequence of time $t_n \to T^+(\vec{\psi})$ on which the space-time kinetic energy inside the light cone vanishes. This is a reformulation that the averaged kinetic energy inside the light cone vanishes, which is well known.

The second step is concerned with sequences of wave maps whose space-time kinetic energy vanishes. Up to a subsequence, one can construct a bubble decomposition i.e extract the harmonic maps, up to an error which tends to 0 in L^∞. This result does not make use of assumption (A3) or (A3'), but only (A1) and (A2).

The bound on the error is insufficient to capture the linear scattering term. However, when this bubble decomposition is combined with the concentration-compactness procedure developed in [12, 13] via linear profile decompositions introduced [1], we manage to derive a sharp scattering theorem *below the threshold in* L^∞. As linear scattering is involved, we do need assumption (A3') here. This result has its own interest, and as a consequence, we can extract the scattering term (for *all times*, not merely a sequence) in the global case. In an analogous way, we can define the regular part $\vec{\phi}$ in the blow up case.

Finally, we revisit the bubble decomposition to prove that the remainder vanishes in the energy space. We rely on channels of energy of the linear problem, a method first introduced by Duyckaerts Kenig and Merle [7, 8, 9, 10] in the context of the 3D nonlinear wave equation. Here we strongly rely on assumption (A3) which makes the linear problem conjugated to the 4D linear wave equation, and on [6] which proves the existence of channels of energy in that case.

Many open problems remain: the first one being to obtain a soliton resolution for all times and not merely a sequence of times. Also one should consider the Yang-Mills case. Finally, the possibilities for the behavior of $\lambda_{j,n}$ are not well understood, and in fact, constructing a solution where $J \geq 2$ is a challenging open question.

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Global solutions and asymptotic behavior for two dimensional gravity water waves

Jean-Marc Delort

(joint work with Thomas Alazard)

Consider an homogeneous and incompressible fluid in a gravity field, occupying a time-dependent domain with a free surface. We assume that the motion is the same in every vertical section and consider the two-dimensional motion in one such section. At time t, the fluid domain, denoted by $\Omega(t)$, is therefore a two-dimensional domain. We assume that its boundary is a free surface described by the equation $y = \eta(t, x)$, so that

$$\Omega(t) = \{ (x, y) \in \mathbb{R} \times \mathbb{R} \,;\, y < \eta(t, x) \}.$$

The velocity field is assumed to satisfy the incompressible Euler equations. Moreover, the fluid motion is assumed to have been generated from rest by conservative forces and is therefore irrotational in character. It follows that the velocity field $v\colon \Omega \to \mathbb{R}^2$ is given by $v = \nabla_{x,y}\phi$ for some velocity potential $\phi\colon \Omega \to \mathbb{R}$ satisfying

$$\Delta_{x,y}\phi = 0, \quad \partial_t\phi + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0, \tag{1}$$

where g is the modulus of the acceleration of gravity ($g > 0$) and where P is the pressure term. Hereafter, the units of length and time are chosen so that $g = 1$.

The problem is then given by two boundary condition on the free surface:

$$\begin{cases} \partial_t\eta = \sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi & \text{on } \partial\Omega, \\ P = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where ∂_n is the outward normal derivative of Ω, so that $\sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi = \partial_y\phi - (\partial_x\eta)\partial_x\phi$. The former condition expresses that the velocity of the free surface coincides with the one of the fluid particles. The latter condition is a balance of forces across the free surface.

Following Zakharov, Craig and Sulem, we work with the trace of ϕ at the free boundary

$$\psi(t,x) := \phi(t,x,\eta(t,x)).$$

To form a system of two evolution equations for η and ψ, one introduces the Dirichlet-Neumann operator $G(\eta)$ that relates ψ to the normal derivative $\partial_n\phi$ of the potential by

$$(G(\eta)\psi)(t,x) = \sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta(t,x)}.$$

Then (η,ψ) solves the so-called Craig–Sulem–Zakharov system

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + \eta + \dfrac{1}{2}(\partial_x\psi)^2 - \dfrac{1}{2(1+(\partial_x\eta)^2)}\big(G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)\big)^2 = 0. \end{cases} \tag{3}$$

It is known that if (η,ψ) is a classical solution of (3), such that (η,ψ) belongs to the space $C^0([0,T];H^s(\mathbb{R}))$ for some $T>0$ and $s>3/2$, then one can define a velocity potential ϕ and a pressure P satisfying (1) and (2). Thus it is sufficient to solve the Craig–Sulem–Zakharov formulation of the water waves equations.

Local existence for (3) has been proved by Sijue Wu [2] (see also [3] for the similar problem for three dimensional fluids i.e. two dimensional interface). Later on, Wu proved almost global existence for (3) i.e. existence of the solution over an interval of length $e^{c/\varepsilon}$ if the Cauchy data are smooth, decay at infinity, and are of size $\varepsilon \ll 1$. For three dimensional fluids, and similar Cauchy data, global existence holds true for small ε, as has been proved independently by Wu [5] and Germain-Masmoudi-Shatah [1]. Moreover these last authors have shown that there is linear scattering.

Our main result is the following statement, dealing with global existence for two dimensional fluids.

Main result. *For small enough initial data sufficiently decaying at infinity, the Cauchy problem for* (3) *is globally in time well-posed. Moreover,* $u = |D_x|^{\frac{1}{2}}\psi + i\eta$ *admits the following asymptotic expansion as* t *goes to* $+\infty$*: There is a continuous function* $\underline{\alpha}\colon \mathbb{R}\to\mathbb{C}$*, depending of* ε *but bounded uniformly in* ε*, such that*

$$u(t,x) = \frac{\varepsilon}{\sqrt{t}}\underline{\alpha}\Big(\frac{x}{t}\Big)\exp\Big(\frac{it}{4|x/t|} + \frac{i\varepsilon^2}{64}\frac{|\underline{\alpha}(x/t)|^2}{|x/t|^5}\log(t)\Big) + \varepsilon t^{-\frac{1}{2}-\kappa}\rho(t,x)$$

where κ is some positive number and ρ is a function uniformly bounded for $t \geq 1$, $\varepsilon \in]0, \varepsilon_0]$.

As an example of small enough initial data sufficiently decaying at infinity, consider

$$\eta|_{t=1} = \varepsilon\eta_0, \quad \psi|_{t=1} = \varepsilon\psi_0, \tag{4}$$

with η_0, ψ_0 in $C_0^\infty(\mathbb{R})$. Then there exists a unique solution

$$(\eta, \psi) \text{ in } C^\infty([1, +\infty[; H^\infty(\mathbb{R}))$$

of (3).

The proof of the main theorem relies on the use of L^2-estimates for the iterated action of the Klainerman vector field $Z = t\partial_t + 2x\partial_x$ on the solution, and on L^∞ optimal bounds for u. The derivation of the L^2-inequalities uses, on the one hand, the formulation of the equation in terms of the "good unknown" of Alinhac, to avoid derivative losses, and, on the other hand, a Shatah paradifferential normal forms method, that allows one to eliminate cubic contributions to the Sobolev energy. To prove L^∞ bounds, one uses in a first step Klainerman-Sobolev inequalities. Because of the critical nature of the problem, this does not give the optimal bounds one needs to perform a bootstrap. In a second step, one exploits the rough estimates provided by the Klainerman-Sobolev inequalities, and an expansion of the solution in terms of semi-classical lagrangian distributions, to deduce from the PDE an ODE satisfied by the solution. One deduces from this ODE the wanted L^∞ estimates, as well as the asymptotic behavior.

Let us mention that a similar global existence result has been obtained independently by Alex Ionescu and Fabio Pusateri.

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Global well - posedness and scattering for the defocusing, mass - critical generalized KdV equation

Benjamin Dodson

This talk is a report on the recent result of [3],

Theorem 0.4. *The mass - critical, defocusing, generalized KdV equation,*

$$u_t + u_{xxx} = \partial_x(u^5) \tag{1}$$

is globally well - posed and scattering for $u(0) \in L^2(\mathbf{R})$.

The aim of the paper is to prove global well - posedness and scattering by use of the concentration compactness method. In this vein previous work included the proof of global well - posedness and scattering for small data [4] using standard Picard iteration and Strichartz - type estimates. Next, [6] proved that a bounded sequence of real - valued functions in L^2 has a profile decomposition of the form

$$u_n = \sum_{j=1}^{J} e^{it_n^j \partial_x^3} (\lambda_n^j)^{-1/2} Re(e^{ix\xi_n^j} \phi^j(\frac{x - x_n^j}{\lambda_n^j})) + w_n^J, \tag{2}$$

where

$$\lim_{J \to \infty} \limsup_{n \to \infty} \| e^{t\partial_x^3} |D_x|^{1/6} w_n^J \|_{L^6_{t,x}(\mathbf{R} \times \mathbf{R})} = 0. \tag{3}$$

Next, [5] proved that a minimal mass blowup solution to (1) on $I \subset \mathbf{R}$ has $x(t) : I \to \mathbf{R}$, $N(t) : I \to (0, \infty)$, $K \subset L^2(\mathbf{R})$ compact such that for all $t \in I$,

$$N(t)^{-1/2} u(t, \frac{x - x(t)}{N(t)}) \in K. \tag{4}$$

Remark: This result assumed global well - posedness and scattering for the one dimensional, defocusing, nonlinear Schrödinger equation, which was later proved by [1].

Remark: See [7] for a converse - type result.

Then global well - posedness and scattering follows by ruling out a solution of the form (4) by using conservation laws and an interaction Morawetz estimate constructed in a similar manner to the interaction Morawetz estimate of [2]. This Morawetz estimate made use of a no - soliton result in the defocusing case that was proved by [7].

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Nondispersive decay for the cubic wave equation

Roland Donninger

(joint work with Anıl Zenginoğlu)

In this joint work with Anıl Zenginoğlu we consider the cubic focusing wave equation

$$(-\partial_t^2 + \Delta_x)v(t,x) + v(t,x)^3 = 0 \tag{1}$$

in three spatial dimensions. Eq. (1) has the conserved energy

$$\tfrac{1}{2}\|\nabla v(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + \tfrac{1}{2}\|\partial_t v(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} - \tfrac{1}{4}\|v(t,\cdot)\|^4_{L^4(\mathbb{R}^3)}$$

and exhibits finite-time blowup as is evidenced by the explicit solution $\tilde{v}(t,x) = \frac{\sqrt{2}}{1-t}$. By time translation and reflection symmetry we obtain from $\tilde{v}$ the explicit solution $v_0(t,x) = \frac{\sqrt{2}}{t}$ which is global for $t \geq 1$ and it decays in a nondispersive manner as $t \to \infty$. Our goal is to study the stability of v_0. However, since v_0 has infinite energy, this cannot be done in the framework of the standard Cauchy problem where one prescribes data on a $t = \text{const}$ surface. Instead, we study a hyperboloidal initial value problem (HIVP) for Eq. (1) where we prescribe data on a spacelike hyperboloid and consider the future evolution in a forward lightcone. To this end we foliate the forward lightcone by spacelike hyperboloids, parametrized by the map $\Phi_T : B_{|T|} \subset \mathbb{R}^3 \to \mathbb{R}^{1+3}$ where $B_{|T|} = \{X \in \mathbb{R}^3 : |X| < |T|\}$ and

$$\Phi_T(X) = \left(-\frac{T}{T^2 - |X|^2}, \frac{X}{T^2 - |X|^2}\right).$$

The parameter T takes values in $(-\infty, 0)$ and $T \to 0-$ corresponds to $t \to \infty$. We prescribe initial data on the hyperboloid $\{\Phi_{-1}(X) : X \in \mathbb{R}^3\} \subset \mathbb{R}^{1+3}$ and consider the future evolution. On each leaf $\Sigma_T := \{\Phi_T(X) : X \in \mathbb{R}^3\}$ we define norms

