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C*-Algebren

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Abstract. C*-algebras play an important role in many modern areas of mathematics, like Noncommutative Geometry and Topology, Dynamical Systems, Harmonic Analysis and others. The conference "C*-algebras" brings together leading experts from those areas in order to strengthen the cooperation and to keep the researchers informed about major developments in the field.

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Introduction by the Organisers

The theory of C*-algebras has its origins in the pioneering work of Murray and von Neumann on "rings of operators" which are now known as von Neumann algebras and which represent noncommutative measure spaces. The start of the field of abstract C*-algebras has been given in 1943 by two fundamental theorems of Gelfand and Naimark: One shows that (unital) commutative C*-algebras are always isomorphic to algebras of continuous functions on (compact) topological spaces, the other shows that every abstract C*-algebra can be represented faithfully as a closed $*$-subalgebra of the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H. The first result shows that general C*-algebras can be viewed philosophically as function spaces of (imaginary) "noncommutative spaces", the second result reflects the high rigidity properties of C*-algebras and provides important tools for their investigation.

The motivation for Gelfand and Naimark to study C*-algebras most likely came at least in part from their interest in studying unitary representations of locally compact groups G, which are in one-to-one correspondence to the $*$-representations of the C*-group algebra $C^*(G)$ which can be attached to G as a suitable C*-completion of the convolution algebra $L^1(G)$. This observation serves as a prototype for a general principle in the field: given a mathematical object (like groups, semigroups, rings, equivalence relations, dynamical systems, groupoids, graphs, metric spaces, to name a few) one can construct a C*-algebra or von Neumann algebra that reflects many of the properties of the given object. Moreover, the topological invariants for C*-algebras, in particular given by K-theory, often provide new invariants for the original objects of study. It is evident that this vast number of possible applications of the theory in different areas of mathematics leads to various specializations of the researchers in the field.

The aim of the Oberwolfach conference "C*-algebras", this year organized by Siegfried Echterhoff (Münster), Mikael Rørdam (Copenhagen), Stefaan Vaes (Leuven) and Dan Voiculescu (Berkeley), is to provide a frame for regular meetings of leading experts in all different directions of the field to keep the contact and to strengthen research cooperations between different areas in the field. At this meeting we had a total of 30 lectures on topics like C*-algebras attached to rings and semigroups, general classification theory of C*-algebras, entropy for dynamical systems and L^2-torsion, graph algebras, Smale spaces, free probability theory, classification of quantum groups, various topics in von Neumann algebra theory including a lecture on classification of crossed product II_1 factors and group-type subfactors, character rigidity of discrete groups, continuous bundles of C*-algebras and von Neumann algebras, amenability of (exotic) groups, automorphism groups of C*-algebras, C*-algebras and model theory, noncommutative Lipschitz geometry for boundaries of hyperbolic spaces and spectral triples, pseudodifferential calculus and groupoid-algebras, perturbation theory for C*-algebras and von Neumann algebras, noncommutative L^p-spaces, and others.

Among the many highlights, we would like to mention in particular the remarkable result due to Hanfeng Li and Andreas Thom (presented by Li) where they give a proof of a conjecture by Deninger on the computation of the entropy for algebraic group actions: it gives a beautiful formula for the entropy of such actions in terms of the Fuglede-Kadison determinant.

Another highlight was presented by Xin Li on semigroup algebras attached to integral domains, which grew out of an extended joint project with Joachim Cuntz and others. Xin Li gives a very detailed description of the structure and the topological invariants of these algebras and he can show that under some extra conditions (e.g. for rings of integers in number fields) the ring can be recovered from the algebra – hence the C*-algebra contains all information of the ring.

Wilhelm Winter gave a report on major progress in dimension theory for C*-algebras and Elliott's classification program. The results show in particular that crossed products by $\mathbb{Z}^n$-actions on finite dimensional compact spaces always have finite nuclear dimension – which implies that for uniquely ergodic actions of $\mathbb{Z}^n$ on finite dimensional compact spaces the crossed product is classifiable by K-theoretic data as soon as it is quasidiagonal (earlier results of Winter and Toms show that the latter assumption is not needed for $\mathbb{Z}$-actions). These results are a cumulation of work by several researchers including Kirchberg, Matui, Rørdam, Sato, Szabo, Tikuisis, Toms and Winter (to name just a few).

Jesse Peterson presented striking results on a conjecture of Alain Connes describing all II_1 factor representations of groups "of higher rank", including his joint work with Andreas Thom confirming the conjecture for special linear groups over certain rings.

Kate Juschenko reported on her joint work with Nekrashevych and de la Salle, providing a unified proof for the amenability of a wide class of discrete groups, including the full topological groups of Cantor minimal systems and groups acting on rooted trees by bounded automorphisms.

A further highlight of the conference was the results presented in the lecture of Yoann Dabrowski. He has succeeded in overcoming some of the enormous difficulties posed by noncommutativity in the stochastic differential equations of free probability theory, related to free entropy and free group factors. Most recently he combined these powerful techniques in joint work with Adrian Ioana with other powerful techniques, those of deformation/rigidity, towards applications to von Neumann algebras.

The organizers thank the participants for a number of wonderful lectures and for a very active and friendly atmosphere during the conference. Looking around in the afternoon break or in the evening we saw people discussing math at all places, and almost all lecture rooms in the library were occupied by small groups discussing their projects and ideas. We hope that the outcome of these discussions will be visible in new publications that will give new input to a possible future conference on C*-algebras.

Workshop: C*-Algebren

Table of Contents

Erik Christensen and Stuart White
Close operator algebras and almost multiplicative maps ... 2425

Yoann Dabrowski (joint with A. Ioana)
Wasserstein Rigidity and applications of non-microstates free entropy to von Neumann algebras ... 2430

Marius Dadarlat (joint with Ulrich Pennig)
Invariants of continuous fields with strongly self-absorbing fibers ... 2432

George A. Elliott (joint with Zhuang Niu)
A canonical Pimsner-Voiculescu embedding ... 2434

Heath Emerson (joint with Bogdan Nica)
Noncommutative Lipschitz geometry and hyperbolic groups ... 2437

Ruy Exel
Graphs, groups and self-similarity ... 2439

Ilijas Farah (joint with B. Hart, L. Robert, A. Tikuisis, A. Toms, W. Winter)
Logic of metric structures and nuclear C-algebras* ... 2440

Uffe Haagerup (joint with Hiroshi Ando, Carl Winsløw)
Ultraproducts, QWEP von Neumann algebras and the Effros-Maréchel topology ... 2443

Ilan Hirshberg (joint with N. C. Phillips)
Rokhlin dimension for finite group actions ... 2444

Cyril Houdayer (joint with Sven Raum)
Baumslag-Solitar groups, relative profinite completions and measure equivalence rigidity ... 2445

Kate Juschenko (joint with V. Nekrashevych and Mikael de la Salle)
Amenability of groups acting by homeomorphisms on compact spaces ... 2448

David Kerr (joint with Martino Lupini, N. Christopher Phillips and Wilhelm Winter)
Borel complexity and automorphism groups ... 2450

Eberhard Kirchberg
C^*-correspondences related to Dini spaces ... 2452

Hanfeng Li (joint with Andreas Thom)
Entropy and L^2-torsion ... 2453

Xin Li
C^*-algebras for semigroups coming from integral domains 2455

Ralf Meyer (joint with Rasmus Bentmann)
Classifying circle actions up to equivariant KK-equivalence 2458

Hervé Oyono-Oyono (joint with Guoliang Yu)
Persistent Approximation Property for C^*-algebras with propagation . . . 2459

Narutaka Ozawa
Continuous bundles of tracial von Neumann algebras 2463

Jesse Peterson
Character rigidity and its consequences 2464

N. Christopher Phillips
Analogs of UHF algebras and Cuntz algebras on L^p spaces 2467

Ian Putnam
A homology theory for Smale spaces 2470

Sven Raum (joint with Moritz Weber)
The classification of orthogonal easy quantum groups 2471

Dimitri Shlyakhtenko (joint with A. Guionnet)
Free monotone transport 2475

Georges Skandalis (joint with Claire Debord)
Pseudodifferential Calculus and Groupoids 2478

Roland Speicher (joint with Serban Belinschi, Tobias Mai)
Polynomials in free variables 2481

Andreas Thom (joint with Philip Dowerk, Abel Stolz, John Wilson)
Ultraproducts of compact simple groups 2483

Aaron Tikuisis (joint with Leonel Robert)
Central sequences, dimension, and $\mathcal{Z}$-stability of C^*-algebras 2485

Stefaan Vaes
Classification of crossed product II_1 factors and group-type subfactors . . 2487

Wilhelm Winter
Classifying crossed products 2490

Makoto Yamashita (joint with Kenny De Commer, Amaury Freslon)
Central approximation properties of free quantum groups 2491

Abstracts

Close operator algebras and almost multiplicative maps

Erik Christensen and Stuart White

1. Close operator algebras — invariants

This section of the report is concerned with close operator algebras. In [7], Kadison and Kastler equipped the collection of operator algebras acting on a fixed Hilbert space $\mathcal{H}$ with a metric d, which arises from restricting the Hausdorff metric on subsets of $\mathcal{B}(\mathcal{H})$ to the unit balls of C^*-subalgebras. Precisely, given C^* or von Neumann algebras $A, B \subset \mathcal{B}(\mathcal{H})$, we have

$$d(A,B) = \max\left(\sup_{\substack{a\in A \\ \|a\|\leq 1}} \inf_{\substack{b\in B \\ \|b\|\leq 1}} \|a-b\|, \sup_{\substack{b\in B \\ \|b\|\leq 1}} \inf_{\substack{a\in A \\ \|a\|\leq 1}} \|b-a\| \right).$$

The main focus of investigation is 'what can be said if $d(A,B)$ is small'?

- Can we show that sufficiently close operator algebras share the same properties and invariants?
- Must such algebras be isomorphic?

In the study of operator algebras, it is always natural to consider matrix amplifications. In our context this leads to the *completely bounded* version of the Kadison-Kastler metric:

$$d_{cb}(A,B) = \sup_n d(A \otimes M_n, B \otimes M_n),$$

where $A, B \subset \mathcal{B}(\mathcal{H})$, and the distance between $A \otimes M_n$ and $B \otimes M_n$ is measured in $\mathcal{B}(\mathcal{H}) \otimes M_n \cong \mathcal{B}(\mathcal{H}^n)$. We also investigate the consequences of "complete closeness", i.e. what can be said when $d_{cb}(A,B)$ is small?

For example, if $d_{cb}(A,B)$ is small, then any projection $p \in A \otimes M_n$ can be suitably approximated by a projection $q \in B \otimes M_n$, leading an isomorphism $K_0(A) \to K_0(B)$ which maps $[p]_0$ to $[q]_0$. This strategy is due to Khoshkam [8], who used it to obtain an isomorphism $K_*(A) \cong K_*(B)$ when $d(A,B)$ is sufficiently small and A is nuclear.

The Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A is a highly refined invariant obtained from equivalence classes of positive elements of $A \otimes \mathcal{K}$. In contrast to projections, where the class in K_0 is invariant under small perturbations, the Cuntz class of a positive element $a \in A \otimes \mathcal{K}$ is sensitive to small modifications in norm. Nevertheless, it is possible to extend Khoshkam's isomorphism to this setting.

Theorem 1 (Perera, Toms, W, Winter, [11])**.** *Let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be C^*-algebras with $d_{cb}(A,B) < 1/42$. Then $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$ in a scale preserving fashion.*

Given Khoshkam's work and Theorem 1 it is natural to ask when close algebras are automatically "completely close". This is resolved in the following theorem.

Theorem 2 (Cameron, C, Sinclair, Smith, W, Wiggins, [1] extending [4]). *The following are equivalent:*

(1) d and d_{cb} are equivalent metrics;
(2) Kadison's similarity problem has a positive answer, i.e. any bounded homomorphism $A \to \mathcal{B}(\mathcal{H})$ whose domain is a C^*-algebra is similar to a *-homomorphism.

This is also true locally: $d_{cb}(A, \cdot)$ and $d(A, \cdot)$ are equivalent if and only if A has the similarity property.

Using Theorem 2, we obtain the following corollaries of Theorem 1.

Corollary ([11]). *Let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be C^*-algebras with $d(A, B)$ sufficiently small and suppose that one of A or B has stable rank one. Then A is stable if and only if B is stable.*

Idea of proof. When one algebra, say A, is stable, it has the similarity property. It follows that $d_{cb}(A, B)$ is small when $d(A, B)$ is small, so that $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$ when $d(A, B)$ is small enough. By a result of Rørdam and Winter, the Cuntz semigroup of a C^*-algebra with stable rank one enjoys a certain cancelation property; further, in the presence of this cancelation property, stability of the C^*-algebra can be determined from the Cuntz semigroup and its scale. □

Corollary ([11]). *Let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be unital with $d_{cb}(A, B)$ sufficiently small (which happens when $d(A, B)$ is sufficiently small and $A \cong A \otimes \mathcal{Z}$). Then the natural affine isomorphism between $T(A)$ and $T(B)$ defined in* [4] *is a homeomorphism. Further, if A has the property that all bounded 2-quasitraces are traces, then so too does B.*

2. Close operator algebras — isomorphisms

We now turn to the question of whether sufficiently close operator algebras must be isomorphic. In the late 1970's EC, Johnson, Raeburn and Taylor established such results in the context of injective von Neumann algebras leading to the following theorem.

Theorem 3. *For all $\varepsilon > 0$, there exists $\delta > 0$ with the property that whenever $M, N \subseteq \mathcal{B}(\mathcal{H})$ are von Neumann algebras with M injective and $d(M, N) < \delta$, there exists a unitary operator $u \in (M \cup N)''$ with $uMu^* = N$ and $\|u - 1\| < \varepsilon$.*

The main strategy for obtaining an isomorphism between the algebras M and N in this theorem is performed in two main steps:

(1) Obtain a completely positive map $T : M \to N$ which is *almost multiplicative* in the sense that the bilinear map $T^\vee : M \times M \to N$ given by $T^\vee(x, y) = T(xy) - T(x)T(y)$ has small norm. If one additionally assumes that N is injective, one can take a conditional expectation $\Phi : \mathcal{B}(\mathcal{H}) \to N$ and set $T = \Phi|_M$.
(2) Prove that every almost multiplicative completely positive map $T : M \to N$ is close to a *-homomorphism $M \to N$.

The second step is an "AMNM" (almost multiplicative near multiplicative) type result, as later formalised by Johnson in [6]. In the context of separable nuclear C^*-algebras, a point norm version of these AMNM techniques can be used to prove the theorem below.

Theorem 4 ([5])**.** *There exists a constant $\gamma_0 > 0$, such that if $A, B \subseteq \mathcal{B}(\mathcal{H})$ are C^*-algebras with A separable and nuclear, and $d(A, B) < \gamma_0$, then there is a unitary $u \in (A \cup B)''$ with $uAu^* = B$.*

A key difference between Theorem 3 and 4 is that the unitary in Theorem 4 can not in general be taken close to 1 due to the examples of Johnson. On the other hand, later work of Johnson shows that the unitary can be taken close to 1 when A is n-subhomogeneous, for some $n \in \mathbb{N}$.

Question. *Exactly which separable nuclear C^*-algebras A have the following property? For all $\varepsilon > 0$, there exists $\delta > 0$, such that whenever A is faithfully represented on $\mathcal{H}$ and $B \subseteq \mathcal{H}$ is another C^*-algebra with $d(A, B) < \delta$, then there exists an isomorphism $\theta : A \to B$ with $\|\theta(x) - x\| \leq \varepsilon\|x\|$ for all $x \in A$.*

Outside the amenable setting, the AMNM technique has not been successfully applied to close algebras — the major obstruction is the first step above. What are methods for obtaining bounded linear maps between close (or completely close) operator algebras?

Instead we consider an alternative approach to obtaining an isomorphism between close algebras. Let M be a II_1 factor. Recall that a *Cartan masa* in M, is a maximal abelian subalgebra $A \subset M$ such that $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates M as a von Neumann algebra. The canonical example arises from the group-measure space construction: if $\alpha : \Gamma \curvearrowright (X, \mu)$ is a free ergodic action of a countable discrete group on a probability space, then $L^\infty(X)$ is a Cartan masa in the II_1 factor $L^\infty(X) \rtimes \Gamma$. In their seminal work, Feldman and Moore classified inclusions $A \subset M$ of Cartan masas up to unitary equivalence. They associated to $A \subseteq M$ a measurable equivalence relation $\mathcal{R}_{A\subseteq M}$ and the (orbit of) a certain class in the cohomology group $H^2(\mathcal{R}_{A\subseteq M}, \mathbb{T})$ and showed that this information classifies the inclusion $A \subseteq M$. In the canonical example where we start from a free ergodic action $\Gamma \curvearrowright (X, \mu)$, the measurable equivalence relation $\mathcal{R}$ is the orbit equivalence relation on X ($x \sim y$ iff there exists $g \in \Gamma$ with $g \cdot x = y$) and the cohomology group $H^2(\mathcal{R}, \mathbb{T})$ is identified with $H^2(\Gamma, L^\infty(X, \mathbb{T}))$.

Theorem 5 (Cameron, C, Sinclair, Smith, W, Wiggins, [2, 3])**.** *Let $A \subset M$ be a Cartan masa in a II_1 factor. Suppose M is represented as a von Neumann algebra on $\mathcal{H}$ and N is another von Neumann algebra on $\mathcal{H}$ such that $d(M, N)$ is sufficiently small. Then there exists a Cartan masa $B \subset N$ with $d(A, B)$ small, and $\mathcal{R}_{A\subset M} \cong \mathcal{R}_{B\subset N} \cong \mathcal{R}$. Further one can choose representative cocycles $\omega_{A\subset M}$ for $A \subset M$ and $\omega_{B\subset N}$ for $B \subset N$ in $Z^2(\mathcal{R}, \mathbb{T})$ to be "uniformly close".*

Thus, given a II_1 factor M containing a Cartan masa A and N as in Theorem 5, if one can additionally find a method for showing that $\omega_{A\subset M}$ and $\omega_{B\subset N}$ are cohomologous, then N must be isomorphic to M. The easiest way of ensuring this happens is when the relevant cohomology group vanishes, yielding the following corollary.

Corollary. *There exists $\gamma_0 > 0$ with the property that whenever $A \subset M$ is a Cartan masa in a II_1 factor such that $H^2(\mathcal{R}_{A\subset M}, \mathbb{T}) = \{1\}$, N is another von Neumann algebra and M, N are represented on the same Hilbert space with $d(M,N) < \gamma_0$, we have $N \cong M$. In particular this happens when M arises from a free ergodic measure preserving action of a free group of rank at least 2.*

Another strategy for proving that $\omega_{A\subset M}$ and $\omega_{B\subset N}$ are cohomologous is to use uniqueness of Cartan results. A countable discrete group Γ is called *Cartan rigid* if any free ergodic probability measure preserving action $\Gamma \curvearrowright (X,\mu)$ has the property that $L^\infty(X)$ is the unique Cartan masa in $L^\infty(X) \rtimes \Gamma$ upto conjugation by an inner automorphism. Recent years have seen remarkable progress in producing examples of Cartan rigid groups — in particular Popa and Vaes have shown that non-elementary hyperbolic groups are Cartan rigid [10].

Corollary. *There exists a constant $\gamma_0 > 0$ with the following property. Let Γ be a Cartan rigid countable discrete group and let $M = L^\infty(X) \rtimes \Gamma$ be a II_1 factor obtained from a free ergodic probability measure preserving action $\Gamma \curvearrowright (X,\mu)$. Let $\Lambda \curvearrowright (Y,\nu)$ be a free ergodic probability measure preserving action of any countable discrete group and $N = L^\infty(Y) \rtimes \Lambda$. If M and N can be represented on the same Hilbert space with $d(M,N) < \gamma_0$, then $M \cong N$.*

Our third strategy uses the additional information in Theorem 5 that $\omega_{A\subset M}$ and $\omega_{B\subset N}$ can be chosen "uniformly close". Consider a free ergodic probability measure preserving action $\Gamma \curvearrowright (X,\mu)$ for which the bounded group cohomology group $H^2_b(\Gamma, L^\infty_{\mathbb{R}}(X)) = 0$ (for example, Monod has shown that any free ergodic probability measure preserving action of $SL_n(\mathbb{Z})$ for $n \geq 3$ enjoys this property [9]). One can use this information as in the first corollary to obtain an isomorphism between $M = L^\infty(X) \rtimes \Gamma$ and any other von Neumann algebra N sufficiently close to M. In fact more is true: under these circumstances an isomorphism $\theta : M \to N$ can be found with $\sup_{\substack{x\in M \\ \|x\|\leq 1}} \|\theta(x) - x\|$ small. In the Corollary below, we are able to extend this further and obtain the first examples of non-injective von Neumann algebras which satisfy essentially the same conclusions as Theorem 3. The purpose of the tensor copy of R is to ensure that the factor M_0 has the similarity property allowing us to use the ideas behind Theorem 2. Indeed, the first step in the proof of the corollary is essentially to remove the tensor copy of R, by showing that N_0 also factorises as $N_0 \cong N \overline{\otimes} R$ in such a way that N is close to $L^\infty(X) \rtimes \Gamma$.

Corollary. *For all $\varepsilon > 0$, there exists $\delta > 0$ with the following property. Let $\Gamma \curvearrowright (X,\mu)$ be a free ergodic probability measure preserving action with $H^2_b(\Gamma, L^\infty_{\mathbb{R}}(X)) = 0$ and define $M_0 = (L^\infty(X) \rtimes \Gamma) \overline{\otimes} R$, where R is the hyperfinite II_1 factor. Suppose M_0 is represented as a von Neumann algebra on $\mathcal{H}$ and N_0 is another von Neumann algebra on $\mathcal{H}$ with $d(M_0, N_0) < \delta$. Then there exists a unitary u on $\mathcal{H}$ with $uM_0u^* = N_0$ and $\|u - 1\| < \varepsilon$.*

3. Almost multiplicative maps

Motivated by the point-norm AMNM techniques used in Theorem 4, in ongoing joint work with Allan Sinclair and Roger Smith, we have become interested in general results for almost multiplicative maps between operator algebras. The prototype is the following theorem of Johnson, which in particular says that every almost multiplicative map from a nuclear C^*-algebra into a von Neumann algebra is close to a homomorphism.

Theorem 6 (Johnson [6])**.** *Let A be an amenable Banach algebra with amenability constant L and B a dual Banach algebra. For $\varepsilon > 0$ and $K > 0$, any bounded linear map $T : A \to B$ with $\|T\| < K$ and $\|T^\vee\| < \varepsilon/(4L + 8K^2L^2)$, there is a bounded homomorphism $S : A \to B$ with $\|S - T\| < \varepsilon$.*

At present we have the following two results, demonstrating that almost multiplicative maps on nuclear C^*-algebras are near to homomorphisms in the point norm topology. The restriction that $\|T\|$ is not too large is a way of ensuring that our maps are almost self-adjoint.

Theorem 7. *Let A be a separable unital nuclear C^*-algebra and B a unital C^*-algebra. Given a bounded unital map $T : A \to B$ with $\|T\| \leq 1.1$ and $\|T^\vee\| \leq 0.01$, then there is a *-homomorphism $\Psi : A \to B$ such that for any finite set X in the unit ball of A, there is a unitary operator $u \in B$ such that*

$$\|T(x) - u^*\Psi(x)u\| \leq 3\big(\|T^\vee\| + \|T\| - 1\big), \quad x \in X. \tag{1}$$

Theorem 8. *Let A be a separable unital C^*-algebra which is the direct limit of semiprojective unital and nuclear C^*-algebras and let B be a unital C^*-algebra. Given a bounded unital map $T : A \to B$ with $\|T^\vee\|(\|T\|^2 + 2) < 0.2$, then there is a *-homomorphism $\Psi : A \to B$ such that for every finite set X of the unit ball of A, there is an invertible positive operator $t \in B$ with $(2\|T\|^2)^{-1}1_B \leq t \leq (2\|T\|^2)1_B$ such that*

$$\|T(x) - t^{-1}\Psi(x)t\| \leq 4\|T\|\|T^\vee\|, \quad x \in X. \tag{2}$$

The key difference between these results is that under the additional assumption that A is the limit of semiprojective nuclear C^*-algebras in Theorem 7, we do not require a universal bound on $\|T\|$; one can still obtain estimates for maps T of relatively large norm, provided $\|T^\vee\|$ is correspondingly small.

References

[1] J. Cameron, E. Christensen, A. Sinclair, R. Smith, S. White, A. Wiggins. A remark on the similarity and perturbation problems. *C. R. Math. Rep. Acad. Sci. Canada*, 35(2), 70–76, 2013.

[2] J. Cameron, E. Christensen, A. Sinclair, R. Smith, S. White, A. Wiggins. Kadison-Kastler stable factors. arXiv:1209.4116.

[3] J. Cameron, E. Christensen, A. Sinclair, R. Smith, S. White, A. Wiggins. Type II_1 factors satisfying the spatial isomorphism conjecture. *Proc. Natl. Acad. Sci. USA*, 109(5), 20338–20343, 2012.

[4] E. Christensen, A. Sinclair, R. Smith, S. White. Perturbations of C^*-algebraic invariants. *Geom. Funct. Anal.*, 20, 368–397, 2010.
[5] E. Christensen, A. Sinclair, R. Smith, S. White, W. Winter. Perturbations of nuclear C^*-algebras. *Acta. Math.*, 230(3), 93–150.
[6] B. Johnson. Approximately multiplicative maps between Banach algebras. *J. London Math. Soc. (2)*, 37, 294–316, 1988.
[7] R. Kadison, D. Kastler. Perturbations of von Neumann algebras I Stability of type. *Amer. Math. J.*, 94(1), 38–54, 1972.
[8] M. Khoshkam. Perturbations of C^*-algebras and K-theory. *J. Oper. Thy*, 12(1):89–99, 1984.
[9] N. Monod. On the bounded cohomology of semi-simple groups, S-arithmetic groups and products. *J. reine angew. Math.*, 640, 167–202, 2010.
[10] S. Popa, S. Vaes, Unique Cartan decomposition for arbitrary actions of hyperbolic groups. *J. reine angew. Math.*, in press, arXiv:1201.2824.
[11] F. Perera, A. Toms, S. White, W. Winter. The Cuntz semigroup and stability of close C^*-algebras. *Anal. PDE*, to appear, arXiv:1210.4533.

Wasserstein Rigidity and applications of non-microstates free entropy to von Neumann algebras

Yoann Dabrowski
(joint work with A. Ioana)

On the one hand, in the framework of Popa's deformation/rigidity Theory, Jesse Peterson [9] introduced a really weak form of rigidity of a II_1 factor he called L^2 rigidity (implied by property Gamma or (T) and non-primeness) using deformations coming from closable derivations valued in the coarse correspondence.

On the other hand, Voiculescu's non-microstates free entropy [14] gives a quantitative estimate on how a specific derivation, the free difference quotient, is approximated by closable derivations (when it may not be closable) and finiteness of non-microstates free entropy should imply similar properties than non-L^2 rigidity from the conjectural equality with microstates free entropy.

In this talk, we explain recent advances in obtaining indecomposibility results for II_1 factors in case of finite non-microstates free entropy or under related assumptions, using variants of L^2-rigidity.

Let us give some background on free entropy. In a fundamental series of papers [12, 13, 14], Voiculescu introduced analogs of entropy and Fisher information in the context of free probability theory. A first microstates free entropy $\chi(X_1, ..., X_n)$ is defined as a normalized limit of the volume of sets of microstates i.e. matricial approximants (in moments) of the n-tuple of self-adjoints X_i living in a finite von Neumann algebra M. Voiculescu and its followers then proved $\chi(X_1, ..., X_n) > -\infty$ implies the von Neumann algebra $W^*(X_1, ..., X_n)$ is a prime full II_1-factor without Cartan subalgebra or property (T) (see [13, 6, 11] and the survey [16]). Recall for instance that a II_1 factor M is *prime* if it cannot be written as the tensor product $M = M_1 \bar{\otimes} M_2$ of two II_1 factors. Also, a maximal abelian von Neumann subalgebra $A \subset M$ is a *Cartan subalgebra* if its normalizer, $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) | uAu^* = A\}$, generates a weakly dense subalgebra of M. Unfortunately, microstates free entropy is hard to compute on concrete examples.

Starting from a definition of a free Fisher information, the squared L^2-norm of the free score functions, the so-called conjugate variable, Voiculescu [14] also defined a non-microstates free entropy $\chi^*(X_1, ..., X_n)$, known by the fundamental work [1] to be greater than the previous microstates entropy, and believed to be equal (at least modulo Connes' embedding conjecture). Moreover it is easier to compute since non-microstates free entropy dimensions $\delta^*(X_1, ..., X_n)$, used to quantify the case of infinite entropy and with similar von Neumann algebraic applications, can be computed in terms of L^2 Betti numbers for (real and imaginary parts of) generators of (finitely generated) groups in which case [8] $\delta^*(\mathbb{C}\Gamma) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$. The general goal of this research is to give applications of non-microstates free entropy to von Neumann algebras using advances in Popa's Deformation/Rigidity Theory and free probability.