$$\|v\|^2_{L^2(\Sigma_T)} := \int_{B_{|T|}} \left| \frac{v \circ \Phi_T(X)}{T^2 - |X|^2} \right|^2 dX$$

$$\|v\|^2_{\dot{H}^1(\Sigma_T)} := \int_{B_{|T|}} \left| \nabla_X \frac{v \circ \Phi_T(X)}{T^2 - |X|^2} \right|^2 dX$$

and set $\|v\|^2_{H^1(\Sigma_T)} := \|v\|^2_{\dot{H}^1(\Sigma_T)} + |T|^{-2}\|v\|^2_{L^2(\Sigma_T)}$. Furthermore, we define the derivative operator ∇_n by

$$\frac{(\nabla_n v) \circ \Phi_T(X)}{T^2 - |X|^2} = \partial_T \frac{v \circ \Phi_T(X)}{T^2 - |X|^2},$$

i.e., ∇_n is a kind of normal derivative to the surfaces Σ_T. Finally, we denote by $D^+(A)$ the future domain of dependence of a subset $A \subset \mathbb{R}^{1+3}$ of Minkowski space. It turns out that

$$E_T(v) := \|v\|^2_{H^1(\Sigma_T)} + \|\nabla_n v\|^2_{L^2(\Sigma_T)}$$

defines a natural energy space for the hyperboloidal initial value problem

$$\text{(2)} \qquad \begin{cases} (-\partial_t^2 + \Delta_x)v(t,x) + v(t,x)^3 = 0, & (t,x) \in D^+(\Sigma_{-1}) \\ v|_{\Sigma_{-1}} = F \\ \nabla_n v|_{\Sigma_{-1}} = G \end{cases}$$

with $F, G : \Sigma_{-1} \to \mathbb{R}$ the prescribed initial data. Note that $E_T(v_0) \simeq |T|^{-1}$ and thus, the nondispersive solution v_0 has *finite* energy in this setting and one can study its stability. Our main result can then be formulated as follows.

Theorem 0.5 (D.-Zenginoğlu [1])**.** *There exists a co-dimension 4 Lipschitz manifold $\mathcal{M}$ of functions in $H^1(\Sigma_{-1}) \times L^2(\Sigma_{-1})$ such that the following holds. For data $(F, G) = (v_0|_{\Sigma_{-1}} + f, \nabla_n v_0|_{\Sigma_{-1}} + g)$ with $(f, g) \in \mathcal{M}$, the HIVP Eq. (2) has a unique solution v which satisfies*

$$|T| E_T(v - v_0) \lesssim |T|^{1-}$$

for all $T \in [-1, 0)$. As a consequence, we have, for any fixed $\delta > 0$,

$$\|v - v_0\|_{L^4(t,2t)L^4(B_{(1-\delta)t})} \lesssim t^{-\frac{1}{2}+}, \quad t \gg 1,$$

i.e., v converges to v_0 in a localized Strichartz sense.

The co-dimension 4 condition is easily understood as a consequence of the symmetries of Eq. (1). The relevant symmetries in this context are time translation (1 dimension) and Lorentz boosts (3 dimensions).

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Dynamics of critical and supercritical focusing wave equations

Thomas Duyckaerts
(joint work with Carlos Kenig, Frank Merle)

Consider the following focusing energy-critical wave equations:

$$\begin{cases} \partial_t^2 u - \Delta u = |u|^{p-1}u, & t \in I \subset \mathbb{R},\ x \in \mathbb{R}^3 \\ (u, \partial_t u)_{\restriction t=0} = (u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1} & , \end{cases} \tag{1}$$

where $p \geq 5$, $s = \frac{3}{2} - \frac{2}{p-1}$, and $\dot{H}^s$ and $\dot{H}^{s-1}$ are the usual homogeneous L^2-based Sobolev spaces on $\mathbb{R}^3$. We will only consider radial, real-valued solutions of the equation. We see (1) as a model focusing dispersive equation. Our goal is to describe the global dynamics, without size restriction on the solutions.

Equation (1) is locally well-posed in $\dot{H}^s \times \dot{H}^{s-1}$ and has the following scale invariance: if u is a solution of (1) and $\lambda > 0$, then u_λ defined by

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x)$$

is also a solution of (1). The Sobolev space $\dot{H}^s \times \dot{H}^{s-1}$ is scale-invariant:

$$\left\| \left(u_\lambda(0), \partial_t u_\lambda(0)\right) \right\|_{\dot{H}^s \times \dot{H}^{s-1}} = \left\| (u_0, u_1) \right\|_{\dot{H}^s \times \dot{H}^{s-1}}.$$

The energy

$$E(u(t), \partial_t u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2\, dx + \frac{1}{2} \int (\partial_t u(t,x))^2\, dx - \frac{1}{p+1} \int |u(t,x)|^{p+1}\, dx$$

of a solution is conserved (at least formally). If $p = 5$, this energy is defined in the scale-invariant space $\dot{H}^1 \times L^2$, and the equation is said to be *energy-critical.* If $p > 5$, then $s > 1$ and the equation is *energy-supercritical.*

Particular solutions of (1) are given by solutions of the stationary equation

$$-\Delta f = |f|^{p-1} f, \quad f \in \dot{H}^s(\mathbb{R}^3). \tag{2}$$

If $p = 5$, (2) has an explicit solution $W = \left(1 + \frac{|x|^2}{3}\right)^{-1/2}$, which is the unique nonzero radial solution of (2) up to scaling and sign-change. If $p > 5$, (2) has no radial nonzero solution. Solutions of (2) are natural candidates to be asymptotic profiles for bounded solutions of (1). Our first result gives a rigorous sense to this statement in the critical case:

Theorem 0.6. *Let u be a radial solution of* (1) *with $p = 5$, and T its maximal (positive) time of existence. Then one of the following holds:*

$$\lim_{t \to T} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} = +\infty \tag{3}$$

or there exist a solution v_L of the linear wave equation, an integer $J \geq 0$, J functions $\lambda_j : [0,T) \to [0,+\infty)$, and J signs $\epsilon_j \in \{-1,+1\}$ such that

(4)

$$\lim_{t \to T} \left\| (u(t), \partial_t u(t)) - \sum_{j=1}^{J} \left(\frac{\epsilon_j}{\lambda_j^{1/2}(t)} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) - (v_L(t), \partial_t v_L(t)) \right\|_{\dot{H}^1 \times L^2} = 0.$$

If $T < \infty$*, then* $J \geq 1$*. Finally, as* $t \to T$*,* $\lambda_1(t) \ll \ldots \ll \lambda_J(t)$*.*

Theorem 0.6 is proven in [4]. It is a soliton resolution result for equation (1) with $p = 5$. Until now, this type of results was only known in some completely integrable cases (see for example the pioneering work of W. Eckhaus and P. Schuur on the Korteweg-De Vries equation).

The study of the dynamics of equation (1) with $p = 5$ started with the celebrated article [6] of C. Kenig and F. Merle, describing the dynamics of solutions such that $E(u_0, u_1) < E(W, 0)$. More recently, a series of remarkable works by J. Krieger, K. Nakanishi and W. Schlag (and also M. Beceanu) are devoted to the construction of the center-stable manifold of W: see [9] and reference therein, and also the Oberwolfach report of K. Nakanishi in the same volume.

When $p = 5$, the existence of radial solutions of (0.6) satisfying (3) is known. Solutions such that (4) holds are known in the following cases:

- $T = +\infty$ and $J = 0$ (scattering to a linear solution).
- $T < \infty$ and $J = 1$: see for example [10].
- $T = +\infty$ and $J = 1$: see [2].

The existence of solutions satisfying (4) with $J \geq 2$ is open.

In the supercritical case, since (2) has no nonzero solution, one can expect that any solution with a critical norm that does not blow up in finite time scatters to a linear solution. The following slightly weaker result is proved in [3]:

Theorem 0.7. *Assume* $p > 5$*. Let* u *be a radial solution of* (1) *with maximal time of existence* T *such that*

$$\limsup_{t \to T} \|(u(t), \partial_t u(t))\|_{\dot{H}^s \times \dot{H}^{s-1}} < \infty.$$

Then $T = +\infty$ *and* u *scatters forward in time to a linear solution.*

We refer to [7] and [8] for previous works on the supercritical *defocusing* wave equation in dimension 3.

Theorem 0.7 says exactly that solutions satisfying an analog of (4) with $J \geq 1$ do not exist in the energy-supercritical case. Let us mention that the energy-subcritical case $3 < p < 5$ is similar: equation (2) has no nonzero solution and the analog of Theorem 0.7 was shown by R. Shen (see [11]).

One of the main ingredient in the proofs of Theorems 0.6 and 0.7 is the *channels of energy method.* It is based on the observation that any nonzero, finite-energy solution u of the linear wave equation satisfies:

$$(5)\quad \exists \eta, R > 0 \text{ s.t. } \forall t > 0 \text{ or } \forall t < 0, \quad \int_{|x| \geq R+|t|} |\nabla u(t,x)|^2 + (\partial_t u(t,x))^2 \, dx \geq \eta.$$

If $p = 5$, by the small data theory, (5) remains valid for solutions of (1) that are small in the energy space. Truncating in the region $\{|x| \leq R\}$, R large, and using finite speed of propagation, one can hope to prove (5) for some large solutions of (1), even in the supercritical case $p > 5$. This property is important, as it can be used to get a contradiction by proving accumulation of the energy in places where it should not be.

This method is quite robust, as it does not rely on the use of conservation laws or monotonicity formulas (such as Morawetz estimates or virial identities) which are not always available.

The proof of Theorem 0.6 and 0.7 also use concentration compactness arguments based on profile decomposition (see [1]), in the spirit of the compactness/rigidity method initiated in [5].

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The weakly nonlinear, big box limit for NLS on the two dimensional torus

Pierre Germain

(joint work with E. Faou, Z. Hani)

I presented in this talk recent work in collaboration with E. Faou and Z. Hani, which we recently uploaded on the arxiv [1]. Start with (NLS) in the weakly nonlinear regime on a big box with periodic boundary conditions:

$$-i\partial_t u + \Delta v = \epsilon^2 |u|^2 u \qquad (t,x) \in \mathbb{R} \times [0,L]^2. \tag{1}$$

The weakly nonlinear regime corresponds to $\epsilon \to 0$, whereas the big box regime corresponds to $L \to \infty$. It is instructive to examine this equation in Fourier space: setting $u(t,x) = \frac{1}{L^2} \sum_{K \in (\mathbb{Z}/L)^2} a_K(t) e^{i(K\cdot x - t|K|^2)}$, the Fourier coefficients (a_K) solve

$$-i\partial_t a_K(t) = \frac{\epsilon^2}{L^4} \sum_{(K_1,K_2,K_3)\in\mathcal{S}(K)} e^{it\Omega} a_{K_1}(t)\overline{a_{K_2}(t)} a_{K_3}(t), \tag{2}$$

where $\Omega = K_1^2 - K_2^2 + K_3^2 - K^2$ and $\mathcal{S}(K) = \{(K_1, K_2, K_3) \in \mathbb{Z}_L^2 : K_1 - K_2 + K_3 = K\}$. In the limit $\epsilon \to 0$, only non-resonant interactions, for which $\Omega = 0$, are expected to have an impact on the dynamics (this can be rigorously proved via a normal form transform). Only keeping these interactions gives the equation

$$-i\partial_t a_K(t) = \frac{\epsilon^2}{L^4} \sum_{(K_1,K_2,K_3)\in\mathcal{R}(K)} a_{K_1}(t)\overline{a_{K_2}(t)}a_{K_3}(t) \tag{3}$$

where $\mathcal{R}(K) = \{(K_1, K_2, K_3) \in \mathbb{Z}_L^2 : K_1 - K_2 + K_3 = K \text{ and } \Omega = 0\ \}$. Defining the operator

$$\mathcal{T}_L(e, f, g)(K) \stackrel{def}{=} \frac{\zeta(2)}{2L^2 \log L} \sum_{\mathcal{R}(K)} e_{K_1} f_{K_2} g_{K_3}, \tag{4}$$

the equation (3) reads

$$-i\partial_t a_K(t) = \frac{2\epsilon^2 \log L}{\zeta(2)L^2} \mathcal{T}_L(f, g, h). \tag{5}$$

The crucial observation is now the following: assume that a_K is the trace on the lattice $(\mathbb{Z}/L)^2$ of, say, a Schwartz function ϕ: $a_K = \phi(K)$. Then as $L \to \infty$

$$\mathcal{T}_L(a_K, a_K, a_K) \stackrel{L\to\infty}{\longrightarrow} \mathcal{T}(\phi, \phi, \phi), \tag{6}$$

where $\mathcal{T}$ is the integral operator acting on say Schwartz functions given by

$$\text{if } \xi \in \mathbb{R}^2, \qquad \mathcal{T}(f, g, h)(\xi) \stackrel{def}{=} \int_{-1}^{1} \int_{\mathbb{R}^2} f(\xi + x)\overline{g(\xi + x + \lambda x^\perp)}h(\xi + \lambda x^\perp)\, dx\, d\lambda \tag{7}$$

(where $\cdot^\perp$ is the rotation around the origin by an angle $\pi/2$). The crucial limit (6) is due to an equirepartition property of points in $\mathcal{R}_K$ in the limit $L \to \infty$. Upon rescaling time, it is now easy to derive from (5) the limiting equation

$$i\partial_t f = \mathcal{T}(f, f, f). \tag{8}$$

This equation turns out to have very striking properties. For instance, it derives from the Hamiltonian $\mathcal{H}$ given by the L^4 Strichartz norm

$$\mathcal{H}(f) = \iint |e^{it\Delta} f|^4\, dx\, dt$$

and admits Gaussians as stationary waves. We refer to [1] for the rigorous derivation of (8) and a study of the properties of this equation.

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Two dimensional water waves in holomorphic coordinates

Mihaela Ifrim
(joint work with John K. Hunter, Daniel Tataru)

This work is concerned with the infinite bottom water wave equation in two space dimensions. We consider this problem expressed in position-velocity potential holomorphic coordinates. Viewing this problem as a quasilinear dispersive equation, we establish two results: (i) local well-posedness in Sobolev spaces, and (ii) almost global solutions for small localized data. Neither of these results are new; they have been recently obtained by Alazard-Burq-Zuily, respectively by Wu using different coordinates and methods. Instead our goal is improve the understanding of this problem by providing a single setting for both results, as well as new, somewhat simpler proofs.

We consider the two dimensional water wave equations with infinite bottom with gravity but without surface tension. This is governed by the incompressible Euler's equations, with boundary conditions on the water surface. Under the additional assumption that the flow is irrotational, the fluid dynamics can be expressed in terms of a one-dimensional evolution of the water surface coupled with the trace of the velocity potential on the surface.

In writing the equations there is a choice to be made, namely that of the parametrization of the free surface. Common choices include Eulerian or Lagrangian coordinates. Instead, we choose to use holomorphic coordinates (t, α) where α corresponds to the holomorphic parametrization of the water domain by the lower half-plane, restricted to the real line.

Written in position-velocity potential holomorphic coordinates, the equations have the form

$$\begin{cases} W_t + F(1 + W_\alpha) = 0, \\ Q_t + FQ_\alpha - iW + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases} \tag{1}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2. \tag{2}$$

Here $W + \alpha$ represents the position of the water surface and Q represents the holomorphic extension of the velocity potential, while P represents the holomorphic projection, i.e. the Fourier multiplier which selects the negative frequencies.

The fully nonlinear system (1)-(2) admits a conserved energy, which has the form

$$E(W, Q) = \int \frac{1}{2}|W|^2 + \frac{1}{2i}(Q\bar{Q}_\alpha - \bar{Q}Q_\alpha) - \frac{1}{4}(\bar{W}^2 W_\alpha + W^2 \bar{W}_\alpha)\, d\alpha.$$

The nonlinear system (1) is degenerate hyperbolic, with double characteristics. For this reason it is more convenient to introduce the operator $A(w, q) = (w, r :=$

$w + Rq$), where $R := \dfrac{Q_\alpha}{1+W_\alpha}$, which diagonalizes the associated linearized equation. In particular for the solution (W_α, Q_α) to the linearized system we have $A(W_\alpha, Q_\alpha) = (\mathbf{W} := W_\alpha, R)$. Motivated by this, we differentiate (1) to obtain a self contained quasilinear system in $(\mathbf{W}, R)$.

As suggested by the above energy, our main spaces are the spaces $\dot{\mathcal{H}}^n$ with norm

$$\|(\mathbf{W}, R)\|^2_{\dot{\mathcal{H}}^n} = \sum_{k=0}^{n} \|\partial_\alpha^k (\mathbf{W}, R)\|^2_{L^2 \times \dot{H}^{\frac{1}{2}}},$$

where $n \geq 1$. Now we are ready to state our main local well-posedness result:

Theorem 0.8. *The system of equations* (1)-(2)*, differentiated and expressed in terms of the new variables* $(\mathbf{W}, R)$*, is locally well-posed in* $\dot{\mathcal{H}}^n(\mathbb{R})$ *for* $n \geq 1$*.*

In terms of Sobolev regularity of the data, this result improves the thresholds in earlier results of Wu [8] and Alazard-Burq-Zuily [1].

Our second goal is to obtain improved lifespan bounds for the small data problem. We propose an alternative approach to the normal form transformation method [6] for two dimensional water waves, which seems to be both simpler and more accurate. Instead of attempting to modify the equation using a normal form transform, we *construct modified energy functionals* which have cubic accuracy. In a simpler context, this method was first introduced by the authors in [4]. The cubic lifespan bound result is as follows:

Theorem 0.9. *Assume that the initial data for the (differentiated) system* (1)-(2) *satisfies*

$$\|(\mathbf{W}(0), R(0))\|_{\dot{\mathcal{H}}^1} \leq \epsilon. \tag{3}$$

Then the solution exists on an ϵ^{-2} *sized time interval* $I_\epsilon = [0, T_\epsilon]$*, and satisfies a similar bound. In addition,whenever the right hand side is finite we have the estimates*

$$\sup_{t \in I_\epsilon} \|(\mathbf{W}(t), R(t))\|_{\dot{\mathcal{H}}^n} \lesssim \|(\mathbf{W}(0), R(0))\|_{\dot{\mathcal{H}}^n}, \text{ for } n \geq 2$$

Assuming some additional localization for the initial data, we are also able to establish almost global existence of solutions. To state the result we need to return to the original set of variables (W, Q). We also take advantage of the scale invariance of the water wave equation. Our weighted norms are for the variables $\partial^k \mathbf{S}^j (W, Q)$, where $S = t\partial_t + 2\alpha\partial_\alpha$ and $\mathbf{S}(W, Q) = ((S-2)W, (S-3)Q)$. We define the weighted energy:

$$\|(W, Q)(t)\|^2_{\mathcal{WH}^n} := \sum_{j+k=1}^{n} \|\mathbf{A}\partial^k \mathbf{S}^j (W, Q)(t)\|^2_{\dot{\mathcal{H}}^0}. \tag{4}$$

Then we have the almost global well-posedness result:

Theorem 0.10. *Let n be large (≥ 7) and ϵ small. Then for each initial data $(W(0), Q(0))$ for the equation* (1) *satisfying*

$$\|(W, Q)(0)\|^2_{\mathcal{WH}^n} \leq \epsilon \tag{5}$$

the solution exists up to time $T_\epsilon = e^{c/\epsilon}$, and satisfies

$$\|(W(t), Q(t))\|^2_{\mathcal{WH}^n} \lesssim \epsilon, \qquad |t| < T_\epsilon. \tag{6}$$

This lifespan bound was originally established by Wu [8]. Here we prove the same result under less restrictive assumptions, and, hopefully, with a simpler proof. We should also mention the recent work of Ionescu-Pausader [5] and Alazard-Delort [2], where global well-posedness is proved for small localized data.

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Strichartz inequalities for 3*D* waves in a strictly convex domain

Oana Ivanovici

(joint work with Gilles Lebeau and Fabrice Planchon)

Consider the wave equation in a domain Ω of dimension $d \geq 2$:

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = 0, & x \in \Omega \\ u(0,x) = \delta_a, \ \partial_t u(0,x) = 0, \end{cases} \tag{1}$$

where $a \in \Omega$, δ_a is the Dirac function and Δ_g denotes the Laplace-Beltrami operator on Ω. If $\partial\Omega \neq \emptyset$ we consider the Dirichlet condition $u|_{\partial\Omega} = 0$.

Let Ω be the free space $\mathbb{R}^d$ with the Euclidian metric $g_{i,j} = \delta_{i,j}$ and $\chi \in C_0^\infty$ be a smooth function supported near 1. If $u_{\mathbb{R}^d}(t,x)$ is the solution to (1) in $\mathbb{R}^d$ then it is given by

$$u_{\mathbb{R}^d}(t,x) = \frac{1}{(2\pi)^d}\int \cos(t|\xi|)e^{i(x-a)\xi}d\xi$$

and it satisfies the classical dispersive estimates:

$$\|\chi(hD_t)u_{\mathbb{R}^d}(t,.)\|_{L^\infty(\mathbb{R}^d)} \le C(d)h^{-d}\min\{1,(h/t)^{\alpha_d}\}. \tag{2}$$

Interpolating between (2) and the energy estimate and using the so called TT^* argument, yields the following Strichartz estimates:

$$h^\beta\|\chi(hD_t)u\|_{L^q([0,T],L^r(\mathbb{R}^d))} \le C\Big(\|u(0,x)\|_{L^2} + \|hD_tu\|_{L^2}\Big). \tag{3}$$

Here $q \in (2,\infty]$, $r \in [2,\infty]$ satisfy $(q,r,d) \neq (2,\infty,3)$, $\frac{2}{q} + \frac{d-1}{r} \le \frac{d-1}{2}$ and

$$\frac{1}{q} = \alpha_d\Big(\frac{1}{2}-\frac{1}{r}\Big), \quad \beta = (d-\alpha_d)\Big(\frac{1}{2}-\frac{1}{r}\Big).$$

Our aim in the present note, based on [3], is to obtain Strichartz estimates inside domains: in this situation, the difficulties arise from the behaviour of the wave flow near the points of the boundary. Before stating our main result, we briefly introduce the Friedlander's model domain of the half space $\Omega_d = \{(x,y)|x > 0, y \in \mathbb{R}^{d-1}\}$ with Laplace operator given by $\Delta_F = \partial_x^2 + (1+x)\Delta_{\mathbb{R}_y^{d-1}}$. Clearly, the manifold Ω with the metric g_F inherited from Δ_F is a strictly convex domain; moreover, (Ω_2, g_F) may be seen as a simplified model for the disk $D(0,1)$ with polar coordinates (r,θ), where $r = 1 - x/2$, $\theta = y$. Our main result is the following:

Theorem 2. [3] *Strichartz inequality holds true for the solution to* (1) *inside* (Ω_d, g_F) *with* $\alpha_d = \frac{d-1}{2} - \frac{1}{6}$.