In a recent joint work with Adrian Ioana [4], we used a finite Fisher information $\Phi^*(X_1, ..., X_n)$ in case $n \geq 3$ (i.e. closable free difference quotient) to get $W^*(X_1, ..., X_n)$ is non-L^2 rigid II_1 factor, and thus, using [9], deduce primeness, and absence of property Γ or (T) in this case. Using recent results in [7, 10], we also obtained absence of Cartan subalgebras in $W^*(X_1, ..., X_n) \otimes B$ for any factor B if $X_1, ..., X_n$ has Lipschitz conjugate variable, or for algebras of the form $W^*(X_1 + \epsilon S_1, ..., X_n + \epsilon S_n)$ for $\epsilon > 0$ and $(S_1, ..., S_n)$ a free semicircular family.

It is crucial in order to get applications under finite non-microstates free entropy, to go beyond the case of a closable free difference quotient. Especially, in this case, there is no semigroup available as used in [9]. Inspired by estimates between non-microstates free entropy and Wasserstein distance [2, 3], we introduce [5] a notion of "Wasserstein rigidity" similar to Peterson's L^2 rigidity but relying on general couplings in the spirit of the definition of Wasserstein distance instead of semi-groups. Of course, we have to impose specific assumptions on various bimodules appearing in these couplings, namely compactness and weak containment in the coarse bimodule. We can exploit free products with amalgamation as a substitute to the semigroup property.

Then, using ideas of our joint work with Adrian Ioana and time reversal of free Stochastic Differential Equations, we can prove [5] that finite free entropy (and $n \geq 3$ or extra non-amenability assumptions) imply non-Wasserstein rigidity. We especially prove that finite non-microstates free entropy implies non-Gamma and primness as soon as $n \geq 3$. We also give applications of finite non-microstates mutual information [15] $i^*(A_1, ..., A_n) < \infty$ under extra assumptions (the main one being non-amenability of one algebra A_i), proving it implies primness of $W^*(A_1, ..., A_n)$. Finally, we obtain variants for entropy relative to a subalgebra B.

References

[1] P. Biane, M. Capitaine, A. Guionnet: *Large deviation bounds for matrix brownian motion,* Invent. Math. **152** (2003), 433-459.

[2] P. Biane, D. Voiculescu. *A Free Probability Analogue of the Wasserstein Metric on the Trace-State Space.* Geom. Func. Anal., **11**:1125–1138, 2001.

[3] Y. Dabrowski: *A non-commutative Path Space approach to stationary free Stochastic Differential Equations*, preprint arXiv:1006.4351.

[4] Y. Dabrowski, A. Ioana: *Unbounded derivations, free dilations and indecomposability results for II_1 factors*, preprint arXiv:1212.6425.
[5] Y. Dabrowski : *Time Reversal of free diffusions and Indecomposability results under finite non-microstates free entropy*, (In preparation).
[6] L. Ge, *Applications of free entropy to finite von Neumann algebras.* II. Ann. of Math. **147** (1998), 143-157.
[7] A. Ioana: *Cartan subalgebras of amalgamated free product II_1 factors*, preprint arXiv:1207.0054.
[8] I. Mineyev and D. Shlyakhtenko. *Non-microstate Free Entropy Dimension for Groups.* Geom. Func. Anal., **15**:476–490, 2005.
[9] J. Peterson: *L^2–rigidity in von Neumann algebras*, Invent. Math. **175** (2009), no. 2, 417–433.
[10] S. Popa, S. Vaes: *Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups*, preprint arXiv:1111.6951.
[11] K. Jung , D. Shlyakhtenko. *All generating sets of all property T von Neumann algebras have free entropy dimension* ≤ 1. J. Noncommut. Geom. **1** (2007), no. 2, 271–279.
[12] D. Voiculescu: *The analogues of entropy and of Fisher's information measure in free probability theory*, II, Invent. Math. **118** (1994), 411-440.
[13] D. Voiculescu: *The analogues of entropy and of Fisher's information measure in free probability theory*, III. Geom. Funct. Anal. **6** (1996), 172-199.
[14] D. Voiculescu: *The analogues of entropy and of Fisher's information measure in free probabilility*, V, Invent. Math. **132** (1998), 189- 227.
[15] D. Voiculescu: *The analogs of entropy and of Fisher's information measure in free probability theory*, VI: *Liberation and Mutual free Information*, Adv. Math., **146** (1999), 101-166.
[16] D. Voiculescu. *Free Entropy.* Bulletin of the London Mathematical Society, **34**(3):257–278, 2002.

Invariants of continuous fields with strongly self-absorbing fibers

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(joint work with Ulrich Pennig)

In a classic paper [1], Dixmier and Douady studied the separable continuous fields of C^*-algebras with fibers isomorphic to the compact operators $\mathcal{K}$. In joint work with U. Pennig we extend significantly the results of Dixmier and Douady to a general theory of continuous fields with fibers $D \otimes \mathcal{K}$ where D is a strongly self-absorbing C*-algebras. The class of strongly self-absorbing C*-algebras, introduced by Toms and Winter [8], is closed under tensor products and contains C*-algebras that are cornerstones of Elliott's classification program of simple nuclear C*-algebras: the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the Jiang-Su algebra $\mathcal{Z}$, the canonical anticommutation relations algebra M_{2^∞} and in fact all UHF-algebras of infinite type. The separable self-absorbing C*-algebras are singled out by a crucial property: there exists an isomorphism $D \to D \otimes D$, which is unitarily homotopic to the map $a \mapsto a \otimes 1_D$, [5]. We prove that

(i) $\mathrm{Aut}(D)$ is contractible in the point-norm topology.

(ii) $\mathrm{Aut}(D \otimes \mathcal{K})$ is well-pointed and it has the homotopy type of a CW-complex.

(iii) The classifying space $B\mathrm{Aut}(D \otimes \mathcal{K})$ has an infinite loop space structure which is compatible with its H-space structure induced by the tensor product.

This gives rise to a generalized cohomology theory $E^*_D(X)$ whose coefficients are given by the homotopy groups of $\mathrm{Aut}(D \otimes \mathcal{K})$ and which we compute.

Theorem. Let $A \neq \mathbb{C}$ be a strongly self-absorbing C-algebra. Then there are isomorphisms of groups*

$$\pi_i(\mathrm{Aut}(D \otimes \mathcal{K})) = \begin{cases} K_0(D)^\times_+ & \text{if } i = 0 \\ K_i(D) & \text{if } i \geq 1 \ . \end{cases}$$

$K_0(D)$ has a natural ring structure. We denote by $K_0(D)^\times$ the invertible elements in this ring and by $K_0(D)^\times_+$ its subgroup consisting of positive elements. We give a necessary and sufficient K-theoretical condition for local triviality.

Theorem. Suppose that X has finite covering dimension and let A be a separable continuous field over X with fibers abstractly isomorphic to $D \otimes \mathcal{K}$. Then A is locally trivial if and only if for each point $x \in X$, there is closed neighborhood V of x and a projection $p \in A(V)$ such that $[p(v)] \in K_0(A(v))^\times$ for all $v \in V$.

The assumption that X is finite dimensional is essential [2], [6].

Using the results (i), (ii), (iii) we show:

Theorem. Let X be a compact metrizable space. The isomorphism classes of locally trivial fields with fibers $D \otimes \mathcal{K}$ form an abelian group under the operation of tensor product. Moreover this group is isomorphic to $E^1_D(X)$, the 1-group of the generalized cohomology theory introduced above in (iii).

Suppose that X is connected with base point x_0. If $D \neq \mathbb{C}$, then there are isomorphisms of multiplicative (abelian) groups

$$[X, \mathrm{Aut}(D \otimes \mathcal{K})] \cong E^0_D(X) \cong K_0(D)^\times_+ \oplus K_0\left(C_0(X \setminus x_0) \otimes D\right).$$

The group operation on the latter group is induced by the multiplication in the ring $K_0(C(X) \otimes D)$. If we assume that D satisfies the Universal Coefficient Theorem in KK-theory, then $K_1(D) = 0$. In this case, assuming that $D \neq \mathbb{C}$ we have

$$E^1_D(X) \cong H^1(X, K_0(D)^\times_+) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, K_0(D)) \ .$$

up to torsion.

If $D = \mathbb{C}$, then of course $E^1_{\mathbb{C}}(X) \cong H^3(X, \mathbb{Z})$, this is the Dixmier-Douady case. An important feature of the general theory is the appearance of characteristic classes in higher dimensions. In general one may use the Atiyah-Hirzebruch spectral sequence for specific computations. In particular, if $D = M_{\mathbb{Q}}$ is the universal UHF-algebra then

$$E^1_{M_{\mathbb{Q}}}(X) \cong H^1(X, \mathbb{Q}^\times_+) \oplus H^3(X, \mathbb{Q}) \oplus H^5(X, \mathbb{Q}) \oplus \cdots \ .$$

If $D = \mathcal{Z}$ is the Jiang-Su algebra and if $H^*(X, \mathbb{Z})$ is torsion free, then

$$E^1_{\mathcal{Z}}(X) \cong H^3(X, \mathbb{Z}) \oplus H^5(X, \mathbb{Z}) \oplus \cdots = \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z}).$$

It is a fact that $H^3(X,\mathbb{Z})$ is a natural direct summand of both $E^1_{\mathcal{Z}}(X)$ and $E^1_{\mathcal{O}_\infty}(X)$ for general compact metrizable spaces. We also show that for any strongly self-absorbing C*-algebra D, if two locally trivial continuous fields with fibers $D\otimes\mathcal{K}$ become isomorphic after tensoring with $\mathcal{O}_\infty$, then they must be isomorphic in the first place.

As a byproduct of our approach we find an operator algebra realization of the classic spectrum $BBU_\otimes$. Let us recall that for a compact connected metric space X the invertible elements of the K-theory ring $K^0(X)$ is an abelian group $K^0(X)^\times$ whose elements are represented by classes of vector bundles of virtual rank ± 1, corresponding to homotopy classes $[X, \mathbb{Z}/2\times BU_\otimes]$. The group operation is induced by the tensor product of vector bundles. Segal has shown that $BU_\otimes$ is in fact an infinite loop space and hence there is a cohomology theory $bu^*_\otimes(X)$ such that $K^0(X)^\times$ is just the 0-group $bu^0_\otimes(X)$ of this theory [7], but gave no geometric interpretation for the higher order groups. Our results lead to a geometric realization of the first group $bu^1_\otimes(X)$ as the isomorphism classes of locally trivial bundles of C*-algebras with fiber the stabilized Cuntz algebra $\mathcal{O}_\infty\otimes\mathcal{K}$ where the group operation corresponds to the tensor product, see [4].

References

[1] J. Dixmier and A. Douady. *Champs continus d'espaces hilbertiens et de C^*-algèbres.* Bull. Soc. Math. France, **91**:227–284, 1963.

[2] M. Dadarlat. *Fiberwise KK-equivalence of continuous fields of C^*-algebras.* J. K-Theory, **3**(2):205–219, 2009.

[3] M. Dadarlat and U. Pennig, *A Dixmier-Douady theory for strongly-selfabsorbing C^*-algebras.* to appear in J. Reine Angew. Math, (arXiv:1302.4468).

[4] M. Dadarlat and U. Pennig, *Unit spectra of K-theory from strongly self-absorbing C^*-algebras.* (arXiv:1306.2583).

[5] M. Dadarlat and W. Winter, *On the KK-theory of strongly self-absorbing C^*-algebras.* Math. Scand., **104**(1):95–107, 2009.

[6] Ilan Hirshberg, Mikael Rørdam, and Wilhelm Winter. *$\mathcal{C}_0(X)$-algebras, stability and strongly self-absorbing C^*-algebras.* Math. Ann., **339**(3):695–732, 2007.

[7] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[8] A. S. Toms and W. Winter, *Strongly self-absorbing C^*-algebras*, Trans. Amer. Math. Soc. **359**(8):3999–4029, 2007.

A canonical Pimsner-Voiculescu embedding

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(joint work with Zhuang Niu)

Abstract. While the Pimsner-Voiculescu embedding of an irrational rotation algebra into an AF algebra, giving rise to an isomorphism of ordered K_0-groups, in fact allows enough variation in the construction to include all such embeddings, there is, it turns out, a rather natural such embedding (although it is not immediately clear which choices in the Pimsner-Voiculescu construction give rise to it). Simply put, it consists of breaking the spectra of the two canonical unitary generators.

1. In [11], Pimsner and Voiculescu constructed an embedding of the irrational rotation C*-algebra A_θ, corresponding to a given irrational number, into an AF algebra—in such a way that the associated map of ordered K_0-groups was an isomorphism.

In fact, the latter property of the embedding was established only somewhat later, in [12], in which it was shown that the order-unit K_0-group of A_θ is $(\mathbb{Z}+\mathbb{Z}\theta, 1)$, with the element θ being the class of the Rieffel projection (see Appendix to [12]). What was actually done in [11], in the first instance, was to embed A_θ (unitally) into the AF algebra B_θ with order-unit K_0-group equal to $(\mathbb{Z}+\mathbb{Z}\theta, 1)$, and in such a way that the K_0-class of the Rieffel projection mapped into θ. (In fact, this property of the embedding follows immediately from the fact that the Rieffel projection in A_θ has trace θ.) In view of the fact stated above about $K_0(A_\theta)$, one sees that the embedding is an order isomorphism at the level of K_0. (So in fact this property is automatic—provided that the AF algebra is B_θ.)

Since the mentioned construction of Pimsner and Voiculescu introduces a number of choices —of inner automorphisms to match up approximately the (very specific) approximate embeddings into the finite-dimensional sub-C*-algebras in an increasing sequence with union dense in B_θ—, it is reasonable to consider how general it might be. In fact, as can be shown using the A$\mathbb{T}$ nature of A_θ (shown in [1]), every unital C*-algebra homomorphism of A_θ into B_θ, giving rise to an order isomorphism of K_0-groups (this in fact being automatic, as pointed out above), is actually equal to the Pimsner-Voiculescu embedding with suitable choices of the inner automorphisms—at least after passing to a subsequence. (Given an arbitrary embedding of A_θ in B_θ, taking 1 into 1 and taking the class of the Rieffel projection into $\theta \in \mathbb{Z}+\mathbb{Z}\theta = K_0(B_\theta)$—actually, since both order-unit K_0-groups are $(\mathbb{Z}+\mathbb{Z}\theta, 1)$, the second property follows from the first (cf. above)—, one sees, as in §1 of [3], that it is approximately unitarily equivalent to (any choice of) the Pimsner-Voiculescu embedding by means of unitaries belonging to certain of the finite-dimensional sub-C*-algebras in the increasing sequence approximating the AF-algebra B_θ. (Of course, any sequence of unitaries can be approximated by one with this last property, but in the present case this is how they arise.) Indeed, the images in B_θ of the canonical unitary generators of A_θ are approximated arbitrarily closely by unitaries in these finite-dimensional subalgebras, and by the uniqueness theorem (based on [1]) described in §1 of [3] for approximate homomorphisms of A_θ into a finite-dimensional C*-algebra, one obtains that these unitaries are approximately unitarily equivalent to the Pimsner-Voiculescu ones—at least at a suitable stage in the sequence.)

2. We have found a particularly natural embedding of A_θ into B_θ (giving rise to an isomorphism of ordered K_0-groups). The description of this embedding is very simple, and involves no choices. Certain well-known automorphisms of A_θ (the flip and the Fourier transform)—but not all automorphisms (for instance, not all those from the canonical torus action)—lift from A_θ to B_θ in this embedding.

The construction consists of a single step: consider the two canonical unitary generators u and v of A_θ (with relation $uv = e^{2\pi i\theta}vu$), and, in the unique finite factor representation of A_θ (recall that θ is irrational), adjoin the spectral projection for u corresponding to the interval $[0, \theta]$, and also the corresponding spectral projection for v. (In other words, consider the C*-algebra of operators generated by u and v and these two spectral projections.) (Alternatively, break the spectrum of the unitaries u and v at a single point, by adjoining the logarithm—cf. [2].) This C*-algebra, denoted by $\mathcal{B}_\theta$ in [2], is isomorphic to the AF algebra B_θ.

3. The proof that $\mathcal{B}_\theta$ is AF for arbitrary (irrational) θ is quite different from the proof for generic θ in [2]. Both arguments use the natural structure of the family $(\mathcal{B}_\theta)$ as an upper semicontinuous field of C*-algebras, when suitable type I C*-algebras are inserted as fibres for rational values of θ, established in [2]. (See Section 8 of [2].) In [2], this semicontinuous field structure is used to extend structural properties from the fibre for a fixed rational θ to the fibres over a whole neighborhood. In [4], it is used to conclude quasidiagonality of the fibre at a fixed irrational θ from the quasidiagonality of the nearby rational fibres—which is easy to check as the rational fibres are of type I and have a very simple structure.

Once quasidiagonality of $\mathcal{B}_\theta$ is known, and in view of the simplicity, nuclearity, and uniqueness of the trace established for arbitrary irrational values of θ in [2] (in Sections 3, 5, and 7 of [2]), it remains only to show that $\mathcal{B}_\theta$ is Jiang-Su stable in order to be able to deduce from a quite recent result of Matui and Sato (Theorem 6.1 of [8]) that $\mathcal{B}_\theta$ belongs to the classifiable class of Winter, Lin, and Lin-Niu ([14], [5], [6]). Since B_θ is known to belong also to this class, it follows that $\mathcal{B}_\theta$ is isomorphic to B_θ.

(One also needs to show that B_θ satisfies the UCT, but this is not difficult.)

4. To show that $\mathcal{B}_\theta$ is Jiang-Su stable, by another recent result of Matui and Sato (Theorem 1.1 of [7]), it is enough to show that this C*-algebra has the Blackadar-Rørdam strict comparability property for positive elements. This is shown in [4] by showing that A_θ is a large subalgebra of $\mathcal{B}_\theta$ in the sense of Putnam-Phillips. (The large subalgebra technique was first introduced by Putnam in [13], and greatly generalized by Phillips in [10].)

5. Certain additional questions are suggested by this work—for instance, the possibility of generalizing the construction to higher-dimensional non-commutative tori (or those based on an even more general discrete abelian group)—and also the question of breaking the spectra of unitary generators in more general crossed products by locally compact abelian groups (as for instance for $\mathbb{R}$ in [9]).

Certainly, our embedding is unique when defined in term of breaking the spectra of the generators of A_θ. What other special properties might it have that might also single it out? (Maximality? Extendibility of certain automorphisms?)

References

[1] G. A. Elliott and D. E. Evans. The structure of the irrational rotation C*-algebra. *Ann. of Math. (2)*, 138(3):477–501, 1993.

[2] G. A. Elliott and Z. Niu. Extended rotation algebras: Adjoining spectral projections to rotation algebras. *J. Reine Angew. Math.*, 665:1–71, 2012.

[3] G. A. Elliott and Z. Niu. Remarks on the Pimsner-Voiculescu embedding. In: Operator Algebra and Dynamics (Nordforsk Network Closing Conference, Faroe Islands, May 2012), *Springer Proceedings in Mathematics & Statistics*, Vol. 58, pp. 125–134, Carlsen, T. M.; Eilers, S.; Restorff, G.; Silvestrov, S. (Eds.), 2013.
[4] G. A. Elliott and Z. Niu. All irrational extended rotation algebras are AF. *Preprint*, 2013.
[5] H. Lin. Localizing the Elliott conjecture at strongly self-absorbing C*-algebras, II. *Crelles Journal*, to appear. *arXiv: 0709.1654v1*, 2007.
[6] H. Lin and Z. Niu. Lifting KK-elements, asymptotic unitary equivalence and classification of simple C*-algebras. *Adv. Math.*, 219:1729–1769, 2008.
[7] H. Matui and Y. Sato. Strict comparison and $\mathcal{Z}$-absorption of nuclear C*-algebras. *Acta Math.*, 209(1):179–196, 2012.
[8] H. Matui and Y. Sato. Decomposition rank of UHF-absorbing C*-algebras. *Preprint*, arXiv:1303.4371, 2013.
[9] P. S. Muhly, I. F. Putnam, and J. Xia. On the K-theory of some C*-algebras of Toeplitz and singular integral operators. *J. Funct. Anal.*, 110(2):161–225, 1992.
[10] N. C. Phillips. Large subalgebras. *Preprint.*
[11] M. Pimsner and D. Voiculescu. Imbedding the irrational rotation C*-algebra into an AF-algebra. *J. Operator Theory*, 4(2):201–210, 1980.
[12] M. Pimsner and D. Voiculescu. Exact sequences for K-groups and Ext-groups of certain cross-product C*-algebras. *J. Operator Theory*, 4(1):93–118, 1980.
[13] I. F. Putnam. The C*-algebras associated with minimal homeomorphisms of the Cantor set. *Pacific J. Math.*, 136(2):329–353, 1989.
[14] W. Winter. Localizing the Elliott conjecture at strongly self-absorbing C*-algebras. *Crelles Journal*, to appear. *arXiv: 0708.0283v3*, 2007.

Noncommutative Lipschitz geometry and hyperbolic groups

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(joint work with Bogdan Nica)

The work in question studies the noncommutative geometry of the C*-algebra crossed-product $C(\partial\Gamma) \rtimes \Gamma$ of a Gromov hyperbolic group Γ acting on its Gromov boundary. A *Fredholm* module over a C*-algebra A is a triple (H, π, F) where $\pi\colon A \to \mathcal{B}(H)$ is a *-representation of A on a Hilbert space H and $F \in \mathcal{B}(H)$ is a bounded operator which is Fredholm in a suitable sense (depending on the parity of the Fredholm module). It is *p-summable* if the commutators $[\pi(a), F]$ are in the Schatten ideal $\mathcal{L}^p(H)$ of compact operators with principal value sequence $\lambda_1 \geq \lambda_2 \geq \dots$ in $l^p(\mathbb{N})$, for a *dense* *-subalgebra of $a \in A$. It is *p+-summable* if the principal values are $\mathrm{O}(n^{-\frac{1}{p}})$ – this implies q-summability for $q > p$. Here, we construct a (uniformly) finitely summable family of Fredholm modules over the crossed-product $A := C(\partial\Gamma) \rtimes \Gamma$, using the Γ-invariant Lipschitz geometry of the boundary $\partial\Gamma$ with respect to a visual metric d_ϵ on the boundary. Each of these Fredholm modules is p-summable for $p > \mathrm{hdim}(\partial\Gamma, d_\epsilon)$, where hdim is the Hausdorff dimension. We prove that every K-homology class for $C(\partial\Gamma) \rtimes \Gamma$ is represented by a Fredholm module in this family. More exactly, we show that there is a dense $*$-subalgebra $\mathcal{A} \subset C(\partial\Gamma) \rtimes \Gamma$, and some $D \geq 0$, such that every K-homology class for $C(\partial\Gamma) \rtimes \Gamma$ is represented by a p-summable Fredholm module over $\mathcal{A}$ for $p > D$. Here D is the *visibility dimension* of the boundary, the infimum over all visibility metrics, of the corresponding Hausdorff dimension.

For instance, if G is a discrete co-compact group of isometries of real hyperbolic n-space, we show that every K-homology class for $C(\partial\Gamma) \rtimes \Gamma$ is represented by an $(n-1)+$-summable Fredholm module over an appropriate dense *-subalgebra of $C(\partial\Gamma) \rtimes \Gamma$.

The proof of these results uses essentially two ingredients. One is the Poincaré duality result of the first author (see [2]), which describes a self-duality isomorphism $\mathrm{K}_*(C(\partial\Gamma) \rtimes \Gamma) \cong \mathrm{K}^{*+1}(C(\partial\Gamma) \rtimes \Gamma)$ induced by Kasparov product with an appropriate K-homology class. The other ingredient is the following construction.

A *visual* metric on the boundary of a hyperbolic group is a metric bi-Lipschitz to $e^{-\epsilon\langle\, ,\, \rangle}$, where $\langle\, ,\, \rangle$ denotes the extension of the Gromov product to the boundary. The group Γ acts by Lipschitz maps with respect to any visual metric. Let μ be the Hausdorff measure with respect to a visual metric. Let H_μ be the Hilbert space of the regular representation $\lambda\colon C(\partial\Gamma) \rtimes \Gamma \to \mathcal{B}(H)$ of the crossed-product corresponding to the probability measure μ. Thus $H_\mu := l^2\big(\Gamma, L^2(\partial\Gamma, \mu)\big)$, Γ acts by $\lambda(g)(\sum_h \xi_h\delta_h) = \sum_h \xi_h\delta_{gh}$, and functions on the boundary act by $\lambda(\phi)(\sum_h \xi_h\delta_h) = \sum_h h(\phi).\xi_h\delta_{gh}$, where the dot denotes multiplication of functions on $\partial\Gamma$. Let $P_{\ell^2\Gamma} \in \mathcal{B}(H_\mu)$ be projection to $l^2\Gamma$ viewed as a subspace of H_μ by considering a function in $l^2\Gamma$ as a function on $\Gamma \times \partial\Gamma$ which is constant in the second coordinate. A remarkable property of the ergodic theory of Γ acting on the probability space $(\partial\Gamma, \mu)$ is that the orbit $\Gamma \cdot \mu \subset \mathrm{Prob}(\partial\Gamma)$ in the compact space of probability measures on the boundary, accumulates only at point masses on $\partial\Gamma$. We show that in fact for elements of the dense *-subalgebra $\mathrm{Lip}(\partial\Gamma, d_\epsilon) \rtimes_{\mathrm{alg}} \Gamma$ of Γ acting on the *-algebra of Lipschitz functions on the boundary, the commutators $[\lambda(a), P_{\ell^2\Gamma}]$ are in $\mathcal{L}^p(H_\mu)$ for $p > \mathrm{hdim}(\partial\Gamma, d_e)$. It follows that the triple $(l^2\big(\Gamma, L^2(\partial\Gamma, \mu)\big), \lambda, P_{\ell^2\Gamma})$ is a Fredholm module over $C(\partial\Gamma) \rtimes \Gamma$ which is p-summable over $\mathrm{Lip}(\partial\Gamma, d_\epsilon) \rtimes_{\mathrm{alg}} \Gamma$ for p at least the Hausdorff dimension of $(\partial\Gamma, d_\epsilon)$.

The Fredholm modules $(l^2\big(\Gamma, L^2(\partial\Gamma, \mu)\big), \lambda, P_{\ell^2\Gamma})$ can be 'twisted' by K-theory classes for $C(\partial\Gamma) \rtimes \Gamma$, by the following procedure. Let $J\colon l^2\Gamma \to l^2\Gamma$ be the unitary operator corresponding to inversion on the group Γ. Extend J by the identity on the kernel of $P_{\ell^2\Gamma}$ to obtain an operator (still denoted J) on H_μ. Then we show that

$$[J\lambda(a)J, \lambda(b)] \in \mathcal{L}^p(H_\mu)$$

for dense $a, b \in A$, which expresses the geometric fact about the Gromov compactification that as one translates a metric ball in the group of fixed size, out to the boundary, the diameters of these sets with respect to any visual metric on $\overline{\Gamma}$ go to zero. The two essentially commutating representations π and $\mathrm{Ad}_J(\lambda)$ of $C(\partial\Gamma) \rtimes \Gamma$ on H_μ can be then used to make Fredholm modules over $\mathrm{Lip}(\partial\Gamma, d_\epsilon) \rtimes_{\mathrm{alg}} \Gamma$ from projections or unitaries in $C(\partial\Gamma) \rtimes \Gamma$. We show that these Fredholm modules have the same summability as the basic Fredholm modules $(l^2\big(\Gamma, L^2(\partial\Gamma, \mu)\big), \lambda, P_{\ell^2\Gamma})$ and that the passage from projection or unitary to the corresponding Fredholm module describes the Poincaré duality isomorphism of [2]. Hence every K-homology class arises in this way.

The last part of the talk announces applications to the reduced C*-algebras of Gromov hyperbolic groups. By restriction, a Fredholm module over $C(\partial\Gamma) \rtimes \Gamma$ gives rise to one over $C_r^*\Gamma$ which has, roughly speaking, the same degree of summability. The induced map $\mathrm{K}^*(C(\partial\Gamma) \rtimes \Gamma) \to \mathrm{K}^*(C_r^*\Gamma)$ can be computed by a variant of the Gysin sequence for boundary actions of [3], and we show that it is surjective in many examples. As a result, we deduce, for example, that if Γ is a classical group of isometries of real hyperbolic n-space, then every K-homology class for $C_r^*\Gamma$ is represented by a Fredholm module over $C_r^*\Gamma$ which is $(n-1)+$-summable over the group ring $\mathbb{C}\Gamma$.