Remark 3. This was proved by M.Blair, H.Smith and C.Sogge in the case $d = 2$ for arbitrary boundary (i.e. without convexity assumption). The above theorem improves all the known results for $d \ge 3$. The case of a general strictly convex boundary is a work in progress with G.Lebeau, F.Planchon and R.Lascar.

In [2], we proved the following dispersive estimate for (Ω_d, g_F), $d \ge 2$:

Theorem 4. *There exists* $T > 0$, $C(d) > 0$ *such that for every* $a \in (0,1]$, $h \in (0,1]$ *and* $t \in (0,T]$ *the solution* $u_a(t,x,y) = \cos(t\sqrt{|\Delta_F|})(\delta_{x=a,y=0})$ *to* (1) *satisfies*

$$|\chi(hD_t)u(t,x)| \le C(d)h^{-d}\min\{1,(h/t)^{\frac{d-2}{2}}\gamma(t,h,a)\}, \tag{4}$$

where

$$\gamma(t,h,a) = \begin{cases} (\frac{h}{t})^{1/2} + a^{1/8}h^{1/4}, \text{ for } a \ge h^{4/7-\epsilon} \\ (\frac{h}{t})^{1/3} + h^{1/4}, \text{ for } a \le h^{1/2}. \end{cases}$$

Moreover, there is a sequence of moments of times $t_n = 4n\sqrt{a}\sqrt{1+a}$ *for which equality holds in* (4)*; for* $t \notin (t_n - \epsilon\frac{a^{1/2}}{n}, t_n + \epsilon\frac{a^{1/2}}{n}) := I_n$, $\gamma(t,h,a)$ *can be bounded by* $(\frac{h}{t})^{1/3}$ *independently of* a.

Remark 5. The estimate (4) means that, compared to the dispersive estimate in the free space (2), there is a loss of a power of $\frac{1}{4}$ of h inside a strictly convex domain, and this is due to micro-local phenomena such as caustics generated in arbitrarily small time near the boundary. Such caustics occur because optical rays sent from a source point under different directions are no longer diverging from each other.

Remark 6. As a corollary to Theorem 4, we immediately obtain Strichartz estimates in any dimension $d \geq 2$ with $\alpha_d = \frac{d-2}{2} + \frac{1}{4}$.

Proof. (of Theorem 2) We distinguish two different regimes:

- In the range $t > h$ and $a < h^{1/2+\epsilon}$, we prove the stronger estimate
$$|\chi(hD_t)u_a(t,x,y)| \leq Ch^{-d}(\frac{h}{t})^{\frac{d-2}{2}}(\frac{h}{t})^{1/3}.$$
This means that in the range $a < h^{1/2+\epsilon}$, one can kill the bad factor $h^{1/4}$ of Theorem 4. The geometry is irrelevant when a is very small, since there are too many singularities in the wave front set and the new estimates are obtained using a finer analysis on the sum of gallery modes (inspired by exponential sum methods). Using the spectral decomposition, we obtain an explicit representation of the Green function as a sum of gallery modes, valid for any a. Taking its Poisson transformation yields a superposition of waves similar to the parametrix we obtained in [2] for $a > h^{4/7-\epsilon}$.
- In the range $a > h^{4/7-\epsilon}$, we observe that the "bad" factor $h^{1/4}$ occurs only near the discrete set of times t_n, with an estimation of $\gamma(t,h,a)$ for t near t_n $(t \in I_n)$ by
$$\gamma(t,h,a) \leq (\frac{h}{t})^{1/2} + h^{1/3} + \frac{a^{1/8}h^{1/4}}{n^{1/4} + h^{-1/12}a^{-1/24}|t^2 - t_n^2|^{1/6}}. \tag{5}$$
Notice also that for $t \notin I_n$, the last factor is $\leq h^{1/3}$. The refinement on $\gamma(t,h,a)$ follows from inspection of the (degenerate) stationary phase argument in [2]

End of the proof: For simplicity restrict to Strichartz with $\alpha_d = \frac{d-1}{2} - \frac{1}{6}$ for $d = 3$. We consider the Green function $G(t,x,y,a) = \chi(hD_t)e^{it\sqrt{|\Delta_F|}}(\delta_{x=a,y=0})$ and for f compactly supported in $(s, a \geq 0, b)$ we set
$$A(f)(t,x,y) = \int G(t-s, x, y-b, a)f(s,a,b)dsdadb.$$
The dispersive exponent is in this case $\alpha_3 = \frac{5}{6}$. We have to prove the end point estimate for $r = \infty$ and $q = 12/5$:
$$h^{2\beta}\|A(f)\|_{L^{12/5}_{t\in[0,1]}L^\infty_{x,y}} \leq C\|f\|_{L_s^{12/7}L^1_{a,b}}, \quad 2\beta = (d - \alpha_d) = 3 - 5/6 = 13/6.$$
We summarise: the swallowtail singularities occur only at $t_n = 4n\sqrt{a}$, $x = a$, $y_n := t_n + O(a^{3/2}n)$; they have an effect on $I_n := (t_n - \frac{\epsilon\sqrt{a}}{n}, t_n + \frac{\epsilon\sqrt{a}}{n})$. outside I_n

there are only cusps with $(\frac{h}{t})^{-1/6}$ loss. The estimate of $\gamma(t,h,a)$ in (5) allows to track precisely where the usual TT^* argument fails.

We write $G(t,x,y,a) = G_0(t,x,y,a) + G_s(t,x,y,a)$ where G_s is the singular part, associated to a cutoff of G in balls centred at the swallowtail singularities

$$|x-a| \le \frac{a}{n^2}, \quad |t - 4n\sqrt{a}\sqrt{1+a}| \le \frac{\sqrt{a}}{n}.$$

Going back to [2], we obtain the following:

Proposition 7.

$$h^{2\beta} \sup_{x,y} |G_0(t,x,y,a)| \le C|t|^{-5/6};$$

$$h^{2\beta} \sup_{x,y} |G_s(t,x,y,a)| \le D(t,a,h), \quad \sup_{a,h} \int_{-1}^{1} |D(t,a,h)|^p dt < \infty, \quad \forall p < 2.$$

Let $A = A_0 + A_s$ corresponding to the previous decomposition. The estimate for A_0 follows easily, since the convolution by $|t|^{-5/6}$ maps $L^{12/7}$ in $L^{12/5}$. By the preceding proposition, $h^{2\beta}A_s$ is bounded from $L^1_s L^1_{a,b}$ into $L^{2-\epsilon}_t L^\infty_{x,y}$. Since the cutoff in balls near the swallowtails singularities is symmetric in (x,a), by duality, $h^{2\beta}A_s$ is bounded from $L^{2+\epsilon}_s L^1_{a,b}$ into $L^\infty_t L^\infty_{x,y}$ and, by interpolation, we get

$$h^{2\beta}A_s \quad \text{is bounded from} \quad L^{12/7}_s \quad \text{into} \quad L^{12-\epsilon}_t L^\infty_{x,y},$$

which is more than enough.

□

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Relaxation of wave maps exterior to a ball to harmonic maps for all data

Andrew Lawrie

(joint work with Carlos Kenig, Wilhelm Schlag)

This report consists of joint work with Carlos Kenig and Wilhelm Schlag. We describe all possible asymptotic dynamics for the 1-equivariant wave-map equation from

$$\mathbb{R}^{1+3}_{t,x} \setminus (\mathbb{R} \times B(0,1)) \to S^3$$

with a Dirichlet condition on the boundary of the ball $B(0,1)$, and data of finite energy. To be specific, consider the Lagrangian

$$\mathcal{L}(U,\partial_t U) = \int_{\mathbb{R}^{1+3}\setminus(\mathbb{R}\times B(0,1))} \frac{1}{2}\big(-|\partial_t U|_g^2 + \sum_{j=1}^{3} |\partial_j U|_g^2\big)\, dtdx$$

where g is the round metric on S^3, and we only consider functions for which the boundary of the cylinder $\mathbb{R}\times B(0,1)$ gets mapped to a fixed point on S^3, say the north pole. Under the usual 1-equivariance assumption the Euler-Lagrange equation associated with this Lagrangian becomes

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \tag{1}$$

where $\psi(t,r)$ measures the angle from the north-pole on S^3. The imposed Dirichlet boundary condition is then $\psi(t,1) = 0$ for all $t\in\mathbb{R}$. In other words, we are considering the Cauchy problem

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0, \quad r \geq 1,\\ &\psi(t,1) = 0, \quad \forall\, t,\\ &\psi(0,r) = \psi_0(r), \quad \psi_t(0,r) = \psi_1(r) \end{aligned} \tag{2}$$

The conserved energy is

$$\mathcal{E}(\psi,\psi_t) = \int_1^\infty \frac{1}{2}\big(\psi_t^2 + \psi_r^2 + 2\frac{\sin^2(\psi)}{r^2}\big) r^2\, dr \tag{3}$$

Any $\psi(t,r)$ of finite energy and continuous dependence on $t \in I := (t_0,t_1)$ must satisfy $\psi(t,\infty) = n\pi$ for all $t\in I$ where $n\in\mathbb{Z}$ is fixed. We can restrict to the case $n \geq 0$ since this covers the entire range $n\in\mathbb{Z}$ by the symmetry $\psi\mapsto -\psi$. We call n the *degree*, and denote by $\mathcal{E}_n$ the connected component of the metric space of all $\vec{\psi} = (\psi_0,\psi_1)$ with $\mathcal{E}(\vec{\psi}) < \infty$ and fixed degree n (of course obeying the boundary condition at $r = 1$), i.e.,

$$\mathcal{E}_n := \{(\psi_0,\psi_1) \mid \mathcal{E}(\psi_0,\psi_1) < \infty,\ \psi_0(1) = 0,\ \lim_{r\to\infty} \psi_0(r) = n\pi\} \tag{4}$$

The advantage of this model lies with the fact that removing the unit ball eliminates the scaling symmetry and also renders the equation subcritical relative to the energy. This subcriticality immediately implies global wellposedness in the energy class. Both of these features are in stark contrast to the same equation on $1+3$-dimensional Minkowski space, which is known to be super-critical and to develop singularities in finite time, see Shatah [8] and also Shatah, Struwe [9].

Another striking feature of this model, which fails for the $1+2$-dimensional analogue, lies with the fact that it admits infinitely many stationary solutions $(Q_n(r),0)$ which satisfy $Q_n(1) = 0$ and $\lim_{r\to\infty} Q_n(r) = n\pi$, for each $n\geq 1$. These solutions have minimal energy in the degree class $\mathcal{E}_n$, and they are the unique stationary solutions in that class.

The natural space to place the solution into for $n = 0$ is the energy space $\mathcal{H}_0 := (\dot{H}_0^1 \times L^2)(\mathbb{R}^3_*)$ with norm

$$\|\vec{\psi}\|^2_{\mathcal{H}_0} := \int_1^\infty (\psi_r^2(r) + \psi_t^2(r))\, r^2\, dr, \qquad \vec{\psi} = (\psi, \psi_t) \tag{5}$$

Here, $\mathbb{R}^3_* := \mathbb{R}^3 \setminus B(0,1)$ and $\dot{H}_0^1(\mathbb{R}^3_*)$ is the completion under the first norm on the right-hand side of (5) of the smooth radial functions on $\{x \in \mathbb{R}^3 \mid |x| > 1\}$ with compact support. For $n \geq 1$, we denote $\mathcal{H}_n := \mathcal{E}_n - (Q_n, 0)$ with "norm"

$$\|\vec{\psi}\|_{\mathcal{H}_n} := \|\vec{\psi} - (Q_n, 0)\|_{\mathcal{H}_0}$$

The point of this notation is that the boundary condition at $r = \infty$ is $\vec{\psi} - (Q_n, 0)(r) \to 0$ as $r \to \infty$.