This result should be compared with a recent observation of Puschnigg, who has shown that lattices of real rank greater than one have *no* non-degenerate Fredholm modules which are summable over $\mathbb{C}\Gamma$; the case of rank-one lattices Γ is therefore quite special from the point of view of the study of their finitely summable Fredholm modules.

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press 1994.

[2] H. Emerson: *Noncommutative Poincaré duality for boundary actions of hyperbolic groups*, J. Reine Angew. Math. 564 (2003), 1–33.

[3] H. Emerson, R. Meyer:*Euler characteristics and Gysin sequences for group actions on boundaries*, Math. Ann. 334 (2006), no. 4, 853–904.

[4] M. Puschnigg: *Finitely summable Fredholm modules over higher rank groups and lattices*, J. K-Theory 8 (2011), no. 2, 223–239.

Graphs, groups and self-similarity

Ruy Exel

The purpose of this presentation is to study a family of C*-algebras generalizing both Katsura [2] algebras and certain algebras introduced by Nekrashevych [4,5] in terms of self-similar groups. This is based on the paper [1] which is joint work with Enrique Pardo.

Let G be a countable discrete group, E be a finite graph with no sources, σ be an action of G on E, and

$$\varphi : G \times E^1 \to G$$

be a one-cocycle for the restriction of σ to E^1, which moreover satisfies

$$\sigma_{\varphi(g,e)}(x) = \sigma_g(x) \ \ \forall g \in G \, \forall e \in E^1 \, \forall x \in E^0.$$

We then define $\mathcal{O}_{G,E}$ to be the universal unital C*-algebra generated by a set

$$\{p_x : x \in E^0\} \cup \{s_e : e \in E^1\} \cup \{u_g : g \in G\},$$

subject to the following relations:

(i) $\{p_x : x \in E^0\} \cup \{s_e : e \in E^1\}$ is a Cuntz-Krieger E-family,

(ii) the map $u : G \to \mathcal{O}_{G,E}$, defined by the rule $g \mapsto u_g$, is a unitary representation of G,

(iii) $u_g s_e = s_{ge} u_{\varphi(g,e)}$, for every $g \in G$, and $e \in E^1$,
(iv) $u_g p_x = p_{gx} u_g$, for every $g \in G$, and $x \in E^0$.

When G is the group of integers, acting trivially on the vertices of the graph, we show that these algebras generalize Katsura's algebras. On the other hand, when the graph is a bouquet of n circles, our algebras include Nekrashevych's algebras defined in terms of self-similar groups.

In the general case $\mathcal{O}_{G,E}$ may be described as a groupoid C*-algebra for an étale groupoid which is remarkably similar to the groupoid associated to the relation of "tail equivalence with lag" on the path space of the graph, as described by Kumjian, Pask, Raeburn and Renault in [3].

We may also describe $\mathcal{O}_{G,E}$ as a Cuntz-Pimsner algebra for a very natural correspondence M over the algebra

$$C(E^0) \rtimes G.$$

As a result we are able to prove that $\mathcal{O}_{G,E}$ is nuclear when G is amenable.

References

[1] R. Exel and E. Pardo, Graphs, groups and self-similarity, arXiv:1307.1120.
[2] T. Katsura, A construction of actions on Kirchberg algebras which induce given actions on their K-groups, *J. reine angew. Math.*, **617** (2008), 27–65.
[3] A. Kumjian, D. Pask, I. Raeburn and J. Renault, Graphs, groupoids, and Cuntz-Krieger algebras, *J. Funct. Anal.,* **144** (1997), 505–541
[4] V. Nekrashevych, Cuntz-Pimsner algebras of group actions, *J. Operator Theory*, **52** (2004), 223–249.
[5] V. Nekrashevych, C*-algebras and self-similar groups, *J. reine angew. Math.*, **630** (2009), 59–123.

Logic of metric structures and nuclear C*-algebras

Ilijas Farah
(joint work with B. Hart, L. Robert, A. Tikuisis, A. Toms, W. Winter)

Many questions about nuclear C*-algebras (e.g., the bootstrap problem, or the Toms–Winter conjecture) are about the existence of nuclear C*-algebras radically different from the ones that can be defined using the conventional means. This is a preliminary report on a work in progress concerned with a possible application of model theory of metric structures ([1], [7]), to construction of such separable nuclear C*-algebras ([4], [14]). For classical ('discrete') model theory and its applications to algebra see e.g., [11]. To a given C*-algebra A one associates an invariant Th(A), the *theory of* A. The information provided by this invariant is in a certain sense orthogonal to the information provided by K-theoretic invariants such as Elliott's invariant. It can be considered as an element of the dual unit ball of a certain real Banach space $\mathbb{V}$, and the set of all theories of separable C*-algebras is weak*-closed.

Formulas and theories. I shall concentrate only on C*-algebras and therefore will refer to [7] instead of the standard reference [1]. We define the set $\mathbb{F}$ of all formulas by recursion. A *atomic formula* is an expression of the form $\|P(x_1, \dots, x_n)\|$ where P is a *-polynomial in non-commuting variables $x_1, \dots, x_n$ for some n. Let $\mathbb{F}$ be the smallest set containing all atomic formulas and satisfying the following.

(1) If ϕ and ψ are in $\mathbb{F}$ and $f \colon \mathbb{R}^2 \to [0, \infty)$ is continuous, then $f(\phi, \psi) \in \mathbb{F}$.
(2) If $\phi \in \mathbb{F}$ and x is a variable then $\sup_x \phi$ and $\inf_x \phi$ are in $\mathbb{F}$.

We let $\mathbb{F}_n$ be the space of all formulas whose free variables are included in $\{x_1, \dots, x_n\}$. For $\phi \in \mathbb{F}_n$ we write $\phi(x_1, \dots, x_n)$ instead of ϕ only when this is necessary. Clearly each $\mathbb{F}_n$ is closed under addition and multiplication by positive reals, and is therefore the positive cone in the unique vector space $\mathbb{V}_n$. A *sentence* is a formula with no free variables.

Interpretation. For $n \in \mathbb{N}$ one defines the *interpretation* $\phi(a_1, \dots, a_n)^A$ for a formula $\phi(x_1, \dots, x_n) \in \mathbb{F}_n$, a C*-algebra A and elements $a_1, \dots, a_n$ in its unit ball by recursion as follows:

(1) $\|P(a_1, \dots a_n)\|^A := \|P(a_1, \dots, a_n)\|$, $f(\phi_1, \dots, \phi_k)^A := f(\phi_1^A, \dots, \phi_k^A)$.
(2) $(\sup_x \psi(x))^A := \sup_{b \in A, \|b\| \leq 1} \psi(b)^A$ and $(\inf_x \psi(x))^A := \inf_{b \in A, \|b\| \leq 1} \psi(b)^A$.

Thus $\phi \in \mathbb{F}_n$ defines a uniformly continuous function ϕ^A on the unit ball of A^n (see [7, Lemma 2.2]). On $\mathbb{F}_0$ we define norm by $\|\phi\| = |\phi^A|$. The *theory of A* is

$$\mathrm{Th}(A) = \{\phi \in \mathbb{F}_0 : \phi^A = 0\}.$$

It is sometimes convenient to identify $\mathrm{Th}(A)$ with the linear functional $\phi \mapsto \phi^A$ and thus equip the set of theories with the weak*-topology (sometimes called *logic topology*). By the compactness and Löwenheim–Skolem theorems for the logic of metric structures ([1, Theorem 5.8 and Theorem 7.3], the set of all theories of separable C*-algebras is weak*-closed.

Examples. (1) $\sup_x \sup_y \|[x, y]\| \in \mathrm{Th}(A)$ if and only if A is abelian.

(2) $\sup_x \sup_y \inf_z (\max(\|x^*x\| - \frac{1}{2}, 0))(\|zx^*xz^* - \frac{1}{2}y^*y\|) \in \mathrm{Th}(A)$ if and only if every positive element of norm $\geq 1/2$ dominates every positive element of norm $\leq 1/2$ in the Cuntz order. This is equivalent to A being simple and purely infinite.

Elementary equivalence. Two algebras are *elementarily equivalent*, in symbols $A \equiv B$, if $\mathrm{Th}(A) = \mathrm{Th}(B)$. By a standard result of Keisler and Shelah ([1, Theorem 5.7]), $A \equiv B$ if and only if A and B have isomorphic ultrapowers. While model-theoretic methods have shown to be useful in the study of 'massive' C*-algebras such as ultrapowers, coronas, and relative commutants (see e.g., [5], [6], or [9]), this report is concerned with nuclear, separable C*-algebras.

If $A \subseteq B$ we say that A is an *elementary submodel* of B, $A \preceq B$, if

$$\phi(a_1, \dots, a_n)^A = \phi(a_1, \dots, a_n)^B.$$

for every n, $\phi \in \mathbb{F}_n$ and all $a_1, \dots, a_n$ in the unit ball of A. By the Fundamental Theorem of Ultrapowers (Łoś's Theorem, e.g., [7]), the diagonal embedding of a

C*-algebra into its ultrapower is elementary. Elementary submodels were implicitly used in the theory of operator algebras in a situation when one starts with a nonseparable C*-algebra with an interesting property and finds a large enough separable subalgebra that inherits this property (cf. [2, II.8.5]). For example, if $A \preceq B$ then every ideal of A is an intersection of an ideal of B with A (cf. [2, II.8.5.6]), and every trace of A extends to a trace of B (cf. [13, Lemma 8]).

A property $\mathcal{P}$ of C*-algebras is *elementary* (or *axiomatizable*) if $A \equiv B$ implies $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are equivalent. Some elementary properties of C*-algebras are being abelian, purely infinite and simple, having real rank n for $n \geq 0$, having stable rank n, for $n \geq 1$, including a unital copy of $M_n(\mathbb{C})$ for $n \geq 2$. Many of the proofs are also elementary. The above implies that in a UHF algebra the theory codes the corresponding generalized integer, and therefore by Glimm's theorem the unital UHF algebras are classified by their theory.

By the abstract classification theory (in particular [8, Proposition 5.1]), in every class of C*-algebras that is not classified by smooth invariants there are elementarily equivalent, but nonisomorphic, separable C*-algebras. This applies for example to AF algebras, Kirchberg algebras, and even stabilized UHF algebras. Embarrassingly, no concrete example of a pair of simple, nuclear, separable C*-algebras that are elementarily equivalent but not isomorphic is known. Many properties of $K_0(A)$ or the Cuntz semigroup of A are encoded in the theory of A. In particular, this applies to the radius of comparison. Therefore, nonisomorphic algebras which agree on all 'reasonable' invariants constructed in [15] are distinguished by their theories. No pair of nonisomorphic nuclear, simple, separable, C*-algebras that have the same theory and the same Elliott invariant is known.

Nuclearity. A nontrivial ultrapower of the CAR algebra is neither simple nor nuclear, and therefore properties of being UHF, nuclear, simple,... are not elementary (cf. [9], [6]). Also, since all nontrivial ultrapowers are tensorially prime ([10]), neither being stable nor being $\mathcal{Z}$-stable is elementary. However, the latter fact is misleading. For separable A one can recover the inf-part of $\mathrm{Th}(A' \cap A^{\mathcal{U}})$ (the relative commutant of A in its ultrapower) from $\mathrm{Th}(A)$, and therefore by standard methods (e.g., [12]) being $\mathcal{Z}$-stable is elementary among separable C*-algebras.

Omiting types. Even among the separable C*-algebras, being nuclear is not elementary. However, nuclearity and related CPAP-like properties can be expressed as 'omitting types properties' (see [1, §12] or [3]), and we have the following.

Proposition 1. *For every A there are either no nuclear C*-algebras $B \equiv A$ or a generic separable C*-algebra elementarily equivalent to A is nuclear.*

This suggests a novel method for constructing nuclear, separable C*-algebras. Starting from an 'interesting' theory, one considers a generic C*-algebra satisfying this theory. Such algebras are constructed from finite pieces of information about its finite subsets, and therefore may look differently from algebras in the bootstrap class. Algebras $C^*_r(F_\infty)$ and $\prod_{\mathcal{U}} M_n(\mathbb{C})$ are not elementarily equivalent to a nuclear C*-algebra. The latter fact implies that the set of theories of separable nuclear C*-algebras is not weak*-closed (it is, however G_δ). If it not known whether the Calkin algebra is elementarily equivalent to a nuclear C*-algebra.

References

[1] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, Vol. II (Z. Chatzidakis et al., eds.), Cambridge University Press, 2008, pp. 315–427.

[2] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[3] K. Carlson, E. Cheung, I. Farah, A. Gerhardt-Bourke, B. Hart, L. Mezuman, N. Sequeira, and A. Sherman, *Omitting types and AF algebras*, preprint, arXiv:1212.3576, 2012.

[4] G.A. Elliott and A.S. Toms, *Regularity properties in the classification program for separable amenable C^*-algebras*, Bull. Amer. Math. Soc. **45** (2008), no. 2, 229–245.

[5] I. Farah and B. Hart, *Countable saturation of corona algebras*, C.R. Math. Rep. Acad. Sci. Canada **35** (2013), 35–56.

[6] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras I: Stability*, Bull. London Math. Soc. (to appear).

[7] ———, *Model theory of operator algebras II: Model theory*, Israel J. Math. (to appear).

[8] I. Farah, A.S. Toms, and A. Törnquist, *The descriptive set theory of C*-algebra invariants*, IMRN (to appear), Appendix with Caleb Eckhardt.

[9] L. Ge and D. Hadwin, *Ultraproducts of C^*-algebras*, Recent advances in operator theory and related topics (Szeged, 1999), Birkhäuser, Basel, 2001, pp. 305–326.

[10] S. Ghasemi, *SAW* algebras and tensor products*, preprint, arXiv:1209.3459, 2012.

[11] D. Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002, An introduction.

[12] H. Matui and Y. Sato, *Strict comparison and Z-absorption of nuclear C*-algebras*, Acta Math. (to appear).

[13] Narutaka Ozawa, *Dixmier approximation and symmetric amenability for C*-algebras*, arXiv preprint arXiv:1304.3523, 2013.

[14] M. Rørdam, *Classification of nuclear C^*-algebras*, Encyclopaedia of Math. Sciences, vol. 126, Springer-Verlag, Berlin, 2002.

[15] A.S. Toms, *Comparison theory and smooth minimal C^*-dynamics*, Comm. Math. Phys. **289** (2009), no. 2, 401–433.

Ultraproducts, QWEP von Neumann algebras and the Effros-Maréchel topology

Uffe Haagerup

(joint work with Hiroshi Ando, Carl Winsløw)

Based on analysis of the Ocneanu and Groh-Raynaud ultraproducts of von Neumann algebras and the Effros-Maréchal topology on the space $\mathrm{vN}(H)$, we prove that the following conditions are equivalent:

(1) M has Kirchberg's quotient weak expectation property (QWEP).
(2) M is in the closure (w.r.t. the Effros-Maréchel topology) of the set of finite dimensional factors in $\mathrm{vN}(H)$.

(3) M admits an embedding into the Ocneanu ultrapower $(R_\infty)^\omega$ of the injective Type III$_1$ factor R_∞ with a normal faithful conditional expectation of $(R_\infty)^\omega$ onto M.
(4) For all $\varepsilon > 0$, $n \in \mathbb{N}$ and $x_1, \ldots, x_n$ in the natural cone $P_M^\natural$ from the standard form of M there is a $k \in \mathbb{N}$ and $a_1, \ldots, a_n$ in $M_k(\mathbb{C})^+$, such that $|\langle x_i, x_j \rangle - \tau_k(a_i a_j)| < \varepsilon$ for $i, j = 1, \ldots, n$, where $\tau_k = \frac{1}{k}\mathrm{Tr}$ is the normalized trace on $M_k(\mathbb{C})$.

References

[1] H. Ando and U. Haagerup, *Ultraproducts of von Neumann algebras,* preprint, arXiv: 1212.5457.
[2] H. Ando, U. Haagerup and C. Winsløw, *Ultraproducts, QWEP von Neumann Algebras, and the Effros-Maréchal Topology,* preprint, arXiv:1306.0460.

Rokhlin dimension for finite group actions

Ilan Hirshberg
(joint work with N. C. Phillips)

Rokhlin dimension for actions of finite groups and $\mathbb{Z}$ on unital C^*-algebras was introduced in [2]. For finite group actions, this generalizes the Rokhlin property, studied by Izumi ([3, 4]) - the Rokhlin property becomes Rokhlin dimension zero in our terminology. There are some straightforward K-theoretic obstructions for a finite group action to have the Rokhlin property in the sense of Izumi. For example, if G acts on A and A has the property that all of its automorphisms are approximately inner, then [1] must be divisible by $|G|$.

In the present work under preparation, we generalize Rokhlin dimension to the non-unital setting, and study more subtle K-theoretic obsructions.

Roughly speaking, the Rokhlin property requires that there be approximately central partitions of unity consisting of projections indexed by the group, which are (approximately) permuted by the group action. Rokhlin dimension d for actions on unital C^*-algebras replaces the projections by $d + 1$ sets of positive elements, such that each set consists of orthogonal elements indexed by the group G and is (approximately) acted on by permutation. However positive elements from different sets ("with different colors") may fail to be orthogonal. For the non-unital case, we require that their sum (almost) acts as a unit on a prescribed finite subset of the C^*-algebra.

The definition of Rokhlin dimension has a commuting tower and noncommuting tower versions - here we focus only on the commuting tower version, and all the results mentioned are about this version only (although permanence of finite nuclear dimension holds for the noncommuting tower version as well).

The generalization to the non-unital case allows us to consider extensions. If $0 \to J \to A \to B \to 0$ is an extention, G acts on A and J is an invariant ideal, then we show that if the action of G on A has finite Rokhlin dimension, then so does the restriction to J and the induced action on B, and conversely, if the restricted action on J and the induced action on B both have finite Rokhlin dimension then so does the action of G on A. We furthermore extend the permanence properties from [2] to the nonunital setting: if G acts on A with finite Rokhlin dimension, then if A has finite nuclear dimension then so does the crossed product, and if A is $\mathcal{Z}$-absorbing then so is the crossed product.

The K-theoretic obstruction we consider is more subtle. If A is a unital separable C^*-algebra, we consider the equivariant K-theory group $K_*^G(A)$. This group can be viewed as a module over the representation ring of G, $R(G)$. We can push results of Atiyah-Segal ([1]) concerning free actions on topological spaces to the case of finite Rokhlin dimension to show that some power of the augmentation ideal $I(G)$ has to annihilate $K_*^G(A)$. Under suitable conditions, this can be used to produce an obstruction. For example, we can show that there is no action of any finite group on the Jiang-Su algebra $\mathcal{Z}$ or the Cuntz algebra $\mathcal{O}_\infty$ with finite Rokhlin dimension.

References

[1] M.F. Atiyah, and G.B. Segal, *Equivariant K-theory and completion*, J. Differential Geometry **3** 1969 1–18.
[2] I. Hirshberg, W. Winter and J. Zacharias, *Rokhlin dimension and C^*-dynamics*, preprint.
[3] M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property. I*, Duke Math. J. **122** (2004), no. 2, 233–280.
[4] M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property. II*, Adv. Math. **184** (2004), no. 1, 119–160.

Baumslag-Solitar groups, relative profinite completions and measure equivalence rigidity

Cyril Houdayer
(joint work with Sven Raum)

The subject of *measured group theory* was introduced by Gromov in [12]. It aims at studying discrete groups from a measure theoretic point of view. One of the fundamental problems in measured group theory is the classification of discrete groups, up to *measure equivalence*. It is of particular interest to classify natural classes of discrete groups. Measured group theory has links to other active fields of mathematics such as von Neumann algebras [20, 21, 22], geometric group theory [23, 17, 13, 14] and ergodic theory [3, 4, 19, 2, 7, 9, 10]. Surveys of current topics in measured group theory can be found in [8, 11].

The *Baumslag-Solitar groups* were introduced in [1] as the first examples of finitely presented non-Hopfian groups. Recall that for all $m, n \in \mathbf{Z} \setminus \{0\}$, the Baumslag-Solitar group $\mathrm{BS}(m,n)$ with parameters (m,n) is defined by the presentation

$$\mathrm{BS}(m,n) = \langle a, t : ta^m t^{-1} = a^n \rangle.$$

The classification of the Baumslag-Solitar groups, up to isomorphism, was obtained in [16]: $\mathrm{BS}(m,n) \cong \mathrm{BS}(p,q)$ if and only if there exists $\varepsilon \in \{-1,1\}$ such that

$\{m, n\} = \{\varepsilon p, \varepsilon q\}$. In this talk, we will always assume that $1 \leq |m| \leq n$. Note that for every $n \geq 1$, $\mathrm{BS}(\pm 1, n)$ is solvable, hence amenable and for every $2 \leq |m| \leq n$, $\mathrm{BS}(m, n)$ is non-amenable. The classification of the Baumslag-Solitar groups, up to quasi-isometry, was an important result in geometric group theory (see [6, 24]). Recently, a partial classification of the von Neumann algebras of the Baumslag-Solitar groups was obtained in [18].

The aim of the present talk is to present new rigidity results for all non-amenable Baumslag-Solitar groups and their direct products in the framework of measured group theory.

In [15], Kida obtained several rigidity results for measure equivalence couplings of non-amenable Baumslag-Solitar groups. He showed in [15, Theorem 1.4] that for a large class of non-amenable Baumslag-Solitar groups, any measurable coupling between such Baumslag-Solitar groups, which is aperiodic on the natural cyclic subgroups, can only exist between Baumslag-Solitar groups which are isomorphic. Recall that a pmp action $\Gamma \curvearrowright (X, \mu)$ of a discrete countable group on a standard probability space is *aperiodic* if every finite index subgroup of Γ acts ergodically on (X, μ). For instance, if the action $\Gamma \curvearrowright (X, \mu)$ is weakly mixing then it is aperiodic.

In the statement of the theorem, we make use of the following notations:

- For all $1 \leq i \leq k$, put $\mathrm{BS}(m_i, n_i) = \langle a_i, t_i : t_i a_i^{m_i} t_i^{-1} = a_i^{n_i} \rangle$.
- For all $1 \leq j \leq l$, put $\mathrm{BS}(p_j, q_j) = \langle b_j, u_j : u_j b_j^{p_j} u_j^{-1} = b_j^{q_j} \rangle$.

Our main result is a stable orbit equivalence (SOE) rigidity result for actions of direct products of arbitrary non-amenable Baumslag-Solitar groups. Our main theorem generalises [15, Theorem 1.4] and moreover provides new SOE rigidity phenomena for Baumslag-Solitar groups.

Theorem. *Let $k, l \in \mathbf{N} \setminus \{0\}$. For every $i \in \{1, \ldots, k\}$ and every $j \in \{1, \ldots, l\}$, let $2 \leq |m_i| \leq n_i$ and $2 \leq |p_j| \leq q_j$. Let*

$$\mathrm{BS}(m_1, n_1) \times \cdots \times \mathrm{BS}(m_k, n_k) \curvearrowright (X, \mu)$$

and

$$\mathrm{BS}(p_1, q_1) \times \cdots \times \mathrm{BS}(p_l, q_l) \curvearrowright (Y, \eta)$$

be stably orbit equivalent free ergodic probability measure preserving actions such that the actions $\langle a_1 \rangle \times \cdots \times \langle a_k \rangle \curvearrowright (X, \mu)$ and $\langle b_1 \rangle \times \cdots \times \langle b_l \rangle \curvearrowright (Y, \eta)$ are aperiodic.

Then $k = l$ and there is a permutation $\sigma \in \mathrm{Sym}(k)$ such that for every $i \in \{1, \ldots, k\}$, we have

- $|m_i| = n_i = |p_{\sigma(i)}| = q_{\sigma(i)}$, *if* $|m_i| = n_i$ *and*
- $m_i = p_{\sigma(i)}$ *and* $n_i = q_{\sigma(i)}$, *if* $|m_i| \neq n_i$.

When for every $i \in \{1, \ldots, k\}$ and every $j \in \{1, \ldots, l\}$, we have $2 \leq |m_i| < n_i$ and $2 \leq |p_j| < q_j$, the conclusion of the main theorem is that $k = l$ and there is a permutation $\sigma \in \mathrm{Sym}(k)$ such that for every $i \in \{1, \ldots, k\}$, $\mathrm{BS}(m_i, n_i) \cong \mathrm{BS}(p_{\sigma(i)}, q_{\sigma(i)})$. Note that in the case when $k = l = 1$, $2 \leq |m| < n$, $2 \leq |p| < q$, n is not a multiple of m and q is not a multiple of p, the main theorem had already been proven in [15, Theorem 1.4]. However, even in this particular case, our proof is new.

References

[1] G. Baumslag and D. M. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201.

[2] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynam. Systems **1** (1981), 431–450.

[3] H. A. Dye, *On groups of measure preserving transformations. I*, Amer. J. Math. **81** (1959), 119–159.

[4] H. A. Dye, *On groups of measure preserving transformations. II*, Amer. J. Math. **85** (1963), 551–576.

[5] M. Elder and G. Willis, *Totally disconnected groups from Baumslag-Solitar groups*, arXiv:1301.4775.

[6] B. Farb and L. Mosher, *A rigidity theorem for the solvable Baumslag-Solitar groups. (With an appendix by Daryl Cooper)*, Invent. Math. **131** (1998), 419–451.

[7] A. Furman, *Orbit equivalence rigidity*, Ann. of Math. **150** (1999), 1083–1108.

[8] A. Furman, *A survey of measured group theory*, In B. Farb and D. Fisher, editors, Geometry, rigidity and group actions, pages 296–374. Chicago, IL: University of Chicago Press, 2011.

[9] D. Gaboriau, *Cost of equivalence relations and groups. (Coût des relations d'équivalence et des groupes)*, Invent. Math. **139** (2000), 41–98.

[10] D. Gaboriau, *ℓ^2 invariants of equivalence relations and groups. (Invariants ℓ^2 de relations d'équivalence et de groupes)*, Publ. Math. Inst. Hautes Étud. Sci. **95** (2002), 93–150.

[11] D. Gaboriau, *Orbit equivalence and measured group theory*, In R. Bhatia et al., editors, Proceedings of the international congress of mathematicians, Hyderabad, India, August 19–27, 2010, volume III, pages 1501–1527. Hackensack, NJ: World Scientific; New Delhi: Hindustan Book Agency, 2011.

[12] M. Gromov, *Geometric group theory. Volume 2: Asymptotic invariants of infinite groups*, volume **182** of London Mathematical Society Lecture Note Series, Cambridge: Cambridge University Press, 1993.

[13] Y. Kida, *Measure equivalence rigidity of the mapping class group*, Ann. of Math. **171** (2010), 1851–1901.

[14] Y. Kida, *Introduction to measure equivalence rigidity of mapping class groups*, In A. Papadopoulos, editor, Handbook of Teichmüller theory. Volume II. Zürich: European Mathematical Society, 2009.

[15] Y. Kida, *Invariants of orbit equivalence relations and Baumslag-Solitar groups*, To appear in Tohoku Math. J.

[16] D. I. Moldavanskii, *Isomorphism of the Baumslag-Solitar groups*, Ukrainian Math. J. **43** (1991), 1569–1571.

[17] N. Monod and Y. Shalom, *Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. **164** (2006), 825–878.

[18] N. Meesschaert and S. Vaes, *Partial classification of the Baumslag-Solitar group von Neumann algebras*, arXiv:1301.0510.

[19] D. S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions*, Bull. Amer. Math. Soc., New Ser. **2** (1980), 161–164.

[20] S. Popa, *Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups*, Invent. Math. **170** (2007), 243–295.

[21] S. Popa, *Deformation and rigidity for group actions and von Neumann algebras*, In M. Sanz-Solé et al., editors, Proceedings of the international congress of mathematicians, Madrid, Spain, August 22–30, 2006, volume I: Plenary lectures and ceremonies, pages 445–477. Zürich: European Mathematical Society, 2007.

[22] S. Popa, *On the superrigidity of malleable actions with spectral gap*, J. Amer. Math. Soc. **21** (2008), 981–1000.
[23] Y. Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. **192** (2004), 119–185.
[24] K. Whyte, *The large scale geometry of the higher Baumslag-Solitar groups*, Geom. Funct. Anal. **11** (2001), 1327–1343.

Amenability of groups acting by homeomorphisms on compact spaces

Kate Juschenko
(joint work with V. Nekrashevych and Mikael de la Salle)

The subject of amenability essentially begins in 1900's with Lebesgue. He asked whether the properties of his integral are really fundamental and follow from more familiar integral axioms. This led to the study of positive, finitely additive and translation invariant measures on different spaces. In particular the study of isometry-invariant measures led to the Banach-Tarski decomposition theorem in 1924. The class of amenable groups was introduced and studied by von Neumann in 1929 and he explained why the paradox appeared only in dimensions greater or equal to three. In 1940's and 1950's a major contribution was made by M. Day in his paper on amenable semigroups.