The exterior equation (2) was proposed by Bizon, Chmaj, and Maliborski [2] as a model in which to study the problem of relaxation to the ground states given by the various equivariant harmonic maps. In the physics literature, this model was introduced in [1] as an easier alternative to the Skyrmion equation. Moreover, [1] stresses the analogy with the damped pendulum which plays an important role in our analysis. Both [2, 1] obtain the existence and uniqueness of the ground state harmonic maps via the phase-plane of the damped pendulum, and they also observed stability of the linearized equation around the harmonic maps. Numerical simulations described in [2] indicated that in each equivariance class and topological class given by the boundary value $n\pi$ at $r = \infty$ *every solution* scatters to the unique harmonic map Q_n that lies in this class. In this paper we verify this conjecture in the 1-equivariant setting, for all degrees and all data.

Our main result is as follows. It should be viewed as a verification of the *soliton resolution conjecture* for this particular case.

Theorem 8. *For any smooth energy data in $\mathcal{E}_n$ there exists a unique global and smooth solution to* (2) *which scatters to the harmonic map* $(Q_n, 0)$.

Scattering here means that on compact regions in space one has $(\psi, \psi_t)(t) - (Q_n, 0) \to (0, 0)$ in the energy topology, or alternatively

$$(\psi, \psi_t)(t) = (Q_n, 0) + (\varphi, \varphi_t)(t) + o_{\mathcal{H}_n}(1) \quad t \to \infty \tag{6}$$

where $(\varphi, \varphi_t) \in \mathcal{H}_0$ solves the linearized version of (2), i.e.,

$$\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{2}{r^2}\varphi = 0, \; r \geq 1, \; \varphi(t, 1) = 0 \tag{7}$$

We would like to emphasize that only the scattering part of Theorem 8 is difficult.

In [7] the second and third authors established this theorem for degree zero, and also proved asymptotic stability of the Q_n for $n \geq 1$. Here we are able to treat data of all sizes in the higher degree case. As in [7] we employ the method of concentration compactness from [5, 6]. The main difference from [7] lies with the rigidity argument. While the virial identity was the key to rigidity in [7] for degree zero (which seems to be impossible for $n \geq 1$), here we follow an alternate route which was developed in a very different context in [3, 4] for the three-dimensional energy critical nonlinear focusing wave equation. To be specific we rely on the *exterior asymptotic energy* arguments developed there. A novel feature of our work is that we elucidate the role of the Newton potential as an obstruction to linear energy estimates exterior to a cone in odd dimensions; in particular we do this for dim = 5, which is what is needed for equivariant wave maps in $\mathbb{R}^3$. It is precisely this feature which allows us to adapt the rigidity blueprint from [3, 4] to the model under consideration.

Finally, let us mention that we expect the methods of this paper to carry over to higher equivariance classes as well.

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ODE-Results for the fractional Laplacian

Enno Lenzmann

(joint work with Rupert Frank and Luis Silvestre)

We consider ground state solutions $Q = Q(|x|) > 0$ for the nonlinear fractional elliptic problem

$$(-\Delta)^s Q + Q - Q^{\alpha+1} = 0 \quad \text{in } \mathbb{R}^N. \tag{1}$$

Here $(-\Delta)^s$ denotes the fractional Laplacian of orders $s \in (0,1)$ and the exponent satisfies $\alpha \in (0, \alpha_*)$, where the critical exponent α_* is given by $\alpha_* = \frac{4s}{N-2s}$ if $s < N/2$ and $\alpha_* = +\infty$ if $s \geq N/2$. The existence of ground state solutions $Q = Q(|x|) > 0$ with $Q \in H^s(\mathbb{R}^N)$ follows from classical variational arguments. Concerning the delicate question about the uniqueness of Q, the important (but analytically rather special) case when $s = 1/2$, $N = 1$, and $\alpha = 1$, was resolved by C. Amick and J. Toland in their celebrated work [1] on the Benjamin-Ono equation. However, until recently, the question of uniqueness and non-degeneracy of Q has been mainly left open (and, for instance, posed as an open problem by Kenig et al. in [6]).

In a joint work of the author with R. Frank in [4], the one-dimensional case concerning uniqueness and non-degeneracy for ground states of equation (1) was completely resolved by developing new (robust) arguments. Here, a key ingredient is a sharp Sturmian-type oscillation estimate for eigenfunctions of fractional Schrödinger operators $H = (-\Delta)^s + V$ on $L^2(\mathbb{R})$. However, the arguments developed in [4] fail short by a factor of 2 in space dimension $N \geq 2$.

In the recent joint work of the author with R. Frank and L. Silvestre in [5], the problem of uniqueness and non-degeneracy of ground states for problem (1) was established for arbitrary space dimension.

Theorem 0.11. *For any $N \geq 1$, $s \in (0,1)$, and $\alpha \in (0, \alpha_*)$, the ground state $Q = Q(|x|) > 0$ for equation (1) is unique and its linearization is non-degenerate, i. e., the linearized operator $L = (-\Delta)^s + 1 - (\alpha+1)Q^\alpha$ satisfies*

$$\ker L = \operatorname{span}\{\partial_{x_1} Q, \ldots, \partial_{x_N} Q\}.$$

The novel key ingredient used in the proof of this result is the following *"radial unique continuation"* property, which provides us with a versatile substitute for classical ODE-type results.

Lemma 0.12. *Let $N \geq 1$ and $s \in (0,1)$. Suppose $u \in L^\infty(\mathbb{R}^N)$ is a radial solution of*

$$(-\Delta)^s u + Vu = 0 \quad \text{in } \mathbb{R}^N,$$

and that u vanishes at infinity. Moreover, if $V \in C^{0,\gamma}(\mathbb{R}^N)$ with some $\gamma > \max\{0, 1-2s\}$ is a radial potential with $V'(r) \geq 0$ for a. e. $r = |x|$, then $u(0) = 0$ implies $u \equiv 0$.

The proof of Lemma 0.12 involves an "energy-type" argument, which hinges on the local monotone quantity

$$\mathcal{H}(r) = \frac{c_s}{2} \int_0^{+\infty} \left((\partial_r U(r,t))^2 - (\partial_t U(r,t))^2 \right) t^{1-2s} \, dt - \frac{1}{2} V(r) u(r)^2, \tag{2}$$

where $c_s > 0$ is some numerical constant and $U : \mathbb{R}^{N+1}_+ \to \mathbb{R}$ denotes the s-harmonic extension of $u : \mathbb{R}^N \to \mathbb{R}$ to the upper half-space $\mathbb{R}^{N+1}_+$. The idea of constructing local monotonicity formula for the fractional Laplacian $(-\Delta)^s$ goes back to the work of X. Cabré, J. Solà-Morales and Y. Sire (see [3, 2]) in the context of so-called layer solutions.

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Low Regularity Local Wellposedness of Chern-Simons-Schrödinger System

Baoping Liu
(joint work with Paul Smith, Daniel Tataru)

We consider the initial value problem for the Chern-Simons-Schrödinger system in two spatial dimension. Mathematically, the model is described by the following system

$$\begin{cases} D_t\phi & = iD_\ell D_\ell \phi + ig|\phi|^2\phi \\ \partial_t A_1 - \partial_1 A_t & = -J_2 \\ \partial_t A_2 - \partial_2 A_t & = J_1 \\ \partial_1 A_2 - \partial_2 A_1 & = -\frac{1}{2}|\phi|^2 \end{cases}$$

where A is the electromagnetic potential, which can be viewed as a one-form on $\mathbb{R}^{2+1}$.

$$D_\alpha = \partial_\alpha + iA_\alpha$$

is the covariant differentiation operator. ϕ is the $2+1$ dimensional complex scalar function, representing a matter field that interacts with a Maxwell-like field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$J_i := \mathrm{Im}(\bar{\phi} D_i \phi).$$

We point out that the nonlinear term corresponds to a two-body δ-function attraction characterized by a positive coupling constant $g > 0$. So this is a *focusing* problem.

Also, scaling symmetry

$$\phi(t,x) \to \lambda\phi(\lambda^2 t, \lambda x), \quad \phi_0(x) \to \lambda\phi_0(\lambda x); \quad \lambda > 0,$$

preserves the charge of the initial data. Hence this problem is L^2-*critical.*

In order to interpret Chern-Simons-Schrödinger system as a well defined time evolution, we need to impose a suitable gauge condition which eliminates the gauge freedom caused by the following transformation.

$$\phi \mapsto e^{-i\theta}\phi \qquad A \mapsto A + d\theta$$

We adopt from [4] the *heat gauge*, which is a variation of the Coulomb gauge. The condition of heat gauge is

$$\nabla \cdot A_x = A_t$$

Combined with the last three equations in the system, we obtain

$$A_1 = H^{-1}A_1(0) - H^{-1}[\mathrm{Re}(\bar{\phi}\partial_2\phi) + \mathrm{Im}(\bar{\phi}\partial_2\phi)] - H^{-1}(A_2|\phi|^2)$$
$$A_2 = H^{-1}A_2(0) + H^{-1}[\mathrm{Re}(\bar{\phi}\partial_1\phi) + \mathrm{Im}(\bar{\phi}\partial_1\phi)] + H^{-1}(A_1|\phi|^2)$$

Here H^{-1} is defined as the Fourier multiplier

$$H^{-1}f := \frac{1}{(2\pi)^3}\int \frac{1}{i\tau + |\xi|^2} e^{i(t\tau + x\cdot\xi)}\tilde{f}(\tau,\xi)d\tau d\xi.$$

It is worth noticing that the main benefit of the heat gauge over the Coulomb gauge is that it improves the regularity of the connection coefficients A_j in the region of high parabolic modulation $|\tau| \gg \xi^2$. This in turn allows for better frequency localizations in the trilinear estimates.

Another important feature is the existence of null structure in the main terms of $A_j\partial_j\phi$.

There are several other key ingredients to handle this problem.

(1) A second iteration in the connection coefficients to prove existence and estimates for $A_j's$.
(2) Use of angular spaces to bound the linear part in right hand side of the Schödinger equation, which act like nonperturbative long-range potentials.
(3) Improved bilinear Strichartz estimate adapted to waves with further localization.
(4) Lateral U^p, V^p spaces that are designed to improve the logarithmic loss from bilinear estimates.

Our main theorem reads

Theorem 9 (L.-Smith-Tataru 2012)**.** *For any small initial data $\phi_0 \in H^s(\mathbb{R}^2)$, $s > 0$, there is a positive time T, depending only on $\|\phi_0\|_{H^s}$, such that the Chern-Simons-Schrödinger system with respect to the heat gauge has a unique solution $\phi(t,x) \in C([0,T], H^s(\mathbb{R}^2))$. In addition, $\phi_0 \mapsto \phi$ is Lipschitz continuous from $H^s(\mathbb{R}^2)$ to $C([0,T], H^s(\mathbb{R}^2))$.*

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Low regularity local well-posedness for quasilinear Schrödinger equations

Jason Metcalfe

(joint work with Jeremy Marzuola, Daniel Tataru)

This is a report on the results of [5, 6] that concern quasilinear Schrödinger equations

$$(1)\qquad \begin{aligned} i\partial_t u + g^{jk}(u,\nabla u)\partial_j\partial_k u &= F(u,\nabla u), \quad u:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}^M,\\ u(0)&=u_0. \end{aligned}$$

Here, in the case of quadratic interactions, we assume

$$g^{jk}(y,z) = I_{d\times d} + O(|y|+|z|), \quad F(y,z) = O(|y|^2+|z|^2), \quad \text{near } (0,0),$$

while for cubic interactions we instead have

$$g^{jk}(y,z) = I_{d\times d} + O(|y|^2+|z|^2), \quad F(y,z) = O(|y|^3+|z|^3), \quad \text{near } (0,0).$$

We shall also permit the ultrahyperbolic case where the identity matrix above can be replaced by one with a different signature.

Local well-posedness, which means existence, uniqueness, and continuous dependence on the initial data, was proved in [3]. See, also, [4] for the ultrahyperbolic case. The current studies seek to improve upon the necessary regularity, in the small data regime, that must be assumed on the initial datum in order to have local well-posedness.

A key difficulty arises in connection with the Mizohata integrability condition (see, e.g., [2]), which requires that for the linear problem

$$(i\partial_t + \Delta_g)v = A_i(x)\partial_i v$$

to be L^2 well-posed one must have that the real part of $A(x)$ is integrable along the Hamiltonian flow of Δ_g. In order to guarantee such for the linearization of (1) in the quadratic case, it is not enough to work in Sobolev spaces. Additional decay must be assumed. The approach of [3], [4] was to work in weighted spaces and to use an "artificial viscosity method".

Our approach is inspired by [1], and decay is imposed by assuming a summability over cubes. To this end, we let $\mathcal{Q}_j$ be a partition of $\mathbb{R}^d$ into cubes of sidelength 2^j. For an associated smooth partition $\{\chi_Q\}$ and for a Banach space U, we define

$$\|u\|^p_{l^p_j U} = \sum_{Q\in\mathcal{Q}_j} \|\chi_Q u\|^p_U.$$

In particular, for the quadratic case, we measure the initial datum in the norm

$$\|u\|_{l^1 H^s}^2 = \sum_{j\geq 0} 2^{2js} \|S_j u\|_{l^1_j L^2}^2,$$

where $\sum_{j\geq 0} S_j(D) = 1$ is a Littlewood-Paley partition. On the other hand, in the cubic case, upon linearization, the Mizohata condition is trivially satisfied due to energy estimates, and here it suffices to take data in standard Sobolev spaces.

Indeed, the main results are:

Theorem 10. *There exists $\varepsilon > 0$ and a Banach space S so that if the initial datum u_0 satisfies $\|u_0\|_S < \varepsilon$, then* (1) *is locally well-posed in S on the time interval $I = [0,1]$.*

- *(Quadratic [5]) For the quadratic case, we take $S = l^1 H^s$ for $s > \frac{d}{2} + 3$.*
- *(Cubic [6]) For the cubic case, we take $S = H^s$ for $s > \frac{d+5}{2}$.*

The iteration spaces and main linear estimate are inspired by the classical local smoothing / energy estimate

$$\sup_{R\geq 0} R^{-1/2} \|D^{1/2} e^{it\Delta} f\|_{L^2_{t,x}([0,1]\times\{|x|\approx R\})} + \|e^{it\Delta} f\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2}.$$

Indeed, we set

$$\|u\|_{X_j} = 2^{j/2}\Big(\sup_l \sup_{Q\in\mathcal{Q}_l} 2^{-l/2} \|u\|_{L^2_{t,x}([0,1]\times Q))}\Big) + \|u\|_{L^\infty_t L^2_x}$$

and

$$\|u\|_{l^1 X^s}^2 = \sum_{j\geq 0} 2^{2js} \|S_j u\|_{l^1_j X_j}^2.$$

The dual-type spaces in which we measure the nonlinearities satisfy the following bound

$$\|f\|_{Y_j} \lesssim \inf_{f = 2^{j/2} f_1 + f_2} \Big[\Big(\inf_l \sum_{Q\in\mathcal{Q}_l} 2^{l/2} \|f_1\|_{L^2_{t,x}([0,1]\times Q)}\Big) + \|f_2\|_{L^1_t L^2_x}\Big]$$

and

$$\|f\|_{l^1 Y^s}^2 = \sum_{j\geq 0} 2^{2js} \|S_j f\|_{l^1_j Y_j}^2.$$

The principal linear estimate, in the quadratic case, then states that solutions to

$$\begin{cases} (i\partial_t + \partial_k g^{kl} \partial_l) u + V\cdot\nabla u = h, \\ u(0) = u_0 \end{cases}$$

satisfy

$$\|u\|_{l^1 X^\sigma} \lesssim \|u_0\|_{l^1 H^\sigma} + \|h\|_{l^1 Y^\sigma}$$

provided that

$$\|g - I\|_{l^1 X^s} \ll 1, \quad \|V\|_{l^1 X^{s-1}} \ll 1, \quad s > \frac{d}{2} + 2, \quad 0 \leq \sigma \leq s.$$

The proof relies on a positive commutator argument. One first passes to a frequency localized set up. A wedge decomposition is employed to further restrict attention to the case that the solution is localized in frequency in a small cone about each coordinate axis. This permits us to trivially handle the ultrahyperbolic case as one need only modify the sign of the multiplier corresponding to those directions where $g(0)$ is negative. About the x_1 axis, e.g., the multiplier is chosen as

$$i2^j\mathcal{M} = m(2^{-l}x_1)\partial_1 + \partial_1 m(2^{-l}x_1),$$

where $m'(s) = \psi^2(s)$, $\psi \in \mathcal{S}$ and ψ is localized to frequencies $\lesssim 1$ and $\psi \sim 1$ on $|s| \leq 1$. Repeating similarly in the other coordinate directions, we obtain via a positive commutator argument an estimate in a cube of sidelength 2^l centered at the origin. Taking a supremum over translates and scales, up to introducing the l^1 summability, the estimate is obtained. The latter is incorporated by introducing cutoffs adapted to a scale that is slightly larger than desired (say M times the desired scale), commuting with the operator, and using M^{-1} as a small parameter to bootstrap the commutators. As M is independent of the frequency scale, passing to the desired summability is trivial.

The iteration to show local well-posedness is then closed with multilinear and Moser-type estimates. For example, we have

$$\|uv\|_{l^1Y^\sigma} \lesssim \|u\|_{l^1X^{s-1}}\|v\|_{l^1X^{\sigma-1}}$$

provided $s > \frac{d}{2} + 2$ and $0 \leq \sigma \leq s$. These are proved using the definitions of the spaces and a usual analysis of the low-high, high-low, and high-high trichotomy.

The arguments in the cubic cases proceed quite similarly. The primary difference is that the l^1 summability is replaced by l^2 summability. In particular, on the initial data, we have $l^2H^s \approx H^s$.

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Stability of line solitons for KP-II

Tetsu Mizumachi

The KP-II equation

$$\text{(KP-II)} \qquad \partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x,y) \in \mathbb{R}^2$$

is a 2-dimensional generalization of the KdV equation that takes slow variations in the transversal direction into account. The KP-II equation describes the motion of shallow water waves with weak surface tension.

I talk on the transversal stability of KdV 1-solitons as solutions for the KP-II equation ([4]). The main difference between stability analysis of KdV 1-solitons or the stability of line solitons with the y-periodic boundary condition is that $\varphi_c(x-2ct)$ does not have the finite L^2-mass because it is not localized in the y-direction. As a consequence, the linearized operator of the KP-II equation around the line soliton has a family of continuous eigenvalues converging to 0 in exponentially weighted space, whereas 0 is an isolated eigenvalue of the linearized KdV operator around a solitary wave in exponentially weighted space.

Since we have continuous spectrum converging to 0, the modulations of the speed and the phase shift of line solitons for the KP-II equation cannot be described by ODEs as KdV ([8]) or the KP-II equation posed on $\mathbb{R}_x \times \mathbb{R}_y/(2\pi\mathbb{Z})$ ([5]).

I find that for the KP-II equation posed on $\mathbb{R}^2$, modulation of the speed parameter $c(t,y)$ and the phase shift $x(t,y)$ cannot be uniform in y and their long time behavior is described by the Burgers equation. Note that similar modulation equations have formally derived for a two spatial dimensional Boussinesq model ([7]).

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On the stability of mKdV breathers

Claudio Muñoz

(joint work with Miguel A. Alejo)

We consider the modified Korteweg-de Vries (mKdV) equation on the real line

$$u_t + (u_{xx} + u^3)_x = 0, \quad u(t,x) \in \mathbb{R}, \quad (t,x) \in \mathbb{R}^2.$$

The associated Cauchy problem is globally well-posed for initial data in $H^1(\mathbb{R})$ [6]. Additionally, the flow map is not uniformly continuous in $H^s(\mathbb{R})$ if $s < \frac{1}{4}$ [7]. In order to prove this last result, Kenig, Ponce and Vega considered a very particular class of solutions of mKdV called *breathers*.

Definition 0.1 ([10]). *Let $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$ be fixed parameters. A breather is a smooth solution of mKdV given explicitly by the formula*

$$B := B(t,x;\alpha,\beta,x_1,x_2) := 2\sqrt{2}\partial_x \Big[\arctan\Big(\frac{\beta}{\alpha} \frac{\sin(\alpha y_1)}{\cosh(\beta y_2)} \Big) \Big],$$

where $y_1 := x + \delta t + x_1$, $y_2 := x + \gamma t + x_2$, $\delta := \alpha^2 - 3\beta^2$, $\gamma := 3\alpha^2 - \beta^2$.

Breathers are *oscillatory bound states*. They are periodic in time (after a suitable space shift) and localized in space. The parameters β and α are scaling parameters, x_1, x_2 are shifts, and $-\gamma$ represents the *velocity* of a breather. Numerical computations [1] showed that breathers are *numerically* stable. Next, in [2] we constructed a Lyapunov functional that controls the dynamics of H^2-perturbations of any breather. Additionally, we consider the H^2 conserved quantity:

$$F[u](t) = \frac{1}{2}\int_{\mathbb{R}} u_{xx}^2(t,x)dx - \frac{5}{2}\int_{\mathbb{R}} u^2 u_x^2(t,x)dx + \frac{1}{4}\int_{\mathbb{R}} u^6(t,x)dx,$$

to prove that **any** breather profile satisfies the elliptic equation

$$B_{(4x)} - 2(\beta^2 - \alpha^2)(B_{xx} + B^3) + (\alpha^2+\beta^2)^2 B + 5BB_x^2 + 5B^2B_{xx} + 3/2\, B^5 = 0.$$

Finally, in [3] we improved our previous results and showed that mKdV breathers are indeed H^1 stable, i.e. stable in the energy space.

Theorem 0.13. *Let $\alpha, \beta > 0$ be fixed scalings. There exist parameters η_0, A_0, depending on α, β only, such that the following holds. Consider $u_0 \in H^1(\mathbb{R})$, and assume that there exists $\eta \in (0, \eta_0)$ such that*

$$\|u_0 - B(0,\cdot;\alpha,\beta,0,0)\|_{H^1(\mathbb{R})} \leq \eta.$$

Then there exist explicit $x_1(t), x_2(t) \in \mathbb{R}$ such that the solution $u(t)$ of the Cauchy problem for the mKdV equation, with initial data u_0, satisfies for all $t \in \mathbb{R}$,

$$\|u(t) - B(t,\cdot;\alpha,\beta,x_1(t),x_2(t))\|_{H^1(\mathbb{R})} \leq A_0\eta.$$

The previous result places breathers as stable objects at the same level of regularity as mKdV solitons, even if they are very different in nature. We recall that solitons are H^1-stable [4, 5]. We also proved [3] that breathers behaving as standard solitons are asymptotically stable in the energy space.

In order to prove the Main Theorem we borrowed some ideas from the work by Merle and Vega [9], where the L^2-stability of KdV solitons was proved. More precisely, we use a Bäcklund transformation [8] associated to mKdV, which links the behavior of breathers, complex-valued solitons and small real-valued solutions of the mKdV equation.