In 1940's the amenability theory shifted into the field of functional analysis, mainly due to the fact that integration agains a positive, finitely additive measure on a space X produces a continuous linear functional μ on $l_\infty(X)$ such that $m(1) = 1 = \|\mu\|$. Currently amenability theory appears in many fields of mathematics, most notably in operator algebras, functional analysis, ergodic theory, probability theory, harmonic analysis. Many conjectures are verified to be true on amenable groups. In many cases when a statement is true for a particular amenable group, for example for $\mathbb{Z}$, it turns out to be true for all amenable groups. In spite of the large list of equivalent definitions of amenability, it is frequently hard and challenging to decide whether a particular group is amenable.

Our recent research develops a technique that can be used to prove amenability of several classes of groups. Since simple groups are building blocks in the group theory it is a natural question to try to find examples of simple amenable groups. The classical example of finitely supported alternating group $A(\infty)$ is simple and amenable, however $A(\infty)$ is not finitely generated. Surprisingly the question of existence of finitely generated simple and amenable group remained open for decades. Recently, in collaboration with N. Monod, [6], we solved this longstanding problem by showing that the full topological group of a Cantor minimal system is amenable. The amenability of this group was previously conjectured by R. Grigorchuk and K. Medynets, [4]. The algebraic properties of the full topological group were studied by H. Matui, [7], who proved that the commutator subgroup of $[[T]]$ is simple and finitely generated for any Cantor minimal subshift $[[T]]$. Two systems (T_1, X_1) and (T_2, X_2) are *flip-conjugate* if T_1 is conjugate to T_2 or T_2^{-1}. By a result of Giordano-Putnam-Skau, [3], the full topological group is a complete invariant of flip-conjugacy for (T, X). Thus combining our result with results of Giordano-Putnam-Skau and Matui we obtain $2^{\aleph_0}$ pairwise non-isomorphic infinite amenable simple finitely generated groups.

Continuing our work on amenability, together with V. Nekrashevych and M. de la Salle, [5], we developed a machinery which produced even more new examples of amenable groups with interesting properties and answered a sequence of open problems. Our proofs are more of probabilistic nature: the main ingredient is to find an action of a group on a discrete set such that all connected components of the Schreier graph of this action are recurrent (as simple random walk).

The main theorem covers amenability of several important classes of groups that act on rooted trees: bounded automorphisms, groups generated by finite automata of linear and quadratic growth. This covers and extends the main results of L. Bartholdi, V. Kaimanovich and V. Nekrashevych, [2], as well as of Amir, Angel and Virag, [1],. Their proof of amenability is very technical. In a sense, our methods provide more direct and unified proof. Using our general approach we also prove amenability of the groups which naturally appear in dynamic: one is a holonomy group of the stable foliation of the Julia set of a Hénnon map, the other is the iterated monodromy group of a mating of two quadratic polynomials. Even though the technique is very general (it covered all known non-elementary amenable groups!) we are convinced that it can be modified to cover many other important examples, which we plan to chase in our future research.

An important and difficult question is to verify that the groups that satisfy the conditions of our main theorem imply *Liouville property*, which is our work in progress. We also plan to develop further the existence of invariant means for the cases when the *Scheier graph* of the action is not recurrent. The question of the existence of means in the transient case is important for understanding amenability of interval exchange transformation group and Thompson group F.

References

[1] G. Amir, O. Angel, B. Virág, *Amenability of linear-activity automaton groups*, Journal of the European Mathematical Society 15 (2013), no. 3, 705–730.

[2] L. Bartholdi, V. Kaimanovich, V. Nekrashevych, *On amenability of automata groups, Duke Mathematical Journal, 154 (2010), no. 3, 575–598.*

[3] Th. Giordano, I. F. Putnam, Ch. F. Skau, *Full groups of Cantor minimal systems,* Israel J. Math. 111 (1999), 285–320.

[4] R. Grigorchuk, K. Medynets, *On algebraic properties of full topological groups,* arXiv:1105.0719

[5] K. Juschenko, V. Nekrashevych, M. de la Salle, *Extensions of amenable groups by recurrent groupoids.* Submitted to Annals of Math., arXiv:1305.2637.

[6] K. Juschenko, N. Monod, *Cantor systems, piecewise translations and simple amenable groups.* To appear in Annals of Math, 2013.

[7] H. Matui, *Some remarks on topological full groups of Cantor minimal systems,* Internat. J. Math. 17 (2006), no. 2, 231–251.

Borel complexity and automorphism groups

David Kerr
(joint work with Martino Lupini, N. Christopher Phillips and Wilhelm Winter)

By encoding classification problems as equivalence relations on standard Borel spaces, one can compare their relative complexity through the mechanism of Borel reduction. This idea has its roots in the concept of smoothness, which Mackey introduced in order to formalize the notion of classifiability in the context of unitary representations of locally compact groups. For equivalence relations E and F on standard Borel spaces X and Y, we say that E is *Borel reducible* to F if there is a Borel map $\theta : X \to Y$ such that, for all $x_1, x_2 \in X$,

$$\theta(x_1)F\theta(x_2) \Leftrightarrow x_1Ex_2.$$

The relation E is *smooth* it is Borel reducible to the relation of equality on $\mathbb{R}$. Resolving a conjecture of Mackey, Glimm showed in a celebrated theorem that a separable C*-algebra is type I precisely when the classification of its irreducible representations is smooth [6].

At a higher level of complexity is the concept of *classification by countable structures*, which for an equivalence relation E means Borel reducibility to the isomorphism relation on the space of countable structures of a countable language [7, Defn. 2.38], or, equivalently, Borel reducibility to the orbit equivalence relation of a Borel action of the infinite permutation group S_∞ on a Polish space [1, Sect. 2.7]. A nonsmooth example of this is Elliott's classification of AF algebras by their ordered K-theory [3].

For a continuous action $G \curvearrowright X$ of a Polish group on a Polish space, Hjorth introduced the notion of *generic turbulence* and showed that it obstructs classification by countable structures for the associated orbit equivalence relation [7, Sect. 3.2]. Generic turbulence occurs for the action of the space of measure-preserving automorphisms of a standard atomless probability space on itself by conjugation [5] and also for the conjugation action $\mathrm{Aut}(R) \curvearrowright \mathrm{Aut}(R)$ where $\mathrm{Aut}(R)$ is the space of automorphisms of the hyperfinite II_1 factor R [8]. In contrast, the relation of conjugacy in the space of homeomorphisms of the Cantor set X is classifiable by countable structures, since this space can be identified with the set of automorphisms of the Boolean algebra of clopen subsets of X. However, when noncommutativity is introduced into the topological setting we have discovered nonclassifiability by countable structures to be a common phenomenon in "zero-dimensional" situations, as the following theorems demonstrate. The topology on the automorphism groups here is the Polish one of pointwise norm convergence.

Theorem 1. *Let A be the Jiang-Su algebra $\mathcal{Z}$, the Cuntz algebra $\mathcal{O}_2$, the Cuntz algebra $\mathcal{O}_\infty$, a UHF algebra of infinite type, or the tensor product of a UHF algebra of infinite type and $\mathcal{O}_\infty$. Then the conjugation action* $\mathrm{Aut}(A) \curvearrowright \mathrm{Aut}(A)$ *is generically turbulent.*

A C*-algebra A is $\mathcal{Z}$*-stable* if $A \otimes \mathcal{Z} \cong A$ (see [4] for a discussion of $\mathcal{Z}$-stability and its relation to other regularity properties).

Theorem 2. *Let A be a separable $\mathcal{Z}$-stable C^*-algebra. Then the orbit equivalence relation of the conjugation action* $\mathrm{Aut}(A) \curvearrowright \overline{\mathrm{Inn}(A)}$ *is not classifiable by countable structures.*

The cutting and swapping arguments which are employed in [8] to prove generic turbulence for the action $\mathrm{Aut}(R) \curvearrowright \mathrm{Aut}(R)$ do not work in the C*-algebraic framework because of the much more stringent nature of the norm topology in comparison with the 2-norm topology. To establish Theorem 1, our strategy is to use the malleability of the tensor product shift automorphism of $A^{\otimes \mathbb{Z}}$ for those C*-algebras A under consideration. Malleability requires that we be able to continuously interchange the factors of a tensor product of shift actions in a way that commutes with this tensor product. Its von Neumann algebra version plays a crucial role in Popa's deformation-rigidity theory [9]. In order to derive Theorem 2 from Theorem 1 we relativize a result of Rosendal [10, Prop. 18] which gives a criterion in terms of periodic approximation for all of the conjugacy classes of a Polish group to be meager.

We also apply the relativization of Rosendal's result to prove the following theorem in the II_1 factor context with respect to the the point-2 norm topology on automorphism groups. A II_1 factor M is *McDuff* if $M \overline{\otimes} R \cong M$.

Theorem 3. *Let M be a separable II_1 factor which is either McDuff or a free product of II_1 factors. Then the orbit equivalence relation of the conjugation action* $\mathrm{Aut}(M) \curvearrowright \overline{\mathrm{Inn}(M)}$ *is not classifiable by countable structures.*

This result applies in particular to the free group factor $L(F_r)$ for every integer $r \geq 2$, since $L(F_r) \cong L(F_{r-1}) * R$ by [2, Thm. 4.1].

References

[1] H. Becker and A. S. Kechris. *The Descriptive Set Theory of Polish Group Actions.* London Mathematical Society Lecture Note Series, 232. Cambridge University Press, Cambridge, 1996.

[2] K. Dykema. On certain free product factors via an extended matrix model. *J. Funct. Anal.* **112** (1993), 31–60.

[3] G. A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra* **38** (1976), 29–44.

[4] G. A. Elliott and A. S. Toms. Regularity properties in the classification program for separable amenable C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)* **45** (2008), 229–245.

[5] M. Foreman and B. Weiss. An anti-classification theorem for ergodic measure-preserving transformations. *J. Eur. Math. Soc.* **6** (2004), 277–292.

[6] J. Glimm. Type I C^*-algebras. *Ann. of Math. (2)* **73** (1961), 572–612.

[7] G. Hjorth. *Classification and Orbit Equivalence Relations.* American Mathematical Society, Providence, RI, 2000.

[8] D. Kerr, H. Li, and M. Pichot. Turbulence, representations, and trace-preserving actions. *Proc. London Math. Soc.* **100** (2010), 459–484.

[9] S. Popa. Deformation and rigidity for group actions and von Neumann algebras. In: International Congress of Mathematicians. Vol. I. Eur. Math. Soc., Zürich, 2007, 445–477.

[10] C. Rosendal. The generic isometry and measure preserving homeomorphism are conjugate to their powers. *Fund. Math.* **205** (2009), 1–27.

C^*-correspondences related to Dini spaces

Eberhard Kirchberg

Topological characterizations of primitive spectra of separable amenable C*-algebras were given several years ago in a long preprint with H. Harnisch based on joint work with M. Rørdam. The shortest formulation is: A sober T_0-space X is homeomorphic to the primitive spectrum (= primitive ideal space) of a separable, amenable C*-algebra if and only if the lattice $\mathcal{O}(X)$ of open subsets of X is order-isomorphic to a sub-lattice $Y \subset \mathcal{O}(P)$ of a locally compact Polish space P that is closed under l.u.b. (unions of families of open sets in Y) and g.l.b. (interiors of intersections of families of open sets in Y). Here, we call the second countable locally (quasi-)compact T_0-spaces X 'Dini spaces', because they are determined by the semi-lattice of its 'Dini functions' $f\colon X \to [0,\infty)$, i.e. functions that satisfy the classical 'Lemma of Dini': Upward directed nets $\{g_\nu\}$ of lower semi-continuous functions on X that converge point-wise to f converge also uniformly to f on X.

Principal open question: *Is every Dini space homeomorphic to the primitive spectrum of a separable, amenable C*-algebra?* An answer is even unknown for the Dini space $X_u := \mathrm{Prim}(C^*(F_2)))$. It is believable —but still unknown if true or not— that each Dini space X is at least homeomorphic to an intersection of an open and a closed subset of X_u.

A Dini space X is called 'coherent' if the intersection of any two compact G_δ subsets of X is a again a compact subset of X. It is a non-trivial result that coherent Dini spaces are homeomorphic to primitive spectra of separable amenable C*-algebras.

Let $S := \{0, 1/n\,;\ n \in \mathbb{N}\} \subset [0,1]$. The Dini space $\mathrm{Prim}(A_0)$ is not coherent if A_0 denotes the AF-algebra of the $a \in \mathrm{C}(S, M_2)$ with $a(0)$ a diagonal matrix.

New observations show that all (abstract) Dini spaces X are naturally related to C*-correspondences of amenable C*-algebras. In this way all Dini spaces become natural parts of the C*-algebra theory. The key result in this direction is the following:

Theorem. *Let X be a Dini space. There exists a stable, separable, amenable C*-algebra $A \cong A \otimes \mathcal{O}_2$ with* coherent *primitive spectrum $Z := \mathrm{Prim}(A)$ and a *-monomorphism $h\colon A \to \mathcal{M}(A)$ such that h and the l.s.c action $\Psi\colon \mathcal{I}(A) \to \mathcal{I}(A)$ of Z on A given by $\Psi(J) := h^{-1}(h(A) \cap M(A,J))$ satisfy:*

- *(i) The lattice $\mathcal{O}(X)$ is order-isomorphic to the image $\Psi(\mathcal{I}(A))$ of Ψ in the lattice $\mathcal{I}(A) \cong \mathcal{O}(Z)$,*
- *(ii) $h(A)A = A$,*
- *(iii) h is unitarily equivalent in $\mathcal{M}(A)$ to its infinite repeat $\delta_\infty \circ h := h \oplus h \oplus \cdots$,*
- *(iv) $\mathcal{M}(h) \circ h$ is approximately equivalent to h in $\mathcal{M}(A)$,*
- *(v) $J \subset \Psi(J)$ for all $J \in \mathcal{I}(A)$.*

I report on applications, outline proofs and ideas for further developments.

Entropy and L^2-torsion

Hanfeng Li
(joint work with Andreas Thom)

The L^2-torsion was introduced by Carey-Mathai [1, 2] and Lück-Rothenberg [11]. For a nice survey, see [10].

Let Γ be a countable discrete group. Denote by $\mathcal{L}\Gamma$ the group von Neumann algebra of Γ, and by tr the canonical trace of $\mathcal{L}\Gamma$ defined by

$$\mathrm{tr}(f) = \langle f\delta_e, \delta_e \rangle ,$$

where δ_e denotes the vector in $\ell^2(\Gamma)$ taking value 1 at the identity element e and 0 everywhere else. For any $f \geq 0$ in $\mathcal{L}\Gamma$, there is a unique Borel probability measure on the interval $[0, \|f\|]$, called the *spectral measure* of f and denoted by μ_f, satisfying

$$\int_0^{\|f\|} x^n \, d\mu_f(x) = \mathrm{tr}(f^n)$$

for all $n = 0, 1, 2, \ldots$. For any $f \in \mathcal{L}\Gamma$, the *Fuglede-Kadison determinant* of f, denoted by $\det_{\mathrm{FK}} f$, is defined by

$$\det{}_{\mathrm{FK}} f = e^{\int_0^{\|f\|} \log(x) \, d\mu_{|f|}(x)} \in [0, +\infty).$$

Lück's modified determinant of f, denoted by $\det'_{\mathrm{FK}} f$, is defined by

$$\det{}'_{\mathrm{FK}} f = e^{\int_{0+}^{\|f\|} \log(x) \, d\mu_{|f|}(x)} \in [0, +\infty).$$

Similarly, one can extend tr to $M_n(\mathcal{L}\Gamma)$, and $\det_{\mathrm{FK}}$ and $\det'_{\mathrm{FK}}$ to $M_{m,n}(\mathcal{L}\Gamma)$ naturally.

Denote by $\mathbb{Z}\Gamma$ the integral group ring of Γ. Lück's *Determinant Conjecture* says that $\det'_{\mathrm{FK}} f \geq 1$ for every $f \in M_{m,n}(\mathbb{Z}\Gamma)$. Elek and Szabó proved this conjecture for sofic groups [5], which include all amenable and residually finite groups.

Let $\mathcal{M}$ be a (left) $\mathbb{Z}\Gamma$-module of type FL, i.e. it admits a finite resolution:

$$0 \to \mathcal{C}_k \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \to \mathcal{M} \to 0 \tag{1}$$

with each $\mathcal{C}_j$ being a finitely generated free $\mathbb{Z}\Gamma$-module. Assume that

$$0 \to \ell^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{C}_k \xrightarrow{\mathrm{id}\otimes\partial_k} \cdots \xrightarrow{\mathrm{id}\otimes\partial_2} \ell^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{C}_1 \xrightarrow{\mathrm{id}\otimes\partial_1} \ell^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{C}_0 \to 0$$

is *weakly acyclic* in the sense that

$$\overline{\mathrm{im}(\mathrm{id} \otimes \partial_{j+1})} = \ker(\mathrm{id} \otimes \partial_j)$$

for all j. Take a basis for each $\mathcal{C}_j$, we may identify ∂_j with the right multiplication by some $f_j \in M_{d_j, d_{j-1}}(\mathbb{Z}\Gamma)$. The L^2-*torsion* of $\mathcal{M}$, denoted by $\rho^{(2)}(\mathcal{M})$, is defined as

$$\rho^{(2)}(\mathcal{M}) = \sum_{j=1}^{k} (-1)^{j+1} \log \det{}'_{\mathrm{FK}} f_j.$$

When Lück's Determinant Conjecture holds, in particular when Γ is sofic, $\rho^{(2)}(\mathcal{M})$ does not depend on the choice of the resolution and the bases of $\mathcal{C}_j$.

Given any countable $\mathbb{Z}\Gamma$-module $\mathcal{M}$, denote by $\widehat{\mathcal{M}}$ the Pontryagin dual of $\mathcal{M}$ as a discrete abelian group. Then $\widehat{\mathcal{M}}$ is a compact metrizable abelian group. The $\mathbb{Z}\Gamma$-module structure of $\mathcal{M}$ corresponds to an action of Γ on $\widehat{\mathcal{M}}$ by continuous automorphisms, called an *algebraic action* of Γ.

When Γ is amenable, one can define the topological entropy for any continuous action of Γ on a compact metrizable space X as follows. For any finite open cover $\mathcal{U}$ of X, denote by $N(\mathcal{U})$ the smallest cardinality of subcovers of $\mathcal{U}$. Take a left Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ of Γ. For any finite open cover $\mathcal{U}$ of X, the Ornstein-Weiss lemma [8, Theorem 6.1] implies that the limit $\lim_{n\to\infty} \frac{\log N(\bigvee_{s\in F_n} s^{-1}\mathcal{U})}{|F_n|}$ exists and does not depend on the choice of the Følner sequence. The *topological entropy* of the action is defined as the supremum of $\lim_{n\to\infty} \frac{\log N(\bigvee_{s\in F_n} s^{-1}\mathcal{U})}{|F_n|}$ for $\mathcal{U}$ ranging over all finite open covers of X. When Γ acts on a probability measure space by measure-preserving transformations, one can also define the measure entropy of the action. For algebraic actions of Γ, Deninger showed that the topological entropy coincides with the measure entropy with respect to the normalized Haar measure [3].

Our main result is the following:

Theorem 1. *Let Γ be a countable discrete amenable group, and $\mathcal{M}$ a (left) $\mathbb{Z}\Gamma$-module for which the L^2-torsion is defined. Then the L^2-torsion of $\mathcal{M}$ is equal to the topological entropy of the corresponding algebraic action of Γ on $\widehat{\mathcal{M}}$.*

The L^2-torsion of Γ is defined as the L^2-torsion of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$, when the latter is defined. A *finite classifying space* for Γ is a connected finite CW-complex with fundamental group Γ and contractible universal covering space. When Γ admits a finite classifying space, its L^2-torsion is defined. As a consequence of Theorem 1, we obtain the following

Corollary 2. *Let Γ be a countable discrete amenable group admitting a finite classifying space. If Γ is nontrivial, then the L^2-torsion of Γ is 0.*

Corollary 2 proves a conjecture of Lück, see Conjecture 9.24 in [9], Conjecture 11.3 in [10] and the remark after Corollary 1.11 in [12]. Wegner [13] proved Corollary 2 for elementary amenable groups using the structure theory of this class of groups.

Another consequence of Theorem 1 is the following

Corollary 3. *Let Γ be a countable discrete amenable group. Let $\mathcal{M}$ be a (left) $\mathbb{Z}\Gamma$-module for which the L^2-torsion is defined. Then the L^2-torsion of $\mathcal{M}$ is nonnegative.*

A special case of Theorem 1 is when $\mathcal{M} = \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ for f a non-zero-divisor in $\mathbb{Z}\Gamma$. One may identify the Pontryagin dual X_f of $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ with the closed subgroup $\{x \in (\mathbb{R}/\mathbb{Z})^\Gamma : xf^* = 0\}$ of $(\mathbb{R}/\mathbb{Z})^\Gamma$, and then the action of Γ on X_f is simply the restriction of the left shift action of Γ to this subgroup. In this case, one obtains

Theorem 4. *Let Γ be a countable discrete amenable group and f be a non-zero-divisor in $\mathbb{Z}\Gamma$. Then the topological entropy of the natural algebraic action of Γ on the Pontryagin dual of $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ is equal to the logarithm of the Fuglede-Kadison determinant of f.*

Theorem 4 proves a conjecture of Deninger [3]. It was previously proved by Yuzvinskiĭ [14] for the case $\Gamma = \mathbb{Z}$, by Lind, Schmidt and Ward [7] for the case $\Gamma = \mathbb{Z}^d$, by Deninger [3] for the case Γ is finitely generated of polynomial growth and f is nonnegative in $\mathcal{L}\Gamma$ and invertible in $\ell^1(\Gamma)$, by Deninger and Schmidt [4] for the case Γ is amenable and residually finite and f is invertible in $\ell^1(\Gamma)$, and by Li [6] for the case Γ is amenable and f is invertible in $\mathcal{L}\Gamma$.

References

[1] A. L. Carey and V. Mathai, *L^2-acyclicity and L^2-torsion invariants*, In: Geometric and Topological Invariants of Elliptic Operators (Brunswick, ME, 1988), pp. 91–118, Contemp. Math., 105, Amer. Math. Soc., Providence, RI, 1990.

[2] A. L. Carey and V. Mathai, *L^2-torsion invariants*, J. Funct. Anal. **110** (1992), no. 2, 377–409.

[3] C. Deninger, *Fuglede-Kadison determinants and entropy for actions of discrete amenable groups*, J. Amer. Math. Soc. **19** (2006), no. 3, 737–758.

[4] C. Deninger and K. Schmidt, *Expansive algebraic actions of discrete residually finite amenable groups and their entropy*, Ergod. Th. Dynam. Sys. **27** (2007), no. 3, 769–786.

[5] G. Elek and E. Szabó, *Hyperlinearity, essentially free actions and L^2-invariants. The sofic property*, Math. Ann. **332** (2005), no. 2, 421–441.

[6] H. Li, *Compact group automorphisms, addition formulas and Fuglede-Kadison determinants*, Ann. of Math. (2) **176** (2012), no. 1, 303–347.

[7] D. Lind, K. Schmidt and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. **101** (1990), 593–629.

[8] E. Lindenstrauss and B. Weiss, *Mean topological dimension*, Israel J. Math. **115** (2000), 1–24.

[9] W. Lück, *L^2-invariants of regular coverings of compact manifolds and CW-complexes*, In: Handbook of Geometric Topology, pp. 735–817, North-Holland, Amsterdam, 2002.

[10] W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-theory*, Springer-Verlag, Berlin, 2002.

[11] W. Lück and M. Rothenberg, *Reidemeister torsion and the K-theory of von Neumann algebras*, K-Theory **5** (1991), no. 3, 213–264.

[12] W. Lück, R. Sauer, and C. Wegner, *L^2-torsion, the measure-theoretic determinant conjecture, and uniform measure equivalence*, J. Topol. Anal. **2** (2010), no. 2, 145–171.

[13] C. Wegner, *L^2-invariants of finite aspherical CW-complexes*, Manuscripta Math. **128** (2009), no. 4, 469–481.

[14] S. A. Yuzvinskiĭ, *Computing the entropy of a group of endomorphisms* (Russian), Sibirsk. Mat. Ẑ. **8** (1967), 230–239, translated in Siberian Math. J. **8** (1967), 172–178.

C*-algebras for semigroups coming from integral domains

Xin Li

Given a left or right cancellative semigroup P, we can construct its left or right reduced semigroup C*-algebra $C^*_\lambda(P)$ or $C^*_\rho(P)$. The construction is analogous to the group case: We just form the left or right regular representation of P

on $\ell^2(P)$ and take the C*-algebra it generates. In the following, we give a survey over recent results about these semigroup C*-algebras for semigroups coming from integral domains.

Let R be an integral domain, i.e. a commutative ring with unit, which has no zero-divisors. We will only consider countable rings. The semigroup of interest is the $ax+b$-semigroup $P = R \rtimes R^\times$. We want to study the semigroup C*-algebras $C^*_\lambda(R \rtimes R^\times)$ and $C^*_\rho(R \rtimes R^\times)$. Let Q be the quotient field of R, and let $G = Q \rtimes Q^\times$ be the $ax+b$-group over Q. P sits as a subsemigroup in G in a canonical way. Moreover, $P \subseteq G$ satisfies the Toeplitz condition from [4], both for the left and right versions. Apart from the Toeplitz condition, we also need the independence condition:

Set $\mathcal{I} := \{(x_1 R) \cap \ldots \cap (x_n R)\colon x_i \in Q^\times\}$. These are divisorial ideals of R. We say that R satisfies independence if for every I, I_1, ..., I_n in $\mathcal{I}$,

$I = \bigcup_{i=1}^n I_i$ implies that $I = I_i$ for some $1 \leq i \leq n$.

For example, rings of algebraic integers in number fields satisfy this condition.

Now let R be the ring of algebraic integers in a number field. Here is a list of known results about $C^*_\lambda(R \rtimes R^\times)$:

- (i) $C^*_\lambda(R \rtimes R^\times) \cong \mathcal{O}_\infty \otimes C^*_\lambda(R \rtimes R^\times)$ has been proven in [2],
- (ii) the primitive ideal space for $C^*_\lambda(R \rtimes R^\times)$ has been computed in [3],
- (iii) K-theory for $C^*_\lambda(R \rtimes R^\times)$ and $C^*_\rho(R \rtimes R^\times)$ has been determined in [2],
- (iv) a classification result for $C^*_\lambda(R \rtimes R^\times)$ has been established in [5],
- (v) KMS-states for a canonical dynamical system $(C^*_\lambda(R \rtimes R^\times), \sigma)$ have been computed in [1].

It turns out that rings of algebraic integers satisfy the independence condition because they are Dedekind domains, i.e. noetherian, integrally closed domains with the property that every nonzero prime ideal is maximal. A natural generalization of Dedekind domains is given by the notion of Krull rings. For instance, a noetherian domain is a Krull ring if and only if it is integrally closed. It was proven in [6] that every Krull ring also satisfies the independence condition. Also, every integral domain which contains an infinite field satisfies independence (see [6]).

Here are generalizations of the results (i), (ii) and (iii):

Theorem 1. Let R be a (countable) integral domain. If R is not a field, and if the Jacobson radical of R is (0), then $C^*_\lambda(R \rtimes R^\times) \cong \mathcal{O}_\infty \otimes C^*_\lambda(R \rtimes R^\times)$.

For a Krull ring R, set $\mathcal{P}(R) = \{$prime ideals of R of height1$\}$, set $\mathcal{P}_{\text{fin}}(R) := \{\mathfrak{p} \in \mathcal{P}(R)\colon [R : \mathfrak{p}^{(i)}] < \infty \text{ for all } i \in \mathbb{N}\}$, and $\mathcal{P}(R) = \mathcal{P}_{\text{fin}}(R) \cup \mathcal{P}_{\text{inf}}(R)$. Here $\mathfrak{p}^{(i)}$ is the i-fold product of $\mathfrak{p}$ with itself (multiplication of divisorial ideals is basically given by multiplication of ideals, but it is a bit more complicated).

Theorem 2. If $\mathcal{P}_{\text{inf}}(R) \neq \emptyset$ or $|\mathcal{P}_{\text{fin}}(R)| = \infty$, then $\text{Prim}(C^*_\lambda(R \rtimes R^\times)) \cong 2^{\mathcal{P}_{\text{fin}}(R)}$.

Here $2^{\mathcal{P}_{\text{fin}}(R)}$ is the power set of $\mathcal{P}_{\text{fin}}(R)$, with the power-cofinite topology. The homeomorphism above can be chosen to be order-preserving.

Theorem 3. Let R be a Krull ring, let $C(R)$ be the divisor class group of R. Then

$$K_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{\mathfrak{k} \in C(R)} K_*(C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes R^*)),$$

$$K_*(C^*_\rho(R \rtimes R^\times)) \cong \bigoplus_{\mathfrak{k} \in C(R)} K_*(C^*(\mathfrak{a}_{\mathfrak{k}^{-1}} \rtimes R^*)).$$

In particular, $K_*(C^*_\lambda(R \rtimes R^\times)) \cong K_*(C^*_\rho(R \rtimes R^\times))$.

Apart from that, we also have the following result:

Theorem 4. Assume that R is an integral domain which is not a field, but which contains an infinite field. Then $C^*_\lambda(R \rtimes R^\times)$ is a (unital) UCT Kirchberg algebra, and

$$\begin{aligned} K_*(C^*_\lambda(R \rtimes R^\times)) &\cong \bigoplus_{[I] \in Q^\times \backslash \mathcal{I}} K_*(C^*_\lambda(I \rtimes Q_I^\times)) \\ [1] &\mapsto [1]_{[R]} \end{aligned}$$

This result applies to all coordinate rings of affine varieties over infinite fields. UCT Kirchberg algebras are completely classified by their K-theory together with the position of the K_0-class of the unit.

All these results are proven in [6].

Remark 1. It is very interesting to compare the left and right reduced semigroup C*-algebras $C^*_\lambda(R \rtimes R^\times)$ and $C^*_\rho(R \rtimes R^\times)$. As C*-algebras, they are completely different, but surprisingly, their K-theory coincide in many cases. It would be desirable to understand this in a better way.

References

[1] J. Cuntz, C. Deninger and M. Laca, *C*-algebras of Toeplitz type associated with algebraic number fields*, Math. Ann. **355** (2013), 1383–1423.

[2] J. Cuntz, S. Echterhoff and X. Li, *On the K-theory of the C*-algebra generated by the left regular representation of an Ore semigroup*, accepted for publication in Journal Euro. Math. Soc.

[3] S. Echterhoff and M. Laca, *The primitive ideal space of the C*-algebra of the affine semigroup of algebraic integers*, Math. Proc. Camb. Phil. Soc. **154** (2013), 119–126.

[4] X. Li, *Nuclearity of semigroup C*-algebras and the connection to amenability*, Advances in Math. **244** (2013), 626–662.

[5] X. Li, *On K-theoretic invariants of semigroup C*-algebras attached to number fields*, arXiv:1212.3199.

[6] X. Li, *Semigroup C*-algebras of $ax + b$-semigroups*, arXiv:1306.5553.

Classifying circle actions up to equivariant KK-equivalence

Ralf Meyer
(joint work with Rasmus Bentmann)

One step in the classification of C*-algebras (with extra structure) is to lift an isomorphism from an invariant to an equivariant KK-equivalence. Previous work by Ralf Meyer and Ryszard Nest allows to do this assuming projective resolutions of length 1. This can be achieved in a few cases by choosing a sufficiently rich invariant. Projective resolutions of length 2 are much more common. In this situation, we may enumerate the possible liftings of a given invariant by a certain Ext^2-group.

This general method is illustrated by the case of circle actions. Let $\mathrm{KK}^{\mathbb{T}}$ be the equivariant Kasparov category $\mathrm{KK}^{\mathbb{T}}$ for the circle group $\mathbb{T}$. The *equivariant bootstrap class* $\mathcal{B}$ in $\mathrm{KK}^{\mathbb{T}}$ consists of those $\mathbb{T}$-actions for which the crossed product belongs to the usual non-equivariant bootstrap class. Let $\mathcal{C}$ be the category of countable, $\mathbb{Z}/2$-graded modules over the representation ring of $\mathbb{T}$. The equivariant K-theory of a $\mathbb{T}$-algebra A is countable $\mathbb{Z}/2$-graded and carries a natural action of the representation ring by exterior tensor product. This gives a functor $F\colon \mathcal{B} \to \mathcal{C}$.

Definition 1. Let M be an object of $\mathcal{C}$. A *lifting* of M is an object $\hat{M}$ of $\mathcal{B}$ together with an isomorphism $\phi\colon F(\hat{M}) \to M$. An *isomorphism* of liftings $(\hat{M}_1, \phi_1) \to (\hat{M}_2, \phi_2)$ is an isomorphism $\psi\colon \hat{M}_1 \to \hat{M}_2$ with $\phi_2 \circ F(\psi) = \phi_1$.

We are going to classify liftings of increasingly complex objects of $\mathcal{C}$. First a trivial case:

Lemma 2. *If $A \in \mathcal{B}$ and $F(A) \cong 0$, then $A \cong 0$.*

This means that we are in a bootstrap-class situation. It also implies that an element in $\mathrm{KK}^{\mathbb{T}}(A_1, A_2)$ for $A_1, A_2 \in \mathcal{B}$ is invertible whenever the induced map $F(A_1) \to F(A_2)$ is so.

Proposition 3. *There is a fully faithful functor F^* from the subcategory of projective objects in $\mathcal{C}$ to $\mathcal{B}$ such that $F(F^*(P)) \cong P$ for all projective P and $\mathrm{KK}^{\mathbb{T}}(F^*(P), B) \cong \mathcal{C}(P, F(B))$ for all $B \in \mathcal{B}$.*

Thus F is a *universal* stable homological functor in the notation of [1]. In this case, [1] provides the following Universal Coefficient Theorem:

Theorem 4. *Let M be an object of $\mathcal{C}$ with a projective resolution of length 1. Then there is $\hat{M} \in \mathcal{B}$ with $F(\hat{M}) \cong M$. If $F(\hat{M}_1) \cong M \cong F(\hat{M}_2)$, then this isomorphism is induced by an invertible element in $\mathrm{KK}^{\mathbb{T}}(\hat{M}_1, \hat{M}_2)$. For any object B of $\mathrm{KK}^{\mathbb{T}}$, there is a natural short exact sequence*

$$\mathrm{Ext}^1_{\mathcal{C}}(\Sigma M, F(B)) \rightarrowtail \mathrm{KK}^{\mathbb{T}}(\hat{M}, B) \twoheadrightarrow \mathrm{Ext}^0_{\mathcal{C}}(M, F(B)).$$

Here Σ denotes the degree shift.

Thus objects with a length-1 projective resolution still have a unique lifting up to isomorphism. But the lifting is no longer natural. Morphisms on the invariant no longer lift uniquely, but the non-uniqueness is controlled by $\mathrm{Ext}^1_{\mathcal{C}}$.

We now come to the new results. In our concrete case, all objects of $\mathcal{C}$ have projective resolutions of length 2. Hence the following theorem always applies:

Theorem 5. *Any object M of $\mathcal{C}$ with a projective resolution of length 2 has a lifting. Isomorphism classes of liftings of M are in bijection with the group $\mathrm{Ext}^2_{\mathcal{C}}(\Sigma M, M)$.*

Corollary 6. *If $\mathrm{Ext}^2_{\mathcal{C}}(\Sigma M, M) = 0$, then M has a unique lifting up to isomorphism.*

If $\hat{M}_1, \hat{M}_2 \in \mathcal{B}$ and $\mathrm{Ext}^2_{\mathcal{C}}(\Sigma F(\hat{M}_1), F(\hat{M}_2)) = 0$, then any isomorphism $F(\hat{M}_1) \cong F(\hat{M}_2)$ lifts to an invertible element in $\mathrm{KK}^{\mathbb{T}}(\hat{M}_1, \hat{M}_2)$.

For instance, if M is concentrated only in even degree, then $\mathrm{Ext}^2_{\mathcal{C}}(\Sigma M, M) = 0$ for parity reasons. Hence such M have a unique lifting. If M is arbitrary and $M_\pm$ are its even and odd parts, then the direct sum of the liftings of M_+ and M_- is a canonical lifting of M. It corresponds to the zero element in $\mathrm{Ext}^2_{\mathcal{C}}(\Sigma M, M)$.

As an example, consider a Cuntz–Krieger algebra A with its usual gauge action. Its equivariant K-theory is the K-theory of the fixed point algebra, which is an AF-algebra. Hence $F(A)$ has no odd part. Thus up to equivariant KK-equivalence the given Cuntz–Krieger algebra is the only lifting of $F(A)$.

In general, we get a complete invariant for objects of $\mathcal{B}$ by taking pairs (M, δ) with $\delta \in \mathrm{Ext}^2_{\mathcal{C}}(\Sigma M, M)$ as objects and isomorphisms $f\colon M_1 \to M_2$ with $f_* f^*(\delta_1) = \delta_2$ as arrows $(M_1, \delta_1) \to (M_2, \delta_2)$. That is, isomorphism classes of objects in $\mathcal{B}$ are in bijection with isomorphism classes of such pairs (M, δ).

We do not have a general recipe for computing, for a given C*-algebra A, the obstruction class $\delta \in \mathrm{Ext}^2_{\mathcal{C}}(\Sigma F(A), F(A))$.

References

[1] R. Meyer, R. Nest, *Homological algebra in bivariant K-theory and other triangulated categories. I*, in Triangulated categories, Cambridge Univ. Press (2010), p. 236–289. DOI 10.1017/CBO9781139107075.006.

Persistent Approximation Property for C^*-algebras with propagation

Hervé Oyono-Oyono
(joint work with Guoliang Yu)

C^*-algebras with propagation show up in the study of elliptic differential operators from the point of view of index. These C^*-algebras encode the large scale properties of the underlying geometric structure. Their K-theory is the receptacle for higher indices. These higher indices are local objects and as such have finite propagation. But when we take their K-theory classes, we lose the information concerning propagation. In [3] was developed a quantitative K-theory for these algebras associated to higher index problem that takes into account the propagation. This quantitative K-theory behaves quite similarly than K-theory regarding to homotopy invariance and can be used as a target for quantitative higher indices. Realizing a K-theory class of a C^*-algebra as an index arises in many problems like the Baum-Connes conjecture or the idempotent conjecture. A necessary condition is that K-theory classes should be approximated in a uniform way by quantitative K-theory classes in a sense that we shall see below. The general framework to model these propagation phenomena is the the notion of filtered C^*-algebras.

Definition 1. A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of linear subspaces such that $A_r \subset A_{r'}$ if $r \leqslant r'$, A_r is stable by involution, $A_r \cdot A_{r'} \subset A_{r+r'}$ and $\bigcup_{r>0} A_r$ is dense in A. If A is unital, we also require that the identity 1 is an element of A_r for every positive number r.

Example 2.

(1) *If Σ is a proper metric discrete space, then $\mathcal{K}(\ell^2(\Sigma))$ is filtered by the family $(\mathcal{K}(\ell^2(\Sigma))_r)_{r>0}$ of compact operators with propagation less than r. Similarly, if A is a C^*-algebra, then $A\otimes\mathcal{K}(\ell^2(\Sigma))$ is filtered by $(A\otimes\mathcal{K}(\ell^2(\Sigma))_r)_{r>0}$*

(2) *Let Σ be a proper discrete metric space, and let H be a separable Hilbert space. Let further $C[\Sigma]_r$ be the space of locally compact operators on $\ell^2(\Sigma)\otimes H$ with propagation less than r. The Roe algebra of Σ is then given by $C^*(\Sigma) = \overline{\cup_{r>0} C[\Sigma]_r} \subset \mathcal{L}(\ell^2(\Sigma) \otimes H)$. By definition, $C^*(\Sigma)$ is filtered by $(C[\Sigma]_r)_{r>0}$.*

(3) *If Γ is a finitely generated group equipped with a word metric, let $B(e,r)$ be for any $r > 0$ the ball of radius r centered at the neutral element. Let us set $\mathbb{C}[\Gamma]_r = \{x \in \mathbb{C}[\Gamma]$ with support in $B(e,r)\}$. Then $C^*_{red}(\Gamma)$ and $C^*_{max}(\Gamma)$ are filtered by $(\mathbb{C}[\Gamma]_r)_{r>0}$. Similarly, if Γ acts on a C^*-algebra A by automorphisms, then $A \rtimes_{red} \Gamma$ and $A \rtimes_{max} \Gamma$ are filtered C^*-algebras.*

Quantitative K-theory is defined in terms of quasi-projections and quasi-unitaries.

Definition 3. If A is a filtered unital C^*-algebra, ε is in $(0, 1/4)$ and $r > 0$

- $q \in A$ is an ε-r projection if $q \in A_r$, $q = q^*$ and $\|q^2 - q\| < \varepsilon$.
- $u \in A$ is an ε-r unitary if $u \in A_r$, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$.

Notice that if q is an ε-r projection, then it has a spectral gap around 1/2. We denote by $\kappa(q)$ the spectral projection associated to q. To define quantitative K-theory, we proceed now as for usual K-theory. Let $A = (A_r)_{r>0}$ be a unital filtered C^*-algebra, let $\mathrm{P}^{\varepsilon,r}(A)$ be the set of ε-r-projections and let $\mathrm{U}^{\varepsilon,r}(A)$ be the set of ε-r-unitaries. We also set $\mathrm{P}^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,r}(M_n(A))$ and $\mathrm{U}^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,r}(M_n(A))$. We define the following equivalence relations:

- on $\mathrm{P}^{\varepsilon,r}_\infty(A) \times \mathbb{N}$, $(p,l) \sim (q,l')$ if there is $k \in \mathbb{N}$ and $h \in \mathrm{P}^{\varepsilon,r}_\infty(C([0,1],A))$ such that $h(0) = \mathrm{diag}(p, I_{k+l'})$ and $h(1) = \mathrm{diag}(q, I_{k+l})$.
- on $\mathrm{U}^{\varepsilon,r}_\infty(A)$, $u \sim v$ if there exist $w \in \mathrm{P}^{\varepsilon,r}_\infty(C([0,1],A))$ such that $w(0) = u$ and $w(1) = v$.

Definition 4. For a unital filtered C^*-algebra $A = (A_r)_{r>0}$, then

- $K_0^{\varepsilon,r}(A) = \mathrm{P}^{\varepsilon,r}(A)/\sim$ and $[p,l]_{\varepsilon,r}$ is the class of (p,l) mod. $\sim$.
- $K_1^{\varepsilon,r}(A) = \mathrm{U}^{\varepsilon,r}(A)/\sim$ and $[u]_{\varepsilon,r}$ is the class of u mod. $\sim$.

We can check that $[p,l]_{\varepsilon,r} + [p',l']_{\varepsilon,r} = [\mathrm{diag}(p,p'), l+l']_{\varepsilon,r}$ provides $K_0^{\varepsilon,r}(A)$ with an abelian group structure and that $[u]_{\varepsilon,r} + [v]_{\varepsilon,r} = [\mathrm{diag}(u,v)]_{\varepsilon,r}$ provides $K_1^{\varepsilon,r}(A)$ with an abelian semi-group structure. Notice that if u is an ε-r-unitary, then $[u]_{2\varepsilon,3r} + [u^*]_{2\varepsilon,3r} = [1]_{2\varepsilon,3r}$. If we equip $\mathbb{C}$ with its obvious filtration, we have an isomorphism $K_0^{\varepsilon,r}(\mathbb{C}) \stackrel{\cong}{\to} \mathbb{Z}$; $[p,l]_{\varepsilon,r} \mapsto \operatorname{rank} \kappa(p) - l$ and $K_1^{\varepsilon,r}(\mathbb{C})$ is trivial. This allows to define quantitative K-theory in the non-unital case as follows:

Definition 5. If A is a non unital filtered C^*-algebra and A^+ its unitarization,

- $K_0^{\varepsilon,r}(A) = \ker:\ K_0^{\varepsilon,r}(A^+) \to K_0^{\varepsilon,r}(\mathbb{C}) \cong \mathbb{Z}$ for the map induced by $A^+ \to \mathbb{C}$; $(a,\lambda) \mapsto \mathbb{C}$;
- $K_1^{\varepsilon,r}(A) = K_1^{\varepsilon,r}(A^+)$;

We have for $0 < \varepsilon \le \varepsilon' < 1/4$ and $0 < r \le r'$ structure homomorphisms

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0(A)$; $[p,l]_{\varepsilon,r} \mapsto [\kappa(p)] - [I_l]$.
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1(A)$; $[u]_{\varepsilon,r} \mapsto [u]$.
- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon',r'}(A)$; $[p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'}$.
- $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \longrightarrow K_1^{\varepsilon',r'}(A)$; $[u]_{\varepsilon,r} \mapsto [u]_{\varepsilon',r'}$.
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r}$ and $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_0^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}$.

Then we can check that for any $0 < \varepsilon < 1/4$ and any y in $K_*(A)$, there exists $r > 0$ and x in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = y$. Moreover for any $\varepsilon \in (0, 1/4)$ there exist ε' in $[\varepsilon, 1/4)$ such that for any $r > 0$ and any x in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = 0$, there exists $r' \geqslant r$ such that $\iota_*^{\varepsilon,\varepsilon',r,r'}(x) = 0$ in $K_*^{\varepsilon',r'}(A)$. In many geometrical situations, realizing all K-theory classes as indices implies that the r' above only depends on r and ε and not on x. This leads us to introduce the following Persistence Approximation Property. Let us consider for a filtered C^*-algebra A and positive numbers ε, ε', and r' such that $0 < \varepsilon \le \varepsilon' < 1/4$ and $0 < r \le r'$ the following statement:

$\mathcal{PA}_*(A, \varepsilon, \varepsilon', r, r')$: for any $x \in K_*^{\varepsilon,r}(A)$, then $\iota_*^{\varepsilon,r}(x) = 0$ in $K_*(A)$ implies that $\iota_*^{\varepsilon,\varepsilon',r,r'}(x) = 0$ in $K_*^{\varepsilon',r'}(A)$.

Theorem 6. *Let Γ be a finitely generated group. Assume that Γ satisfies the Baum-Connes conjecture with coefficients and admits a cocompact universal example for proper action. Then for some universal $\lambda > 1$, any $\varepsilon \in (0, \frac{1}{4\lambda})$, any C^*-algebra A acted upon by Γ and any $r > 0$, there exists $r' \geqslant r$ such that $\mathcal{PA}_*(A \rtimes_r \Gamma, \varepsilon, \lambda\varepsilon, r, r')$ holds.*

Typical examples that satisfy these assumptions are hyperbolic groups [1], or fundamental groups of compact oriented 3-manifolds [2]. We focus on the case of $A = C_0(\Gamma, B)$, equipped with the diagonal action of Γ by left translations on $C_0(\Gamma)$ and trivial on B. Then we can identify $C_0(\Gamma, B) \rtimes_r \Gamma$ and $B \otimes \mathcal{K}(\ell^2(\Gamma))$ as filtered C^*-algebras. As a corollary of the previous theorem, we get

Corollary 7. *Under the assumptions of previous theorem, then for some universal* $\lambda > 1$*, any* $\varepsilon \in (0, \frac{1}{4\lambda})$*, any* C^**-algebra* B *and any* $r > 0$*, there exists* $r' \geqslant r$ *such that* $\mathcal{PA}_*(B \otimes \mathcal{K}(\ell^2(\Gamma)), \varepsilon, \lambda\varepsilon, r, r')$ *holds.*

Notice that this statement in purely geometric. In view of this, we consider for a discrete metric space Σ with bounded geometry the following geometric statement:

$$\mathcal{PA}_\Sigma(\varepsilon, \varepsilon', r, r') : \text{ for any } C^*\text{-algebra } A, \text{ then } \mathcal{PA}_*(A \otimes \mathcal{K}(\ell^2(\Sigma), \varepsilon, \varepsilon', r, r') \text{ holds.}$$

The following concept is the geometrical counterpart of the existence of a cocompact universal example for proper actions.

Definition 8. A discrete metric space Σ is uniformly coarsely contractible if for every $r > 0$ there exists $r' > r$ such that any compact substet of $P_r(\Sigma)$ lies in a contractible compact subset of of $P_{r'}(\Sigma)$.

We are now in position to state the analogue of theorem 7 in a geometric setting.

Theorem 9. *Let* Σ *be a discrete metric space with bounded geometry. Assume that* Σ *is uniformly coarsely contractible and embeds coarsely in a Hilbert space. Then for any* $\varepsilon \in (0, \frac{1}{4})$ *and any* $r > 0$*, there exists* ε' *in* $[\varepsilon, 1/4)$ *and* $r' \geqslant r$ *such that* $\mathcal{PA}_\Sigma(\varepsilon, \varepsilon', r, r')$ *holds.*

In particular, if Σ is hyperbolic in the Gromov sense, then Σ satisfies the assumptions of the this theorem. The Persistence Approximation Property is indeed related to the Novikov conjecture in the following way. Recall first that the Novikov conjecture is implied by the coarse Baum-Connes conjecture.

Theorem 10. *Let* Σ *be a discrete metric space with bounded geometry such that*

- *for every* $r > 0$ *there exists* $r' \geqslant r$ *such that for any finite subset* F *of* Σ*, then* $P_r(F)$ *lies in a contractible compact subset of of* $P_{r'}(F)$*;*
- *for any* $\varepsilon \in (0, \frac{1}{4})$*, any* $r > 0$ *and any finite subset* F *of* Σ*, there exists* ε' *in* $[\varepsilon, 1/4)$ *and* $r' \geqslant r$ *such that* $\mathcal{PA}_F(\varepsilon, \varepsilon', r, r')$ *holds.*

Then the coarse Baum-Connes conjecture holds for Σ*.*

References

[1] V. Lafforgue, Vincent, *La conjecture de Baum-Connes à coefficients pour les groupes hyperboliques*, Noncommut. Geom. **6** (2012), no. 1, 1–197
[2] M. Matthey, H Oyono-Oyono, W. Pistch. *Homotopy invariance of higher signatures and 3-manifold groups*, Bull. Soc. Math. France 136 (2008), no. 1, 1-25.
[3] H. Oyono-Oyono, G. Yu. *On a quantitative operator* K*-theory*, to appear at Ann. Math. Fourier.

Continuous bundles of tracial von Neumann algebras

Narutaka Ozawa

I gave a talk based on the work [4]. I presented the following results and gave a sketch of the proof and discussed the difficulty that arises when one deals with infinite-dimensional compact topological space K.

Recently there has been considerable progress in the classification theory for finite simple nuclear C*-algebras. It was realized that some regularity property is necessary for C*-algebras to be classifiable by K-theoretic invariants. Several regularity properties have been proposed and a conjecture is made that the main three should be in fact equivalent. We state it for the stably finite case, where the tracial state space $T(A)$ is non-empty by Haagerup's theorem.

Conjecture (Toms–Winter)**.** For a separable unital simple nuclear stably finite infinite-dimensional C*-algebra A, the following are equivalent.

(1) $A \cong A \otimes \mathcal{Z}$.
(2) A has strict comparison.
(3) A has finite decomposition rank.

The implication (1) ⇒ (2) has been proved by Rørdam ([5]), and (3) ⇒ (1) has been proved by Winter ([8]). Matui and Sato ([3]) have proved (2) ⇒ (1), provided that

$$(*) \qquad \mathbb{M}_k \hookrightarrow A' \cap \ell_\infty(\mathbb{N}, A)/\{(a_n)_{n=1}^\infty : \lim_n \|a_n\|_{2,\mathrm{u}} = 0\},$$

where $\|a\|_{2,\mathrm{u}} = \sup_{\tau \in T(A)} \tau(a^*a)^{1/2}$ is the uniform 2-norm. The assertion $(*)$ is true if the extreme boundary $\partial T(A)$ is finite ([3]), or more generally $\partial T(A)$ is compact and finite-dimensional ([2, 6, 7]). To have a better insight into this problem, we will look at the C*-completion M of A with respect to the uniform 2-norm:

$$\begin{aligned} M &:= \{(a_n)_{n=1}^\infty \in \ell_\infty(\mathbb{N}, A) : \text{ a Cauchy sequence in } \|\cdot\|_{2,\mathrm{u}}\}/\sim \\ &= \text{SOT-closure}\,(\pi_E(A)) \subset \mathbb{B}(L^2(A, E)), \end{aligned}$$

where $E \colon A \to C(T(A))$ is the evaluation map, which is completely positive. The notion of continuous bundles of tracial von Neumann algebras naturally arises in this context.

Theorem 1. *Assume $\partial T(A)$ is compact. Then, M is a continuous* W*-*bundle over $\partial T(A)$.*

Here, for a compact topological space K, we say a C*-algebra M is a (tracial) *continuous* W*-*bundle* over K if there are a unital embedding $C(K) \subset M$ and a faithful conditional expectation from M onto $C(K)$ which is tracial: $E(ab) = E(ba)$ for all $a, b \in M$ (which implies $C(K)$ is central), and the closed unit ball of M is complete with respect to the uniform 2-norm $\|a\|_{2,\mathrm{u}} = \|E(a^*a)\|^{1/2}$ (i.e., $\pi_E(M) \subset \mathbb{B}(L^2(M, E))$ is SOT-closed). The simplest non-trivial example of a continuous W*-bundle is

$$C_\sigma(K, \mathcal{R}) := \{f \in \ell_\infty(K, \mathcal{R}) : \ f \text{ is continuous from } K \text{ into } L^2(\mathcal{R})\}$$

where $\mathcal{R}$ denotes the hyperfinite II_1 factor. For $\lambda \in K$, let $\tau_\lambda := \mathrm{ev}_\lambda \circ E$ and π_λ its GNS representation. Then, we have the following.

Theorem 2. *For any continuous* W^**-bundle* M *over* K*, the following holds.*

(1) $\pi_\lambda(M)$ *is a von Neumann algebra.*
(2) Every "continuous" $f\colon K \ni \lambda \mapsto f(\lambda) \in \pi_\lambda(M)$ *gives rise to an element* $a \in M$ *such that* $f(\lambda) = \pi_\lambda(a)$*.*
(3) Assume that M *is* $\|\cdot\|_{2,\mathrm{u}}$*-separable and* $\pi_\lambda(M)$ *is hyperfinite for every* λ*. Then,* $M \hookrightarrow C_\sigma(K,\mathcal{R})$ *as a* W^**-bundle. They are moreover isomorphic if in addition* $\pi_\lambda(M) \cong \mathcal{R}$ *and* $\dim K < +\infty$*.*

The last statement is a W^*-analogue of a result of Dadarlat–Winter ([1]) for C^*-bundles. In fact, we have the following classification result of continuous W^*-bundles.

Theorem 3. *Let* M *be a* $\|\cdot\|_{2,\mathrm{u}}$*-separable continuous* W^**-bundle over* K *such that* $\pi_\lambda(M) \cong \mathcal{R}$ *for every* $\lambda \in K$*. Then,* $M \cong C_\sigma(K,\mathcal{R})$ *as a* W^**-bundle if and only if* M *has the property* (Γ) *as a* W^**-bundle.*

The last condition holds true if $\dim K < +\infty$ (and hence Theorem 2.(3)), but it remains unclear for the case $\dim K = +\infty$.

References

[1] M. Dadarlat and W. Winter; *Trivialization of $C(X)$-algebras with strongly self-absorbing fibres.* Bull. Soc. Math. France **136** (2008), 575–606.

[2] E. Kirchberg and M. Rørdam; *Central sequence* C^**-algebras and tensorial absorption of the Jiang-Su algebra.* J. Reine Angew. Math., to appear. arXiv:1209.5311

[3] H. Matui and Y. Sato; *Strict comparison and $\mathcal{Z}$-absorption of nuclear* C^**-algebras.* Acta Math. **209** (2012), 179–196.

[4] N. Ozawa; *Dixmier approximation and symmetric amenability for* C^**-algebras.* J. Math. Sci. Univ. Tokyo, to appear. arxiv:1304.3523

[5] M. Rørdam; *The stable and the real rank of $\mathcal{Z}$-absorbing* C^**-algebras.* Internat. J. Math. **15** (2004), 1065–1084.

[6] Y. Sato; *Trace spaces of simple nuclear* C^**-algebras with finite-dimensional extreme boundary.* Preprint. arXiv:1209.3000

[7] A. Toms, S. White, and W. Winter; *$\mathcal{Z}$-stability and finite dimensional tracial boundaries.* Preprint. arXiv:1209.3292

[8] W. Winter; *Nuclear dimension and $\mathcal{Z}$-stability of pure* C^**-algebras.* Invent. Math. **187** (2012), 259–342.

Character rigidity and its consequences

Jesse Peterson

A character on a group Γ is a class function τ of positive type which is normalized so that $\tau(e) = 1$. The set of characters forms a convex space and the extreme points are naturally in bijective correspondence (via the GNS-construction) to unitary representations which generated von Neumann algebra is a finite factor.

In 1964, Thoma [7] initiated the systematic study of characters on infinite discrete groups, classifying all extremal characters for the group of finite permutations of the natural numbers. We'll say a group Γ is character rigid if the only extremal characters correspond to either the left regular representation or else a finite dimensional representation. The first example of character rigid groups were found by Kirillov [4] who showed this property for the groups $PSL_n(k)$ where $n \geq 3$, and k is an arbitrary infinite field. More recently Bekka [1] has shown that, in fact, the group $PSL_n(\mathbb{Z})$ is also character rigid for $n \geq 3$, giving the first such example for an irreducible lattice in a higher rank semi-simple group. This is significant since it was conjectured by Connes (based on the rigidity theorems of Mostow, Margulis, and Zimmer) that all such lattices are character rigid (see the discussion in [3]).