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Threshold manifold of global dynamics for the critical wave equation

Kenji Nakanishi

(joint work with Joachim Krieger, Wilhelm Schlag)

This talk is on the joint work [8] with Joachim Krieger and Wilhelm Schlag. We study the global dynamics of the focusing critical wave equation

$$\ddot{u} - \Delta u = f'(u) = |u|^{2^*-2}u, \quad f(u) := |u|^{2^*}/2^*, \quad 2^* := 2d/(d-2),$$

where $u(t,x) : I \times \mathbb{R}^d \to \mathbb{R}$ for some interval $I \subset \mathbb{R}$ and $d = 3, 4, 5$. The conserved energy is given in terms of $\vec{u} := (u, \dot{u}) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) =: \mathcal{H}$ at each t by

$$E(\vec{u}) := \|\dot{u}\|_{L^2_x}^2/2 + \|\nabla u\|_{L^2_x}^2/2 - \|f(u)\|_{L^1_x}$$

Both the equation and E are invariant for the scaling $u_\lambda(t,x) := \lambda^{d/2-1}u(\lambda t, \lambda x)$. The equation admits an explicit static solution: the ground state

$$W(x) := [1 + |x|^2/(d(d-2))]^{1-d/2} \sim \langle x \rangle^{2-d}, \quad -\Delta W = f'(W),$$

then $Sta(W) := \{W_\lambda(x-c)\}_{\lambda>0, c\in\mathbb{R}^d}$ is the family of ground states. Its scaling derivative gives a kernel to the linearized operator

$$0 = \partial_\lambda|_{\lambda=1}(-\Delta W - f'(W_\lambda)) = LW'$$

$$L := -\Delta - f''(W)$$

$$W' := \partial_\lambda\big|_{\lambda=1} W_\lambda = (x \cdot \nabla + d/2 - 1)W \notin L^2(\mathbb{R}^d) \quad (d \le 4)$$

The slow decay of W' gives a serious difficulty for the dispersive estimates around the ground states $\{W_\lambda\}$.

There has been a lot of work on the above equation. Recently, Duyckaerts, Kenig and Merle [4] gave a complete classification of asymptotic behavior of all radial solutions in $\mathcal{H}(\mathbb{R}^3)$: let $T \in (0, \infty]$ be the maximal existence time of any radial solution u. Then either $\limsup_{t \to T-0} \|\vec{u}(t)\| = \infty$ with $T < \infty$, or

$$\vec{u}(t) = \sum_{j=1}^{N} \vec{W}_{\lambda_j(t)} + \vec{v}(t) + o(1) \quad \text{in } \mathcal{H} \quad (t \to T-0) \tag{1}$$

for some $N \in \{0, 1, 2, \dots, \}$, $\lambda_1(t), \dots, \lambda_N(t)$ and some free solution v.

The next question is where and how these solutions in (1) arise in $\mathcal{H}$. While the existence for $N > 1$ is still an open question, the existence for $N = 1$ is known, even before [4], both for $T < \infty$ and for $T = \infty$, see [1, 2, 3, 5, 9, 10].

Our previous work [6, 7] classified the global dynamics into 4 sets according to scattering and blowup, in $E(\vec{u}) < E(W) + e^2$ and *away from* $Sta(W)$. For subcritical equations [11, 12, 13], we have classification into 9 sets without the latter restriction, where the solutions staying near the ground states constitute a manifold splitting $\mathcal{H}$ into the blowup and the scattering regions. The distinction between the subcritical and critical cases is the blowup in (1). It turns out that they are on the same manifold as those with $T = \infty$, which separates the global dynamics in $\mathcal{H}^\varepsilon := \{\varphi \in \mathcal{H} \mid E(\varphi) < E(W) + \varepsilon^2\}$.

Theorem 11. [8] $0 < \exists\varepsilon \ll \exists\delta \ll 1$, $Sta(W) \subset \exists\mathcal{M} \subset \mathcal{H}(\mathbb{R}^d)$: *$C^1$ manifold with codimension 1, connected, unbounded and invariant for the flow, translation and scaling. Every solution u in $\mathcal{H}^\varepsilon$ with maximal existence time $T > 0$ satisfies one of the following.*

(1) $\vec{u}(0) \notin \pm\mathcal{M}$, $T = \infty$ *and u scatters as* $t \to \infty$.
(2) $\vec{u}(0) \notin \pm\mathcal{M}$, $T < \infty$ *and* $\liminf_{t \to T-0} \operatorname{dist}(\vec{u}(t), Sta(W)) > \delta$.
(3) $\vec{u}(0) \in \pm\mathcal{M}$, $\limsup_{t \to T-0} \operatorname{dist}(\vec{u}(t), Sta(W)) \lesssim \sqrt{E(\vec{u}) - E(W)} (< e)$.

A small neighborhood of each point on $\mathcal{M}$ is C^1-diffeo to a ball where (1) and (2) correspond to the lower and upper halves. Let $\mathcal{M}^\dagger := \{(\varphi_1, -\varphi_2) \mid \varphi \in \mathcal{M}\}$ be the time inversion, then $\mathcal{M} \cap \mathcal{M}^\dagger$ is a bounded connected C^1 manifold of codimension 2, while $\mathcal{M} \cap -\mathcal{M}^\dagger = \emptyset$.

In particular, $\mathcal{M}$ is on the boundary of both (1) and (2). In the subcritical cases, we conclude that $\pm\mathcal{M}$ is exactly the boundary between (1) and (2), since both (1) and (2) are open. In the critical case, it is an open question if the set (2) is open. The above result does not treat those solutions in (1) with higher energy. Adding radiation to the above $\mathcal{M}$ allows us to include at least those with $N = 1$ and $T = \infty$.

Theorem 12. [8] $\mathcal{M} \subset \exists\widetilde{\mathcal{M}} \subset \mathcal{H}(\mathbb{R}^d)$: *$C^1$ manifold of codimension 1, connected and invariant for the flow, translation, scaling and the Lorentz transform, containing all the ground state solitons $Sol(W)$ generated by the Lorentz transforms from $Sta(W)$. $Sol(W) \subset \exists O_1$, $\bar{O}_1 \subset \exists O_2$ both open, and every solution u from O_1 satisfies one of the following.*

(1) $\vec{u}(0) \notin \pm\widetilde{\mathcal{M}}$, $T = \infty$ *and u scatters as* $t \to \infty$.

(2) $\vec{u}(0) \notin \pm\widetilde{\mathcal{M}}$, $T < \infty$ *and* $\vec{u}(t) \notin O_2$ *near* $t = T$.
(3) $\vec{u}(0) \in \pm\widetilde{\mathcal{M}}$ *and* $\vec{u}(t) \in O_1$ *near* $t = T$.

If u *is a solution satisfying* $\vec{u}(t) = S(t) + \vec{v}(t) + o(1)$ *in* $\mathcal{H}$ *as* $t \to \infty$ *for some* $S : [0, \infty) \to Sol(W)$ *and some free solution* v*, then* $\vec{u}(0) \in \widetilde{\mathcal{M}}$.

Can we also include those solutions of (1) with $N = 1$ and $T < \infty$ on a threshold manifold? The following simple observation shows that it is impossible.

Proposition 13. [8] *Let* $d = 3$*. For any* $A > 0$*, there is a blowup in the form* (1) *with* $N = 1$*, such that the distance in* $\mathcal{H}$ *to any forward global solution is bigger than* A.

See [8] for more detail and the proofs.

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Global solutions of the Euler-Maxwell two-fluid system in 3D

Benoit Pausader
(joint work with Yan Guo, Alexandru Ionescu)

We consider the Euler-Maxwell system in 3D

$$\begin{aligned}
\partial_t n_e + \mathrm{div}(n_e v_e) &= 0 \\
n_e m_e \left(\partial_r v_e + v_e \nabla v_e\right) + \delta p_e &= -n_e e \left(E + \frac{v_e}{c} \times B\right) \\
\partial_t n_i + \mathrm{div}(n_i v_I) &= 0 \\
n_i M_i \left(\partial_t v_i + v_i \nabla v_i\right) + \nabla p_i &= Z n_i e \left(E + \frac{v_i}{c} \times B\right) \\
\partial_t B + c\nabla \times E &= 0 \\
\partial E - c\nabla \times B &= 4\pi e \left(n_e v_e - Z n_i v_i\right)
\end{aligned}$$

together with the elliptic equations

$$\mathrm{div}(B) = 0 \qquad \mathrm{div}(E) = 4\pi e(Z n_i - n_e).$$

This models the behavior of a plasma consisting of a fluid of electrions (modeled by a density $n_e \geq 0$ and a velocity $v_e \in \mathbb{R}^3$) and a fluid of ions (modeled by n_i, v_i) interacting with the electromagnetic field (E, B) produced by their charges. We shoiw that a small, smooth, irrotational and localized perturbation goes back to an equilibrium as $t \to \infty$ while remaining smooth. The main challenge is to study quadratic interactions which are strongly resonant. The key information is that the quadratic phase is nondegenerate on the resonant set.

The Gross-Pitaevskii hierarchy on the three-dimensional torus

Vedran Sohinger
(joint work with Philip Gressman, Gigliola Staffilani)

We study the Gross-Pitaevskii hierarchy on the spatial domain $\mathbb{T}^3$. We first prove a conditional uniqueness result for the hierarchy. The conditional uniqueness is proved in a class of density matrices which contains the factorized solutions that come from the nonlinear Schrödinger equation. We then add randomness into the model by means of the collision operator. As a result of the randomization, we can obtain estimates for a wider range of regularity exponents. By applying these estimates, we can analyze the limiting behavior of randomized Duhamel expansions in low regularities.

1. The Gross-Pitaevskii hierarchy

For a fixed spatial domain $\Lambda = \mathbb{T}^d$ or $\Lambda = \mathbb{R}^d$, the *Gross-Pitaevskii hierarchy on* Λ is defined to be a sequence $(\gamma^{(k)}(t))_k$ of functions $\gamma^{(k)} : \mathbb{R} \times \Lambda^k \times \Lambda^k \to \mathbb{C}$, also known as *time-dependent density matrices of order* k, which solve the following infinite system of linear PDEs:

$$(1) \qquad \begin{cases} i\partial_t\gamma^{(k)} + (\Delta_{\vec{x}_k} - \Delta_{\vec{x}'_k})\gamma^{(k)} = \sum_{j=1}^k B_{j,k+1}(\gamma^{(k+1)}) \\ \gamma^{(k)}|_{t=0} = \gamma_0^{(k)}. \end{cases}$$

Here, $\Delta_{\vec{x}_k} := \sum_{j=1}^k \Delta_{x_j}$ and $\Delta_{\vec{x}'_k} := \sum_{j=1}^k \Delta_{x'_j}$. The operator $B_{j,k+1}$ is called the *collision operator* and it is given by:

$$B_{j,k+1}(\gamma^{(k+1)})(\vec{x}_k;\vec{x}'_k) := \int_\Lambda dx_{k+1}\big(\delta(x_j-x_{k+1})-\delta(x'_j-x_{k+1})\big)\gamma^{(k+1)}(\vec{x}_k,x_{k+1};\vec{x}'_k,x_{k+1}).$$

We also write this expression as $B^+_{j,k+1}(\gamma^{(k+1)})(\vec{x}_k;\vec{x}'_k) - B^-_{j,k+1}(\gamma^{(k+1)})(\vec{x}_k;\vec{x}'_k)$. The Gross-Pitaevskii hierarchy is related to the nonlinear Schrödinger equation. Namely, suppose that ϕ solves the cubic nonlinear Schrödinger equation on Λ:

$$\begin{cases} i\partial_t\phi + \Delta\phi = |\phi|^2\phi \\ \phi|_{t=0} = \phi_0. \end{cases}$$

Then, $\gamma^{(k)}(t,\vec{x}_k;\vec{x}'_k) = |\phi\rangle\langle\phi|^{\otimes k}(t,\vec{x}_k;\vec{x}'_k) := \prod_{j=1}^k \phi(t,x_j)\overline{\phi(t,x'_j)}$ solves (1). These are defined to be the *factorized solutions.*

Throughout our work, we will consider the case when the spatial domain $\Lambda = \mathbb{T}^3$.