In my talk I discussed joint work with Andreas Thom [6] where we extend Kirillov and Bekka's results to the PSL_2 setting. For some consequences of this phenomenon we refer the reader to [5] in the previous report.

Theorem 1. *Let R be either an infinite field, or else a ring of algebraic integers with infinitely many units, than $PSL_2(R)$ is character rigid.*

The proof consists of two parts. The first part is to conclude that if a representation generates a II_1 factor, then the representation is mixing when restricted to the abelian subgroup of diagonal matrices. The second part is then to conclude that the representation is actually the left regular representation.

As motivation for the first part of the argument we present here a proof of the Howe-Moore property for $SL_2(\mathbb{R})$, which has a number of similar ideas.

Theorem 2 ([2, 8])**.** *Suppose $\pi : SL_2(\mathbb{R}) \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representation which does not have invariant vectors, then the representation is mixing, i.e., $\lim_{g\to\infty}\langle \pi(g)\xi, \eta\rangle = 0$, for all $\xi, \eta \in \mathcal{H}$.*

Proof. Suppose $\pi : SL_2(\mathbb{R}) \to \mathcal{U}(\mathcal{H})$ is not mixing. Hence, there exist $g_n \in SL_2(\mathbb{R})$ such that $g_n \to \infty$, but such that $\pi(g_n)$ does not converge to 0 in the weak operator topology. Taking a subsequence we may assume that $WOT\text{-}\lim_{n\to\infty} \pi(g_n) = T \neq 0$. We may write $SL_2(\mathbb{R}) = O(2) \cdot A \cdot O(2)$ where A denotes the subgroup of positive diagonal matrices, and hence $g_n = k_n a_n k_n'$ where $k_n, k_n' \in O(2)$ and $a_n \in A$. Taking a subsequence we may assume that $k_n \to k$, and $k_n' \to k'$, and so $\pi(k_n) \to \pi(k)$, and $\pi(k_n') \to \pi(k')$ in the strong operator topology. We then have $WOT\text{-}\lim_{n\to\infty} \pi(a_n) = \pi(k)^* T \pi(k')^*$. Set $S = \pi(k)^* T \pi(k')^* \neq 0$.

Taking a further subsequence, and possibly replacing a_n with a_n^{-1} we may further assume that $a_n = \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n^{-1} \end{pmatrix}$ where $\alpha_n \to \infty$. We denote by V the subgroup of upper triangular matrices with diagonal entries equal to 1. If $b = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in V$ then $a^{-1}ba = \begin{pmatrix} 1 & \alpha_n^{-2}\beta \\ 0 & 1 \end{pmatrix} \to \mathrm{id}$. Hence

$$\pi(b)S = \lim_{n\to\infty} \pi(ba_n) = \lim_{n\to\infty} \pi(a_n)\pi(a_n^{-1}ba_n) = S.$$

Taking the inverse transpose also gives $\pi(b^t)S^* = S^*$. However, $S, S^* \in \pi(A)''$, which is abelian. Hence, $SS^* = S^*S$, and $\pi(g)SS^* = \pi(g)S^*S = S^*S$ for all $g \in \langle V, V^t\rangle = SL_2(\mathbb{R})$. Hence, any non-zero vector in the range of S is a non-zero $SL_2(\mathbb{R})$-invariant vector. □

We now present the first part of the proof of Theorem 1 in the case of fields.

Lemma 3. *Let k be an infinite field. Suppose $\pi : SL_2(k) \to \mathcal{U}(M)$ is a representation, with $M = \pi(SL_2(k))''$ a non-trivial finite factor with trace τ, then $\pi_{|A}$ is mixing.*

Proof. Let p (resp. q) be the projection onto the space of V (resp. V^t)-invariant vectors. If we fix $a \in A \setminus \{\pm\mathrm{id}\}$, then for $b \in V$ we have

$$\tau(ap^\perp) = \tau(b^{-1}ap^\perp b) = \langle (a^{-1}b^{-1}ab)p^\perp, a^{-1}\rangle_\tau. \tag{1}$$

However, $\{a^{-1}b^{-1}ab \mid b \in V\} = \left\{ \left(\begin{smallmatrix} 1 & (\alpha^{-2}-1)\beta \\ 0 & 1 \end{smallmatrix}\right) \mid \beta \in k \right\} = V$. Using Equation 1 and taking convex combinations then gives $\tau(ap^\perp) = \langle pp^\perp, a^{-1}\rangle = 0$. Taking the inverse transpose and applying the same argument also shows $\tau(aq^\perp) = 0$.

If $q \wedge p = 0$, then we would have a non-trivial invariant vector which would contradict the fact that M is a non-trivial finite factor. Thus we have $q \wedge p = 0$ and so it follows that $\ker(E(p^\perp + q^\perp)) = \{0\}$ where E is the trace preserving conditional expectation onto $\pi(A)''$. We then have that $x = E(p^\perp + q^\perp)) \geq 0$ is $\pi(A)$-cyclic for the representation restricted to $L^2(\pi(A)'')$, and we also have $\tau(\pi(a)x) = 0$ for $a \neq \pm\mathrm{id}$. It easily follows that $\pi(a) = E(\pi(a)) \to 0$ in the weak operator topology as $a \to \infty$ in A. In fact, our argument gives the stronger conclusion that in fact π is equivalent to a multiple of the left regular representation when we restrict to A. □

Using the previous lemma we may now present the proof of Theorem 1 for the case of fields.

proof of Theorem 1 for fields. Let $\pi : SL_2(k) \to \mathcal{U}(M)$ be a representation with $M = \pi(SL_2(k))''$ a non-trivial finite factor having trace τ.

Every $g \in SL_2(k) \setminus \{\pm\mathrm{id}\}$ is conjugate in $GL_2(k)$ to a matrix of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \beta \end{smallmatrix}\right)$ for some $\beta \in k$. Thus, (by possibly conjugating the representation) it is enough to show that $\tau(\pi(g)) = 0$ for every matrix g of this form.

A simple computation gives

$$\tau\left(\begin{smallmatrix} 0 & 1 \\ -1 & \beta \end{smallmatrix}\right) = \tau\left(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & \beta \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\right) = \tau\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\begin{pmatrix} \alpha^{-2} & -\beta \\ 0 & \alpha^2 \end{pmatrix}\right). \tag{2}$$

However, as $\alpha \to \infty$, the matrices $\begin{pmatrix} \alpha^{-2} & -\beta \\ 0 & \alpha^2 \end{pmatrix}$ are asymptotically τ-orthogonal. Indeed, this follows since we know from Lemma 3 that $\tau_{|A} \in c_0(A)$, and we have

$$\begin{pmatrix} \alpha_1^{-2} & -\beta \\ 0 & \alpha_1^2 \end{pmatrix}^{-1}\begin{pmatrix} \alpha_2^{-2} & -\beta \\ 0 & \alpha_2^2 \end{pmatrix} = \begin{pmatrix} \alpha_1^2\alpha_2^{-2} & \beta(\alpha_1^2-\alpha_2^2) \\ 0 & \alpha_1^{-2}\alpha_2^2 \end{pmatrix} \sim \begin{pmatrix} \alpha_1^2\alpha_2^{-2} & 0 \\ 0 & \alpha_1^{-2}\alpha_2^2 \end{pmatrix}.$$

Taking convex combinations in Equation 2 then gives $\tau\left(\begin{smallmatrix} 0 & 1 \\ -1 & \beta \end{smallmatrix}\right) = 0$. □

References

[1] B. Bekka, *Operator-algebraic supperrigidity for $SL_n(\mathbb{Z})$, $n \geq 3$*, Invent. Math. **169** (2007), no. 2, 401-425.

[2] R. E. Howe, C. C. Moore, *Asymptotic properties of unitary representations*, J. Funct. Anal. **32** (1979), no. 1, 72-96.

[3] V. Jones, *Ten problems*, *Mathematics: Frontiers and Perspectives*, (2000), 79-91.
[4] A.A. Kirillov, *Positive definite functions on a group of matrices with elements from a discrete field*, Soviet. Math. Dokl. **6** (1965), 707–709.
[5] J. Peterson, *Character rigidity and its consequences*, Oberwolfach report no. 42/2013.
[6] J. Peterson, A. Thom, *Character rigidity for special linear groups*, arXiv.org:1303.4007, preprint 2013.
[7] E. Thoma, *Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe*, Math. Z. **85** (1964), 40-61.
[8] R. J. Zimmer, *Orbit spaces of unitary representations, ergodic theory, and simple Lie groups*, Annals of Math. **106** (1977), 573-588.

Analogs of UHF algebras and Cuntz algebras on L^p spaces

N. Christopher Phillips

We give analogs of UHF and Cuntz C*-algebras which act on L^p spaces. Surprisingly, there seems to be a substantial theory, of which so far only a little is known, and for which our algebras are interesting classes of examples.

Even for $p = 2$, there is something new. We have a class of nonselfadjoint operator algebras which look like UHF algebras, but which are isomorphic (and similar) to UHF algebras if and only if they are amenable as Banach algebras. They are thus new examples for a question about which very little seems to be known, namely whether every amenable closed subalgebra of $L(H)$ is similar to a C*-algebra. See Theorem 6. (Without separability, this is false [1].)

All algebras will be unital. All measure spaces will be σ-finite. The tensor product of two L^p spaces will be taken to be $L^p(X,\mu) \otimes_p L^p(Y,\nu) = L^p(X \times Y,\, \mu \times \nu)$. The spatial L^p operator tensor product $A \otimes_p B$ of closed subalgebras $A \subset L(L^p(X,\mu))$ and $B \subset L(L^p(Y,\nu))$ is the closed subalgebra generated by all $a \otimes b \in L\big(L^p(X \times Y,\, \mu \times \nu)\big)$ for $a \in A$ and $b \in B$. (It depends on how the algebras are represented. This can probably be fixed by using matrix normed algebras.)

Many difficulties will be suppressed. For example, injective contractive homomorphisms need not have closed range. Probably also direct limits of simple algebras need not be simple.

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1. Analogs of UHF algebras on L^p spaces

Let $p \in [1,\infty)$. For $d \in \mathbb{Z}_{>0}$, we let $l_d^p = l^p\big(\{1,2,\ldots,d\}\big)$. We further let $M_d^p = L\big(l_d^p\big)$ with the usual operator norm, and we algebraically identify M_d^p with the algebra M_d of $d \times d$ complex matrices in the standard way.

Let $d = (d(n))_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\{2,3,4,\ldots\}$. Set $r(n) = d(1)d(2)\cdots d(n)$ for $n \in \mathbb{Z}_{\geq 0}$. Set $X_n = \{1,2,\ldots,d\}$ and let μ_n be normalized counting measure.

Let $(Z_n, \mathcal{D}_n, \lambda_n)$ be the product measure space $\prod_{k=n+1}^{\infty}(X_n, \mathcal{B}_n, \mu_n)$. Take

$$B_n = M_{d(1)}^p \otimes_p M_{d(2)}^p \otimes_p \cdots \otimes_p M_{d(n)}^p \otimes_p 1_{L^p(Z_n, \lambda_n)},$$

acting on $\left(\bigotimes_{k=1}^n L^p(X_k, \mu_k)\right) \otimes_p L^p(Z_n, \lambda_n) = L^p(Z_0, \lambda_0)$. Now take $B = \overline{\bigcup_{n=0}^{\infty} B_n}$. We call this algebra the *spatial L^p UHF algebra of type d.*

Theorem 5 (See Theorem 4.2 and Theorem 4.8 [4])**.** Suppose d_1 and d_2 are two sequences as above, and $p_1, p_2 \in [1, \infty)$. Let B_1 and B_2 be the corresponding algebras constructed as above. Then the following are equivalent:

(1) B_1 and B_2 are isometrically isomorphic as Banach algebras.
(2) B_1 and B_2 are isomorphic as Banach algebras.
(3) $p_1 = p_2$ and the "supernatural numbers" associated with d_1 and d_2 in the usual way are the same.

The condition on supernatural numbers is, as usual, equivalent to having the same K-theory (including the location of [1]). The algebras are also simple.

We generalize the construction. Let $K = (K_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of compact subsets of the diagonal invertible matrices in $M_{d(n)}^p$ such that $1 \in K_n$ for all n. (For $p = 2$, we need not restrict to diagonal matrices.) Choose countable families $(s_n(i))_{i \in I_n}$ which are dense in K_n. Let $(X_n, \mathcal{B}_n, \mu_n)$ be an atomic probability space such that $L^p(X_n, \mu_n) \cong \bigoplus_{i \in I_n} l_{d(n)}^p$ (L^p direct sum). Let $\rho_n(a) = \bigoplus_{i \in I_n} s_n(i) a s_n(i)^{-1} \in L(L^p(X_n, \mu_n))$ for $a \in M_{d(n)}$. Now let B_n be the subalgebra of $L(L^p(Z_0, \lambda_0))$ generated by all $\rho_1(a_1) \otimes \rho_2(a_2) \otimes \cdots \otimes \rho_n(a_n) \otimes 1$ with $a_k \in M_{d(k)}$ for $k = 1, 2, \ldots, n$. It is still isomorphic to $M_{r(n)}$, but has a different (larger) norm than $M_{r(n)}^p$. Take $B = \overline{\bigcup_{n=0}^{\infty} B_n}$.

We prove that B is simple (Theorem 3.13 of [4]) and has a unique normalized trace (Corollary 3.12 of [4]), and that $K_*(B)$ is the same as for the UHF C*-algebra constructed using the same sequence d (combine Example 3.8 of [4] and Theorem 6.4 of [6]).

The following result is part of Theorem 4.10 of [5].

Theorem 6. Let B be constructed as above. Let D be the algebra obtained the same way except replacing the functions s_n by the functions $t_n(i) = 1$ for all $i \in I_n$. Then the following are equivalent:

(1) B is isomorphic (not necessarily isometrically) to D.
(2) There is an invertible operator s such that $sBs^{-1} = D$.
(3) B is amenable as a Banach algebra (Definition 2.1.9 of [7]).
(4) B is symmetrically amenable as a Banach algebra (Definition 2.1 of [2]).
(5) The tensor flip on $B \otimes_p B$ is approximately inner.
(6) There is a uniform bound on the norms of the obvious maps $M_{r(n)}^p \to B_n$.

The algebra D in Theorem 6 is isometrically isomorphic to the spatial L^p UHF algebra of type d. For any fixed d and p, there are uncountably many nonisomorphic possibilities for B. (See Theorem 5.14 of [5].)

2. Analogs of Cuntz algebras on L^p spaces

Let $d \in \{2, 3, \ldots\}$. Let L_d be the universal complex algebra given by generators $s_1, s_2, \ldots, s_d$ and $t_1, t_2, \ldots, t_d$ such that $t_j s_k = \delta_{j,k} \cdot 1$ for all j, k, and $\sum_{j=1}^{d} s_j t_j = 1$. Then $\mathcal{O}_d$ is the C*-algebra generated by any unital representation ρ of L_d on a Hilbert space satisfying the regularity condition $\rho(t_j) = \rho(s_j)^*$ for all j. On L^p spaces, we use a different regularity condition. It can be expressed in different ways; here are three of them. (For these and many more, see Theorem 7.7 of [3].)

Theorem 7. Let $d \in \{2, 3, \ldots\}$ and let $p \in (1, \infty) \setminus \{2\}$. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let $\rho \colon L_d \to L(L^p(X, \mu))$ be a unital homomorphism. Then the following are equivalent:

(1) $\|\rho(s_j)\|$, $\|\rho(t_j)\| \leq 1$ for all $j = 1, \ldots, d$, and the map from M_d^p which for $j, k = 1, 2, \ldots, d$ sends the standard matrix unit $e_{j,k}$ to $\rho(s_j t_k)$ is contractive.

(2) For $j = 1, 2, \ldots, d$, we have $\|\rho(s_j)\|$, $\|\rho(t_j)\| \leq 1$ and there is a measurable subset $X_j \subset X$ such that the range of $\rho(s_j)$ is $L^p(X_j, \mu)$.

(3) For every numbers $\lambda_1, \lambda_2, \ldots, \lambda_d \in \mathbb{C}$ with $\|(\lambda_1, \lambda_2, \ldots, \lambda_d)\|_p = 1$, the operator $\sum_{j=1}^{d} \lambda_j \rho(s_j)$ is an isometry.

We call ρ *spatial* if it satisfies the equivalent conditions of Theorem 7. There is then a uniqueness theorem, whose proof is completely different from the C* case.

Theorem 8. Let $d \in \{2, 3, \ldots\}$ and let $p \in (1, \infty) \setminus \{2\}$. Let ρ_1 and ρ_2 be two spatial representations of L_d. Then there is an isometric isomorphism $\varphi \colon \overline{\rho_1(L_d)} \to \overline{\rho_2(L_d)}$ such that $\varphi(\rho_1(s_j)) = \rho_2(s_j)$ and $\varphi(\rho_1(t_j)) = \rho_2(t_j)$ for $j = 1, 2, \ldots, d$.

We can therefore define the Banach algebra $\mathcal{O}_d^p$ to be $\overline{\rho(L_d)}$ for any spatial representation ρ. It is simple and purely infinite (Theorem 5.14 of [4]), amenable as a Banach algebra (Corollary 5.18 of [4]), and has the same K-theory as when $p = 2$ (Theorem 7.19 of [6]). For distinct $p_1, p_2 \in [1, \infty)$ and any d_1 and d_2, there is no nonzero continuous homomorphism from $\mathcal{O}_{d_1}^{p_1}$ to $\mathcal{O}_{d_2}^{p_2}$ (Theorem 9.2 of [3]).

Cuntz's original calculation of the K-theory seems not to work for $\mathcal{O}_d^p$. Instead, we adapt the crossed product method. Thus, in [6] we define the reduced crossed product $F_{\mathrm{r}}^p(G, A, \alpha)$ for an isometric action α of a second countable locally compact group G on a nondegenerate closed subalgebra $A \subset L(L^p(X, \mu))$, and prove a number of its properties. We then show, as in the C* case, that there is an action of $\mathbb{Z}$ on a suitably stabilized spatial L^p UHF algebra of type d^∞ whose reduced L^p operator crossed product is a stabilization of $\mathcal{O}_d^p$. See Section 7 of [6]. We then use a Pimsner-Voiculescu exact sequence (Theorem 6.15 of [6]) to compute $K_*\big(\mathcal{O}_d^p\big)$.

References

[1] I. Farah and N. Ozawa, *A nonseparable amenable operator algebra which is not isomorphic to a C*-algebra*, preprint (arXiv: 1309.2145 [math.OA]).

[2] B. E. Johnson, *Symmetric amenability and the nonexistence of Lie and Jordan derivations*, Math. Proc. Cambridge Philos. Soc. **120**(1996), 455–473.

[3] N. C. Phillips, *Analogs of Cuntz algebras on L^p spaces*, preprint (arXiv: 1201.4196v1 [math.FA]).
[4] N. C. Phillips, *Simplicity of UHF and Cuntz algebras on L^p spaces*, preprint (arXiv: 1309.0115v2 [math.FA]).
[5] N. C. Phillips, *Amenability and uniqueness of UHF algebras on L^p spaces*, preprint (arXiv: 1309.3694v2 [math.FA]).
[6] N. C. Phillips, *Crossed products of L^p operator algebras and the K-theory of Cuntz algebras on L^p spaces*, preprint (arXiv: 1309.6406v1 [math.FA]).
[7] V. Runde, *Lectures on Amenability*, Springer-Verlag Lecture Notes in Math. no. 1774, Springer-Verlag, Berlin, 2002.

A homology theory for Smale spaces

Ian Putnam

Smale spaces, as defined by David Ruelle [8], are topological dynamical systems which display hyperbolicity: locally, the space has a natural structure as a product of two subsets called the local stable set (where the dynamics contracts) and the local unstable set (where it expands). These systems include Anosov diffeomorphisms, various strange attractors considered by R.F. Williams, shifts of finite type (which are precisely the zero-dimensional examples) and, most importantly, the basic sets for Smale's Axiom A systems [9, 2].

Smale spaces have an abundance of periodic points and counting them, for specific periods, provides interesting information about the dynamics. In the case of a shift of finite type (SFT), the number of periodic points of period n is just the trace of the adjacency matrix of the graph underlying the SFT, raised to the power n. This is a fairly easy result, but it bears some similarity to the Lefschetz formula for smooth maps of manifolds. This naturally raises the question: is there some type of homology theory for SFT's which underlies this formula? Positive answers were given by Bowen and Franks [3] and separately by Krieger [4]. Krieger's approach was to associate a C*-algebra to the stable (or unstable) equivalence relation (which is an AF-algebra with a stationary Bratteli diagram) and then take its K-theory.

Returning to more general Smale spaces, Bowen's seminal result [1] in the subject was that every irreducible Smale space is the image of an SFT under a finite-to-one factor map. This first indicates that SFT's have a fundamental role to play in the theory and secondly, this result can be used as a tool for the study of Smale spaces.

Manning [5] used Bowen's result to provide a formula for the number of periodic points in a general Smale space. His idea was first to find an SFT as in Bowen's theorem and then construct a sequence of systems which record when the map from the SFT to the Smale space is n-to-one, for varying values on n. Each of these systems is itself an SFT and Manning's formula calculates the periodic point data for the Smale space based on the sequence of Krieger's invariant for these.

This is even more suggestive of the Lefschetz formula and that there should exist some sort of homology theory for Smale spaces. This was conjectured Bowen [2] in the late 1970's. We present such a theory [7].

One of the difficulties with Krieger's invariant is that it has rather subtle functorial properties. This has been worked out by people in symbolic dynamics. We say that a factor map between two Smale spaces is *s-bijective* if, when restricted to a stable set in the domain, it is a bijection to a stable set in the range. Krieger's invariant is covariant for s-bijective maps and contravariant for u-bijective maps.

The key step in our proof is to obtain an improved version of Bowen's theorem. Bowen's theorem basically says that a Smale space X can be covered by a totally disconnected one (an SFT). The improved version is that X can be covered by one where the stable sets are the same as those in X (and so the covering map will be s-bijective), while the unstable sets are totally disconnected. Similarly, we can also find a Smale space which covers by a u-bijective map whose stable sets are totally disconnected [6].

Having these two systems covering X, we emulate Bowen's construction by considering systems which record when the two maps are L-to-one and M-to-one, for varying values of L, M. These are all SFT's and we construct a double complex by considering their Krieger invariants. The homology of this complex turns out to be independent of the choice of covers and provides a Lefschetz formula as conjectured by Bowen [7].

References

[1] R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. **92** (1970), 725-747.

[2] R. Bowen, *On Axiom A Diffeomorphisms*, AMS-CBMS Reg. Conf. **35**, Providence, 1978.

[3] R. Bowen and J. Franks, *Homology for zero-dimensional basic sets*, Ann. of Math. **106** (1977), 73-92.

[4] W. Krieger, *On dimension functions and topological Markov chains*, Inventiones Math. **56** (1980), 239-250.

[5] A. Manning, *Axiom A diffeomorphisms have rational zeta functions*, Bull. London Math. Soc. **3** (1971), 215-220.

[6] I. F. Putnam, *Lifting factor maps to resolving maps*, Israel J. Math. **146** (2005), 253-280.

[7] I.F. Putnam, *A homology theory for Smale spaces*, Memoirs A.M.S., to appear.

[8] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Math. and its Appl. **5**, Addison-Wesley, Reading, 1978.

[9] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817.

The classification of orthogonal easy quantum groups

Sven Raum

(joint work with Moritz Weber)

Easy quantum groups were introduced in [2] by Banica and Speicher, in the context of Wang's free quantum groups [6]. Via Speicher's partitions, this class of quantum groups has a natural link to free probability theory. The classification of orthogonal easy quantum groups was started right away in [2] and it was continued in [1, 8]. In joint work with Moritz Weber, we completed the classification of all orthogonal easy quantum groups [4, 5].

1. Partition quantum groups (or easy quantum groups)

Easy quantum groups form a particular class of compact quantum groups. They are defined combinatorially in the framework of Woronowicz's theory of compact matrix quantum groups. The name 'easy quantum group' was introduced in [2]. We prefer to refer to their class of quantum groups as *partition quantum groups* or more precisely as *orthogonal partition quantum groups.*

1.1. **Woronowicz's compact matrix quantum groups.** *Compact matrix quantum groups* (CMQG) were defined by Woronowicz in [9]. They form an analogue of compact Lie groups in the setting of non-commutative topology. A compact matrix quantum group is a unital C*-algebra A with an element $u \in \mathrm{M}_n(A)$ such that

- A is generated by the entries of u,
- there is a *-homomorphism $\Delta : A \to A \otimes A$ such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all $1 \leq i, j \leq n$,
- u is unitary and its transpose u^t is invertible.

The matrix u is called the *fundamental corepresentation* of (A, u). We also write $A = \mathrm{C}(G)$ to indicate that A should be thought of as the (non-commutative) algebra of functions on some quantum group G.

A morphism ϕ between CMQGs A and B is a morphism of the underlying C*-algebras satisfying $(\phi \otimes \mathrm{id})(u_A) = u_B$. If there is a morphism $\mathrm{C}(G) \to \mathrm{C}(H)$ between CMGQs, we say that H is a quantum subgroup of G.

A *unitary corepresentation matrix* of a compact matrix quantum group A is a unitary $v \in \mathrm{M}_k(A)$ such that $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$ for all $i, j \in \{1, \ldots, k\}$. If $v \in \mathrm{M}_k(A)$ and $w \in \mathrm{M}_l(A)$ are unitary corepresentation matrices of A, then a linear map $T : \mathbb{C}^k \to \mathbb{C}^l$ is an *intertwiner* from v to w if $Tv = wT$. The space of all such intertwiners in denoted by $\mathrm{Hom}(v, w)$. The *tensor product* $u \otimes v \in \mathrm{M}_{kl}(A)$ of two corepresentation $v \in \mathrm{M}_k(A)$ and $w \in \mathrm{M}_l(A)$ is defined as their Kronecker tensor product $(u \otimes v)_{(i,m),(j,n)} = u_{ij} v_{mn}$.

A CMQG (A, u) is called orthogonal if $u_{ij} = u_{ij}^*$ for all i, j. Writing $\overline{u} = (u_{ij}^*)$ for the entrywise conjugation of u, we see that (A, u) is orthogonal if and only if $u = \overline{u}$. By the work of Woronowicz on Tannaka-Krein duality for compact quantum groups [10], an orthogonal CMQG is uniquely determined by the spaces of intertwiners $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$, $k, l \in \mathbb{N}$.

1.2. **Partition quantum groups.** A partition consists of k upper points, l lower points and strings connecting these points. In [2], Banica and Speicher introduced a way to associate with a partition p on k upper and l lower points a linear map $T_p : (\mathrm{C}^n)^{\otimes k} \to (\mathrm{C}^n)^{\otimes l}$. They then introduced the class of easy quantum groups, which we call partition quantum groups in what follows. An orthogonal compact matrix quantum group (A, u) is called a *partition quantum group* if there are sets

$\mathcal{C}(k,l)$ of partitions on k upper and l lower points such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{T_p \mid p \in \mathcal{C}(k,l)\}$ for all $k, l \in \mathbb{N}$.

2. The classification of partition quantum groups

Partition quantum groups have several interesting aspects. We will concentrate however our presentation on their complete classification, which has been achieved by a combination of work in [2, 1, 8, 4, 5].

2.1. The classification results of Banica-Speicher, Banica-Curran-Speicher and Weber. Banica and Speicher [2] and Weber [8] completely classified partition groups and orthogonal free quantum groups. A partition group is a partition quantum group (A, u) such that A is commutative. An orthogonal free quantum group is an (orthogonal) partition quantum group whose intertwiners are described by non-crossing partitions. There are 6 partition groups and 7 free quantum groups.

Of particular importance is the free hyperoctahedral quantum group, defined as

$$\mathrm{C}(H_n^+) = \mathrm{C}^*(u_{ij}, 1 \leq i,j \leq n \mid u = \overline{u} \text{ is unitary and all } u_{ij} \text{ are partial isometries}).$$

It corresponds to the easy group $H_n = \mathbb{Z}/2\mathbb{Z}^{\oplus n} \rtimes S_n$ satisfying

$$\mathrm{C}(H_n) = \mathrm{C}^*(u_{ij}, 1 \leq i,j \leq n \mid u = \overline{u} \text{ is unitary, all } u_{ij} \text{ are partial isometries and all } u_{ij} \text{ commute}).$$

An easy quantum group (A, u) is called hyperoctahedral if it is a quantum subgroup of H_n^+, that is there is a morphism $\mathrm{C}(H_n^+) \to A$ of CMQGs, and the unitary corepresentation matrix u is irreducible. Hyperoctahedral easy quantum groups are of particular importance, since in [1, 8] all non-hyperoctahedral easy quantum groups are classified: there are exactly 13 of them.

2.2. Classification of hyperoctahedral partition quantum groups. In the classification of hyperoctahedral partition quantum groups one relation describes a dividing line between two cases. If (A, u) is a CMQG, then

$$u_{ij}^2 u_{kl} = u_{kl} u_{ij}^2 \quad \text{for all } i,j,k,l$$

says that all squares of entries of u are central in A. We say that a hyperoctahedral partition quantum group A is in the semi-direct product case if it satisfies the relation above.

2.3. The semi-direct product case. The main result of [5] shows that the hyperoctahedral partition quantum groups in the semi-direct product case are completely determined by their diagonal subgroup. They are isomorphic to bicrossed products.

If (A, u) is a CMQG consider the quotient $\pi : A \to A/(u_{ij}, \text{ for all } i \neq j)$. The diagonal subgroup of A is the discrete group generated by the images $\pi(u_{ii})$ of the diagonal entries of u. It is equipped with the generating set $g_i = \pi(u_{ii})$.

Theorem 1. *Let G be a hyperoctahedral partition quantum group in the semi-direct product case. Denote by Γ the diagonal subgroup of $\mathrm{C}(G)$ and its generators by g_i. Furthermore denote by (p_{ij}) the fundamental corepresentation of $\mathrm{C}(S_n)$ obtained from the natural embedding $S_n \hookrightarrow O_n$. Write $\mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(S_n)$ for the CMQG whose C^*-algebra is $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}(S_n)$ and whose fundamental corepresentation is $(u_{g_i} p_{ij})$. Then $\mathrm{C}(G) \cong \mathrm{C}^*(\Gamma) \bowtie \mathrm{C}(S_n)$ as CMQGs.*

2.4. **An additional countable one parameter series.** In [4] we identify all remaining hyperoctahedral partition quantum groups, therefore completing the classification of partition quantum groups. We remark that $\mathrm{C}^*((\mathbb{Z}/2\mathbb{Z})^{*n}) \bowtie \mathrm{C}(S_n)$ is the maximal hyperoctahedral partition quantum group in the semi-direct product case. All other quantum groups A in the semi-direct product case satisfy $\mathrm{C}^*(\mathbb{Z}/2\mathbb{Z}^{*n}) \bowtie \mathrm{C}(S_n) \to A \to \mathrm{C}(H_n)$.

Theorem 2. *There is a countable chain of pairwise different CMGQs*

$$\mathrm{C}(H_n^+) \to A_1 \to A_2 \to \cdots \to \mathrm{C}^*(\mathbb{Z}/2\mathbb{Z}^{*n}) \bowtie \mathrm{C}(S_n)$$

such that every hyperoctahedral partition quantum group which is not in the semi-direct product case is isomorphic to some A_i.

References

[1] Teodor Banica, Stephen Curran, and Roland Speicher. Classification results for easy quantum groups. Pac. J. Math. **247** No 1 (2010), 1–26.

[2] Teodor Banica and Roland Speicher. Liberation of orthogonal Lie groups. Adv. Math., **222** No 4 (2009), 1461–1501.

[3] Claus Köstler and Roland Speicher. A noncommutative de Finetti theorem: invariance under quantum permutations is equivalent to freeness with amalgamation. Commun. Math. Phys. **291** No 2 (2009) 473–490.

[4] Sven Raum and Moritz Weber. The full classification of partition quantum groups. In preparation.

[5] Sven Raum and Moritz Weber. A connection between easy quantum groups, varieties of groups and reflection groups. ArXiv:1212.4742.

[6] Shuzhou Wang. Free products of compact quantum groups. Commun. Math. Phys. **167** No 3 (1995), 671–692.

[7] Shuzhou Wang. Quantum symmetry groups of finite spaces. Commun. Math. Phys. **195** No 1 (1998), 195–211.

[8] Moritz Weber. On the classification of easy quantum groups. Adv. Math. **245** No 1 (2013), 500–533.

[9] Stanisław. L. Woronowicz. Compact matrix pseudogroups. Commun. Math. Phys. **111** (1987), 613–665.

[10] Stanisław. L. Woronowicz. Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups. Invent. Math. **93** No 1 (1988), 35–76

Free monotone transport

Dimitri Shlyakhtenko
(joint work with A. Guionnet)

In this talk we discuss the construction of a free analog of a *monotone transport map.*

In the classical setting, a *transport map* is a (Borel) map $F : \mathbb{R}^n \to \mathbb{R}^n$ with the property that F transforms one given probability measure into another:

$$F_*\mu_0 = \mu_1.$$

Of course, such a map always exists but is non-unique except in very special cases. Nonetheless, a theorem of Brenier [Bre91, Vil03] states that under regularity assumptions on μ_0, μ_1, there is a *unique* transport map F which in addition has the property that $F = \nabla G$ for a convex function G. Such an F is called monotone because its Jacobian derivative $JF = (\partial_j F_i)_{ij} = \operatorname{Hess} G$ is positive semi-definite.

There is much interest in obtaining non-commutative versions of such transport maps. Given two von Neumann algebras $(M_0, \tau_0) = W^*(X_1, \ldots, X_n)$ and $(M_1, \tau_1) = W^*(Y_1, \ldots, Y_n)$, we wish to construct $F = (F_1, \ldots, F_n) \in M_0$ with the property that the map $Y_j \mapsto F_j$ extends to a trace-preserving embedding $M_1 \to M_0$.

Unlike the classical case, one can only expect such a transport map to exist under substantial regularity assumptions on the non-commutative laws of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ [Oza04]. A natural category of such laws are *free Gibbs laws.*

Let $\mathcal{A} = \mathcal{A}^{(A)}$ be the Banach algebra of non-commutative power series with the norm

$$\left\| \sum_k \sum_{i_1,\ldots,i_k} \alpha(k; i_1, \ldots, i_k) X_{i_1} \ldots X_{i_k} \right\|_A = \sum_k \sum_{i_1,\ldots,i_k} |\alpha(k; i_1, \ldots, i_k)| A^k$$

and let $\mathcal{A} \otimes \mathcal{A}$ be its tensor square (for the projective tensor product). Denote by

$$\partial_j : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

Voiculescu's non-commutative difference quotient derivation [Voi98], defined on a monomial P by

$$\partial_j P = \sum_{P = RX_jQ} P \otimes Q$$

(the sum is taken over all decompositions of the monomial P as a product RX_jQ). Similarly, consider the cyclic gradient map $\mathcal{D} : \mathcal{A} \to \mathcal{A}^n$ [Voi02a] given by

$$\mathcal{D}P = (\mathcal{D}_1 P, \ldots, \mathcal{D}_n P)$$

where

$$\mathcal{D}_j P = \sum_{P = RX_jQ} QR.$$

We say that the non-commutative law of $X_1, \dots, X_n \in (M, \tau)$ is a free Gibbs law with potential $V \in \mathcal{A}^{(A)}$ ($A > \max_j \|X_j\|$) if the following equation holds:

$$\tau(\mathcal{D}_j P) = \tau \otimes \tau(\partial_j P), \qquad \forall P \in \mathcal{A}. \tag{1}$$

The free Gibbs state for a given potential need not be unique. However, in the case that $V = \frac{1}{2}\sum X_j^2$, the associated free Gibbs law is exactly Voiculescu's free semicircle law. More generally, it is known that if $V = \frac{1}{2}\sum X_j^2 + W$ and $\|W\|_A$ is sufficiently small, then there is a unique free Gibbs law with this potential [GMS06, Gui06]. Thus in these cases we write τ_V for the unique free Gibbs state with potential V.

It is of interest to determine the isomorphism classes of the von Neumann algebra $W^*(\tau_V)$ and the C^*-algebra $C^*(\tau_V)$ generated in the GNS representation of the algebra $\mathcal{A}$ associated to τ_V. It is a conjecture due to Voiculescu [Voi06, p. 240] that $W^*(\tau)$ is a free group factor whenever τ is a free Gibbs state with a polynomial potential.

Our main theorem states that this is indeed the case if $V = \frac{1}{2}\sum X_j^2 + W$ and $\|W\|_A$ is small enough. In fact, we prove the following theorem about the existence of monotone transport between the free semicircle law and free Gibbs laws of this form:

Theorem A. Let $S_1, \dots, S_n$ be a free semicircular family, and consider the free group factor $M_0 = W^*(S_1, \dots, S_n) \cong L(\mathbb{F}_n)$ with trace τ_0.

Assume that $\max_j \|S_j\| < A' < A$, $V = \frac{1}{2}\sum X_j^2 + W \in \mathcal{A}^{(A)}$ and $\|W\|_A$ is small enough. Then there exists an element $g \in \mathcal{A}^{(A')}$ so that if we set $G = \frac{1}{2}\sum X_j^2 + g$, then:

- If we set $F = (F_1, \dots, F_n) = \mathcal{D}G$, then $Y_j = F_j(S_1, \dots, S_n)$, then the law of $(Y_1, \dots, Y_n)$ is the free Gibbs state τ_V.
- The map F is monotone, in the sense that $\mathcal{J}F = (\partial_j F_i)_{i,j} \in M_{n\times n}(M_0 \otimes M_0)$ is positive semi-definite.
- The von Neumann algebra $M_1 = W^*(\tau_V) \cong W^*(Y_1, \dots, Y_n) = M_0$.

This theorem has an interesting consequence if we particularize to a specific potential. By an observation of Dabrowski [Dab10, Theorem 34], for each $2 \le n < \infty$ there exists a number $q_0 > 0$ with the following property. Let $|q| < q_0$ and let $S_1^{(q)}, \dots, S_n^{(q)}$ be the q-deformed semicircular family of Bozejko-Speicher [BS91], and let $M_q(n)$ be the corresponding von Neumann algebra. Then there exists W_q so that the law of this family is the free Gibbs law with potential $V_q = \frac{1}{2}\sum X_j^2 + W_q$.

Theorem B. For all $|q| < q_0 = q_0(n)$, $M_q(n) \cong L(\mathbb{F}_n)$.

The same statement holds for C^*-algebras, as well.

We give a quick outline of the proof of Theorem A. Let us introduce the notation $\mathcal{J}F = (\partial_j F_i)_{i,j}$ for the non-commutative Jacobian of a map F. Then the equation (1) is equivalent to the statement that

$$\mathcal{J}^*(I) = \mathcal{D}V$$

where the adjoint is taken in L^2. Thus, assuming that $S_1, \ldots, S_n$ are free semicircular variables, we seek to construct $F = \mathcal{D}G$ so that F has free Gibbs law with potential V, i.e., we have

$$\mathcal{J}_F^*(I) = \mathcal{D}V(F)$$

where the index F refers to differentiation is with respect to $F_1, \ldots, F_n$. Applying chair rule, we conclude that we want

$$\mathcal{J}^*((\mathcal{J}F)^{-1}) = \mathcal{D}(V)(F).$$

This is a consequence of the following equation for G (recall that $F = \mathcal{D}G$):

$$(1 \otimes \tau + \tau \otimes 1) Tr \log \mathcal{J}\mathcal{D}G = \mathcal{S}\left[V(\mathcal{D}G(X)) - \frac{1}{2}\sum X_j^2\right] \tag{2}$$

where $\mathcal{S}$ denotes cyclic symmetrization of non-commutative power series.

This equation can be solved by iteration in the algebra $\mathcal{A}$.

Equation (2) turns out to be the free analog of the Monge-Ampere equation. Indeed, taking the logarithm of both sides of the classical Monge-Ampere equation for the transport map $H = \nabla\Psi$ from the measure $Z^{-1}\exp(-\frac{1}{2}\sum x_j^2)dx_1 \ldots dx_n$ to the measure $Z_V^{-1}\exp(-V(x_1, \ldots, x_n))dx_1 \ldots dx_n$ yields the equation

$$\log|\det J\nabla\Psi| = Tr(\log J\nabla\Psi) = V(\nabla\Psi) - \frac{1}{2}\sum x_j^2$$

(here J denotes the Jacobian).

Free Gibbs states describe limits of certain random matrix models. More precisely, for $V = \frac{1}{2}\sum X_j^2 + W$ as above, let $\mu_V^{(N)}$ be the measure on n-tuples of $N \times N$ self-adjoint matrices having the following density with respect to Lebesgue measure restricted to matrices with operator norm no bigger than A:

$$(A_1, \ldots, A_n) \mapsto Z_{N,V}^{-1}\exp(-N\mathrm{Tr}(V(A_1, \ldots, A_n)).$$

Let $\tau_V^{(N)}$ be the trace on $\mathcal{A}$ given by

$$\tau_V^{(N)}(P) = \mathbb{E}_{\mu_V^{(N)}}\left(\frac{1}{N}\mathrm{Tr}(P)\right).$$

Then $\tau_V^{(N)} \to \tau_V$ weakly [GMS06].

It turns out that also our transport map is a limit of transport maps.

Theorem C. Let F be as in Theorem A. For each N, let $F^{(N)}$ be the map on n-tuples of $N \times N$ matrices given by applying F in the sense of non-commutative functional calculus, and let $\hat{F}^{(N)}$ be the (entry-wise) monotone transport map between the Gaussian measure and the measure $\mu_V^{(N)}$. Then

$$\|\hat{F}^{(N)} - F^{(N)}\|_{L^2(\mu_V^{(N)})} \to 0$$

as $N \to \infty$.

References

[Bre91] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math **44** (1991), 375–417.

[BS91] M. Bożejko and R. Speicher, *An example of a generalized Brownian motion*, Commun. Math. Phys. **137** (1991), 519 – 531.

[Dab10] Y. Dabrowski, *A free stochastic partial differential equation*, arXiv.org:1008:4742, 2010.

[GMS06] A. Guionnet and E. Maurel-Segala, *Combinatorial aspects of matrix models*, ALEA Lat. Am. J. Probab. Math. Stat. **1** (2006), 241–279.

[Gui06] A. Guionnet, *Random matrices and enumeration of maps*, Proceedings Int. Cong. Math. **3** (2006), 623–636.

[GS12] *Free Monotone Transport*, arXiv.org:1204.2182, 2012

[Oza04] N. Ozawa, *There is no separable universal II_1-factor*, Proc. Amer. Math. Soc. **132** (2004), 487–490.

[Vil03] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, Vol. 58, AMS, Providence, RI, 2003.

[Voi98] ———, *The analogues of entropy and of Fisher's information measure in free probability, V*, Invent. Math. **132** (1998), 189–227.

[Voi02a] ———, *Cyclomorphy*, Int. Math. Research Notices **No. 6** (2002), 299–332.

[Voi02b] ———, *Free entropy*, Bull. LMS. **34** (2002), no. 3, 257–278.

[Voi06] ———, *Symmetries arising from free probability theory*, Frontiers in Number Theory, Physics, and Geometry I (Pierre Cartier, Bernard Julia, Pierre Moussa, and Pierre Vanhove, eds.), Springer Berlin Heidelberg, 2006, pp. 231–243.

Pseudodifferential Calculus and Groupoids

Georges Skandalis

(joint work with Claire Debord)

The aim of our work is to replace pseudodifferential calculus by convolution on a suitable Lie groupoid. Our main result is that the algebra of order 0 pseudodifferential operators is Morita equivalent to a convolution algebra on a groupoid.

1. Order 0 Pseudodifferential Operators on a Manifold

Let M be a smooth compact manifold. Pseudodifferential operators on M are crucial in the Atiyah-Singer index theorem.

- They are used to show elliptic operators are Fredholm.
- They are needed in order to construct the analytic index as a K-theory map (or a corresponding KK-element) $K_0(C_0(T^*M)) \to \mathbb{Z}$.

Order 0 pseudodifferential operators give rise to a short exact sequence

$$0 \to \mathcal{K} \longrightarrow \Psi_0^*(M) \xrightarrow{\sigma_0} C(S^*M) \to 0 \tag{1}$$

The analytic index can then be constructed as a map of relative K-theory groups $K_0(C_0(T^*M)) = K(i) \to K(\sigma_0) = \mathbb{Z}$, using the commuting diagram

$$\begin{array}{ccc} \Psi_0^*(M) & \xrightarrow{\sigma_0} & C(S^*M) \\ {\scriptstyle m}\uparrow & & \| \\ C(M) & \xrightarrow{i} & C(S^*M) \end{array}$$

Connes (see [4]) explained how to construct the analytic index without any mention of pseudodifferential calculus, just by using his *tangent groupoid* G_{TM} defined by gluing $M \times M \times \mathbb{R}_+^*$ with the total space TM of the tangent bundle - the *deformation to the normal cone* of the diagonal inclusion $M \to M \times M$.

Indeed, the analytic index is the connecting map to the exact sequence

$$0 \to \mathcal{K} \otimes C_0(\mathbb{R}_+^*) \longrightarrow C^*(G_{TM}) \xrightarrow{ev_0} C^*(TM) \simeq C_0(T^*M) \to 0.$$

Even better, restricting this groupoid to $[0,1]$, we obtain a diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{K} \otimes C_0(]0,1]) & \longrightarrow & C^*(G_{TM}[0,1]) & \xrightarrow{ev_0} & C_0(T^*M) & \to 0 \\ & & & {\scriptstyle ev_1} \downarrow & & & \\ & & & \mathcal{K} & & & \end{array}$$

Since $\mathcal{K} \otimes C_0(]0,1])$ is K-contractible, ev_0 is invertible in K-theory.

Proposition. $\quad \mathrm{ind}_a = (ev_1)_* \circ (ev_0)_*^{-1}$.

This follows immediately from the existence of the equivalent of exact sequence (1) for general Lie groupoids (see below).

2. Generalization to Lie Groupoids

Let G be a Lie groupoid with objects a compact manifold M. Denote by A the total space of the Lie algebroid of G, *i.e.* the normal bundle to the inclusion $M \to G$ as units and A^* its dual bundle; let also SA^* denote the associated sphere bundle, *i.e.* the space of half lines in A^*. Alain Connes ([3]) extended the exact sequence (1) to the case of holonomy groupoids. It appeared quite naturally that Connes' construction only used the (longitudinal) smooth structure of the holonomy groupoid and was easily extended (by Nistor-Weinstein-Xu [8] and Montubert-Pierrot [7]) to any (longitudinally) smooth groupoid. We thus obtain an exact sequence

$$0 \to C^*(G) \longrightarrow \Psi^*(G) \xrightarrow{\sigma_0} C(S^*A) \to 0. \tag{2}$$

As in the single manifold case, this extension encodes the analytic index, which is a map $K_*(C_0(A^*)) \to K_*(C^*(G))$.

The generalization of Connes' tangent groupoid is also quite easy ([6, 7]) and gives rise to the so-called *adiabatic groupoid* $G_{ad} = G \times \mathbb{R}_+^* \cup A \times \{0\}$ - an exact sequence

$$0 \to C^*(G) \otimes C_0(\mathbb{R}_+^*) \longrightarrow C^*(G_{ad}) \xrightarrow{ev_0} C_0(A^*) \to 0 \tag{3}$$

and a "pseudodifferential operator free" description of the analytic index ([7]).

3. "Gauge Adiabatic Groupoid"

There is a natural smooth action of $\mathbb{R}_+^*$ on the adiabatic groupoid G_{ad}: it acts on $G \times \mathbb{R}_+^*$ by rescaling the $\mathbb{R}_+^*$ component and on A by homotheties. We thus obtain a crossed product groupoid $G_{ad} \rtimes \mathbb{R}_+^*$ that we may call Gauge Adiabatic

Groupoid and denote by G_{ga}. Taking crossed products in the exact sequence (3), we obtain a new exact sequence

$$0 \to C^*(G) \otimes \mathcal{K} \longrightarrow C^*(G_{ga}) \xrightarrow{ev_0} C_0(A^*) \rtimes \mathbb{R}_+^* \to 0.$$

Now, taking out the 0-section M of A^*, we find an ideal $C_0(A^* \setminus M) \rtimes \mathbb{R}_+^* \simeq C(S^*A) \otimes \mathcal{K}$ of $C_0(A^*) \rtimes \mathbb{R}_+^*$. Taking the pre-image of this ideal, we obtain an ideal J of the algebra $C^*(G_{ga})$ of kernels on the groupoid G_{ga} and an exact sequence

$$0 \to C^*(G) \otimes \mathcal{K} \longrightarrow J \xrightarrow{ev_0} C(S^*A) \otimes \mathcal{K} \to 0. \tag{4}$$

The similarity of exact sequences (2) and (4) is not only apparent !

Theorem. There is a Morita equivalence between $\Psi^*(G)$ and J, under which the ideal $C^*(G)$ of $\Psi^*(G)$ corresponds to the ideal $C^*(G) \otimes \mathcal{K}$ of J.

4. On the proof of Morita Equivalence

This theorem was proved by Aastrup-Melo-Monthubert-Schrohe in [1] in the case of the ordinary pseudodifferential calculus (*i.e.* the pair groupoid $M \times M$), by showing that the exact sequences (2) and (4) define the same element of KK^1 and then using Voiculescu's theorem.

We construct an explicit bimodule $\mathcal{E}$. Let us sketch this construction here:

The bimodule $\mathcal{E}$ is the completion of a space $\mathcal{E}^\infty$ of smooth compactly supported functions (half densities to be precise) on G_{ad} whose Fourier transform vanishes to infinite order on $M \times \{0\}$. An element f of $\mathcal{E}^\infty$ is a family $(f_t)_{t \in \mathbb{R}_+}$ where $f_0 \in C_c^\infty(A)$ and $f_t \in C_c^\infty(G) \subset C^*(G)$ for $t \neq 0$. A key result, which we believe has its own interest, is the following:

Proposition. For $f = (f_t)_{t \in \mathbb{R}_+} \in \mathcal{E}^\infty$, the integral $\displaystyle\int_0^\infty f_t \frac{dt}{t}$ converges strictly to an element $P \in \Psi^*(G)$ with principal symbol $\sigma_P(x,\xi) = \displaystyle\int_0^\infty \hat{f}_0(x, t\xi) \frac{dt}{t} \cdot$

For $f, g \in \mathcal{E}^\infty$, we put $\langle f | g \rangle = \displaystyle\int_0^\infty f_t^* * g_t \frac{dt}{t} \in \Psi^*(G)$. The bimodule $\mathcal{E}$ is the completion of $\mathcal{E}^\infty$ with respect to the norm $\|f\|_{\mathcal{E}} = \|\langle f | f \rangle\|^{1/2}$.

The right action is given by the following Lemma:

Lemma. Let $f = (f_t)_{t \in \mathbb{R}_+} \in \mathcal{E}^\infty$.

(1) For $g \in C_c^\infty(G)$, the family $(f_t * g)_{t \in \mathbb{R}_+^*}$ is a smooth function from $\mathbb{R}_+$ to $C_c^\infty(G)$ with rapid decay when $t \to 0$.

(2) For P a ("classical") pseudodifferential operator on G of order 0, the family $(f_t * P)_{t \in \mathbb{R}_+^*}$ extends to an element g of $\mathcal{E}^\infty$ (up to a replacement of the compact support requirement by a rapid decay one - see [2]) defined by $\hat{g}_0 = \hat{f}_0 \sigma_P$.

The left action of J is easy: $C_c^\infty(G_{ad})$ acts by left multiplication and $\mathbb{R}_+^*$ by scaling. We thus get an action of $C_c^\infty(G_{ga})$ and, after completion, of $C^*(G_{ga})$ on $\mathcal{E}$, *i.e.* a morphism $\theta : C^*(G_{ga}) \to \mathcal{L}(\mathcal{E})$.

Finally:

Proposition. $\theta(J) = \mathcal{K}(\mathcal{E})$.

See [5] for details.

References

[1] J. Aastrup, S.T. Melo, B. Monthubert and E. Schrohe, Boutet de Monvel's Calculus and Groupoids I. J. Noncommut. Geom. **4 no. 3**, (2010), 313–329.

[2] P. Carrillo Rouse, A Schwartz type algebra for the tangent groupoid. *In* K-theory and noncommutative geometry. EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, (2008), 181–199.

[3] A. Connes, Sur la théorie non commutative de l'intégration. *In* Algèbres d'opérateurs, Springer Lect. Notes in Math. **725**, (1979), 19–143

[4] A. Connes, *Non commutative geometry,* Academic Press, Inc. San Diego, CA (1994).

[5] C. Debord and G. Skandalis, Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and Pseudodifferential calculus. arXiv:1307.6320

[6] M. Hilsum and G. Skandalis, Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes). Ann. Sci. École Norm. Sup. **4 t. 20** 20 (1987), 325–390.

[7] B. Monthubert and F. Pierrot, Indice analytique et Groupoïdes de Lie. C.R. Acad. Sci. Paris, Sér. I, **325 no. 2**, (1997), 193–198.

[8] V. Nistor, A. Weinstein and P. Xu, Pseudodifferential operators on differential groupoids. Pacific J. Math. **189**, (1999), 117–152.

Polynomials in free variables

Roland Speicher

(joint work with Serban Belinschi, Tobias Mai)

For a selfadjoint random variable $x = x^*$ in some C^*-probability space (A, φ) (i.e., A is a unital C^*-algebra and φ a state on A) we denote by μ_x the distribution of x, i.e., the probability measure on the real line determined by $\int t^n d\mu_x(t) = \varphi(x^n)$ for all natural n. Usually, this measure is calculated out of the Cauchy transform $G(z) = \varphi[(z-x)^{-1}]$ of x; this is a well-defined and analytic function for all z in the complex upper half plane.

Consider now m selfadjoint variables $x_1, \dots, x_m$ which are free, with respect to the given φ. Let p be a selfadjoint polynomial in m non-commuting variables. Then we are interested in the distribution of $p(x_1, \dots, x_m)$.

Problem: How can we calculate the distribution $\mu_{p(x_1,\dots,x_m)}$ out of the distributions $\mu_{x_1}, \dots, \mu_{x_m}$ of the single variables?

For simple polynomials – like $p(x,y) = x+y$, $p(x,y) = xy$ (this is not really selfadjoint, but should be read as $p(x,y) = \sqrt{x}y\sqrt{x}$, if x is positive), or $p(x,y) = xy+yx$ – this problem was addressed and solved by Voiculescu (for the sum and the product) and by Nica and Speicher (for the anti-commutator). However, general polynomials have been elusive up to now.

We show how general selfadjoint polynomials can be treated by using the linearization trick, in order to reformulate the problem of a general polynomial as a problem of an operator-valued *linear* polynomial. First striking uses of such a linearization idea in free probability and random matrix theory are due to Haagerup and Thorbjornsen [5, 4], a more streamlined version which also preserves selfadjointness (which is relevant for our purposes) is due to Anderson [1].

As an example of this consider the polynomial $p(x,y) = xy + yx + x^2$. The linearization gives us an operator-valued linear polynomial

$$\hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}.$$

The crucial relation between p and $\hat{p}$ is that we can recover the Cauchy transform of p, $\varphi[(z-p)^{-1}]$, as the $(1,1)$-entry of the operator-valued Cauchy transform of $\hat{p}$, $\mathrm{id} \otimes \varphi[(b-\hat{p})^{-1}]$, where b is the matrix which has z as its $(1,1)$-entry and zeroes otherwise. By [1] such a selfadjoint linearization $\hat{p}$ exist for any selfadjoint polynomial p.

The calculation of the the operator-valued Cauchy transform of $\hat{p}$ can now be achieved by invoking operator-valued free probability. By writing

$$\hat{p} = \begin{pmatrix} 0 & x & \frac{x}{2} \\ x & 0 & 0 \\ \frac{x}{2} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & -1 \\ y & -1 & 0 \end{pmatrix}$$

we see that $\hat{p}$ is the sum of two matrix-valued variables. It is an easy (but crucial) observation that the freeness between x and y with respect to φ gives freeness between those two matrices with respect to $\mathrm{id} \otimes \varphi$. Since the distribution of x determines the matrix-valued distribution of the first matrix and the distribution of y determines the matrix-valued distribution of the second matrix, the determination of the matrix-valued distribution of $\hat{p}$ is now a problem about the sum of operator-valued free variables, i.e., about operator-valued convolutions.

In [2] we have developed, relying on subordination ideas, the analytic theory of such operator-valued convolutions so far that it can be used to give a satisfying and useful description of the operator-valued Cauchy transform of $\hat{p}$. (One should note that the equations for such Cauchy transforms are rarely explicityly solvable, thus it is important to have fixed point characteriztions of them which guarantee a good analytic and numerical behavior.) In the following we work in an operator-valued C^*-probability space, i.e, we have a unital C^*-subalgebra $B \subset A$, and a conditional expectation $E : A \to B$. The operator-valued Cauchy transform of a selfadjoint random variable $x \in A$ is then given by $G_x(b) = E[(b-x)^{-1}]$ for all $b \in \mathbb{H}^+(B)$, where $\mathbb{H}^+(B) := \{b \in B \mid (b-b^*)/(2i) > 0\}$ is the "complex upper half plane" in B. The main result from [2] is then as follows.