2. An unconditional uniqueness result

We say that a sequence $\Gamma(t) = (\gamma^{(k)}(t))_k$ is *syymetric* if the following identities hold for all $t \in \mathbb{R}, k \in \mathbb{N}, (\vec{x}_k,\vec{x}'_k) \in \Lambda^k \times \Lambda^k$:

i) $\gamma^{(k)}(t,\vec{x}_k;\vec{x}'_k) = \overline{\gamma^{(k)}(t,\vec{x}'_k;\vec{x}_k)}$

ii) $\gamma^{(k)}(t,x_{\sigma(1)},\ldots,x_{\sigma(k)};x'_{\sigma(1)},\ldots,x'_{\sigma(k)}) = \gamma^{(k)}(t,x_1,\ldots,x_k;x'_1,\ldots,x'_k)$ for all $\sigma \in S_k$.

Given $\alpha \in \mathbb{R}$, let $\mathcal{A}_\alpha$ denote the class of symmetric $\Gamma(t) = (\gamma^{(k)}(t))_k$ such that there exist continuous, positive functions $\sigma, f : \mathbb{R} \to \mathbb{R}$ with the property that for all $k \in \mathbb{N}$, $j = 1,\ldots,k$, $t \in \mathbb{R}$, one has:

$$\int_{t-\sigma(t)}^{t+\sigma(t)} \|S^{(k,\alpha)}B_{j,k+1}(\gamma^{(k+1)})(s)\|_{L^2(\Lambda^k\times\Lambda^k)}ds \leq f^{k+1}(t).$$

Here, $S^{(k,\alpha)} := \prod_{j=1}^k (1-\Delta_{x_j})^{\frac{\alpha}{2}}(1-\Delta_{x'_j})^{\frac{\alpha}{2}}$ denotes differentiation of order α on density matrices of order k.

The first result that we prove is:

Theorem 2.1. *(*[9]*) Solutions to the Gross-Pitaevskii hierarchy* (1) *are unique in the class $\mathcal{A}_\alpha$ for $\alpha > 1$. $\mathcal{A}_\alpha$ is non-empty when $\alpha \geq 1$.*

This result is joint work with Philip Gressman and with Gigliola Staffilani.

2.1. **Main ideas of the proof.** The proof of the uniqueness result is based on the combinatorial analysis developed by Klainerman and Machedon [14] in the study of the problem on $\mathbb{R}^3$. In particular, we use the *boardgame argument* to reduce the number of Duhamel terms. Then, we need to prove a spacetime estimate in order to bound the obtained expression. The exact nature of the analysis will be different due to the fact that we are now working on a periodic domain.

This first result needed to apply this strategy is the following *"Strichartz-type estimate"*:

Lemma 2.2. *(Spacetime estimate) For $\alpha > 1$, there exists $C_1 = C_1(\alpha) > 0$ such that, for all $k \in \mathbb{N}$, and for all $j \in \{1, \ldots, k\}$, one has:*

$$\|S^{(k,\alpha)} B_{j,k+1} \mathcal{U}^{(k+1)}(t) \gamma_0^{(k+1)}\|_{L^2([0,2\pi] \times \Lambda^k \times \Lambda^k)} \leq C_1 \|S^{(k+1,\alpha)} \gamma_0^{(k+1)}\|_{L^2(\Lambda^{k+1} \times \Lambda^{k+1})}.$$

The range $\alpha > 1$ is sharp in this estimate.

In the above lemma, $\mathcal{U}^{(k)}(t)\gamma_0^{(k)} := e^{it\sum_{j=1}^k \Delta_{x_j}} \gamma_0^{(k)} e^{-it\sum_{j=1}^k \Delta_{x'_j}}$ denotes the *free evolution* operator. Lemma 2.2 is proved by counting lattice points using the notion of the *determinant of a lattice* and applying the analysis developed in the work of P. McMullen [16]. The sharpness of the condition $\alpha > 1$ is obtained by the construction of an explicit counterexample to the estimate when $\alpha = 1$. This is in contrast to the $\mathbb{R}^3$ case [14], where the analogous estimate holds for all $\alpha \geq 1$.

The second result states that the class $\mathcal{A}_\alpha$ contains the physically relevant factorized solutions:

Lemma 2.3. *(Verification of the spacetime bound) The factorized solution $\Gamma(t) = (|\phi\rangle\langle\phi|^{\otimes k}(t))$ belongs to $\mathcal{A}_\alpha$ whenever $\alpha \geq 1$.*

An analogue of Lemma 2.3 on $\mathbb{R}^3$ was proved in [14] by the use of Strichartz estimates on $\mathbb{R}^3$. It is not possible to directly adapt this argument to the periodic setting by directly using the known Strichartz estimates on $\mathbb{T}^3$. We can circumvent this difficulty by using the multilinear analysis of Herr, Tataru, and Tzvetkov [11], which is applied in the atomic space framework developed previously by Koch and Tataru [12].

Our uniqueness result is a potential first step in the program of the rigorous derivation of the nonlinear Schrödinger equation from N-body Schrödinger dynamics in the periodic setting. A rigorous derivation was given in the in the non-periodic setting in the work of Erdős, Schlein, and Yau [4, 5, 6, 7, 8]. We note that this program was carried through on $\mathbb{T}^2$ by Kirkpatrick, Schlein and Staffilani [13].

3. Randomization and the Gross-Pitaevskii hierarchy

This section is joint work with Gigliola Staffilani. We examine what happens when we add randomness into the model in the hope of extending the deterministic analysis to a wider range of regularity exponents. This heuristic is based on the probabilistic analysis of nonlinear dispersive equations pioneered in the previous work of Lebowitz, Rose and Speer [15], Zhidkov [18], Bourgain [1, 2], and Burq and Tzvetkov [3] and others.

Let us fix $(h_\xi(\omega))_{\xi \in \mathbb{Z}^3}$ to be a sequence of independent, identically distributed standard Bernoulli random variables and we define the *randomized collision operator* $[B_{j,k+1}]^\omega$ as follows:

$$([B^+_{1,k+1}]^\omega \gamma_0^{(k+1)})\widehat{\ }(\vec{\xi}_k; \vec{\xi}'_k) :=$$

$$\sum_{\xi_{k+1},\, \xi'_{k+1} \in \mathbb{Z}^3} h_{\xi_1}(\omega) \cdot h_{\xi_1 - \xi_{k+1} + \xi'_{k+1}}(\omega) \cdot h_{\xi_{k+1}}(\omega) \cdot h_{\xi'_{k+1}}(\omega)$$

$$\cdot \widehat{\gamma_0}^{(k+1)}(\xi_1 - \xi_{k+1} + \xi'_{k+1}, \xi_2, \dots, \xi_k, \xi_{k+1}; \xi'_1, \dots, \xi'_k, \xi'_{k+1}).$$

The general $[B^\pm_{j,k+1}]^\omega$ is defined similarly. For such a collision operator, we can prove the following estimate:

Theorem 3.1. *([17]) For $\alpha > \frac{3}{4}$, there exists $C_2 = C_2(\alpha)$ such that for all $k \in \mathbb{N}$ and $j \in \{1, 2, \dots, k\}$:*

$$\big\|S^{(k,\alpha)}[B_{j,k+1}]^\omega \gamma_0^{(k+1)}\big\|_{L^2(\Omega \times \Lambda^k \times \Lambda^k)} \leq C_2 \big\|S^{(k+1,\alpha)} \gamma_0^{(k+1)}\big\|_{L^2(\Lambda^{k+1} \times \Lambda^{k+1})}.$$

We note that in the above theorem, the norm doesn't involve any integrals in time, which is a stronger result. Moreover, we note that, in the context of randomized collision operators, the range of admissible regularity exponents has been extended to $\alpha > \frac{3}{4}$.

We can use the randomized collision operator and the result of Theorem 3.1 in order to study the *independently randomized Gross-Pitaevskii hierarchy:*

$$\begin{cases} i\partial_t \gamma^{(k)} + (\Delta_{\vec{x}_k} - \Delta_{\vec{x'}_k})\gamma^{(k)} = \sum_{j=1}^k [B_{j,k+1}]^{\omega_{k+1}}(\gamma^{(k+1)}) \\ \gamma^{(k)}|_{t=0} = \gamma_0^{(k)}. \end{cases} \tag{2}$$

Here, we use the notation $[B_{j,k+1}]^{\omega_{k+1}}$ to denote that the randomizations at each level k are mutually independent. We consider homogeneous initial data, i.e. we take $\gamma_0^{(k)} = 0$ for all k. For $\sigma^{(k)}_{n;\, \omega_{k+1}, \dots, \omega_{k+n}} : \mathbb{R}_t \times \Lambda^k \times \Lambda^k \to \mathbb{C}$, which denote appropriate Duhamel expansions in (2) of length n, we can show the following result:

Theorem 3.2. *([17]) Suppose that $\alpha > \frac{3}{4}$ and $k \in \mathbb{N}$. There exists $T > 0$ such that:*

$$\sup_{t \in [0,T]} \big\|S^{(k,\alpha)} \sigma^{(k)}_{n;\, \omega_{k+1}, \omega_{k+2}, \dots, \omega_{n+k}}(t)\big\|_{L^2\left(\prod_{m \geq 2} \Omega_m;\, L^2(\Lambda^k \times \Lambda^k)\right)} \to 0$$

as $n \to \infty$.

In [17], we also study the *dependently randomized Gross-Pitaevskii hierarchy*, which is given by:

$$\begin{cases} i\partial_t \gamma^{(k)} + (\Delta_{\vec{x}_k} - \Delta_{\vec{x'}_k})\gamma^{(k)} = \sum_{j=1}^k [B_{j,k+1}]^{\omega}(\gamma^{(k+1)}) \\ \gamma^{(k)}\big|_{t=0} = \gamma_0^{(k)}. \end{cases} \tag{3}$$

This hierarchy has the advantage that, in this context, one can construct factorized solutions. It has the disadvantage that it is no longer possible to iteratively apply the randomized estimate from Theorem 3.1. We can again consider Duhamel expansions $\sigma^{(k)}_{n;\,\omega}$ constructed similarly as before, but now we have to add some *non-resonance conditions*. These conditions resemble the *Wick ordering* used in [2]. We can then prove estimates for Duhamel expansions of arbitrary length, thus avoiding a direct application of the bound from Theorem 3.1. For (3), we can prove a result similar to Theorem 3.2. More precisely, we can prove:

Theorem 3.3. *Suppose that $\alpha \geq 0$ and $k \in \mathbb{N}$. There exists $T > 0$ such that:*

$$\sup_{t\in[0,T]} \left\| S^{(k,\alpha)} \sigma^{(k)}_{n;\,\omega}(t) \right\|_{L^2\left(\Omega\times\mathbb{T}^{3k}\times\mathbb{T}^{3k}\right)} \to 0$$

as $n \to \infty$.

We note that in Theorem 3.3, the range of α has been extended to $\alpha \geq 0$ and that the norm in which one has convergence is simpler.

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