Theorem (Belinschi, Mai, Speicher 2013): Let x and y be selfadjoint operator-valued random variables which are free over B. Then there exists a Fréchet analytic map $\omega\colon \mathbb{H}^+(B) \to \mathbb{H}^+(B)$ so that $G_{x+y}(b) = G_x(\omega(b))$ for all $b \in \mathbb{H}^+(B)$. Moreover, if $b \in \mathbb{H}^+(B)$, then $\omega(b)$ is the unique fixed point of the map

$$f_b\colon \mathbb{H}^+(B) \to \mathbb{H}^+(B), \quad f_b(w) = h_y(h_x(w)+b)+b,$$

and $\omega(b) = \lim_{n\to\infty} f_b^{\circ n}(w)$ for any $w \in \mathbb{H}^+(B)$. In the above we have used the notation $h(b) := \frac{1}{G(b)} - b$.

Having reduced the problem of polynomials in free variables to operator-valued free additive convolution it is conceivable that further progress on analytic properties of operator-valued convolution will result in understanding qualitative properties of arbitrary selfadjoint polynomials in free variables. This will be pursued in future investigations. In this context we want to point out that the absence of atoms for the distribution of selfadjoint polynomials in free semicirculars (or more generally, in free variables without atoms) has been established recently by Shlyakhtenko and Skoufranis [6], by quite different methods. It is an open problem to derive these and similar results based on the above description. Another direction for future work is the extension of the present approach to general, not necessarily selfadjoint polynomials (where the eigenvalue distribution will be replaced by the Brown measure). This will be pursued in [3].

References

[1] G. Anderson, *Convergence of the largest singular value of a polynomial in independent Wigner matrices*, Preprint, 2011, arXiv:1103.4825v2.

[2] S. Belinschi, T. Mai and R. Speicher *Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem*, Preprint, 2013, archiv:1303.3196.

[3] S. Belinschi, P. Sniady and R. Speicher, in preparation.

[4] U. Haagerup, H. Schultz and S. Thorbjørnsen, *A random matrix approach to the lack of projections in* $C^*_{\text{red}}(\mathbb{F}_2)$, Adv. Math. **204** (2006), 1–83.

[5] U. Haagerup and S. Thorbjørnsen, *A new application of Random Matrices:* $\text{Ext}(C^*_{\text{red}}(F_2))$ *is not a group*, Annals of Mathematics **162** (2005), 711–775.

[6] D. Shlyakhtenko and P. Skoufranis, *Freely independent random variables with non-atomic distributions*, Preprint, 2013, archiv:1305.1920.

Ultraproducts of compact simple groups

Andreas Thom

(joint work with Philip Dowerk, Abel Stolz, John Wilson)

In this talk I report about joint works with various co-authors – some published and some work in progress. The first main result is:

Theorem 1 (Stolz-Thom). *Let $(G_n)_n$ be a sequence of non-abelian finite simple groups and let $\omega \in \beta\mathbb{N}$ be an ultrafilter. The lattice of normal subgroups of the ultraproduct $\prod_\omega G_n$ is linearly ordered by inclusion. Moreover, $\prod_\omega G_n$ has a unique simple quotient.*

For alternating groups, the first part of the result was known by work of Ellis-Hachtman-Schneider-Thomas [2], the second part by work of Elek-Szabo [1]. An essential ingredient in our approach is a seminal result by Liebeck-Shalev from [3]. We do not want to go into the details of the work Liebeck-Shalev [3], but its main result basically says that a conjugation invariant subset S of a non-abelian finite simple group G generates the group as fast as it possibly can, i.e. in

$O(\log |G|/\log |S|)$ steps. In order to extend the previous theorem to ultraproducts of compact simple groups, we need to generalize the work of Liebeck-Shalev.

We denote the normalized trace on $M_n\mathbb{C}$ by $\tau\colon M_n\mathbb{C} \to \mathbb{C}$. Let $u \in U(n)$, we set

$$\sigma_t(u) = \inf\{\|(\lambda - u)p\| \mid \tau(p) \geq 1 - t, \lambda \in S^1\}, \quad t \in [0,1]$$

and call these numbers the projective singular values. It is obvious that

$$\sigma_t(uv) = \sigma_t(vu), \quad \text{and} \quad \sigma_t(u) = \sigma_t(u^{-1}).$$

A basic consequence of the Ky-Fan inequalities is the inequality

$$\sigma_{t+s}(uv) \leq \sigma_t(u) + \sigma_s(v),$$

whenever both sides are defined. We denote by $PU(n)$ the projective unitary group and by $[u]$ the conjugacy class of the image of u in $PU(n)$. It is now elementary to see that $v \in ([u] \cup [u^{-1}])^k$ implies $\sigma_{kt}(v) \leq k \cdot \sigma_t(u)$. Our second main result is a weak converse to this – again it basically says that any conjugacy class in $PU(n)$ generates $PU(n)$ almost as fast as it possibly can.

Theorem 2 (Stolz-Thom)**.** *There exists a constant $C > 0$ such that for all $n, k \in \mathbb{N}$ and all $u, v \in PU(n)$ with $\sigma_{kt}(v) \leq k \cdot \sigma_t(u)$, we have $v \in ([u] \cup [u^{-1}])^m$ for all $m \geq Ck^2$.*

The proof relies on a combination of combinatorics and Lie theory and results obtained in seminal work of Nikolov-Segal [4]. The main point in the preceding theorem is that it is dimension independent, i.e. C does not depend on n. Based on this, we can prove the third main result.

Theorem 3 (Stolz-Thom)**.** *Let $(G_n)_n$ be a sequence of non-abelian compact simple groups and let $\omega \in \beta\mathbb{N}$ be an ultrafilter. The lattice of normal subgroups of the ultraproduct $\prod_\omega G_n$ is distributive. Moreover, $\prod_\omega G_n$ has a unique simple quotient.*

The unique maximal quotients that appear in the study of ultraproducts $\prod_\omega G_n$ are rather interesting groups. It is an open question whether every countable group embeds into the unique simple quotient of $\prod_\omega A_n$ resp. $\prod_\omega PU(n)$. This amounts to the question whether every countable group is sofic resp. hyperlinear, see [5]. These problems are notoriously difficult and our study is motivated by the desire to understand the fine structure of these simple quotients of ultraproducts.

In joint work with John Wilson we studied the isomorphism problem for the unique simple quotients, a problem much harder then the isomorphism problem for the usual ultraproducts. Our main results are:

Theorem 4 (Thom-Wilson)**.** *Let $(G_n)_n$ be a sequence of classical finite simple groups, e.g. for example $G_n = PSL_n(p_n^{k_n})$. Then, the unique simple quotients of $\prod_\omega G_n$ remembers the characteristic $\lim_{n\to\omega} p_n \in \mathbb{N} \cup \{\infty\}$.*

Moreover, the unique simple quotient of $\prod_\omega G_n$ is not isomorphic to the unique simple quotient of $\prod_\omega A_n$.

The precise result is more complicated to state. Currently we are able to distinguish various families appearing in the classification of finite simple groups.

In joint work with Philip Dowerk we studied extensions of Theorem 2 to other groups of functional analytic type. The generalization of Theorem 2 to projective unitary groups of general II_1-factors is work in progress, similarly for groups of unitary operators of the form $1 + k \in B(\ell^2\mathbb{N})$, for some k in a prescribed operator ideal. Let us finish by explaining one more result in this direction. We denote by $\mathcal{C}(\ell^2\mathbb{N})$ the Calkin algebra and by $PU_1(\mathcal{C}(\ell^2\mathbb{N}))$ the maximal connected subgroup of its projective unitary groups – equipped with its natural length function

$$\ell(u) = \inf\{\|\lambda - u\| \mid \lambda \in S^1\}.$$

Again, for $u \in PU_1(\mathcal{C}(\ell^2\mathbb{N}))$ we denote by $[u]$ its conjugacy class.

Theorem 5 (Dowerk-Thom)**.** *There exists a constant $C > 0$, such that for all $u, v \in PU_1(\mathcal{C}(\ell^2\mathbb{N}))$ with $\ell(v) \leq k\ell(u)$ we have $v \in ([u] \cup [u^{-1}])^m$ for all $m \geq Ck^2$.*

The proof relies on Theorem 2. If one drops the assumption that u and v lie in the connected component of the identity in $PU(\mathcal{C}(\ell^2\mathbb{N}))$, then the Fredholm index has to be taken into account.

References

[1] G. Elek and E. Szabo, *Hyperlinearity, essentially free actions and L^2-invariants.* The sofic property, Math. Ann. 332 (2005), no. 2, 421-441.

[2] P. Ellis, S. Hachtman, S. Schneider and S. Thomas, *Ultraproducts of finite alternating groups*, RIMS Kokyu-roku (2008), no. 1619, 1-7.

[3] M. Liebeck and A. Shalev, *Diameters of finite simple groups: sharp bounds and applications*, Ann. of Math. (2) 154 (2001), no. 2, 383-406.

[4] N. Nikolov and D. Segal, *Generators and commutators in finite groups; abstract quotients of compact groups*, Invent. Math. 190 (2012), no. 3, 513-602.

[5] V. Pestov, *Hyperlinear and sofic groups: a brief guide, Bull.* Symbolic Logic 14 (2008), no. 4, 449-480.

[6] A. Stolz and A. Thom, *On the lattice of normal subgroups in ultraproducts of compact simple groups*, to appear in Proceedings of LMS.

Central sequences, dimension, and $\mathcal{Z}$-stability of C^*-algebras

Aaron Tikuisis

(joint work with Leonel Robert)

This talk concerned the potential equivalence of different regularity properties for C^*-algebras. It was shown, by Rørdam [4] and Toms [7], building on work of Villadsen [8], that amenability is not a strong enough condition for classification; certain new regularity properties are being studied, with the aim of developing a new, stronger amenability-type notion. While there are candidates for what this new notion might be, there is no sweeping statement about many definitions being equivalent (as is the case for amenability).

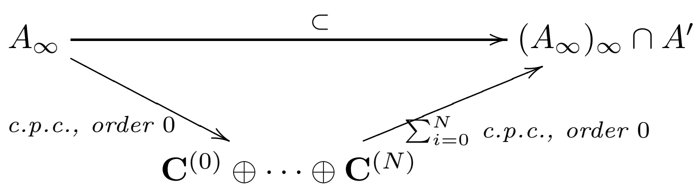
The two candidates that I focused on are finite nuclear dimension [11] and amenability+$\mathcal{Z}$-stability [1, 5]. That these properties are equivalent, for simple, unital, non-type I, separable C^*-algebras, is part of the Toms-Winter conjecture. However, it is worthwhile to conjecture that they are equivalent even without assuming the algebras are simple and unital (as long as we ask that no ideal has type I representations).

Jointly with Wilhelm Winter, I have been involved in showing that $\mathcal{Z}$-stability implies finite nuclear dimension [6], for a special class of C^*-algebras; our result is complemented by a recent result of the same nature by Matui and Sato (but for a very different special class of C^*-algebras) [2].

It is known in considerably more generality that finite nuclear dimension implies $\mathcal{Z}$-stability (although there are known obstructions that we must assume away). Arguments in this direction were pioneered by Winter [9, 10]. As it turns out, these arguments revolve around comparison and divisibility properties of the central sequence algebra (either implicitly or explicitly). (In the nonunital case, one should use Kirchberg's central sequence algebra $\mathbf{F}(A) := (\prod_\omega A \cap A')/\{x \in \prod_\omega A | xA = Ax = 0\}$.) $\mathcal{Z}$-stability, it turns out, is equivalent to M-comparison and N-almost-divisibility of $\mathbf{F}(A)$. The problem of showing that A is $\mathcal{Z}$-stable when it has finite nuclear dimension then comes down, largely, to exploring the extent to which these properties transfer from A to $\mathbf{F}(A)$.

The following lemma, reminiscent of the definition of nuclear dimension, allows certain regularity properties to be pass from A to $\mathbf{F}(A)$ (especially comparison properties):

Lemma. *(Robert-T* [3]*) Let A be a* C**-algebra of finite nuclear dimension n, and let $N := 2n+1$. Then there exist hereditary subalgebras $C^{(0)}, \ldots, C^{(N)}$ of A_∞ and maps making the following diagram commute:*



L. Robert and I have proven that, if A has finite nuclear dimension, then it is $\mathcal{Z}$-stable if and only if $\mathbf{F}(A)$ has two full, orthogonal elements [3]. We have moreover shown that finite nuclear dimension implies $\mathcal{Z}$-stability in the following cases: (i) the C*-algebra has no purely infinite subquotients and its primitive ideal space has a basis of compact open sets, (ii) the C*-algebra has no purely infinite quotients and its primitive ideal space is Hausdorff. The stumbling block to going beyond these cases, at present, is producing full orthogonal elements, first in A, and then centrally, in $\mathbf{F}(A)$.

Slides from the talk may be found on my website:
http://homepages.abdn.ac.uk/a.tikuisis/.

References

[1] Xinhui Jiang and Hongbing Su. On a simple unital projectionless C^*-algebra. *Amer. J. Math.*, 121(2):359–413, 1999.

[2] Hiroki Matui and Yasuhiko Sato. Decomposition rank of UHF-absorbing C^*-algebras, 2013. arXiv preprint math.OA/1303.4371.
[3] Leonel Robert and Aaron Tikuisis. Nuclear dimension and $\mathcal{Z}$-stability of non-simple C*-algebras, 2013. arXiv preprint math.OA/1308.2941.
[4] Mikael Rørdam. A simple C^*-algebra with a finite and an infinite projection. *Acta Math.*, 191(1):109–142, 2003.
[5] Mikael Rørdam and Wilhelm Winter. The Jiang-Su algebra revisited. *J. Reine Angew. Math.*, 642:129–155, 2010.
[6] Aaron Tikuisis and Wilhelm Winter. Decomposition rank of $\mathcal{Z}$-stable C^*-algebras. arXiv preprint math.OA/1210.1386, 2012. arXiv preprint math.OA/1210.1386.
[7] Andrew S. Toms. On the classification problem for nuclear C^*-algebras. *Ann. of Math. (2)*, 167(3):1029–1044, 2008.
[8] Jesper Villadsen. Simple C^*-algebras with perforation. *J. Funct. Anal.*, 154(1):110–116, 1998.
[9] Wilhelm Winter. Decomposition rank and $\mathcal{Z}$-stability. *Invent. Math.*, 179(2):229–301, 2010.
[10] Wilhelm Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras. *Invent. Math.*, 187(2):259–342, 2012.
[11] Wilhelm Winter and Joachim Zacharias. The nuclear dimension of C^*-algebras. *Adv. Math.*, 224(2):461–498, 2010.

Classification of crossed product II_1 factors and group-type subfactors

Stefaan Vaes

We give an overview of several classification results for crossed product II_1 factors, i.e. II_1 factors of the form $M = B \rtimes \Gamma$, given a trace preserving action of a countable group Γ on the tracial von Neumann algebra (B, τ).

The main question is to decide when two such crossed products are isomorphic, in particular in the cases where $B = L^\infty(X, \mu)$ is abelian or $B = R$, the hyperfinite II_1 factor. Closely related to this question, is the classification problem for the group-type subfactors $R^H \subset R \rtimes K$, associated in [BH95] to outer actions of finite groups H and K on the hyperfinite II_1 factor R.

1. Crossed products with $B = L^\infty(X, \mu)$

Trace preserving actions $\Gamma \curvearrowright (B, \tau)$ are exactly given by probability measure preserving (pmp) actions $\Gamma \curvearrowright (X, \mu)$. If this action is moreover free and ergodic, then $M = L^\infty(X) \rtimes \Gamma$ is a II_1 factor and $B = L^\infty(X)$ is a *Cartan subalgebra* of M. This means that $B \subset M$ is a maximal abelian subalgebra whose normalizer $\mathcal{N}_M(B) = \{u \in \mathcal{U}(M) \mid uBu^* = B\}$ generates M as a von Neumann algebra.

Cartan subalgebras play a crucial role in the classification of II_1 factors because of the following result of Singer. Assume that $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ are free ergodic pmp actions. Then the existence of an isomorphism $\pi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ satisfying $\pi(L^\infty(X)) = L^\infty(Y)$ is equivalent with the *orbit equivalence* of the actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$.

So if one can prove that a crossed product $L^\infty(X) \rtimes \Gamma$ has a *unique Cartan subalgebra* up to unitary conjugacy, the classification problem is reduced to the classification of the corresponding orbit equivalence relations.

Theorem 1 ([PV11]). *Let $\mathbb{F}_n \curvearrowright (X, \mu)$ be any free ergodic pmp action of the free group $\mathbb{F}_n$, $2 \leq n \leq \infty$. Then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \mathbb{F}_n$ up to unitary conjugacy.*

Combining Theorem 1 with the results of [Ga99], it follows that crossed products $L^\infty(X) \rtimes \mathbb{F}_n$ and $L^\infty(Y) \rtimes \mathbb{F}_m$ are never isomorphic if $n \neq m$, independently of the choice of free ergodic pmp actions.

Note that Theorem 1 was proven before in [OP07] for *profinite* actions of the free group $\mathbb{F}_n$, i.e. inverse limits of actions on finite sets.

A group satisfying the conclusion of Theorem 1 is called $\mathcal{C}$-rigid. Over the last two years, several families of groups were shown to be $\mathcal{C}$-rigid. This includes, among others, all nonelementary hyperbolic groups (see [PV12]) and arbitrary free product groups $\Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2$, $|\Gamma_2| \geq 3$ (see [Io12]).

In [PV11], Theorem 1 is deduced from a much more general structural theorem for arbitrary tracial crossed products $M = B \rtimes \mathbb{F}_n$. Using Popa's *intertwining-by-bimodules* (see [Po03]) and the concept of *relative amenability* (see [OP07]), the following general dichotomy is proven in [PV11]. If A is a von Neumann subalgebra of $M = B \rtimes \mathbb{F}_n$ and if A is amenable relative to B, then at least one of the following properties holds : A intertwines into B inside M, or the normalizer $\mathcal{N}_M(A)''$ of A inside M stays amenable relative to B. By [PV12], the same dichotomy holds for hyperbolic groups.

2. Crossed products with $B = R$

Consider actions $\Gamma \curvearrowright^\alpha R$ and $\Lambda \curvearrowright^\beta R$ by outer automorphisms. Then the existence of an isomorphism $\pi : R \rtimes \Gamma \to R \rtimes \Lambda$ satisfying $\pi(R) = R$ is equivalent with the *cocycle conjugacy* of the actions, i.e. the existence of an automorphism $\pi \in \mathrm{Aut}(R)$, a group isomorphism $\delta : \Gamma \to \Lambda$ and a family of unitaries $(v_g)_{g \in \Gamma}$ satisfying $\beta_{\delta(g)} = \pi \circ (\mathrm{Ad}\, v_g) \circ \alpha_g \circ \pi^{-1}$ and $v_{gh} = v_g \alpha_g(v_h)$ for all $g, h \in \Gamma$.

The dichotomy result in the previous section then implies the following.

Theorem 2 ([PV12]). *Let Γ, Λ be nonelementary hyperbolic groups and $\Gamma \curvearrowright^\alpha R$, $\Lambda \curvearrowright^\beta R$ actions by outer automorphisms. Then $R \rtimes \Gamma$ is isomorphic with $R \rtimes \Lambda$ if and only if $\Gamma \cong \Lambda$ and the actions α, β are cocycle conjugate.*

In particular, $R \rtimes \mathbb{F}_n$ is never isomorphic to $R \rtimes \mathbb{F}_m$ if $n \neq m$.

3. Group-type subfactors and non outer conjugate actions

Let $\Gamma \curvearrowright R$ be an action by outer automorphisms and assume that Γ is generated by the finite subgroups $H, K < \Gamma$ satisfying $H \cap K = \{e\}$. Following [BH95], these data give rise to the *group-type subfactor* $R^H \subset R \rtimes K$ of finite Jones index ([Jo82a]). These group-type subfactors $R^H \subset R \rtimes K$ are irreducible, have index $|H|\,|K|$ and have a standard invariant that only depends on $H, K < \Gamma$.

In [Po92], Popa proved the fundamental result that every strongly amenable standard invariant arises from precisely one hyperfinite subfactor. This is no longer true in the nonamenable case. In [BNP06], it was shown that there are uncountably many nonisomorphic group-type subfactors that all have index 6 and the same standard invariant. This result is deduced from a theorem in [Po01] saying that infinite property (T) groups Γ admit uncountably many non outer conjugate actions $\Gamma \curvearrowright R$. Taking such a Γ generated by finite subgroups H, K, the corresponding group-type subfactors all have the same standard invariant, but are nonisomorphic.

In [BV13], these results are refined in two ways. We first prove that all nonamenable groups Γ admit uncountably many non outer conjugate actions.

Theorem 3 ([BV13])**.** *Let Γ be any fixed nonamenable group. For every torsion free amenable group Λ, realize the hyperfinite II_1 factor as*

$$R = (M_2(\mathbb{C})^{\Gamma\times\Lambda} \mathbin{\overline{\otimes}} M_2(\mathbb{C})^{\Lambda}) \rtimes \Lambda$$

where Λ acts diagonally by Bernoulli shifts and where the infinite tensor products are taken w.r.t. the tracial state on $M_2(\mathbb{C})$. Define the action $(\alpha_g^{\Lambda})_{g\in\Gamma}$ by Bernoulli shifts. Then $(\alpha_g^{\Lambda_1})_{g\in\Gamma}$ and $(\alpha_g^{\Lambda_2})_{g\in\Gamma}$ are outer conjugate if and only if $\Lambda_1 \cong \Lambda_2$.

Note that in [Jo82b], it was already shown that all nonamenable groups Γ admit at least two non outer conjugate actions. Note also that by [Oc85], the actions $(\alpha_g^{\Lambda})_{g\in\Gamma}$ are all cocycle conjugate if Γ is a fixed *amenable* group and the Λ vary.

For more specific nonamenable groups Γ, we construct in [BV13] even more non outer conjugate actions $(\alpha_g^{\Delta})_{g\in\Gamma}$, labeled by ergodic measure preserving automorphisms Δ of the interval $[0,1]$. We prove that $(\alpha_g^{\Delta})_{g\in\Gamma}$ and $(\alpha_g^{\Delta'})_{g\in\Gamma}$ are outer conjugate actions if and only if Δ and Δ' are conjugate automorphisms, i.e. there exists an $S \in \mathrm{Aut}(X,\mu)$ such that $\Delta' = S \circ \Delta \circ S^{-1}$.

The previous paragraph applies in particular to the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ and the associated group-type subfactors. So we get :

Theorem 4 ([BV13])**.** *To every ergodic measure preserving automorphisms Δ of the interval $[0,1]$ is associated a group-type subfactor $S(\Delta)$ such that $S(\Delta) \cong S(\Delta')$ if and only Δ and Δ' are conjugate automorphisms. All $S(\Delta)$ have index 6 and the same standard invariant, given as the free composition of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.*

References

[BH95] D. Bisch and U. Haagerup, *Composition of subfactors: new examples of infinite depth subfactors*, Ann. Sci. École Norm. Sup. **29** (1996), 329–383.

[BNP06] D. Bisch, R. Nicoara and S. Popa, *Continuous families of hyperfinite subfactors with the same standard invariant*, Internat. J. Math. **18** (2007), 255–267.

[BV13] A. Brothier and S. Vaes, *Families of hyperfinite subfactors with the same standard invariant and prescribed fundamental group*, preprint.

[Ga99] D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), 41–98.

[Io12] A. Ioana, *Cartan subalgebras of amalgamated free product II_1 factors*, preprint.

[Jo82a] V.F.R Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[Jo82b] V.F.R. Jones, *A converse to Ocneanu's theorem*, J. Operator Theory **10** (1983), 61–63.

[Oc85] A. Ocneanu, *Actions of discrete amenable groups on von Neumann algebras*, Lecture Notes in Mathematics **1138**, Springer-Verlag, Berlin, 1985.

[OP07] N. Ozawa and S. Popa, *On a class of II_1 factors with at most one Cartan subalgebra*, Ann. Math. **172** (2010), 713–749.

[Po92] S. Popa, *Classification of amenable subfactors of type II*, Acta Math. **172** (1994), 163–255.
[Po01] S. Popa, *Some rigidity results for non-commutative Bernoulli shifts*, J. Funct. Anal. **230** (2006), 273–328.
[Po03] S. Popa, *Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, I.* Invent. Math. **165** (2006), 369–408.
[PV11] S. Popa and S. Vaes, *Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups*, to appear in Acta Math.
[PV12] S. Popa and S. Vaes, *Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups*, to appear in J. Reine Angew. Math.

Classifying crossed products

Wilhelm Winter

In this talk I introduce a new method to prove classifiability of simple, nuclear C*-algebras via embeddings into classifiable C*-algebras, see [4]. The main technical theorem says that, if A is a separable, simple, unital C*-algebras with finite nuclear dimension and nonempty trace space, and if the identity map approximately factors through TAF (or TAI) C*-algebras at least uniformly on traces, then A is again TAF (or TAI).

As a first application, this yields a somewhat simpler proof of a special case of a beautiful recent result of Matui and Sato (see [2]), namely that separable, simple, unital, monotracial and quasidiagonal C*-algebras with finite nuclear dimension are rationally TAF, hence (in the presence of the UCT) classifiable.

Upon combining the main theorem with a recent result of Szabó (see [3]) and with a result on quasidiagonality of crossed products by Lin (see [1]), transformation group C*-algebras associated to free minimal $\mathbb{Z}^d$-actions on the Cantor set with compact space of ergodic measures are classified by their ordered K-theory. In fact, the respective statement holds for finite dimensional compact metrizable spaces, provided that projections of the crossed products separate tracial states.

References

[1] H. Lin, *AF-embeddings of the crossed products of AH-algebras by finitely generated abelian groups*, IMRP **3** (2008).
[2] H. Matui and Y. Sato, *Decomposition rank of UHF-absorbing* C**-algebras*, arXiv preprint math.OA/1303.4371 (2013).
[3] G. Szabó, *The Rokhlin dimension of topological $\mathbb{Z}^m$-actions*, arXiv preprint math.OA/1308.5418 (2013).
[4] W. Winter, *Classifying crossed product* C**-algebras*, arXiv preprint math.OA/1308.4084 (2013).

Central approximation properties of free quantum groups

Makoto Yamashita
(joint work with Kenny De Commer, Amaury Freslon)

We show that the discrete duals of the free orthogonal quantum groups have the Haagerup property and the completely contractive approximation property. It follows that the reduced C*-algebras and the von Neumann algebras of these quantum groups have the corresponding approximation properties. Analogous results also hold for the free unitary quantum groups and the quantum automorphism groups of finite-dimensional C*-algebras.

Our results generalize previous works by Brannan and Freslon for the unimodular case. These are the first systematic examples of nonamenable and nonunimodular discrete quantum groups with such approximation properties. One feature of our approach is that we do not rely on the rapid decay property, but instead use a holomorphic family of completely bounded multipliers, inspired by the proof of the CCAP of free groups by Pytlik and Szwarc.

We show the above approximation properties using the holomorphic family of multipliers

$$\left(\frac{\mathrm{Tr}(Q_{d/2}^z)}{\mathrm{Tr}(Q_{d/2})}\right)_{d\in\mathbb{N}} \in Z(\ell_\infty \mathbb{F}\mathrm{O}_F) \quad (z \in \mathbb{D}),$$

where the $Q_{d/2}$ is the component of the Woronowicz character in the irreducible representation of spin $d/2$. The first step of the proof is to reduce the problem to the case of $\mathrm{SU}_q(2)$ by means of the monoidal equivalence. For $\mathrm{SU}_q(2)$, the complete positivity of this multiplier for $z \in (-1, 1)$ follows from the work of Voigt on the Baum–Connes conjecture. Then, the structure of $\mathcal{T} \simeq C^*(\alpha) \subset C(\mathrm{SU}_q(2))$ allows us to refine his result and to obtain a holomorphic family of central bounded functionals on $C(\mathrm{SU}_q(2))$ realizing the above multipliers. When the value of z is small enough, it is easy to see that the cubic power of these multipliers are approximated by the finitely supported ones in the completely bounded norm. Finally, the holomorphicity implies that this phenomenon has to occur for any z.

We thus obtain a one parameter family of completely positive central multipliers which are in the closure of finitely supported ones, and at the same time approximating the identity. Such multipliers can be induced to discrete quantum subgroups. Borrowing an argument of Ricard and Xu, we also show that this condition is preserved under taking free products. These considerations show that the free unitary groups and free automorphisms groups also have the same approximation properties as the free orthogonal ones.

References

[1] K. De Commer, A. Freslon, M. Yamashita. CCAP for the discrete quantum groups $\mathbb{F}\mathrm{O}_F$. preprint (2013), available at `arXiv:1306.6064`.

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